Design and Analysis of Algorithm

Divide and Conquer strategy

(Matrix Multiplication by Strassen's Algorithm)

Lecture -24

Overview

- *Learn the implementation techniques of "divide and conquer" in the context of the Strassen's Matrix multiplication with analysis.*
- $Conventional\ strategy \Rightarrow\ \mathrm{O}(n^3)$.
- $Divide\ and\ Coques\ strategy \Rightarrow\ \mathrm{O}(n^3)$.
- $Strassen's\ strategy \Rightarrow\ \mathrm{O}(n^{2.81})$.

Matrix Multiplication

- Problem definition:

Input: Two $n \times n$ (square) matrices, $A = (a_{ij})$ and $B = (b_{ij})$.

Output: $n \times n$ matrix $C = (c_{ij})$, where $C = A \cdot B$, i.e.,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

for $i, j = 1, 2, \ldots, n$.

Need to compute n^2 entries of C. Each entry is the sum of n values.

Matrix Multiplication

- Conventional strategy:

SQUARE-MAT-MULT(A, B, n)

```
let C be a new n × n matrix
for i = 1 to n
    for j = 1 to n
        c_ij = 0
        for k = 1 to n
            c_ij = c_ij + a_ik · b_kj
return C
```

Analysis: Three nested loops, each iterates n times, and innermost loop body takes constant time $\Rightarrow \Theta(n^3)$.

Matrix Multiplication

- Question is

  $Is\ \Theta(n^3)$ is the best or we can multiply the matrix in $o(n^3)$ time?

  (i.e. can we solve it in $< \Theta(n^3)$)
- Let's see with Divide and Conquer strategy………

Matrix Multiplication

- Divide-and-conquer strategy :
  - As with the other divide-and-conquer algorithms, assume that n is a power of 2(i.e. 2^n).
  - Partition each of A,B, C into four $\frac{\boldsymbol{n}}{\boldsymbol{2}} \times \frac{\boldsymbol{n}}{\boldsymbol{2}}$ matrices:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

For multiplication we can write $C = A \,.\, B$ as

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Matrix Multiplication

- Divide-and-conquer strategy :

For multiplication we can write $C = A \,.\, B$ as

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Which create four equations. They are

$$C_{11} = A_{11} \,.\, B_{11} + A_{12} \,.\, B_{21}$$
$$C_{12} = A_{11} \,.\, B_{12} + A_{12} \,.\, B_{22}$$
$$C_{21} = A_{21} \,.\, B_{11} + A_{22} \,.\, B_{21}$$
$$C_{22} = A_{21} \,.\, B_{12} + A_{22} \,.\, B_{22}$$

Each of these equations multiplies two $\frac{n}{2} \times \frac{n}{2}$ matrices and then adds their $\frac{n}{2} \times \frac{n}{2}$ products.

Matrix Multiplication

- Divide-and-conquer strategy :

By using the equations of previous slide we ca write the Divide and conquer algorithm.

REC-MAT-MULT (A, B, n)

Let C be a n x n matrix

If n== 1

$C_{11} = A_{11} \text{ x } B_{11}$

else partition A,B, and C into $\frac{\boldsymbol{n}}{\mathbf{2}} \times \frac{\boldsymbol{n}}{\mathbf{2}}$ submatrices.

$C_{11} = \text{REC}-\text{MAT}-\text{MULT}(A_{11}, B_{11}, N/2) + \text{REC}-\text{MAT}-\text{MULT}(A_{12}, B_{21}, N/2)$

$C_{12} = \text{REC}-\text{MAT}-\text{MULT}(A_{11}, B_{12}, N/2) + \text{REC}-\text{MAT}-\text{MULT}\ (A_{12}, B_{22}, N/2)$

$C_{21} = \text{REC}-\text{MAT}-\text{MULT}(A_{21}, B_{11}, N/2) + \text{REC}-\text{MAT}-\text{MULT}(A_{22}, B_{21}, N/2)$

$C_{22} = \text{REC}-\text{MAT}-\text{MULT}(A_{21}, B_{12}, N/2) + \text{REC}-\text{MAT}-\text{MULT}\ (A_{22}, B_{22}, N/2)$

Return C

Matrix Multiplication

- Analysis of Divide-and-conquer strategy :

Let T(n) be the time to multiply two $\frac{n}{2} \times \frac{n}{2}$ matrices.

Base Case: n=1. Perform one scalar multiplication: $\Theta(1)$**.**

Recursive Case: n>1

- Dividing takes b $\Theta(1)$ time, using index calculations.
- Conquering makes 8 recursive calls, each multiplying $\frac{n}{2} \times \frac{n}{2}$ matrices.

  (i.e. $8T(n/2)$)
- Combining Takes $\Theta(n^2)$ time to add $\frac{n}{2} \times \frac{n}{2}$ matrices four items.

Hence the Recurrence is

$$T(n) = \begin{cases} \Theta(1) & if\ n = 1 \\ 8T\left(\frac{n}{2}\right) + \Theta(n^2) & if n > 1 \end{cases}$$

The Complexity is $\Theta(n^3)$ (Apply Master Method)

Matrix Multiplication

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Can we do better?

Matrix Multiplication

- Strassen's strategy :

The Idea:

- Make the recursion tree less bushy.
- Perform only 7(seven) recursive multiplications of n/2 x n/2 matrices, rather than 8(Eight).

Matrix Multiplication

- Strassen's strategy :

The Algorithm:

1. As in the recursive method, partition each of the matrices into four $\frac{n}{2} \times \frac{n}{2}$ submatrices. Time: $\Theta(1)$
2. Compute 7 matrix products P, Q, R, S, T, U, V for each $\frac{n}{2} \times \frac{n}{2}$.
3. Compute $\frac{n}{2} \times \frac{n}{2}$ submatrices of C by adding and subtracting various combinations of the P_i . Time: $\Theta(n^2)$.

Matrix Multiplication

- Strassen's strategy :

  Details of Step 2:

  Compute 7 matrix products:

  $P=(A_{11}+A_{22}).(B_{11}+B_{22})$

  $Q=(A_{21}+A_{22}).B_{11}$

  $R=A_{11}.(B_{12}-B_{22})$

  $S=A_{22}.(B_{21}-B_{11})$

  $T=(A_{11}+A_{12}).B_{22}$

  $U=(A_{21}-A_{11}).(B_{11}+B_{12})$

  $V=(A_{12}-A_{22}).(B_{21}+B_{22})$

Matrix Multiplication

- Strassen's strategy :

  Details of Step 3:

  Compute C with 4 adding and subtracting :

  $\boldsymbol{C_{11} = P + S - T + V}$

  $\boldsymbol{C_{12} = R + T}$

  $\boldsymbol{C_{21} = Q + S}$

  $\boldsymbol{C_{22} = P + R - Q + U}$

Matrix Multiplication

- Strassen's strategy :

  Analysis:

  The Recurrence is

$$T(n) = \begin{cases} \Theta(1) & if\ n = 1 \\ 7T\left(\frac{n}{2}\right) + \Theta(n^2) & if n > 1 \end{cases}$$

  The Complexity is $\Theta(n^{\log_2 7}) = \Theta(n^{2.81})$ (By using Master Method)

Matrix Multiplication

Example 1

- Compute Matrix multiplication of the following two matrices with the help of Strassen's strategy

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Matrix Multiplication

Example 1

- Compute Matrix multiplication of the following two matrices with the help of Strassen's strategy

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Ans:

A_{11}=1, A_{12}=2, A_{21}=3, and A_{22}=4

B_{11}=5, B_{12}=6, B_{21}=7, and B_{22}=8

Matrix Multiplication

Example 1

Calculate the value of P, Q, R, S, T, U and V

$P=(A_{11} + A_{22}).\ (B_{11}+B_{22}) = (1+4)(5+8)=5 \text{ x } 13=65$

$Q=(A_{21} + A_{22}).\ B_{11}= (3+4)5=7 \text{ x } 5=35$

$R= A_{11}.(B_{12} - B_{22})=1(6\text{-}8)=1 \text{ x } \text{-}2=\text{-}2$

$S= A_{22}.(B_{21} - B_{11}) = 4(7\text{-}5)=4 \text{ x } 2=8$

$T=(A_{11} + A_{12}).\ B_{22}=(1+2)8=3 \text{ x } 8=24$

$U= (A_{21} - A_{11}).(B_{11} + B_{12})=(3\text{-}1)(5+6)=2 \text{ x } 11=22$

$V=(A_{12} - A_{22}).(B_{21} + B_{22})=(2\text{-}4)(7+8)=\text{-}2 \text{ x } 15=\text{-}30$

Matrix Multiplication

Example 1

Compute $\boldsymbol{C_{11}, C_{12}, C_{21}, and\ C_{22}}$:

$C_{11} = P + S - T + V$= 65+8-24-30=19

$C_{12} = R + T$=-2+24=22

$C_{21} = Q + S$=35+8=43

$C_{22} = P + R - Q + U$=65-2-35+22=50

Hence,

$$C = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Matrix Multiplication

Example 2

Compute Matrix multiplication of the following two matrices with the help of Strassen's strategy.

$$A = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 3 & 1 & 2 & 5 \\ 3 & 2 & 1 & 4 \\ 5 & 2 & 6 & 7 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 5 & 4 & 2 & 3 \\ 1 & 4 & 0 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

Matrix Multiplication

Example 2

First we partition the input matrices into sub matrices as shown below:

$$A_{11} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}, A_{12} = \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix} \qquad B_{11} = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}, B_{12} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix}, A_{22} = \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \qquad B_{21} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, B_{22} = \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix}$$

Matrix Multiplication

Example 2

Calculate the value of P, Q, R, S, T, U and V

$P = (A_{11} + A_{22}).(B_{11}+B_{22})$

$$= \begin{bmatrix} 5 & 6 \\ 9 & 8 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 9 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 64 & 45 \\ 90 & 67 \end{bmatrix}$$

$Q=(A_{21} + A_{22}).\ B_{11}$

$$= \begin{bmatrix} 4 & 6 \\ 11 & 9 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 38 & 28 \\ 67 & 47 \end{bmatrix}$$

Matrix Multiplication

Example 2

R= $A_{11}.(B_{12} - B_{22})$

$$= \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 \\ 7 & 2 \end{bmatrix}$$

S= $A_{22}.(B_{21} - B_{11})$

$$= \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -5 \\ -20 & 4 \end{bmatrix}$$

Matrix Multiplication

Example 2

T=$(A_{11}+A_{12}).\ B_{22}$

$$=\begin{bmatrix}4 & 3\\5 & 6\end{bmatrix}\begin{bmatrix}0 & 2\\4 & 1\end{bmatrix}$$

$$=\begin{bmatrix}12 & 11\\24 & 16\end{bmatrix}$$

U= $(A_{21}-A_{11}).(B_{11}+B_{12})$

$$=\begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}5 & 3\\7 & 7\end{bmatrix}$$

$$=\begin{bmatrix}-5 & -3\\17 & 13\end{bmatrix}$$

V=$(A_{12}-A_{22}).(B_{21}+B_{22})$

$$=\begin{bmatrix}-1 & -3\\-4 & -2\end{bmatrix}\begin{bmatrix}1 & 6\\7 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-22 & -15\\-18 & -30\end{bmatrix}$$

Matrix Multiplication

Example 2

Now, Compute $\boldsymbol{C_{11}}, \boldsymbol{C_{12}}, \boldsymbol{C_{21}}, \boldsymbol{and}\ \boldsymbol{C_{22}}$:

$$C_{11} = P + S - T + V$$

$$C_{11} = \begin{bmatrix} 64 & 45 \\ 90 & 67 \end{bmatrix} + \begin{bmatrix} -9 & -5 \\ -20 & 4 \end{bmatrix} - \begin{bmatrix} 12 & 11 \\ 24 & 16 \end{bmatrix} + \begin{bmatrix} -22 & -15 \\ -18 & -30 \end{bmatrix} = \begin{bmatrix} 21 & 14 \\ 28 & 25 \end{bmatrix}$$

$$C_{12} = R + T$$

$$C_{12} = \begin{bmatrix} 8 & 4 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} 12 & 11 \\ 24 & 16 \end{bmatrix} = \begin{bmatrix} 20 & 15 \\ 31 & 18 \end{bmatrix}$$

Matrix Multiplication

Example 2

Now, Compute $\boldsymbol{C_{11}}, \boldsymbol{C_{12}}, \boldsymbol{C_{21}}, \boldsymbol{and}\ \boldsymbol{C_{22}}$:

$$C_{21} = Q + S$$

$$C_{21} = \begin{bmatrix} 38 & 28 \\ 67 & 47 \end{bmatrix} + \begin{bmatrix} -9 & -5 \\ -20 & 4 \end{bmatrix} = \begin{bmatrix} 29 & 23 \\ 47 & 51 \end{bmatrix}$$

$$C_{22} = P + R - Q + U$$

$$C_{22} = \begin{bmatrix} 64 & 45 \\ 90 & 67 \end{bmatrix} + \begin{bmatrix} 8 & 4 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 38 & 28 \\ 67 & 47 \end{bmatrix} + \begin{bmatrix} -5 & -3 \\ 17 & 13 \end{bmatrix} = \begin{bmatrix} 29 & 18 \\ 47 & 35 \end{bmatrix}$$

Matrix Multiplication

Example 2

So the values of $\boldsymbol{C_{11}, C_{12}, C_{21}, and\ C_{22}}$ are:

$C_{11} = \begin{bmatrix} 21 & 14 \\ 28 & 25 \end{bmatrix}$, $C_{12} = \begin{bmatrix} 20 & 15 \\ 31 & 18 \end{bmatrix}$, $C_{21} = \begin{bmatrix} 29 & 23 \\ 47 & 51 \end{bmatrix}$ and $C_{22} = \begin{bmatrix} 29 & 18 \\ 47 & 35 \end{bmatrix}$

Hence the resultant Matrix C is =

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} 21 & 14 & 20 & 15 \\ 28 & 25 & 31 & 18 \\ 29 & 23 & 29 & 18 \\ 47 & 51 & 47 & 35 \end{bmatrix}$$

Thank u