Maximum Operations on a Set with Multiples Constraint

1 Problem Statement

Given a positive integer k and a set S of all integers from l to r (inclusive), you can perform the following operation any number of times:

1. Choose a number x from S such that there are at least k multiples of x in S (including x itself).
2. Remove x from S.

The goal is to find the maximum number of such operations that can be performed.

2 Key Observations

2.1 Multiples Count

For any $x \in S$, the number of multiples of x in S is given by:

$$m(x) = \left\lfloor \frac{r}{x} \right\rfloor - \left\lfloor \frac{l-1}{x} \right\rfloor$$

This counts how many numbers in $[l, r]$ are divisible by x.

2.2 Special Case for $x = 1$

The number 1 is a special case because every number in S is a multiple of 1. Thus:

$$m(1) = r - l + 1$$

If $m(1) \geq k$, then 1 can be removed in one operation.

2.3 General Case for $x > 1$

For $x > 1$, the number of multiples $m(x)$ must be at least k. The maximum x that can have $m(x) \geq k$ is $\left\lfloor \frac{r}{k} \right\rfloor$, since for $x > \frac{r}{k}$, $\left\lfloor \frac{r}{x} \right\rfloor < k$.

3 Algorithm

3.1 Steps

1. If $l = r$:
   - If $k \leq 1$, the answer is 1 (the single element can be removed).
   - Otherwise, the answer is 0.
2. If $k = 1$:
   - Every number in S can be removed, so the answer is $r - l + 1$.
3. For $k > 1$:
   - If $l = 1$:
     - Count numbers $x \in [2, \lfloor \frac{r}{k} \rfloor]$.
     - Add 1 if $m(1) = r - l + 1 \geq k$.
   - If $l > 1$:
     - Count numbers $x \in [l, \lfloor \frac{r}{k} \rfloor]$.

4 Complexity Analysis

The algorithm processes each test case in constant time $O(1)$, as it involves simple arithmetic operations and comparisons. The overall complexity for t test cases is $O(t)$.

5 Example Calculations

5.1 Example 1: $l = 3, r = 9, k = 2$

$$m(3) = \left\lfloor \frac{9}{3} \right\rfloor - \left\lfloor \frac{2}{3} \right\rfloor = 3 \geq 2$$

$$m(4) = \left\lfloor \frac{9}{4} \right\rfloor - \left\lfloor \frac{2}{4} \right\rfloor = 2 \geq 2$$

$$m(5) = \left\lfloor \frac{9}{5} \right\rfloor - \left\lfloor \frac{2}{5} \right\rfloor = 1 < 2$$

Numbers $x = 3$ and $x = 4$ can be removed. $m(1) = 7 \geq 2$, but since $l > 1$, 1 is not in S. Thus, the answer is 2.

5.2 Example 2: $l = 1, r = 10^9, k = 2$

$$\left\lfloor \frac{10^9}{2} \right\rfloor = 5 \times 10^8$$

Count numbers $x \in [2, 5 \times 10^8]$: $5 \times 10^8 - 2 + 1 = 5 \times 10^8 - 1$.

$$m(1) = 10^9 \geq 2$$

Add 1 for $x = 1$. Total operations: 5×10^8.