UNIT-2

Simulation of Continious and Discrete System

⊛. Continious System Models :-

A continious system is one in which important activities of the system completes smoothly without any delay, i.e, no queue of events, no sorting of time simulation etc. When a continious system is modeled mathematically, its variables representing the attributes are controlled by continuous functions. The methods of applying simulation to ~~continious~~ continuous models can be developed by showing their application to models where the differential equations are linear and have constant coefficients, and then generalizing to more complex questions.

Continuous Simulation → Continuous simulation concerns the modeling over time of a system by a representation in which state variables change continuously with respect to time.

The simplest differential equation models have one or more linear differential equations with constant coefficients. It is then often possible to solve the model without the use of simulation. However, when non-linearities are introduced in the model, it frequently becomes impossible, or very difficult to model these systems.

⊛. Analog Computers:

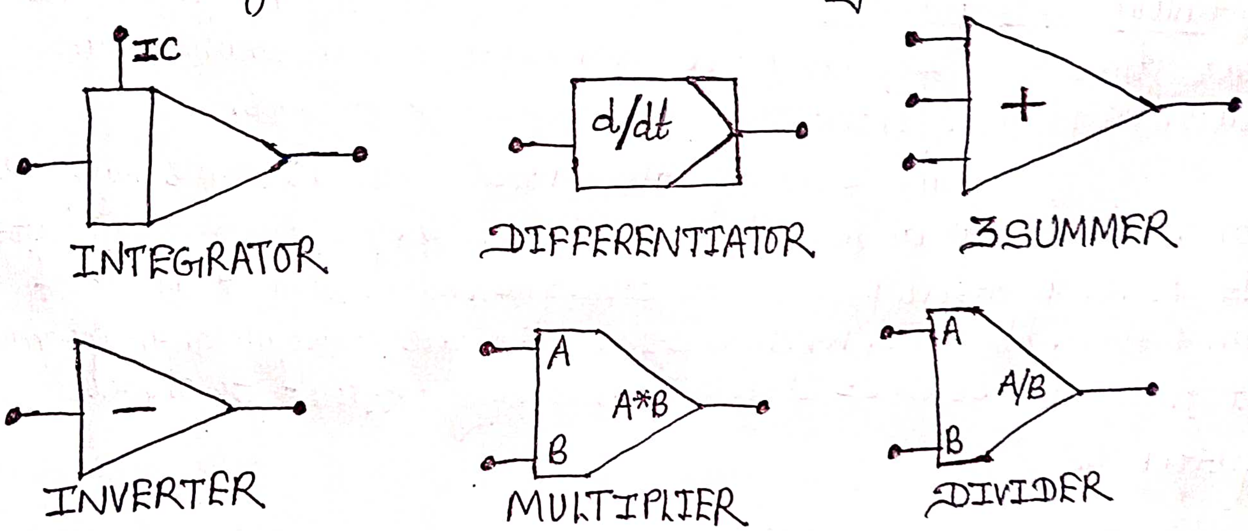
Analog computers are those computers that are unified with devices like adder and integral so as to simulate the continuous mathematical model of the system, which generates continuous outputs. The electronic analog computer based on the use of high gain dc amplifiers are widely used analog computer. In such analog computer, voltages are equated to mathematical variables.

The coefficient of the model equations are obtained by using the scale factors. The circuit can be arranged to produce integrator that provides integral w.r.t time of a single input voltage or sum of input voltages. Sign inverter is used to reverse the sign of the input as per the requirement of the model equation. The analog computer provides limited accuracy.

Analog Methods: [Imp]

Analog method of system simulation is for the use of analog computer and other analog devices in the simulation of continuous system. The analog computation is sometimes called differential analyser. Electronics analog computers for simulation are based on the use of high gain dc amplifiers. In such analog computer, voltages are equated to mathematical variables. The proper configurations can handle addition of several input voltages each representing the input variables. The analog computers provide limited accuracy.

The general method to apply analog computers for the simulation of continuous system models involves following components:



Example:- Automobile Suspension Problem:

The general method by which analog computers are applied can be demonstrated using second order differential equation:

$$Mx'' + Dx' + Kx = K F(t).$$

Solving the equation for the highest order derivative gives;

$$Mx'' = K F(t) - Dx' - Kx$$

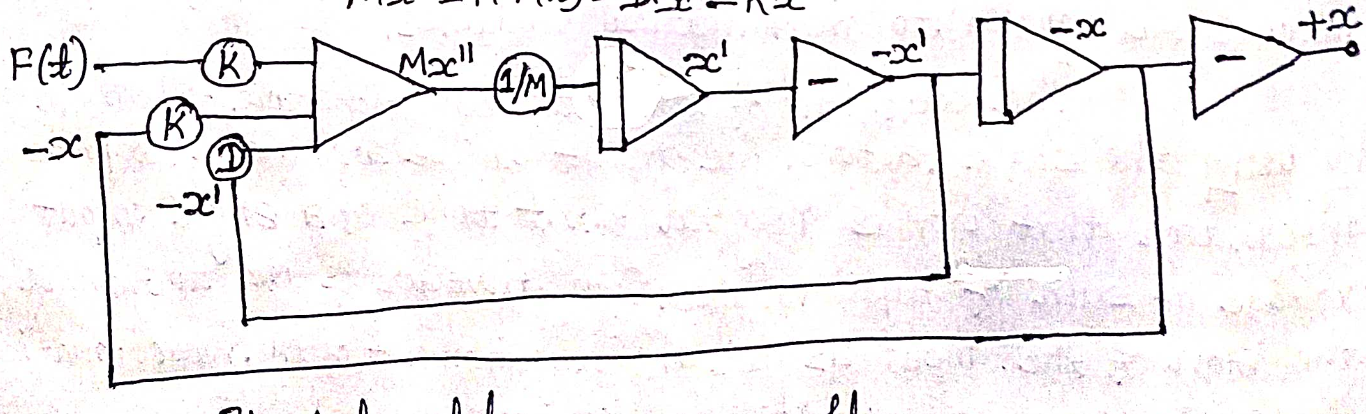


Fig: Automobile suspension problem.

Suppose a variable representing the input $F(t)$ is supplied, assume there exist variables representing $-x$ and $-x'$. These three variables can be scaled and added to produce Mx''. Integrating it with a scale factor $1/M$ produces x'. Changing sign produces $-x'$, further integrating produces $-x$, a futher sign inverter is included to produce $+x$ as output.

⊛. Hybrid Simulation: [Imp]

In most cases, the system under study is clearly either continuous or discrete in nature, and it is the determining factor in deciding whether to use an analog or digital computer for system simulation. But, in case of hybrid computers the system is of neither a pure continuous nor a pure discrete nature. The simulation provided by such hybrid computers is known as hybrid simulation. The major difficulty in use of hybrid simulation is that it requires high speed converters to transform signals from analog to digital and vice versa.

⊛. Digital-Analog Simulators:

Digital-Analog simulators indicates the use of programming languages in digital computer to simulate the continuous system. The language is composed of macro-instructions which are able to act as adder, integrator and sign-changer. A program is written to link these macro-instructions as per the necessity. More powerful techniques of applying digital computers to the simulation of continuous systems have been developed. As a result, digital-analog simulators are out now in extensive use.

⊛. Feedback Systems: [Imp]

A significant factor in the performance of many systems is that a coupling occurs between the input and output of the system. The term feedback is used to describe the phenomenon.

A home heating system controlled by a thermostat is a simple example of a feedback system. The system has a furnace whose purpose is to heat a room, and the output of the system can be measured as room tempreature. Depending upon whether the tempreature is below or above the thermostat setting the furnace will be turned on or off, so that information is being feedback from the output to the input. In this-case there are only two states, either the furnace is on or off.

⊛. Discrete Event Simulation:

Model used in discrete system simulation has a set of numbers to represent the state of the system. A number used to represent some aspect of the system state is called a state description. Some state descriptions range over values that have physical significance. Other represent conditions such as the flag denoting whether a break in work is due.

As the simulation proceeds, the state descriptors change value. We define a discrete event as a set of circumtances, that causes the instantaneous change in one or more system state descriptors. It is possible that two different events occur simultaneously or are modelled as being simultaneous so that not all changes of state descriptors occuring simultaneously necessarily belong to a single event.

⊛. Representation of time:

The passage of time is recorded by a number referred to as clock time. It is usually set to zero at the begining of a simulation and subsequently indicates how many units of simulated time has passed since the begining. As a rule, there is no direct connection between simulated time and the time taken to carry out the computations.

The simulation even when carried out by a high speed digital computer could easily take several thousands times as long as the actual system operation. On the other hand, for the simulation of an economic system where events have been aggregated to occur once a year, a hundred years of operation could easily be performed in a few minutes of calculations.

⊛Methods For Updating Clock Time: [Imp]

Clock time is updated on the following two models:

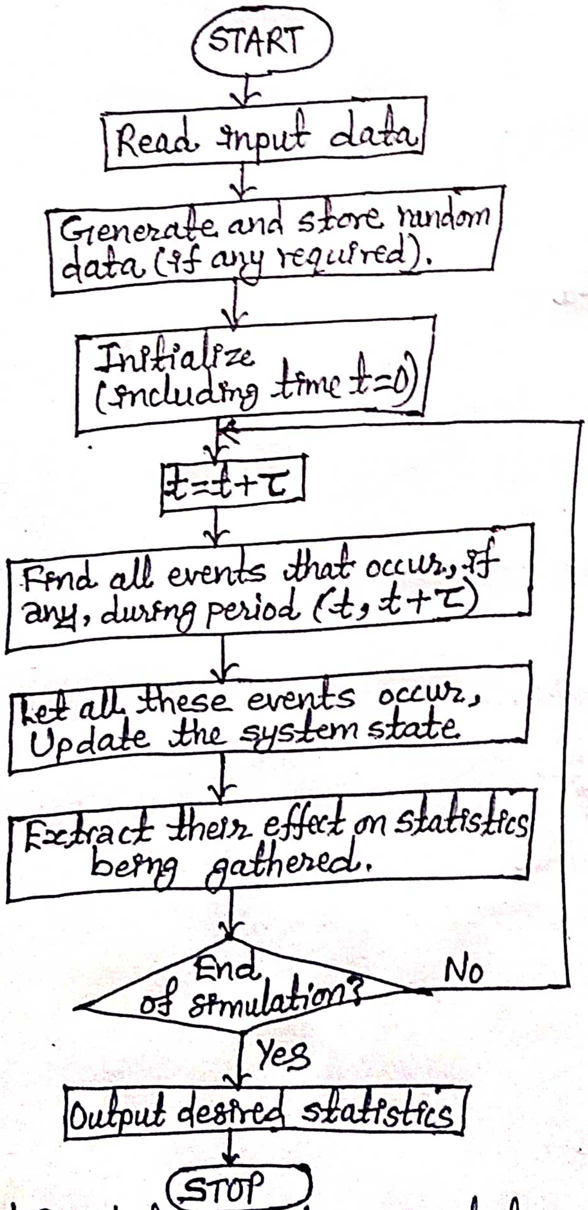
i) Interval oriented (OR Fixed time-step model):

In this model the timer simulated by the computer is updated at a fixed time interval. The system is checked to see if any event has taken place during that interval. All the events which take place during the time interval are considered to have occurred simultaneously at the end of the interval.

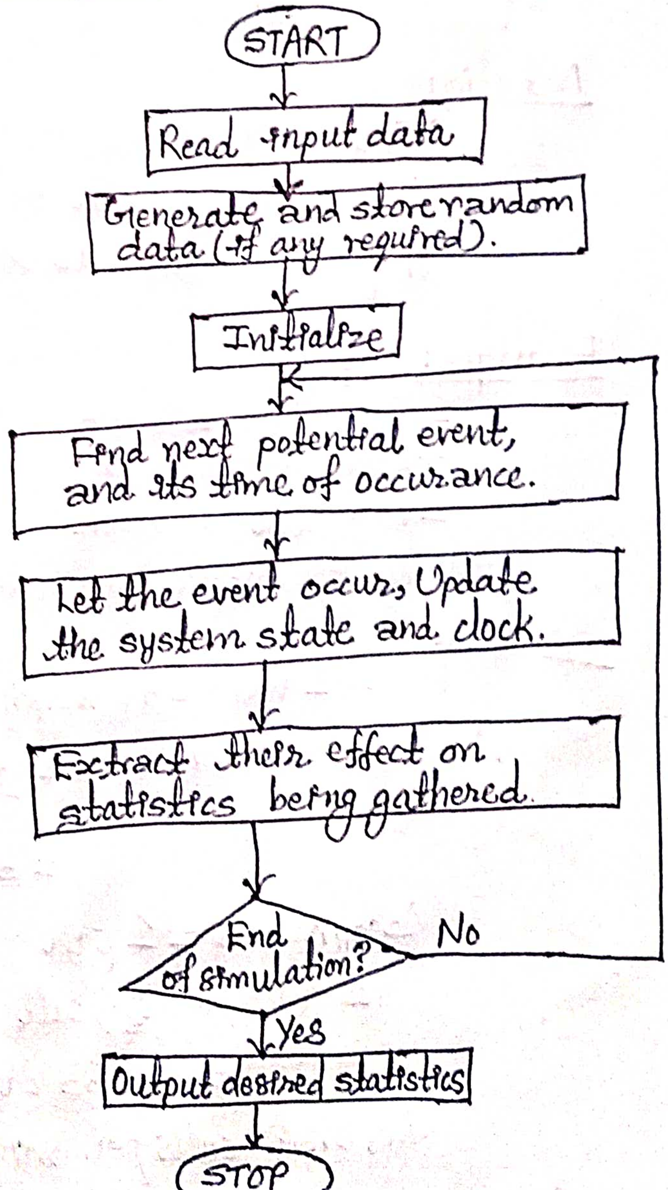
ii) Event oriented (OR Event-to-event model):

In this model the computer advances the time to the ~~occurance~~ occurrence of the next event. So it shifts from one event to the next event and the system state does not change in between. A track of the current time is kept when something interesting happens to system. It is also known as the next-event model.

⇒ The flowcharts for both models is shown below:



i) Fixed time-step simulation



ii) Next-event simulation

⊛. Arrival Processes:-

The simulation time starts when first customer arrives to the system. The arrival time is random and hence inter-arrival time is also random. If arrivals vary stochastically, it is necessary to define the probability function of the inter-arrival times. Two or more arrivals may be simultaneous. If n variables are simultaneous then (n-1) of them have zero inter-arrival times.

Customer	Inter-arrival time	Arrival time on clock
1	—	0
2	2	2
3	4	6
4	1	8
5	2	9
6	6	15

Notation

τ_a = mean inter-arrival time.

λ = mean arrival time

They are related as, $\lambda = \frac{1}{\tau_a}$

Example: Office

- Five days a week
- Eight hours a day
- 800 calls a week.

Model the office using time of minutes.

Solution:

Here, 1 week = 40 working hours.

$\therefore$ Inter-arrival time $= \frac{40 \times 60}{800}$

$= 3$ minutes.

3 minutes = 1 call

1 min $= \left(\frac{1}{3}\right)$ call

$= 0.333$ calls

i.e, 0.333 calls per minutes.

⊛. Poisson Distribution:

A poisson distribution variable N with mean $\alpha > 0$ has pmf;

$$P(n) = P(N=n) = \frac{e^{-\alpha} \cdot \alpha^{n}}{n!}, \quad n = 0, 1, 2, \ldots$$

where, α is the mean rate and must be positive.

Poisson Process:

Consider random events that can be described by a counting function $N(t)$ defined for all $t \geq 0$. This counting function represents the number of events that occurred in $[0, t]$. For each interval $[0, t]$ the value $N(t)$ is an observation of a random variable where the only possible values are the integers $0, 1, 2, \ldots$

The counting process $\{N(t) : t \geq 0\}$ is said to be a Poisson process with mean rate λ if the following assumptions are fulfilled:

i) $N(0) = 0$

ii) It has independent increments

iii) No. of events in any interval of length t is a Poisson random variable with parameter λt.

Therefore,
$$\boxed{P(N(t)=n) = \frac{e^{-\lambda t}(\lambda t)^{n}}{n!}}$$

Generation of Poisson variate:

- Procedures for generating a Poisson variate N:

Step1: Set $n=0$, $P=1$

Step2: Generate a random number R_{n+1} and replace P by $P \cdot R_{n+1}$.

Step3: If $P < e^{-\alpha}$, then accept $N=n$. Otherwise reject the current n, increase n by one, and return to step 2.

Note: If $N=n$, then $n+1$ random numbers are requested. So the average number is given by $E(N+1) = \alpha + 1$.

Example:- Generate 3 Poisson variates with mean $\alpha = 0.2$ for the random numbers $R = 0.4357, 0.4146, 0.8353, 0.9952, 0.8004$.

Solution:

Given: $e^{-\alpha} = e^{-0.2} = 0.8187$

Step1: Set $n=0, P=1$

Step2: $R_1 = 0.4357$, $P = P.R_{n+1} = 1 * R_1 = 1 \times 0.4357 = 0.4357$

Step3: $P = 0.4357$

$e^{-\alpha} = 0.8187$

Since $P < e^{-\alpha}$, accept $N=0$ and goto step1, for second random number.

Step1: Set $n=0, P=1$

Step2: $R_1 = 0.4146$, $P = 1 * R_1 = 0.4146$

Step3: $P = 0.4146$, $e^{-\alpha} = 0.8187$

Since, $P < e^{-\alpha}$, accept $N=0$ and goto step 1.

reject greater या equals to
भए हुन्छ । reject भए step1
मा नगई n+1 गरि step2
मा जाने

Step1: set $n=0, P=1$

Step2: $R_1 = 0.8353$, $P = 1 * R_1 = 0.8353$

Step3: Since $P > e^{-\alpha}$, reject $n=0$ and return to step2 with $n=1$.

Step2: $n=1$, Generating random no. R_{n+1} and replacing P by $P.R_{n+1}$.

$R_2 = 0.9952$

$P = R_1 * R_2 = 0.8353 * 0.9952 = 0.8313$

Step3: Since $P > e^{-\alpha}$, reject $n=1$ and return to step2 with $n=2$.

Step2: $n=2$, $R_3 = 0.8004$

$P = R_1 * R_2 * R_3 = 0.8353 * 0.9952 * 0.8004$

$= 0.6654$

Step3: Since $P < e^{-\alpha}$, accept $N=2$.

So all the numbers are done so the summary will be as follows:-

n	R_{n+1}	P	Accept/Reject	Result
0	0.4357	0.4357	$P < e^{-\alpha}$ (Accept)	$N=0$
0	0.4146	0.4146	$P < e^{-\alpha}$ (Accept)	$N=0$
0	0.8353	0.8353	$P \geq e^{-\alpha}$ (Reject)	
1	0.9952	0.8313	$P \geq e^{-\alpha}$ (Reject)	
2	0.8004	0.6654	$P < e^{-\alpha}$ (Accept)	$N=2$

⊛. Non-stationary Poisson Process:

→ theory upto arrival rate 1 maybe imp asked in model set.

A non-stationary Poisson process is a Poisson process for which the arrival rate varies with time. More specifically, it can be defined as follows:

The counting process $N(t)$ is a non-stationary Poisson process if:

i) The process has independent increments.

ii) $Pr[N(t+dt) - N(t) \begin{cases} = 0] = 1 - \lambda(t)\, dt. \\ = 1] = \lambda(t)\, dt \\ > 1] = 0 \end{cases}$

where, $\lambda(t)$ = the arrival rate at time t.

dt = differential sized interval.

The definition is identical to the stationary Poisson process, with the exception that the arrival rate $\lambda(t)$ is now a function of time.

⇒ A counting process $N(t)$ is a stationary Poisson process with rate λ if

i) The process has independent increments.

ii) The process has stationary increments.

iii) $Pr[N(t+dt) - N(t) \begin{cases} = 0] = 1 - \lambda dt \\ = 1] = \lambda dt \\ > 1] = 0 \end{cases}$

⇒ A non-stationary Poisson process can be transformed into a stationary Poisson process with arrival rate 1.

Generation of non-stationary Poisson process:

Step 1: Let $\lambda^* = \max_{0 \le t \le T} \lambda(t)$ be the maximum of the arrival rate function and set $t = 0$ and $i = 1$.

Step 2: Generate E from the exponential distribution with rate λ^* and let $t = t + E$ (this is the arrival time of the stationary Poisson process).

Step 3: Generate random number R from the $U(0,1)$ distribution.

If $R \le \lambda(t)/\lambda^*$ then, $T_i = t$ and $i = i+1$.

Step 4: Go to step 2.

Example: For the arrival-rate function table given below, generate the first two arrival times.

t (min)	Mean Time between Arrivals (min)	Arrival rate $\lambda(t)$ (arrivals/min)
0	15	1/15
60	12	1/12
120	7	1/7
180	5	1/5
240	8	1/8
300	10	1/10
360	15	1/15
420	20	1/20
480	20	1/20

Solution:

Step1: $\lambda^* = max_{0 \le t \le T} \lambda(t) = 1/5$, $t = 0$, and $i = 1$.

$\therefore E = -5 \ln(R)$

Step2: For random number $R = 0.2130$, $E = -5 \ln(0.213) = 13.13$ and $t = 0 + 13.13 = 13.13$

Step3: Generate $R = 0.8830$. Since $R = 0.8830 \nleq \lambda(13.13)/\lambda^* = (1/15)/(1/5) = 1/3$, do not generate the arrival.

Step4: Go to step 2.

Step2: For random number $R = 0.5530$, $E = -5 \ln(0.553) = 2.96$. and $t = 13.13 + 2.96 = 16.09$

Step3: Generate $R = 0.0240$. Since $R = 0.0240 \le \lambda(16.09)/\lambda^* = (1/15)/(1/5) = 1/3$, set $T_1 = t = 16.09$ and $i = i + 1 = 2$.

Step4: Go to step 2.

Step2: For random number $R = 0.0001$, $E = -5 \ln(0.0001) = 46.05$ and $t = 16.09 + 46.05 = 62.14$.

Step3: Generate $R = 0.1443$. Since $R = 0.1443 \le \lambda(62.14)/\lambda^* = (1/12)/(1/5)$ set $T_1 = t = 62.14$ and $i = i + 1 = 3$.

Step4: Go to step 2.

⊛. Batch arrivals :

If arriving customers to a queue occur in "batches" such as busloads, then we can model this by a point process $\Psi = \{t_n\}$ in which the arrival times of customers can coincide: $t_0 \leq t_1 \leq t_2 \leq \ldots$, where $\lim_{n \to \infty} t_n = \infty$. Since the limit is infinite, we conclude that the inequalities with a finite number of equalities in between.

For example: $0 = t_0 = t_1 = t_2 < 1 = t_3 = t_4 = t_5 = t_6 < 3 = t_7 = t_8 < t_9 \ldots$ means that a batch of size 3 occured at the origin, followed by a batch of size 4 at time $t = 1$ followed by a batch of size 2 at time $t = 3$, and so on.

⊛. Models of Gathering statistics:

Commonly used statistics parameters (included in report).

i) Counts → Number of entities of particular type or number of times some event occured.

ii) Summary measures → Extreme values, mean values, standard deviation.

iii) Utilization → fraction of time some entity is engaged.

iv) Occupancy → Fraction of a group of entities in use on the average.

v) Distribution → Queue length, waiting times etc.

vi) Transit times → Time taken for an entity to move from one part of the system to some other part.

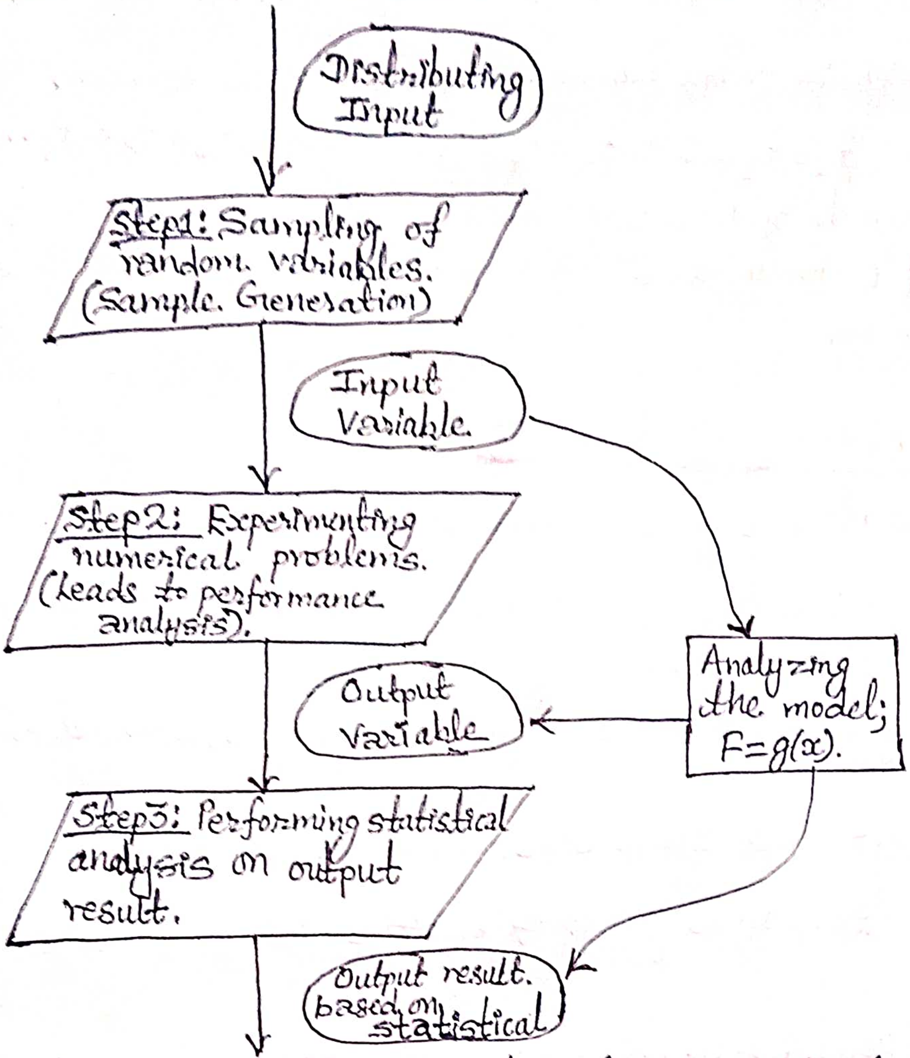
⊛. Monte Carlo Simulation: [Imp]

Monte Carlo simulation is a computerized mathematical technique to generate random sample data based on some known distribution for numerical experiments. This method is applied to risk quantitative analysis and decision making problems. This method is used by the professionals of various profiles such as finance, project management, energy, manufacturing, engineering, research & development, transportation etc.

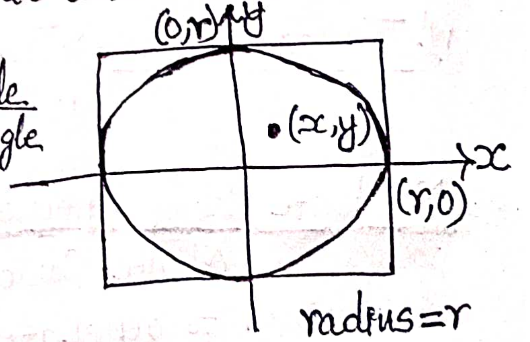
Following are the three important characteristics of Monte-Carlo method:

i) The output must generate random samples.
ii) The input distribution must be known.
iii) The result must be known while performing an experiment.

Flowchart for Monte Carlo Simulation:



Example: Determining the value of PI using Monte Carlo Method:



$$\frac{\text{Area of quadrant of circle}}{\text{Area of rectangle}} = \frac{\text{No. of points inside the circle}}{\text{No. of points inside the rectangle}}$$

or, $$\frac{\frac{1}{4}\pi r^2}{r^2} = \frac{n}{N}$$

$$\therefore \pi = \frac{4n}{N}$$

We use random number generation to determine the sample points that lie inside or outside the curve. Let (x_0, y_0) be an initial guess for the sample point then from a linear congruential method of random number generation:

$$X_{i+1} = (ax_i + c) \bmod m \;\&\; Y_{i+1} = (ay_i + c) \bmod m$$

where, a & c are constants, m is the upper limit of generated random number. If $y <= y_i$ then increment n.