Unit → 4

DETERMINANT

Determinant is a function defined on a set of square matrices, which associates every square matrix to a unique number.

i.e., $\det : A \rightarrow \det A$.

$A \xrightarrow{\ \ ||\ \ } |A|$

Note → For a square matrix of order 1×1, its determinant is the number (scalar) itself. For example $A_{1\times1} = [-5]_{1\times1}$

then, $\det A = |-5|$
$= -5$.

Working / Finding process of determinant:

i) For 2×2 matrix:

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Then its determinant is defined as: $\det(A) = |A| = a_{11}\cdot a_{22} - a_{12}\cdot a_{21}$

ii) Likewise for 3×3 matrix:

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

a_{11} means 1st row र 1st column लाई छोडेर अनि a_{22}, a_{23}, a_{32}, a_{33} यो दुई को multiply finally -ve sign र यो दुई को multiply similarly for others.

$= a_{11}(a_{22}\cdot a_{33} - a_{23}\cdot a_{32}) - a_{12}(a_{21}\cdot a_{33} - a_{23}\cdot a_{31}) + a_{13}(a_{21}\cdot a_{32} - a_{22}\cdot a_{31})$

OR $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$

Example:- Compute the determinant of $A = \begin{bmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{bmatrix}$.

$\det(A) = \begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 1 & 4 & -1 \end{vmatrix}$

$= 2(-1-8) - (-4)(-3-2) + 3(12-1)$

$= -18 - 20 + 33$

$= -5$

⊛. Co-factor expansion:

Let $A=[a_{ij}]$ be a matrix, the $(i,j)^{th}$ cofactor of A denoted by C_{ij} and given by $C_{ij}=(-1)^{i+j}\det(A_{ij})$.

Then,

$\det(A)=a_{11}C_{11}+a_{12}C_{12}+\dots+a_{1n}C_{1n}$ which is known as cofactor expansion accorss the first row of A_1.

This concept leads to following theorem:

Theorem 1: The determinant of an nxn matrix of A can be computed by a cofactor expansion across the i^{th} row as:

$$\det(A)=a_{i1}C_{i1}+a_{i2}C_{i2}+\dots+a_{in}C_{in}.$$

where, $C_{ij}=(-1)^{i+j}\det(A_{ij})$.

& The cofactor expansion across the j^{th} column is

$$\det(A)=a_{1j}\cdot C_{1j}+a_{2j}C_{2j}+\dots+a_{nj}C_{nj}.$$

where $C_{ij}=(-1)^{i+j}\det(A_{ij})$.

Example: Using cofactor expansion, compute the determinant of A where

$$A=\begin{bmatrix}1 & 5 & 0\\ 2 & 0 & 0\\ 3 & -2 & 3\end{bmatrix}.$$

Sol^n

Here, $\det(A)=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}$

$$=a_{11}(-1)^{1+1}\det(A_{11})+a_{12}(-1)^{1+2}\det(A_{12})+a_{13}(-1)^{1+3}\det(A_{13})$$

$$=1\begin{vmatrix}0 & 0\\ -2 & 3\end{vmatrix}-5\begin{vmatrix}2 & 0\\ 3 & 3\end{vmatrix}+0\begin{vmatrix}2 & 0\\ 3 & -2\end{vmatrix}$$

$$=0-30+0$$

$$=-30.$$

Theorem 2: If A is a triangular matrix, then det(A) is the product of the entries on the main diagonal of A.

Then $\det(A)=(a_{11})(a_{22})\dots(a_{nn})$.

Example: Find det(A) where $A=\begin{bmatrix}2 & 0 & 0\\ 3 & 5 & 0\\ 1 & 0 & 1\end{bmatrix}$

Sol^n

Given matrix is a triangular matrix. So, det(A) is the product of diagonal elements.

$$\det(A)=\begin{bmatrix}2 & 0 & 0\\ 3 & 5 & 0\\ 1 & 0 & 1\end{bmatrix}=2\times5\times1=10$$

Properties of determinants:

1) **Row operations:-** Let A be a square matrix.

ⓐ. If a multiple of one row of A is added to another row to produce a matrix B then $\det(A) = \det(B)$.

ⓑ If two rows of A are interchanged to produce B then $\det(A) = -\det(B)$

ⓒ. If one row of A is multiplied by k (scalar) to produce B then $k.\det(A) = \det(B)$.

Example: By using row operation, compute $\det(A)$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 3 & 7 & 4 \end{bmatrix}$.

Sol^n

Here, $\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 3 & 7 & 4 \end{vmatrix}$

$R_3 \to R_3 - 3R_1$

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & -4 \\ 0 & 1 & -5 \end{vmatrix}$$

$R_2 \to \frac{1}{5} R_2$

$$\det(A) = \overset{k}{5}\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -4/5 \\ 0 & 1 & -5 \end{vmatrix}$$

This is not matrix this is determinant so if we divide any row by integer then we should multiply also that whole det..

$R_3 \to R_3 - R_2$ then,

$$\det(A) = 5\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & -4/5 \\ 0 & 0 & -21/5 \end{vmatrix}$$

scalar k outside det. product of diagonal element

The determinant is triangular. So, $\det(A) = (5)(1)(1)(-21/5)$
$= -21$

Theorem: A square matrix A is invertible if and only if $\det(A) \neq 0$.

Q. Use determinants to find out matrix is invertible or not, $\begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}$.

Sol^n

Here, $\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 15 & 9 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix}$ [Applying $R_1 \to R_1 - 5R_2$]

$= -(1)\begin{vmatrix} 15 & 9 \\ 5 & 3 \end{vmatrix}$

$= (-1)(45-45) = 0$ This means given matrix is not invertible.

Q. Explore the effect of an elementary row operation on the determinant of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$.

Solⁿ

Here,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and $$\begin{vmatrix} a & b \\ kc & kd \end{vmatrix} = kad - kbc = k(ad - bc)$$

$$= k \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

The determinant is multiplied by a scalar k is same as one row of the determinant is multiplied by the scalar k.

Note: If the elementary row replacement in the matrix i.e $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ then the determinant will be 1.

Q. Verify that $\det(EA) = (\det E)(\det A)$, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Solⁿ

Let $E = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then,

$$EA = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ ka+c & kb+d \end{bmatrix}$$

$$\therefore \det(EA) = \begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix}$$

$$= (kab + ad) - (kab + bc)$$

$$= ad - bc.$$

$$\det(A) = \begin{vmatrix} a & d \\ c & b \end{vmatrix} = ad - bc$$

& $$\det E = \begin{vmatrix} 1 & 0 \\ k & 1 \end{vmatrix} = 1$$

Thus $\det(E) . \det(A) = (1)(ad - bc) = ad - bc = \det(EA)$.

Q. Use the determinant to decide if v_1, v_2, v_3, v_4 are linearly independent or not, when $v_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 5 \\ 3 \\ -5 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -7 \\ 6 \\ 4 \end{bmatrix}, v_4 = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -2 \end{bmatrix}$.

Solution

Here,

$$\det[v_1\ v_2\ v_3\ v_4] = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix}$$

$R_4 \to R_4 + R_2$

$$= \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix}$$

$$= (-2) \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix}$$

$R_2 \to R_2 - 3R_1$

$$= -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= -2(1) \begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix}$$

$$= -2(0+15)$$
$$= -30 \neq 0.$$

This means the given column vectors are linearly dependent.

Column Operations:

Example: Evaluate the column operations, $\det(A) = \begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$

Soln Here, $\det(A) = \begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix}$

Performing $C_2 \to C_2 - 5C_1$ and $C_3 \to C_3 + 3C_1$ then

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 3 & -18 & 12 \\ 2 & 3 & -1 \end{vmatrix}$$

Performing $C_3 \to C_3 + \frac{12}{18} C_2$ then,

$$\det(A) = \begin{vmatrix} 1 & 0 & 0 \\ 3 & -18 & 0 \\ 2 & 3 & 1 \end{vmatrix}$$

This is triangular matrix. So, $\det(A) = (1)(-18)(1)$
$= -18$

Theorem: (Multiplicative Property)

If A and B are $n\times n$ matrices then $\det(AB) = \det A \cdot \det(B)$.

Example 1:- Show that $\det(AB) = \det(A).\det(B)$ holds for matrices.

Soln Here,

Given, $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$

Now, $AB = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$

$= \begin{bmatrix} 33 & 10 \\ 12 & 5 \end{bmatrix}$

Then, $\det(AB) = \begin{vmatrix} 33 & 10 \\ 12 & 5 \end{vmatrix} = 165 - 120 = 45$

Next, $\det(A).\det(B) = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} 6 & 1 \\ 3 & 2 \end{vmatrix} = (5).(9) = 45$

Thus, $\det(AB) = \det(A).\det.(B)$.

Example 2:→ If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. show that $\det(A+B) = \det(A) + \det(B)$ if and only if $a+d=0$.

Solution Since, $\det.A = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$\det.B = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Here, $A+B = \begin{bmatrix} 1+a & b \\ c & 1+d \end{bmatrix}$

Then $\det.(A+B) = \begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix} = 1+a+d+ad-bc$

Suppose $a+d=0$, then,

$\det(A) + \det(B) = \det(A+B)$

$\Rightarrow 1+ad-bc = 1+a+d+ad-bc$

$\Rightarrow a+d=0.$

Cramer's Rule, Volume and Linear Transformations:

Cramers Rule: let A be an invertible $n\times n$ matrix. For any b in R^n, the unique solution x of $Ax=b$ has entries given by,

$$x_i = \frac{\det(A_i(b))}{\det(A)} \quad \text{for } i=1,2,\dots,n.$$

Example 1: By using Cramer's rule, solve the system of equations

$$3x_1 - 2x_2 = 6$$
$$-5x_1 + 4x_2 = 8.$$

Soln

Taking the given system as in $Ax=b$ and choosing it as,

$$A=\begin{bmatrix}3 & -2\\ -5 & 4\end{bmatrix},\ A_1(b)=\begin{bmatrix}6 & -2\\ 8 & 4\end{bmatrix},\ A_2(b)=\begin{bmatrix}3 & 6\\ -5 & 8\end{bmatrix}$$

$$\det(A)=\begin{vmatrix}3 & -2\\ -5 & 4\end{vmatrix}=12-10=2\neq 0.$$

So, the system has unique solution and the process is possible. If the system would have $\det=0$ then the system does not has unique solution.

Therefore by Cramer's rule,

$$x_1=\frac{\det A_1(b)}{\det A}=\frac{24+16}{2}=20$$

$$x_2=\frac{\det A_2(b)}{\det A}=\frac{24+30}{2}=27.$$

Thus $x_1=20$, $x_2=27$ be the solution of given system.

Example:-2: Using Cramer rule determine the value of s for which the system has unique solution.

$$3sx_1 - 2x_2 = 4$$
$$-6x_1 + sx_2 = 1,$$

Soln

Here,

$$A=\begin{bmatrix}3s & -2\\ -6 & s\end{bmatrix},\ X=\begin{bmatrix}x_1\\ x_2\end{bmatrix} \text{ and } b=\begin{bmatrix}4\\ 1\end{bmatrix}$$

Then,

$$A_1(b)=\begin{bmatrix}4 & -2\\ 1 & s\end{bmatrix} \text{ and } A_2(b)=\begin{bmatrix}3s & 4\\ -6 & 1\end{bmatrix}$$

Therefore, $\det A_1(b) = 4s+2 = 2(2s+1)$

and, $\det A_2(b) = 3s+24 = 3(s+8)$

also $\det(A) = 3s^2 - 12 = 3(s-2)(s+2)$

Now by Cramer's rule

$$x_1 = \frac{\det A_1(b)}{\det A} = \frac{2(2s+1)}{3(s-2)(s+2)}$$

$$x_2 = \frac{\det A_2(b)}{\det A} = \frac{s+8}{(s-2)(s+2)}$$

Hence, system has unique solution when $s \neq 2$ and $s \neq -2$.

Formula for finding A^{-1},

Let A be an invertible $n \times n$ matrix. Then,

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A).$$

Example: Find the inverse of the matrix $\begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

Solution

Given,

$$A = \begin{bmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Here, $\det(A) = \begin{vmatrix} 3 & 5 & 4 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$

$= 3(0-1) - 5(1-2) + 4(1-0)$

$= -3 + 5 + 4$

$= 6 \neq 0$, so the inverse of A exists.

The Co-factors of A are:

> **Note:** Co-factor of an element a_{ij} in the determinant $|A|$ is diagonal as $(-1)^{i+j} M_{ij}$.
>
> i.e, $(-1)^{i+j} \cdot \det A_{ij}$.
>
> It is denoted by C_{ij}.

$C_{11} = (-1)^2 \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1.$ $\qquad C_{12} = (-1)^3 \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1.$

$C_{13} = (-1)^4 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1.$ $\qquad C_{21} = (-1)^3 \begin{vmatrix} 5 & 4 \\ 1 & 1 \end{vmatrix} = -1$

$C_{22} = (-1)^4 \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = -5.$ $\qquad C_{23} = (-1)^5 \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = 7$

$C_{31} = (-1)^4 \begin{vmatrix} 5 & 4 \\ 0 & 1 \end{vmatrix} = 5.$ $\quad C_{32} = (-1)^5 \begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} = 1.$

$C_{33} = (-1)^6 \begin{vmatrix} 3 & 5 \\ 1 & 0 \end{vmatrix} = -5.$

Then, adj.(A) = Transpose of matrix of cofactors of A.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & -5 & 7 \\ 5 & 1 & -5 \end{bmatrix}$$

$$\text{adj.}(A) = \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$$

Now, using inverse formula,

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{adj}(A)$$

$$= \frac{1}{6} \begin{bmatrix} -1 & -1 & 5 \\ 1 & -5 & 1 \\ 1 & 7 & -5 \end{bmatrix}$$

⊛. Determinants as Area or Volume:

(i) If A is 2×2 matrix, the area of the pallelogram determined by the columns of A is $|\det(A)|$
i.e. positive value of det(A).

(ii) If A is 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det(A)|$
i.e. positive value of det (A).

Example 1: Find the area of the pallelogram whose vertices are $(0,-2), (6,-1), (-3,1), (3,2)$.

Soln Given vertices of pallelogram are $(0,-2), (6,-1), (-3,1), (3,2)$.
Now, translate the vertices so as one vertex becomes at origin as;

$(0,-2) + (0,2) = (0,0)$
$(6,-1) + (0,2) = (6,1)$
$(-3,1) + (0,2) = (-3,3)$
$(3,2) + (0,2) = (3,4)$

Then the pallelogram is shifted with vertices $(0,0), (6,1), (-3,3), (3,4)$.
So, the pallelogram is determined by the columns of

$$A = \begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix}$$

Then, $\det(A) = \begin{vmatrix} 6 & -3 \\ 1 & 3 \end{vmatrix} = 18 + 3 = 21$

Thus the area of pallelogram is $|21| = 21$.

Example 2: Find the volume of the ~~pall~~ parallelepiped with one vertex at origin and the adjcent vertices at $(1,4,0), (-2,-5,2)$ and $(-1,2,-1)$.

Solution

Since the one vertex of the parallelopiped is at origin and the adjcent vertices are at $(1,4,0), (-2,-5,2)$ and $(-1,2,-1)$.

Then,

$$\det(A) = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} 1 & -2 & -1 \\ 4 & -5 & 2 \\ 0 & 2 & -1 \end{vmatrix} \end{array}$$

$$= 1\begin{vmatrix} -5 & 2 \\ 2 & -1 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ 0 & -1 \end{vmatrix} - 1\begin{vmatrix} 4 & -5 \\ 0 & 2 \end{vmatrix}$$

$$= 1 - 8 - 8$$

$$= -15.$$

Thus, the volume of the parallepiped with $|-15| = 15$.

⊛. Linear transformations in determinants:-

Let $T: R^2 \to R^2$ be the linear transformation determined by a 2×2 matrix A. If s is a pallelogram in R^2 then,

$$\text{area of } T(s) = |\det(A)| \cdot \{\text{area of } s\}.$$

Likewise, if $T: R^3 \to R^3$ be determined by 3×3 matrix A and if s is a parallelepiped in R^3 then,

$$\text{Volume of } T(s) = |\det(A)| \cdot \{\text{volume of } s\}.$$

Example 1: Let S be a parallelogram determined by vectors $b_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $b_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$; and let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$. Compute the area of image of S under mapping $x \to Ax$.

Solution :

Given that S is the pallelogram determined by vectors $b_1 = \begin{bmatrix}1\\3\end{bmatrix}$ and $b_2 = \begin{bmatrix}5\\1\end{bmatrix}$.

So, $\det(S) = \begin{vmatrix}1 & 5\\3 & 1\end{vmatrix} = 1-15 = -14$

Thus, Area of $S = |-14| = 14$.

And given that $A = \begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}$

Then, $\det(A) = \begin{vmatrix}1 & -1\\0 & 2\end{vmatrix} = 2$.

Therefore, the area of S under the mapping $x \to Ax$ is,

$$\begin{aligned}\text{area of image of } S &= \text{Area of } T(S)\\ &= |\det A| \cdot \{\text{area of } S\}\\ &= 2\times 14\\ &= 28 \text{ sq. unit.}\end{aligned}$$

Example 2:- Let a and b are positive numbers. Find the area of the region E bounded by the ellipse whose equation is $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$.

Solution

Let, $A = \begin{bmatrix}a & 0\\0 & b\end{bmatrix}$

Let, $u = \begin{bmatrix}u_1\\u_2\end{bmatrix}$

and $x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

Let E be the image of unit disk D under a linear transformation T determined by matrix A with $Au = x$. Then,

$$u_1 = \frac{x_1}{a}, \quad u_2 = \frac{x_2}{b}.$$

Since u_1, u_2 lies in the unit disk with $u_1^2 + u_2^2 \leq 1$ if and only if x is in E with $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1$.

Then,

$$\begin{aligned}\text{area of ellipse} &= \text{area of } T(D)\\ &= |\det(A)| \cdot \{\text{area of } D\}\\ &= ab.\pi \quad [\because D \text{ is an unit disk}]\\ &= \pi ab.\end{aligned}$$

Example 3: Let the four vertices $O(0,0)$, $A(1,0)$, $B(0,1)$ and $C(1,1)$ of a unit square be represented by 2×4 matrix: $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Investigate and interpret geometrically the effect of pre-multiplication of this matrix by the 2×2 matrix $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

Solution.

The matrix represented to a square having vertices at $O(0,0)$, $A(1,0)$, $B(0,1)$ and $C(1,1)$ is $S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

and given matrix is $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$

Therefore, the effect of pre-multiplication of S by A is,

$$S' = A.S = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4 & -1 & 3 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

This means the vertices of the effect of the square A are $O'(0,0)$, $A'(4,2)$, $B'(-1,1)$ and $C'(3,3)$.

Here,

$$(\text{area of } S).(\text{area of } A).$$

$$= |\det(S)| \cdot |\det(A)|.$$

$$= \left| \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right| \cdot \left| \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} \right|$$

$$= |1| . |6|$$

$$= 6.$$

and area of $S' = \left| \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} \right|$.

$$= |(4) - (-2)|$$

$$= |6|$$

$$= 6.$$