# Unit → 6

# Vector Spaces Continued:

**The dimension of Nul A:** The dimension of Nul A is the number of free variable in the equation of $Ax=0$.

**The dimension of Col A:** The dimension of Col A is the number of pivot column in A.

**Example 1:** Find the dimensions of the null space and column space of $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

**Solution:**

First reduce the augmented matrix $[A \ \ 0]$ to echelon form:

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three free variables $x_2, x_4$ and $x_5$. Hence the dimension of Nul A is 3. Also dim Col A = 2 because A has two pivot columns.

**Example 2:** Find a basis and dimension of the subspace

$$H = \left\{ \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} : a, b, c \in R \right\}$$

**Solution:**

We have,

$$H = \begin{bmatrix} 3a+6b-c \\ 6a-2b-2c \\ -9a+5b+3c \\ -3a+b+c \end{bmatrix} = a\begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix} + b\begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} + c\begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$$

$$= av_1 + bv_2 + cv_3$$

where, $v_1 = \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$

which shows that H is linear combination of $v_1, v_2, v_3$. Clearly $v_1 \neq 0$, $v_2$ is not multiple of $v_1$, but $v_3$ is a multiple of $v_1$. So by spanning set theorem $\{v_1, v_2\}$ also spans H and since it is linearly independent. So, it is a basis for H and dimension of H (dim H) = 2.

## ⊛ Rank (Row Space):

($\sim \to$ equivalent sign)

Let A be an m×n matrix. Each row of A has n entries and thus can be identified with a vector in $R^n$. The set of all linear combinations of the row vectors is called the row space of A. and is denoted by row A.

**Example:** Find the bases for the row space and the column space, and the null space of the matrix:

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & 17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

**Solution:**

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exam मा reduce गरेको पनि देखाउने यसमा धेरै पाना नभएर step काटिएको हो।

To find the bases for the row space and the column space. We have to reduce A to echelon form. then we get,

$$\sim A = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore the basis for row space of the matrix A is $\{(1,3,-5,1,5), (0,1,-2,2,-7), (0,0,0,-4,20)\}$.

For the column space observe from B that the pivots are in columns $1^{st}$ $2^{nd}$ and $4^{th}$. Hence, therefore, the basis for Col A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$.

**For Nul A:** We need to change in reduced echelon form of matrix A.

$$\text{So, } \sim A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

यहाँ directly लेखिएको हो exam मा reduce गरेर देखाउने

∴ General solution of $Ax=0$ is,

$x_1 + x_3 + x_5 = 0$

$x_2 - 2x_3 + 3x_5 = 0$

$x_3$ is free

$x_4 - 5x_5 = 0$

$x_5$ is free.

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$\therefore \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$ is basis for Nul A.

# Change of Basis:

Let $B=\{b_1,b_2,\dots,b_n\}$ and $C=\{c_1,c_2,\dots,c_n\}$ are basis for $R^n$. Then change of co-ordinate matrix from B to C is denoted by $\underset{C\leftarrow B}{P}$ and defined by $\underset{C\leftarrow B}{P} = [[b_1]_C\ [b_2]_C \dots [b_n]_C]$.

and $[x]_C = \underset{C\leftarrow B}{P} [x]_B$.

It means the matrix $\underset{C\leftarrow B}{P}$ convert B-coordinates into C-ordinate.

Note: $\underset{C\leftarrow B}{P} = \left[\underset{B\leftarrow C}{P}\right]^{-1}$

Example 1: Let $B=\{b_1,b_2\}$ and $C=\{c_1,c_2\}$ where $b_1=\begin{bmatrix}-9\\1\end{bmatrix}$, $b_2=\begin{bmatrix}-5\\-1\end{bmatrix}$, $c_1=\begin{bmatrix}1\\-4\end{bmatrix}$, $c_2=\begin{bmatrix}3\\-5\end{bmatrix}$ are the two basis for $R^2$, then.

i) Find the change of co-ordinate matrix from C to B.

ii) Find the change of co-ordinate matrix from B to C.

Solution:

(i) For the change of coordinate matrix from B to C.

$$[[b_1]_C\ [b_2]_C] = \underset{C\leftarrow B}{P}$$

For $[b_1]_C$, let $b_1 = xc_1 + yc_2$

$$\Rightarrow \begin{bmatrix}-9\\1\end{bmatrix} = x\begin{bmatrix}1\\-4\end{bmatrix} + y\begin{bmatrix}3\\-5\end{bmatrix}$$

i.e, $-9 = x+3y$ — (i)

and $1 = -4x-5y$ — (ii).

Solving, we have, $x=6$ and $y=-5$.

Therefore $[b_1]_C = \begin{bmatrix}6\\-5\end{bmatrix}$.

Again, for $[b_2]_C$, let $b_2 = xc_1+yc_2$

$$\Rightarrow \begin{bmatrix}-5\\-1\end{bmatrix} = x\begin{bmatrix}1\\-4\end{bmatrix} + y\begin{bmatrix}3\\-5\end{bmatrix}$$

i.e, $x+3y=-5$ — (iii)

and $-4x-5y=-1$ — (iv).

Solving we have $x=4$ and $y=-3$.

Therefore $[b_2]_C = \begin{bmatrix}4\\-3\end{bmatrix}$

Thus, $\underset{C \leftarrow B}{P} = \left[ [b_1]_C \; [b_2]_C \right] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}.$

Again,

$$\underset{B \leftarrow C}{P} = \left[ \underset{C \leftarrow B}{P} \right]^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 & -2 \\ 5/2 & 3 \end{bmatrix}.$$

## ⊛. Co-ordinate mapping:

Let, $T: V \to \mathbb{R}^n$ be a transformation. Let $B = \{b_1, b_2, \dots, b_n\}$ be a basis for V then there exists unique set of scalars $c_1, c_2, \dots, c_n$ such that, $x = c_1 b_1 + c_2 b_2 + \dots + c_n b_n$.

Then the vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ is called the co-ordinate vector of X relative to the basis B, denoted by $[X]_B$.

i.e, $[X]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Same theory of co-ordinate vector

So there exists a rule which associates every member in V to unique member in $\mathbb{R}^n$. $T: V \to \mathbb{R}^n$, called the co-ordinate mapping.

$$X \xrightarrow[\ [\ ]_B\ ]{T} [X]_B.$$

## ⊛. One-to-one transformation:

Let $u, v \in V$ such that $[u]_B = [v]_B$

Let, $[u]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, $[v]_B = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$

Now, $u = c_1 b_1 + c_2 b_2 + \dots + c_n b_n$

But, $[u]_B = [v]_B$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$\therefore u = d_1 b_1 + d_2 b_2 + \dots + d_n b_n$

$\Rightarrow u = v$

This shows that the transformation is one-to-one.

**Example:-** Consider two bases $B=\{b_1, b_2\}$ and $C=\{c_1, c_2\}$ for a vector space such that,

$$b_1 = 4c_1 + c_2 \quad \& \quad b_2 = -6c_1 + c_2$$

Suppose $x = 3b_1 + b_2$

i.e, $[X]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ Find $[X]_C$.

**Sol^n**.

$$[X]_C = \underset{C \leftarrow B}{P} [X]_B ,$$

read as change of matrix from B to C

where, $\underset{C \leftarrow B}{P} = \left[ [b_1]_C \ [b_2]_C \right]$

$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$$

$$[X]_C = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\therefore [X]_C = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

**Q.** Let $B=\{b_1, b_2, \dots, b_n\}$, $C=\{c_1, c_2, \dots, c_n\}$ be two bases of a vector space $V$ of dimension $n$. To show $\{[b_1]_C, [b_2]_C, \dots, [b_n]_C\}$ is linearly independent.

**Sol^n**

For $b_1, b_2, \dots, b_n$ are basis in $B$.

$\Rightarrow b_1, \dots, b_n$ are linearly independent.

So, $x_1 b_1 + x_2 b_2 + \dots + x_n b_n = 0$ —— (i)

$\Rightarrow x_1 = \dots = x_n = 0$

Operating $[\,]_C$ in both sides of (i)

$$[x_1 b_1 + x_2 b_2 + \dots + x_n b_n]_C = [0]_C$$

$x_1 [b_1]_C + x_2 [b_2]_C + \dots + x_n [b_n]_C = 0$, Since $[\,]_C$ is linear.

Since, $x_1 = \dots = x_n = 0$.

$[b_1]_C, \dots [b_n]_C$ are linearly independent.

i.e, columns of $\underset{C \leftarrow B}{P} = \left[ [b_1]_C \cdots [b_n]_C \right]$ are linearly independent,

Q. Let $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ and consider the bases for $\mathbb{R}^2$ given by $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$.

i) Find the change of co-ordinate matrix from C to B.

ii) Find the change of co-ordinate matrix from B to C.

Soln

Here, $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $c_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $c_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$.

$B = \{b_1, b_2\}$ & $C = \{c_1, c_2\}$ are the bases for $\mathbb{R}^2$.

i) $\underset{B \leftarrow C}{P} = ?$

i.e. $\left[ [c_1]_B \; [c_2]_B \right] = ?$

Consider the augmented matrix $[b_1 \; b_2 : c_1 \; c_2]$

$$= \left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right]$$

$\underbrace{R_2 \to R_2 + 3R_1}$

$$\left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ 0 & -2 & -12 & -8 \end{array}\right]$$

$\underbrace{R_2 \to -\frac{1}{2} R_2}$

$$\left[\begin{array}{cc|cc} 1 & -2 & -7 & -5 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

$\underbrace{R_1 \to R_1 + 2R_2}$

$$\left[\begin{array}{cc|cc} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

$$\therefore \underset{B \leftarrow C}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}.$$

Since, the columns of $\underset{B \leftarrow C}{P}$ are linearly independent.

It is invertible.

$$\left[\underset{B \leftarrow C}{P}\right]^{-1} = \underset{C \leftarrow B}{P}$$

$$\text{determinant} \begin{vmatrix} 5 & 3 \\ 6 & 4 \end{vmatrix} = 5 \times 4 - 6 \times 3$$

$$= 20 - 18$$

$$= 2.$$

$\frac{1}{\text{determinant}} \times \text{adj. matrix}$ for finding inverse

$$\therefore \underset{C \leftarrow B}{P} = \left[\underset{B \leftarrow C}{P}\right]^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}$$