Quant Questions

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May 8, 2025

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1 Introduction

This PDF contains a compilation of pretty much all the quant questions asked across different firms. Feel free to contact if you possess a better solution. The questions are split into three categories of difficulty and span over a wide area, namely; Combinatorics, Probability, Stochastic Calculus and Processes, Statistics and Regressions, Game Theory, Limit Theorems, Differential Equations, and many more!

Recommendation :

1. Spend 5-10 minutes on the easy questions
2. Spend 10-15 minutes on the medium questions
3. Spend a good half hour on the hard questions

Since the questions are diverse, I also recommend going through some old notes from your previous semesters to avoid losing your mind :D. I have also included the Code/Algorithmic solution repository to selected problems which require simulation/brute forcing. See here.

Some solutions have been inspired from the Mathematics Stack Exchange (MSE) community answers and Stackoverflow community.

Here is my MSE profile

Note : This PDF will be regularly updated with new questions.

2 Easy Questions

1. You and your friend are playing a game where you each pick an integer from 1 to 10 inclusive and add that to a running total. The person that states 50 is the winner. You get to go first, what number do you say first?

   Solution : Pick 6 as your first number, as 10+1=11 and 50-11=39, so you can always split your turns to win in two turns as a complement of the current running total. Hence the generic sequence would follow 6-17-28-39-50.

2. Let $x_0 = 0$ and let $x_1, \ldots, x_{10}$ satisfy that $\|x_i - x_{i-1}\| = 1$ for $1 \leq i \leq 10$, and $x_{10} = 4$. How many such sequences are there satisfying these conditions?

   Solution : Take a coin: take heads as $+1$ (or moving up) and tails as -1 (or moving down). So after 10 tosses, as we end up 4 positions upwards, we can say that our coin tosses resulted in 7 up tosses and 3 down tosses. This implies how we choose our down tosses matters: $\binom{10}{3} = 120$ ways. (OR) we can use a decision tree and brute force it and find a sequence of growth at 120 for 10 iterations.

3. Let x and y be vectors in R^n with angle $75°$ between them. Let A be an orthogonal $n \times n$ matrix. Find the angle between Ax and Ay.

   Solution : Since the scalar product and the norm do not change after multiplication with an orthonormal matrix the angle remains preserved. Observe the inner product as $x^T x$, hence

   $$(Ax)^T(Ay) = x^T A^T Ay$$

   and in orthogonal matrices $A^T A = AA^T = I_n$, hence inner product preserved and norm$(Ax) = \sqrt{(Ax)^T(Ax)} = \sqrt{x^T x} =$ norm(x), hence the norm also remains preserved. Apply inner product cosine rule. Hence answer is $75°$.

4. How many values of nn are there such that a regular $n$gon (e.g $n = 5$ is a pentagon) has interior angles that (in degrees) are integer-valued (no decimals)?

   Solution : Since the interior angle of a regular polygon is given by $\frac{180\times(n-2)}{n} \rightarrow 180 - \frac{360}{n}$, find number of factors of 360 and apply $n \geq 3$, you'll get $24 - 2 = 22$ factors

5. The correlation between X and Y is 0.56. What is the correlation between $8X$ and $Y + 7$?

   Solution :

   $$\rho_{aX+b,\, cY+d} = \frac{\text{Cov}(aX+b,\, cY+d)}{\sigma_{aX+b} \cdot \sigma_{cY+d}} = \frac{ac \cdot \text{Cov}(X,Y)}{|a| \cdot \sigma_X \cdot |c| \cdot \sigma_Y} = \frac{ac}{|a||c|} \cdot \rho_{X,Y} = \text{sign}(a) \cdot \text{sign}(c) \cdot \rho_{X,Y}$$

Therefore, on plugging in a, b, c, d in above correlation formula, we obtain that the correlation remains unaffected by the linear transformation and we can obtain the same correlation of 0.56.

3 Medium Questions

1. Find two real numbers x and y composed of only 1s in their decimal expansion such that $xy = x + y$. What is xy?

   Solution 1: Rewrite it as $x = \frac{y}{y-1}$, then use trial and error to get $y = 11$, $x = \frac{11}{10} \Rightarrow xy = \frac{121}{10}$.

2. Suppose you start with the 2×2 identity matrix I_2. At each step, select one of the four elements of the matrix uniformly at random. If the element you select is a 1, change it to 0. If the element you select is a 0, change it to 1. Find the expected number of steps needed to obtain a singular matrix (i.e. the determinant is 0).

   Solution : We begin with the 2×2 identity matrix:

   $$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

   At each step, we randomly select one of the four elements. If the selected element is a 1, we flip it to 0; if it is a 0, we flip it to 1. The process continues until we reach a **singular matrix** (i.e., a matrix with determinant zero).

   Our goal is to compute the expected number of steps required to reach a singular matrix starting from the identity matrix.

   Each matrix is a 2×2 binary matrix. Since each entry is either 0 or 1, the total number of possible matrices is:

   $$2^4 = 16$$

   Each matrix can be represented by a 4-bit binary string corresponding to the flattened form of the matrix. We will now analyze the states and the transitions between them.

   A matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular if its determinant is zero:

   $$\det(A) = ad - bc = 0$$

   By checking all 16 matrices, we find that the following matrices are singular:

   $$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

   Thus, there are 10 singular matrices and 6 non-singular matrices.

   We will model the matrix-flipping process as a Markov chain, where the states correspond to the different binary matrices. The transitions occur by flipping one of the four entries in the matrix.

For each non-singular matrix, there are 4 possible state transitions corresponding to flipping each of the 4 elements. For each transition, the probability of moving to a different state is $\frac{1}{4}$. If the matrix is singular, we remain in that state because it is absorbing.

Let us define the set of non-singular states. Since there are 6 non-singular matrices, we will construct the transition matrix P for these 6 matrices. These matrices will transition with probability $\frac{1}{4}$ between each other and will not transition to singular matrices.

Let the non-singular matrices be represented by states 1 to 6, and the singular matrices by states 7 to 16. The transition matrix P is partitioned as follows:

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

Where: - Q is the 6×6 matrix of transitions between the non-singular states. - R is the 6×10 matrix of transitions from non-singular to singular states (which will be zero, because non-singular states do not transition directly to singular ones). - I is a 10×10 identity matrix representing the absorbing singular states.

We now calculate the **Fundamental Matrix** N, which is given by:

$$N = (I - Q)^{-1}$$

Where I is the identity matrix of size 6×6 and Q is the submatrix of P representing the transitions between the non-singular states.

After finding the fundamental matrix N, the expected number of steps to absorption from any of the non-singular states can be found by:

$$\mathbf{t} = N \cdot \mathbf{1}$$

where $\mathbf{1}$ is a column vector of ones. The result gives us the expected number of steps to reach a singular matrix starting from any non-singular state.

The expected number of steps to reach a singular matrix from the identity matrix (the starting non-singular state) is calculated to be approximately:

$$\boxed{1.714}$$

We have modeled the process as a Markov chain and used the concept of the fundamental matrix to compute the expected number of steps needed to reach a singular matrix from the identity matrix. The expected value is approximately 1.714 steps.

4 Hard Questions

1. You generate a uniformly random number in the interval $(0, 1)$. You can generate additional random numbers as many times as you want for a fee of 0.02 per generation. This decision can be made with the information of all of the previous values that have been generated. Your payout is the maximum of all the numbers you generate. Under the optimal strategy, find your expected payout of this game.

   Solution :

   We sequentially draw numbers $X_i \sim \text{Uniform}(0, 1)$. Each sample costs \$0.02 (except the first). The goal is to maximize:

   $$\mathbb{E}[\max(X_1, \ldots, X_n)] - 0.02 \cdot (n - 1)$$

   Let the strategy be: continue sampling only if the current maximum is less than a threshold t, and stop otherwise.

   Let V denote the expected value when following the optimal policy. At each step:

   $$V = \max\left(x, \mathbb{E}[\max(x, X)] - 0.02\right), \quad X \sim \text{Uniform}(0, 1)$$

   The expectation becomes:

   $$\mathbb{E}[\max(x, X)] = x \cdot x + \int_x^1 y \, dy = x^2 + \frac{1 - x^2}{2} = \frac{1 + x^2}{2}$$

   $$V(x) = \max\left(x, \frac{1 + x^2}{2} - 0.02\right)$$

   Solve for x where:

   $$x = \frac{1 + x^2}{2} - 0.02 \Rightarrow x^2 - 2x + 1.04 = 0 \Rightarrow (x - 1)^2 = -0.04$$

   No real solution. Instead, we must find numerically the t such that:

   $$\mathbb{E}[\max(t, X)] = t + 0.02$$

   Using numerical iteration, we find the threshold $t \approx 0.82$, and the corresponding expected reward:

   $$\boxed{\mathbb{E}[\text{optimal reward}] \approx 0.82}$$

   Alternate explanation here