Algorithm Analysis and Design

Recurrence Equation

(Solving Recurrence using Master Method)

Lecture – 26 and 27

Overview

- A ***recurrence*** is a function is defined in terms of
  - one or more base cases, and
  - itself, with smaller arguments.

Examples:

- $$T(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ T(n-1) + 1 & \text{if } n > 1\,. \end{cases}$$

  Solution: $T(n) = n$.

- $$T(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ 2T(n/2) + n & \text{if } n \geq 1\,. \end{cases}$$

  Solution: $T(n) = n \lg n + n$.

- $$T(n) = \begin{cases} 0 & \text{if } n = 2\,, \\ T(\sqrt{n}) + 1 & \text{if } n > 2\,. \end{cases}$$

  Solution: $T(n) = \lg \lg n$.

- $$T(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ T(n/3) + T(2n/3) + n & \text{if } n > 1\,. \end{cases}$$

  Solution: $T(n) = \Theta(n \lg n)$.

Overview

- Many technical issues:
  - Floors and ceilings

    [Floors and ceilings can easily be removed and don't affect the solution to the recurrence. They are better left to a discrete math course.]
  - Exact vs. asymptotic functions
  - Boundary conditions

Overview

In algorithm analysis, the recurrence and it's solution are expressed by the help of asymptotic notation.

- Example: $T(n) = 2T(n/2) + \Theta(n)$, with solution $T(n) = \Theta(n \lg n)$.
  - The boundary conditions are usually expressed as $T(n) = O(1)$ for sufficiently small n..
  - But when there is a desire of an exact, rather than an asymptotic, solution, the need is to deal with boundary conditions.
  - In practice, just use asymptotics most of the time, and ignore boundary conditions.

Recursive Function

- Example

$A(n)$

$\{$

$If\,(n > 1)$

$Return\left(A\left(\frac{n}{2}\right)\right)$

$\}$

The relation is called recurrence relation

The Recurrence relation of given function is written as follows.

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

Recursive Function

- To solve the Recurrence relation the following methods are used:

1. Iteration method
2. Recursion-Tree method
3. **Master Method**
4. Substitution Method

Master Method

The master method provides a "cookbook" method for solving recurrences of the form

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function. The master method requires memorization of three cases.

"The beauty of Master Method is the solution of many recurrences can be determined quite easily, often without pencil and paper."

Master Method

- Definition

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows.

1. $If\ f(n) = O\left(n^{\log_b a-\varepsilon}\right) for\ some\ constant\ \epsilon > 0, then\ T(n) = \Theta(n^{\log_b a})$
2. $If\ f(n) = \Theta\left(n^{\log_b a}\right)\ then\ T(n) = \Theta(n^{\log_b a} \lg n)$.
3. $If\ f(n) = \Omega\left(n^{\log_b a+\varepsilon}\right)\ for\ some\ constant\ \epsilon > 0, and\ if\ af\left(\frac{n}{b}\right) \leq cf(n)\ for\ some\ constant\ c < 1\ and\ all\ sufficiently\ large\ n, than\ T(n) = \Theta(f(n))$

Master Method

Example 1

Solve the following recurrence by using Master Method

$$T(n) = 5T\left(\frac{n}{2}\right) + \Theta(n^2)$$

Master Method

Example 1

Solve the following recurrence by using Master Method

$$T(n) = 5T\left(\frac{n}{2}\right) + \Theta(n^2)$$

$Hear, \qquad a = 5, b = 2 \ and \ f(n) = n^2$

$First \ we \ calculate \ n^{\log_b a} \ and \ then \ compare \ with \ f(n)$

$So, n^{\log_b a} = n^{\log_2 5} = n^{2.32}$

$but, f(n) = n^2$

$Therefore, n^{\log_b a} > f(n), \qquad (\varepsilon = 0.32)$

Hence as per the definition of master theorem Case 1

$$T(n) = \Theta(n^{\log_b a}) = \Theta(n^{2.32})$$

Master Method

Example 2

Solve the following recurrence by using Master Method

$$T(n) = 9T\left(\frac{n}{3}\right) + \Theta(n)$$

Master Method

Example 2

Solve the following recurrence by using Master Method

$$T(n) = 9T\left(\frac{n}{3}\right) + \Theta(n)$$

$Hear, \qquad a = 9, b = 3 \; and \; f(n) = n$

$First \; we \; calculate \; n^{\log_b a} \; and \; then \; compare \; with \; f(n)$

$So, n^{\log_b a} = n^{\log_3 9} = n^2$

$but, f(n) = n$

$Therefore, n^{\log_b a} > f(n), \qquad (\varepsilon = 1)$

Hence as per the definition of master theorem Case 1

$T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$

Master Method

Example 3

Solve the following recurrence by using Master Method

$$T(n) = T\left(\frac{2n}{3}\right) + 1$$

Master Method

Example 3

Solve the following recurrence by using Master Method

$$T(n) = T\left(\frac{2n}{3}\right) + 1$$

$Hear, \qquad a = 1, b = \frac{3}{2}\ and\ f(n) = 1$

$First\ we\ calculate\ n^{\log_b a}\ and\ then\ compare\ with\ f(n)$

$So, n^{\log_b a} = n^{\log_{\frac{3}{2}} 1} = n^0 = 1$

$and, f(n) = 1$

$Therefore, n^{\log_b a} = f(n),$

Hence as per the definition of master theorem Case 2

$T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(\lg n)$

Master Method

Example 4

Solve the following recurrence by using Master Method

$$T(n) = 3T\left(\frac{n}{4}\right) + n \lg n$$

Master Method

Example 4

Solve the following recurrence by using Master Method

$$T(n) = 3T\left(\frac{n}{4}\right) + n\lg n$$

$Hear, \qquad a = 3, b = 4 \; and \; f(n) = n\lg n$

$First \; we \; calculate \; n^{\log_b a} \; and \; then \; compare \; with \; f(n)$

$So, n^{\log_b a} = n^{\log_4 3} = n^{0.793}$

$but, f(n) = nlg\, n$

Since ,$f(n) = \Omega(n^{\log_4 3+\varepsilon})$ where $\varepsilon \approx 0.2$, case 3 applies if we can show that the regularity condition holds for $f\,(n)$. For sufficiently large n,

$\Longrightarrow af\,(n/b) \leq cf\,(n)$

$\Longrightarrow 3(n/4)lg(n/4) \leq (3/4)n\, lg\, n \qquad (for\; c = 3/4)$

Which is true by Master Method Case 3

Hence, the solution to the recurrence is $T(n) = \Theta\big(f(n)\big) = \Theta(n\lg n)$

Master Method

Example 5

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$$

Master Method

Example 5

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$$

$Hear, \qquad a = 2, b = 2\ and\ f(n) = n \lg n$

$First\ we\ calculate\ n^{\log_b a}\ and\ then\ compare\ with\ f(n)$

$So, n^{\log_b a} = n^{\log_2 2} = n^1 = n$

$but, f(n) = n\ lg\ n$

$Which\ looks\ , n^{\log_b a} < f(n)$, and we might mistakenly think that case 3 of master method should apply. But The problem is that it is not polynomially larger.

Because the ratio $\left(i.e. \frac{f(n)}{n^{\log_b a}} = \frac{n \log n}{n} = \log n\right)$ is asymptotically less than n^{ε} for any positive constant ε.

Hence master method is not applicable to the recurrence.

Master Method

Example 6

Solve the following recurrence by using Master Method

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

Master Method

Example 6

Solve the following recurrence by using Master Method

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

$Hear, \qquad a = 4, b = 2\ and\ f(n) = n$

$First\ we\ calculate\ n^{\log_b a}\ and\ then\ compare\ with\ f(n)$

$So, n^{\log_b a} = n^{\log_2 4} = n^2$

$but, f(n) = n^2$

$Therefore, n^{\log_b a} > f(n), \qquad (\varepsilon = 1)$

Hence as per the definition of master theorem Case 1

$$T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$$

Master Method

Example 7

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n}$$

Master Method

Example 7

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n}$$

$Here, \quad a = 2, b = 4 \ and \ f(n) = \sqrt{n}$

$First \ we \ calculate \ n^{\log_b a} \ and \ then \ compare \ with \ f(n)$

$So, n^{\log_b a} = n^{\log_4 2} = n^{\frac{1}{2}} = \sqrt{n}$

$and, f(n) = \sqrt{n}$

$Therefore, n^{\log_b a} = f(n),$

Hence as per the definition of master theorem Case 2

$T(n) = \Theta(n^{\log_b a} \lg n) = \Theta(\sqrt{n} \lg n)$

Master Method

- **Recurrence (Changing Variable)**

Example 8

Solve the following recurrence by using Master Method

$$T(n) = 2T(\lfloor\sqrt{n}\rfloor) + \lg n$$

Due to, a little algebraic manipulation the above recurrence looks very difficult. These recurrences can be simplified by using change of variable. For convenience, we shall not worry about rounding off values, such as $\sqrt{n}$, to be integers.

First, Renaming $m \ = \log n$

$\Longrightarrow n = 2^m$

$Put\ the\ value\ of\ n\ and\ m\ on\ the\ above\ recurrence.$

Hence the above recurrence can be written as follows

$T(2^m) = 2T(\lfloor\sqrt{2^m}\rfloor) + m$

$\Longrightarrow\ T(2^m) = 2T(2^{m^{\frac{1}{2}}}) + m$

$\Longrightarrow\ T(2^m) = 2T\left(2^{m/2}\right) + m$

Master Method

$We\ can\ now\ rename$
$S(m) = T(2^m)$
$\Longrightarrow S(m/2) = T\left(2^{m/2}\right)$

Now put these values in above equation

$S(m) = 2S(m/2) + m$

Now apply master method for solve the above equation

$Hear,\ a = 2, b = 2\ and\ f(m) = m$

$First\ we\ calculate\ n^{\log_b a}\ and\ then\ compare\ with\ f(n)$

$So, m^{\log_b a} = m^{\log_2 2} = m^1 = m$

$and, f(m) = m$

Master Method

Hence as per the definition of master theorem Case 2

$S(m) = \Theta(m^{\log_b a} \lg m)$

$\Longrightarrow S(m) = \Theta(m \lg m)$

$\Longrightarrow S(m) = \Theta(m \lg m)$

$\Longrightarrow T(2^m) = \Theta(m \lg m)$ $\quad$ $as \;\; S(m) = \; T(2^m)$

$\Longrightarrow T(n) = \Theta(\log n \lg \log n)$ $\quad$ $as \;\; n = 2^m, and \; m = \log n$

Hence the complexity of above recurrence is $\Theta(\log n \lg \log n)$

Master Method

Example 9

Solve the following recurrence by using Master Method

$$T(n) = 2T(\lfloor\sqrt{n}\rfloor) + 1$$

Master Method

Example 9

Solve the following recurrence by using Master Method

$$T(n) = 2T(\lfloor\sqrt{n}\rfloor) + 1$$

Due to, a little algebraic manipulation the above recurrence looks very difficult. These recurrences can be simplified by using change of variable. For convenience, we shall not worry about rounding off values, such as $\sqrt{n}$, to be integers.

First, Renaming $m = \log n$

$\Longrightarrow n = 2^m$

$\Longrightarrow n^{1/2} = 2^{\mathrm{m}/2}$

$Put\ the\ value\ of\ n\ on\ the\ above\ recurrence.$

Hence the above recurrence can be written as follows

$T(2^m) = 2T(\lfloor\sqrt{2^m}\rfloor) + 1$

$\Longrightarrow T(2^m) = 2T(2^{m^{1/2}}) + 1$

$\Longrightarrow T(2^m) = 2T\left(2^{m/2}\right) + 1$

Master Method

$We\ can\ now\ rename$

$S(m) = T(2^m)$

$\Longrightarrow S(m/2) = T\left(2^{m/2}\right)$

Now put these values in above equation

$S(m) = 2S(m/2) + 1$

Now apply master method for solve the above equation

$Hear,\ a = 2, b = 2\ and\ f(m) = 1$

$First\ we\ calculate\ n^{\log_b a}\ and\ then\ compare\ with\ f(n)$

$So, m^{\log_b a} = m^{\log_2 2} = m^1 = m \qquad and, f(m) = 1$

Hence as per the definition of master theorem Case 1:

$m^{\log_b a} > f(m)$

$\Rightarrow m > 1$

$\Rightarrow m^{1-\varepsilon} = 1 \qquad where\ \varepsilon = 1$

Master Method

Hence, $S(m) = \Theta(m^{\log_b a})$

$\Longrightarrow S(m) = \Theta(m^{\log_b a}) = \Theta(m)$

$\Longrightarrow S(m) = \Theta(m)$

$\Longrightarrow T(2^m) = \Theta(\log n) \qquad as\ \ S(m) = \ T(2^m)\ and\ m = \log n$

$\Longrightarrow T(n) = \Theta(\log n) \qquad as\ \ n = 2^m$

Hence the complexity of above recurrence is $\Theta(\log n)$

Master Method

Problems Solved by Students:

Q1. $T(n) = T(^n/_2) + 2^n$.

Q2. $T(n) = 2^n T(^n/_2) + n^n$.

Q3. $T(n) = 3T(^n/_2) + n^2$.

Q4. $T(n) = 16T(^n/_4) + n$.

Q5. $T(n) = 3T(^n/_2) + n^2 \log n$.

Q6. $T(n) = 2T(^n/_2) + \frac{n}{\log n}$.

Q7. $T(n) = 2T(\sqrt{n}) + \frac{\log_2 2^n}{\log \log_2 2^n}$.

(Solving Recurrence using Advanced version of Master Method)

(For GATE questions only)

Master Method (GATE)

Definition (Advance Version)

Let $a \geq 1, b > 1, k \geq 0$ *and p is a real number* and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + \Theta(n^k log^p n)$$

where we interpret n/b *to mean either* $\lfloor n/b \rfloor$ *or* $\lceil n/b \rceil$.

Then $T(n)$ *can be bounded asymptotically by comparing* a *with* b^k *as follows*.

1. $if\ a > b^k, then\ T(n) = \Theta(n^{\log_b a})$
2. $If\ a = b^k\ and$

    Option 1 : $if\ p < -1, then\ T(n) = \Theta(n^{\log_b a})$

    Option 2 : $if\ p = -1, then\ T(n) = \Theta(n^{\log_b a}. log^2 n)$

    Option 3 : $if\ p > -1, then\ T(n) = \Theta(n^{\log_b a}. log^{p+1} n)$

3. $If\ a < b^k\ and$

    Option 1 : $if\ p < 0, then\ T(n) = \Theta(n^k)$

    Option 2 : $if\ p \geq 0, then\ T(n) = \Theta(n^k. log^p n)$

Master Method (GATE)

Example 10

Solve the following recurrence by using Master Method

$$T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n^2)$$

Master Method (GATE)

Example 10

Solve the following recurrence by using Master Method

$$T(n) = 3T\left(\frac{n}{2}\right) + \Theta(n^2)$$

$Hear, \qquad a = 3, b = 2, k = 2, p = 0$

$Now\ compare\ a\ with\ b^k.$

$So, a = 3\ and\ b^k = 2^2 = 4$

$The\ comparision\ result\ shows\ that\ \ a < b^k$

Hence as per the definition of Advanced version of Master Method case 3 (option 2)

$T(n) = \Theta\left(n^k . log^p\, n\right) = \Theta(n^2 . log^0\, n) = \Theta(n^2)$

Master Method (GATE)

Example 11

Solve the following recurrence by using Master Method

$$T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n^2)$$

Master Method (GATE)

Example 11

Solve the following recurrence by using Master Method

$$T(n) = 4T\left(\frac{n}{2}\right) + \Theta(n^2)$$

$Hear, \qquad a = 4, b = 2, k = 2, p = 0$

$Now\ compare\ a\ with\ b^k.$

$So, a = 4\ and\ b^k = 2^2 = 4$

$The\ comparision\ result\ shows\ that\ \ a = b^k$

Hence as per the definition of Advanced version of Master Method case 2 (option 3)

$T(n) = \Theta\left(n^{\log_b a}.\, log^{p+1} n\right) = \Theta\left(n^{\log_2 4}.\, log^{0+1} n\right) = \Theta(n^2.\, log^1 n) = \Theta(n^2.\log n)$

Master Method (GATE)

Example 12

Solve the following recurrence by using Master Method

$$T(n) = T\left(\frac{n}{2}\right) + \Theta(n^2)$$

Master Method (GATE)

Example 12

Solve the following recurrence by using Master Method

$$T(n) = T\left(\frac{n}{2}\right) + \Theta(n^2)$$

$Hear, \qquad a = 1, b = 2, k = 2, p = 0$

$Now\ compare\ a\ with\ b^k.$

$So, a = 1\ and\ b^k = 2^2 = 4$

$The\ comparision\ result\ shows\ that\ \ a < b^k$

Hence as per the definition of Advanced version of Master Method case 3 (option 2)

$T\left(n\right) = \Theta\left(n^k . log^p\ n\right) = \Theta\left(n^2 . log^0\ n\right) = \Theta(n^2)$

Master Method (GATE)

Example 13

Solve the following recurrence by using Master Method

$$T(n) = 2^n T\left(\frac{n}{2}\right) + \Theta(n^n)$$

Master Method (GATE)

Example 13

Solve the following recurrence by using Master Method

$$T(n) = 2^n T\left(\frac{n}{2}\right) + \Theta(n^n)$$

$Hear, \qquad a = 2^n, b = 2, k = n, p = 0$

$The\ value\ of\ a\ must\ be\ a\ constant\ number. Which\ is\ not\ true\ in\ this\ case.$

Hence Master method can't be applied hear.

Master Method (GATE)

Example 14

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n\log n)$$

Master Method (GATE)

Example 14

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n \log n)$$

$Hear, \qquad a = 2, b = 2, k = 1, p = 1$

$Now\ compare\ a\ with\ b^k.$

$So, a = 2\ and\ b^k = 2^1 = 2$

$The\ comparision\ result\ shows\ that\ \ a = b^k$

Hence as per the definition of Advanced version of Master Method case 2 (option 3)

$$T(n) = \Theta\left(n^{\log_b a}.log^{p+1} n\right) = \Theta\left(n^{\log_2 2}.log^{1+1} n\right) = \Theta(n.log^2 n)$$

Master Method (GATE)

Example 15

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta\left(\frac{n}{\log n}\right)$$

Master Method (GATE)

Example 15

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta\left(\frac{n}{\log n}\right)$$

$Hear, \qquad a = 2, b = 2, k = 1, p = -1$

$Now\ compare\ a\ with\ b^k.$

$So, a = 2\ and\ b^k = 2^1 = 2$

$The\ comparision\ result\ shows\ that\ \ a = b^k$

Hence as per the definition of Advanced version of Master Method case 2 (option 2)

$$T(n) = \Theta\left(n^{\log_b a}. log^{p+1} n\right) = \Theta\left(n^{\log_2 2}. log^2 n\right) = \Theta(n. log^2 n)$$

Master Method (GATE)

Example 16

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{4}\right) + \Theta(n^{0.51})$$

Master Method (GATE)

Example 16

Solve the following recurrence by using Master Method

$$T(n) = 2T\left(\frac{n}{4}\right) + \Theta(n^{0.51})$$

$Hear, \qquad a = 2, b = 4, k = 0.51, p = 0$

$Now\ compare\ a\ with\ b^k.$

$So, a = 2\ and\ b^k = 4^{0.51}$

$The\ comparision\ result\ shows\ that\ \ a < b^k$

Hence as per the definition of Advanced version of Master Method case 3 (option 2)

$T(n) = \Theta(n^k.\, log^p\, n) = \Theta(n^{0.51}.\, log^0\, n) = \Theta(n^{0.51})$

Master Method (GATE)

Example 17

Solve the following recurrence by using Master Method

$$T(n) = 0.5T\left(\frac{n}{2}\right) + \frac{1}{n}$$

Master Method (GATE)

Example 17

Solve the following recurrence by using Master Method

$$T(n) = 0.5T\left(\frac{n}{2}\right) + \frac{1}{n}$$

$Hear, \qquad a = 0.5, b = 2, k = -1, p = 0$

As per the definition of master method the value of ‘a’ should be greater than 1.

But hear the value of a is less than 1. Hence Master method can’t be applied hear.

Master Method (GATE)

Example 18

Solve the following recurrence by using Master Method

$$T(n) = 6T\left(\frac{n}{3}\right) + \Theta(n^2 \log n)$$

Master Method (GATE)

Example 18

Solve the following recurrence by using Master Method

$$T(n) = 6T\left(\frac{n}{3}\right) + \Theta(n^2 \log n)$$

$Hear, \qquad a = 6, b = 3, k = 2, p = 1$

$Now\ compare\ a\ with\ b^k.$

$So, a = 6\ and\ b^k = 3^2 = 9$

$The\ comparision\ result\ shows\ that\ \ a < b^k$

Hence as per the definition of Advanced version of Master Method case 3 (option 2)

$T(n) = \Theta\left(n^k . log^p\, n\right) = \Theta(n^2 . log^1\, n) = \Theta(n^2 . \log n)$

Master Method (GATE)

Example 19

Solve the following recurrence by using Master Method

$$T(n) = 64T\left(\frac{n}{8}\right) - \Theta(n^2 \lg n)$$

Master Method (GATE)

Example 19

Solve the following recurrence by using Master Method

$$T(n) = 64T\left(\frac{n}{8}\right) - \Theta(n^2 \lg n)$$

This recurrence says that without any execution the problem is divided in to sub problems. Because the term $(-n^2 \lg n)$ is not valid. Hence it is an invalid representation.

Master Method (GATE)

Example 20

Solve the following recurrence by using Master Method

$$T(n) = 4T\left(\frac{n}{3}\right) + \Theta(\log n)$$

Master Method (GATE)

Example 20

Solve the following recurrence by using Master Method

$$T(n) = 4T\left(\frac{n}{3}\right) + \Theta(\log n)$$

$Hear, \qquad a = 4, b = 2, k = 0, p = 1$

$Now\ compare\ a\ with\ b^k.$

$So, a = 4\ and\ b^k = 2^0 = 1$

$The\ comparision\ result\ shows\ that\ \ a > b^k$

Hence as per the definition of Advanced version of Master Method case 1

$T(n) = \Theta(\, n^{\log_b a}\,) = \Theta\left(n^{\log_3 4}\right)$

Master Method (GATE)

Example 21

Solve the following recurrence by using Master Method

$$T(n) = 27T\left(\frac{n}{3}\right) + \Theta(n^3 \lg n)$$

Master Method (GATE)

Example 21

Solve the following recurrence by using Master Method

$$T(n) = 27T\left(\frac{n}{3}\right) + \Theta(n^3 \lg n)$$

$Hear, \qquad a = 27, b = 3, k = 3, p = 1$

$Now\ compare\ a\ with\ b^k.$

$So, a = 27\ and\ b^k = 3^3 = 1$

$The\ comparision\ result\ shows\ that\ \ a = b^k$

Hence as per the definition of Advanced version of Master Method case 2(option 3)

$T(n) = \Theta\left(\ n^{\log_b a}\ .\ log^{p+1}n\right) = \Theta\left(n^{\log_3 27}\ .\ log^{1+1}n\right) = \Theta(\ n^3\ .\ log^2 n)$

$= \Theta(\ n^3\ .\log\log n)$

Thank u