THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR TIME-DEPENDENT CONVECTION-DIFFUSION SYSTEMS

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Abstract. In this paper, we study the Local Discontinuous Galerkin methods for nonlinear, time-dependent convection-diffusion systems. These methods are an extension of the Runge-Kutta Discontinuous Galerkin methods for purely hyperbolic systems to convection-diffusion systems and share with those methods their high parallelizability, their high-order formal accuracy, and their easy handling of complicated geometries, for convection dominated problems. It is proven that for scalar equations, the Local Discontinuous Galerkin methods are L^2-stable in the nonlinear case. Moreover, in the linear case, it is shown that if polynomials of degree k are used, the methods are k-th order accurate for general triangulations; although this order of convergence is suboptimal, it is sharp for the LDG methods. Preliminary numerical examples displaying the performance of the method are shown.

1. Introduction. In this paper, we study the Local Discontinuous Galerkin (LDG) methods for nonlinear, convection-diffusion systems of the form

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{F}(\mathbf{u}, D\,\mathbf{u}) = 0, \ \text{in } (0, T) \times \Omega,$$

where $\Omega \subset \mathbb{R}^d$ and $\mathbf{u} = (u_1, \ldots, u_m)^t$. The LDG methods are an extension of the Runge-Kutta Discontinuous Galerkin (RKDG) methods for the nonlinear hyperbolic system

$$\partial_t \mathbf{u} + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0, \ \text{in } (0, T) \times \Omega,$$

introduced by the authors [12,13,14,15,16], and further developed by Atkins and Shu [2], Bassi and Rebay [4], Bey and Oden [7], Biswas, Devine, and Flaherty [8], deCougny *et al.* [17], Devine *et al.* [19, 20], Lowrie, Roe and van Leer [30], and by Özturan *et al.* [33]. The RKDG methods are constructed by applying the *explicit* time discretizations introduced by Shu [37] and Shu and Osher [38,39] to a space discretization that uses discontinuous basis functions. Since the space discretization is highly local in character and produces easily invertible, block-diagonal mass matrices and since the time-marching scheme is explicit, the RKDG method is a highly parallelizable method; see Biswas, Devine, and Flaherty [8]. Moreover, it is not only a formally high-order accurate method that can easily handle complicated geometries, but it satisfies a cell entropy inequality that enforces a nonlinear L^2-stability property even without the slope limiters typical of this method; see Jiang and Shu [27].

Extensions of the RKDG method to hydrodynamic models for semiconductor device simulation have been constructed by Chen *et al.* [9], [10]. In these extensions, approximations of the derivatives of the discontinuous approximate solution are obtained by using a simple projection into suitable finite elements spaces. This projection requires the inversion of global mass matrices, which in [9] and [10] are 'lumped' in order to maintain the high parallelizability of the method. Since in [9]

and [10] polynomials of degree one are used, the 'mass lumping' is justified; however, if polynomials of higher degree were used, the 'mass lumping' needed to enforce the full parallelizability of the method could cause a degradation of the formal order of accuracy. Fortunately, this is not an issue with the methods proposed by Bassi and Rebay [5] (see also Bassi *et al* [6]) for the compressible Navier-Stokes equations. In these methods, the original idea of the RKDG method is applied to *both u and D u* which are now considered as *independent* unknowns. Like the RKDG methods, the resulting methods are highly parallelizable methods of high-order accuracy which are very efficient for time-dependent, convection-dominated flows. The LDG methods are a generalization of these methods.

The basic idea to construct the LDG methods is to *suitably rewrite* the convection-diffusion system into a larger, degenerate, first-order system and then discretize it by the RKDG method. By a careful choice of this rewriting, nonlinear stability can be achieved even without slope limiters, just as the RKDG method in the purely hyperbolic case; see Jiang and Shu [27]. In the linear case, the stability result leads to the sub-optimal rate of $(\Delta x)^k$ for the $L^\infty(0, T; L^2)$-norm of the error if polynomials of degree at most k are used. However, these estimates are sharp, as numerical evidence reported in Bassi *et al.* [6] and in this paper indicate. In the purely hyperbolic case, the rate of convergence of $(\Delta x)^{k+1/2}$ is recovered, as expected. Indeed, this is the same rate of convergence obtained for the original Discontinuous Galerkin method (introduced by Reed and Hill [35]) for purely hyperbolic case by Johnson and Pitkaränta [28] and confirmed to be optimal by Peterson [34]. LeSaint and Raviart [29] proved a rate of convergence of $(\Delta x)^k$ for general triangulations and of $(\Delta x)^{k+1}$ for Cartesian grids; Richter [36] obtained the optimal rate of convergence of $(\Delta x)^{k+1}$ for some structured two-dimensional non-Cartesian grids. The technique for proving the error estimates used in this paper is different from the techniques used in the above mentioned papers. It is very simple and relies, as ex-

pected, on a straightforward combination of (i) the L^2-stability of the LDG method and of (ii) the approximation properties of the finite element spaces.

The LDG methods introduced in this paper are very different from the so-called Discontinuous Galerkin (DG) method for parabolic problems introduced by Jamet [26] and studied by Eriksson, Johnson, and Thomée [25], Eriksson and Johnson [21, 22, 23, 24], and more recently by Makridakis and Babuška [31]. In the DG method, the approximate solution is discontinuous only in time, not in space; in fact, the space discretization is the standard Galerkin discretization with *continuous* finite elements. This is in strong contrast with the space discretizations of the LDG methods which use *discontinuous* finite elements. To emphasize this difference, we call the methods developed in this paper the **Local** Discontinuous Galerkin methods. We also must emphasize that the large number of degrees of freedom and the restrictive conditions of the size of the time step for explicit time-discretizations, render the LDG methods inefficient for diffusion-dominated problems; in this situation, the use of methods with continuous-in-space approximate solutions is recommended. However, as for the successful RKDG methods for purely hyperbolic problems, the extremely local domain of dependency of the LDG methods allows a very efficient parallelization that by far compensates for the extra number of degrees of freedom in the case of convection-dominated flows.

Many researchers have worked in the devising and analysis of numerical methods for convection-dominated flows. In particular, Dawson [18] and, more recently, Arbogast and Wheeler [1] have developed and analyzed methods that share several properties with the LDG methods: They use discontinuous-in-space approximations, they are locally conservative, and they approximate the diffusive fluxes with independent variables (by using a mixed method). We refer the reader interested in numerical methods for convection-dominated flows to [18] and [1] and the references therein.

Another numerical method that uses discontinuous approximations is the one proposed and studied by Baker *et al.* [3]. This method, however, is not for convection-dominated flows but for the Stokes problem. The advantage of using discontinuous approximations in this setting is that this allows for a pointwise verification of the incompressibility condition at the interior of each triangle. Optimal estimates are obtained.

In this paper, we restrict ourselves to the semidiscrete LDG methods for convection-diffusion problems with periodic boundary conditions. Our aim is to clearly display the most distinctive features of the LDG methods in a setting as simple as possible. The fully discrete methods for convection-diffusion problems in bounded domains will be treated in a forthcoming paper. This paper is organized as follows: In §2, we introduce the LDG methods for the simple one-dimensional case $d = 1$ in which

$$\mathbf{F}(u, Du) = f(u) - a(u)\, \partial_x u,$$

u is a scalar and $a(u) \geq 0$ and show some preliminary numerical results displaying the performance of the method. In this simple setting, the main ideas of how to devise the method and how to analyze it can be clearly displayed in a simple way. Thus, the L^2-stability of the method is proven in the general nonlinear case and the rate of convergence of $(\Delta x)^k$ in the L$^\infty(0, T; \text{L}^2)$-norm for polynomials of degree $k \geq 0$ in the linear case is obtained; this estimate is sharp. In §3, we extend these results to the case in which u is a scalar and

$$\mathbf{F}_i(u, Du) = f_i(u) - \sum_{1 \leq j \leq d} a_{ij}(u)\, \partial_{x_j} u,$$

where a_{ij} defines a positive semidefinite matrix. Again, the L^2-stability of the method is proven for the general nonlinear case and the rate of convergence of $(\Delta x)^k$ in the L$^\infty(0, T; \text{L}^2)$-norm for polynomials of degree $k \geq 0$ and arbitrary triangulations is proven in the linear case. In this case, the multidimensionality of the

problem and the arbitrariness of the grids increase the technicality of the analysis of the method which, nevertheless, uses the same ideas of the one-dimensional case. In §4, the extension of the LDG method to multidimensional systems is briefly described and concluding remarks are offered.

2. The LDG methods for the one-dimensional case. In this section, we present and analyze the LDG methods for the following simple model problem:

$$\partial_t\, u + \partial_x\, (f(u) - a(u)\, \partial_x\, u) = 0 \quad \text{in } (0, T) \times (0, 1), \tag{2.1a}$$

$$u(t = 0) = u_0, \qquad\qquad \text{on } (0, 1), \tag{2.1b}$$

with periodic boundary conditions.

a. General formulation and main properties. To define the LDG method, we introduce the new variable $q = \sqrt{a(u)}\, \partial_x\, u$ and rewrite the problem (2.1) as follows:

$$\partial_t\, u + \partial_x\, (f(u) - \sqrt{a(u)}\, q) = 0 \quad \text{in } (0, T) \times (0, 1), \tag{2.2a}$$

$$q - \partial_x\, g(u) = 0 \qquad\qquad \text{in } (0, T) \times (0, 1), \tag{2.2b}$$

$$u(t = 0) = u_0, \qquad\qquad \text{on } (0, 1), \tag{2.2c}$$

where $g(u) = \int^u \sqrt{a(s)}\, ds$. The LDG method for (2.1) is now obtained by simply discretizing the above system with the Discontinuous Galerkin method.

To do that, we follow [13] and [14]. We define the flux $\mathbf{h} = (\, h_u, h_q\,)^t$ as follows:

$$\mathbf{h}(u, q) = (\, f(u) - \sqrt{a(u)}\, q\,, \, -g(u)\,)^t. \tag{2.3}$$

For each partition of the interval $(0, 1)$, $\{\, x_{j+1/2}\,\}_{j=0}^N$, we set $I_j = (x_{j-1/2}, x_{j+1/2})$, and $\Delta x_j = x_{j+1/2} - x_{j-1/2}$ for $j = 1, \ldots, N$; we denote the quantity $\max_{1 \leq j \leq N} \Delta x_j$ by Δx. We seek an approximation $\mathbf{w}_h = (u_h, q_h)^t$ to $\mathbf{w} = (u, q)^t$ such that for each time $t \in [0, T]$, both $u_h(t)$ and $q_h(t)$ belong to the finite dimensional space

$$V_h = V_h^k = \{v \in L^1(0, 1) : v|_{I_j} \in P^k(I_j), \; j = 1, \ldots, N\}, \tag{2.4}$$

where $P^k(I)$ denotes the space of polynomials in I of degree at most k. In order to determine the approximate solution (u_h, q_h), we first note that by multiplying (2.2a), (2.2b), and (2.2c) by arbitrary, smooth functions v_u, v_q, and v_i, respectively, and integrating over I_j, we get, after a simple formal integration by parts in (2.2a) and (2.2b),

$$\int_{I_j} \partial_t u(x,t)\, v_u(x)\, dx - \int_{I_j} h_u(\mathbf{w}(x,t))\, \partial_x v_u(x)\, dx$$
$$+ h_u(\mathbf{w}(x_{j+1/2}, t))\, v_u(x_{j+1/2}^-) - h_u(\mathbf{w}(x_{j-1/2}, t))\, v_u(x_{j-1/2}^+) = 0, \tag{2.5a}$$

$$\int_{I_j} q(x,t)\, v_q(x)\, dx - \int_{I_j} h_q(\mathbf{w}(x,t))\, \partial_x v_q(x)\, dx$$
$$+ h_q(\mathbf{w}(x_{j+1/2}, t))\, v_q(x_{j+1/2}^-) - h_q(\mathbf{w}(x_{j-1/2}, t))\, v_q(x_{j-1/2}^+) = 0, \tag{2.5b}$$

$$\int_{I_j} u(x,0)\, v_i(x)\, dx = \int_{I_j} u_0(x)\, v_i(x)\, dx. \tag{2.5c}$$

Next, we replace the smooth functions v_u, v_q, and v_i by test functions $v_{h,u}$, $v_{h,q}$, and $v_{h,i}$, respectively, in the finite element space V_h and the exact solution $\mathbf{w} = (u,q)^t$ by the approximate solution $\mathbf{w}_h = (u_h, q_h)^t$. Since this function is discontinuous in each of its components, we must also replace the nonlinear flux $\mathbf{h}(\mathbf{w}(x_{j+1/2}, t))$ by a numerical flux $\hat{\mathbf{h}}(\mathbf{w})_{j+1/2}(t) = (\hat{h}_u(\mathbf{w}_h)_{j+1/2}(t), \hat{h}_q(\mathbf{w}_h)_{j+1/2}(t))$ that will be suitably chosen later. Thus, the approximate solution given by the LDG method is defined as the solution of the following weak formulation:

$$\int_{I_j} \partial_t u_h(x,t)\, v_{h,u}(x)\, dx - \int_{I_j} h_u(\mathbf{w}_h(x,t))\, \partial_x v_{h,u}(x)\, dx$$
$$+ \hat{h}_u(\mathbf{w}_h)_{j+1/2}(t)\, v_{h,u}(x_{j+1/2}^-) - \hat{h}_u(\mathbf{w}_h)_{j-1/2}(t)\, v_{h,u}(x_{j-1/2}^+) = 0, \qquad \forall\, v_{h,u} \in P^k(I_j), \tag{2.6a}$$

$$\int_{I_j} q_h(x,t)\, v_{h,q}(x)\, dx - \int_{I_j} h_q(\mathbf{w}_h(x,t))\, \partial_x v_{h,q}(x)\, dx$$
$$+ \hat{h}_q(\mathbf{w}_h)_{j+1/2}(t)\, v_{h,q}(x_{j+1/2}^-) - \hat{h}_q(\mathbf{w}_h)_{j-1/2}(t)\, v_{h,q}(x_{j-1/2}^+) = 0, \qquad \forall\, v_{h,q} \in P^k(I_j), \tag{2.6b}$$

$$\int_{I_j} u_h(x,0)\, v_{h,i}(x)\, dx = \int_{I_j} u_0(x)\, v_{h,i}(x)\, dx, \qquad \forall\, v_{h,i} \in P^k(I_j). \tag{2.6c}$$

It only remains to choose the numerical flux $\hat{\mathbf{h}}(\mathbf{w}_h)_{j+1/2}(t)$. We use the notation:

$$[p] = p^+ - p^-, \quad \text{and} \quad \overline{p} = \frac{1}{2}(p^+ + p^-),$$

and $p^\pm_{j+1/2} = p(x^\pm_{j+1/2})$. To be consistent with the type of numerical fluxes used in the RKDG methods, we consider numerical fluxes of the form

$$\hat{\mathbf{h}}(\mathbf{w}_h)_{j+1/2}(t) \equiv \hat{\mathbf{h}}(\mathbf{w}_h(x^-_{j+1/2}, t), \mathbf{w}_h(x^+_{j+1/2}, t)),$$

that (i) are locally Lipschitz and consistent with the flux $\mathbf{h}$, (ii) allow for a local resolution of q_h in terms of u_h, (iii) reduce to an E-flux (see Osher [32]) when $a(\cdot) \equiv 0$, and that (iv) enforce the L^2-stability of the method.

To reflect the convection-diffusion nature of the problem under consideration, we write our numerical flux as the sum of a convective flux and a diffusive flux:

$$\hat{\mathbf{h}}(\mathbf{w}^-, \mathbf{w}^+) = \hat{\mathbf{h}}_{conv}(\mathbf{w}^-, \mathbf{w}^+) + \hat{\mathbf{h}}_{diff}(\mathbf{w}^-, \mathbf{w}^+). \tag{2.7a}$$

The convective flux is given by

$$\hat{\mathbf{h}}_{conv}(\mathbf{w}^-, \mathbf{w}^+) = \left(\hat{f}(u^-, u^+), 0\right)^t, \tag{2.7b}$$

where $\hat{f}(u^-, u^+)$ is any locally Lipschitz E-flux consistent with the nonlinearity f, and the diffusive flux is given by

$$\hat{\mathbf{h}}_{diff}(\mathbf{w}^-, \mathbf{w}^+) = \left(-\frac{[g(u)]}{[u]} \overline{q}, -\overline{g(u)}\right)^t - \mathbb{C}_{diff}[\mathbf{w}], \tag{2.7c}$$

where

$$\mathbb{C}_{diff} = \begin{pmatrix} 0 & c_{12} \\ -c_{12} & 0 \end{pmatrix}, \tag{2.7d}$$

$$c_{12} = c_{12}(\mathbf{w}^-, \mathbf{w}^+) \quad \text{is locally Lipschitz}, \tag{2.7e}$$

$$c_{12} \equiv 0 \quad \text{when } a(\cdot) \equiv 0. \tag{2.7f}$$

We claim that this flux satisfies the properties (i) to (iv).

Let us prove our claim. That the flux $\hat{\mathbf{h}}$ is consistent with the flux $\mathbf{h}$ easily follows from their definitions, (2.3) and (2.7). That $\hat{\mathbf{h}}$ is locally Lipschitz follows from the fact that $\hat{f}(\cdot, \cdot)$ is locally Lipschitz and from (2.7d); we assume that $f(\cdot)$ and $a(\cdot)$ are locally Lipschitz functions, of course. Property (i) is hence satisfied.

That the approximate solution q_h can be resolved element by element in terms of u_h by using (2.6b) follows from the fact that, by (2.7c), the flux $\hat{h}_q = -\overline{g(u)} - c_{12}\,[\,u\,]$ is independent of q_h. Property (ii) is hence satisfied.

Property (iii) is also satisfied by (2.7f) and by the construction of the convective flux.

To see that the property (iv) is satisfied, let us first rewrite the flux $\hat{\mathbf{h}}$ in the following way:

$$\hat{\mathbf{h}}(\mathbf{w}^-, \mathbf{w}^+) = \big(\frac{[\,\phi(u)\,]}{[\,u\,]} - \frac{[\,g(u)\,]}{[\,u\,]}\,\overline{q}, \; -\overline{g(u)}\,\big)^t - \mathbb{C}\,[\,\mathbf{w}\,],$$

where

$$\mathbb{C} = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12} & 0 \end{pmatrix}, \quad c_{11} = \frac{1}{[\,u\,]}\Big(\frac{[\,\phi(u)\,]}{[\,u\,]} - \hat{f}(u^-, u^+)\Big).$$

with $\phi(u)$ defined by $\phi(u) = \int^u f(s)\,ds$. Since $\hat{f}(\cdot, \cdot)$ is an E-flux,

$$c_{11} = \frac{1}{[\,u\,]^2}\int_{u^-}^{u^+}\big(f(s) - \hat{f}(u^-, u^+)\big)\,ds \geq 0,$$

and so, by (2.7d), the matrix $\mathbb{C}$ is semipositive definite. The property (iv) follows from this fact and from the following result.

Proposition 2.1. (L^2-stability) *We have,*

$$\frac{1}{2}\int_0^1 u_h^2(x, T)\,dx + \int_0^T\int_0^1 q_h^2(x, t)\,dx\,dt + \Theta_{T,\mathbb{C}}([\mathbf{w}_h]) \leq \frac{1}{2}\int_0^1 u_0^2(x)\,dx,$$

where

$$\Theta_{T,\mathbb{C}}([\mathbf{w}_h]) = \int_0^T \sum_{1 \leq j \leq N}\Big\{[\mathbf{w}_h(t)]^t\mathbb{C}\,[\mathbf{w}_h(t)]\Big\}_{j+1/2}\,dt.$$

This result will be proven in section §2.c. Thus, this shows that the flux $\hat{\mathbf{h}}$ given by (2.7) does satisfy the properties (i) to (iv)- as claimed.

Now, we turn to the question of the quality of the approximate solution defined by the LDG method. In the linear case $f' \equiv c$ and $a(\cdot) \equiv a$, from the above stability result and from the the approximation properties of the finite element space V_h, we can prove the following error estimate. We denote the $\mathrm{L}^2(0,1)$-norm of the ℓ-th derivative of u by $|u|_\ell$.

Theorem 2.2. (L^2-error estimate) *Let* $\mathbf{e}$ *be the approximation error* $\mathbf{w} - \mathbf{w}_h$. *Then we have,*

$$\left\{ \int_0^1 |e_u(x,T)|^2 \, dx + \int_0^T \int_0^1 |e_q(x,t)|^2 \, dx \, dt + \Theta_{T,\mathbb{C}}([\mathbf{e}]) \right\}^{1/2} \le C \, (\Delta x)^k,$$

where $C = C(k, |u|_{k+1}, |u|_{k+2})$. *In the purely hyperbolic case* $a = 0$, *the constant* C *is of order* $(\Delta x)^{1/2}$ *and in the purely parabolic case* $c = 0$, *the constant* C *is of order* Δx *for even values of* k *for uniform grids and for* $\mathbb{C}$ *identically zero.*

This result will be proven in section §2.d. The above error estimate gives a suboptimal order of convergence, but it is sharp for the LDG methods. Indeed, Bassi *et al* [6] report an order of convergence of order $k+1$ for even values of k and of order k for odd values of k for a steady state, purely elliptic problem for uniform grids and for $\mathbb{C}$ identically zero. Our numerical results for a purely parabolic problem give the same conclusions; see Table 5 in the section §2.b.

Our error estimate is also sharp in that the optimal order of convergence of $k+1/2$ is recovered in the purely hyperbolic case, as expected. This improvement of the order of convergence is a reflection of the *semipositive definiteness* of the matrix $\mathbb{C}$, which enhances the stability properties of the LDG method. Indeed, since in the purely hyperbolic case

$$\Theta_{T,\mathbb{C}}([\mathbf{w}_h]) = \int_0^T \sum_{1 \le j \le N} \left\{ [u_h(t)]^t \, c_{11} \, [u_h(t)] \right\}_{j+1/2} dt,$$

the method enforces a control of the jumps of the variable u_h, as shown in Proposition 2.1. This additional control is reflected in the improvement of the order of accuracy from k in the general case to $k + 1/2$ in the purely hyperbolic case.

However, this can only happen in the purely hyperbolic case for the LDG methods. Indeed, since $c_{11} = 0$ for $c = 0$, the control of the jumps of u_h is not enforced in the purely parabolic case. As indicated by the numerical experiments of Bassi *et al.* [6] and those of section §2.b below, this can result in the effective degradation of the order of convergence. To remedy this situation, the control of the jumps of u_h in the purely parabolic case can be easily enforced by letting c_{11} be strictly positive if $|c| + |a| > 0$. Unfortunately, this is not enough to guarantee an improvement of the accuracy: an additional control on the jumps of q_h is required! This can be easily achieved by allowing the matrix $\mathbb{C}$ to be *symmetric and positive definite* when $a > 0$. In this case, the order of convergence of $k + 1/2$ can be easily obtained for the general convection-diffusion case. However, this would force the matrix entry c_{22} to be nonzero and the property (ii) of local resolvability of q_h in terms of u_h would not be satisfied anymore. As a consequence, the high parallelizability of the LDG would be lost.

The above result shows how strongly the order of convergence of the LDG methods depend on the choice of the matrix $\mathbb{C}$. In fact, the numerical results of section §2.b in uniform grids indicate that with yet another choice of the matrix $\mathbb{C}$, see (2.9), the LDG method converges with the optimal order of $k + 1$ in the general case. The analysis of this phenomenon constitutes the subject of ongoing work.

b. Preliminary numerical results.

In this section we provide preliminary numerical results for the schemes discussed in this paper. We will only provide results for the following one dimensional, linear

convection diffusion equation

$$\partial_t u + c\, \partial_x u - a\, \partial_x^2 u = 0 \quad \text{in } (0, T) \times (0, 2\,\pi),$$

$$u(t = 0, x) = \sin(x), \qquad \text{on } (0, 2\,\pi),$$

where c and $a \geq 0$ are both constants; periodic boundary conditions are used. The exact solution is $u(t, x) = e^{-at}\sin(x - ct)$. We compute the solution up to $T = 2$, and use the LDG method with $\mathbb{C}$ defined by

$$\mathbb{C} = \begin{pmatrix} \frac{|c|}{2} & -\frac{\sqrt{a}}{2} \\ \frac{\sqrt{a}}{2} & 0 \end{pmatrix}. \tag{2.9}$$

We notice that, for this choice of fluxes, the approximation to the convective term cu_x is the standard upwinding, and that the approximation to the diffusion term $a\,\partial_x^2 u$ is the standard three point central difference, for the P^0 case. On the other hand, if one uses a central flux corresponding to $c_{12} = -c_{21} = 0$, one gets a spread-out five point central difference approximation to the diffusion term $a\,\partial_x^2 u$.

The LDG methods based on P^k, with $k = 1, 2, 3, 4$ are tested. Elements with equal size are used. Time discretization is by the third-order accurate TVD Runge-Kutta method [38], with a sufficiently small time step so that error in time is negligible comparing with spatial errors. We list the L_∞ errors and numerical orders of accuracy, for u_h, as well as for its derivatives suitably scaled $\Delta x^m \partial_x^m u_h$ for $1 \leq m \leq k$, at the center of of each element. This gives the complete description of the error for u_h over the whole domain, as u_h in each element is a polynomial of degree k. We also list the L_∞ errors and numerical orders of accuracy for q_h at the element center.

In all the convection-diffusion runs with $a > 0$, accuracy of at least $(k + 1)$-th order is obtained, for both u_h and q_h, when P^k elements are used. See Tables 1 to 3. The P^4 case for the purely convection equation $a = 0$ seems to be not in the asymptotic regime yet with $N = 40$ elements (further refinement with $N = 80$ suffers

from round-off effects due to our choice of non-orthogonal basis functions), Table 4. However, the absolute values of the errors are comparable with the convection dominated case in Table 3.

Table 1. The heat equation $a = 1$, $c = 0$. L_∞ errors and numerical order of accuracy, measured at the center of each element, for $\Delta x^m \partial_x^m u_h$ for $0 \leq m \leq k$, and for q_h.

k	variable	$N = 10$ error	$N = 20$ error	order	$N = 40$ error	order
1	u	4.55E-4	5.79E-5	2.97	7.27E-6	2.99
	$\Delta x\, \partial_x u$	9.01E-3	2.22E-3	2.02	5.56E-4	2.00
	q	4.17E-5	2.48E-6	4.07	1.53E-7	4.02
2	u	1.43E-4	1.76E-5	3.02	2.19E-6	3.01
	$\Delta x\, \partial_x u$	7.87E-4	1.03E-4	2.93	1.31E-5	2.98
	$(\Delta x)^2\, \partial_x^2 u$	1.64E-3	2.09E-4	2.98	2.62E-5	2.99
	q	1.42E-4	1.76E-5	3.01	2.19E-6	3.01
3	u	1.54E-5	9.66E-7	4.00	6.11E-8	3.98
	$\Delta x\, \partial_x u$	3.77E-5	2.36E-6	3.99	1.47E-7	4.00
	$(\Delta x)^2\, \partial_x^2 u$	1.90E-4	1.17E-5	4.02	7.34E-7	3.99
	$(\Delta x)^3\, \partial_x^3 u$	2.51E-4	1.56E-5	4.00	9.80E-7	4.00
	q	1.48E-5	9.66E-7	3.93	6.11E-8	3.98
4	u	2.02E-7	5.51E-9	5.20	1.63E-10	5.07
	$\Delta x\, \partial_x u$	1.65E-6	5.14E-8	5.00	1.61E-9	5.00
	$(\Delta x)^2\, \partial_x^2 u$	6.34E-6	2.04E-7	4.96	6.40E-9	4.99
	$(\Delta x)^3\, \partial_x^3 u$	2.92E-5	9.47E-7	4.95	2.99E-8	4.99
	$(\Delta x)^4\, \partial_x^4 u$	3.03E-5	9.55E-7	4.98	2.99E-8	5.00
	q	2.10E-7	5.51E-9	5.25	1.63E-10	5.07

Table 2. The convection diffusion equation $a = 1$, $c = 1$. L_∞ errors and numerical order of accuracy, measured at the center of each element, for $\Delta x^m \partial_x^m u_h$ for $0 \leq m \leq k$, and for q_h.

k	variable	$N = 10$ error	$N = 20$ error	order	$N = 40$ error	order
1	u	6.47E-4	1.25E-4	2.37	1.59E-5	2.97
	$\Delta x\, \partial_x u$	9.61E-3	2.24E-3	2.10	5.56E-4	2.01
	q	2.96E-3	1.20E-4	4.63	1.47E-5	3.02
2	u	1.42E-4	1.76E-5	3.02	2.18E-6	3.01
	$\Delta x\, \partial_x u$	7.93E-4	1.04E-4	2.93	1.31E-5	2.99
	$(\Delta x)^2\, \partial_x^2 u$	1.61E-3	2.09E-4	2.94	2.62E-5	3.00
	q	1.26E-4	1.63E-5	2.94	2.12E-6	2.95
3	u	1.53E-5	9.75E-7	3.98	6.12E-8	3.99
	$\Delta x\, \partial_x u$	3.84E-5	2.34E-6	4.04	1.47E-7	3.99
	$(\Delta x)^2\, \partial_x^2 u$	1.89E-4	1.18E-5	4.00	7.36E-7	4.00
	$(\Delta x)^3\, \partial_x^3 u$	2.52E-4	1.56E-5	4.01	9.81E-7	3.99
	q	1.57E-5	9.93E-7	3.98	6.17E-8	4.01
4	u	2.04E-7	5.50E-9	5.22	1.64E-10	5.07
	$\Delta x\, \partial_x u$	1.68E-6	5.19E-8	5.01	1.61E-9	5.01
	$(\Delta x)^2\, \partial_x^2 u$	6.36E-6	2.05E-7	4.96	6.42E-8	5.00
	$(\Delta x)^3\, \partial_x^3 u$	2.99E-5	9.57E-7	4.97	2.99E-8	5.00
	$(\Delta x)^4\, \partial_x^4 u$	2.94E-5	9.55E-7	4.95	3.00E-8	4.99
	q	1.96E-7	5.35E-9	5.19	1.61E-10	5.06

Table 3. The convection dominated convection diffusion equation $a = 0.01$, $c = 1$. L_∞ errors and numerical order of accuracy, measured at the center of each element, for $\Delta x^m \partial_x^m u_h$ for $0 \le m \le k$, and for q_h.

k	variable	$N = 10$	$N = 20$		$N = 40$	
		error	error	order	error	order
1	u	7.14E-3	9.30E-4	2.94	1.17E-4	2.98
	$\Delta x\, \partial_x u$	6.04E-2	1.58E-2	1.93	4.02E-3	1.98
	q	8.68E-4	1.09E-4	3.00	1.31E-5	3.05
2	u	9.59E-4	1.25E-4	2.94	1.58E-5	2.99
	$\Delta x\, \partial_x u$	5.88E-3	7.55E-4	2.96	9.47E-5	3.00
	$(\Delta x)^2\, \partial_x^2 u$	1.20E-2	1.50E-3	3.00	1.90E-4	2.98
	q	8.99E-5	1.11E-5	3.01	1.10E-6	3.34
3	u	1.11E-4	7.07E-6	3.97	4.43E-7	4.00
	$\Delta x\, \partial_x u$	2.52E-4	1.71E-5	3.88	1.07E-6	4.00
	$(\Delta x)^2\, \partial_x^2 u$	1.37E-3	8.54E-5	4.00	5.33E-6	4.00
	$(\Delta x)^3\, \partial_x^3 u$	1.75E-3	1.13E-4	3.95	7.11E-6	3.99
	q	1.18E-5	7.28E-7	4.02	4.75E-8	3.94
4	u	1.85E-6	4.02E-8	5.53	1.19E-9	5.08
	$\Delta x\, \partial_x u$	1.29E-5	3.76E-7	5.10	1.16E-8	5.01
	$(\Delta x)^2\, \partial_x^2 u$	5.19E-5	1.48E-6	5.13	4.65E-8	4.99
	$(\Delta x)^3\, \partial_x^3 u$	2.21E-4	6.93E-6	4.99	2.17E-7	5.00
	$(\Delta x)^4\, \partial_x^4 u$	2.25E-4	6.89E-6	5.03	2.17E-7	4.99
	q	3.58E-7	3.06E-9	6.87	5.05E-11	5.92

Table 4. The convection equation $a = 0$, $c = 1$. L_∞ errors and numerical order of accuracy, measured at the center of each element, for $\Delta x^m \partial_x^m u_h$ for $0 \le m \le k$.

k	variable	$N = 10$	$N = 20$		$N = 40$	
		error	error	order	error	order
1	u	7.24E-3	9.46E-4	2.94	1.20E-4	2.98
	$\Delta x \, \partial_x u$	6.09E-2	1.60E-2	1.92	4.09E-3	1.97
2	u	9.96E-4	1.28E-4	2.96	1.61E-5	2.99
	$\Delta x \, \partial_x u$	6.00E-3	7.71E-4	2.96	9.67E-5	3.00
	$(\Delta x)^2 \, \partial_x^2 u$	1.23E-2	1.54E-3	3.00	1.94E-4	2.99
3	u	1.26E-4	7.50E-6	4.07	4.54E-7	4.05
	$\Delta x \, \partial_x u$	1.63E-4	2.00E-5	3.03	1.07E-6	4.21
	$(\Delta x)^2 \, \partial_x^2 u$	1.52E-3	9.03E-5	4.07	5.45E-6	4.05
	$(\Delta x)^3 \, \partial_x^3 u$	1.35E-3	1.24E-4	3.45	7.19E-6	4.10
4	u	3.55E-6	8.59E-8	5.37	3.28E-10	8.03
	$\Delta x \, \partial_x u$	1.89E-5	1.27E-7	7.22	1.54E-8	3.05
	$(\Delta x)^2 \, \partial_x^2 u$	8.49E-5	2.28E-6	5.22	2.33E-8	6.61
	$(\Delta x)^3 \, \partial_x^3 u$	2.36E-4	5.77E-6	5.36	2.34E-7	4.62
	$(\Delta x)^4 \, \partial_x^4 u$	2.80E-4	8.93E-6	4.97	1.70E-7	5.72

Finally, to show that the order of accuracy could really degenerate to k for P^k, as was already observed in [6], we rerun the heat equation case $a = 1, c = 0$ with the central flux

$$\mathbb{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This time we can see that the global order of accuracy in L_∞ is only k when P^k is used with an odd value of k.

Table 5. The heat equation $a = 1$, $c = 0$. L_∞ errors and numerical order of accuracy, measured at the center of each element, for $\Delta x^m \partial_x^m u_h$ for $0 \leq m \leq k$, and for q_h, using the central flux.

k	variable	$N = 10$	$N = 20$		$N = 40$	
		error	error	order	error	order
1	u	3.59E-3	8.92E-4	2.01	2.25E-4	1.98
	$\Delta x\, \partial_x u$	2.10E-2	1.06E-2	0.98	5.31E-3	1.00
	q	2.39E-3	6.19E-4	1.95	1.56E-4	1.99
2	u	6.91E-5	4.12E-6	4.07	2.57E-7	4.00
	$\Delta x\, \partial_x u$	7.66E-4	1.03E-4	2.90	1.30E-5	2.98
	$(\Delta x)^2\, \partial_x^2 u$	2.98E-4	1.68E-5	4.15	1.03E-6	4.02
	q	6.52E-5	4.11E-6	3.99	2.57E-7	4.00
3	u	1.62E-5	1.01E-6	4.00	6.41E-8	3.98
	$\Delta x\, \partial_x u$	1.06E-4	1.32E-5	3.01	1.64E-6	3.00
	$(\Delta x)^2\, \partial_x^2 u$	1.99E-4	1.22E-5	4.03	7.70E-7	3.99
	$(\Delta x)^3\, \partial_x^3 u$	6.81E-4	8.68E-5	2.97	1.09E-5	2.99
	q	1.54E-5	1.01E-6	3.93	6.41E-8	3.98
4	u	8.25E-8	1.31E-9	5.97	2.11E-11	5.96
	$\Delta x\, \partial_x u$	1.62E-6	5.12E-8	4.98	1.60E-9	5.00
	$(\Delta x)^2\, \partial_x^2 u$	1.61E-6	2.41E-8	6.06	3.78E-10	6.00
	$(\Delta x)^3\, \partial_x^3 u$	2.90E-5	9.46E-7	4.94	2.99E-8	4.99
	$(\Delta x)^4\, \partial_x^4 u$	5.23E-6	7.59E-8	6.11	1.18E-9	6.01
	q	7.85E-8	1.31E-9	5.90	2.11E-11	5.96

c. Proof of the nonlinear stability. In this section, we prove the the nonlinear stability result of Proposition 2.1. To do that, we first show how to obtain the corresponding stability result for the exact solution and then mimic the argument to obtain Proposition 2.1.

The continuous case as a model. We start by rewriting the equations (2.5a) and (2.5b), in *compact form*. If in equations (2.5a) and (2.5b) we replace $v_u(x)$ and

$v_q(x)$ by $v_u(x,t)$ and $v_q(x,t)$, respectively, add the resulting equations, sum on j from 1 to N, and integrate in time from 0 to T, we obtain that

$$\mathcal{B}(\mathbf{w},\mathbf{v}) = 0, \qquad \forall \text{ smooth } \mathbf{v}(t), \quad \forall t \in (0,T), \tag{2.10a}$$

where

$$\mathcal{B}(\mathbf{w},\mathbf{v}) = \int_0^T \int_0^1 \partial_t u(x,t)\, v_u(x,t)\, dx\, dt + \int_0^T \int_0^1 q(x,t)\, v_q(x,t)\, dx\, dt$$
$$- \int_0^T \int_0^1 \mathbf{h}(\mathbf{w}(x,t))^t\, \partial_x\, \mathbf{v}(x,t)\, dx\, dt, \tag{2.10b}$$

using the fact that $\mathbf{h}(\mathbf{w}(x,t))^t\, \partial_x \mathbf{w}(x,t) = \partial_x(\,\phi(u) - g(u)\,q\,)$ is a complete derivative, we see that

$$\mathcal{B}(\mathbf{w},\mathbf{w}) = \frac{1}{2}\int_0^1 u^2(x,T)\, dx + \int_0^T \int_0^1 q^2(x,t)\, dx\, dt - \frac{1}{2}\int_0^1 u_0^2(x)\, dx, \tag{2.11}$$

and that $\mathcal{B}(\mathbf{w},\mathbf{w}) = 0$, by (2.10a), we immediately obtain the following L^2-stability result:

$$\frac{1}{2}\int_0^1 u^2(x,T)\, dx + \int_0^T \int_0^1 q^2(x,t)\, dx\, dt = \frac{1}{2}\int_0^1 u_0^2(x)\, dx.$$

This is the argument we have to mimic in order to prove Proposition 2.1.

The discrete case. Thus, we start by finding a *compact form* of equations (2.6a) and (2.6b). If we replace $v_{h,u}(x)$ and $v_{h,q}(x)$ by $v_{h,u}(x,t)$ and $v_{h,q}(x,t)$ in the equations (2.6a) and (2.6b), add them up, sum on j from 1 to N and integrate in time from 0 to T, we obtain

$$\mathcal{B}_h(\mathbf{w}_h,\mathbf{v}_h) = 0, \qquad \forall\, \mathbf{v}_h(t) \in V_h^k \times V_h^k, \quad \forall t \in (0,T). \tag{2.12a}$$

where

$$\mathcal{B}_h(\mathbf{w}_h,\mathbf{v}_h) = \int_0^T \int_0^1 \partial_t u_h(x,t)\, v_{h,u}(x,t)\, dx\, dt + \int_0^T \int_0^1 q_h(x,t)\, v_{h,q}(x,t)\, dx\, dt$$
$$- \int_0^T \sum_{1 \le j \le N} \hat{\mathbf{h}}(\mathbf{w}_h)_{j+1/2}^t(t)\, [\,\mathbf{v}_h(t)\,]_{j+1/2}\, dt$$
$$- \int_0^T \sum_{1 \le j \le N} \int_{I_j} \mathbf{h}(\mathbf{w}_h(x,t))^t\, \partial_x\, \mathbf{v}_h(x,t)\, dx\, dt. \tag{2.12b}$$

Next, we obtain an expression for $\mathcal{B}_h(\mathbf{w}_h, \mathbf{w}_h)$. It is contained in the following result.

Lemma 2.3. *We have*

$$\mathcal{B}_h(\mathbf{w}_h, \mathbf{w}_h) = \frac{1}{2} \int_0^1 u_h^2(x, T) \, dx + \int_0^T \int_0^1 q_h^2(x, t) \, dx \, dt + \Theta_{T,\mathbb{C}}([\mathbf{w}_h]) - \frac{1}{2} \int_0^1 u_h^2(x, 0) \, dx,$$

where $\Theta_{T,\mathbb{C}}([\mathbf{w}_h])$ is defined in Proposition 2.1.

Next, since $\mathcal{B}_h(\mathbf{w}_h, \mathbf{w}_h) = 0$, by (2.12a), we get the inequality

$$\frac{1}{2} \int_0^1 u_h^2(x, T) \, dx + \int_0^T \int_0^1 q_h^2(x, t) \, dx \, dt + \Theta_{T,\mathbb{C}}([\mathbf{w}_h]) = \frac{1}{2} \int_0^1 u_h^2(x, 0) \, dx$$

from which Proposition 2.1 easily follows, since

$$\frac{1}{2} \int_0^1 u_h^2(x, 0) \, dx \le \frac{1}{2} \int_0^1 u_0^2(x) \, dx,$$

by (2.5c). It remains to prove Lemma 2.3.

Proof Lemma 2.3. After setting $\mathbf{v}_h = \mathbf{w}_h$ in (2.12b), we get

$$\mathcal{B}(\mathbf{w}_h, \mathbf{w}_h) = \frac{1}{2} \int_0^1 u_h^2(x, T) \, dx + \int_0^T \int_0^1 q_h^2(x, t) \, dx \, dt + \int_0^T \Theta_{diss}(t) \, dt - \frac{1}{2} \int_0^1 u_h^2(x, 0) \, dx,$$

where

$$\Theta_{diss}(t) = - \sum_{1 \le j \le N} \left\{ \hat{\mathbf{h}}(\mathbf{w}_h)_{j+1/2}^t(t) \, [\, \mathbf{w}_h(t) \,]_{j+1/2} + \int_{I_j} \mathbf{h}(\mathbf{w}_h(x, t))^t \, \partial_x \, \mathbf{w}_h(x, t) \, dx \right\}.$$

It only remains to show that $\int_0^T \Theta_{diss}(t) \, dt = \Theta_{T,\mathbb{C}}([\mathbf{w}_h])$.

To do that, we proceed as follows. Since

$$\mathbf{h}(\mathbf{w}_h(x, t))^t \, \partial_x \, \mathbf{w}_h(x, t) = (\, f(u_h) - \sqrt{a(u_h)} \, q_h \,) \, \partial_x \, u_h - g(u_h) \, \partial_x \, q_h$$

$$= \partial_x \left(\int^{u_h} f(s) \, ds - g(u_h) \, q_h \right)$$

$$= \partial_x \left(\phi(u_h) - g(u_h) \, q_h \right)$$

$$\equiv \partial_x \, H(\mathbf{w}_h(x, t)),$$

we get

$$\Theta_{diss}(t) = \sum_{1 \le j \le N} \left\{ [H(\mathbf{w}_h(t))]_{j+1/2} - \hat{\mathbf{h}}(\mathbf{w}_h)^t_{j+1/2}(t) [\mathbf{w}_h(t)]_{j+1/2} \right\}$$

$$\equiv \sum_{1 \le j \le N} \left\{ [H(\mathbf{w}_h(t))] - \hat{\mathbf{h}}(\mathbf{w}_h)^t(t) [\mathbf{w}_h(t)] \right\}_{j+1/2}$$

Since, by the definition of H,

$$[H(\mathbf{w}_h(t))] = [\phi(u_h(t))] - [g(u_h(t)) q_h(t)]$$

$$= [\phi(u_h(t))] - [g(u_h(t))] \overline{q}_h(t) - [q_h(t)] \overline{g(u_h(t))},$$

and since $(\hat{h}_u, \hat{h}_q)^t = \hat{\mathbf{h}}$, we get

$$\Theta_{diss}(t) = \sum_{1 \le j \le N} \left\{ [\phi(u_h(t))] - [g(u_h(t))] \overline{q}_h(t) - [u_h(t)] \hat{h}_u \right\}_{j+1/2}$$

$$+ \sum_{1 \le j \le N} \left\{ -[q_h(t)] \overline{g(u_h)(t)} - [q_h(t)] \hat{h}_q \right\}_{j+1/2},$$

This is the crucial step to obtain the L^2-stability of the LDG methods, since the above expression gives us key information about the form that the flux $\hat{\mathbf{h}}$ should have in order to make $\Theta_{diss}(t)$ a nonnegative quantity and hence enforce the L^2-stability of the LDG methods. Thus, by taking $\hat{\mathbf{h}}$ as in (2.7a), we get

$$\Theta_{diss}(t) = \sum_{1 \le j \le N} \left\{ [\mathbf{w}_h(t)]^t \mathbb{C} [\mathbf{w}_h(t)] \right\}_{j+1/2},$$

and the result follows. This completes the proof. $\square$

This completes the proof of Proposition 2.1.

c. The error estimate in the linear case. In this section, we prove the error estimate of Theorem 2.2 which holds for the linear case $f'(\cdot) \equiv c$ and $a(\cdot) \equiv a$. To do that, we first show how to estimate the error between the solutions $\mathbf{w}_\nu = (u_\nu, q_\nu)^t$, $\nu = 1, 2$, of

$$\partial_t u_\nu + \partial_x (f(u_\nu) - \sqrt{a(u_\nu)} q_\nu) = 0 \quad \text{in } (0, T) \times (0, 1),$$

$$q_\nu - \partial_x g(u_\nu) = 0 \qquad\qquad \text{in } (0, T) \times (0, 1),$$

$$u_\nu(t = 0) = u_{0,\nu}, \qquad\qquad \text{on } (0, 1).$$

Then, we mimic the argument in order to prove Theorem 2.2.

The continuous case as a model. By the definition of the form $\mathcal{B}(\cdot, \cdot)$, (2.10b), we have, for $\nu = 1, 2$,

$$\mathcal{B}(\mathbf{w}_\nu, \mathbf{v}) = 0, \qquad \forall \text{ smooth } \mathbf{v}(t), \quad \forall t \in (0, T).$$

Since in this case, the form $\mathcal{B}(\cdot, \cdot)$ is bilinear, from the above equation we obtain the so-called *error equation*:

$$\mathcal{B}(\mathbf{e}, \mathbf{v}) = 0, \qquad \forall \text{ smooth } \mathbf{v}(t), \quad \forall t \in (0, T),$$

where $\mathbf{e} = \mathbf{w}_1 - \mathbf{w}_2$. Now, from (2.11), we get that

$$\mathcal{B}(\mathbf{e}, \mathbf{e}) = \frac{1}{2} \int_0^1 e_u^2(x, T) \, dx + \int_0^T \int_0^1 e_q^2(x, t) \, dx \, dt - \frac{1}{2} \int_0^1 e_u^2(x, 0) \, dx,$$

and since $e_u(x, 0) = u_{0,1}(x) - u_{0,2}(x)$ and $\mathcal{B}(\mathbf{e}, \mathbf{e}) = 0$, by the *error equation*, we immediately obtain the error estimate we sought:

$$\frac{1}{2} \int_0^1 e_u^2(x, T) \, dx + \int_0^T \int_0^1 e_q^2(x, t) \, dx \, dt = \frac{1}{2} \int_0^1 (u_{0,1}(x) - u_{0,2}(x))^2 \, dx.$$

To prove Theorem 2.2, we only need to obtain a discrete version of this argument.

The discrete case. Since,

$$\mathcal{B}_h(\mathbf{w}_h, \mathbf{v}_h) = 0, \qquad \forall \, \mathbf{v}_h(t) \in V_h \times V_h, \quad \forall t \in (0, T),$$

$$\mathcal{B}_h(\mathbf{w}, \mathbf{v}_h) = 0, \qquad \forall \, \mathbf{v}_h(t) \in V_h \times V_h, \quad \forall t \in (0, T),$$

by (2.12a) and by equations (2.5a) and (2.5b), respectively, we immediately obtain our *error equation*:

$$\mathcal{B}_h(\mathbf{e}, \mathbf{v}_h) = 0, \qquad \forall \, \mathbf{v}_h(t) \in V_h \times V_h, \quad \forall t \in (0, T),$$

where $\mathbf{e} = \mathbf{w} - \mathbf{w}_h$. Now, according to the continuous case argument, we should consider next the quantity $\mathcal{B}_h(\mathbf{e}, \mathbf{e})$; however, since $\mathbf{e}$ is not in the finite element space, it

is more convenient to consider $\mathcal{B}_h(\mathbb{P}_h(\mathbf{e}), \mathbb{P}_h(\mathbf{e}))$, where $\mathbb{P}_h(\mathbf{e}(t)) = (\mathbb{P}_h(e_u(t)), \mathbb{P}_h(e_q(t)))$ is the so-called L^2-projection of $\mathbf{e}(t)$ into the finite element space $V_h^k \times V_h^k$. The L^2-projection of the function p into V_h, $\mathbb{P}_h(p)$, is defined as the only element of the finite element space V_h such that

$$\int_0^1 (\mathbb{P}_h(p)(x) - p(x)) \, v_h(x) \, dx = 0, \qquad \forall \, v_h \in V_h. \tag{2.13}$$

Note that, in fact $u_h(t=0) = \mathbb{P}_h(u_0)$, by (2.6c).

Thus, by Lemma 2.3, we have

$$\mathcal{B}_h(\mathbb{P}_h(\mathbf{e}), \mathbb{P}_h(\mathbf{e})) = \frac{1}{2} \int_0^1 |\mathbb{P}_h(e_u(T))(x)|^2 \, dx + \int_0^T \int_0^1 |\mathbb{P}_h(e_q(t))(x)|^2 \, dx \, dt$$
$$+ \Theta_{T,\mathbb{C}}([\mathbb{P}_h(\mathbf{e})]) - \frac{1}{2} \int_0^1 |\mathbb{P}_h(e_u(0))(x)|^2 \, dx,$$

and since

$$\mathbb{P}_h(e_u(0)) = \mathbb{P}_h(u_0 - u_h(0)) = \mathbb{P}_h(u_0) - u_h(0) = 0,$$

by (2.6c) and (2.13), and

$$\mathcal{B}_h(\mathbb{P}_h(\mathbf{e}), \mathbb{P}_h(\mathbf{e})) = \mathcal{B}_h(\mathbb{P}_h(\mathbf{e}) - \mathbf{e}, \mathbb{P}_h(\mathbf{e})) = \mathcal{B}_h(\mathbb{P}_h(\mathbf{w}) - \mathbf{w}, \mathbb{P}_h(\mathbf{e})),$$

by the *error equation*, we get

$$\frac{1}{2} \int_0^1 |\mathbb{P}_h(e_u(T))(x)|^2 \, dx + \int_0^T \int_0^1 |\mathbb{P}_h(e_q(t))(x)|^2 \, dx \, dt$$
$$+ \Theta_{T,\mathbb{C}}([\mathbb{P}_h(\mathbf{e})]) = \mathcal{B}_h(\mathbb{P}_h(\mathbf{w}) - \mathbf{w}, \mathbb{P}_h(\mathbf{e})). \tag{2.14}$$

Note that since in our continuous model, the right-hand side is zero, we expect the term $\mathcal{B}(\mathbb{P}_h(\mathbf{w}) - \mathbf{w}, \mathbb{P}_h(\mathbf{e}))$ to be small.

Estimating the right-hand side. To show that this is so, we must suitably treat the term $\mathcal{B}(\mathbb{P}_h(\mathbf{w}) - \mathbf{w}, \mathbb{P}_h(\mathbf{e}))$.

Lemma 2.4. *For* $\mathbf{p} = \mathbb{P}_h(\mathbf{w}) - \mathbf{w}$, *we have*

$$\mathcal{B}_h(\mathbf{p}, \mathbb{P}_h(\mathbf{e})) \leq \frac{1}{2}\Theta_{T,\mathbb{C}}(\overline{\mathbf{p}}) + \frac{1}{2}\int_0^T \int_0^1 |\mathbb{P}_h(e_q(t))(x)|^2 \, dx \, dt$$

$$+ (\Delta x)^{2k} \int_0^T C_1(t) \, dt + (\Delta x)^k \int_0^T C_2(t) \left\{ \int_0^1 |\mathbb{P}_h(e_u(t))(x)|^2 \, dx \right\}^{1/2} dt,$$

where

$$C_1(t) = 2 c_k^2 \left\{ \left(\frac{(|c| + c_{11})^2}{c_{11}} \Delta x \, |c_{12}|^2 \, d_k^2 \right) |u(t)|_{k+1}^2 + a (\Delta x)^{2(\hat{k}-k)} |u(t)|_{\hat{k}+1}^2 \right\},$$

$$C_2(t) = 2 c_k \left\{ \sqrt{a} \, |c_{12}| \, |u(t)|_{k+2}^2 + a (\Delta x)^{2(\hat{k}-k)} |u(t)|_{\hat{k}+2}^2 \right\}.$$

where the constants c_k *and* d_k *depend solely on* k, *and* $\hat{k} = k$ *except when the grids are uniform and* k *is even, in which case* $\hat{k} = k + 1$.

Note how c_{11} appears in the denominator of $C_1(t)$. However, $C_1(t)$ remains bounded as c_{11} goes to zero since the convective numerical flux is an E-flux.

To prove this result, we will need the following auxiliary lemmas. We denote by $|u|_{H^{(k+1)}(J)}^2$ the integral over J of the square of the $(k+1)$-the derivative of u.

Lemma 2.5. *For* $\mathbf{p} = \mathbb{P}_h(\mathbf{w}) - \mathbf{w}$, *we have*

$$|\overline{p_u}_{j+1/2}| \leq c_k (\Delta x)^{\hat{k}+1/2} |u|_{H^{(\hat{k}+1)}(J_{j+1/2})},$$

$$|[p_u]_{j+1/2}| \leq c_k (\Delta x)^{k+1/2} |u|_{H^{(k+1)}(J_{j+1/2})},$$

$$|\overline{p_q}_{j+1/2}| \leq c_k \sqrt{a} (\Delta x)^{\hat{k}+1/2} |u|_{H^{(\hat{k}+2)}(J_{j+1/2})},$$

$$|[p_q]_{j+1/2}| \leq c_k \sqrt{a} (\Delta x)^{k+1/2} |u|_{H^{(k+2)}(J_{j+1/2})},$$

where $J_{j+1/2} = I_j \cup I_{j+1}$, *the constant* c_k *depends solely on* k, *and* $\hat{k} = k$ *except when the grids are uniform and* k *is even, in which case* $\hat{k} = k + 1$.

Proof. The two last inequalities follow from the first two and from the fact that $q = \sqrt{a} \, \partial_x u$. The two first inequalities with $\hat{k} = k$ follow from the definitions of $\overline{p_u}$ and $[p_u]$ and from the following estimate:

$$|\mathbb{P}_h(u)(x_{j+1/2}^{\pm}) - u_{j+1/2}| \leq \frac{1}{2} c_k (\Delta x)^{k+1/2} |u|_{H^{(k+1)}(J_{j+1/2})},$$

where the constant c_k depends solely on k. This inequality follows from the fact that $\mathbb{P}_h(u)(x^\pm_{j+1/2}) - u_{j+1/2} = 0$ when u is a polynomial of degree k and from a simple application of the Bramble-Hilbert lemma; see Ciarlet [11].

To prove the inequalities in the case in which $\hat{k} = k + 1$, we only need to show that if u is a polynomial of degree $k + 1$ for k even, then $\overline{p_u} = 0$. It is clear that it is enough to show this equality for the particular choice

$$u(x) = \big((x - x_{j+1/2})/(\Delta x/2)\big)^{k+1}.$$

To prove this, we recall that if P_ℓ denotes the Legendre polynomials of order ℓ: (i) $\int_{-1}^1 P_\ell(s) P_m(s)\, ds = \frac{2}{2\ell+1}\delta_{\ell m}$, (ii) $P_\ell(\pm 1) = (\pm 1)^\ell$, and (iii) $P_\ell(s)$ is a linear combination of odd (even) powers of s for odd (even) values of ℓ. Since we are assuming that the grid is uniform, $\Delta x_j = \Delta x_{j+1} = \Delta x$, we can write, by (i),

$$\mathbb{P}_h(u)(x) = \sum_{0 \le \ell \le k} \frac{2\ell+1}{2}\left\{\int_{-1}^1 P_\ell(s)\, u(x_j + \tfrac{1}{2}\Delta x\, s)\, ds\right\} P_\ell(\frac{x - x_j}{\Delta x/2}),$$

for $x \in I_j$. Hence, for our particular choice of u, we have

$$\overline{p_u}_{j+1/2} = \frac{1}{2}\sum_{0 \le \ell \le k}\frac{2\ell+1}{2}\int_{-1}^1 P_\ell(s)\left\{(s-1)^{k+1} P_\ell(1) + (s+1)^{k+1} P_\ell(-1)\right\} ds$$

$$= \frac{1}{2}\sum_{0 \le \ell, i \le k}\frac{2\ell+1}{2}\binom{k+1}{i}\int_{-1}^1 P_\ell(s)\, s^i\left\{(-1)^{k+1-i} P_\ell(1) + P_\ell(-1)\right\} ds$$

$$= \frac{1}{2}\sum_{0 \le \ell, i \le k}\frac{2\ell+1}{2}\binom{k+1}{i}\int_{-1}^1 P_\ell(s)\, s^i\left\{(-1)^{k+1-i} + (-1)^\ell\right\} ds$$

by (ii). When the factor $\left\{(-1)^{k+1-i} + (-1)^\ell\right\}$ is different from zero, $|k+1-i+\ell|$ is even and since k is also even, $|i - \ell|$ is odd. In this case, by (iii),

$$\int_{-1}^1 P_\ell(s)\, s^i\, ds = 0,$$

and so $\overline{p_u}_{j+1/2} = 0$. This completes the proof. $\square$

We will also need the following result that follows from a simple scaling argument; see Ciarlet [11].

Lemma 2.6. *We have*

$$| \, [\mathbb{P}_h(p)]_{j+1/2} \, | \leq d_k \, (\Delta x)^{-1/2} \, \| \, \mathbb{P}_h(p) \, \|_{L^2(J)},$$

where $J_{j+1/2} = I_j \cup I_{j+1}$ and the constant d_k depends solely on k.

We are now ready to prove Lemma 2.4.

Proof of Lemma 2.4. To simplify the notation, let us set $\mathbf{v}_h = \mathbb{P}_h \mathbf{e}$. By the definition of $\mathcal{B}_h(\cdot, \cdot)$, we have

$$
\begin{aligned}
\mathcal{B}_h(\mathbf{p}, \mathbf{v}_h) &= \int_0^T \int_0^1 \partial_t p_u(x, t) \, v_{h,u}(x, t) \, dx \, dt + \int_0^T \int_0^1 p_q(x, t) \, v_{h,q}(x, t) \, dx \, dt \\
&\quad - \int_0^T \sum_{1 \leq j \leq N} \hat{\mathbf{h}}(\mathbf{p})_{j+1/2}^t(t) \, [\, \mathbf{v}_h(t) \,]_{j+1/2} \, dt \\
&\quad - \int_0^T \sum_{1 \leq j \leq N} \int_{I_j} \mathbf{h}(\mathbf{p}(x, t))^t \, \partial_x \, \mathbf{v}_h(x, t) \, dx \, dt \\
&= - \int_0^T \sum_{1 \leq j \leq N} \hat{\mathbf{h}}(\mathbf{p})_{j+1/2}^t(t) \, [\, \mathbf{v}_h(t) \,]_{j+1/2} \, dt,
\end{aligned}
$$

by the definition of the L^2-projection (2.13).

Now, recalling that $\mathbf{p} = (p_u, p_q)^t$ and that $\mathbf{v}_h = (v_u, v_q)^t$, we have

$$
\begin{aligned}
\hat{\mathbf{h}}(\mathbf{p})^t \, [\, \mathbf{v}_h(t) \,] &= (c \, \overline{p_u} - c_{11} \, [\, p_u \,]) \, [\, v_u \,] \\
&\quad + (-\sqrt{a} \, \overline{p_q} - c_{12} \, [\, p_q \,]) \, [\, v_u \,] \\
&\quad + (-\sqrt{a} \, \overline{p_u} + c_{12} \, [\, p_u \,]) \, [\, v_q \,] \\
&\equiv \theta_1 + \theta_2 + \theta_3.
\end{aligned}
$$

By Lemmas 2.5 and 2.6,

$$| \, \theta_1 \, | \leq c_k \, (\Delta x)^{k+1/2} \, | \, u \, |_{H^{k+1}(J)} \, (| \, c \, | + c_{11}) \, | \, [\, v_u \,] \, |,$$

$$| \, \theta_2 \, | \leq c_k \, d_k \, (\Delta x)^k \, \big(a \, | \, u \, |_{H^{\hat{k}+2}(J)} \, (\Delta x)^{\hat{k}-k} + \sqrt{a} \, | \, c_{12} \, | \, | \, u \, |_{H^{k+2}(J)} \big) \big) \, \| \, v_u \, \|_{L^2(J)},$$

$$| \, \theta_3 \, | \leq c_k \, d_k \, (\Delta x)^k \, \big(\sqrt{a} \, | \, u \, |_{H^{\hat{k}+1}(J)} \, (\Delta x)^{\hat{k}-k} + | \, c_{12} \, | \, | \, u \, |_{H^{k+1}(J)} \big) \big) \, \| \, v_q \, \|_{L^2(J)}.$$

This is the crucial step for obtaining our error estimates. Note that the treatment of θ_1 is very different than the treatment of θ_2 and θ_3. The reason for this difference is that the upper bound for θ_1 can be controlled by the form $\Theta_{T,\mathbb{C}}([\mathbf{v}_h])$- we recall that $\mathbf{v}_h = \mathbb{P}_h(\mathbf{e})$. This is not the case for the upper bound for θ_2 because $\Theta_{T,\mathbb{C}}[\mathbf{v}_h] \equiv 0$ if $c = 0$ nor it is the case for the upper bound for θ_3 because $\Theta_{T,\mathbb{C}}[\mathbf{v}_h]$ does not involve the jumps $[v_q]$!

Thus, after a suitable application of Young's inequality and simple algebraic manipulations, we get

$$\hat{\mathbf{h}}(\mathbf{p})^t \left[\mathbf{v}_h(t)\right] \leq \frac{1}{2}\, c_{11}\, [v_u]^2 + \frac{1}{4}\|\, v_q\,\|^2_{L^2(J)} + \frac{1}{2}\, C_1(t)\,(\Delta x)^{2k} + \frac{1}{2}\, C_2(t)\,(\Delta x)^k\, \|\, v_u\,\|_{L^2(J)}.$$

Since

$$\mathcal{B}_h(\mathbf{p}, \mathbf{v}_h) \leq \int_0^T \sum_{1 \leq j \leq N} \left|\,\hat{\mathbf{h}}(\mathbf{p})^t_{j+1/2}(t)\, [\,\mathbf{v}_h(t)\,]_{j+1/2}\,\right| dt,$$

and since $J_{j+1/2} = I_j \cup I_{j+1}$, the result follows after simple applications of the Cauchy-Schwartz inequality. This completes the proof. $\square$

Conclusion. Combining the equation (2.14) with the estimate of Lemma 2.4, we easily obtain, after a simple application of Gronwall's lemma,

$$\left\{ \int_0^1 |\,\mathbb{P}_h(e_u(T))(x)\,|^2\, dx + \int_0^T \int_0^1 |\,\mathbb{P}_h(e_q(t))(x)\,|^2\, dx\, dt + \Theta_{T,\mathbb{C}}([\mathbb{P}_h(\mathbf{e})]) \right\}^{1/2}$$

$$\leq (\Delta x)^k \int_0^T \sqrt{C_1(t)}\, dt + (\Delta x)^k \int_0^T C_2(t) \left\{ \int_0^1 |\,\mathbb{P}_h(e_u(t))(x)\,|^2\, dx \right\}^{1/2} dt.$$

Theorem 2.2 follows easily from this inequality, Lemma 2.6, and from the following simple approximation result:

$$\|\, p - \mathbb{P}_h(p)\,\|_{L^2(0,1)} \leq g_k\,(\Delta x)^{k+1}\,|\, p\,|_{H^{(k+1)}(0,1)}$$

where g_k depends solely on k; see Ciarlet [11].

3. The LDG methods for the multi-dimensional case. In this section, we consider the LDG methods for the following convection-diffusion model problem

$$\partial_t u + \sum_{1 \le i \le d} \partial_{x_i} \left(f_i(u) - \sum_{1 \le j \le d} a_{ij}(u) \, \partial_{x_j} u \right) = 0 \quad \text{in } (0, T) \times (0, 1)^d, \quad (3.1\text{a})$$

$$u(t = 0) = u_0, \qquad\qquad\qquad \text{on } (0, 1)^d, \qquad\qquad (3.1\text{b})$$

with periodic boundary conditions. Essentially, the one-dimensional case and the multidimensional case can be studied in exactly the same way. However, there are two important differences that deserve explicit discussion. The first is the treatment of the matrix of entries $a_{ij}(u)$, which is assumed to be *symmetric, semipositive definite* and the introduction of the variables q_ℓ, and the second is the treatment of arbitrary meshes.

To define the LDG method, we first notice that, since the matrix $a_{ij}(u)$ is assumed to be symmetric and semipositive definite, there exists a symmetric matrix $b_{ij}(u)$ such that

$$a_{ij}(u) = \sum_{1 \le \ell \le d} b_{i\ell}(u) \, b_{\ell j}(u). \qquad\qquad (3.2)$$

Then we define the new scalar variables $q_\ell = \sum_{1 \le j \le d} b_{\ell j}(u) \, \partial_{x_j} u$ and rewrite the problem (3.1) as follows:

$$\partial_t u + \sum_{1 \le i \le d} \partial_{x_i} \left(f_i(u) - \sum_{1 \le \ell \le d} b_{i\ell}(u) \, q_\ell \right) = 0 \quad \text{in } (0, T) \times (0, 1)^d, \qquad (3.3\text{a})$$

$$q_\ell - \sum_{1 \le j \le d} \partial_{x_j} g_{\ell j}(u) = 0, \quad \ell = 1, \dots d, \qquad \text{in } (0, T) \times (0, 1)^d, \qquad (3.3\text{b})$$

$$u(t = 0) = u_0, \qquad\qquad\qquad \text{on } (0, 1)^d, \qquad\qquad (3.3\text{c})$$

where $g_{\ell j}(u) = \int^u b_{\ell j}(s) \, ds$. The LDG method is now obtained by discretizing (3.3) by the Discontinuous Galerkin method.

We follow what was done in §2. So, we set $\mathbf{w} = (u, \mathbf{q})^t = (u, q_1, \cdots, q_d)^t$ and, for each $i = 1, \cdots, d$, introduce the flux

$$\mathbf{h}_i(\mathbf{w}) = \left(f_i(u) - \sum_{1 \le \ell \le d} b_{i\ell}(u) \, q_\ell, -g_{1i}(u), \cdots, -g_{di}(u) \right)^t. \qquad (3.4)$$

We consider triangulations of $(0,1)^d$, $\mathcal{T}_{\Delta x} = \{K\}$, made of non-overlapping polyhedra. We require that for any two elements K and K', $\overline{K} \cap \overline{K}'$ is either a face e of both K and K' with nonzero $(d-1)$-Lebesgue measure $|e|$, or has Hausdorff dimension less than $d-1$. We denote by $\mathcal{E}_{\Delta x}$ the set of all faces e of the border of K for all $K \in \mathcal{T}_{\Delta x}$. The diameter of K is denoted by Δx_K and the maximum Δx_K, for $K \in \mathcal{T}_{\Delta x}$ is denoted by Δx. We require, for the sake of simplicity, that the triangulations $\mathcal{T}_{\Delta x}$ be regular, that is, there is a constant independent of Δx such that

$$\frac{\Delta x_K}{\rho_K} \le \sigma \quad \forall\, K \in \mathcal{T}_{\Delta x},$$

where ρ_K denotes the diameter of the maximum ball included in K.

We seek an approximation $\mathbf{w}_h = (u_h, \mathbf{q}_h)^t = (u_h, q_{h1}, \cdots, q_{hd})^t$ to $\mathbf{w}$ such that for each time $t \in [0, T]$, each of the components of $\mathbf{w}_h$ belong to the finite element space

$$V_h = V_h^k = \{\, v \in L^1((0,1)^d) : \ v|_K \in P^k(K) \ \forall\, K \in \mathcal{T}_{\Delta x}\}, \tag{3.5}$$

where $P^k(K)$ denotes the space of polynomials of total degree at most k. In order to determine the approximate solution $\mathbf{w}_h$, we proceed exactly as in the one-dimensional case. This time, however, the integrals are made on each element K of the triangulation $\mathcal{T}_{\Delta x}$. We obtain the following weak formulation on each element K of the triangulation $\mathcal{T}_{\Delta x}$:

$$\int_K \partial_t\, u_h(x,t)\, v_{h,u}(x)\, dx - \sum_{1 \le i \le d} \int_K h_{i\,u}(\mathbf{w}_h(x,t))\, \partial_{x_i} v_{h,u}(x)\, dx$$
$$+ \int_{\partial K} \hat{h}_u(\mathbf{w}_h, \mathbf{n}_{\partial K})(x,t)\, v_{h,u}(x)\, d\,\Gamma(x) = 0, \qquad \forall\, v_{h,u} \in P^k(K), \tag{3.6a}$$

For $\ell = 1, \cdots, d$:

$$\int_K q_{h\ell}(x,t)\, v_{h,q_\ell}(x)\, dx - \sum_{1 \le j \le d} \int_K h_{j\,q_\ell}(\mathbf{w}_h(x,t))\, \partial_{x_j} v_{h,q_\ell}(x)\, dx$$
$$+ \int_{\partial K} \hat{h}_{q_\ell}(\mathbf{w}_h, \mathbf{n}_{\partial K})(x,t)\, v_{h,q_\ell}(x)\, d\,\Gamma(x) = 0, \quad \forall\, v_{h,q_\ell} \in P^k(K), \tag{3.6b}$$

$$\int_K u_h(x,0)\, v_{h,i}(x)\, dx = \int_K u_0(x)\, v_{h,i}(x)\, dx, \qquad \forall\, v_{h,i} \in P^k(K),$$

$$(3.6c)$$

where $\mathbf{n}_{\partial K}$ denotes the outward unit normal to the element K at $x \in \partial K$. It remains to choose the numerical flux $(\hat{h}_u, \hat{h}_{q_1}, \cdots, \hat{h}_{q_d})^t \equiv \hat{\mathbf{h}} \equiv \hat{\mathbf{h}}(\mathbf{w}_h, \mathbf{n}_{\partial K})(x,t)$.

As in the one-dimensional case, we require that the fluxes $\hat{\mathbf{h}}$ be of the form

$$\hat{\mathbf{h}}(\mathbf{w}_h, \mathbf{n}_{\partial K})(x) \equiv \hat{\mathbf{h}}(\mathbf{w}_h(x^{int_K}, t), \mathbf{w}_h(x^{ext_K}, t); \mathbf{n}_{\partial K}),$$

where $\mathbf{w}_h(x^{int_K})$ is the limit at x taken from the interior of K and $\mathbf{w}_h(x^{ext_K})$ the limit at x from the exterior of K, and consider fluxes that (i) are locally Lipschitz, conservative, that is,

$$\hat{\mathbf{h}}(\mathbf{w}_h(x^{int_K}), \mathbf{w}_h(x^{ext_K}); \mathbf{n}_{\partial K}) + \hat{\mathbf{h}}(\mathbf{w}_h(x^{ext_K}), \mathbf{w}_h(x^{int_K}); -\mathbf{n}_{\partial K}) = 0,$$

and consistent with the flux

$$\sum_{1 \le i \le d} \mathbf{h}_i\, n_{\partial K, i},$$

(ii) allow for a local resolution of each component of $\mathbf{q}_h$ in terms of u_h *only*, (iii) reduce to an E-flux when $a(\cdot) \equiv 0$, and that (iv) enforce the L^2-stability of the method.

Again, we write our numerical flux as the sum of a convective flux and a diffusive flux:

$$\hat{\mathbf{h}} = \hat{\mathbf{h}}_{conv} + \hat{\mathbf{h}}_{diff},$$

where the convective flux is given by

$$\hat{\mathbf{h}}_{conv}(\mathbf{w}^-, \mathbf{w}^+; \mathbf{n}) = \left(\hat{f}(u^-, u^+; \mathbf{n}), 0 \right)^t,$$

where $\hat{f}(u^-, u^+; \mathbf{n})$ is any locally Lipschitz E-flux which is conservative and consistent with the nonlinearity

$$\sum_{1 \le i \le d} f_i(u)\, n_i,$$

and the diffusive flux $\hat{\mathbf{h}}_{diff}(\mathbf{w}^-, \mathbf{w}^+; \mathbf{n})$ is given by

$$\Big(- \sum_{1 \le i,\ell \le d} \frac{[\, g_{i\ell}(u)\,]}{[\, u\,]} \, \overline{q_\ell}\, n_i, \; - \sum_{1 \le i \le d} \overline{g_{i1}(u)}\, n_i, \cdots, \; - \sum_{1 \le i \le d} \overline{g_{id}(u)}\, n_i \,\Big)^t - \mathbb{C}_{diff}\,[\,\mathbf{w}\,],$$

where

$$\mathbb{C}_{diff} = \begin{pmatrix} 0 & c_{12} & c_{13} & \cdots & c_{1d} \\ -c_{12} & 0 & 0 & \cdots & 0 \\ -c_{13} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{1d} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$c_{1j} = c_{1j}(\mathbf{w}^-, \mathbf{w}^+) \quad \text{is locally Lipschitz for } j = 1, \cdots, d,$$

$$c_{1j} \equiv 0 \quad \text{when } a(\cdot) \equiv 0 \quad \text{for } j = 1, \cdots, d.$$

We claim that this flux satisfies the properties (i) to (iv).

To prove that properties (i) to (iii) are satisfied is now a simple exercise. To see that the property (iv) is satisfied, we first rewrite the flux $\hat{\mathbf{h}}$ in the following way:

$$\Big(- \sum_{1 \le i,\ell \le d} \frac{[\, g_{i\ell}(u)\,]}{[\, u\,]} \, \overline{q_\ell}\, n_i, \; - \sum_{1 \le i \le d} \overline{g_{i1}(u)}\, n_i, \cdots, \; - \sum_{1 \le i \le d} \overline{g_{id}(u)}\, n_i \,\Big)^t - \mathbb{C}\,[\,\mathbf{w}\,],$$

where

$$\mathbb{C} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1d} \\ -c_{12} & 0 & 0 & \cdots & 0 \\ -c_{13} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -c_{1d} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$c_{11} = \frac{1}{[\, u\,]} \Big(\sum_{1 \le i \le d} \frac{[\, \phi_i(u)\,]}{[\, u\,]} \, n_i - \hat{f}(u^-, u^+; \mathbf{n}) \Big),$$

where $\phi_i(u) = \int^u f_i(s)\,ds$. Since $\hat{f}(\cdot, \cdot; \mathbf{n})$ is an E-flux,

$$c_{11} = \frac{1}{[\, u\,]^2} \int_{u^-}^{u^+} \Big(\sum_{1 \le i \le d} f_i(s)\, n_i - \hat{f}(u^-, u^+; \mathbf{n}) \Big)\, ds \ge 0,$$

and so the matrix $\mathbb{C}$ is semipositive definite. The property (iv) follows from this fact and from the following multidimensional version of Proposition 2.1.

Proposition 3.1. (L^2-stability) *We have,*

$$\frac{1}{2}\int_{(0,1)^d} u_h^2(x,T)\, dx + \int_0^T\int_{(0,1)^d} |\,\mathbf{q}_h(x,t)\,|^2\, dx\, dt + \Theta_{T,\mathbb{C}}([\mathbf{w}_h]) \leq \frac{1}{2}\int_{(0,1)^d} u_0^2(x)\, dx,$$

where

$$\Theta_{T,\mathbb{C}}([\mathbf{w}_h]) = \int_0^T \sum_{e\in\mathcal{E}_{\Delta x}} \int_e [\mathbf{w}_h(x,t)]^t\mathbb{C}\,[\mathbf{w}_h(x,t)]\, d\,\Gamma(x)\, dt.$$

We can also prove the following error estimate. We denote the integral over $(0,1)^d$ of the sum of the squares of all the derivatives of order $(k+1)$ of u by $|\,u\,|_{k+1}^2$.

Theorem 3.2. (L^2-error estimate) *Let* $\mathbf{e}$ *be the approximation error* $\mathbf{w}-\mathbf{w}_h$. *Then we have, for arbitrary, regular grids,*

$$\left\{\int_{(0,1)^d} |\,e_u(x,T)\,|^2\, dx + \int_0^T\int_{(0,1)^d} |\,\mathbf{e}_q(x,t)\,|^2\, dx\, dt + \Theta_{T,\mathbb{C}}([\mathbf{e}])\right\}^{1/2} \leq\, C\,(\Delta x)^k,$$

where $C = C(k,|\,u\,|_{k+1},|\,u\,|_{k+2})$. *In the purely hyperbolic case* $a_{ij} = 0$, *the constant* C *is of order* $(\Delta x)^{1/2}$. *In the purely parabolic case* $c = 0$, *the constant* C *is of order* Δx *for even values of* k *and of order* 1 *otherwise for Cartesian products of uniform grids and for* $\mathbb{C}$ *identically zero provided that the local spaces* Q^k *are used instead of the spaces* P^k, *where* Q^k *is the space of tensor products of one dimensional polynomials of degree* k.

The algebraic manipulations needed to prove this result are a straightforward extension to the multidimensional case of the manipulations of the proof of the corresponding one-dimensional result, Theorem 2.2. The approximation properties of the finite element spaces V_h that extend the results of Lemmas 2.5 and 2.6 are the following. Let e denote a face of the element K and let us denote by K_e the element sharing the face e with K, then

$$\|\,\mathbb{P}_h(u)^\pm - u\,\|_{L^2(e)} \leq \frac{1}{2}c_k\,(\Delta x\,)^{k+1/2}\,|\,u\,|_{H^{(k+1)}(K\cup K_e)},$$

where $\mathbb{P}_h(u)^{\pm}$ is either the value of $\mathbb{P}_h(u)$ at e from the interior of K or from its exterior, and

$$\| \, [\mathbb{P}_h(p)] \, \|_{L^2(e)} \le d_k \, (\Delta x)^{-1/2} \, \| \, \mathbb{P}_h(p) \, \|_{L^2(K \cup K_e)},$$

where $[\mathbb{P}_h(p)]$ denotes the jump at e of $\mathbb{P}_h(p)$. Finally, we also use the following result:

$$\| \, p - \mathbb{P}_h(p) \, \|_{L^2(0,1)^d} \le g_k \, (\Delta x)^{k+1} \, |\, p \,|_{H^{(k+1)}(0,1)^d},$$

All these approximation results can be found in Ciarlet [11], for example.

4. Concluding remarks. In this paper, we have considered the so-called LDG methods for convection-diffusion problems. For scalar problems in multidimensions, we have shown that they are L^2-stable and that in the linear case, they are of order k if polynomials of order k are used. We have also shown that this estimate is sharp and have displayed the strong dependence of the order of convergence of the LDG methods on the choice of the numerical fluxes.

The LDG methods for multidimensional systems, like for example the compressible Navier-Stokes equations and the equations of the hydrodynamic model for semiconductor device simulation, can be easily defined by simply applying the procedure described for the multidimensional scalar case to each component of **u**. In practice, especially for viscous terms which are not symmetric but still semipositive definite, such as for the compressible Navier-Stokes equations, we can use $\mathbf{q} = (\partial_{x_1} u, ..., \partial_{x_d} u)$ as the auxiliary variables. Although with this choice, the L^2-stability result will not be available theoretically, this would not cause any problem in practical implementation; see Bassi and Rebay [5] and Bassi *et al* [6].

The main advantage of these methods is their extremely high parallelizability and their high-order accuracy which render them suitable for computations of convection-dominated flows. Indeed, although the LDG method have a large

amount of degrees of freedom per element, and hence more computations per element are necessary, its extremely local domain of dependency allows a very efficient parallelization that by far compensates for the extra amount of local computations.

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