CS 591, Lecture 11
Data Analytics: Theory and Applications
Boston University

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ℓ_1 heavy hitters : State-of-the-art

Larsen, Nelson, Nguyen, Thorup [Larsen et al., 2016]
Another suggestion for the class project (contains a cool graph clustering problem, probably of independent interest)

Today we will use Jelani'sexposition for Count-Min sketch from 2016 TUM Summer School.

ℓ_1 point queries

Setting: Strict Turnstile

- $x \in \mathbb{R}^n$
- $x \leftarrow 0$ initially
- At step i we see an update (i, Δ) which causes the change

$$x_i \leftarrow x_i + \Delta.$$

- Δ can be negative, $x_i \geq 0$ at all times for all $i \in [n]$
  When x_is can also be negative this is called the general turnstile model.

- $\textsc{Query}$(i): Return value $\tilde{x}_i$ in the range $x_i \pm \varepsilon \cdot \|x\|_1$.

Heavy Hitters

- HeavyHitter: Return a set $L \subseteq [n]$ such that
  1. $|x_i| \geq \varepsilon \|x\|_1 \Rightarrow i \in L$
  2. $|x_i| < \frac{\varepsilon}{2} \|x\|_1 \Rightarrow i \notin L$

Today, we present the CountMin sketch [Cormode and Muthukrishnan, 2005], which solves ℓ_1 point query in the general turnstile model.

- C code available by G. Cormode

Python implementation

```
from streamlib import CountMin
cm = CountMin()
cm.processBatch([0, 0,1, 1, 1, 2,4])
for i in xrange(5):
  print  i,'␣', cm.estimate(i)
```

Observation

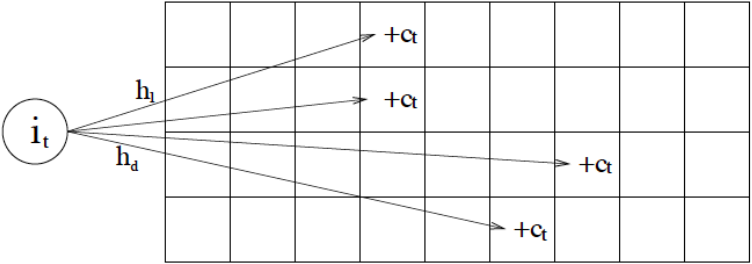
Claim: If we can solve point query in small space then we can solve heavy hitters in small space as well (though not necessarily efficient run-time).

- Run point query with $\varepsilon/4$ on each $i \in [n]$
- Output the set of indices i for which we had large estimate of x_i, i.e. at least $(3\varepsilon/4)\|x\|_1$

Count-Min sketch

1. We store hash functions $h_1, \ldots h_L : [n] \to [t]$, each chosen independently from a 2-wise independent family.
2. We store counters $C_{a,b}$ for $a \in [s]$, $b \in [t]$ with $s = \lceil 2/\varepsilon \rceil$, $t = \lceil \log_2(1/\delta) \rceil$.
3. Upon an update (i, Δ), we add Δ to all counters $C_{a,h_a(i)}$ for $a = 1, \ldots, s$.
4. To answer *query*(i), we output $\min_{1 \le a \le s} C_{a,h_a(i)}$.

Count-Min sketch



Source: Count-Min Sketch by G. Cormode

Note that our total memory consumption, in words is $m = O(st) = O(\varepsilon^{-1} \log(1/\delta))$.

Count-Min sketch

Claim: $query(i) = x_i \pm \varepsilon\|x\|_1$ w.p $\geq 1 - \delta$.

Proof: Fix i, let $Z_j = 1$ if $h_r(j) = h_r(i)$, $Z_j = 0$ otherwise. Now note that for any $r \in [s]$, $C_{r,h_r(i)} = x_i + \underbrace{\sum_{j \neq i} x_j Z_j}_{E}$. We have $\mathbb{E}(E) = \sum_{j \neq i} |x_j| \mathbb{E} Z_j = \sum_{j \neq i} |x_j|/t \leq \varepsilon/2 \cdot \|x\|_1$. Thus by Markov's inequality, $\mathbb{P}(E > \varepsilon\|x\|_1) < 1/2$. Thus by independence of the s rows of the CountMin sketch, $\mathbb{P}(\min_r C_{r,h_r(i)} > x_i + \varepsilon\|x\|_1) < 1/2^L = \delta$.

Count-Min sketch

Theorem

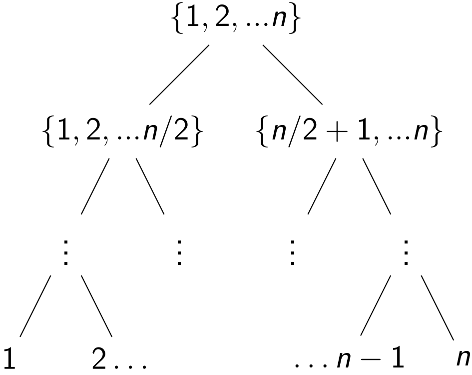
There is an algorithm solving the ℓ_1 ε-heavy hitter problem in the strict turnstile model with failure probability δ, space $O(\varepsilon^{-1}\log(n/\delta))$, update time $O(\log(n/\delta))$, and query time $O(n\log(n/\delta))$.

Proof.

We can instantiate a point query data structure with failure probability δ/n. Then we point query every $i \in [n]$ and include in our outpust list L only those i for which query returned a value at least $(3\varepsilon/4)\|x\|_1$. $\square$

Question: Can we improve the query time?

Count-Min sketch: Speeding up Query Time



Count-Min sketch: Speeding up Query Time

- Tree has $1 + \lg n$ levels
- We store in memory is $1 + \lg n$ CM sketches, one per level
- Upon an update, we feed that update to the appropriate coordinate at the CM sketch at every level.

  For α-HHs, and final failure probability δ, each CM sketch has error parameter $\varepsilon = \alpha/4$ and failure probabliity $\eta = \delta\alpha/(4 \lg n)$.
- Insight: The value at any ancestor of a node is at least as big as the value at that node.

Count-Min sketch: Speeding up Query Time

- Insight: The value at any ancestor of a node is at least as big as the value at that node.
- There can only be at most $2/\alpha$ indices that are $\alpha/2$ heavy hitters
- We move down the tree starting from the root (definitely a HH)
- At each level j of the tree, we keep track of a list L_j of heavy hitters at that level (definitely anything that is α-HH, nothing below $\frac{\alpha}{2} - HH$.

Count-Min sketch: Speeding up Query Time

- For any node in L_j we point query its two children.
- If a child has point query output at least $(3\alpha/4)\|x\|_1$, we include it in L_{j+1}.
- Finally, our final output list L is simply the list corresponding to the bottom-most level of the tree.

Count-Min sketch: Speeding up Query Time

For failure probability δ:

- Words of space: $O(\varepsilon^{-1} \lg n \lg((\lg n)/(\alpha\delta)))$
- Update time: $O(\lg n \lg(1/\eta)) = O(\lg n \lg((\lg n)/(\alpha\delta)))$
- Query time: $O(\varepsilon^{-1} \lg n \lg((\lg n)/(\alpha\delta)))$

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For failure probability δ

- Words of space: $O(\varepsilon^{-1} \lg(n/\delta))$
- Update time: $O(\lg(n/\delta))$
- Query time: $O(\varepsilon^{-1} \lg(n/\delta)\, poly(\lg n))$

references I

Cormode, G. and Muthukrishnan, S. (2005).
An improved data stream summary: the count-min sketch and its applications.
J. Algorithms, 55(1):58–75.

Larsen, K. G., Nelson, J., Nguyễn, H. L., and Thorup, M. (2016).
Heavy hitters via cluster-preserving clustering.
In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science (FOCS)*.