Discrete-time Fourier Analysis

TRAN Hoang Tung

Information and Communication Technology (ICT) Department
University of Science and Technology of Hanoi (USTH)

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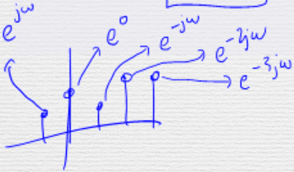
Definition

need examples!

The Discrete-time Fourier Transform (DTFT)

n: time

$$\boxed{X(e^{j\omega})} \triangleq \mathcal{F}[x(n)] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \rightarrow \left(e^{j\omega}\right)^{-n}$$



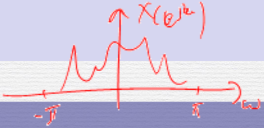
$X(\omega)$

ω: frequency

$X(e^{j\omega})$

better $\rightarrow$ $X(e^{j\omega})$

Two Properties



Periodicity

$$X(e^{j\omega}) \overset{?}{=} X(e^{j[\omega+2\pi]})$$

Implication: We need only one period of $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{+\infty}^{+\infty} x(n) \cdot e^{-jn\omega}$$

$$X(e^{j[\omega+2\pi]}) = \sum_{-\infty}^{+\infty} x(n) \cdot e^{-jn[\omega+2\pi]}$$

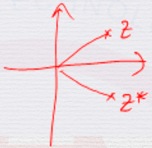
$$= \sum_{-\infty}^{+\infty} x(n) \cdot e^{-jn\omega} \cdot \underbrace{e^{-jn2\pi}}_{=1} = X(e^{j\omega})$$

Two Properties

Periodicity

$$X(e^{j\omega}) = X(e^{j[\omega+2\pi]})$$

Implication: We need only one period of $X(e^{j\omega})$



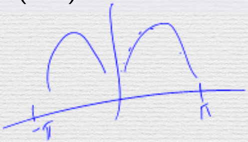
Symmetry

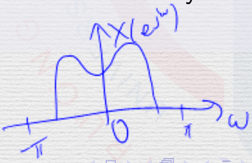
For real-valued $x(n)$

$(e^{j\omega})^* = e^{-j\omega}$

$$X(e^{-j\omega}) = X^*(e^{j\omega})$$

Implication: We now need to consider only a half period of $X(e^{j\omega})$





1 The Discrete-time Fourier Transform (DTFT)

2 DTFT Properties

3 Frequency Presentation of LTI Systems

4 Sampling & Reconstruction

FT

time frequency

+ * shift

Properties (1)

Linearity

$$\underbrace{\mathcal{F}[a_1x_1(n) + a_2x_2(n)]}_{F[x(n)]} = a_1\underbrace{\mathcal{F}[x_1(n)]}\; + a_2\underbrace{\mathcal{F}[x_2(n)]}$$

$$X(e^{j\omega}) = a_1 \cdot X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

$$F[x(n-5)] = X(e^{j\omega}) \cdot \ldots$$

$$e^{j5}$$

Properties (1)

Linearity

$$\mathcal{F}[a_1x_1(n) + a_2x_2(n)] = a_1\mathcal{F}[x_1(n)] + a_2\mathcal{F}[x_2(n)]$$

Time-shifting

$$\mathcal{F}[x(n-k)] = X(e^{j\omega})e^{-j\omega k}$$

Properties (1)

Linearity

$$\mathcal{F}[a_1x_1(n) + a_2x_2(n)] = a_1\mathcal{F}[x_1(n)] + a_2\mathcal{F}[x_2(n)]$$

Time-shifting

$$\mathcal{F}[x(n-k)] = X(e^{j\omega})e^{-j\omega k}$$

Frequency-shifting

$$\mathcal{F}[x(n)e^{j\omega_0 n}] = X(e^{j(\omega-\omega_0)})$$

Properties (2)

Conjugation

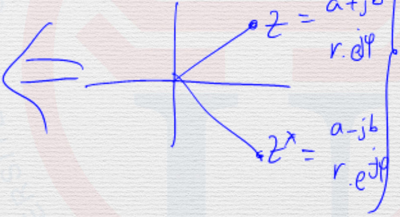
$$\mathcal{F}[x^*(n)] = X^*(e^{-j\omega})$$

$$X^*(e^{-j\omega}) = \left\{ \sum x(n) \cdot e^{j\omega n} \right\}^*$$

$$(a+b)^* = a^* + b^*$$

$$(ab)^* = a^* \cdot b^*$$

$$(r_1 \cdot e^{j\varphi_1} \cdot r_2 \cdot e^{j\varphi_2})$$



Properties (2)

Conjugation

$$\mathcal{F}[x^*(n)] = X^*(e^{-j\omega})$$

Folding

$$\mathcal{F}[x(-n)] = X(e^{-j\omega})$$

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Conjugation

$$\mathcal{F}[x^*(n)] = X^*(e^{-j\omega})$$

Folding

$$\mathcal{F}[x(-n)] = X(e^{-j\omega})$$

Convolution

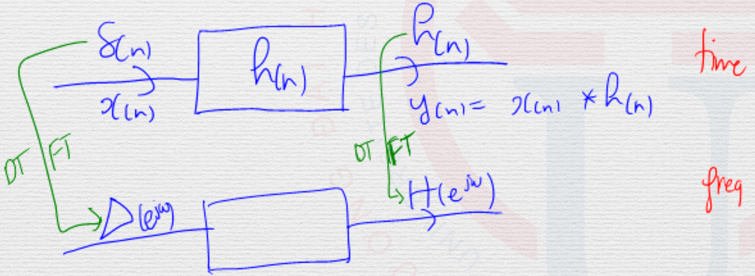
$$\mathcal{F}[x_1(n) * x_2(n)] = \mathcal{F}[x_1(n)]\mathcal{F}[x_2(n)] = X_1(e^{j\omega})X_2(e^{j\omega})$$

Frequency Response

Definition

The DTFT of an impulse response is called the frequency response (or transfer function) of an LTI system

$$H(e^{j\omega}) \triangleq \mathcal{F}[h(n)] = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}$$



Response to a Complex Exponential $e^{j\omega_0 n}$

$$x(n) \;\; e^{j\omega_0 n} \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n) \;\; H(e^{j\omega_0})\cdot e^{j\omega_0 n}$$

response : $y(n) = h(n) * x(n)$

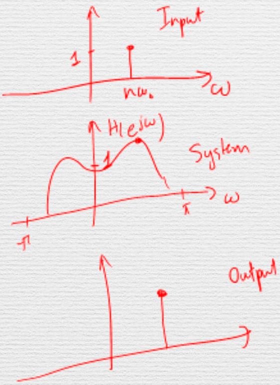
$$= h(n) * e^{j\omega_0 n}$$

$$= \sum_{k=-\infty}^{+\infty} h(k)\cdot e^{j\omega_0 (n-k)}$$

$$= \sum_{k=-\infty}^{+\infty} h(k)\cdot e^{-j\omega_0 k} \cdot e^{jn\omega_0}$$

$$= H(e^{j\omega_0})\cdot \underbrace{e^{jn\omega_0}}$$

$$= H(e^{j\omega_0})\cdot \; x(n)$$



Response to a Complex Exponential $e^{j\omega_0 n}$

$$y(n) = H(e^{j\omega_0})e^{j\omega_0 n}$$

Response to Sinusoidal $x(n) = Acos(\omega_0 n \theta_0)$

Response to Sinusoidal $x(n) = Acos(\omega_0 n \theta_0)$

$$y(n) = A|H(e^{j\omega_0})|cos(\omega_0 n + \theta_0 + \angle H(e^{j\omega_0}))$$

This response is called the steady-state response, denoted by $y_{ss}(n)$.

Given an LTI system

$$y(n) + \sum_{l=1}^{N} a_l y(n-l) = \sum_{m=0}^{M} b_m x(n-m)$$

$y(n) - 4y(n-3) + 3y(n-1) = x(n)$

$Y(e^{j\omega}) + \sum_{l=1}^{N} a_l \cdot Y(e^{j\omega}) \cdot e^{-j\omega l} = \sum_{m=0}^{M} b_m X(e^{j\omega}) e^{-j\omega m}$

$\Leftrightarrow Y(e^{j\omega}) \left[1 + \sum \ldots\right] = X(e^{j\omega}) \left[\sum \ldots\right]$

$\Leftrightarrow \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum \ldots}{1 + \sum \ldots} = H(e^{j\omega})$

transfer function

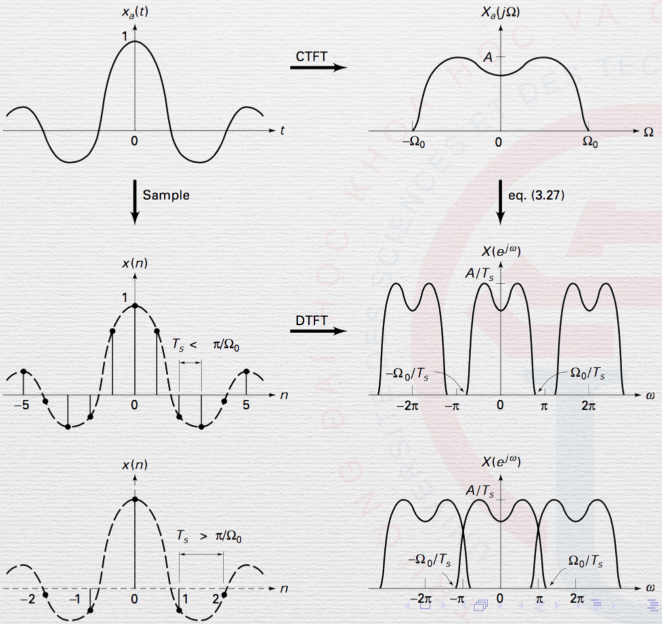
Given an LTI system

$$y(n) + \sum_{l=1}^{N} a_l y(n-l) = \sum_{m=0}^{M} b_m x(n-m)$$

its frequency response is

$$H(e^{j\omega}) = \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{1 + \sum_{l=1}^{N} a_l e^{-j\omega l}}$$

Sampling


$x_a(t)$
1
0
t
CTFT
$X_a(j\Omega)$
A
$-\Omega_0$
0
Ω_0
Ω
Sample
eq. (3.27)
$x(n)$
1
$T_s < \pi/\Omega_0$
−5
0
5
n
DTFT
$X(e^{j\omega})$
A/T_s
$-\Omega_0/T_s$
Ω_0/T_s
-2π
$-\pi$
0
π
2π
ω
$x(n)$
$T_s > \pi/\Omega_0$
−2
−1
0
1
2
n
$X(e^{j\omega})$
A/T_s
$-\Omega_0/T_s$
Ω_0/T_s
-2π
$-\pi$
0
π
2π
ω

Reconstruction