We build the mixed elements for the DFR scheme with:

Solution Points: We use points within a reference triangle, excluding the edges, for a Lagrangian element of O(K) to store the solution. If we need derivatives, or interpolated quantities (Flux), we use the solution points element.

Flux Points: We use a customized Raviart-Thomas (RT) vector element of O(K+1) to store the vector Flux function computed from the solution values. The RT element is of order O(K+1) and is a combination of the points from the solution element for the interior, and points along the three triangle edges. The custom RT basis is established using a procedure outlined in: "A Direct Flux Reconstruction Scheme for Advection-Diffusion Problems on Triangular Grids" by Romero, Witherden and Jameson (2017). A complete RT basis, [B], is used together with unit basis vectors, [w], to satisfy the following equation:

[B_j(r_i) dot w_i] [C] = [delta_i_j]

=> solve for [C], the coefficients defining the custom RT basis

[C] is the vector of coefficients defining the basis using the basis vectors [w] and [B].

The [w] directions of the custom RT element basis are defined such that:

w([r]) = w(edge_locations) = unit normals on each of three edges

w([r]) = w(interior) = unit normals in the two primary geometry directions (r and s)

For order K there are:

- (K+1) locations on each edge, for a total of 3(K+1) edge basis functions.

- (K)(K+1) locations in the interior, half for the w_r direction and half for the w_s direction

- Total: (K+3)(K+1) basis functions for the custom RT_K element

Notes:

1) The number of interior points matches the Lagrangian element in 2D at order (K-1). A Lagrange element at order (K) has N_p = (K+1)(K+2)/2 degrees of freedom, so an order (K-1) element has (K)(K+1)/2 DOF. Considering that we need a term for each of the two interior directions at each interior point, we need exactly 2*N_p DOF at order (K-1) for the interior of the custom RT element, resulting in (K)(K+1) terms.

2) Note (1) confirms that the custom element requires exactly the same number of interior points (K)(K+1)/2 as a Lagrange element of order (K-1), which means we can use the custom RT element for the DFR approach, which needs to provide a O(K+1) element to preserve the gradient at O(K). We will use the solution points from the Lagrange element at O(K) to construct the interior of the O(K+1) RT element without requiring interpolation of the solution points, as they already reside at the same geometric locations.

(3) To create the custom RT element, we initialize the solution element, then define the custom RT element from the interior point locations of the solution element to ensure that they are colocated.

(4) To use the custom RT element:

a. calculate the solution, calculate the flux vector field from the solution at the solution points
b. transfer the flux vector field values to the DFR element interior
c. interpolate flux values at from the interior of the RT element to the locations on the triangle edges
d. use the method of characteristics to calculate the corrected flux using the neighbor element's edge flux combined with the edge flux from this element
e. calculate the gradient of the vector flux field using the custom RT element
f. transfer the gradient values from the RT element to the solution element for use in advancing the solution in differential form (directly)

By calculating the flux gradient in a way that yields an O(K) polynomial on the solution points, we can use the differential form of the equations directly for the solution, rather than using the traditional Galerkin approach of repeated integration by parts to obtain an equation with only first derivatives. This simplifies the solution process, resulting in a more efficient computational approach, in addition to making it easier to solve more complex equations with the identical formulation.