Unit → 5
Vector Spaces

Not more imp. focus on questions asked in exam

Definition → A non-empty set V (say) of objects called the vectors with the two binary operations: addition and multiplication by a scalar satisfying following properties:-

For all $u, v, w \in V$.

i) Closure: $u+v \in V$
ii) Commutativity: $u+v = v+u$
iii) Associativity: $u+(v+w) = (u+v)+w$
iv) Existance of additive identity: $\exists 0$ in V; called the zero vector such that $0+v = v$
v) Existence of inverse: $\exists -v \in V : v+(-v) = 0$

Abelian group.

$\forall a, b \in$ field (i.e, set of scalars) on which V is defined

vi) $au \in V$.
vii) $a(u+v) = au + av$
viii) $(a+b)u = au + bu$
ix) $a(bu) = (ab)u = b(au)$
x) $1.u = u$.

Examples: (i) $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R}\}$ is a vector space.
(ii) The set of polynomials constitutes a vector space.
(iii) $M_{m \times n} = \{(a_{ij})_{m \times n} ; a_{ij} \in \mathbb{R}\}$, set of the matrices of order $m \times n$ is a vector space under addition and multiplication by a scalar operations.

Sub-space: [refer example no.1 after this]

Definition → A sub-space U of a vector space say V over a field $\mathbb{K}$ is a non-empty subset of V such that it is also a vector space over the field $\mathbb{K}$.

OR

A non-empty subset U of a vector space V over field $\mathbb{K}$ is said to be subspace of V if it satisfies the following conditions;

$\forall u, v \in U$ and $a \in \mathbb{K}$,

i) $u+v \in U$ ii) $au \in U$ iii) $0 \in U$.

A subset U of a vector space V is said to subspace of V if $\forall\ u, v \in U$ and $a, b \in \mathbb{k}$,

$$au + bv \in U.$$

⊛. Linear Combination:

A linear combination of a set $S = \{v_1, v_2, \ldots, v_n\}$ of vectors in a vector space V (say) is any vector $c_1v_1 + c_2v_2 + \ldots + c_nv_n$ in V for any set of scalars c_i's, where $i = 1, \ldots, n$.

⊛. Linear span or linear hall:

A linear span of a set $S = \{v_1, \ldots, v_n\}$ of vectors in a vector space V (say) is the set of all possible linear combinations of the given set of vectors denoted by span S or $l\{v_1, \ldots v_n\}$.

So, $l\{v_1, \ldots, v_n\} = \{v : v = c_1v_1 + c_2v_2 + \ldots + c_nv_n\}$, c_i's $\in \mathbb{k}$.

Statement → Linear span of a given set of vectors $\{v_1, \ldots, v_n\}$ in a vector space V (say) is the subspace of V i.e. $l\{v_1, v_2, \ldots v_n\}$ is subspace of V; $\{v_1, \ldots, v_n\} \subseteq V$.

⊛ Examples related to these topics:

Example 1: Show $V = \mathbb{R}^3 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, x, y, z \in \mathbb{R} \right\}$ is a vector space over $\mathbb{R}$ and the set $W = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, s, t \text{ are real} \right\}$ is a subspace of V.

solution:

i) Taking, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, is an zero element in W.

ii) For all $\alpha, \beta \in \mathbb{R}$ and $w_1 = \begin{bmatrix} s_1 \\ t_1 \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} s_2 \\ t_2 \\ 0 \end{bmatrix} \in W$ then,

$$\alpha w_1 + \beta w_2 = \begin{bmatrix} \alpha s_1 + \beta s_2 \\ \alpha t_1 + \beta t_2 \\ 0 \end{bmatrix} \in W.$$

Hence, W is a subspace of V.

Example 2: Let $W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$ prove that W is not subspace of R^2 by showing that it is not closed under scalar multiplication.

Solution:

Since $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in W$ and $c = -1$ then,

$$cu = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \notin W$$

$\therefore$ W is not subspace of R^2.

Example 3: Let $V = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, a, b, c \in R \right\}$ is a vector space over the field R. Then show $W = \left\{ \begin{bmatrix} s \\ t \\ 4 \end{bmatrix}, s, t \in R \right\}$ is not subspace of V.

Solution:

Since for $w_1 = \begin{bmatrix} s_1 \\ t_1 \\ 4 \end{bmatrix}, w_2 = \begin{bmatrix} s_2 \\ t_2 \\ 4 \end{bmatrix} \in W$, and $\alpha, \beta \in R$ then,

$$\alpha w_1 + \beta w_2 = \begin{bmatrix} \alpha s_1 + \beta s_2 \\ \alpha t_1 + \beta t_2 \\ 4(\alpha + \beta) \end{bmatrix} \notin W.$$

Therefore, W is not a subspace of V.

Example 4: Let v_1 and v_2 in a vector space V. Define $H = \text{Span}\{v_1, v_2\} = \{\alpha v_1 + \beta v_2, \alpha, \beta \in R\}$ then H is a subspace of V.

Solution: Taking $\alpha = \beta = 0$ then $0 \in H$.

And, taking $\alpha, \beta \in K$ then for all $w_1, w_2 \in H$ with

$$w_1 = \alpha_1 v_1 + \beta_1 v_2$$
$$w_2 = \alpha_2 v_1 + \beta_2 v_2$$

Then,

$$\alpha w_1 + \beta w_2 = (\alpha \alpha_1 + \beta \alpha_2) v_1 + (\alpha \beta_1 + \beta \beta_2) v_2$$
$$= \alpha_3 v_1 + \beta_3 v_2 \in H.$$

where, $\alpha \alpha_1 + \beta \alpha_2 = \alpha_3, \alpha \beta_1 + \beta \beta_2 = \beta_3 \in K$

Therefore, H is a subspace of V.

Example 5: Let $H = \{(a-3b, b-a, a, b) ; a \text{ and } b \text{ in } R\}$. Show that H is a subspace of R^4.

Solution:

Since $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

$= a v_1 + b v_2$

where, $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

This shows that $H = \text{Span}\{v_1, v_2\}$, where v_1 and v_2 are the vectors from R^4. Thus H is a subspace of R^4. Since we have theorem that: If $v_1, \dots, v_p$ are in a vector space V, then $\text{Span}\{v_1, \dots, v_p\}$ is a subspace of V.

Example 6: For what values of h will y be in the subspace of R^3 spanned by v_1, v_2, v_3 if $v_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$.

Solution:

Let y be in a subspace of R^3 spanned by v_1, v_2 and v_3 then it is possible to find α, β, and $\gamma \in R$ such that

$$y = \alpha v_1 + \beta v_2 + \gamma v_3$$

$$\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + \beta \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + \gamma \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

i.e, the system $Ax = b$, $A = \begin{bmatrix} 1 & 5 & -3 \\ -1 & -4 & 1 \\ -2 & -7 & 0 \end{bmatrix}$, $b = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$ is consistent.

For this we have to reduce into row-echelon form.

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right]$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + 2R_1$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right]$$

Applying $R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & h-5 \end{array}\right]$$

Given that the given system is consistent. This means we have, $h - 5 = 0$

$$\Rightarrow h = 5.$$

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⊛. Null Space:

Let $A_{m\times n}$ be an $m\times n$ matrix. Null space of the matrix A, denoted by N(A) or Null(A) is the set of vectors X in $\mathbb{R}^n$ such that $AX=0$.

i.e, $N(A) = \{X / AX=0\}$.

Example 1: Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \end{bmatrix}_{2\times3}$ is a matrix of order 2×3.

Then, N(A) is the set of those vectors $X \in \mathbb{R}^3$ such that $AX=0$.

i) Find N(A).

Solution:

Let $X \in \mathbb{R}^3$: $AX=0$

so, that, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then, $AX=0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + 2x_2 - 3x_3 = 0$$
$$2x_1 + 4x_2 - 6x_3 = 0.$$

Now, we can find the solution by making augmented matrix of these system of equations.

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here x_1 is basic variable, x_2 and x_3 are free variables

$x_1 + 2x_2 - 3x_3 = 0$

x_2 is free

x_3 is free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} //$$

$$\therefore X = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} x_3 \quad \text{Ans:}$$

Example 2: Let $A=\begin{bmatrix}1 & -3 & -2\\ -5 & 9 & 1\end{bmatrix}$ then determine if $u=\begin{bmatrix}5\\3\\-2\end{bmatrix}$ belongs to null space of A.

Solution:

Here, $$Au=\begin{bmatrix}1 & -3 & -2\\ -5 & 9 & 1\end{bmatrix}\begin{bmatrix}5\\3\\-2\end{bmatrix}=\begin{bmatrix}5-9+4\\ -25+27-2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

This means, u is in Nul A.

Example 3: Find the spanning set of the null space of the matrix.
$$A=\begin{bmatrix}-3 & 6 & -1 & 1 & -7\\ 1 & -2 & 2 & 3 & -1\\ 2 & -4 & 5 & 8 & -4\end{bmatrix}$$

Solution:

To find the spanning set for the null space of the matrix A we have to solve the equation $Ax=0$. and find the set of vectors such that.
$$x=y_1u+y_2v+y_3w \quad \text{where } x\in R^5,\ y_1,y_2,y_3\in K.$$

Now,
$$Ax=0.$$

Equivalent sign

$$\sim\begin{bmatrix}-3 & 6 & -1 & 1 & -7 & : & 0\\ 1 & -2 & 2 & 3 & -1 & : & 0\\ 2 & -4 & 5 & 8 & -4 & : & 0\end{bmatrix}$$

Applying $R_2\leftrightarrow R_1$, we have
$$\sim\begin{bmatrix}1 & -2 & 2 & 3 & -1 & : & 0\\ -3 & 6 & -1 & 1 & -7 & : & 0\\ 2 & -4 & 5 & 8 & -4 & : & 0\end{bmatrix}$$

Applying $R_2\to R_2+3R_1$ and $R_3\to R_3-2R_1$ then
$$\sim\begin{bmatrix}1 & -2 & 2 & 3 & -1 & : & 0\\ 0 & 0 & 5 & 10 & -10 & : & 0\\ 0 & 0 & 1 & 2 & -2 & : & 0\end{bmatrix}$$

$$\sim\begin{array}{c}\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}\\ \begin{bmatrix}1 & -2 & 2 & 3 & -1 & : & 0\\ 0 & 0 & 1 & 2 & -2 & : & 0\\ 0 & 0 & 0 & 0 & 0 & : & 0\end{bmatrix}\end{array}$$

Column मा हेर्ने
first column $= x_1$
Second column $= x_2$
$\vdots$
last column x_n हुन्छ
जुन column मा pivot element हुन्छ त्यो basic अरु चाहिँ free variable

Here, x_2, x_4 and x_5 are free variables and x_1, x_3 are basic variables.

So, $x_1-2x_2-x_4+3x_5=0 \quad \Rightarrow x_1=2x_2+x_4-3x_5$

x_2 is free $\quad \Rightarrow x_2=x_2$

$x_3+2x_4-2x_5=0 \quad \Rightarrow x_3=-2x_4+2x_5$

x_4 is free $\quad \Rightarrow x_4=x_4$

x_5 is free $\quad \Rightarrow x_5=x_5$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 u + x_4 v + x_5 w$$

Therefore, every element of Nul A can be expressed as a linear combination of u, v, w. Hence spanning set of Nul $A = \{u, v, w\}$.

⊛ Column Space:

Let A be $m \times n$ matrix $[a_1 \ a_2 \ a_3 \dots a_n]$ then column space of A is denoted by Col A and defined by the space generated by the columns of A.

i.e, $\text{Col } A = \text{Span}\{a_1, a_2, \dots, a_n\}$.

Example: Find a matrix A such that $W = \text{Col } A$ where

$$W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} : a, b \in R \right\}.$$

Solution:

Here, $W = \left\{ a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} : a, b \in R \right\}$

$$= \text{span} \left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus, the matrix $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

Q. Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

(a) If the column space of A is a sub-space of R^k, $k = ?$

(b) If the null space of A is sub-space of R^k, $k = ?$

Solution:

Since the column of A has entries three so Col A is a sub-space of R^3 i.e, $k = 3 \leftarrow$ Column भन्दा

Similarly, the null space of A is sub-space of R^4 so $k = 4$.

row भन्दा

row र column हिसाबले उल्टो।

Q. Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$ find a non-zero vector in Col A. and Nul A.

Solution:

Given, $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

Any column of A say, $\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$ is in Col A.

For Nul A $[A \ \ 0] \sim \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ By reducing A into RREF

$\therefore$ Here, x_3 free variable. So,

$x_1 = -9x_3$, $x_2 = 5x_3$, $x_4 = 0$

$\therefore X = \begin{bmatrix} -9x_3 \\ 5x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix}$

Let $x_3 = 1$

$\therefore x = (-9, 5, 1, 0)$ is non-zero vector in Nul A.

Q. Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$.

(a). Determine if u is in null A. Could u be in Col A?

(b). Determine if v is in col.A. Could v be in Nul A?

Solution:

(a). If u is in Nul A when $Au = 0$.

Here,

$$Au = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, u is not a solution of $Ax = 0$. So u is not in Nul A.

(b). To confirm the vector v is in Col A it is sufficient to show that system of linear equation $Ax = v$ is consistent if

$$[A \ \ v] = \begin{bmatrix} 2 & 4 & -2 & 1 & : 3 \\ -2 & -5 & 7 & 3 & : -1 \\ 3 & 7 & -8 & 6 & : 3 \end{bmatrix}$$

By applying elementary row operation.

$$\sim \begin{bmatrix} 2 & 4 & -2 & 1 : 3 \\ 0 & 1 & -5 & -4 : -2 \\ 0 & 0 & 0 & 17 : 1 \end{bmatrix}$$

At this point it is clear that the equation $Ax = v$ is consistent.

So, v is in Col. A.

⊛. Kernel and Range of Linear Transformation:

Let $T: V \to W$ be a linear transformation then.

$T(x) = Ax$, where A is a matrix associate with linear transformation T.

$$\boxed{\begin{aligned} \ker T &= \{x \in V : T(x) = 0\} \\ &= \{x \in V : Ax = 0\} \\ &= \text{Nul } A. \end{aligned}}$$

$$\boxed{\begin{aligned} \text{(Image of } T\text{) i.e., Im}T \text{ or Range of } T &= \{T(x) : \forall x \in V\} \\ &= \{Ax : \forall x \in V\} \\ &= \text{Col } A. \end{aligned}}$$

⊛ Hence, kernal of linear transformation T is Nul A and range of transformation T is Col A, where A is matrix associate with linear transformation T.

Example: Let $T: R^3 \to R^3$ be a linear transformation defined by $T(x, y, z) = (x, y, -2y)$. Find (i) $\ker T$ (ii) ImT.

Solution,

Given, $$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -2y \end{pmatrix} = \begin{pmatrix} x + 0y + 0z \\ 0x + y + 0z \\ 0x - 2y + 0z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= A\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

$\therefore A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix}$ is matrix ~~with~~ associate with linear transformation T.

We know that $\ker T = \text{Nul } A$

For Nul A: $$[A \,.\, 0] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is reduced echelon form, x_1 & x_2 are basic while x_3 is free.

$x_1 = 0$
$x_2 = 0$
$x_3 = \text{free}$

Thus, $\text{Nul } A = \left\{\begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} : x_3 \in R\right\}$ $\therefore \ker T = \left\{\begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} : x_3 \in R\right\}.$

For Col A:

Col A = span $\{a_1, a_2, a_3\}$, where a_1, a_2 and a_3 are 1^{st}, 2^{nd} and 3^{rd} column of A.

$$= \left\{ a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\-2\end{pmatrix} + c\begin{pmatrix}0\\0\\0\end{pmatrix} : a, b, c \in R \right\}$$

$$= \left\{ \begin{pmatrix}a\\b\\-2b\end{pmatrix} : a, b \in R \right\}$$

$$\therefore \text{Im}T = \left\{ \begin{pmatrix}a\\b\\-2b\end{pmatrix} : a, b \in R \right\}$$

Basis:

Let H be a subspace of a vector space V. An indexed set vectors $B = \{b_1, b_2, \dots b_p\}$ in V is a basis for H. If.

i) the set $\{b_1, b_2, \dots, b_p\}$ is linearly independent.

ii) $H = \text{span}\{b_1, b_2, \dots, b_p\}$.

Example - Prove that the set of vectors $(3,0,-1), (0,1,2), (1,-1,1)$ form a basis of R^3.

Solution:

Here we have to show that v_1, v_2, v_3 are linearly independent and they span R^3.

For linearly independent, $Ax=0$.

$$\begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -8 & 0 \end{bmatrix}$$

No basic variable so having a trivial solution. Thus v_1, v_2, v_3 are linearly independent.

For v_1, v_2, v_3 span R^3. $\quad A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix}$$

Each row has pivot so, column of A span R^3.

$\therefore \{v_1, v_2, v_3\}$ span R^3.

Thus $\{v_1, v_2, v_3\}$ is basic for R^3.

Note: The pivot columns of matrix A form a basis for Col A.

Example 1: Find a basis for Col B, where $A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ is a reduced echelon form. continuing after 4-5 steps.

Here, pivot column of A is 1st, 3rd and 5th column.

Thus, basis for col $A = \left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \\ 5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \\ 2 \\ 8 \end{pmatrix} \right\}$

Example 2: Find the basis for the set of vectors in R^3 in the plane $x-3y+2z=0$.

Solution:

Given, $x-3y+2z=0$.

or, $\begin{bmatrix} 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

i.e, $AX = 0$

where, $A = \begin{bmatrix} 1 & -3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

So, $\begin{bmatrix} A & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 & 0 \end{bmatrix}$

Here y and z are free variables.

So, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\therefore \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is basis.