UNIT-3
MATRIX ALGEBRA

Matrix → It is a rectangular array of numbers enclosed between round or square brackets.

Matrix multiplication → Let, $A_{m\times n}$ and $B_{n\times p}$ be two matrices such that the number of columns in the first matrix equals to the number of rows in the second matrix. Then the product of A and B, AB is defined as;

The order of $A\times B$ is $m\times p$; the number of rows in first matrix followed by number of columns in the second matrix.

i.e, $A_{m\times n} \cdot B_{n\times p}$

$= AB_{m\times p}$

Example: let $A = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ 1 & 2 \end{bmatrix}_{3\times 2}$ & $B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & -1 \end{bmatrix}_{2\times 3}$

where, $b_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, b_3 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Now, $AB = A[b_1 \ b_2 \ b_3]$

$AB = [Ab_1 \ Ab_2 \ Ab_3]$ ——— (P)

Here,

$$Ab_1 = 1\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

$$Ab_2 = 0\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \\ 8 \end{bmatrix}$$

$$Ab_3 = 2\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 0 \end{bmatrix}$$

Now substituting these values in (P)

$$AB = \begin{bmatrix} 3 & -4 & 5 \\ 4 & 0 & 8 \\ -1 & 8 & 0 \end{bmatrix}_{3\times 3}.$$

③. Matrix product as composition mapping:

Let $A_{m\times n}$, $B_{n\times p}$ and $X_{p\times 1}$ be the matrices of the order $m\times n$, $n\times p$ and $p\times 1$ respectively. Then the product of B and X i.e, image of X under the transformation multiplication by B is a matrix, BX of order $n\times 1$.

When the matrix BX is multiplied by A, A(BX) is an image of BX under the mapping: multiplication by A.

Symbolically: $X \xrightarrow[\text{by } B]{\text{multiplication}} BX \xrightarrow[\text{by } A]{\text{multiplication}} A(BX)$.

Understanding diagram

$$\begin{tikzcd} X \arrow[r, "\text{multiplication by } B"] \arrow[rr, bend right, "\text{multiplication by } AB"'] & BX \arrow[r, "\text{multiplication by } A"] & A(BX) \end{tikzcd}$$

i.e, $A(BX) = (AB)X$.

So, the matrix multiplication to a column matrix (vector) is composition mapping.

④ Invertible matrix:

Definition → $A_{n\times n}$ square matrix is said to be an invertible if there exists another matrix $B_{n\times n}$ (say), such that $AB = BA = I_n$.

The matrix B is called an inverse of A, denoted by A^{-1} and written as, $A^{-1} = B$.

Determination of an inverse of a matrix by row reduction algorithm

Let $A_{n\times n}$ be an $n\times n$ square matrix. Consider the augmented matrix $[A \ I]$.

i.e, $$\left[\begin{array}{ccc|ccc} a_{11} & a_{12}\cdots & a_{1n} & 1 & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ a_{n1} & a_{n2}\cdots & a_{nn} & 0 & \cdots & 1 \end{array}\right]$$

Now, reduce the coefficient matrix of A into the identity matrix by the row operations. Let the augmented matrix assures (takes) the form. $[I \ B]$ then, $B = A^{-1}$.

Uniqueness of inverse:

Let $A_{n\times n}$ be an invertible matrix with an inverse $B_{n\times n}$ (say). Then, from the definition, $AB=BA=I \ldots$ (i).

Claim: B is unique (i.e, inverse of A is unique).

If possible suppose that $C_{n\times n}$ is also an inverse of A.

Now,

$C = IC$

or, $C = (BA).C$

or, $C = B(AC)$, by associativity

or, $C = BI$ $\left(\because C \text{ is an inverse of } A \atop CA = AC = I\right)$

or, $C = B$.

Matrix product (Column row expansion):

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}_{3\times2}$ be two matrices in which the product AB can be defined.

Now, consider $Col_1 A . row_1 B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}_{2\times1} . \begin{bmatrix} 1 & -1 \end{bmatrix}_{1\times2}$

$= \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}_{2\times2}$

$Col_1 A . row_1 B = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}_{2\times2}$

Also, $Col_2 A . row_2 B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{2\times1} \begin{bmatrix} 0 & 1 \end{bmatrix}_{1\times2} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}_{2\times2}$

& $Col_3 A . row_3 B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times1} \begin{bmatrix} 2 & 1 \end{bmatrix}_{1\times2} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}_{2\times2}$

$\therefore Col_1 A . row_1 B + Col_2 A . row_2 B + Col_3 A . row_3 B = \begin{bmatrix} 2+0+2 & -2-1+1 \\ 3+0+0 & -3+1+0 \end{bmatrix}$

$$\therefore \sum_{k=1}^{3} Col_k A . row_k B = \begin{bmatrix} 4 & -2 \\ 3 & -2 \end{bmatrix}$$

Partioned matrix:

Let $A = \begin{bmatrix} 2 & 1 & -1 & 4 & 0 & 5 \\ -1 & 4 & 6 & 3 & 1 & 7 \\ 0 & 4 & 2 & 0 & 1 & 3 \end{bmatrix}_{3\times6}$ be a 3x6 matrix.

Let, it be partioned into sub matices as follows:-

$$A = \left[\begin{array}{cc|cc|cc} 2 & 1 & -1 & 4 & 0 & 5 \\ -1 & 4 & 6 & 3 & 1 & 7 \\ \hline 0 & 4 & 2 & 0 & 1 & 3 \end{array}\right]$$

हामिलाई चाहिए अनुसार जस्तो नि partition गर्न मिल्छ ।

Now,

$A_{11} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$, $A_{12} = \begin{bmatrix} -1 & 4 \\ 6 & 3 \end{bmatrix}$, $A_{13} = \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix}$

$A_{21} = \begin{bmatrix} 0 & 4 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 2 & 0 \end{bmatrix}$, $A_{23} = \begin{bmatrix} 1 & 3 \end{bmatrix}$

Here,

$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$ is the partioned matrix.

Partioned matrix is the matrix whose elements are considered to be it's sub-matices.

Scalar product → The scalar product of a partition matrix is obtained by multiplying each block of the matrix with the scalar.

Sum → The sum of two partition matices is obtained by adding the corresponding blocks (sub-matrices) provided that the matices are of the same order and way of partition is also same.

Multiplication → Let A and B be two partioned matrices in which the number of columns in the first matrix equals to number of rows in the second matrix and partition of columns of A should be exactly the same to the row partition of B. Then the product is obtained by the sum of the products of block matrices as the element in ordinary way.

V. Imp

For example :-

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & 2 & 7 & -1 \end{array}\right]_{3\times5}$$

& $$B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right]_{5\times2}$$

Now, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}_{2\times2}$, $B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}_{2\times1}$.

& $$AB_{2\times1} = \begin{bmatrix} A_{11}B_{11} + A_{12}\cdot B_{21} \\ A_{21}\cdot B_{11} + A_{22}\cdot B_{21} \end{bmatrix}_{2\times2} \quad \text{—— (1)}$$

Here,

$$A_{11}\cdot B_{11} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 12+6-3 & 8-3+7 \\ 6-15+6 & 4+5-14 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 12 \\ -3 & -5 \end{bmatrix}$$

$$A_{12}\cdot B_{21} = \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0-20 & 0-8 \\ -3-5 & 9-2 \end{bmatrix}$$

$$= \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

$$A_{21}\cdot B_{11} = \begin{bmatrix} 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0+8-6 & 0-4+14 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 10 \end{bmatrix}$$

$$A_{22}\cdot B_{21} = \begin{bmatrix} 7 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -7-5 & 21-2 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 19 \end{bmatrix}$$

Again,

$$A_{11}\cdot B_{11} + A_{12}\cdot B_{21} = \begin{bmatrix} 15 & 12 \\ -3 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 15-20 & 12-8 \\ -3-8 & -5+7 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ -11 & 2 \end{bmatrix}$$

& $A_{21}\cdot B_{11} + A_{22}\cdot B_{21} = \begin{bmatrix} 2 & 10 \end{bmatrix} + \begin{bmatrix} -12 & 19 \end{bmatrix}$

$$= \begin{bmatrix} 2-12 & 10+19 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & 29 \end{bmatrix}$$

Finally using these in (i) we get multiplied partioned matrix as follows:-

$$AB = \left[\begin{array}{cc} 5 & 4 \\ -11 & 2 \\ \hline -10 & 29 \end{array}\right]_{3\times 2}$$

⊛. Inverse of partioned matrix:

Let, $A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$ be a partioned matrix with the block matrices $A_{11}, A_{12}, O, A_{22}$ where A_{11} is a square matrix of order $p\times p$, A_{22} is of order $q\times q$ (say). Then it's inverse A^{-1} is given by

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & -(A_{11}^{-1} A_{12} A_{22}^{-1}) \\ O & A_{22}^{-1} \end{bmatrix}$$ such that $A\cdot A^{-1} = \begin{bmatrix} I_p & O \\ O & I_q \end{bmatrix}$.

Example:- Find the inverse of $A = \left[\begin{array}{cc|cc} 1 & 3 & 9 & 0 \\ 2 & 4 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & -2 \end{array}\right]$.

Soln

Given,

$$A = \left[\begin{array}{cc|cc} 1 & 3 & 9 & 0 \\ 2 & 4 & 0 & 1 \\ \hline 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & -2 \end{array}\right]$$

Here, $A_{11} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ & $A_{22} = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$

Now,

$$|A_{11}| = 4 - 6 = -2 \neq 0$$

$$\text{adj}.\ A_{11} = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$A_{11}^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$$

Again, $|A_{22}| = 0 - 4 = -4 \neq 0$

$$\text{adj}.\ A_{22} = \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix}$$

$$A_{22}^{-1} = \frac{1}{-4} \begin{bmatrix} -2 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

Here,

$$A_{12} \cdot A_{22}^{-1} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 9/2 & 9/2 \\ 1/2 & 0 \end{bmatrix}$$

&

$$-(A_{11}^{-1} \cdot A_{12} \cdot A_{22}^{-1}) = -\left\{ \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 9/2 & 9/2 \\ 9/2 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} -9+3 & -9 \\ 9/2 - 1/4 & 9/2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 \\ -17/4 & -9/2 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} A_{11}^{-1} & -(A_{11}^{-1} A_{12} A_{22}^{-1}) \\ 0 & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} -2 & 3/2 & 6 & 9 \\ 1 & -1/2 & -17/4 & -9/2 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

(*) LU factorization (V.Imp)

$A = LU$

L → Lower triangular matrix of echelon form

U → Upper triangular matrix = echelon form

While solving first we obtain echelon form of given matrix and as soon as we get pivot element we divide pivot column below pivot element (including pivot element) by pivot element, while finding L.
For clear understanding below are the some examples.

U.2075

Example:-1: Find the LU factorization of the matrix $\begin{bmatrix} 2 & 5 \\ 6 & -7 \end{bmatrix}$.

Solution

Let $A = \begin{bmatrix} 2 & 5 \\ 6 & -7 \end{bmatrix}$

Now, we reduce A into echelon form

$R_2 \to R_2 + (-3) R_1$.

$$A = \begin{bmatrix} 2 & 5 \\ 0 & -22 \end{bmatrix}$$ Echelon form.

First pivot column is $\begin{bmatrix} 2 \\ 6 \end{bmatrix} \div 2$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

& Second pivot column is $[-22] \div -22$

$$= 1$$

Now,

$$L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$ ← Upper triangular so 0 in place of 5

$$U = \begin{bmatrix} 2 & 5 \\ 0 & -22 \end{bmatrix}$$ ← (= A) echelon form

$$\therefore LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ 0 & -22 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 5-0 \\ 6+0 & 15-22 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 6 & -7 \end{bmatrix} = A$$ //

Example 2: Find the LU factorization of $A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$

Soln

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$$

$R_2 \to R_2 + (-3)R_1$, $R_3 \to R_3 + R_1/2$

pivot element आउने निकाल्ने

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & -6 & 10 & -1 \end{bmatrix}, \text{ First pivot column is } \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \div 2 = \begin{bmatrix} 1 \\ 3 \\ -1/2 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \text{ Second pivot column is } \begin{bmatrix} 3 \\ -6 \end{bmatrix} \div 3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

−4 लाई divide गरेनौं किनकि upper triangular भइसकेकोले 0 भैहाल्छ

Now,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix}, \text{ Third pivot column is } [5] \div 5 = 1.$$

$$U = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\therefore LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1/2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+0 & -4+0+0 & 4+0+0 & -2+0+0 \\ 6+0+0 & -12+3+0 & 12-5+0 & -6+3+0 \\ -1+0+0 & 2-6+0 & -2+10+0 & 1-6+5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & & 0 \end{bmatrix} = A //$$

Example 3: Find the LU factorization of $\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}_{4\times 5}$

Soln

Let $A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + (-1)R_1$, $R_4 \to R_4 + 3R_1$

$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}$, First pivot column is $\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \div 2$

$= \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}$

$R_3 \to R_3 + 3R_2$, $R_4 \to R_4 + (-4)R_2$

$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}$, Second pivot column is $\begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \div 3$

$= \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$

$R_4 \to R_4 + (-2)R_3$

$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$ Third pivot column is $\begin{bmatrix} 2 \\ 4 \end{bmatrix} \div 2$

$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

& Fourth pivot column is $[5] \div 5$
$= 1$.

$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$ & $U = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$

$$\therefore LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$= A$ //

* Solve the system $2x_1 + x_2 + 5x_3 = 1$
$-4x_1 + 4x_3 = 2$
$6x_1 + 2x_2 + 3x_3 = 4$

Soln

The system is equivalent to the matrix equation, $AX = b$ .. (i)
where, $A = \begin{bmatrix} 2 & 1 & 5 \\ -4 & 0 & 4 \\ 6 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $b = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Let, $A = LU$

So, eqn (i) becomes

$(LU)X = b$

$L(UX) = b$ — (ii).

again let

$UX = y$ — (iii)

Then eqn (ii) becomes

$Ly = b$ — (iv)

Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

So, from (iv)

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

or, $\begin{bmatrix} y_1+0+0 \\ -2y_1+y_2+0 \\ 3y_1-\frac{1}{2}y_2+y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

$\Rightarrow$ $y_1 = 1$

$y_2 = 2+2y_1$
$= 2+2\times1$
$= 4$

& $y_3 = 4-3\times y_1+\frac{1}{2}y_2$
$= 4-3\times1+\frac{1}{2}\times4$
$= 4-3+\frac{4}{2}$
$= 3$

$\therefore y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

But we have to find x_1, x_2, x_3 so, from (iii)

$\begin{bmatrix} 2 & 1 & 5 \\ 0 & 2 & 14 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

$\Rightarrow 2x_1+x_2+5x_3 = 1$ —— (a)
$\Rightarrow 2x_2+14x_3 = 4$ —— (b)
& $-5x_3 = 3$
or $x_3 = -\frac{3}{5}$ —— (c)

Using value of x_3 in (b).

$2x_2+14\left(\frac{-3}{5}\right) = 4$

or, $x_2 = \frac{4+\frac{42}{5}}{2}$

or, $x_2 = \frac{124}{5}$

Again, using value of x_2 and x_3 in (a).

$2x_1+\frac{124}{5}+5\times\left(-\frac{3}{5}\right) = 1$

or, $x_1 = \frac{5}{218}$