we can simply write the general solution as the real component of the linear combination of solutions with positive, imaginary exponents (i.e. $\boldsymbol{u}e^{i\omega t}$) as $\boldsymbol{X}$ must be real at all times. At this juncture, we proceed with a short linear algebra interlude.

Eigenvectors

Let $\boldsymbol{A}$ be a $n \times n$ matrix. A non-null vector $\boldsymbol{u}$ is known as an eigenvector of $\boldsymbol{A}$ if

$$\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}$$

for some scalar λ. λ is termed an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{u}$ is known as an eigenvector associated with the eigenvalue λ. The eigenvalues can be determined as follows.

$$\boldsymbol{A}\boldsymbol{u} - \lambda \boldsymbol{u} = \boldsymbol{0}$$

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{u} = \boldsymbol{0},$$

where $\boldsymbol{0}$ is a vector of zeroes with n rows and $\boldsymbol{I}$ is the identity matrix of order n. The identity matrix of order n is a $n \times n$ square[4] matrix whose top-left to bottom-right diagonal entries are one — all other entries are zero. As its nomenclature implies, multiplying a square matrix $\boldsymbol{X}$ by the identity matrix of the same dimensions simply returns $\boldsymbol{X}$ ($\boldsymbol{X}\boldsymbol{I} = \boldsymbol{X}$ and $\boldsymbol{I}\boldsymbol{X} = \boldsymbol{X}$). Therefore, we have simply expressed $\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u}$ in writing the second equation.

Now, consider the following definition: an inverse $\boldsymbol{X}^{-1}$ of a square matrix $\boldsymbol{X}$ is defined as a matrix such that the matrix multiplications $\boldsymbol{X}\boldsymbol{X}^{-1} = \boldsymbol{I}$ and $\boldsymbol{X}^{-1}\boldsymbol{X} = \boldsymbol{I}$ (i.e. they yield the identity matrix of the same dimensions as $\boldsymbol{X}$). Suppose that an inverse of $(\boldsymbol{A} - \lambda \boldsymbol{I})$ exists. Then by multiplying this inverse to both sides of the previous equation, we obtain the trivial solution

$$\boldsymbol{u} = \boldsymbol{0},$$

which is contrary to what we want, as an eigenvector is not a null vector by definition, and the null case is not physically meaningful in the case of coupled oscillations. Thus, in order for non-trivial solutions of $\boldsymbol{u}$ to exist, $(\boldsymbol{A} - \lambda \boldsymbol{I})$ must be non-invertible or singular. In linear algebra, this is equivalent to saying that the determinant of this term is zero.

[4] A square matrix is simply one with an identical number of rows and columns.

Determinants

The determinant of a $n \times n$ square matrix, $\boldsymbol{X}$, is a quantity that can be computed recursively as follows. Given that

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix},$$

$$\det(\boldsymbol{X}) = \begin{cases} x_{11} \quad \text{if} \quad n = 1 \\ x_{i1}Y_{i1} + x_{i2}Y_{i2} + \cdots + x_{in}Y_{in} \quad \text{for} \quad n \geq 2, \end{cases}$$

where i refers to that particular row. Y_{ij} is defined as

$$Y_{ij} = (-1)^{i+j}\det(\boldsymbol{Z}_{ij}).$$

$\boldsymbol{Z}_{ij}$ is the matrix obtained by removing the ith row and jth column from $\boldsymbol{X}$, i.e.

$$\boldsymbol{Z}_{ij} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1(j-1)} & x_{1(j+1)} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2(j-1)} & x_{2(j+1)} & \dots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} & x_{(i-1)2} & \dots & x_{(i-1)(j-1)} & x_{(i-1)(j+1)} & \dots & x_{(i-1)n} \\ x_{(i+1)1} & x_{(i+1)2} & \dots & x_{(i+1)(j-1)} & x_{(i+1)(j+1)} & \dots & x_{(i+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{n(j-1)} & x_{n(j+1)} & \dots & x_{nn} \end{pmatrix}.$$

The recursive definition of $\det(\boldsymbol{X})$ for $n \geq 2$ above is known as the co-factor expansion along row i, where i is an arbitrary integer $1 \leq i \leq n$. In fact, the determinant can also be calculated via a co-factor expansion along any column j.

$$\det(\boldsymbol{X}) = \sum_{i=1}^{N} x_{ij}Y_{ij} \quad \text{for} \quad n \geq 2.$$

Let us now evaluate the determinants of two concrete examples to clarify this esoteric definition. The most common form of matrices would be the

2×2 matrix. If we let $\boldsymbol{X}$ be

$$\boldsymbol{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then by performing a co-factor expansion along the first row,

$$\det(\boldsymbol{X}) = aY_{11} + bY_{12} = a \cdot (-1)^{1+1}|d| + b(-1)^{1+2}|c| = ad - bc,$$

where the vertical lines denote taking the determinant of the matrix they enclose.

Problem: Determine the determinant of the following matrix

$$\boldsymbol{X} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Performing a co-factor expansion along the first row,

$$\begin{aligned} \det(\boldsymbol{X}) &= 1 \cdot \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \\ &= (3 \cdot 2 - 1 \cdot 0) - 2 \cdot (1 \cdot 2 - 0 \cdot 0) \\ &= 2. \end{aligned}$$

Incidentally, there is an efficient memorization scheme for the determinant of a 3×3 matrix known as Sarrus' rule.

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} \begin{aligned} &= x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} \\ &\quad -x_{31}x_{22}x_{13} - x_{32}x_{23}x_{11} - x_{33}x_{21}x_{12} \end{aligned}$$

This sum can be visualized by replicating the first two columns on the right of the original block of numbers and taking the sum of the products along the bolded diagonals, minus the sum of the products along the dashed diagonals in Fig 10.11.

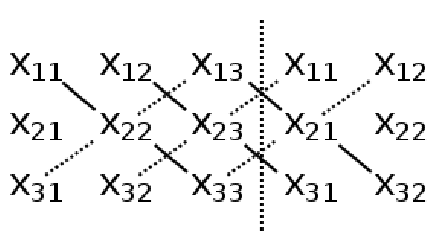


Figure 10.11: Sarrus' rule

Evaluating Eigenvalues and Normal Frequencies

Returning to our main topic of eigenvectors, the eigenvalues associated with a matrix $\boldsymbol{A}$ can be computed by setting the determinant of $\boldsymbol{A} - \lambda \boldsymbol{I}$ to be zero, so

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0.$$

This will generate a nth order polynomial for λ which has n roots by the fundamental theorem of algebra. In the case of coupled oscillators, by observing Eq. (10.13), we have

$$\left(\boldsymbol{A} + \omega^2 \boldsymbol{I}\right) \boldsymbol{u} = \boldsymbol{0}, \tag{10.14}$$

which has non-trivial solutions only if

$$\det\left(\boldsymbol{A} + \omega^2 \boldsymbol{I}\right) = 0.$$

That is, the squared negative of the normal frequencies are the eigenvalues of the matrix $\boldsymbol{A}$. Let us consider the specific spring-mass oscillators in the previous section. The equations of motion produced are

$$\begin{aligned} \ddot{x}_1 &= -\frac{2k}{m} x_1 + \frac{k}{m} x_2, \\ \ddot{x}_2 &= \frac{k}{m} x_1 - \frac{2k}{m} x_2. \end{aligned}$$

Therefore, the matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{pmatrix}.$$

Then, we require

$$\begin{gathered} \begin{vmatrix} -\frac{2k}{m} + \omega^2 & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} + \omega^2 \end{vmatrix} = 0 \\ \left(-\frac{2k}{m} + \omega^2\right)^2 - \left(\frac{k}{m}\right)^2 = \left(\omega^2 - \frac{3k}{m}\right)\left(\omega^2 - \frac{k}{m}\right) = 0 \\ \omega^2 = \frac{k}{m} \quad \text{or} \quad \frac{3k}{m}. \end{gathered}$$

Thus, the eigenvalues of $\boldsymbol{A}$ are $-\frac{k}{m}$ and $-\frac{3k}{m}$ while the normal frequencies are $\sqrt{\frac{k}{m}}$ and $\sqrt{\frac{3k}{m}}$.

Evaluating Eigenvectors and Normal Modes

Now that we have computed the eigenvalues of $\boldsymbol{A}$, we can now determine the eigenvectors associated with an eigenvalue λ_i by substituting $\lambda = \lambda_i$ back into the equation $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{u} = \boldsymbol{0}$. Then, we can solve the resultant matrix equation (some variables will still be expressed in terms of the others) to obtain a general solution for $\boldsymbol{u}$ that satisfies the equation. This general solution, which is expressed in terms of a linear combination of independent vectors, is known as the eigenspace associated with the eigenvalue λ_i. The eigenspace is usually denoted as $\boldsymbol{E}_{\lambda_i}$ but we shall denote it as $\boldsymbol{E}_{-\lambda_i}$ (as $\lambda_i = -\omega_i^2$ where ω_i is the ith normal frequency) for our purposes. The independent vectors which appear in the linear combination are the basis eigenvectors associated with the eigenvalue λ_i, as substituting any linear combination of them for $\boldsymbol{u}$ in $\boldsymbol{Au}$ would result in $\lambda_i\boldsymbol{u}$. Furthermore, in the context of coupled oscillators, the basis eigenvectors associated with eigenvalue $\lambda_i = -\omega_i^2$ turn out to be the normal modes associated with the normal frequency ω_i. Do not worry too much about what these terms mean for now and consider the following specific example. In the case of the coupled spring-mass oscillators, we substitute the various values for ω^2 that we have found, into Eq. (10.14).

When $\lambda_1 = -\omega_1^2 = -\frac{k}{m}$, we obtain

$$\begin{pmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0.$$

Solving gives

$$u_1 = u_2.$$

Therefore, the eigenspace for $\boldsymbol{u}$ associated with λ_1 is the collection of vectors

$$\boldsymbol{E}_{\frac{k}{m}} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

where C_1 is a scalar. That is, any vector $\boldsymbol{u}$ of this form would be an eigenvector associated with the eigenvalue λ_1. Evidently, the only basis eigenvector associated with the eigenvalue λ_1 is $(1, 1)$. Similarly, when $\lambda_2 = -\omega_2^2 = -\frac{3k}{m}$,

$$\begin{pmatrix} \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{k}{m} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$u_2 = -u_1.$$

The eigenspace associated with λ_2 is the collection of vectors

$$\boldsymbol{E}_{\frac{3k}{m}} = C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for some scalar C_2. The only basis eigenvector associated with the eigenvalue λ_2 is $(1, -1)$. Finally, the general solution for the displacements of the masses, $\boldsymbol{X}$, is obtained by concatenating the various $\boldsymbol{E}_{\omega_i^2} e^{i\omega_i t}$'s. We will not include $\boldsymbol{E}_{\omega_i^2} e^{-i\omega_i t}$ as we will take the real component of the combination later to obtain the physical solution for $\boldsymbol{X}$ (see paragraph below Eq. (10.13)). The expression obtained from patching is

$$\boldsymbol{E}_{\frac{k}{m}} e^{i\sqrt{\frac{k}{m}}t} + \boldsymbol{E}_{\frac{3k}{m}} e^{i\sqrt{\frac{3k}{m}}t} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{k}{m}}t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{3k}{m}}t}.$$

If we let $C_1 = D_1 e^{i\phi_1}$ and $C_2 = D_2 e^{i\phi_2}$ where D_1, D_2, ϕ_1 and ϕ_2 are real constants, taking the real component of the above expression yields

$$\boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} D_1 \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} D_2 \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right).$$

As seen from above, the basis eigenvectors associated with eigenvalue λ_i now function as the normal modes of the normal frequency ω_i. The above expression is the most general solution for $\boldsymbol{X}$ as we have 4 constants to accommodate the 4 initial conditions (positions and velocities of both masses). As a last remark, this method of finding the eigenvalues is not foolproof. If there are repeated eigenvalues, we may not be able to find sufficient linearly independent solutions. Then, we would need to guess other forms of solutions. In the specific case where $\omega^2 = 0$ is a possibility, we should guess polynomials of degree one (i.e. $\boldsymbol{X} = \boldsymbol{u}(c_0 + c_1 t)$) as $\omega^2 = 0$ insinuates that the second derivative of $\boldsymbol{X}$ is zero.

Problems

Simple Harmonic Motion

1. *Pendulum Clock**

A pendulum clock, which has a period of one second when connected to a fixed pivot, is attached to the ceiling of a lift at rest. The lift then undergoes an upwards acceleration a for t seconds. Immediately afterwards, it is slowed down with deceleration a until it stops. Would the clock still be accurate at this juncture? For instance, if the time taken for the whole journey is 10s but the pendulum only oscillates 9 times, the clock would be slower by 1s and is no longer accurate.

2. *Dropping a Mass**

Consider a spring-mass system of mass m and spring constant k on a frictionless, horizontal table. If the initial amplitude is A and another mass m is dropped vertically onto the oscillating mass and sticks with it when its displacement is $\frac{A}{2}$, determine the final amplitude of oscillation A'.

3. *Kinematic Quantities**

Given that the speeds of an oscillating particle at displacements x_1 and x_2 are v_1 and v_2 respectively, determine the amplitude and angular frequency of the oscillation.

4. *Physical Pendulum**

A Physics student measures the period of an arbitrary physical pendulum about a certain pivot to be T. Then, he identifies another pivot on the opposite side of the center of mass that gives the same period. If the two points are separated by a distance l, can he determine the gravitational field strength g of the Earth, assuming that it is uniform throughout the pendulum?

5. *Cavendish Experiment**

The Cavendish experiment was performed to determine the universal gravitational constant G. Two identical small balls of mass m are connected by a light rod with length L and lie on a frictionless table. The center of the rod is connected to the ceiling via a vertical torsion wire. The torsion constant of the wire, κ, is defined as the restoring torque per unit angular twist of the wire.

(1) Find the period T of this torsion pendulum in terms of the above parameters (besides G) when the rod is rotated.
(2) Now, place two identical large balls of mass M at two diametrically opposite points on the perimeter of a circle of diameter L about the center of the rod (i.e. the small balls lie on the same circle). When the system is at equilibrium, the rod has rotated an angle θ and the distance between the center of a small ball and its adjacent large ball is $r \ll L$. Determine G in terms of L, r, M, T and θ.
(3) Suppose that the small balls are perturbed by a small angle from the equilibrium position. Will they oscillate about the equilibrium position? If so, determine the angular frequency of such oscillations in terms of κ, θ, m, L and r.

6. *Particle in Potential**

A particle of mass m is acted on by a one-dimensional potential energy given by

$$U(x) = U_0(-ax^2 + bx^4),$$

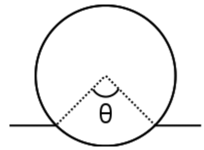
where U_0, a and b are positive constants. Determine the equilibrium x-coordinates of the particle and classify them as stable or unstable. If an equilibrium position is stable, determine the angular frequency of small oscillations about it.

7. *V-Shape Rails**

Two particles of common mass m are constrained to move along two rails, which subtend an angle 2θ, that form a V-shape. The particles are connected by a spring with spring constant k. What is the angular frequency of oscillations for the motion where the spring remains perpendicular to the symmetrical axis of the rails?

8. *Floating Cylinder**

A cylinder of density d, radius r and length l is floating on water of density ρ as shown in the diagram on the next page. Write an expression for the equilibrium value of the angle θ, subtended by the wetted portion of the cylinder, as labeled in the diagram on the next page. If the cylinder is now pressed slightly downwards, determine the angular frequency of small oscillations. Feel free to express your answer in terms of the equilibrium angle θ.

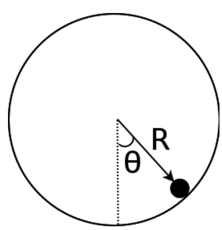


9. *Two Circles**

A small circle of radius r is attached to the circumference of a large circle of radius R. If the surface mass density of the circles is σ, determine the angular frequency of small oscillations about the equilibrium position if the center of the large circle is pivoted.

10. *Two Spheres***

A spherical ball of radius r, mass m and uniform mass density rolls without slipping in the interior of a sphere with mass M, radius R and uniform mass density near the bottom of the sphere, solely in the θ direction. The large sphere cannot translate but it may rotate. What is the angular frequency of small oscillations of the ball about the bottom of the large sphere?

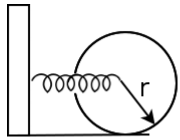


11. *Masses and String***

A mass m is undergoing circular motion about a hole on a horizontal table at radius r_0. A string, passing through the hole, is attached to m and another mass M which hangs vertically. If mass m is given a slight radial push, determine the angular frequency of small oscillations in the radial direction.

12. *Non-slip Oscillation***

Referring to the figure on the next page, a cylinder of mass m and radius r lies with its cylindrical axis in the plane of the horizontal ground. A spring of spring constant k and relaxed length l is attached to the center of the cylinder at one end and a fixed wall at the other end. If the cylinder does not slip with the ground, determine the angular frequency of oscillations.



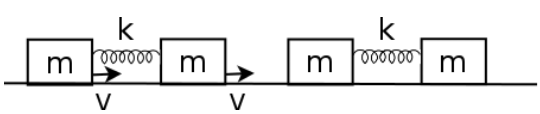
13. ***Particle in a Sphere******

A particle of mass m is currently undergoing circular motion at angular frequency $\omega_0 > \sqrt{\frac{g}{R}}$ in the interior of a massive sphere of radius R. Let θ be the angle between the vertical axis, passing through the center of the sphere and pointing downwards, and the position of the particle in spherical coordinates, taking a positive value in the anti-clockwise direction. Suppose that the particle is given a slight push in the θ direction, determine the angular frequency of small oscillations in the θ direction.

Damped and Coupled Oscillators

14. ***Colliding Couples*****

Two point masses of mass m are connected by a spring of spring constant k and relaxed length l. The two masses both have an initial velocity v and the spring between them stays at its relaxed length. These masses then travel towards an identical set-up (consisting of two masses connected by a spring) on a frictionless, horizontal table. If these four masses are aligned and undergo perfectly elastic, head-on collisions, determine the equations of motion of the masses after the first collision and before the second collision. Determine the elapsed time between the first and second collisions and show that there will only be a total of two collisions.



15. ***Colliding Masses*****

A particle of mass M approaches two initially stationary particles of common mass $m = 2\text{kg}$ that are connected by a spring of spring constant $k = 1\text{N/m}$, at an initial velocity v_0. The collision is one-dimensional, elastic and instantaneous. Determine the minimum value of M for which M will again collide