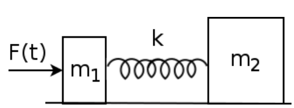
with the system comprising the other particles. As you will obtain a transcendental equation, an approximate value is fine. How much time will elapse between the two collisions for this particular value of M?

16. *Spring-Mass with Friction***

A horizontal spring-mass system is placed on a rough table, with one end of the spring attached to a vertical wall. The massless spring has spring constant k and rest length l_0 while the load has mass m and static and kinetic friction coefficients μ relative to the table. Initially, the mass m is placed right next to the wall such that the length of the spring is virtually zero. Subsequently, m is released such that it begins to undergo a one-dimensional oscillatory motion. If we define the origin to be at the fixed end of the spring and the x-axis to be along the direction of motion of m, sketch the graph of the x-coordinate x of the mass against the elapsed time t. Thus, determine the total number of oscillation cycles that m completes before coming to a stop.

17. *Stabilizer***

A heavy bob is often used to stabilize buildings in the event of earthquakes. Let us consider a related problem. Two masses m_1 and m_2 are stationary on a horizontal, frictionless plane and are connected by a spring of spring constant k. Suppose a force $F(t) = f\cos\omega t$ is exerted on m_1, in the direction of the line joining the two masses. Determine the value of k for which the particular solution to the equation of motion of m_1 yields an oscillation of zero amplitude. There is no damping.



18. *Masses on Hoop****

Three masses, one with mass m, and two with mass $2m$ are constrained to move along a massive circular hoop of radius R. Three springs that are wrapped around the hoop connect adjacent masses. The springs between mass m and the two masses $2m$ have spring constant $2k$ while the spring between the two masses of mass $2m$ has spring constant k. Find the normal modes of oscillation and the displacements of the masses from their equilibrium positions as functions of time under arbitrary initial conditions.

Solutions

1. Pendulum Clock*

The frequency of a pendulum, under an effective gravity g_{eff}, is $f = \frac{1}{2\pi}\sqrt{\frac{g_{eff}}{l}}$. When the lift is accelerating upwards, the pendulum experiences an inertial force ma downwards where m is its mass and hence lives in a world with effective gravity $g + a$. Similarly, when the lift is decelerating, the effective gravity is $g - a$. Since $f \propto \sqrt{g_{eff}}$, the elapsed time that the pendulum clock would have recorded during this experiment is

$$t \cdot \sqrt{\frac{g+a}{g}} + t \cdot \sqrt{\frac{g-a}{g}} \neq 2t,$$

where $2t$ is the actual time elapsed. Therefore, the clock is no longer accurate.

2. Dropping a Mass*

The total mechanical energy is initially

$$E = \frac{1}{2}kA^2.$$

When the mass m is at a state with displacement $\frac{A}{2}$, its potential and kinetic energies are respectively

$$U = \frac{1}{8}kA^2,$$
$$T = E - U = \frac{3}{8}kA^2.$$

Thus, its speed at this instant is

$$v = \sqrt{\frac{3kA^2}{4m}}.$$

The final speed of the two masses after the collision is given by the conservation of momentum to be

$$v' = \sqrt{\frac{3kA^2}{16m}}.$$

Thus, the final kinetic energy is

$$T' = \frac{1}{2} \cdot 2mv'^2 = \frac{3kA^2}{16}.$$

The total mechanical energy afterwards is

$$E = U + T' = \frac{5kA^2}{16}.$$

At the amplitude A', the kinetic energy of the mass is zero. Thus,

$$\frac{1}{2}kA'^2 = E$$

$$A' = \sqrt{\frac{5}{8}}A.$$

3. Kinematic Quantities*

$$|v_1| = \omega\sqrt{A^2 - x_1^2},$$

$$|v_2| = \omega\sqrt{A^2 - x_2^2}.$$

Dividing the first equation by the second and squaring, we get

$$\frac{v_1^2}{v_2^2} = \frac{A^2 - x_1^2}{A^2 - x_2^2}.$$

Solving,

$$A = \sqrt{\frac{v_2^2 x_1^2 - v_1^2 x_2^2}{v_2^2 - v_1^2}}.$$

Substituting this expression for A into either of the first two equations,

$$\omega = \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}.$$

4. Physical Pendulum*

Applying Eq. (10.6), the period of a physical pendulum is

$$T = 2\pi\sqrt{\frac{I_{pivot}}{Mr_{CM}g}},$$

where I_{pivot} is the moment of inertia of the pendulum about the pivot, M is the total mass of the physical pendulum and r_{CM} is the distance between the pivot and the center of mass of the pendulum. Let the distances between the two pivots and the center of mass be r and $l-r$ respectively. Then,

$$2\pi\sqrt{\frac{I_{CM}+Mr^2}{Mrg}}=T,$$

$$2\pi\sqrt{\frac{I_{CM}+M(l-r)^2}{M(l-r)g}}=T,$$

where I_{CM} is the moment of inertia of the pendulum about its center of mass. Eliminating I_{CM},

$$\frac{T^2Mrg}{4\pi^2}-Mr^2=\frac{T^2M(l-r)g}{4\pi^2}-M(l-r)^2$$

$$\implies g=\frac{4\pi^2 l}{T^2}.$$

5. Cavendish Experiment*

Due to symmetry, the center of the rod is the instantaneous center of rotation. The moment of inertia of the rod and the small balls about the center is $I=2\cdot m\frac{L^2}{4}=\frac{mL^2}{2}$. When the rod has rotated for an angle θ, the restoring torque is $-\kappa\theta$. Therefore, the equation of motion of the rod is

$$I\ddot{\theta}=-\kappa\theta$$

$$\ddot{\theta}=-\frac{2\kappa}{mL^2}\theta,$$

which indicates a simple harmonic motion of period

$$T=2\pi\sqrt{\frac{mL^2}{2\kappa}}.$$

For the second part, the torque produced by the gravitational force of the large balls balances the torsion torque at equilibrium. Note that we only consider the gravitational force on a small ball due to the nearer large ball. The other gravitational force is comparatively negligible, since $r \ll L$. Balancing torques,

$$\frac{GMm}{r^2}\cdot L=\kappa\theta$$

$$G=\frac{\kappa\theta r^2}{MmL}=\frac{2\pi^2r^2L\theta}{MT^2}.$$

Let $\theta_M=\theta_0+\frac{2r}{L}$ denote the angular position of the large balls, relative to the original position of the rod. θ_0 is the angle θ in the previous section (we reserve the variable θ for the following definition). When the rod has experienced an angular displacement θ ($\theta_M - \theta \ll \theta_M$), the separation between the centers of neighboring small and large balls is approximately $\frac{L}{2}(\theta_M - \theta)$. Therefore, the equation of motion of the rod is

$$I\ddot{\theta} = 2 \cdot \frac{GMm}{\frac{L^2}{4}(\theta_M - \theta)^2} \cdot \frac{L}{2} - \kappa\theta$$
$$\ddot{\theta} = \frac{8GM}{L^3(\theta_M - \theta)^2} - \frac{2\kappa}{mL^2}\theta.$$

When $\theta = \theta_0$, the system is in equilibrium such that

$$\frac{8GM}{L^3 \cdot \frac{4r^2}{L^2}} - \frac{2\kappa}{mL^2}\theta_0 = 0.$$

Now, suppose $\theta = \theta_0 + \varepsilon$. The equation of motion becomes

$$\ddot{\varepsilon} = \frac{8GM}{L^3\left(\frac{2r}{L} - \varepsilon\right)^2} - \frac{2\kappa}{mL^2}(\theta_0 + \varepsilon)$$
$$\ddot{\varepsilon} = \frac{2GM}{r^2L\left(1 - \frac{L\varepsilon}{2r}\right)^2} - \frac{2\kappa}{mL^2}(\theta_0 + \varepsilon).$$

Performing a first-order binomial expansion,

$$\begin{aligned}\ddot{\varepsilon} &= \frac{2GM}{r^2L}\left(1 + \frac{L\varepsilon}{r}\right) - \frac{2\kappa}{mL^2}(\theta_0 + \varepsilon) \\ &= -\left(\frac{2\kappa}{mL^2} - \frac{2GM}{r^3}\right)\varepsilon = -\left(\frac{2\kappa}{mL^2} - \frac{2\kappa\theta_0}{mLr}\right)\varepsilon,\end{aligned}$$

where we have performed the cancellation of some terms based on the previous equilibrium equation and substituted $\frac{GM}{r^2} = \frac{\kappa\theta_0}{mL}$. The above indicates a simple harmonic motion of angular frequency

$$\omega = \sqrt{\frac{2\kappa(r - \theta_0 L)}{mL^2 r}}$$

if $r > \theta_0 L$. Otherwise, the rod will not exhibit simple harmonic motion.

6. Particle in Potential*

Since a conservative force is the negative potential energy gradient, the equilibrium positions correspond to the locations where $U'(x) = 0$.

$$U'(x) = U_0(-2ax + 4bx^3) = U_0 x(4bx^2 - 2a).$$

The equilibrium positions are thus $x = 0$, $x = -\sqrt{\frac{a}{2b}}$ and $x = \sqrt{\frac{a}{2b}}$. Computing the second derivative,

$$U''(x) = U_0(12bx^2 - 2a).$$

Since $U''(0) = -2aU_0 < 0$, the equilibrium position $x = 0$ corresponds to a potential energy maximum which indicates that a slight deviation tends to be amplified by the conservative force (which is directed towards lower values of potential energy). Hence, $x = 0$ is unstable but on the other hand, $U''(\pm\sqrt{\frac{a}{2b}}) = 4aU_0 > 0$ which indicates that $x = -\sqrt{\frac{a}{2b}}$ and $x = \sqrt{\frac{a}{2b}}$ are stable equilibria. The angular frequency of oscillations about these positions is

$$\omega = \sqrt{\frac{U''\left(\pm\sqrt{\frac{a}{2b}}\right)}{m}} = \sqrt{\frac{4aU_0}{m}}.$$

7. V-Shape Rails*

Suppose that both masses are shifted along the rails by a displacement x from their equilibrium positions. The spring would have stretched or contracted by an additional $2x\sin\theta$, beyond its length when the two masses are at equilibrium. Therefore, the equation of motion of one mass at this juncture is

$$m\ddot{x} = -2k\sin\theta x \cdot \sin\theta,$$

where we multiply by $\sin\theta$ to obtain the component of force along the rail that it lies along. Since

$$\ddot{x} = -\frac{2k\sin^2\theta}{m}x,$$

the angular frequency of small oscillations is

$$\omega = \sqrt{\frac{2k\sin^2\theta}{m}}.$$

An alternative method for this question would start with the potential energy of the system when each mass is at a distance x from the point of connection of the rails.

$$U(x) = \frac{1}{2}k(2x\sin\theta - l_0)^2,$$

where l_0 is the rest length of the spring. The first derivative is

$$U'(x) = k(2x\sin\theta - l_0)\cdot 2\sin\theta,$$

which shows that the equilibrium x-coordinate is $\frac{l_0}{2\sin\theta}$. The second derivative of this is

$$U''(x) = 4k\sin^2\theta.$$

Therefore, the angular frequency of small oscillations about the equilibrium position is

$$\omega = \sqrt{\frac{U''\left(\frac{l_0}{2\sin\theta}\right)}{2m}} = \sqrt{\frac{2k\sin^2\theta}{m}}.$$

Note that we have to use $2m$ instead of m here as $U(x)$ is the potential energy of the entire system.

8. Floating Cylinder*

The volume of the cylinder submerged in water is the area of the sector (multiplied by l) minus the area of the isosceles triangle, with sides r that subtend angle θ (multiplied by l).

$$V = \frac{\theta}{2}r^2 l - \frac{1}{2}r^2 l\sin\theta.$$

The cylinder is in equilibrium when the upthrust balances its weight.

$$\rho\left(\frac{\theta}{2}r^2 l - \frac{1}{2}r^2 l\sin\theta\right)g = \pi r^2 l d g$$

$$\theta - \sin\theta = \frac{2\pi d}{\rho}.$$

When the cylinder is displaced by a vertical small distance ε from its equilibrium position, the net force that it experiences (which opposes its deviation) is equal to ρg multiplied by the change in the submerged volume of the cylinder. The latter is equal to the length of the horizontal chord on the cross-section of the cylinder along the water level, $2r\sin\frac{\theta}{2}$, multiplied by ε

(the vertical displacement) and l. Therefore, the equation of motion of the cylinder is

$$\pi r^2 l d\ddot{\varepsilon} = -2r\sin\frac{\theta}{2}\cdot l\cdot\rho g\varepsilon$$

$$\ddot{\varepsilon} = -\frac{2\rho g\sin\frac{\theta}{2}}{\pi r d}\varepsilon,$$

which indicates a simple harmonic motion of angular frequency

$$\omega = \sqrt{\frac{2\rho g\sin\frac{\theta}{2}}{\pi r d}}.$$

9. Two Circles*

The moment of inertia of the smaller circle about the center of the larger circle is $\frac{1}{2}\sigma\pi r^4 + \sigma\pi r^2 R^2$ by the parallel axis theorem. Thus, the total moment of inertia of the system about the pivot is

$$I = \frac{1}{2}\sigma\pi R^4 + \frac{1}{2}\sigma\pi r^4 + \sigma\pi r^2 R^2 = \frac{1}{2}\sigma\pi(r^2+R^2)^2.$$

The net external torque on this system is that due to the weight of the smaller circle.

$$\tau = -\sigma\pi r^2 gR\sin\theta$$

$$\implies I\ddot{\theta} = -\sigma\pi r^2 gR\sin\theta.$$

Using the small angle approximation $\sin\theta\approx\theta$,

$$\ddot{\theta} = -\frac{2gr^2R}{(r^2+R^2)^2}\theta.$$

The angular frequency of small oscillations about $\theta = 0$ is

$$\omega = \frac{\sqrt{2gr^2R}}{r^2+R^2}.$$

10. Two Spheres**

Let θ be the angle that the line joining the center of the spheres makes with the vertical, and let it be positive in the anti-clockwise direction. Let ϕ and ψ be the angles that the ball and large sphere have rotated about their centers respectively, also positive anti-clockwise. Since $(R-r)\dot{\theta}$ is the velocity of the center of the ball, $(R-r)\dot{\theta}+r\dot{\phi}$ is the velocity of the point on the ball that is in contact with the large sphere. Therefore, the non-slip condition is

$$(R-r)\dot{\theta}+r\dot{\phi}=R\dot{\psi}$$
$$\implies (R-r)\ddot{\theta}+r\ddot{\phi}=R\ddot{\psi}.$$

Let f be the friction force on the ball in the anti-clockwise direction due to the large sphere. Applying Newton's second law to the ball,

$$f-mg\sin\theta=m(R-r)\ddot{\theta}.$$

Applying $\tau=I\alpha$ to the spheres about their respective centers,

$$f=\frac{2}{5}mr\ddot{\phi},$$
$$-f=\frac{2}{5}MR\ddot{\psi}.$$

Then,

$$r\ddot{\phi}=-\frac{M}{m}R\ddot{\psi}.$$

Substituting this into the non-slip condition,

$$R\ddot{\psi}=\frac{m(R-r)}{m+M}\ddot{\theta},$$
$$f=-\frac{2mM(R-r)}{5(m+M)}\ddot{\theta}.$$

Substituting this into the equation obtained from Newton's second law,

$$\frac{m(R-r)(5m+7M)}{5(m+M)}\ddot{\theta}=-mg\sin\theta.$$

Using the small angle approximation $\sin\theta\approx\theta$,

$$\ddot{\theta}=-\frac{5(m+M)g}{(R-r)(5m+7M)}\theta.$$

The angular frequency of small oscillations about $\theta=0$ is thus

$$\omega=\sqrt{\frac{5(m+M)g}{(R-r)(5m+7M)}}.$$

11. Masses and String**

Let T be the tension in the string and let r be the radial coordinate of m with respect to the hole. Then, the equations of motion of m and M are

$$-T = m(\ddot{r} - r\dot{\theta}^2),$$

$$T - Mg = M\ddot{r},$$

by the conservation of string.

$$\implies -Mg = (m+M)\ddot{r} - mr\dot{\theta}^2.$$

Observe that the angular momentum of mass m about the hole is conserved as it only experiences a radial force. Then,

$$L = mr^2\dot{\theta}$$

for some constant L.

$$-Mg = (m+M)\ddot{r} - \frac{L^2}{mr^3}.$$

When $r = r_0$, $\ddot{r} = 0$, hence

$$Mg = \frac{L^2}{mr_0^3}.$$

This will be useful in canceling terms later. Next, express the radial coordinate r as $r_0 + \varepsilon$ where ε is a slight displacement from the equilibrium position. Then,

$$-Mg = (m+M)\ddot{\varepsilon} - \frac{L^2}{mr_0^3\left(1 - \frac{\varepsilon}{r_0}\right)^3}.$$

Performing a binomial expansion,

$$-Mg = (m+M)\ddot{\varepsilon} - \frac{L^2}{mr_0^3}\left(1 - \frac{3\varepsilon}{r_0}\right).$$

Substituting $\frac{L^2}{mr_0^3} = Mg$,

$$\ddot{\varepsilon} = -\frac{3Mg}{(m+M)r_0}\varepsilon.$$

The angular frequency of small oscillations is thus

$$\omega = \sqrt{\frac{3Mg}{(m+M)r_0}}.$$