Mukesh Patel School of Technology Management & Engineering

Basic Science and Humanities Department

BTI (All programs), Semester VI

Linear Algebra and Differential Equations

Tutorial Manual

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Rank of a matrix & System of linear equations

1. Reduce the following matrices to Row Echelon form and find its rank.

a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 3 & 5 & 9 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ d) $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ 5 & -12 & -1 & 6 \end{bmatrix}$

e) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ f) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ g) $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 1 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$

h) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$ i) $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

[Ans: 3, 3, 2, 2, 2, 3, 4, 3, 2]

2. Solve $2x_1 - 2x_2 + 4x_3 + 3x_4 = 9,\ \ x_1 - x_2 + 2x_3 + 2x_4 = 6,\ \ 2x_1 - 2x_2 + x_3 + 2x_4 = 3,\ \ x_1 - x_2 + x_4 = 2$

[Ans. No solution]

3. Solve $x + y + 2z = 8, -x - 2y + 3z = 1, 3x - 7y + 4z = 10.$

[Ans. $x = 3, y = 1, z = 2$]

4. Solve $5x + 3y + 7z = 4, 3x + 26y + 2z = 9, 7x + 2y + 10z = 5.$

[Ans. $x = \frac{7-16k}{11}, y = \frac{3+k}{11}, z = k$]

5. Solve $x + 2y + 3z = 0, 3x + 4y + 4z = 0, 7x + 10y + 12z = 0.$

[Ans. $x = y = z = 0$]

6. Solve $x + y - 3z + 2w = 0, 2x - y + 2z - 3w = 0, 3x - 2y + z - 4w = 0, -4x + y - 3z + w = 0$

[Ans: $x = y = z = w = 0$]

7. Solve $x + 3y + 2z = 0, 2x - y + 3z = 0, 3x - 5y + 4z = 0, x + 17y + 4z = 0.$

[Ans. $x = 11k, y = k,\ z = -7k$]

Vector Space, Linear independence of vectors, basis, dimension

1) Show that the set of all skew symmetric matrices of order $m \times n$ is a vector subspace of $M_{m\times n}(\mathbb{R})$.

2) Find the set of all solutions of $x+2y+z=0$ and show that it is a subspace of $\mathbb{R}^3$.

3) Express the vector (5, 9, 5) as linear combinations of (2, 1, 4), (3, 2, 5) & (1, -1, 3) in R^3

   Ans: (5, 9, 5) = 3(2, 1, 4) + 1 (3, 2, 5) + (- 4) (1, -1, 3)

4) Determine whether (2, - 1, -8) is in the linear span of S = { (1, 2, 1), (1, 1, -1), (4, 5, -2) } $\subseteq R^3$ **Ans:** No

5) Determine whether (0, 0, 0) is in the linear span of S = { (1, 2, 1), (1, 1, -1), (4, 5, -2) } $\subseteq R^3$ **Ans:** Yes

6) Determine whether the set S = { (1, 1, 1), (4, 4, 0), (3, 0, 0) } spans R^3 **Ans:** Yes

7) Find a basis for the following subspaces W of R^2
   a) W is x-3y=0
   b) W is the line passing through the points (1,4) and (0,0).
   c) W is the line passing through (0,0) parallel to the vector (2,1)
   d) W is the linear span of {(1,-2), (2,-4)}
   e) W is $\{(x,3x) \mid x \in R\}$
   f) W is the row space of $\begin{pmatrix} 1 & -1 \\ 3 & -3 \\ 5 & -2 \end{pmatrix}$
   g) W is the null space of $\begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$
   h) W is the column space of $\begin{pmatrix} -1 & 2 & -3 & 4 \\ 2 & -5 & 0 & 2 \end{pmatrix}$

8) Find a basis for the following subspaces W of R^3
   a) W is $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$
   b) W is $x=-2t; y=t; z=3t$, t is any real number.
   c) W is the line passing through origin and parallel to the vector (2,4,-1)
   d) W is *x-3y+2z=0*
   e) W is the solution space of the equations
      2x-2y+z=0
      x+3y-4z=0
   f) W is $\{(x,y,2x-y) \mid x,y \in R\}$

g) W is the null space of $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & 2 \\ 4 & 5 & 6 \end{pmatrix}$

h) W is the row space of $\begin{pmatrix} 1 & 2 & 5 \\ -3 & 0 & 6 \\ -1 & -3 & 5 \\ 0 & 1 & 4 \end{pmatrix}$

i) W is the column space of $\begin{pmatrix} 1 & 1 & -2 \\ -3 & 2 & 3 \\ 2 & -4 & 8 \end{pmatrix}$

j) W is the subspace spanned by the vectors (0,1,2), (1,-2,5), (3,1,0), (2,-4,0).

9) Show that the set $S = \{ (1, 0, 0), (1, 1, 0), (1, 1, 1) \}$ is a basis of vector space $\mathbb{R}^3$.

10) Determine whether { (1,1,1), (1,2,3), (2,3,4)} form a basis of R^3. If not find the dimension of the subspace they span.

11) Determine a basis and dimension of the solution space of following homogeneous systems

a)
$$\begin{aligned} x+y+z&=0 \\ 3x+2y-2z&=0 \\ 4x+3y-z&=0 \\ 6x+5y+z&=0 \end{aligned}$$

Ans:

B = { (4,- 5,1) }

dim = 1

b)
$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 0 \\ -x_1 - x_2 + 4x_3 - x_4 &= 0 \\ -2x_1 - 4x_2 + 7x_3 - 2x_4 &= 0 \end{aligned}$$

Ans:

B = { (1,0,0,-1) }

Dim=1

Linear transformation, Matrix associated with linear transformation

Show that the following maps are not linear:

1. $T:\mathbb{R}^2 \to \mathbb{R}^2, T(x,y) = (x+1, y)$
2. $T:\mathbb{R} \to \mathbb{R}, T(x) = |x|$
3. Draw the image of the unit square with corners $(0,0),(1,0),(1,1)$ and $(0,1)$ under the following linear maps and explain the geometric effect of T on the rectangle mentioned above.

a) $T(x,y) = \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}, \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \right)$

b) $T(x,y) = (x, 4y)$

c) $T(x,y) = (2x, 2y)$

4. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(3,1)=(1,2)$, $T(0,1)=(1,1)$. Compute $T(1,0)$.
5. Let $T:\mathbb{R}^2 \to \mathbb{R}^2, T(x,y)=\left(2x-3y,-x+y\right)$. Find the matrix associated with T with respect to the standard bases.

   $Ans:\begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix}$
6. Let $T:\mathbb{R}^3 \to \mathbb{R}^2, T(x,y,z)=\left(2x-z,y+2z\right)$. Find the matrix M associated with T with respect to the standard bases. Verify that $M\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}=T\left(2,-1,3\right)$.

   $Ans:\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$
7. Let $T:\mathbb{R}^2 \to \mathbb{R}^3, T(x,y)=\left(x-y,2x+3y,3x+2y\right)$. Find the matrix associated with T with respect to the bases : $\left\{(1,0),(1,1)\right\}$ of $\mathbb{R}^2$ and the standard basis of $\mathbb{R}^3$.

   $Ans:\begin{pmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 5 \end{pmatrix}$
8. Let $T:\mathbb{R}^3 \to \mathbb{R}^2, T(x,y,z)=\left(x+y+z,x-2y+3z\right)$. Let $B=\left\{e_3,e_2,e_1\right\}$ and $C=\left\{e_2,e_1\right\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively. Find the matrix M with respect to B and C. Verify that $M\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}=T\left(1,2,3\right)$.

   $Ans:\begin{pmatrix} 3 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
9. Find the matrix associated with T, where T is reflection about $y-axis$.

   $Ans:\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
10. For the following matrices given below, write the corresponding linear transformations $T:\mathbb{R}^2 \to \mathbb{R}^2$. Also draw the image of the rectangle with corner points (0,0), (2,0), (2,1) and (0,1). In each case explain the geometric effect of T on this rectangle.

    a) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

    b) $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

    c) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Range and kernel of a linear map, rank and nullity,
Composition of linear maps & Inverse of a linear transformation

1. Find the range and kernel of $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z)=(x-y,x+z)$.
   $Ans: Ker\,T=\{(-z,-z,z)\mid z\in\mathbb{R}\}, Range=\{(r,s)\mid r,s\in\mathbb{R}\}$.
2. Find the range and kernel of $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by
   $T(x,y,z)=(x-y,x+y,2x-3y)$.
   $Ans: Ker\,T=\{(0,0,z)\big| z\in\mathbb{R}\}, Range=\left\{\left(r,s,\frac{5r-s}{2}\right)\mid r,s\in\mathbb{R}\right\}$.
3. Find rank and nullity of $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z)=(x-y,y-z,z-x)$ and verify rank – nullity theorem.
   $Ans: Ker\,T=\{(x,x,x)\mid x\in\mathbb{R}\}, Range=\{(r,s,-r-s)\mid r,s\in\mathbb{R}\}, nullity=1, rank=2$
4. Find rank and nullity of $T:\mathbb{R}^2 \to \mathbb{R}$ defined by $T(x,y)=(2,-1)\cdot(x,y)$ and verify rank – nullity theorem.
   $Ans: Ker\,T=\{(x,2x)\mid x\in\mathbb{R}\},\ Range=\mathbb{R}\ nullity=1, rank=1$
5. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y)=(x+2y,2x-y)$ and $S:\mathbb{R}^2 \to \mathbb{R}$ be defined by $S(a,b)=a-2b$. Find $S\circ T(x,y)$.
   $Ans: S\circ T(x,y)=-3x+4y$
6. Let $T:\mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x,y)=(x,x-y,y)$ and $S:\mathbb{R}^3 \to \mathbb{R}^2$ be defined by $S(a,b,c)=(a-2b,a+c)$. Find $S\circ T(x,y)$. Find matrices representing all these transformations. Verify that the matrix representing $S\circ T$ is the product of matrices representing S and T.
   $$Ans: S\circ T(x,y)=(-x+2y,x+y), M_S=\begin{pmatrix}1&-2&0\\1&0&1\end{pmatrix}, M_T=\begin{pmatrix}1&0\\1&-1\\0&1\end{pmatrix}, M_{S\circ T}=\begin{pmatrix}-1&2\\1&1\end{pmatrix}$$
7. Let T_1 and T_2 represent linear maps on $\mathbb{R}^2$. If T_1 represents reflection about x- axis and T_2 represents expansion in the direction of y-axis by a factor of 2, write the linear transformation $T_1\circ T_2$ and $T_2\circ T_1$.
8. Let T_1 and T_2 represent linear maps on $\mathbb{R}^2$. If T_1 represents reflection about y- axis and T_2 represents expansion in the direction of x-axis by a factor of 2, write the linear transformation $T_1\circ T_2$ and $T_2\circ T_1$.
9. Draw the image of the unit square with corner points (0,0), (1,0), (1,1) and (0,1) under the map $T=T_1\circ T_2$ where T_1 is rotation by an angle $\frac{\pi}{4}$ and T_2 is reflection about the line x=0.

10. Show that $\mathrm{T}:R^2\to R^2$ be defined by $\mathrm{T}(x,y)=(x+y,x-y)$ is invertible.

11. Show that $T: R^3 \to R^3$ be defined by $T(x,y,z)=(x-y, x+z, x+y+2z)$ is invertible.
12. Determine whether $T: R^3 \to R^3$ be defined by $T(x,y,z)=(x+y+z, y, x+z)$ is invertible. Ans: T is not invertible.
13. Show that $T: R^3 \to R^3$ defined by $T(x,y,z)=(2x-y+z, x+y, 3x+y+z)$ is invertible. Find the matrix associated with T^{-1}.
14. Show that $T: R^3 \to R^3$ defined by $T(x,y,z)=(3x-2z, y, 3x+4y)$ is invertible.
15. Let $T: R^2 \to R^2$ for which $T(3,1)=(2,-4)$ and $T(1,1)=(0,2)$. Find an explicit formula for $T(x,y)$. Also find if T is invertible. If it is invertible, find the matrix associated with T^{-1}.

    Ans: $T(x,y)=(\mathrm{x}-y, 5-3x)$. T is invertible, $T^{-1}(x,y)=\frac{1}{2}(5x+y, 3x+y)$.

Eigenvalues and eigenvectors: Symmetric, skew-symmetric and orthogonal matrices

1. Find eigenvalues and eigenvectors of $A=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}$

   [Ans. -1,-6 and $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\-1\end{bmatrix}$]

2. Find eigenvalues and eigenvectors of $A=\begin{bmatrix}3 & 1 & 4\\ 0 & 2 & 6\\ 0 & 0 & 5\end{bmatrix}$

   [Ans. 2,3,5 and $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}3\\2\\1\end{bmatrix}$]

3. Find eigenvalues and eigenvectors of $A=\begin{bmatrix}1 & 2 & 2\\ 0 & 2 & 1\\ -1 & 2 & 2\end{bmatrix}$

   [Ans. 1,2,2 and $\begin{bmatrix}1\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}$]

4. Find eigenvalues and eigenvectors of $A=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & -3 & 3\end{bmatrix}$

   [Ans. 1,1,1 and $\begin{bmatrix}1\\1\\1\end{bmatrix}$]

5. Find eigenvalues and eigenvectors of $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$

   [Ans. 5,-3,-3 and $\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}3\\0\\1\end{bmatrix}$]

6. Find eigenvalues and eigenvectors of $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$

   [Ans. -2,3,6 and $\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}$]

7. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

[Ans. 1,1,1 and $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$]

8. Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & -2 & -5 \\ 4 & -1 & -5 \\ -2 & -1 & -3 \end{bmatrix}$

[Ans. 5,2,2 and $\begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$]

9. If $A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix}$ then find eigen values of $4A^{-1} + 3A + 2I$. [Ans. 9, 15]

10. If $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ then find eigen values and eigen vectors of $A^3 + I$.

[Ans: 2, 2, 126 and $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$]

11. If $A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{bmatrix}$ then find eigen values and eigen vectors of A^2.

[Ans: 2, 4, 4 and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$]

12. If the product of two eigen values of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16, find the third eigen value. [Ans. 2]

13. Determine the algebraic multiplicity and geometric multiplicity for $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

[Ans. For $\lambda = 1$, A.M.=2 & G.M.=2; For $\lambda = 3$, A.M.=1 & G.M.=1]

14. Find Eigenvalues and eigenvectors for following Symmetric matrix

a. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 5 \\ 0 & 5 & 4 \end{bmatrix}$ [Ans. $\lambda = -1, 1, 9$ and $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$]

b. $A = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$ [Ans. $\lambda = 0, 0, 9$ and $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$]

c. $A=\begin{bmatrix}3&2&4\\2&0&2\\4&2&3\end{bmatrix}$ [Ans. $\lambda=-1,-1,8$ and $\begin{bmatrix}-1\\2\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}2\\1\\2\end{bmatrix}$]

15. Show that for given skew symmetric matrix $A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, the eigenvalues are purely imaginary.

16. Find eigenvalues of the skew symmetric matrix, $A=\begin{bmatrix}0&9&-12\\-9&0&20\\12&-20&0\end{bmatrix}$.

[Ans. $0,25i,-25i$]

17. Find eigenvalues of the orthogonal matrix, $A=\begin{bmatrix}2/3&1/3&2/3\\-2/3&2/3&1/3\\1/3&2/3&2/3\end{bmatrix}$

[Ans. $-1,\frac{5+i\sqrt{11}}{6},\frac{5-i\sqrt{11}}{6}$]

18. Find Eigen values and Eigen vectors for the following matrices:

a. $\begin{bmatrix}9&-1\\5&7\end{bmatrix}$ [Ans. $\lambda=8+2i,8-2i$ and $\begin{bmatrix}1+2i\\5\end{bmatrix},\begin{bmatrix}1-2i\\5\end{bmatrix}$]

b. $\begin{bmatrix}-3&-2\\5&-1\end{bmatrix}$ [Ans. $\lambda=-2+3i,-2-3i$ and $\begin{bmatrix}2\\-1-3i\end{bmatrix},\begin{bmatrix}2\\-1+3i\end{bmatrix}$]

c. $\begin{bmatrix}6&-13\\1&0\end{bmatrix}$ [Ans. $\lambda=3+2i,3-2i$ and $\begin{bmatrix}3+2i\\1\end{bmatrix},\begin{bmatrix}3-2i\\1\end{bmatrix}$]

d. $\begin{bmatrix}3&0&0\\0&0&3\\0&-3&0\end{bmatrix}$ [Ans. $\lambda=3,3i,-3i$ and $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\i\end{bmatrix},\begin{bmatrix}0\\i\\1\end{bmatrix}$]

19. Find Eigen values and Eigen vectors for the following orthogonal matrix

$\begin{bmatrix}3/5&-4/5\\4/5&3/5\end{bmatrix}$ [Ans. $\lambda=\frac{3+4i}{5},\frac{3-4i}{5}$ and $\begin{bmatrix}i\\1\end{bmatrix},\begin{bmatrix}-i\\1\end{bmatrix}$]

Similar Matrices, Diagonalisation & Cayley-Hamilton Theorem

1. Show that the matrix $A=\begin{bmatrix}8 & -8 & -2\\ 4 & -3 & -2\\ 3 & -4 & 1\end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix. [Ans. $M=\begin{bmatrix}4 & 3 & 2\\ 3 & 2 & 1\\ 2 & 1 & 1\end{bmatrix}, D=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$]

2. Show that the matrix $A=\begin{bmatrix}4 & 2 & -2\\ -5 & 3 & 2\\ -2 & 4 & 1\end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix. [Ans. $M=\begin{bmatrix}2 & 1 & 0\\ 1 & 1 & 1\\ 4 & 2 & 1\end{bmatrix}, D=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 5\end{bmatrix}$]

3. Show that the matrix $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix. [Ans. $M=\begin{bmatrix}1 & -2 & 3\\ 2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}, D=\begin{bmatrix}5 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & -3\end{bmatrix}$]

4. Show that the matrix $A=\begin{bmatrix}-17 & 18 & -6\\ -18 & 19 & -6\\ -9 & 9 & 2\end{bmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix. [Ans. $M=\begin{bmatrix}2 & 1 & -1\\ 2 & 1 & 0\\ 1 & 0 & 3\end{bmatrix}, D=\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$]

5. Show that the matrix $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$ is not similar to a diagonal matrix.

6. Show that the matrix $A=\begin{bmatrix}1 & 3 & -2\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}$ is not similar to a diagonal matrix.

7. Apply Cayley-Hamilton theorem to $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$ and deduce that $A^8=625I$

Apply Cayley-Hamilton theorem to the following matrices and obtain the inverse

8. $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ [Ans. $A^{-1} = \dfrac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$]

9. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ [Ans. $A^{-1} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}$]

10. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ [Ans. $A^{-1} = \dfrac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$]

11. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ [Ans. $A^{-1} = -\dfrac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ -6 & 2 & 8 \\ 3 & -1 & -5 \end{bmatrix}$]

Find the characteristic equation of the matrix A and hence find A^{-1} and A^4.

12. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & -2 \end{bmatrix}$ [Ans. $A^{-1} = \begin{bmatrix} -1 & 2 & 1 \\ 2 & -2 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 3 & -10 \\ -12 & -2 & 23 \end{bmatrix}$]

13. $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ [Ans. $A^{-1} = \dfrac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$, $A^4 = \begin{bmatrix} 41 & 0 & -40 \\ 0 & 16 & 0 \\ -40 & 0 & 41 \end{bmatrix}$]

14. $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ [Ans. $A^{-1} = \dfrac{1}{4}\begin{bmatrix} 12 & 4 & 6 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$, $A^4 = \begin{bmatrix} -88 & -168 & -264 \\ 192 & 416 & 144 \\ 56 & 72 & 472 \end{bmatrix}$]

15. Find the characteristic equation of the matrix A given below and hence, find the matrix represented by $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$, where

$A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$. [Ans. $\begin{bmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{bmatrix}$]

16. Find the characteristic equation of the matrix A given below and hence, find the matrix represented by $A^9 - 6A^8 + 10A^7 - 3A^6 + A + I$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

[Ans. $\begin{bmatrix} 2 & 2 & 3 \\ -1 & 4 & 1 \\ 1 & 0 & 3 \end{bmatrix}$]

Exact Differential Equation, Non Exact Differential equation reducible to Exact form

Solve the following Differential equations:

1. $(y^2e^{xy^2}+4x^3)\,dx+(2xye^{xy^2}-3y^2)\,dy=0$ **(ans:** $e^{xy^2}+x^4-y^3=c$**)**

2. $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$ given that $y(0)=4$ **(ans:** $x+ye^{\frac{x}{y}}=4$**)**

3. $\left(2xy\cos x^2-2xy+1\right)dx+\left(\sin x^2-x^2\right)dy=0$ **(ans:** $y\sin x^2-x^2y+x=c$ **)**

4. $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}\left(x+xy^2\right)dy=0$ **(ans:** $x^6+3x^4y+x^4y^3=c$**)**

5. $(2x\log x-xy)dy+2ydx=0$ **(ans:** $2y\log x-\frac{y^2}{2}=c$**)**

6. $(xy^2-e^{\frac{1}{x^3}})dx-x^2ydy=0$ **(ans:** $\frac{e^{\frac{1}{x^3}}}{3}-\frac{y^2}{2x^2}=c$**)**

7. $\left(6x^2+4y^3+12y\right)dx+3x\left(1+y^2\right)dy=0$ **(ans:** $x^6+3x^4y+x^4y^3=c$**)**

8. $\left(2xy^4e^y+2xy^3+y\right)dx+\left(x^2y^4e^y-x^2y^2-3x\right)dy=0$ **(ans:** $x^2e^y+\frac{x^2}{y}+\frac{x}{y^3}=c$**)**

9. $y\left(xy+e^x\right)dx-e^x\,dy=0$ **(ans:** $\frac{x^2}{2}+\frac{e^x}{y}=c$**)**

10. $(3x^2y^4+2xy)dx+(2x^3y^3-x^2)dy=0$ **(ans:** $x^2+x^3y^3=cy$**)**

11. $y(x^2y+e^x)dx-e^xdy=0$ **(ans:** $\frac{x^3}{3}+\frac{e^x}{y}=c$**)**

12. $y(1+xy)dx+x(1+xy+x^2y^2)dy=0$ **(ans:** $\frac{1}{2x^2y^2}+\frac{1}{xy}-\log y=c$**)**

13. $y(xy+2x^2y^2)dx+x(xy+x^2y^2)dy=0$ **(ans:** $\log\left(x^2y\right)-\frac{1}{xy}=c$**)**

14. $y(x+y)dx-x\left(y-x\right)dy=0$ **(ans:** $\frac{1}{2}\log\left(xy\right)-\frac{y}{2x}=c$**)**

15. $(x^2-xy+y^2)dx-xy\,dy=0$ **(ans:** $\log\left(x-y\right)+\frac{y}{x}=c$**)**

16. $(3xy^2-y^3)dx-(2x^2y-xy^2)dy=0$ **(ans:** $\frac{x^3}{cy^2}=e^{\frac{-y}{x}}$ **)**

Linear Differential Equations, Bernoulli's Equation, Orthogonal trajectories

1. $x\cos x\dfrac{dy}{dx}+y(x\sin x+\cos x)=1$ **(ans:** $xy\sec x=\tan x+c$ **)**

2. $(1-x^2)\dfrac{dy}{dx}+2xy=x\sqrt{1-x^2}$ **(ans:** $y=\sqrt{1-x^2}+c\left(1-x^2\right)$ **)**

3. $x(x-1)\dfrac{dy}{dx}-(x-2)y=x^3(2x-1)$ **(ans:** $y=x^3+\dfrac{cx^2}{x-1}$ **)**

4. $(1+x+xy^2)dy+(y+y^3)dx=0$ **(ans:** $xy+\tan^{-1}y=c$ **)**

5. $\dfrac{dy}{dx}+(2x\tan^{-1}y-x^3)(1+y^2)=0$ **(ans:** $\tan^{-1}y=\dfrac{\left(x^2-1\right)}{2}+ce^{-x^2}$ **)**

6. $\dfrac{dy}{dx}=xy+y^2e^{(-x^2/2)}.\log x$ **(ans:** $\dfrac{-e^{x^2/2}}{y}=x\log x-x+c$ **)**

7. $\dfrac{dy}{dx}+\left(\dfrac{x}{1-x^2}\right)y=x\sqrt{y}$ **(ans:** $\sqrt{y}+\dfrac{1}{3}\left(1-x^2\right)=c\left(1-x^2\right)^{\frac{1}{4}}$ **)**

8. $xy\left(1+xy^2\right)\dfrac{dy}{dx}=1$ **(ans:** $\dfrac{1}{x}=2-y^2-ce^{-\frac{y^2}{2}}$ **)**

9. $4x^2y\dfrac{dy}{dx}=3x\left(3y^2+2\right)+\left(3y^2+2\right)^3$ **(ans:** $\dfrac{-1}{\left(3y^2+2\right)^2}=\dfrac{3}{7}x^{-1}+cx^{-9/2}$ **)**

10. $4xy\dfrac{dy}{dx}=(y^2+3)+x^3(y^2+3)^3$ **(ans:** $\dfrac{-x}{(y^2+3)^2}=\dfrac{x^4}{4}+c$ **)**

11. Find the orthogonal trajectories of each of the following family of curves:

a. $x-4y=c$

b. $x^2+y^2=c^2$

c. $x^2-y^2=c$

d. $y^2=cx^3$

e. $y=c\left(\sec x+\tan x\right)$

f. $y^2=\dfrac{x^3}{a-x}$

g. $y=cx^2$

12. Given $x^2+3y^2=cy$ find that member of the orthogonal trajectories which passes through point (1,2).

13. Find the constant e such that $y^3=c_1x$ and $x^2+ey^2=c_2$ are orthogonal to each other.

Second order linear differential equations with variable coefficients, Method of variation of parameters

Q1. Solve the following linear Differential Equation using operator method:

1. $\left(D^2+4D+4\right)y=\sin x$ [Ans: $y=\left(c_1+c_2x\right)e^{-2x}+\dfrac{3\sin x-4\cos x}{25}$]
2. $\left(D^2+4\right)y=5x^2+\sin 2x$ [Ans: $y=\left(C_1\cos 2x+C_2\sin 2x\right)+\frac{5}{8}\left(2x^2-1\right)-\frac{x\cos 2x}{4}$]
3. $\left(D^2+3D+2\right)y=x+x^2$ [Ans: $y=c_1e^{-x}+c_2e^{-2x}+\dfrac{x^2}{2}-x+1$]
4. $\left(D^2-D\right)y=2x+1+4\cos x+2e^x$ [Ans: $y=c_1+c_2e^{-x}-xe^x-x^2-3x-2\sin x-2\cos x$]
5. $\left(D^2-1\right)y=e^x\left(1+x\right)^2$ [Ans: $y=c_1e^x+c_2e^{-x}+\dfrac{xe^x}{12}\left(3+3x+2x^2\right)$]
6. $\left(D^2+D-6\right)y=e^{2x}\sin 3x$ [Ans: $y=c_1e^{2x}+c_2e^{-3x}-\dfrac{e^{2x}}{102}\left(5\cos 3x+3\sin 3x\right)$]

Q.2 Solve by using method of undetermined coefficients:

1. $\left(D^2+2D+4\right)y=2x^2+3e^{-x}$

2. $\left(D^2+1\right)y=\sin x$

3. $\left(D^2+1\right)y=2\cos x$

4. $\left(D^2-5D+6\right)y=e^{3x}+\sin x$

5. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+3y=x^3+\cos x$

6. $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}=e^x\sin x$

Q3.Solve the following by method of variation of parameters :

1. $(D^2+1)y=\tan x$ [**Ans.:** $y=C_1\cos x+C_2\sin x-\cos x[\log(\sec x+\tan x)]$]
2. $\left(D^2+1\right)y=\operatorname{cosec} x\cot x$ [Ans.: $C_1\sin x+C_2\cos x-\cos x\log(\sin x)-\cos x-x\sin x$]
3. $\dfrac{d^2y}{dx^2}+y=\dfrac{1}{1+\sin x}$ [**Ans.:** $y=C_1\cos x+C_2\sin x-1-x\cos x+\sin x\log(1+\sin x)+\sin x$]
4. $(D^2+1)y=3x-8\cot x$ [**Ans.:** $y=C_1\cos x+C_2\sin x-8\sin x\log(\operatorname{cosec} x-\cot x)+3x$]
5. $(D^2-1)y=\dfrac{2}{1+e^x}$ [Ans.: $y=C_1e^x+C_2e^{-x}-1+(e^x-e^{-x})\log(1+e^x)$]

Cauchy-Euler equation; Power series solutions, Legendre polynomials

Q1. Solve the following linear Differential equation:

1. $x^2\dfrac{d^2y}{dx^2}+2x.\dfrac{dy}{dx}-20y=(x+1)^2$ $\quad\left[y=C_1x^4+C_2x^{-5}-\dfrac{1}{14}x^2-\dfrac{1}{9}x-\dfrac{1}{20}\right]$

2. $x^2\dfrac{d^2y}{dx^2}-2x.\dfrac{dy}{dx}-4y=x^2+2\log x$ $\quad\left[y=C_1x^4+\dfrac{C_2}{x}-\dfrac{1}{6}x^2-\dfrac{1}{2}\left[\log x-\dfrac{3}{4}\right]\right]$

3. $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+5y=x^2\sin(\log x)$ $\quad\left[y=x^2\left[C_1\sin(\log x)+C_2\cos(\log x)\right]-\dfrac{x^2}{2}\log x\cos(\log x)\right]$

4. $x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+4y=\cos\{\log(x)\}+x\sin\{\log(x)\}$

   $\left[\textbf{Ans.}: y=x\left[C_1\cos(\sqrt{3}\log x)+C_2\sin(\sqrt{3}\log x)\right]+\dfrac{x}{2}\sin(\log x)+\dfrac{1}{13}[3\cos(\log x)-2\sin(\log x)]\right]$

5. $x^2\dfrac{d^2y}{dx^2}+x.\dfrac{dy}{dx}-4y=x^3$ $\quad\left[\textbf{Ans.}: y=C_1x^2+C_2\left(\dfrac{1}{x^2}\right)+\dfrac{1}{5}x^3\right]$

Q2. Find the power series solution in powers of x of the differential equation

1. $xy'-(x+2)y-2x^2-2x=0$ $\quad$ Ans: $y=2x+c_2x^2e^x$

2. $y''+xy'+(x^2+2)y=0$

   Ans: $y=c_0\left[1-x^2+\dfrac{1}{4}x^4+\ldots\ldots\right]+c_1\left[x-\dfrac{1}{2}x^3+\dfrac{3}{40}x^5+\ldots\ldots\right]$

3. $(x^2-1)y''+3xy'+xy=0 \quad with\ y(0)=4, y'(0)=6$

   Ans: $y=4+6x+\dfrac{11}{3}x^3+\dfrac{1}{2}x^4+\dfrac{11}{4}x^5\ldots.$

Partial differential equations

Form the partial differential equations by eliminating the arbitrary constants from the following:

1. $z=(x^2+a)(y^2+b)$
   [**Ans.** $pq=4xyz$]
2. $2z=(ax+y)^2+b$
   [**Ans.** $px+qy=q^2$]
3. $ax^2+by^2+cz^2=1$
   [**Ans.** $z(xp+yq)=1-z^2$]

Form the partial differential equations by eliminating the arbitrary functions from the following:

1. $z=F(x^2-y^2)$
   [**Ans.** $py+qx=0$]
2. $z=x+y+f(xy)$
   [**Ans.** $px-qy=x-y$]
3. $z=f\left(\dfrac{xy}{z}\right)$
   [**Ans.** $px=qy$]

Solutions of Partial Differential Equations by the Method of Direct Integration

1. $\dfrac{\partial^2 z}{\partial x \partial y}=\cos x \cos y$ [**Ans.** $z=\sin x \sin y+f(x)+\phi(y)$]
2. Solve $\dfrac{\partial^2 z}{\partial x^2}+z=0$, given that when $x=0,\ z=e^y\ and\ \dfrac{\partial z}{\partial x}=1$

   [**Ans.** $z=e^y\cos y+\sin x$]
3. Solve $\dfrac{\partial^2 z}{\partial y^2}=z$, given that when $y=0,\ z=e^x\ and\ \dfrac{\partial z}{\partial y}=e^{-x}$

   [**Ans.** $z=e^y\cosh x+e^{-y}\sinh x$]
4. Solve $\dfrac{\partial^2 z}{\partial x^2}=z$, given that when $x=0,\ z=e^y\ and\ \dfrac{\partial z}{\partial x}=e^{-y}$

   [**Ans.** $z=e^x\cosh y+e^{-x}\sinh y$]

Separation of variables method

1. Solve by the method of separation of variables $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$

   [**Ans.** $z = \left[Ae^{\left(1+\sqrt{1+k}\right)}x + Be^{\left(1-\sqrt{1+k}\right)}x \right] e^{-ky}$]

2. Solve by the method of separation of variables $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, *where* $u(0, y) = 8e^{-3y}$

   [**Ans.** $u = 8e^{-12-3y}$]

3. Solve by the method of separation of variables $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$, *where* $u(x, 0) = 4e^{-x}$

   [**Ans.** $u = 3e^{-5x-3y} + 2e^{-3x-2y}$]