Unit → 8
Orthogonality and Least Squares:

⊛. Scalar (or inner) product:

Definition → Let $u=(u_1, u_2, \dots, u_n)$ and $v=(v_1, v_2, \dots, v_n)$ then the scalar product of u and v is denoted by $u.v$ and defined as $u.v = u_1v_1 + u_2v_2 + \dots + u_nv_n$. This product is also known as dot product.

Note: Let, $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be two column matrices representing the vectors in $\mathbb{R}^n$.

Then, the inner product $u.v = u^T.v$

i.e, $u.v$ = matrix product of u^T (transpose of u) and v.

⊛. Properties of Inner product:

Let, u and v be any two vectors in $\mathbb{R}^n$. Then,

i) $u.v = v.u$ (commutative)

ii) $(u+v).w = u.w + v.w$ (distributive).

iii) $(cu).v = c.(u.v) = u.(c.v)$

iv) $u.u \geq 0$ and $u.u = 0$ if and only if $u=0$.

Norm of a vector (Length of a vector):

The length or norm of a vector v is a non-negative scalar $\|v\| = \sqrt{v.v} = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$

where, $v = (v_1, v_2, \dots, v_n)$.

✓ Note that, this definition implies $\|v\|^2 = v.v$

Unit Vector:

Definition → A vector having length 1, is called a unit vector. Mathematically, if v be a vector in $\mathbb{R}^n$ then its unit vector is, $\dfrac{v}{\|v\|}$.

Example: Find the unit vector along the vector $v=(-2,1,0)$ and verify it.

Solution: Let $v=(-2,1,0)$

Then, $\|v\| = \sqrt{(-2)^2 + 1^2 + 0^2}$

$= \sqrt{4+1+0}$

$= \sqrt{5}$.

Therefore, the unit vector of v is $\frac{v}{||v||} = \frac{(-2,1,0)}{\sqrt{5}} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$.

Verification:

Let $u = \frac{v}{||v||} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$.

Now, the length of u is, $||u|| = \sqrt{\left(\frac{-2}{\sqrt{5}}\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2 + (0)^2}$

$$= \sqrt{\frac{4}{5} + \frac{1}{5} + 0}$$

$$= \sqrt{\frac{4+1}{5}}$$

$$= \sqrt{1}$$

$$= 1$$

Thus, $\frac{v}{||v||} = \left(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$ be unit vector along the vector v.

Normalization of a vector:

Definition → Let v be a vector in $\mathbb{R}^n$. Set $u = \frac{v}{||v||}$ then process of creating u is called normalizing v.

Distance between two vectors:

Definition → Let u and v are in $\mathbb{R}^n$, then the distance between u and v is the length between them. It is denoted by $dis(u,v)$ and define as,

$$dis(u,v) = ||u - v||.$$

Example 1: If $u = (2,3)$ and $v = (3,-1)$ then find the distance between them.

Solution: Given, $u = (2,3)$ and $v = (3,-1)$.

Then, $u - v = (2,3) - (3,-1) = (-1,4)$.

Now distance between u and v is,

$$||u - v|| = \sqrt{(-1)^2 + (4)^2} = \sqrt{1+16} = \sqrt{17}$$

Example 2: Find the distance between $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $z = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

Solution: Let $u = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $z = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

Then, $u - z = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -6 \end{bmatrix}$.

So, $(u-z)\cdot(u-z) = \begin{bmatrix}4\\-4\\-6\end{bmatrix}\cdot\begin{bmatrix}4\\-4\\-6\end{bmatrix} = 16+16+36 = 72.$

Now, the distance between u and z is

$$dis(u,z) = \|u-z\| = \sqrt{(u-z)\cdot(u-z)}$$
$$= \sqrt{72}$$
$$= 6\sqrt{2}.$$

Orthogonal Vectors:

Two vectors u and v in $\mathbb{R}^n$ are orthogonal to each other if $u\cdot v = 0$.

Example: Show that the vectors $u=(2,-3,3)$ and $v=(12,3,-5)$ are orthogonal.

Solution:

Given, $u=(2,-3,3)$ and $v=(12,3,-5)$

Now,
$$u\cdot v = (2,-3,3)\cdot(12,3,-5)$$
$$= 24-9-15$$
$$= 0$$

This means u and v are orthogonal.

The Pythagorean Theorem:

Statement → Two vectors u and v are orthogonal if and only if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

Proof

First suppose that u and v are orthogonal.

Therefore $u\cdot v = 0$ —— (P).

Since $\|u\|^2 = u\cdot u$.

So,
$$\|u+v\|^2 = (u+v)\cdot(u+v)$$
$$= u\cdot(u+v) + v\cdot(u+v)$$
$$= u\cdot u + u\cdot v + v\cdot u + v\cdot v$$
$$= \|u\|^2 + 0 + 0 + \|v\|^2 \quad [\text{using (P)}]$$
$$= \|u\|^2 + \|v\|^2.$$

Conversely suppose that $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

$$\Rightarrow (u+v)\cdot(u+v) = \|u\|^2 + \|v\|^2$$
$$\Rightarrow u\cdot u + u\cdot v + v\cdot u + v\cdot v = \|u\|^2 + \|v\|^2$$
$$\Rightarrow \|u\|^2 + u\cdot v + v\cdot u + \|v\|^2 = \|u\|^2 + \|v\|^2$$
$$\Rightarrow u\cdot v + v\cdot u = 0$$
$$\Rightarrow 2u\cdot v = 0$$
$$\Rightarrow u\cdot v = 0$$

This means the vectors u and v are orthogonal.

⊛. Angles in R^2 (or R^3):

If u and v are two vectors in R^2 (or R^3) and if θ be the angle between these vectors, then the dot product of u and v be defined as,

$$u.v = \|u\|\|v\|\cdot\cos\theta$$

$$\Rightarrow \cos\theta = \frac{u.v}{\|u\|\cdot\|v\|}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{u.v}{\|u\|\cdot\|v\|}\right)$$

Thus, the angle θ between any two vectors u and v is defined as,

$$\theta = \cos^{-1}\left(\frac{u.v}{\|u\|.\|v\|}\right).$$

Example: Find the angle between the vectors $(1,0,-1)$ and $(-1,1,-1)$.

Solution:

Let $u=(1,0,-1)$ and $v=(-1,1,-1)$.

Now, $\|u\| = \sqrt{(1,0,-1).(1,0,-1)}$

$= \sqrt{1+0+1}$

$= \sqrt{2}$

$\|v\| = \sqrt{(-1,1,-1)\cdot(-1,1,-1)}$

$= \sqrt{(-1)(-1)+1\times1+(-1)\times(-1)}$

$= \sqrt{3}$.

$u.v = -1+0+1$

$= 0$.

$\because \cos\theta = \frac{u.v}{\|u\|.\|v\|}$

or, $\cos\theta = \frac{0}{\sqrt{2}.\sqrt{3}}$

or, $\theta = \cos^{-1}(0)$

or, $\theta = 90°$.

⊛ Orthogonal Sets:

A set of vectors $\{u_1, u_2, \dots, u_p\}$ in $\mathbb{R}^n$, is said to be an orthogonal set if $u_i.u_j = 0$ for $i \neq j$ for $i, j = 1, 2, \dots, p$.

Example1: Examine a set of vectors $\{u_1, u_2, u_3\}$ is an orthogonal set where $u_1 = (2,-7,-1)$, $u_2 = (-6,-3,9)$ and $u_3 = (3,1,-1)$?

Solution:

Given, $u_1 = (2,-7,-1)$, $u_2 = (-6,-3,9)$, $u_3 = (3,1,-1)$.

Now, $u_1.u_2 = (2,-7,-1).(-6,-3,9) = -12+21-9 = 0$

$u_2.u_3 = (-6,-3,9).(3,1,-1) = -18-3-9 = -30 \neq 0$

$u_1.u_3 = (2,-7,-1).(3,1,-1) = 6-7+1 = 0$.

This shows that the set $\{u_1, u_2, u_3\}$ is not an orthogonal set.

Example 2: Show that $\{(3,1,1),(-1,2,1),(-\frac{1}{2},-2,\frac{7}{2})\}$ is an orthogonal basis for R^3.

Solution:

Let, $u_1=(3,1,1)$, $u_2(-1,2,1)$, $u_3=(-\frac{1}{2},-2,\frac{7}{2})$.

Here,
$$u_1 \cdot u_2 = (3,1,1).(-1,2,1) = -3+2+1=0.$$
$$u_2 \cdot u_3 = (-1,2,1).(-\frac{1}{2},-2,\frac{7}{2}) = \frac{1}{2}-4+\frac{7}{2}=0.$$
$$u_1 \cdot u_3 = (3,1,1)\cdot(-\frac{1}{2},-2,\frac{7}{2}) = -\frac{3}{2}-2+\frac{7}{2}=0.$$

Therefore, $\{u_1,u_2,u_3\}$ is an orthogonal set.

Also,
$$\|u_1\| = u_1 \cdot u_1 = (3,1,1).(3,1,1) = 9+1+1=11 \neq 0$$
$$\|u_2\| = u_2 \cdot u_2 = (-1,2,1).(-1,2,1) = 1+4+1=6 \neq 0.$$
$$\|u_3\| = u_3 \cdot u_3 = (-\frac{1}{2},-2,\frac{7}{2}) = \frac{1}{4}+4+\frac{49}{4} = \frac{64}{4} = 16 \neq 0.$$

Since every orthogonal set of non-zero vectors is a basis for the subspace of the space.
Here $\{u_1,u_2,u_3\}$ is an orthogonal set of vectors, so $\{u_1,u_2,u_3\}$ is a basis for $\mathbb{R}^3$ and therefore, is an orthogonal basis for $\mathbb{R}^3$.

⊛ An Orthogonal Projection:

Example: Find the orthogonal projection of $\begin{bmatrix}-1\\1\end{bmatrix}$ onto the line $\begin{bmatrix}-1\\3\end{bmatrix}$ and the origin.

Solution,

Let. $y=\begin{bmatrix}-1\\1\end{bmatrix}$, $u=\begin{bmatrix}-1\\3\end{bmatrix}$ for projection onto the line.

Then,
$$y \cdot u = \begin{bmatrix}-1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\3\end{bmatrix} = (-1)(-1)+(1)(3) = 1+3=4$$
$$u \cdot u = \begin{bmatrix}-1\\3\end{bmatrix} \cdot \begin{bmatrix}-1\\3\end{bmatrix} = (-1)(-1)+(3)(3) = 1+9 = 10.$$

Now, the orthogonal projection $\hat{y}$ of y onto u is,
$$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) \cdot u$$
$$= \left(\frac{4}{10}\right)\begin{bmatrix}-1\\3\end{bmatrix}$$
$$= \frac{2}{5}\begin{bmatrix}-1\\3\end{bmatrix}.$$

⊛. Orthonormal sets:

Orthonormal set → An orthogonal set of unit vectors, is called an orthonormal set.

Orthonormal basis → If every vector of an orthogonal basis of unit vectors then the basis is called orthonormal basis.

Note: An $m \times n$ matrix U has orthonormal columns if and only if $U^TU = I$.

Example: Let $U = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$. Then show that U has orthonormal columns if and only if $U^TU = I$

Solution: Let $U = [u_1 \ u_2]$

where,

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \text{ and } u_2 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

Then,

$$U^TU = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 + 1/2 + 0 & \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 \\ \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 & 4/9 + 4/9 + 1/9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I.$$

Next,

$$u_1 \cdot u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{2} + \frac{1}{2} + 0 = 1$$

$$u_2 \cdot u_2 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$$

$$u_1 \cdot u_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0.$$

and similarly, $u_2 \cdot u_1 = u_1 \cdot u_2 = 0$.

This shows that the columns u_1 and u_2 are orthonormal columns of U.

The Gram Schmidt Process:

The Gram Schmidt process is a simple process or algorithm to obtain an orthogonal or orthonormal basis for any non-zero subspace of $\mathbb{R}^n$.

Example: Let $x_1 = (1,-4,0,1)$ and $x_2 = (7,-7,-4,1)$. If $W = \text{Span}\{x_1, x_2\}$ then construct an orthogonal basis for W by using Gram-Schmidt process.

Solution: Given $x_1 = (1,-4,0,1)$ and $x_2 = (7,-7,-4,1)$. Also let $W = \text{Span}\{x_1, x_2\}$. Then W is a subspace of R^4. Let $v_1 = x_1$. By Gram-Schmidt process we construct vectors v_2 so that $\{v_1, v_2\}$ is an orthogonal basis for W.

Take $v_1 = x_1 = (1,-4,0,1)$

and
$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$$
$$= x_2 - \left(\frac{x_2 \cdot x_1}{x_1 \cdot x_1}\right) x_1 \quad [\because v_1 = x_1].$$
$$= x_2 - \frac{(7,-7,-4,1) \cdot (1,-4,0,1)}{(1,-4,0,1) \cdot (1,-4,0,1)} x_1$$
$$= x_2 - \frac{7+28+0+1}{1+16+0+1} x_1$$
$$= (7,-7,-4,1) - \frac{36}{18} \cdot (1,-4,0,1)$$
$$= (5,1,-4,-1).$$

This, $\{v_1, v_2\}$ is an orthogonal set of non-zero vectors in W. Since W is defined by a basis of two vectors. So, the set $\{v_1, v_2\}$ is an orthogonal basis for W.

Remember that:

$$v_1 = x_1$$
$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$$
$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} \cdot v_2$$
$$\vdots$$
so on to v_p

Then $\{v_1, \ldots, v_p\}$ is an orthogonal basis for W

⊛. The QR-Factorization Algorithm:

If A is an $m \times n$ matrix with linearly independent columns then A can be factored as $A = QR$ where Q is an $m \times n$ matrix whose columns form an orthonormal basis for col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Example. Find QR-factorization of a matrix A where.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution:

Let the columns of A are x_1, x_2, x_3.

So, $x_1 = (1,1,1,1)$, $x_2 = (0,1,1,1)$, $x_3 = (0,0,1,1)$.

Let $v_1 = x_1 = (1,1,1,1)$

Take $v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1$

$$= (0,1,1,1) - \frac{(0,1,1,1).(1,1,1,1)}{(1,1,1,1).(1,1,1,1)} (1,1,1,1)$$

$$= (0,1,1,1) - \frac{3}{4}(1,1,1,1)$$

$$= \frac{1}{4}(-3,1,1,1).$$

Set $v_2' = (-3,1,1,1)$.

Also, $v_3 = x_3 - \frac{x_3 v_1}{v_1 \cdot v_1} \cdot v_1 - \frac{x_3 \cdot v_2'}{v_2' \cdot v_2'} \cdot v_2'$

$$= (0,0,1,1) - \frac{2}{4}(1,1,1,1) - \frac{2}{12}(-3,1,1,1).$$

$$= \frac{1}{3}(0,-2,1,1).$$

Set $v_3' = (0,-2,1,1)$.

Thus, $\{v_1, v_2', v_3'\}$ be an orthogonal basis. Then let $\{u_1, u_2, u_3\}$ be normalize of the orthogonal basis.

So, $u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1,1)}{2}$

$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{(-3,1,1,1)}{\sqrt{12}}$$

$$u_3 = \frac{v_3'}{\|v_3'\|} = \frac{(0,-2,1,1)}{\sqrt{6}}$$

Let Q be the matrix whose columns are u_1, u_2, u_3. Then

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

Since we have $A = QR$, by QR-factorization theorem. Then,

$$Q^TA = Q^T(QR) = Q^TQR = IR = R.$$

Now,

$$R = Q^TA$$

$$\Rightarrow R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}.$$

Least Squares Problems:

If A is $m \times n$ matrix and b is in $\mathbb{R}^m$ then a least square solution of $Ax=b$ is an $\hat{x}$ in $\mathbb{R}^n$ such that

$$\|b - A\hat{x}\| \le \|b - Ax\|. \text{ for all } x \text{ in } \mathbb{R}^n.$$

Note: The least-squars solution of $Ax=b$ satisfies the equation

$$A^TAx = A^Tb \text{ — (i)}.$$

The matrix eqn represents a system of equations called the normal equations for $Ax=b$. A solution of (i) is often denoted by $\hat{x}$.

Example 1: Find the least-squares solution of the inconsistent system $Ax=b$ for $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$.

Solution:- Here, $A^TA = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}.$

Since the least-squars solution of $Ax=b$ satisfies the equation $A^TAx = A^Tb$.

Therefore $\hat{x} = (A^TA)^{-1}(A^Tb)$ — (1)

Here, $|A^TA| = \begin{vmatrix} 17 & 1 \\ 1 & 5 \end{vmatrix} = 84 \neq 0.$

Then,
$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

Therefore ① becomes,
$$\hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix}$$
$$= \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

__Example 2:__ Determine the least-square error in the least-squares solution of $Ax=b$ where A and b are $\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$.

__Solution:-__

A & b use in this are same as in example 1

From previous example we have $\hat{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Then,
$$A\hat{x} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}.$$

So,
$$b - A\hat{x} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix}$$

and $||b - A\hat{x}|| = \sqrt{(-2)^2 + (-4)^2 + (8)^2} = \sqrt{84}.$

Thus, the least square error is $\sqrt{84}$.

__Note:-__ $||Av|| < ||Au||$ shows v is the least square solution of $Ax=b$.