Computing the Divergence of the Vector Field $\mathbf{F}$

1. Structure of the Basis Functions

We express the vector field $\mathbf{F}$ in terms of the Raviart-Thomas basis functions:

$$\mathbf{F}(r,s)=\sum_{j=1}^{N_{\text{int}}} c_j^{(E4)}\boldsymbol{\psi}_j^{(E4)}(r,s)+c_j^{(E5)}\boldsymbol{\psi}_j^{(E5)}(r,s)+\sum_{k=1}^{N_{\text{edge}}} c_k^{(n)}\boldsymbol{\psi}_k^{(n)}(r,s)$$

where:

- $N_{\text{int}}=\frac{(P+1)(P+2)}{2}$ is the number of interior DOFs (two per interior node).
- $N_{\text{edge}}=3(P+1)$ is the number of edge DOFs (one per edge node for normal continuity).

The total number of DOFs is:

$$N=N_{\text{int}}+N_{\text{edge}}=(P+1)(P+3)$$

2. Interpolation Equations

Each DOF provides a linear constraint, forming a system of equations.

(a) Interior Nodes

For each interior node (r_i,s_i), we impose:

$$\mathbf{F}(r_i,s_i)\cdot\mathbf{E4}(r_i,s_i)=d_i^{(E4)}$$

$$\mathbf{F}(r_i,s_i)\cdot\mathbf{E5}(r_i,s_i)=d_i^{(E5)}$$

Expanding using only interior basis functions:

$$\sum_{j=1}^{N_{\text{int}}} c_j^{(E4)}\boldsymbol{\psi}_j^{(E4)}(r_i,s_i)\cdot\mathbf{E4}(r_i,s_i)+\sum_{j=1}^{N_{\text{int}}} c_j^{(E5)}\boldsymbol{\psi}_j^{(E5)}(r_i,s_i)\cdot\mathbf{E5}(r_i,s_i)=d_i^{(E4)}$$

$$\sum_{j=1}^{N_{\text{int}}} c_j^{(E4)}\boldsymbol{\psi}_j^{(E4)}(r_i,s_i)\cdot\mathbf{E5}(r_i,s_i)+\sum_{j=1}^{N_{\text{int}}} c_j^{(E5)}\boldsymbol{\psi}_j^{(E5)}(r_i,s_i)\cdot\mathbf{E5}(r_i,s_i)=d_i^{(E5)}$$

(b) Edge Nodes

At each edge node (r_k,s_k), the normal component is enforced:

$$\mathbf{F}(r_k,s_k)\cdot\mathbf{n}_k=g_k$$

Expanding using only edge basis functions:

$$\sum_{j=1}^{N_{\text{edge}}} c_j^{(n_k)}\boldsymbol{\psi}_j^{(n_k)}(r_k,s_k)\cdot\mathbf{n}_k=g_k$$

(c) Zero Cross-Terms

Since interior basis functions $\boldsymbol{\psi}_j^{(E4)}$, $\boldsymbol{\psi}_j^{(E5)}$ are constructed only within the element interior, they satisfy:

$$\boldsymbol{\psi}_j^{(E4)} \cdot \mathbf{n}_k = 0, \quad \boldsymbol{\psi}_j^{(E5)} \cdot \mathbf{n}_k = 0$$

This means that the edge constraints only involve edge basis functions.

3. Matrix System

Now, we can express the full system in block matrix form:

$$\begin{bmatrix} A_{\text{int}} & 0 \\ 0 & A_{\text{edge}} \end{bmatrix} \begin{bmatrix} \mathbf{c}_{\text{int}} \\ \mathbf{c}_{\text{edge}} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{\text{int}} \\ \mathbf{b}_{\text{edge}} \end{bmatrix}$$

where:

- A_{int} represents interior projection equations (purely involving $\mathbf{E4}$, $\mathbf{E5}$).
- A_{edge} represents edge normal constraints (purely involving $\mathbf{n}$).
- The block structure arises because interior and edge basis functions are independent.

The unknown coefficients are split:

$$\mathbf{c} = \begin{bmatrix} \mathbf{c}_{\text{int}} \\ \mathbf{c}_{\text{edge}} \end{bmatrix}$$

The right-hand side is:

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_{\text{int}} \\ \mathbf{b}_{\text{edge}} \end{bmatrix}$$

4. Computing the Divergence

To compute the divergence of $\mathbf{F}$:

$$\nabla \cdot \mathbf{F}(r, s) = \sum_{j=1}^{N} c_j \nabla \cdot \boldsymbol{\psi}_j(r, s)$$

This can be represented in matrix form:

$$\text{div}(\mathbf{F}) = D\mathbf{c}$$

where:

- D is the divergence matrix with entries $D_{ij} = \nabla \cdot \boldsymbol{\psi}_j(r_i, s_i)$.
- $\mathbf{c}$ is the vector of coefficients combining interior and edge contributions.

Since the divergence matrix D only depends on the basis functions, it can be precomputed, and the divergence field at specific points is obtained through this matrix-vector multiplication.