P, NP, NP-Hard & NP-complete Problems

Lecture – 71-72

Objectives

- P, NP, NP-Hard and NP-Complete
- Solving 3-CNF Sat problem
- Discussion of Gate Questions

Types of Problems

- Trackable
- Intrackable
- Decision
- Optimization

Trackable : Problems that can be solvable in a reasonable(polynomial) time.

Intrackable : Some problems are *intractable*, as they grow large, we are unable to solve them in reasonable time.

Tractability

- What constitutes reasonable time?
  - Standard working definition: *polynomial time* on an input of size n the worst-case running time is $O(n^k)$ for some constant k
  - $O(n^2)$, $O(n^3)$, $O(1)$, $O(n \lg n)$, $O(2^n)$, $O(n^n)$, $O(n!)$
  - Polynomial time: $O(n^2)$, $O(n^3)$, $O(1)$, $O(n \lg n)$
  - Not in polynomial time: $O(2^n)$, $O(n^n)$, $O(n!)$
- Are all problems solvable in polynomial time?
  - No(Turing's **"Halting Problem"** is not solvable by any computer, no matter how much time is given.)

Optimization/Decision Problems

- Optimization Problems
  - An optimization problem is one which asks, “What is the optimal solution to problem X?”
  - Examples:
    - 0-1 Knapsack
    - Fractional Knapsack
    - Minimum Spanning Tree
- Decision Problems
  - An decision problem is one with yes/no answer
  - Examples:
    - Does a graph G have a MST of weight $\leq$ W?

Optimization/Decision Problems

- An optimization problem tries to find an optimal solution
- A decision problem tries to answer a yes/no question
- Many problems will have decision and optimization versions
  - Eg: Traveling salesman problem
    - optimization: find hamiltonian cycle of minimum weight
    - decision: is there a hamiltonian cycle of weight $\leq k$

Till now all the algorithm we have learned are classified and written below in two categories:

Polynomial based Algorithm

- Linear Search
- Binary Search
- Insertion Sort
- Merge Sort
- Matrix Multiplication
- etc...

Exponential based Algorithm

- 0/1 Knapsack
- Travelling Salesman
- Sum of Subset
- N-Queen
- Graph Coloring
- Hamiltonian Cycle

P, NP, NP-Hard, NP-Complete

-Definitions

The Class *P*

P: the class of problems that have polynomial-time deterministic algorithms.

- That is, they are solvable in $O(p(n))$, where $p(n)$ is a polynomial on n
- *A* deterministic algorithm is (essentially) one that always computes the correct answer

Sample Problems in P

- Fractional Knapsack
- MST (Minimum Spanning Tree)
- Sorting
- Others ?

The class *NP*

NP: the class of decision problems that are solvable in polynomial time on a *nondeterministic* machine (or with a nondeterministic algorithm)

- (*A determinstic* computer is what **we know**)
- *A nondeterministic* computer is one that can "**guess**" the right answer or solution

- Think of a nondeterministic computer as a parallel machine that can freely spawn ***an infinite number*** of processes
- Thus *NP* can also be thought of as the class of problems **"whose solutions can be verified in polynomial time"**
- Note that *NP* stands for "Nondeterministic Polynomial-time"

Sample Problems in NP

- Fractional Knapsack
- MST
- Others?
  - Traveling Salesman
  - Graph Coloring
  - Satisfiability (SAT)
    - the problem of deciding whether a given Boolean formula is satisfiable

P And NP Summary

- **P** = set of problems that can be solved in polynomial time
  - Examples: Fractional Knapsack, …
- **NP** = set of problems for which a solution can be verified in polynomial time
  - Examples: Fractional Knapsack,…, TSP, CNF SAT, 3-CNF SAT
- Clearly $\mathbf{P} \subseteq \mathbf{NP}$
- Open question: Does $\mathbf{P} = \mathbf{NP}$? Or $\mathbf{P} \neq \mathbf{NP}$?

NP-hard

- What does NP-hard mean?
  - A lot of times you can solve a problem by reducing it to a different problem. I can reduce Problem B to Problem A if, given a solution to Problem A, I can easily construct a solution to Problem B. (In this case, "easily" means "in polynomial time.").
- A problem is **NP-hard** if all problems in NP are polynomial time reducible to it, ...
- Ex:- Hamiltonian Cycle (HC)

Every problem in NP is reducible to HC in polynomial time. Ex:- TSP is reducible to HC.

B A

Example: lcm(m, n) = (m * n) / gcd(m, n),

NP-complete problems

- *A* problem is **NP-complete** if the problem is both
  - NP-hard, and
  - NP.

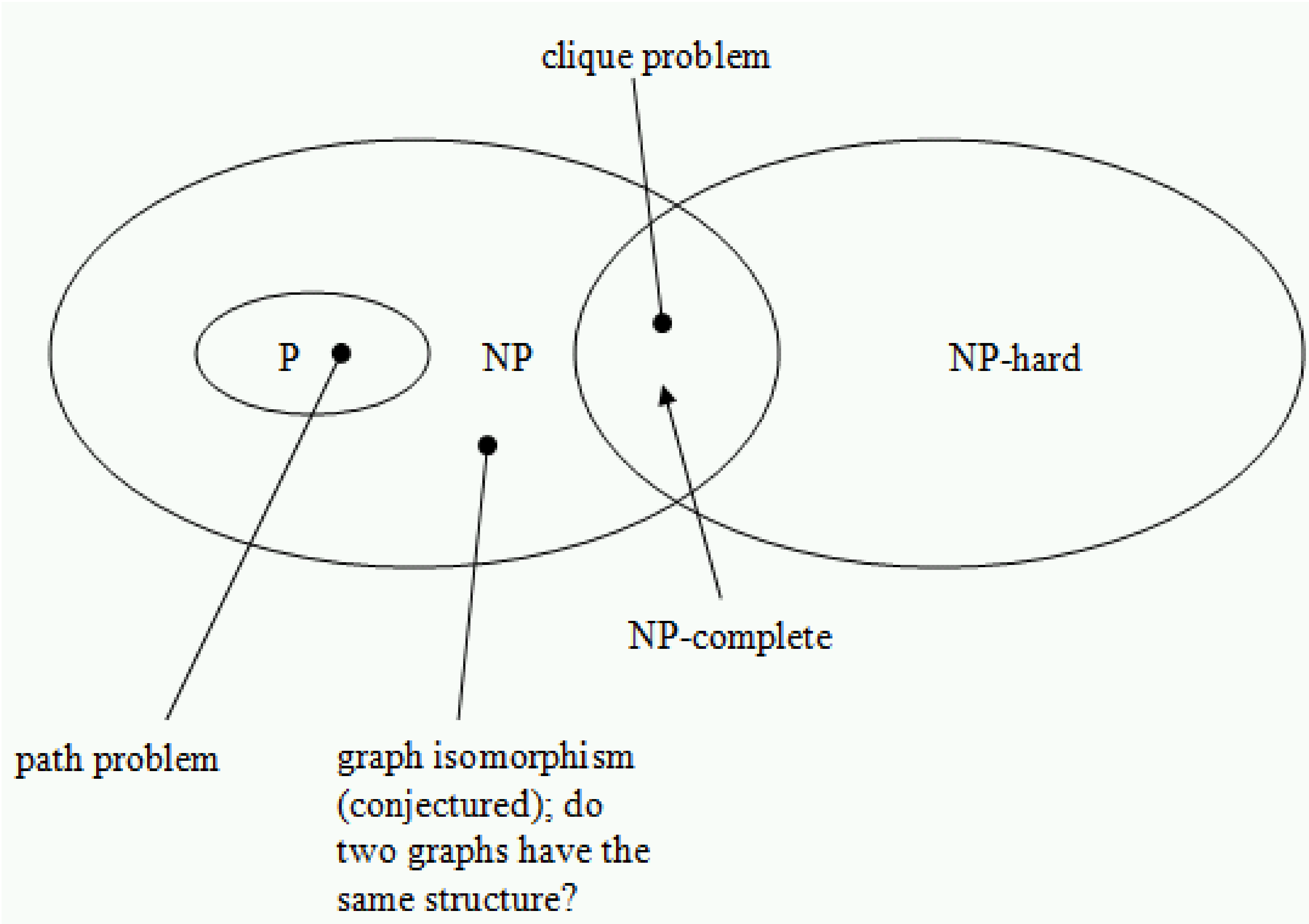
Reduction

- A problem R can be *reduced* to another problem Q if any instance of R can be rephrased to an instance of Q, the solution to which provides a solution to the instance of R
  - This rephrasing is called a *transformation*
- Intuitively: If R reduces in polynomial time to Q, R is "no harder to solve" than Q
- Example: lcm(m, n) = (m * n) / gcd(m, n), lcm(m,n) problem is reduced to gcd(m, n) problem

NP-Hard and NP-Complete

- If R is *polynomial-time reducible* to Q, we denote this R $\leq_p$Q
- Definition of NP-Hard and NP-Complete:
  - If all problems R ∈ **NP** are *polynomial-time* reducible to Q, then Q is *NP-Hard*
  - We say Q is *NP-Complete* if Q is NP-Hard and Q ∈ **NP**
- If R $\leq_p$Q and R is NP-Hard, Q is also NP-Hard

Relationship between P, NP, NP-Hard and NP-Complete



Summary

- P is set of problems that can be solved by a deterministic Turing machine in Polynomial time.
- NP is set of problems that can be solved by a Non-deterministic Turing Machine in Polynomial time. P is subset of NP (any problem that can be solved by deterministic machine in polynomial time can also be solved by non-deterministic machine in polynomial time) but $P \neq NP$.

Summary

- Some problems can be translated into one another in such a way that a fast solution to one problem would automatically give us a fast solution to the other.

- There are some problems that every single problem in NP can be translated into, and a fast solution to such a problem would automatically give us a fast solution to every problem in NP. This group of problems are known as **NP- Complete**. Ex:- Clique Problem

Summary

- A problem is NP-hard if an algorithm for solving it can be translated into one for solving any NP- problem (nondeterministic polynomial time) problem. NP-hard therefore means "at least as hard as any NP Problem" although it might, in fact, be harder.

First NP-complete problem (Circuit Satisfiability)

Problem definition

- Boolean combinational circuit
  - Boolean combinational elements, wired together
  - Each element, inputs and outputs (binary)
  - Limit the number of outputs to 1.
  - Called *logic gates*: NOT gate, AND gate, OR gate.
  - *true table*: giving the outputs for each setting of inputs
  - true assignment: a set of boolean inputs.
  - satisfying assignment: a true assignment causing the output to be 1.

First NP-complete problem (Circuit Satisfiability)

- Circuit satisfying problem: given a boolean combinational circuit composed of AND, OR, and NOT, is it stisfiable?
- CIRCUIT-SAT={<*C*>: *C* is a satisfiable boolean circuit}
- Implication: in the area of computer-aided hardware optimization, if a subcircuit always produces 0, then the subcircuit can be replaced by a simpler subcircuit that omits all gates and just output a 0.

First NP-complete problem (Circuit Satisfiability)

Two instances of circuit satisfiability problems

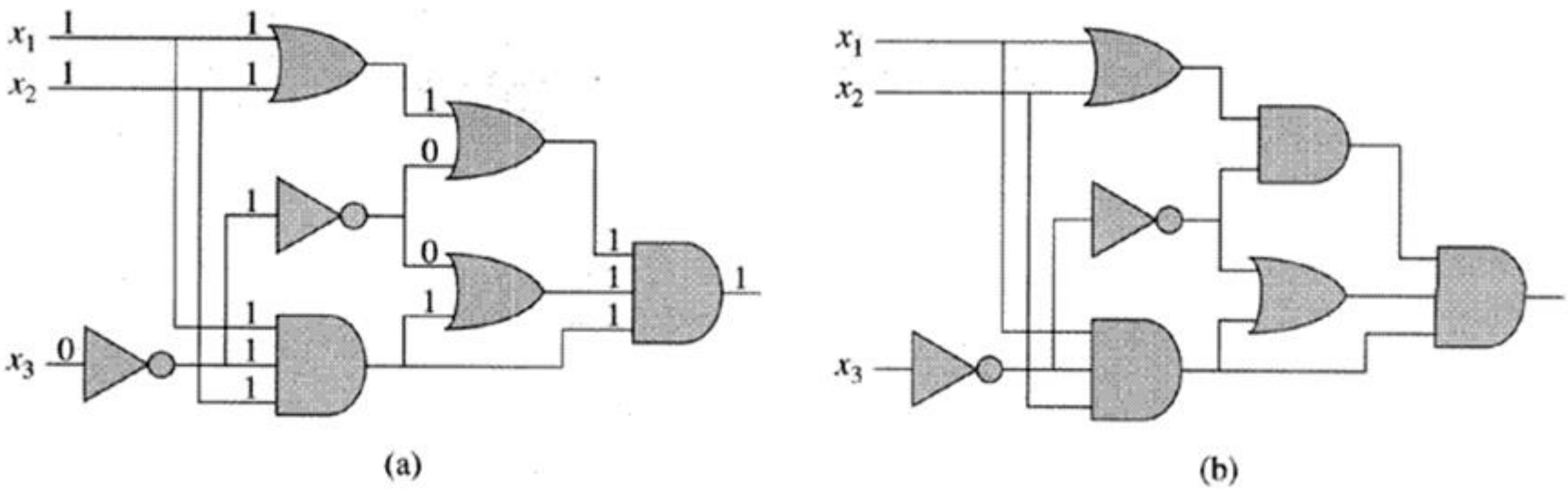


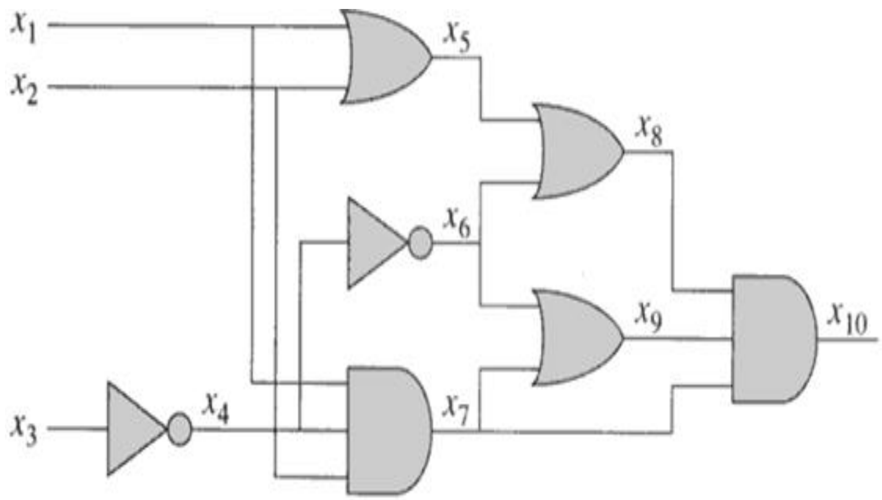
Figure: Two instances of the circuit-satisfiability problem

a) The assignment $(x_1 = 1, x_2 = 1, x_3 = 0)$ to the input of this circuit causes the output of the circuit to be 1. The circuit is therefore satisfiable.
b) No assignment to the input of the circuit can cause the output of the circuit to be 1. The circuit is therefore unsatisfiable.

Solving circuit-satisfiability problem

- Intuitive solution:
  - for each possible assignment, check whether it generates 1.
  - suppose the number of inputs is k, then the total possible assignments are 2^k. So the running time is $\Omega(2^k)$. When the size of the problem is $\Theta(k)$, then the running time is not polynomial.

Example of reduction of CIRCUIT-SAT to SAT



$$
\begin{aligned}
\Phi = x_{10} & \wedge (x_{10} \leftrightarrow (x_7 \wedge x_8 \wedge x_9)) \\
& \wedge (x_9 \leftrightarrow (x_6 \vee x_7)) \\
& \wedge (x_8 \leftrightarrow (x_5 \vee x_6)) \\
& \wedge (x_7 \leftrightarrow (x_1 \wedge x_2 \wedge x_4)) \\
& \wedge (x_6 \leftrightarrow \neg x_4) \\
& \wedge (x_5 \leftrightarrow (x_1 \vee x_2)) \\
& \wedge (x_4 \leftrightarrow \neg x_3)
\end{aligned}
$$

REDUCTION: $\Phi = x_{10} = x_7 \wedge x_8 \wedge x_9 = (x_1 \ \wedge x_2 \wedge x_4) \wedge (x_5 \ \vee x_6 \ (\wedge \ (x_6 \vee x_7) = (x_1 \ \wedge x_2 \wedge x_4) \wedge ((x_1 \ \vee x_2) \vee \neg \ x_4) \wedge (\neg \ x_4 \vee (x_1 \wedge x_2 \wedge x_4 \ldots\ldots$=((

Conversion to 3 CNF

- The result is that in Φ', each clause has at most three literals
- Change each clause into conjunctive normal form as follows:
  - Construct a true table, (small, at most 8 by 4)
  - Write the disjunctive normal form for all true-table items evaluating to 0
  - Using DeMorgan law to change to CNF.
- The resulting Φ'' is in CNF but each clause has 3 or less literals.
- Change 1 or 2-literal clause into 3-literal clause as follows:
  - If a clause has one literal l, change it to $(l \vee p \vee q) \wedge (l \vee p \vee \neg q) \wedge (l \vee \neg \mathrm{p} \vee q) \wedge (l \vee \neg p \vee \neg q)$.
  - If a clause has two literals $(l_1 \vee l_2)$change it $(l_1 \vee l_2 \vee \neg p) \wedge (l_1 \vee l_2 \vee \neg p)$

Example of a polynomial-time reduction:

Now reduce the 3CNF-satisfiability problem to the CLIQUE problem $(i.e. 3\Phi \leq_P Clique)$

3CNF formula:

$$(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge \overbrace{(x_3 \vee \overline{x_5} \vee x_6)}^{\text{clause}} \wedge (\overset{\text{literal}}{x_3} \vee \overline{x_6} \vee x_4) \wedge (x_4 \vee x_5 \vee x_6)$$

A 3 CNF has k clauses, and each clause has three literals.

Language:

3CNF-SAT $=\{w : w$ is a satisfiable 3CNF formula$\}$

What is a Clique?

A clique V' is an undirected graph G=(V,E) i.e. $V' \subseteq V$.
Each pair of which is connected by an edge in E $(i.e.\ e_i \in E)$
The size of a clique is the number of vertices it contains.

Definition of Clique problem : In a clique problem each node is connected to each other nodes of that graph.

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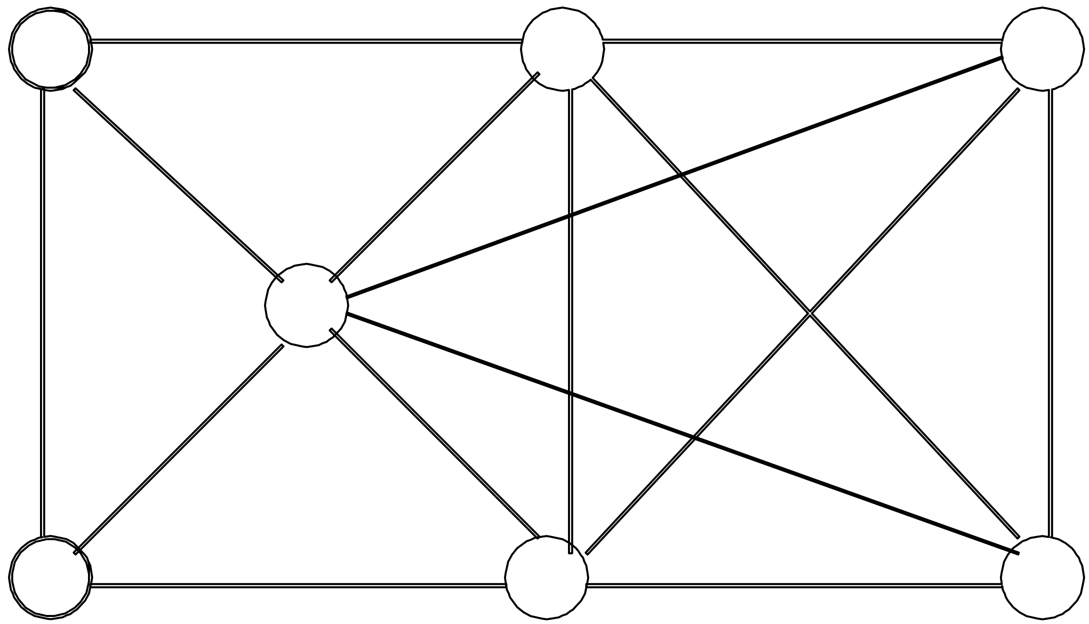
Language:

CLIQUE = $\{ < G, k >$:graph G contains a k-clique$\}$

A 5-clique in graph G



Definition of Clique Problem : In a clique problem each node is connected to each other nodes of that graph.

Language:

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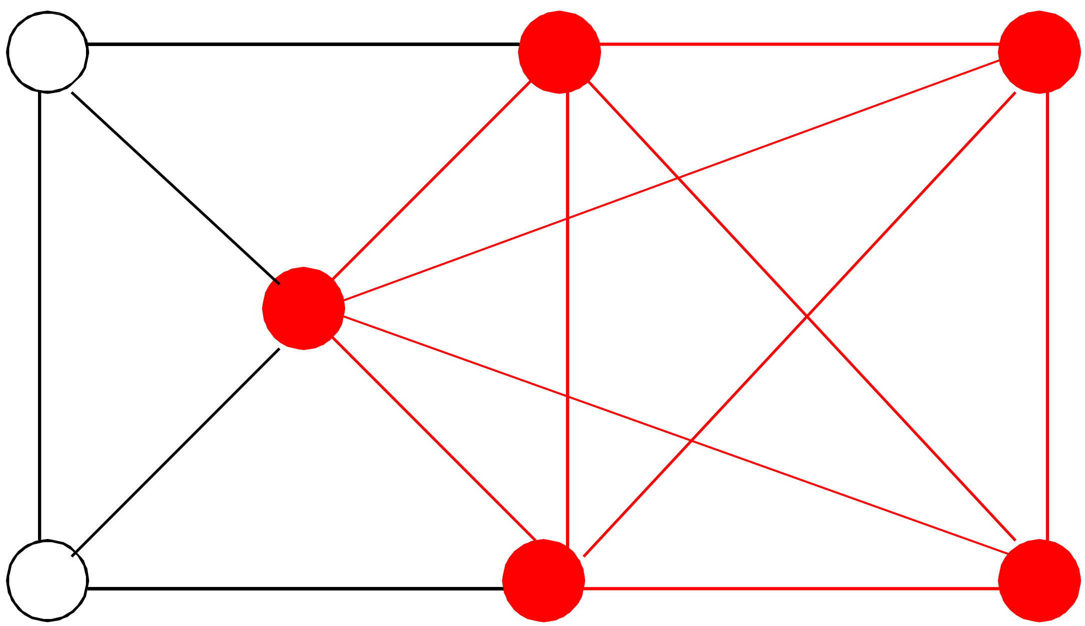
A 5-clique in graph G

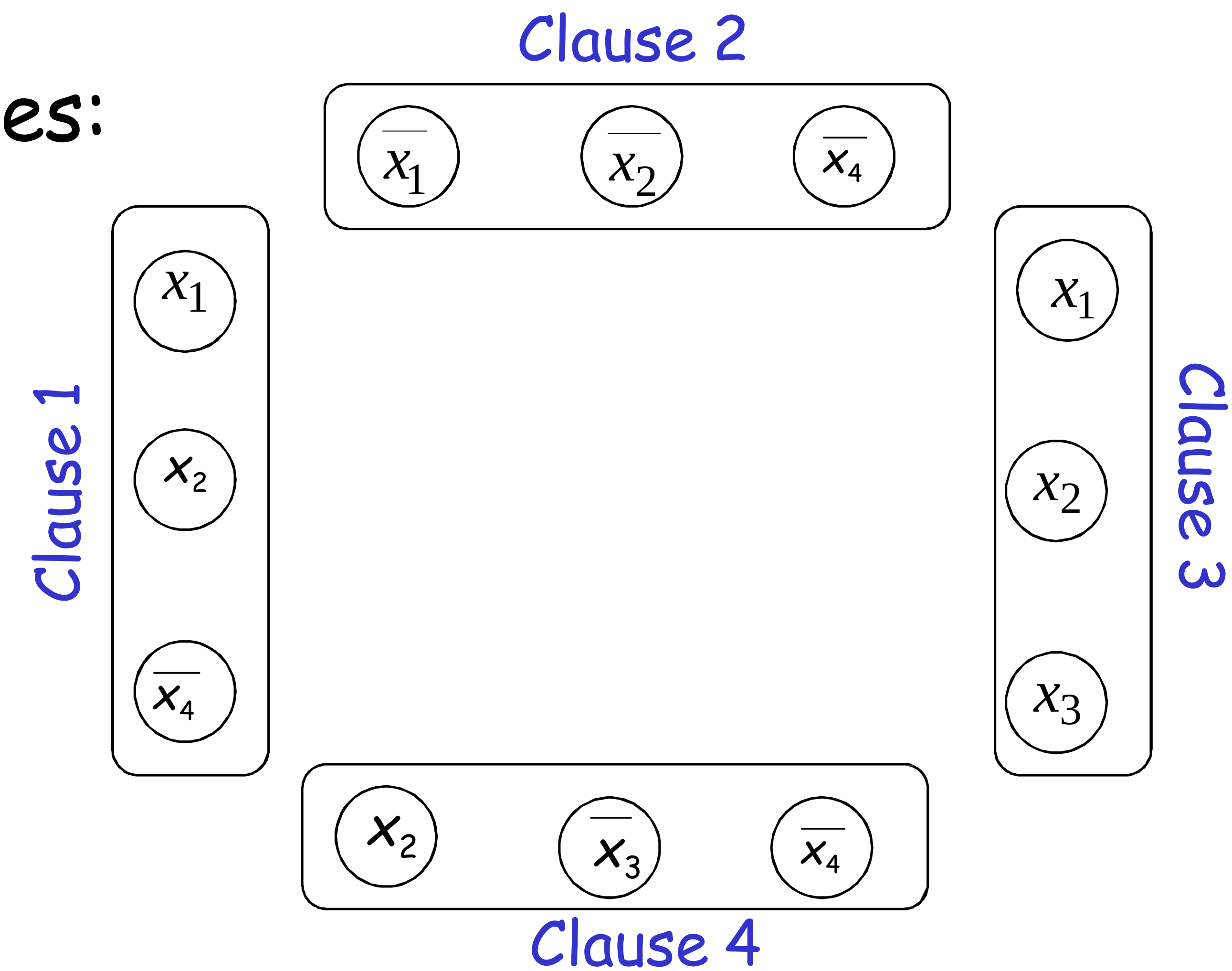


Transform formula to graph.

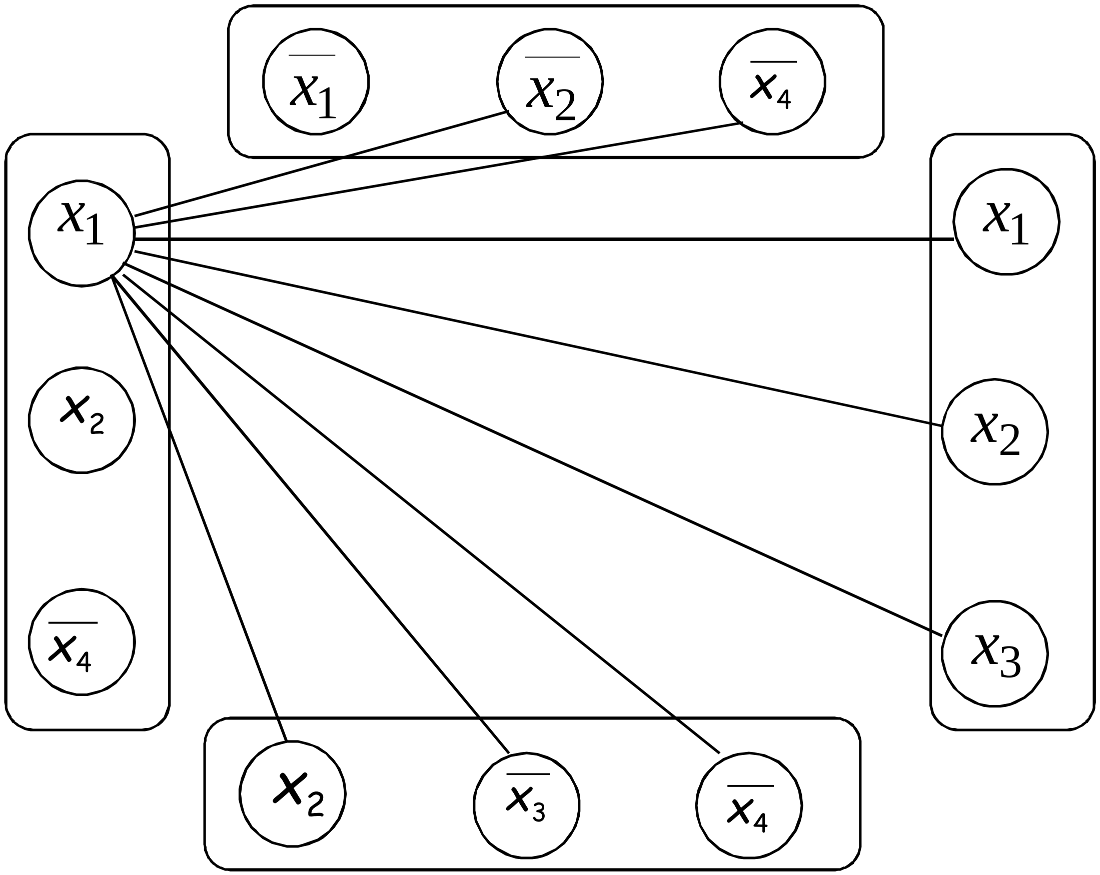
Example:

$$(x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4})(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$

Create Nodes:

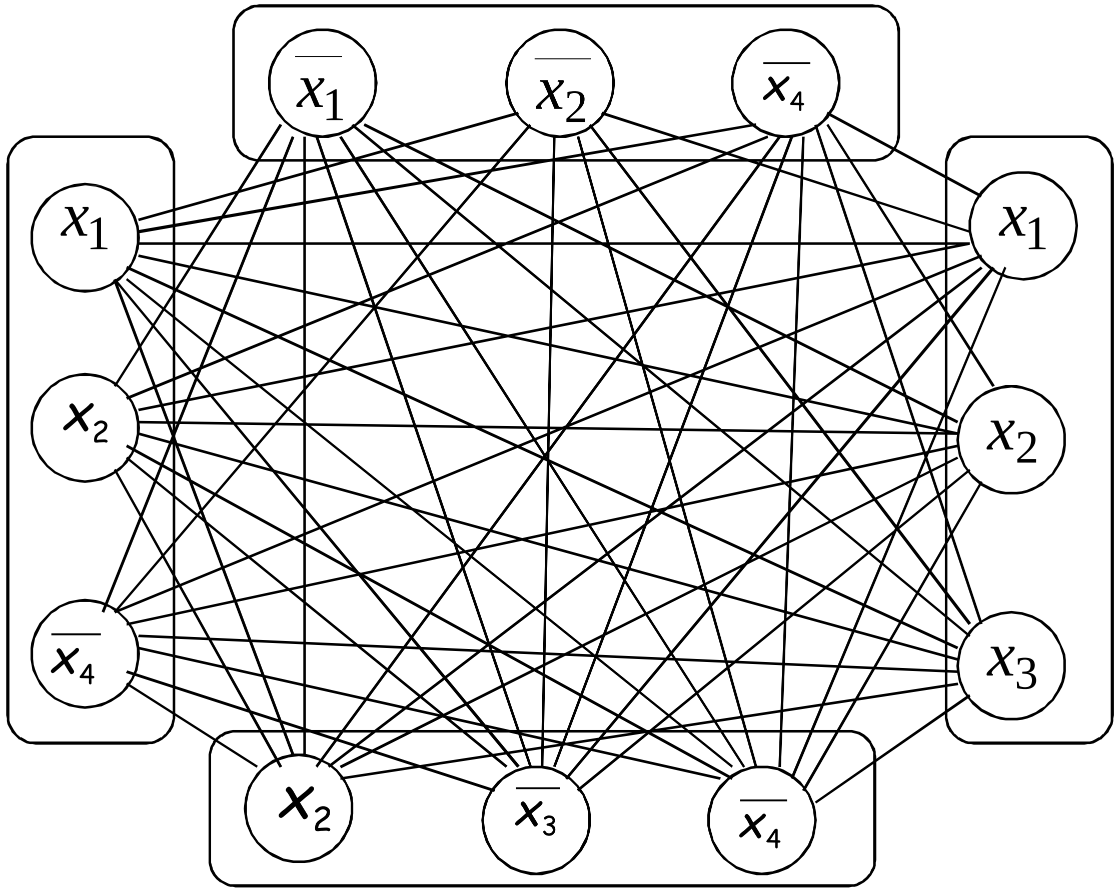


$$(x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4})(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



Add link from a literal ξ to a literal in every other clause, except the complement $\overline{\xi}$

$$(x_1 \vee x_2 \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4})(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$



Resulting Graph

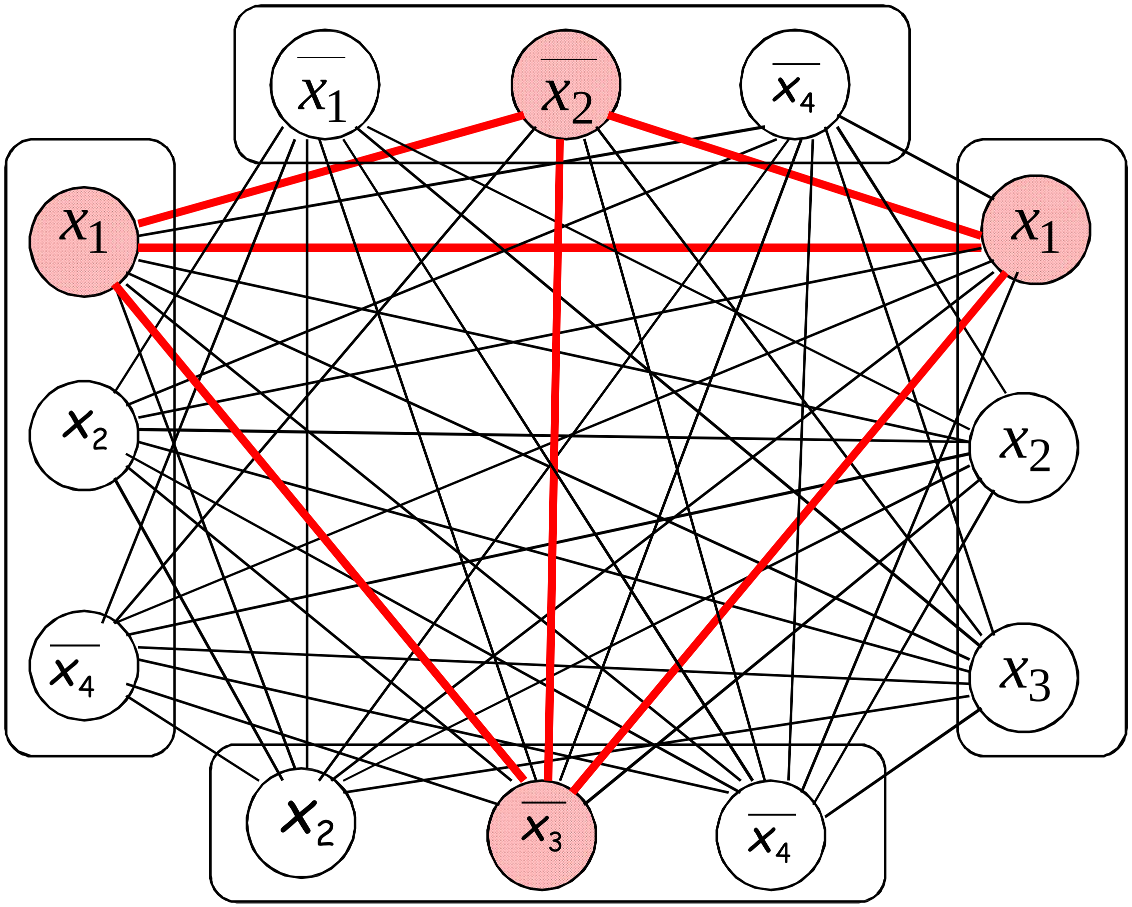
$$(x_1 \lor x_2 \lor \overline{x_4}) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_4})(x_1 \lor x_2 \lor x_3) \land (x_2 \lor \overline{x_3} \lor \overline{x_4})$$

$x_1 = 1$

$x_2 = 0$

$x_3 = 0$

$x_4 = 1$

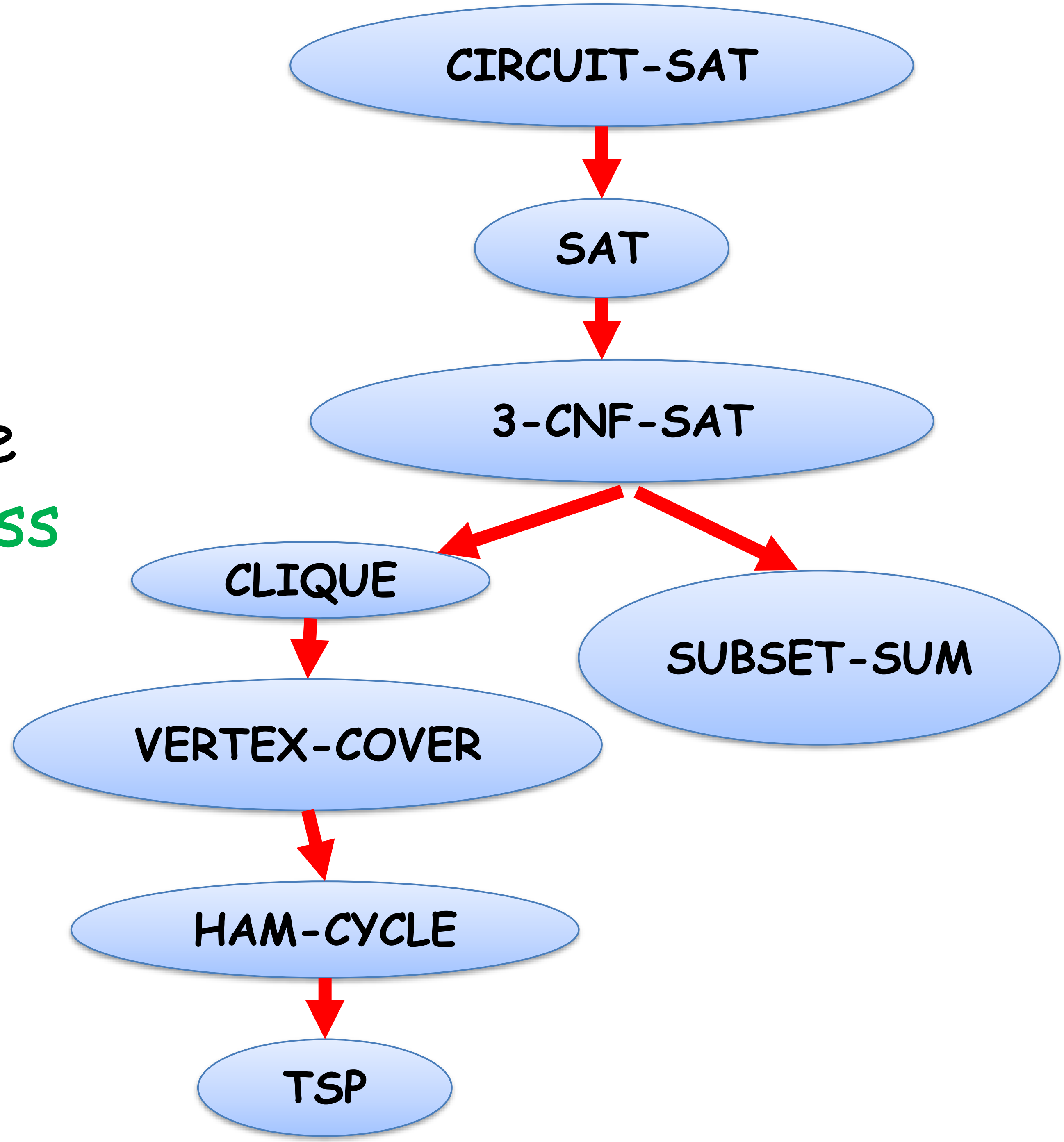


The formula is satisfied if and only if the Graph has a 4-clique

The objective is to find a clique of size 4, where 4 is the number of clauses.

End of Proof

The side figure outline the structure of the NP- Completeness proof by using reduction methodology



Theorem:

If: a. Language A is NP-complete

b. Language B is in NP

c. A is polynomial time reducible to B

Then: B is NP-complete

Corollary: CLIQUE is NP-complete

Proof:

a. 3CNF-SAT is NP-complete

b. CLIQUE is in NP

c. 3CNF-SAT is polynomial reducible to CLIQUE (shown earlier)

Apply previous theorem with

A=3CNF-SAT and B=CLIQUE

Previous Gate Questions

1 MCQ (Single Correct Answer)

Q. No. 1

GATE CSE 2015 Set 2

Consider two decision problems Q_1, Q_2 such that Q_1 reduces in polynomial time to 3-SAT and 3-SAT reduces in polynomial time to Q_2. Then which one of the following is consistent with the above statement?

A Q_1 is NP, Q_2 is NP hard.

B Q_2 is NP, Q_1 is NP hard.

C Both Q_1 and Q_2 are in NP.

D Both Q_1 and Q_2 are NP hard.

Q. No. 2

Which of the following statements are TRUE?

1. The problem of determining whether there exists a cycle in an undirected graph is in P.

2. The problem of determining whether there exists a cycle in an undirected graph is in NP.

3. If a problem A is NP−Complete, there exists a non-deterministic polynomial time algorithm to solve A.

A 1 , 2 and 3

B 1 and 2 only

C 2 and 3 only

D 1 and 3 only

Q. No. 3

GATE CSE 2009

Let π_A be a problem that belongs to the class NP. Then which one of the following is TRUE?

- A. There is no polynomial time algorithm for π_A
- B. If π_A can be solved deterministically in polynomial time, then P = NP
- C. If π_A is NP-hard, then it is NP-complete.
- D. π_A may be undecidable.

Q. No. 4

GATE CSE 2006

Let S be an NP-complete problem and Q and R be two other problems not known to be in NP. Q is polynomial time reducible to S and S is polynomial-time reducible to R. Which one of the following statements is true?

A R is NP-complete

B R is NP-hard

C Q is NP-complete

D Q is NP-hard

Q. No. 5

GATE CSE 2004

The problems 3-SAT and 2-SAT are

- (A) both in P
- (B) both NP-complete
- (C) NP-complete and in P respectively
- (D) undecidable and NP-complete respectively

Q. No. 6

GATE CSE 2003

Ram and Shyam have been asked to show that a certain problem Π is NP-complete. Ram shows a polynomial time reduction from the 3-SAT problem to Π, and Shyam shows a polynomial time reduction from Π to 3-SAT. Which of the following can be inferred from these reductions ?

A. Π is NP-hard but not NP-complete

B. Π is in NP, but is not NP-complete

C. Π is NP-complete

D. Π is neither NP-hard, nor in NP

Q. No. 7

GATE CSE 2014 Set 3

Consider the decision problem 2CNFSAT defined as follows:

{ Φ | Φ is a satisfiable propositional formula in CNF with at most two literal per clause}

For example, $\Phi = (x_1 \vee x_2) \wedge \left(x_1 \vee \overline{x}_3\right) \wedge (x_2 \vee x_4)$ is a Boolean formula and it is in 2CNFSAT.

The decision problem 2CNFSAT is

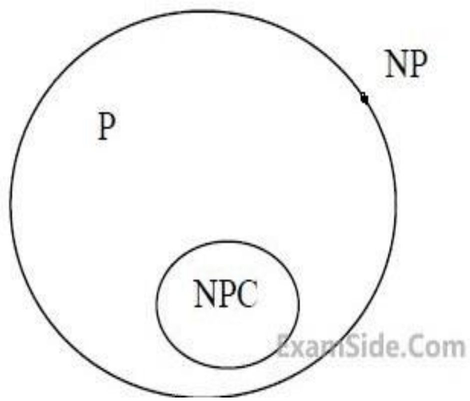
- A NP-Complete.
- B Solvable in polynomial time by reduction to directed graph reachability.
- C Solvable in constant time since any input instance is satisfiable.
- D NP-Hard, but not NP-complete.

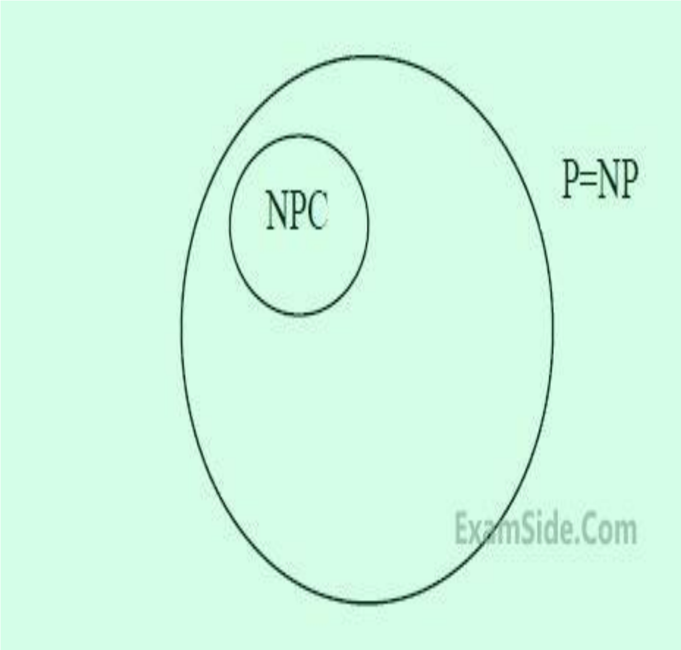
Q. No. 8

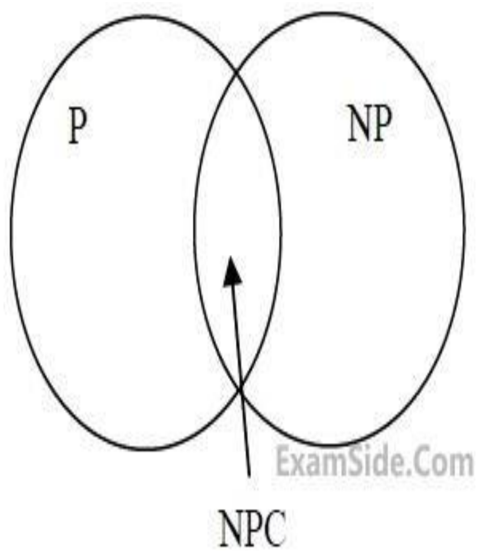
GATE CSE 2014 Set 1

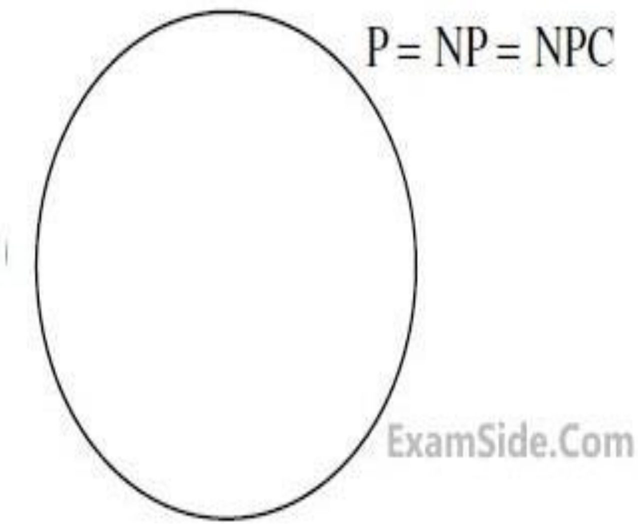
Suppose a polynomial time algorithm is discovered that correctly computes the largest clique in a given graph. In this scenario, which one of the following represents the correct Venn diagram of the complexity classes P, NP and NP Complete (NPC)?

A









Q. No. 10

GATE CSE 2006

Let $SHAM_3$ be the problem of finding a Hamiltonian cycle in a graph G = (V, E) with |V| divisible by 3 and $DHAM_3$ be the problem of determining if a Hamiltonian cycle exists in such graphs. Which one of the following is true?

There exist: search problem

- (A) Both $DHAM_3$ and $SHAM_3$ are NP-hard
- (B) $SHAM_3$ is NP-hard, but $DHAM_3$ is not
- (C) $DHAM_3$ is NP-hard, but $SHAM_3$ is not
- (D) Neither $DHAM_3$ nor $SHAM_3$ is NP-hard

Explanation: The problem of finding whether there exist a Hamiltonian Cycle or not is NP Hard and NP Complete Both.

Finding a Hamiltonian cycle in a graph G = (V,E) with V divisible by 3 is also NP Hard.

Q. No. 11

GATE CSE 1992

Which of the following problems is not NP-hard?

- A. Hamiltonian circuit problem
- B. The 0/1 Knapsack problem
- C. Finding bi-connected components of a graph
- D. The graph coloring problem

Thank You