The z-Transform

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Lesson Objectives

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1. calculate the z-Transform of any given signals/systems
2. understand basic properties of the z-Transform
3. make a connection with difference equations

The Direct z-Transform

Definition

The z-transform of a discrete-time signal $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable.

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Notations

The z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv Z[x(n)]$$

and the relationship between $x(n)$ and $X(z)$ is indicated by

$$x(n) \xleftrightarrow{z} X(z)$$

Region Of Convergence (ROC)

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ROC of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

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2. $x_3(n) = [1, 2, 5, 0, \underset{\uparrow}{0}]$

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3. $x_4(n) = [2, 1, \underset{\uparrow}{2}, 5]$

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ROC of *finite-duration* signals

Entire z-plane, except possibly the point $z = 0$ and/or $z = \infty$

Region Of Convergence (ROC)

What about *infinite-duration* signals?

Determine the z-transform of the signal:

$$x(n) = \left[\underset{\uparrow}{1}, \left(\frac{1}{2}\right), \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots, \left(\frac{1}{2}\right)^n, \ldots\right]$$

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Solution

$$X(z) = \frac{1}{a - \frac{1}{2}z^{-1}}, \;\text{ROC: } |z| > \frac{1}{2}$$

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Plot the ROC?

ROC Outside a Circle

Exercise

Determine the z-transform of the signal:

$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

ROC Outside a Circle

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$$x(n) = \alpha^n u(n) = \begin{cases} \alpha^n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0. \end{cases}$$

Solution

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \text{ ROC: } |z| > |\alpha|$$

ROC Inside a Circle

Exercise

Determine the z-transform of the signal:

$$x(n) = -\alpha^n u(-n-1) = \begin{cases} 0 & \text{if } n \geq 0 \\ -\alpha^n & \text{if } n < 0. \end{cases}$$

ROC Inside a Circle

Exercise

Determine the z-transform of the signal:

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Solution

$$X(z) = -\frac{1}{1 - \alpha z^{-1}}, \text{ ROC: } |z| < |\alpha|$$

ROC

Exercise

Determine the z-transform of:

$$x(n) = \alpha^n u(n) + \beta^n u(-n-1)$$

ROC

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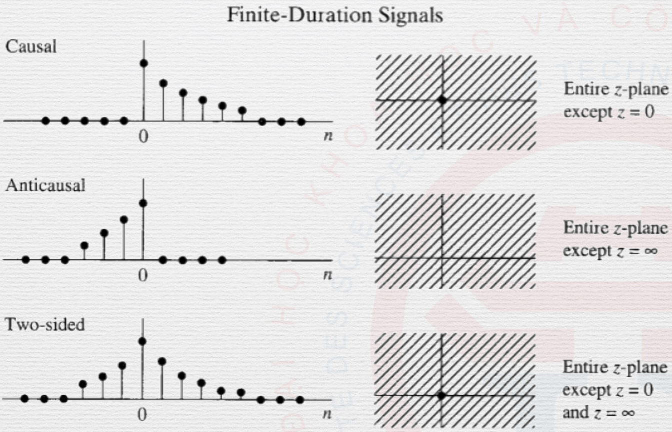
Solution

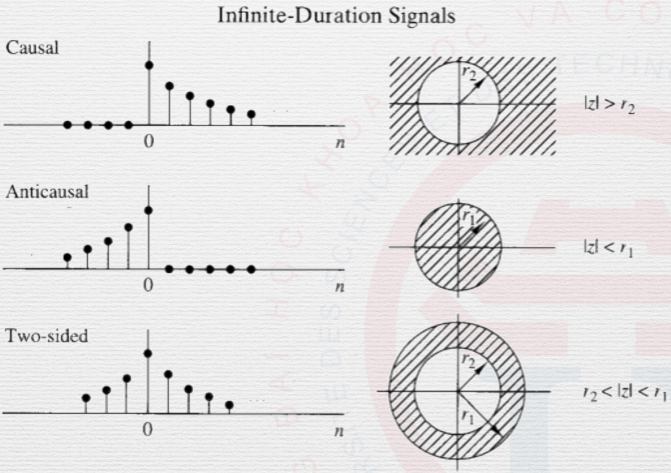
There are two cases:

- $|\beta| < |\alpha|$: two ROCs do not overlap, $X(z)$ does not exist
- $|\beta| > |\alpha|$: ROC of $X(z)$ is $|\beta| > |z| > |\alpha|$

ROC Summary


Finite-Duration Signals
Causal
0
n
Entire z-plane
except z = 0
Anticausal
0
n
Entire z-plane
except z = ∞
Two-sided
0
n
Entire z-plane
except z = 0
and z = ∞

ROC Summary



Infinite-Duration Signals
Causal
0
n
$|z| > r_2$
Anticausal
0
n
$|z| < r_1$
Two-sided
0
n
$r_2 < |z| < r_1$

Peroperties

The z-transform is a very powerful tool for the study of discrete-time signals thanks to some of its very important properties:

- Linearity
- Time shifting
- Scaling in the z-domain
- Time reversal
- Differentiation in the z-domain
- Convolution

Properties

Linearity

If $x_1(n) \xleftrightarrow{z} X_1(z)$ and $x_2(n) \xleftrightarrow{z} X_2(z)$ then

$$x(n) = a_1x_1(n) + a_2x_2(n) \xleftrightarrow{z}$$

Properties

Linearity

If $x_1(n) \xleftrightarrow{z} X_1(z)$ and $x_2(n) \xleftrightarrow{z} X_2(z)$ then

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Time Shifting

If

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

then

$$x(n-k) \overset{z}{\longleftrightarrow}$$

Properties

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If $x_1(n) \xleftrightarrow{z} X_1(z)$ and $x_2(n) \xleftrightarrow{z} X_2(z)$ then

$$x(n) = a_1x_1(n) + a_2x_2(n) \xleftrightarrow{z} X(z) = a_1X_1(z) + a_2X_2(z)$$

Time Shifting

If

$$x(n) \xleftrightarrow{z} X(z)$$

then

$$x(n-k) \xleftrightarrow{z} z^{-k}X(z)$$

Digital Signal Processing

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Objectives

The z-Transform

Properties of the z-Transform

Rational z-Transform

Scaling in the z-domain

For any constant a, if

$$x(n) \xleftrightarrow{z} X(z), \qquad ROC : r_1 < |z| < r_2$$

then

$$a^n x(n) \xleftrightarrow{z}$$

Scaling in the z-domain

For any constant a, if

$$x(n) \xleftrightarrow{z} X(z), \qquad ROC: r_1 < |z| < r_2$$

then

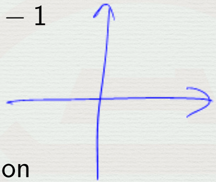
$$a^n x(n) \xleftrightarrow{z} X(a^{-1}z), \qquad ROC: |a|r_1 < |z| < |a|r_2$$

Exercise

Exercise

Determine the z-transform of the signal:

$$x(n) = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases}$$

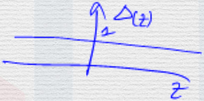


using

- Direct approach: z-transform definition
- Indirect approach: z-transform properties

$\delta(n) \Longrightarrow \Delta(z) = \sum_{-\infty}^{+\infty} x(n) \cdot z^{-n} = 1 \cdot z^{-0} = 1$

$\delta(n-1) \longrightarrow \Delta_1(z) = z^{-1} \cdot \Delta(z) = z^{-1}$



$\delta(n-k) \longrightarrow \Delta_k(z) = z^{-k}$

Properties

Time Reversal

If

$$x(n) \xleftrightarrow{z} X(z), \qquad ROC: r_1 < |z| < r_2$$

then

$$x(-n) \xleftrightarrow{z}$$

Properties

Time Reversal

If

$$x(n) \overset{z}{\longleftrightarrow} X(z), \qquad ROC: r_1 < |z| < r_2$$

then

$r_1 < |z^{-1}| < r_2$

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}), \qquad ROC: \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Differentiation in the z-domain

If

$$x(n) \overset{z}{\longleftrightarrow} X(z) = \left(\sum x(n) . z^{-n}\right)'$$

then

$$nx(n) \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$$

$$\frac{dz^n}{dz} = (z^n)' = n . z^{n-1}$$

Convolution Property

Recall: Discrete Convolution

$$y[n] \equiv (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

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Convolution of Two Sequences

If $x_1(n) \xleftrightarrow{z} X_1(z)$ and $x_2(n) \xleftrightarrow{z} X_2(z)$ then

$$x(n) = x_1(n) * x_2(n) \xleftrightarrow{z}$$

Convolution Property

Recall: Discrete Convolution

$$y[n] \equiv (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Convolution of Two Sequences

If $x_1(n) \overset{z}{\longleftrightarrow} X_1(z)$ and $x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$ then

$$x(n) = x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z)$$

Computing the Convolution using z-Transform

Step-by-step

1. Compute the z-transform of the two signals
2. Multiply the two z-transfroms $X(z) = X_1(z)X_2(z)$
3. Find the inverse z-transform of $X(z)$

Computing the Convolution using z-Transform

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1. Compute the z-transform of the two signals
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Exercise

Compute the convolution $x(n)$ of the signals:

$$x_1(n) = [1, \underset{\uparrow}{-2}, 1]$$

$$x_2(n) = \begin{cases} 1, & 0 \leq n \leq 5 \\ 0, & \text{elsewhere} \end{cases}$$

Digital Signal Processing

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Properties of the z-Transform

Rational z-Transform

$$X_1(z) = z - 2 + z^{-1}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

$$X_1(z) \cdot X_2(z) = (z - 2 + z^{-1})(1 + z^{-1} + \dots + z^{-5})$$

$$
\begin{aligned}
= \; & z - 2 + z^{-1} \\
& 1 - 2z^{-1} + z^{-2} \\
& z^{-1} - 2z^{-2} + z^{-3} \\
& z^{-2} - 2z^{-3} + z^{-4} \\
& z^{-3} - 2z^{-4} + z^{-5} \\
& z^{-4} - 2z^{-5} + z^{-6}
\end{aligned}
$$

$$X(z) = 1.z - 1.z^0 - z^{-5} + z^{-6} \quad \left(= \sum x(n) \cdot z^{-n}\right)$$

$$\Rightarrow x(n) = [1 \;\; \underset{\uparrow}{-1} \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; -1 \;\; 1]$$

Rational z-Transform

Definition

$X(z)$ is a rational function (a ratio of two polynomials in z)

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$\frac{1 + 3z^{-1} - z^{-7}}{3 + z^{-3} + z^{-6}}$$

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Poles and Zeros

$\notin$ ROC

- Poles: values of z for which $X(z) = \infty$ → roots of $A(z)$
- Zeros: values of z for which $X(z) = 0$ → roots of $B(z)$

$$x^2 - 3x + 2 = (x-2)(x-1)$$

$$\rightarrow \begin{cases} x = 1 \\ x = 2 \end{cases}$$

Exercise

Exercise

Determine the pole-zero plot for the signal

$$\underline{x(n)} = \underline{a^n u(n)}, \qquad a > 0$$

$$X(z) \Rightarrow \begin{cases} \text{zeros} \\ \text{poles} \end{cases}$$

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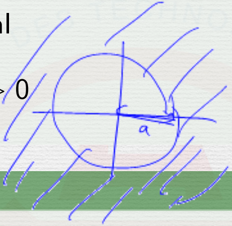

Solution

$$X(z) = \frac{1}{a - az^{-1}} = \frac{z}{z-a}, \qquad ROC : |z| > a$$

$X(z)$ has one zero at $z_1 = 0$ and one pole at $p_1 = a$. Note that the pole is not included in the ROC.