Unit-10
Ring and Field:

⊛ Ring:

Definition → An algebraic structure $(R, +, \times)$ with the two binary operations addition $(+)$ and multiplication $(\times)$ that satisfies the following conditions is called a ring.

i) Closure for addition

$$a+b \in R, \forall a,b \in R.$$

ii) Associativity

$$a+(b+c)=(a+b)+c, \forall a,b,c \in R.$$

iii) Existence of identity

$$\exists\, 0 \in R \text{ such that } 0+a=a+0=a, \forall a \in R.$$

iv) Existence of inverse

$$\exists -a \in R$$

$$a+(-a)=(-a)+a=0 \ \forall a \in R.$$

v) Commutativity

$$a+b=b+a \quad , \forall a,b \in R.$$

group

Abelian group

vi) Associativity for multiplication

$$a\cdot(b\cdot c)=(a\cdot b)\cdot c \quad \forall a,b,c \in R.$$

viii) Distributivity for multiplication over addition:

ⓐ Left distributive: $a\cdot(b+c)=a\cdot b+a\cdot c \quad \forall a,b,c \in R.$

ⓑ Right distributive: $(a+b)\cdot c=a\cdot c+b\cdot c \quad \forall a,b,c \in R.$

Note: First four conditions show that R is a group under addition, & the first five conditions shows that R is an abelian group.

OR

✓ Ring can also be defined as an algebraic structure $(R, +, \times)$ such that:

ⓐ R is an abelian group under $+$.

ⓑ Associativity holds for multiplication.

ⓒ. Multiplication is distributive from left as well as right.

Commutative ring → A ring $(R,+,\times)$ is said to be commutative ring if multiplication operation is commutative.

Examples for commutative ring

① $(\mathbb{Z},+,\times)$ is a ring

Sol^n

For we have, $\mathbb{Z}$ is a non-empty set.

i) $a+b \in \mathbb{Z} \quad \forall a,b \in \mathbb{Z}$.

ii) $a+(b+c)=(a+b)+c \quad , \forall a,b,c \in \mathbb{Z}$.

iii) $\exists 0 \in \mathbb{Z} : 0+a=a+0=a \quad \forall a \in \mathbb{Z}$.

iv) $\exists -a \in \mathbb{Z} : a+(-a)=(-a)+a=0 \quad \forall a \in \mathbb{Z}$.

v) $a+b=b+a \quad \forall a,b \in \mathbb{Z}$.

vi) $ab=ba \quad \forall a,b \in \mathbb{Z}$.

vii) $a(bc)=(ab)c \quad \forall a,b \in \mathbb{Z}$.

viii) ⓐ. $a(b+c)=ab+bc$

ⓑ $(a+b)\cdot c=ab+bc$

It is commutative ring since $ab=ba \; \forall a,b \in \mathbb{Z}$.

②. The set of real numbers with the binary operations: $+,\times$ is a ring. i.e, $(\mathbb{R},+,\times)$ is a ring.

③. The set of rational numbers with the two binary operations addition, $+$ and multiplication $\times$, is a ring.

④. **Null (zero) ring** → The set $\{0\}$ with the two binary operation $+,\times$ constitutes a ring called null ring.

⊛ Related Questions:

①. Show $\mathbb{Z}_7=\{0,1,2,3,4,5,6\}$ is a ring under the binary operation addition modulo $(+_7)$ and multiplication modulo $(\times_7)$, 4.

Proof: The composition table for addition modulo and multiplication modulo.

$+_7$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

max. num 6 हो त्यसैले 6 भन्दा माथि हुनु भएमा 7 subtract गर्ने. यसमा 1+6=7 आको >6 त्यसैले 7-7=0 गरेको similarly for others >6.

$\times_7$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

हुनुपर्ने $3\times5=15$ which is $>$ 2 times 6.
So, $7\times2=14$
subtracting from 15 we get 1 as remainder.
So, if >6 divide by 7 and write the remainder.

Here the set is abelian group for addition modulo 7.

i) Closure $\rightarrow \forall a,b \in Z_7,\ a+_7 b \in Z_7$.

ii) Associativity $\rightarrow$

~~$2+_7(3+_7 4) = (2+_7 3)+_7 4$~~

$$(a+_7 b)+_7 c = a+_7(b+_7 c)\ \forall a,b,c \in Z_7.$$

iii) Additive identity $\rightarrow$ 0 is the additive identity.

iv) Existence of additive inverse $\rightarrow \forall a \in Z_7$

$$\exists\, a \in Z_7 : a+(-a)=0.$$

v) Commutativity holds $\rightarrow a+_7 b = b+_7 a$.

vi) Closed for multiplication $\rightarrow a\times_7 b \in Z_7,\ \forall a,b \in Z_7$.

vii) Associtivity for multiplication $\rightarrow$

$$a\times_7(b\times_7 c) = (a\times_7 b)\times_7 c,\ \forall a,b,c \in Z_7.$$

viii) Distributivity for multiplication over addition:

Left: $a\times_7(b+_7 c) = a\times_7 b + a\times_7 c$.

Right: $(a+_7 b)\times_7 c = a\times_7 c + b\times_7 c\ \forall a,b,c \in Z_7$.

Q. Evaluate: $(12)(14)$ in Z_{21}.

Solution,

we have, $12\times14 = 168$
$= 8\times21+0$
$= 0$

Z_{21} दिइएकोले 21 ले multiply गरि remainder add गरेको र ans remainder लेखेको.

Q. Evaluate the sum $(1,2)+(3,5)$ in $Z_3\times Z_7$.

Solution. $(1,2)+(3,5)$ in $Z_3\times Z_7$
$= (1+3,\ 2+5)$
$= (4,\ 7)$
$= (1,0)$

Z_3 र Z_7 दिइएको छ $>$ 3 र 7 आयो। (यसैले 3 र 7 ले seperately divide गरि remainder लेखेको.

Q. Compute the product in the given ring:

ⓐ. $(12)(6) \in \mathbb{Z}_{25}$

ⓑ $(20)(-8) \in \mathbb{Z}_{26}$

ⓒ $(-3,5)(2,-4) \in \mathbb{Z}_4 \times \mathbb{Z}_{11}$

Solution:

ⓐ. $(12)(6) \in \mathbb{Z}_{25}$.

we have, $12 \times 6 = 72$ — 25 ले expand गरेको।

$= 25 \times 2 + 22$

$= 22$

ⓑ $(20)(-8) \in \mathbb{Z}_{26}$

we have, $(20).(-8) = -160$

$= -6 \times 26 + (-4)$

$= -4$

$= -4 + 26$ — negative value भएकोले positive बनाउन 26 add गरेको।

$= 22$

ⓒ. $(-3,5)(2,-4) = (-6,-20)$

$= (-2,-9)$ → negative छ Z_4 र Z_{11} दिएको छ त्यसैले 4 र 11 जोडेको

$= (2,2)$

⊛ Properties of ring: (Not more imp).

Let $a, b \in (R,+,\times)$ 0 be an additive identity of the ring.

then,

(i) $a.0 = 0.a = 0$.

(ii) $a(-b) = (-a).b = -(a.b)$

(iii) $(-a).(-b) = -(a.b)$.

(i) Proof:

we have, $a.0 = a.(0+0)$ $(\because 0 = 0+0)$

or, $a.0 = a.0 + a.0$ (distributivity property).

Since, $0, a \in R$

$a.0 \in R$

$0 + a.0 = a.0 + a.0$

$\Rightarrow 0 = a.0$

i.e, $a.0 = 0$ (right cancellation law)

$0.a = (0+0).a$

or, $0.a = 0.a + 0.a$ (right distributivity).

or, $0 + 0.a = 0.a + 0.a$

$\Rightarrow 0 = 0.a$ (right cancellation law).

(ii) Proof

we have,

$$0.b = 0$$

$$[a+(-a)].b = 0.$$

$$\Rightarrow a.b + (-a).b = 0$$

$$\Rightarrow (-a).b = -(a.b) \quad \text{—(i).}$$

Also,

$$a.0 = 0$$

$$a\cdot[b+(-b)] = 0$$

$$\Rightarrow a.b + a(-b) = 0.$$

$$\Rightarrow a\cdot(-b) = -(a.b) \quad \text{—(ii).}$$

From (i) and (ii) we get,

$$a(-b) = (-a)b = -(a.b)$$

(iii). Proof:

we know that, $(+a).(-b) = -(a.b)$

using $-a$ for a

$$(-a).(-b) = -[(-a).b]$$

$$= -[-(a.b)]$$

$$= ab.$$

Zero divisor:

Let us consider a ring $(M_2(\mathbb{Z},+,\times))$. In the ring $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an identity element.

Let $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in (M_2(\mathbb{Z},+,\times))$ be non-zero elements such that their product is zero. The ring is called the ring with zero divisor.

Definition → A ring $(R,+,\times)$ is a ring with zero divisor.

Ring with no zero divisor → Let $(\mathbb{Z},+,\times)$ be a ring with no zero divisor because, $ab=0 \Rightarrow$ either $a=0$ or $b=0$ or both i.e, $ab=0$ only when at least one is zero.

Integral domain → A Ring $(R,+,\times)$ is said to be integral domain if and only if (iff)

(i) R is commutative ring.

(ii) R has an identity element for multiplication.

(iii) R has no zero divisors.

THE END

Best of Luck

Note:

⇒ Practice provided model questions and additional 4 sets also.

❤️

If my notes really helped you,then you can support me on esewa for my hardwork.

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