COM S 229, Spring 2015
Programming Project 0
Uniquely Listing the Rational Numbers

Georg Cantor is most famous for his clever proofs of the cardinalities of infinite sets. Let's count the rationals: Start with $\frac{0}{1}$, increment the numerator, and count each fraction, so we have $\frac{0}{1}$ is 1, $\frac{1}{1}$ is 2, $\frac{2}{1}$ is 3, $\frac{3}{1}$ is 4, and so on. Using this algorithm, we'll never reach $\frac{1}{2}$, so one of two things is true: either the rationals are not countable ($|\mathbb{Q}| > \aleph_0$), or we've devised a bad algorithm. In fact, Cantor showed that the rationals are countable, so we have done it wrong. If instead of incrementing the numerator, we increment the denominator, we'll run into the same issue. Obviously, any algorithm to count the rational numbers is going to have to deal with both the numerator and the denominator together. It's a problem that, while possible, you probably won't work out a solution in your head.

Cantor solved it graphically, drawing (part of) an infinite matrix containing every rational number and defining a path that will eventually reach any element in the matrix:

$$
\begin{array}{c|cccccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\
\hline
1 & \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \cdots \\
2 & \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \frac{2}{6} & \frac{2}{7} & \frac{2}{8} & \cdots \\
3 & \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \frac{3}{6} & \frac{3}{7} & \frac{3}{8} & \cdots \\
4 & \frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \frac{4}{6} & \frac{4}{7} & \frac{4}{8} & \cdots \\
5 & \frac{5}{1} & \frac{5}{2} & \frac{5}{3} & \frac{5}{4} & \frac{5}{5} & \frac{5}{6} & \frac{5}{7} & \frac{5}{8} & \cdots \\
6 & \frac{6}{1} & \frac{6}{2} & \frac{6}{3} & \frac{6}{4} & \frac{6}{5} & \frac{6}{6} & \frac{6}{7} & \frac{6}{8} & \cdots \\
7 & \frac{7}{1} & \frac{7}{2} & \frac{7}{3} & \frac{7}{4} & \frac{7}{5} & \frac{7}{6} & \frac{7}{7} & \frac{7}{8} & \cdots \\
8 & \frac{8}{1} & \frac{8}{2} & \frac{8}{3} & \frac{8}{4} & \frac{8}{5} & \frac{8}{6} & \frac{8}{7} & \frac{8}{8} & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

The matrix is indexed by the counting numbers, with a row for every numerator and a column for every denominator. As drawn, it doesn't include zero; fractions with a numerator of zero are rational, so you could use the integers instead of the counting numbers for numerators, or you could just add 1 to the count, since $\frac{0}{n} = 0$, which is what I choose to do and why I left zero out of the matrix.

Take some time to study this matrix and the curve and convince yourself that the matrix contains every rational number (save zero), and that the curve will reach them all. Since the curve moves discretely (i.e., one cell at a time), and it hits every cell, the rationals are countable.

Note the main diagonal, which contains $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots$. You'll also notice that the matrix contains $\frac{2}{3}$, $\frac{4}{6}$, and $\frac{12}{18}$. Obviously, each number is not uniquely counted. If we count down the main diagonal, we'll find that the matrix has $\aleph_0$ ones, and yet the whole matrix has $\aleph_0$ elements!

Cantor's proof is elegant and simple enough to introduce the concepts to children with a sense of wonder about mathematics, but the lack of uniqueness is unsatisfying. Cantor proved the countability of the rationals at the end of the 19^{th} century, and it wasn't until 1999 that Neil Calkin and Herbert Wilf found an algorithm which lists the rationals uniquely, which they presented in their paper *Recounting the rationals* https://www.math.upenn.edu/˜wilf/website/recounting.pdf.

You are to write two programs in C. Using the method and ordering described by Calkin and Wilf, the first program will iteratively calculate and list the first 100,000,000 numbers, one per line, of the hyperbinary sequence (the sequence defined in bullet 1 on the first page of Calkin and Wilf's paper). The second program will take an integer command line parameter, n, then recursively calculate and print the n^{th} rational number as defined by Calkin and Wilf.

Your solution should be in C and (like all assignments for this course) should meet the requirements specified in the syllabus (i.e., build with a makefile, be submitted in a tarball, include a readme file, etc.).