

Let's consider the RT1 example: three interior points map to each edge with two nodes. Three parametrized values transfer via the orthogonal projection to three points along the tangent line of one edge. I now want to obtain the values of the existing 1D polynomial basis that best fit these three point values using least squares - how do I form the matrix problem and solution given my current Vandermonde matrix for the edge polynomial?

ChatGPT said:

Given your RT1 element, you have:

- Three interior points that project orthogonally onto an edge.
- A 1D polynomial basis on the edge (formed using Jacobi polynomials).
- An inverse Vandermonde matrix that maps nodal values to polynomial coefficients.

Your goal is to fit the three projected values using the 1D edge basis in a least-squares sense.

## Step 1: Define the System

You are given three projected values  $U_i$  (from the RT interior) at locations  $s_0^i$  along the edge. The best-fit function in your edge polynomial basis (which consists of two nodes for RT1) is:

$$U_{\text{edge}}(s_0) = \sum_{k=0}^N c_k P_k(s_0)$$

where  $P_k(s_0)$  are the 1D Jacobi basis polynomials and  $c_k$  are the coefficients we seek.

To solve this, we minimize the least-squares error:

$$\sum_{i=1}^3 \left( \sum_{k=0}^N c_k P_k(s_0^i) - U_i \right)^2.$$

This results in a linear system:

$$\mathbf{A}\mathbf{c} = \mathbf{b}$$

where:

- $A_{ik} = P_k(s_0^i)$  (evaluation of basis polynomials at the projected locations).
- $c_k$  are the unknown coefficients.
- $b_i = U_i$  (values projected from the interior).

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## Step 2: Construct the Matrix System

Given:

- Three projected values:

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

- A 1D basis with two functions  $P_0(s_0)$ ,  $P_1(s_0)$  (since RT1 edges have two nodes).
- Three sampling locations  $s_0^i$  along the edge.

The least-squares matrix is:

$$\mathbf{A} = \begin{bmatrix} P_0(s_0^1) & P_1(s_0^1) \\ P_0(s_0^2) & P_1(s_0^2) \\ P_0(s_0^3) & P_1(s_0^3) \end{bmatrix}$$

and the right-hand side is:

$$\mathbf{b} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}.$$

The least-squares solution is:

$$\mathbf{c} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}.$$

This provides the best-fit coefficients  $c_k$  in the Jacobi polynomial basis.

### Step 3: Use the Vandermonde Matrix for Nodal Values

Since you already have the **Vandermonde matrix**  $\mathbf{V}$  for the edge, which relates **nodal values to Jacobi coefficients**, we can compute the **nodal values**  $U_{\text{edge, nodes}}$  from the fitted coefficients.

Using:

$$U_{\text{edge, nodes}} = \mathbf{V}\mathbf{c}$$

you obtain the **edge nodal values** needed for flux calculations.

### Final Steps for Implementation

1. Compute the basis values  $P_k(s_0^i)$  at the three projected locations.
2. Assemble the least-squares matrix  $\mathbf{A}$  and RHS  $\mathbf{b}$ .
3. Solve for the best-fit coefficients:

$$\mathbf{c} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{b}$$

4. Convert to nodal values using the edge **Vandermonde matrix**:

$$U_{\text{edge, nodes}} = \mathbf{V}\mathbf{c}.$$

### Projection of values from Interior nodes to parameterized edge

A **convenient expression** for projecting values from **(r, s) space** to a **tangential parameter**  $s'$  along an edge originating at  $(r_0, s_0)$  can be derived as follows:

### General Form for Projection to a Tangential Parameter $s'$

Given:

- A reference point  $(r_0, s_0)$  on the edge.
- A **tangential direction vector**  $\mathbf{t} = (t_r, t_s)$ , which is the **unit tangent** along the edge.
- A general interior point  $(r, s)$ .

We define the **tangential projection** as:

$$s' = (r - r_0)t_r + (s - s_0)t_s$$

where:

- $s'$  is the **scalar parameter** measuring displacement **along the edge tangent** starting from  $(r_0, s_0)$ .
- $\mathbf{t} = (t_r, t_s)$  is the **unit tangent vector** of the edge.

## Alternative Expression Using a Normalized Tangent

If the edge is parameterized by an arc-length parameterization, we can normalize  $s'$  so that it lies in a canonical range (e.g.,  $[-1, 1]$ ) by dividing by the edge length  $L$ :

$$s' = \frac{(r - r_0)t_r + (s - s_0)t_s}{L}$$

where:

$$L = \sqrt{(r_1 - r_0)^2 + (s_1 - s_0)^2}$$

for an edge between endpoints  $(r_0, s_0)$  and  $(r_1, s_1)$ .

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Given:

- A point  $(r, s)$  in the plane.
- A reference point  $(r_0, s_0)$  on the line.
- A **unit tangent vector**  $\mathbf{t} = (t_1, t_2)$ , defining the line direction.

## Projection Formula

The **scalar parameter** ( $\lambda$ ) measuring displacement along the tangent is:

$$\lambda = (r - r_0)t_1 + (s - s_0)t_2$$

Using this, the **projected point**  $(r_{\text{proj}}, s_{\text{proj}})$  is:

$$r_{\text{proj}} = r_0 + \lambda t_1$$

$$s_{\text{proj}} = s_0 + \lambda t_2$$

## Why This Works

- This **projects**  $((r, s))$  **orthogonally onto the line**, preserving alignment with the tangent.
- The computed  $(r_{\text{proj}}, s_{\text{proj}})$  **lies directly on the line** defined by  $(r_0, s_0)$  and  $\mathbf{t}$ .