Lecture Notes for Sep 16, 2025 - Substitution Method

These notes are incomplete. Please watch the Sep 16 lecture and finish them.

This topic is **very difficult** to understand since it's an **abstract concept**. Take your time to read and **understand** it. Talk to *Dr. Otte* if you still don't get it. Otherwise refer to the Sep 16 lecture on Canvas.

This is the **second** of the 3 methods to getting the runtime of an algorithm: - Induction Method - **Substitution Method** - Master Method

Overview

The **substitution method** is based on **induction**. It **verifies** a perceived solution, not **solve a recurrence**.

The substitution method **shows** the lower **or** upper bound. If you want both, do the upper and lower bounds individually.

Comparing the Induction and Substitution Methods

For these examples, we'll be proving:

something (fill in later)

Induction Method

Proof:

$$\sum_{i=1}^{u} = \frac{u}{2}(u+1)$$

Base Case:

$$\text{holds for } i = 1$$
$$\sum_{i=1}^{1} i = \frac{1}{2}(1+1) = 1$$
$$\text{This is correct!}$$

Inductive Hypothesis:

$$\text{assume holds for } k \leq n$$
$$=> \sum_{i=1}^{k} i = \frac{k}{2}(k+1)$$

Inductive Step:

$$\text{...holds for successor}$$
$$=> \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$$
$$\text{also } \sum_{i=1}^{k+1} i = \frac{k+1}{2}(k+2)$$
$$\text{These need to be equal!}$$

Substitution:

Now, we **substitute** the inductive step with the hypothesis.

$$\text{also } \sum_{i=1}^{k+1} i = \frac{k+1}{2}(k+2)$$

Completion:

$$\text{show equality}$$
$$\frac{k}{2}(k+1) + (k+1) = \frac{k+1}{2}(k+2)$$

Substitution Method

Proof:

$$T(n) = 2T(\frac{n}{2}) + n$$

Where:

$$T(n) = O(n \lg n)$$
$$T(n) \leq cn \lg n$$

Base Case:

see book *(grumble grumble)*

Inductive Hypothesis:

This holds for $k < n$, specifically for $k = \frac{n}{2}$

$$=> T(\frac{n}{2}) \leq c\frac{n}{2} \lg \frac{n}{2}$$

Inductive Step:

$$T(n) = 2T(\frac{n}{2}) + n$$

We have an implicit **predecessor-successor** relationship via recurrence.

- Predecessor: $T(\frac{n}{2})$
- Successor: $T(n)$

Substitution:

Per **inductive hypothesis**, substitute $T(\frac{n}{2})$.

(there's more here I missed)

Completion:

This shows that the inequality holds.

$$T(n) \leq cn \lg \frac{n}{2} + n \leq cn \lg n$$

(there's more here I missed)

Template Solution

Example:

Prove that $T(n) = 2T(\frac{n}{2}) + n => T(n) = O(n \lg n)$

Proof:

1. Obtain the $\frac{n}{2}$ from the recurrence.
2. Rewrite $T(n) = O(n \lg n)$ as $T(n) \leq cn \lg n$

Make sure you do step 2! It's important!

Inductive Hypothesis:

1. Set $k = \frac{n}{2}$ *(chosen from proof)*
2. Substitute k into the simplified solution.

$$=> T(\frac{n}{2}) \leq c(\frac{n}{2}) \lg n$$

This is used for the **upper bound** proof.

Substitution:

Replace the **recurive term** with the **non-recursive term**. **Substitute** the $c(\frac{n}{2}) \lg n$ into the original equation, then **simplify**.

$$T(n) = 2T(\frac{n}{2}) + n$$

$$T(n) = 2c(\frac{n}{2}) \lg n + n$$

Completion:

$$T(n) \leq cn \lg \frac{n}{2} + n \leq cn \lg n$$

$$\lg \frac{n}{2} + \frac{1}{c} \leq \lg n$$

$$-1 + \frac{1}{c} \leq 0$$

$$\frac{1}{c} \leq 1$$

$$c \geq 1$$