UNIT-4

Markov Chains

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A Markov Chain is a mathematical system that undergoes transitions from one state to another, between a finite or countable number of possible states. It is a random process characterized as memoryless: the next state depends on current state and not in the sequence of events that preceded it. This specific kind of "memorylessness" is called the Markov property. Markov chains have many applications as statistical models of real-world processes.

Formal Definition: A Markov chain is a sequence of random variables X1, X2, X3, ... with the Markov property, namely that, given the present state, the future and past states are independent.

i.e, $P_r(X_{n+1}=x \mid X_1=x_1, X_2=x_2, \ldots, X_n=x_n) = P_r(X_{n+1}=x \mid X_n=x_n)$.

Features of Markov Chain:

i) The outcome of each experiment is one of a set of discrete states.

ii) The outcome of an experiment depends only on the present state and not on any past state.

iii) The transition probabilities remain constant from one transition to the next.

Example 1: Weather

(a) If it rains today there is 40% probability of raining tomorrow.

(b) If it does not rains today there is 20% probability of raining tomorrow.

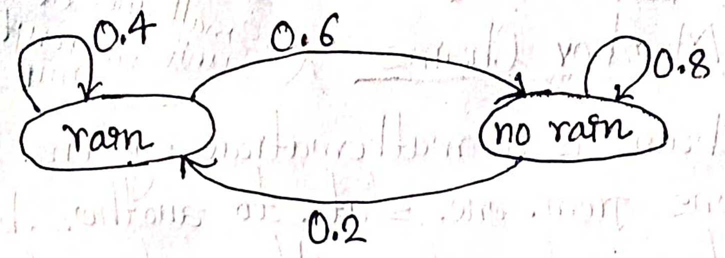
Here,

i.e, 40% in decimal

Total probability is always 1. So, 1-0.4 = 0.6

$$\text{Transition matrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}$$

Now, we have two states rain and no rain. So, we can draw transition diagram as follows:



Example 2: An insurance company classifies drivers as low-risk if they are accident free for one year. Past records indicate that 98% of the drivers in the low-risk category (L) will remain in that category the next year, and 78% of the drivers who are not in the low risk category (L') one year will be in the low-risk category the next year.

(a) Find the transition matrix or probability matrix.

(b) If 90% of the drivers in the community are in the low-risk category this year, what is the probability that a driver choosen at random from the community will be in the low-risk category the next year? The year after next?

Solution:

(a) The transition matrix is as follows:

$$P = \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix}$$

(b). Given, $X_0 = [0.90 \quad 0.10]$

Now, Probability of low-risk next year is given by;

$$X_1 = X_0 \cdot P$$

$$= [0.90 \quad 0.10] \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix}$$

$$= [0.96 \quad 0.04]$$

So, there is chance of 96% low-risk next year and 4% not in low-risk next year.

Similarly, Probability of low-risk the year after next is;

$$X_2 = X_1 \cdot P$$

$$= [0.96 \quad 0.04] \begin{bmatrix} 0.98 & 0.02 \\ 0.78 & 0.22 \end{bmatrix}$$

$$= [0.972 \quad 0.028]$$

General rule for n is;
$X_n = X_{n-1} \cdot P$

⊛ Applications of Markov Chain:

i) Physics: Markovian systems appear extensively in thermodynamics and statistical mechanics, whenever probabilities are used to represent unknown or unmodelled details of the system, if it can be assumed that the dynamics are time invariant and that no relevant history need be considered which is not already included in the state description.

ii) Queuing theory: Markov chains are the basis for the analytical treatment of queues. This makes them critical for optimizing the performance of telecommunication networks where messages must often complete for limited resources such as bandwidth. Numerous queuing models use continuous-time Markov chains.

iii) Internet applications: The page rank of a webpage as used by Google is defined by a Markov chains. It is the probability to be at page i in the stationary distribution on the following Markov chain on all known web pages. Markov models have also been used to analyze web navigation behaviour of users.

iv) Statistics: Markov chain methods also have became very important for generating sequences of random numbers to accurately reflect very complicated desired probability distributions, via a process called Markov Chain Monte Carlo (MCMC).