1. Discuss methods of Half-interval and Newton's for solving the non-linear equation f(x) =0. Illustrate the methods by figures and compare them stating their advantages and disadvantages.

Half-Interval method:

Suppose that $f(x)$ is continuous function in the interval $[a_0, b_0]$ and $f(a_0)\, f(b_0) < 0$, then by intermediate value theorem, there exists a root of $f(x)$ in the interval (a_0, b_0). We calculate the first approximation of this root as $c_0 = \frac{(a_0+b_0)}{2}$. If $f(c_0) = 0$, then c_0 is the root of $f(x)$. If not then we bisect the interval $[a_0, b_0]$ into two equal length sub-intervals $[a_0, c_0]$& $[c_0, b_0]$ and set $a_1 = a_0, b_1 = c_0$ if $f(a_0)f(c_0) < 0$ and $a_1 = c_0, b_1 = b_0$ if $f(c_0)f(b_0) < 0$. The second approximation of the root is now calculated as $c_1 = \frac{(a_1+b_1)}{2}$. If $f(c_1) = 0$, then c_1 is the root of $f(x)$. If not then we again bisect the interval $[a_1, b_1]$ into two equal length sub-intervals $[a_1, c_1]$& $[c_1, b_1]$ and set $a_2 = a_1, b_2 = c_1$ if $f(a_1)f(c_1) < 0$ & $a_2 = c_1, b_2 = b_1$ if $f(c_1)f(b_1) < 0$ and then calculate the third approximation as $c_2 = \frac{(a_2+b_2)}{2}$ and continuing the above process.

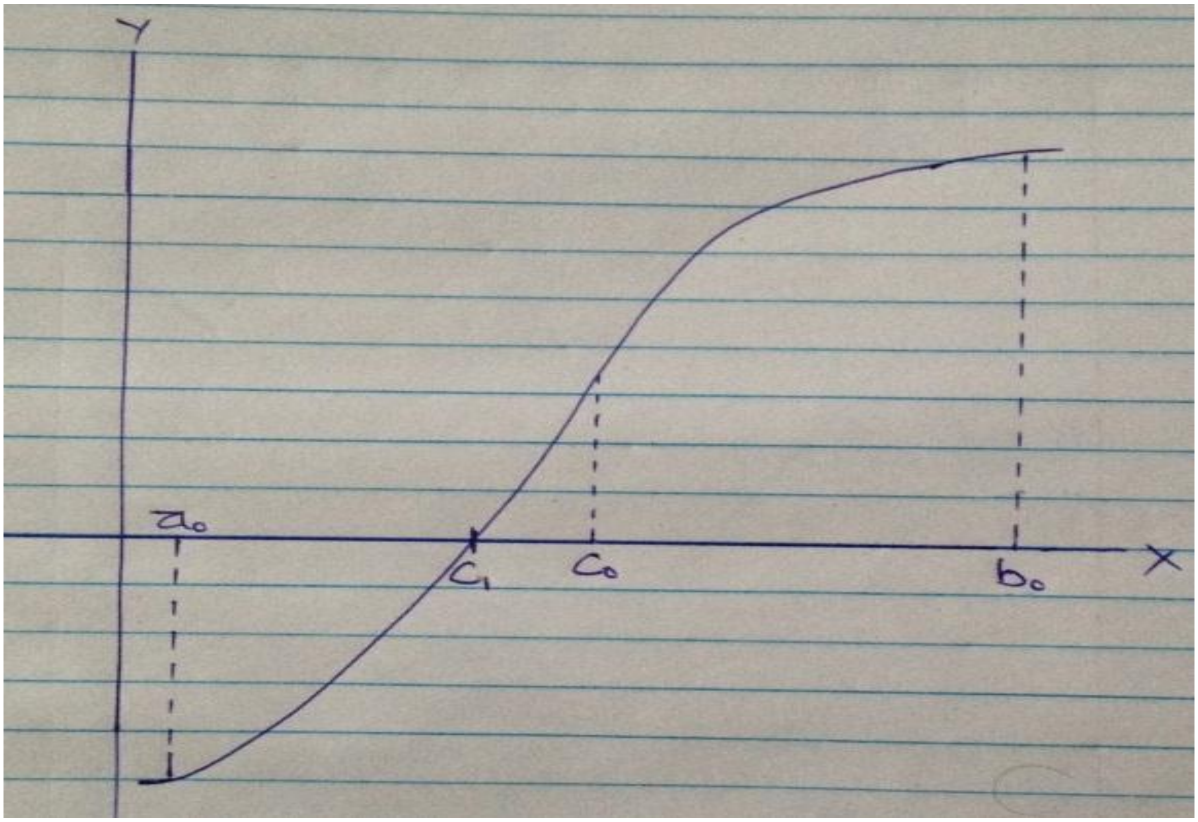


Fig: Half Interval Method

This process of calculating the approximations $c_0, c_1, c_2, \ldots$ is repeated until we find a root of $f(x)$ or a satisfactory approximation of it.

Advantages:

- This method is guaranteed to work for any continuous function $f(x)$ on the interval $[a, b]$ with $f(a)f(b) < 0$.
- The number of iterations required to achieve a specified accuracy is known in advance.

Disadvantage:

- The method converges slowly, i.e., it requires more iterations to achieve the same accuracy when compared with some other methods for solving non-linear equations.

Newton's Method:

Let $f(x)$ be a differentiable function and let x_0 be an initial points which is sufficiently close to the root of $f(x)$. Let $(x_1, 0)$ be the point of intersection of the x-axis and the tangent drawn to the curve $f(x)$ at $\big(x_0, f(x_0)\big)$. Newton's method takes this point as the first approximation for the root of $f(x)$. To calculate this point we note that the slope of the tangent to $f(x)$ at $x = x_0$ is equal to the slope of the line through the points $(x_1, 0)$ and $\big(x_0, f(x_0)\big)$. i.e.

$$f'(x_0) = \frac{(f(x_0) - 0)}{x_0 - x_1} \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

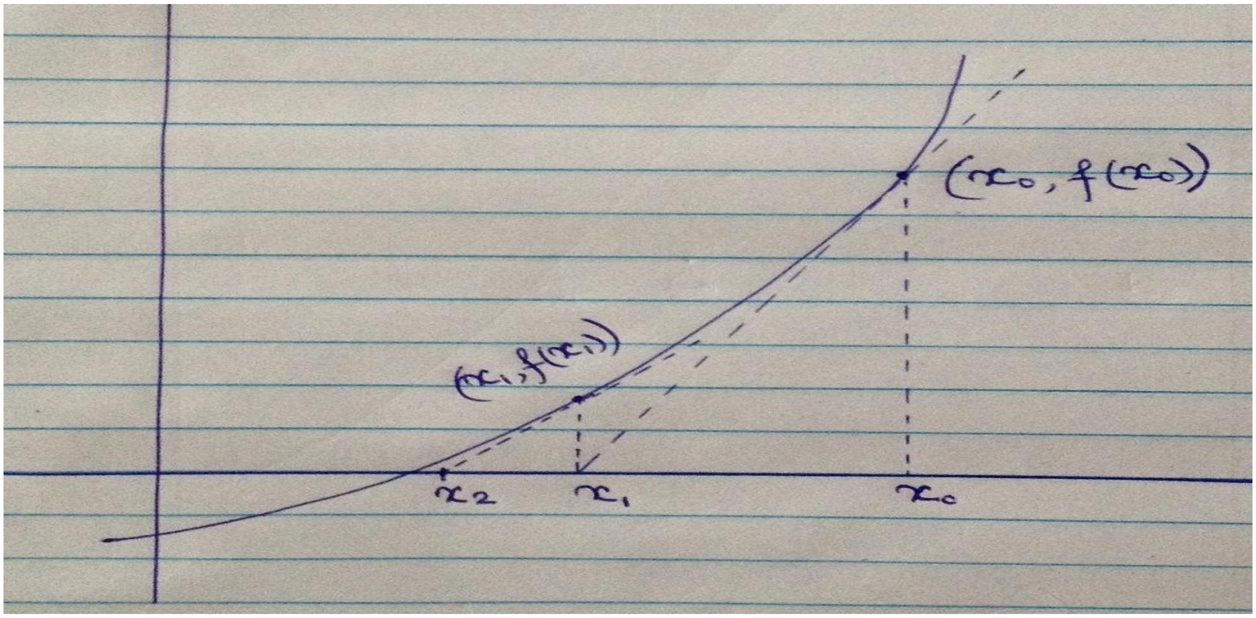


Fig: Newton-Raphson Method

If $f(x_1) = 0$, then x_1 is the required root of $f(x)$. If not, then we take the point of intersection $(x_2, 0)$ of the x-axis and the tangent to the $f(x)$ at $x = x_1$ as the next approximation of the root. As above, we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, the $(n+1)^{th}$ approximation of the root of $f(x)$ is given by the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ; \quad n \geq 0$$

We continue to calculate the approximations $x_1, x_2, x_3, \ldots$ using the above formula until we find the root or its satisfactory approximation.

Advantages:

- Unlike the incremental search and bisection methods, the Newton-Raphson method isn't fooled by singularities.
- Also, it can identify repeated roots, since it does not look for changes in the sign of $f(x)$ explicitly.
- It can find complex roots of polynomials, assuming you start out with a complex value for x_1.
- For many problems, Newton-Raphson converges quicker than either bisection or incremental search.

Disadvantages:

- The Newton-Raphson method only works if you have a functional representation of $f'(x)$. Some functions may be difficult or impossible to differentiate. You may be able to work around this by approximating the derivative $f'(x) \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$.
- The Newton-Raphson method is not guaranteed to find a root.

2. Derive the equation for Lagrange's interpolating polynomial and find the value of f(x) at x=1 for the following:

x	**-1**	**-2**	**2**	**4**
f(x)	**-1**	**-9**	**11**	**69**

Solution: Here,

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x+2)(x-2)(x-4)}{(-1+2)(-1-2)(-1-4)}$$
$$= \frac{1}{15}(x+2)(x-2)(x-4)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x+1)(x-2)(x-4)}{(-2+1)(-2-2)(-2-4)}$$
$$= -\frac{1}{24}(x+1)(x-2)(x-4)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x+1)(x+2)(x-4)}{(2+1)(2+2)(2-4)} = -\frac{1}{24}(x+1)(x+2)(x-4)$$

$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+1)(x+2)(x-2)}{(4+1)(4+2)(4-2)} = \frac{1}{60}(x+1)(x+2)(x-2)$$

And Lagrange's interpolating polynomial is given by:

$$P_3(x) = f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x) + f_3 l_3(x)$$

$$P_3(x) = (-1)\frac{1}{15}(x+2)(x-2)(x-4) + (-9)\frac{(-1)}{24}(x+1)(x-2)(x-4)$$
$$+ 11\frac{(-1)}{24}(x+1)(x+2)(x-4) + 69\frac{1}{60}(x+1)(x+2)(x-2)$$

$$\therefore P_3(x) = x^3 + x + \frac{1}{4}$$

Now, the value of $f(x)$ at $x = 1$ is given as;

$$P_3(1) = 1^3 + 1 + \frac{1}{4} = \frac{9}{4}$$

3. Write Newton-cotes integration formulas in basic form for x=1, 2, 3 and give their composite rules. Evaluate $\int_0^1 e^{-x^2}\,dx$ using the Gaussian integration three point formula.

To find the value of $\int_a^b f(x)dx$ numerically using the Newton-Cotes method, we first of all divide the interval $[a, b]$ into n equal parts of length h by points $x_i = a + ih, i = 0, 1, 2, \dots, n$ where $h = \frac{(b-a)}{n}$. Then $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ forms a partition of $[a, b]$. Let $P_n(x)$ be the interpolating polynomial of $f(x)$ interpolating at $n+1$ points $(x_i, f_i), i = 0, 1, 2, \dots, n$ where $f_i = f(x_i)$. Then $P_n(x)$ is given by the formula

$$P_n(x) = f_0 + S\Delta f_0 + \frac{S(S-1)}{2}\Delta^2 f_0 + \cdots + \frac{S(S-1)\dots(S-n+1)}{n!}\Delta^n f_0$$

Where, $S = \frac{x-x_0}{n}$ & $\Delta^j f_0 = \Delta^{j-1} f_1 - \Delta^{j-1} f_0$ are the j^{th} forward differences.

We now approximate the value of $\int_a^b f(x)dx$ by $\int_a^b P_n(x)dx$.

Therefore, $\int_a^b f(x)dx \approx \int_a^b P_n(x)dx$

$$= \int_a^b \left[f_0 + S\Delta f_0 + \frac{S(s-1)}{2!}\Delta^2 f_0 + \cdots + \frac{S(s-1)\ldots(S-n+1)}{n!}\Delta^n f_0\right]dx$$

Which is the Newton-Cotes formula for numerically evaluated $\int_a^b f(x)dx$.

Numerical:

Let $x = \frac{(1-0)y+1+0}{2} = 0.5y + 0.5$

Then the limit of integration are changed from $(0, 1)$ to $(-1, 1)$ so that

$$\int_0^1 e^{-x^2} dx = \frac{1-0}{2}\int_{-1}^1 e^{-(0.5y+0.5)^2} dy$$

Using the Gaussian 3-point formula, we get

$$\int_{-1}^1 e^{-(0.5y+0.5)^2} dy$$

$$= 0.55556 \times e^{-(0.5\times(-0.77460)+0.5)^2} + 0.88889 \times e^{-(0.5\times0+0.5)^2} + 0.55556 \times e^{-(0.5\times0.77460+0.5)^2}$$

$$= 0.54855 + 0.69227 + 0.25282 = 1.49364$$

$$\therefore \int_0^1 e^{-x^2} dx = \frac{1-0}{2} \times 1.49364 = 0.74682$$

4. Solve the following system of algebraic linear equation using Gauss-Jordan algorithm.

$$\begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 4 & -3 & 0 & 1 \\ 6 & 1 & -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -7 \\ 6 \end{pmatrix}$$

The augmented matrix of the system is as follow:

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

Interchanging first row with second row: $R1 \leftrightarrow R2$

$$\begin{bmatrix} 2 & 2 & 3 & 2 & -2 \\ 0 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

Normalize the first row: $R1 \rightarrow \frac{1}{2}R1$

$$\begin{bmatrix} 1 & 1 & \frac{3}{2} & 1 & -1 \\ 0 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

Eliminate x_1 from 2nd, 3rd and 4th row: $R2 \rightarrow R2$; $R3 \rightarrow R3 - 4R1$; $R4 \rightarrow R4 - 6R1$

$$\begin{bmatrix} 1 & 1 & \frac{3}{2} & 1 & -1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & -7 & -6 & -3 & -3 \\ 0 & -5 & -15 & -11 & 12 \end{bmatrix}$$

Normalize the second row: $R2 \rightarrow \frac{1}{2}R2$

$$\begin{bmatrix} 1 & 1 & \frac{3}{2} & 1 & -1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & -7 & -6 & -3 & -3 \\ 0 & -5 & -15 & -11 & 12 \end{bmatrix}$$

Eliminate x_2 from 1st, 3rd and 4th row: $R1 \rightarrow R1 - R2$; $R3 \rightarrow R3 + 7R2$; $R4 \rightarrow R4 + 5R2$

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -6 & \frac{1}{2} & -3 \\ 0 & 0 & -15 & -\frac{17}{2} & 12 \end{bmatrix}$$

Normalize the third row: $R3 \rightarrow -\frac{1}{6}R3$

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{2} \\ 0 & 0 & -15 & -\frac{17}{2} & 12 \end{bmatrix}$$

Eliminating x_3 from 1st, 2nd and 4th row: $R1 \to R1 - \frac{3}{2}R3\,;R2 \to R2\,;R4 \to R4 + 15R3$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{8} & -\frac{7}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{117}{12} & \frac{39}{2} \end{bmatrix}$$

Normalize the fourth row: $R4 \to -\frac{12}{117}R4$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{5}{8} & -\frac{7}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Eliminating x_4 from 1st, 2nd and 3rd row: $R1 \to R1 - \frac{5}{8}R4\,;R2 \to R2 - \frac{1}{2}R4\,;R3 \to R3 + \frac{1}{12}R4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Therefore, the solution is $x_1 = -\frac{1}{2}\,;x_2 = 1\,;x_3 = \frac{1}{3}\,;x_4 = -2$

5. Write an algorithm and computer program to solve system of linear equation using Gauss-Seidel iterative method.

<u>Algorithm:</u>

<u>Input:</u>

A diagonally dominant system of linear equations $Ax = b$

<u>Process:</u>

FOR $i = 1$ TO n SET $x_i = \frac{b_i}{a_{ii}}$

BEGIN: SET $key = 0$

FOR $i = 1$ TO n

{ SET $sum = b_i$

FOR $j = 1$ TO n AND $j = 1$

{ SET $sum = sum - (a_{ij} * x_j)$

}

SET $dummy = sum/a_{ii}$

IF $key = 0$ AND $\left|\frac{dummy-x_i}{dummy}\right| > error$

THEN

SET $key = 1$

SET $x_i = dummy$

}

IF $key = 1$ THEN

GOTO BEGIN

<u>Output:</u>

Approximate solution $x_i; i = 1, 2, 3, \dots, n$ of $Ax = b$

<u>Computer program:</u>

```
#include<iostream.h>

#include<conio.h>
```

```
#include<iomanip.h>
#include<math.h>
#define MAXIT 50
#define EPS 0.000001
void gaseid(int n, float a[10][10], float b[10], float x[10], int *count, int *status);
void main()
{       float a[10][10], b[10], x[10];
        int i, j, n, count, status;
        cout<<"** SOLUTION BY GUASS SEIDEL ITERATION METHOD **"<<endl;
        cout<<"input the size of the system:"<<endl;
        cin>>n;
        cout<<"input coefficients, a(i.j)"<<endl;
        cout<<"one row on each line"<<endl;
        for(i=1; i<=n; i++)
        for(j=1; j<=n; j++)
        cin>>a[i][j];
        cout<<"input vector b:"<<endl;
        for(i=1; i<=n; i++)
        cin>>b[i];
        gaseid(n, a, b, x, &count, &status);
        if(status==2)
        {       cout<<"no CONVERGENCE in "<<MAXIT<<"
iterations."<<endl<<endl<<endl;
        }
        else
        {       cout<<"SOLUTION VECTOR X"<<endl;
                for(i=1; i<=n; i++)
                cout<<setw(15.6)<<x[i]<<endl;
```

```
		cout<<"iterations= "<<count;
	}
	getch();
}
void gaseid(int n, float a[10][10], float b[10], float x[10], int *count, int *status)
{	int i, j, key;
	float sum, x0[10];
	for(i=1; i<=n; i++)
	x0[i]=b[i]/a[i][i];
	*count=1;
	begin:
	key=0;
	for(i=1; i<=n; i++)
	{	sum=b[i];
		for(j=1; j<=n; j++)
		{	if(i==j)
			continue;
			sum=sum-a[i][j]*x0[j];
		}
		x[i]=sum/a[i][i];
		if(key==0)
		{	if(fabs((x[i]-x0[i])/x[i])>EPS)
			key=1;
		}
	}
	if(key==1)
	{	if(*count==MAXIT)
		{	*status=2;
```

```
                return;
            }
            else
            {       *status=1;
                for(i=1; i<=n; i++)
                x0[i]=x[i];
            }
            *count=*count+1;
            goto begin;
        }
        return;
}
```

6. Explain the Picard's proves of successive approximation. Obtain a solution up to the fifth approximation of the equation $\frac{d^2y}{dx} = y + x$ such that $y = 1$ when $x = 0$ using Picard's process of successive approximation.

Suppose that we are given a differential equation of the form $\frac{dy}{dx} = f(x, y)\,; y(x_0) = y$.

Then $dy = f(x, y)dx$

Integrating both sides of above equation in the interval (x_0, x), we get

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f\big(t, y(t)\big)dt$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f\big(t, y(t)\big)dt$$

$$y(x) = y(x_0) + \int_{x_0}^{x} f\big(t, y(t)\big)dt \quad \dots\dots\dots (i)$$

Now to solve equation (i), we use the method of iteration as follows:

We replace $y(t)$ on the right of equation (i) by y, and calculate the first approximation $y_1(x)$ of $y(x)$ as $y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)dt$

The second approximation $y_2(x)$ of $y(x)$ is calculated by substituting $y(t)$ on the right of equation (i) as $y_2(x) = y_0 + \int_{x_0}^{x} f\big(t, y_1(t)\big)dt$

Proceeding similarly, the n^{th} approximation of $y(x)$ is given by the iteration $y_n(x) = y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)dt$

This iterative method of solving the differential equation is known as Picard's Method.

Numerical:

For $\frac{dy}{dx} = y + x$ when $y = 1$ & $x = 0\,; i.e. y(x) = y(0) = 1$

Picard's iteration method is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)dt = 1 + \int_{0}^{x} f\big(t, y_{n-1}(t)\big)dt$$

When $x = 1$, we get

$$y_1(x) = 1 + \int_0^x f\big(t, y_0(t)\big)dt = 1 + \int_0^x (t+1)dt = 1 + \left[\frac{(t+1)^2}{2}\right]_0^x = 1 + \frac{(x+1)^2}{2} - \frac{1}{2}$$

$$= 1 + x + \frac{x^2}{2}$$

When $x = 2$, we get

$$y_2(x) = 1 + \int_0^x f\big(t, y_1(t)\big)dt = 1 + \int_0^x \left(t + 1 + t + \frac{t^2}{2}\right)dt = 1 + \left[t + t^2 + \frac{t^3}{6}\right]_0^x$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

When $x = 3$, we get

$$y_3(x) = 1 + \int_0^x f\big(t, y_2(t)\big)dt = 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{6}\right)dt = 1 + \left[t + t^2 + \frac{t^3}{3} + \frac{t^4}{24}\right]_0^x$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

When $x = 4$, we get

$$y_4(x) = 1 + \int_0^x f\big(t, y_3(t)\big)dt = 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{24}\right)dt$$

$$= 1 + \left[t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{120}\right]_0^x = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

When $x = 5$, we get

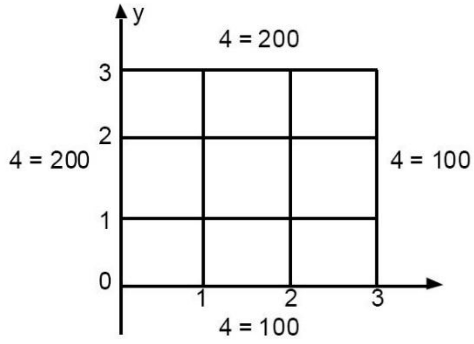
$$y_5(x) = 1 + \int_0^x f(t, y_4(t))dt = 1 + \int_0^x \left(t + 1 + t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{120}\right)dt$$

$$= 1 + \left[t + t^2 + \frac{t^3}{3} + \frac{t^4}{12} + \frac{t^5}{60} + \frac{t^6}{720}\right]_0^x = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720}$$

7. Derive a difference equation to represent a Laplace's equation. Solve the following Laplace equation:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ **Within $0 \le x \le 3, 0 \le y \le 3$**

For the rectangular plate given as:



Difference equation to represent Laplace's equation:

Let $u = u(x, y)$ be a function of two independent variables x & y. Then by Taylor's formula:

$$u(x + h, y) = u(x, y) + hu_x(x, y) + \frac{h^2}{2!}u_{xx}(x, y) + \frac{h^3}{3!}u_{xxx}(x, y) + \cdots \quad \ldots\ldots\ldots (i)$$

$$u(x - h, y) = u(x, y) - hu_x(x, y) + \frac{h^2}{2!}u_{xx}(x, y) - \frac{h^3}{3!}u_{xxx}(x, y) + \cdots \quad \ldots\ldots\ldots (ii)$$

$$u(x, y + k) = u(x, y) + ku_y(x, y) + \frac{k^2}{2!}u_{yy}(x, y) + \frac{k^3}{3!}u_{yyy}(x, y) + \cdots \quad \ldots\ldots\ldots (iii)$$

$$u(x, y - k) = u(x, y) - ku_y(x, y) + \frac{k^2}{2!}u_{yy}(x, y) - \frac{k^3}{3!}u_{yyy}(x, y) + \cdots \quad \ldots\ldots\ldots (iv)$$

Adding equations (i) & (ii) and ignoring the terms containing h^4 and higher powers, we get

$$u(x + h, y) + u(x - h, y) = 2u(x, y) + h^2u_{xx}(x, y)$$

$$or, u_{xx}(x, y) = \frac{1}{h^2}[u(x + h, y) - 2u(x, y) + u(x - h, y)] \quad \ldots\ldots\ldots (A)$$

Adding equations (iii) & (iv) and ignoring the terms containing k^4 and higher powers, we get

$$u(x, y+k) + u(x, y-k) = 2u(x, y) + k^2 u_{yy}(x, y)$$

$$or, u_{yy}(x, y) = \frac{1}{k^2}[u(x, y+k) - 2u(x, y) + u(x, y-k)] \quad \dots\dots\dots (B)$$

Now if $u_{xx} + u_{yy} = 0$ is the given Laplace's equation, then from equation (A) & (B) we have

$$\frac{1}{h^2}[u(x+h, y) - 2u(x, y) + u(x-h, y)] + \frac{1}{k^2}[u(x, y+k) - 2u(x, y) + u(x, y-k)] = 0$$

Choosing $h = k$, we get

$$u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h) - 4u(x, y) = 0$$

$$\therefore u(x, y) = \frac{1}{4}[u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h)]$$

is the difference equation for Laplace's equation.

Numerical:

From the difference equation for Laplace's equation, we have

$$200 + 200 + u_2 + u_3 - 4u_1 = 0 \Rightarrow -4u_1 + u_2 + u_3 = -400 \quad \dots\dots\dots (i)$$

$$200 + 100 + u_4 + u_1 - 4u_2 = 0 \Rightarrow u_1 - 4u_2 + u_4 = -300 \quad \dots\dots\dots (ii)$$

$$u_1 + 200 + 100 + u_4 - 4u_3 = 0 \Rightarrow u_1 - 4u_3 + u_4 = -300 \quad \dots\dots\dots (iii)$$

$$u_2 + u_3 + 100 + 100 - 4u_4 = 0 \Rightarrow u_2 + u_3 - 4u_4 = -200 \quad \dots\dots\dots (iv)$$

Solving the equations $(i), (ii), (iii)$ & (iv), we get

$u_1 = 175$

$u_2 = u_3 = 150$

$u_4 = 125$

(OR) 7. Derive a difference equation to represent Poisson's equation. Solve the Poisson's equation $\nabla^2 f = 2x^2y^2$ over the square to main $0 \le x \le 3, 0 \le y \le 3$ with $f = 0$ on the boundary and $h = 1$.

Difference equation to represent Poisson's equation:

Let $u = u(x, y)$ be a function of two independent variables x & y. Then by Taylor's formula:

$$u(x+h,y) = u(x,y) + hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) + \frac{h^3}{3!}u_{xxx}(x,y) + \cdots \quad \dots\dots\dots (i)$$

$$u(x-h,y) = u(x,y) - hu_x(x,y) + \frac{h^2}{2!}u_{xx}(x,y) - \frac{h^3}{3!}u_{xxx}(x,y) + \cdots \quad \dots\dots\dots (ii)$$

$$u(x,y+k) = u(x,y) + ku_y(x,y) + \frac{k^2}{2!}u_{yy}(x,y) + \frac{k^3}{3!}u_{yyy}(x,y) + \cdots \quad \dots\dots\dots (iii)$$

$$u(x,y-k) = u(x,y) - ku_y(x,y) + \frac{k^2}{2!}u_{yy}(x,y) - \frac{k^3}{3!}u_{yyy}(x,y) + \cdots \quad \dots\dots\dots (iv)$$

Adding equations (i) & (ii) and ignoring the terms containing h^4 and higher powers, we get

$$u(x+h,y) + u(x-h,y) = 2u(x,y) + h^2u_{xx}(x,y)$$

$$or, u_{xx}(x,y) = \frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)] \quad \dots\dots\dots (A)$$

Adding equations (iii) & (iv) and ignoring the terms containing k^4 and higher powers, we get

$$u(x,y+k) + u(x,y-k) = 2u(x,y) + k^2u_{yy}(x,y)$$

$$or, u_{yy}(x,y) = \frac{1}{k^2}[u(x,y+k) - 2u(x,y) + u(x,y-k)] \quad \dots\dots\dots (B)$$

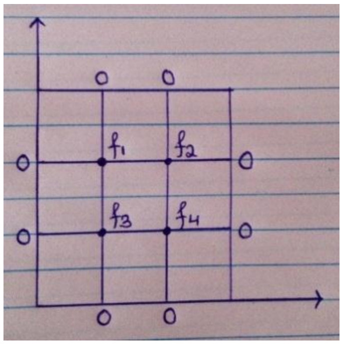
Now if $u_{xx} + u_{yy} = g(x,y)$ is the given Poisson's equation, then from equation (A) & (B) choosing $h = k$ we have,

$$u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y) = h^2g(x,y)$$

which is the difference equation for Poisson's equation.

Numerical:

The domain is divided as follows with $f = 0$ at the boundary



Now, from the difference equation for the Poisson's equation, we have

$0 + 0 + f_2 + f_3 - 4f_1 = 1^2 \times 2 \times 1^2 \times 2^2$

$or, f_2 + f_3 - 4f_1 = 8 \quad \ldots\ldots\ldots (i)$

$0 + 0 + f_1 + f_4 - 4f_2 = 1^2 \times 2 \times 2^2 \times 2^2$

$or, f_1 + f_4 - 4f_2 = 32 \quad \ldots\ldots\ldots (ii)$

$0 + 0 + f_1 + f_4 - 4f_3 = 1^2 \times 2 \times 1 \times 1$

$or, f_1 + f_4 - 4f_3 = 2 \quad \ldots\ldots\ldots (iii)$

$0 + 0 + f_2 + f_3 - 4f_4 = 1^2 \times 2 \times 2^2 \times 1$

$or, f_2 + f_3 - 4f_4 = 8 \quad \ldots\ldots\ldots (iv)$

Solving these equations, we get

$f_1 = -\frac{11}{2}$

$f_2 = -\frac{43}{4}$

$f_3 = -\frac{13}{4}$

$f_4 = -\frac{11}{2}$