Robin Hartshorne

Algebraic Geometry

Springer

Preface

This book provides an introduction to abstract algebraic geometry using the methods of schemes and cohomology. The main objects of study are algebraic varieties in an affine or projective space over an algebraically closed field; these are introduced in Chapter I, to establish a number of basic concepts and examples. Then the methods of schemes and cohomology are developed in Chapters II and III, with emphasis on applications rather than excessive generality. The last two chapters of the book (IV and V) use these methods to study topics in the classical theory of algebraic curves and surfaces.

The prerequisites for this approach to algebraic geometry are results from commutative algebra, which are stated as needed, and some elementary topology. No complex analysis or differential geometry is necessary. There are more than four hundred exercises throughout the book, offering specific examples as well as more specialized topics not treated in the main text. Three appendices present brief accounts of some areas of current research.

This book can be used as a textbook for an introductory course in algebraic geometry, following a basic graduate course in algebra. I recently taught this material in a five-quarter sequence at Berkeley, with roughly one chapter per quarter. Or one can use Chapter I alone for a short course. A third possibility worth considering is to study Chapter I, and then proceed directly to Chapter IV, picking up only a few definitions from Chapters II and III, and *assuming* the statement of the Riemann–Roch theorem for curves. This leads to interesting material quickly, and may provide better motivation for tackling Chapters II and III later.

The material covered in this book should provide adequate preparation for reading more advanced works such as Grothendieck [EGA], [SGA], Hartshorne [5], Mumford [2], [5], or Shafarevich [1].

Acknowledgements

In writing this book, I have attempted to present what is essential for a basic course in algebraic geometry. I wanted to make accessible to the nonspecialist an area of mathematics whose results up to now have been widely scattered, and linked only by unpublished "folklore." While I have reorganized the material and rewritten proofs, the book is mostly a synthesis of what I have learned from my teachers, my colleagues, and my students. They have helped in ways too numerous to recount. I owe especial thanks to Oscar Zariski, J.-P. Serre, David Mumford, and Arthur Ogus for their support and encouragement.

Aside from the "classical" material, whose origins need a historian to trace, my greatest intellectual debt is to A. Grothendieck, whose treatise [EGA] is the authoritative reference for schemes and cohomology. His results appear without specific attribution throughout Chapters II and III. Otherwise I have tried to acknowledge sources whenever I was aware of them.

In the course of writing this book, I have circulated preliminary versions of the manuscript to many people, and have received valuable comments from them. To all of these people my thanks, and in particular to J.-P. Serre, H. Matsumura, and Joe Lipman for their careful reading and detailed suggestions.

I have taught courses at Harvard and Berkeley based on this material, and I thank my students for their attention and their stimulating questions.

I thank Richard Bassein, who combined his talents as mathematician and artist to produce the illustrations for this book.

A few words cannot adequately express the thanks I owe to my wife, Edie Churchill Hartshorne. While I was engrossed in writing, she created a warm home for me and our sons Jonathan and Benjamin, and through her constant support and friendship provided an enriched human context for my life.

For financial support during the preparation of this book, I thank the Research Institute for Mathematical Sciences of Kyoto University, the National Science Foundation, and the University of California at Berkeley.

August 29, 1977
Berkeley, California

ROBIN HARTSHORNE

Introduction

The author of an introductory book on algebraic geometry has the difficult task of providing geometrical insight and examples, while at the same time developing the modern technical language of the subject. For in algebraic geometry, a great gap appears to separate the intuitive ideas which form the point of departure from the technical methods used in current research.

The first question is that of language. Algebraic geometry has developed in waves, each with its own language and point of view. The late nineteenth century saw the function-theoretic approach of Riemann, the more geometric approach of Brill and Noether, and the purely algebraic approach of Kronecker, Dedekind, and Weber. The Italian school followed with Castelnuovo, Enriques, and Severi, culminating in the classification of algebraic surfaces. Then came the twentieth-century "American" school of Chow, Weil, and Zariski, which gave firm algebraic foundations to the Italian intuition. Most recently, Serre and Grothendieck initiated the French school, which has rewritten the foundations of algebraic geometry in terms of schemes and cohomology, and which has an impressive record of solving old problems with new techniques. Each of these schools has introduced new concepts and methods. In writing an introductory book, is it better to use the older language which is closer to the geometric intuition, or to start at once with the technical language of current research?

The second question is a conceptual one. Modern mathematics tends to obliterate history: each new school rewrites the foundations of its subject in its own language, which makes for fine logic but poor pedagogy. Of what use is it to know the definition of a scheme if one does not realize that a ring of integers in an algebraic number field, an algebraic curve, and a compact Riemann surface are all examples of a "regular scheme of dimension one"? How then can the author of an introductory book indicate the inputs to algebraic geometry coming from number theory, commutative algebra, and complex analysis, and also introduce the reader to the main objects of study, which are algebraic varieties in affine or projective space, while at the same time developing the modern language of schemes and cohomology? What choice of topics will convey the meaning of algebraic geometry, and still serve as a firm foundation for further study and research?

My own bias is somewhat on the side of classical geometry. I believe that the most important problems in algebraic geometry are those arising from old-fashioned varieties in affine or projective spaces. They provide the geometric intuition which motivates all further developments. In this book, I begin with a chapter on varieties, to establish many examples and basic ideas in their simplest form, uncluttered with technical details. Only after that do I develop systematically the language of schemes, coherent sheaves, and cohomology, in Chapters II and III. These chapters form the technical heart of the book. In them I attempt to set forth the most important results, but without striving for the utmost generality. Thus, for example, the cohomology theory is developed only for quasi-coherent sheaves on noetherian schemes, since this is simpler and sufficient for most applications; the theorem of "coherence of direct image sheaves" is proved only for projective morphisms, and not for arbitrary proper morphisms. For the same reasons I do not include the more abstract notions of representable functors, algebraic spaces, étale cohomology, sites, and topoi.

The fourth and fifth chapters treat classical material, namely nonsingular projective curves and surfaces, but they use techniques of schemes and cohomology. I hope these applications will justify the effort needed to absorb all the technical apparatus in the two previous chapters.

As the basic language and logical foundation of algebraic geometry, I have chosen to use commutative algebra. It has the advantage of being precise. Also, by working over a base field of arbitrary characteristic, which is necessary in any case for applications to number theory, one gains new insight into the classical case of base field **C**. Some years ago, when Zariski began to prepare a volume on algebraic geometry, he had to develop the necessary algebra as he went. The task grew to such proportions that he produced a book on commutative algebra only. Now we are fortunate in having a number of excellent books on commutative algebra: Atiyah–Macdonald [1], Bourbaki [1], Matsumura [2], Nagata [7], and Zariski–Samuel [1]. My policy is to quote purely algebraic results as needed, with references to the literature for proof. A list of the results used appears at the end of the book.

Originally I had planned a whole series of appendices—short expository accounts of some current research topics, to form a bridge between the main text of this book and the research literature. Because of limited time and space only three survive. I can only express my regret at not including the others, and refer the reader instead to the Arcata volume (Hartshorne, ed. [1]) for a series of articles by experts in their fields, intended for the nonspecialist. Also, for the historical development of algebraic geometry let me refer to Dieudonné [1]. Since there was not space to explore the relation of algebraic geometry to neighboring fields as much as I would have liked, let me refer to the survey article of Cassels [1] for connections with number theory, and to Shafarevich [2, Part III] for connections with complex manifolds and topology.

Because I believe strongly in active learning, there are a great many exercises in this book. Some contain important results not treated in the main text. Others contain specific examples to illustrate general phenomena. I believe that the study of particular examples is inseparable from the development of general theories. The serious student should attempt as many as possible of these exercises, but should not expect to solve them immediately. Many will require a real creative effort to understand. An asterisk denotes a more difficult exercise. Two asterisks denote an unsolved problem.

See (I, §8) for a further introduction to algebraic geometry and this book.

Terminology

For the most part, the terminology of this book agrees with generally accepted usage, but there are a few exceptions worth noting. A *variety* is always irreducible and is always over an algebraically closed field. In Chapter I all varieties are quasi-projective. In (Ch. II, §4) the definition is expanded to include *abstract varieties,* which are integral separated schemes of finite type over an algebraically closed field. The words *curve, surface,* and *3-fold* are used to mean varieties of dimension 1, 2, and 3 respectively. But in Chapter IV, the word *curve* is used only for a nonsingular projective curve; whereas in Chapter V a *curve* is any effective divisor on a nonsingular projective surface. A *surface* in Chapter V is always a nonsingular projective surface.

A *scheme* is what used to be called a prescheme in the first edition of [EGA], but is called scheme in the new edition of [EGA, Ch. I].

The definitions of a *projective morphism* and a *very ample invertible sheaf* in this book are not equivalent to those in [EGA]—see (II, §4, 5). They are technically simpler, but have the disadvantage of not being local on the base.

The word *nonsingular* applies only to varieties; for more general schemes, the words *regular* and *smooth* are used.

Results from algebra

I assume the reader is familiar with basic results about rings, ideals, modules, noetherian rings, and integral dependence, and is willing to accept or look up other results, belonging properly to commutative algebra or homological algebra, which will be stated as needed, with references to the literature. These results will be marked with an A: e.g., Theorem 3.9A, to distinguish them from results proved in the text.

The basic conventions are these: All rings are commutative with identity element 1. All homomorphisms of rings take 1 to 1. In an integral domain or a field, $0 \neq 1$. A *prime ideal* (respectively, *maximal ideal*) is an ideal $\mathfrak{p}$ in a ring A such that the quotient ring $A/\mathfrak{p}$ is an integral domain (respectively, a field). Thus the ring itself is not considered to be a prime ideal or a maximal ideal.

A *multiplicative system* in a ring A is a subset S, containing 1, and closed under multiplication. The *localization* $S^{-1}A$ is defined to be the ring formed by equivalence classes of fractions a/s, $a \in A$, $s \in S$, where a/s and a'/s' are said to be *equivalent* if there is an $s'' \in S$ such that $s''(s'a - sa') = 0$ (see e.g. Atiyah–Macdonald [1, Ch. 3]). Two special cases which are used constantly are the following. If $\mathfrak{p}$ is a prime ideal in A, then $S = A - \mathfrak{p}$ is a multiplicative system, and the corresponding localization is denoted by $A_{\mathfrak{p}}$. If f is an element of A, then $S = \{1\} \cup \{f^n \mid n \geqslant 1\}$ is a multiplicative system, and the corresponding localization is denoted by A_f. (Note for example that if f is nilpotent, then A_f is the zero ring.)

References

Bibliographical references are given by author, with a number in square brackets to indicate which work, e.g. Serre, [3, p. 75]. Cross references to theorems, propositions, lemmas within the same chapter are given by number in parentheses, e.g. (3.5). Reference to an exercise is given by (Ex. 3.5). References to results in another chapter are preceded by the chapter number, e.g. (II, 3.5), or (II, Ex. 3.5).

CHAPTER I

Varieties

Our purpose in this chapter is to give an introduction to algebraic geometry with as little machinery as possible. We work over a fixed algebraically closed field k. We define the main objects of study, which are algebraic varieties in affine or projective space. We introduce some of the most important concepts, such as dimension, regular functions, rational maps, nonsingular varieties, and the degree of a projective variety. And most important, we give lots of specific examples, in the form of exercises at the end of each section. The examples have been selected to illustrate many interesting and important phenomena, beyond those mentioned in the text. The person who studies these examples carefully will not only have a good understanding of the basic concepts of algebraic geometry, but he will also have the background to appreciate some of the more abstract developments of modern algebraic geometry, and he will have a resource against which to check his intuition. We will continually refer back to this library of examples in the rest of the book.

The last section of this chapter is a kind of second introduction to the book. It contains a discussion of the "classification problem," which has motivated much of the development of algebraic geometry. It also contains a discussion of the degree of generality in which one should develop the foundations of algebraic geometry, and as such provides motivation for the theory of schemes.

2 Projective Varieties

To define projective varieties, we proceed in a manner analogous to the definition of affine varieties, except that we work in projective space.

Let k be our fixed algebraically closed field. We defined *projective n-space* over k, denoted $\mathbf{P}_k^n$, or simply $\mathbf{P}^n$, to be the set of equivalence classes of $(n + 1)$-tuples $(a_0, \ldots, a_n)$ of elements of k, not all zero, under the equivalence relation given by $(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$ for all $\lambda \in k$, $\lambda \neq 0$. Another way of saying this is that $\mathbf{P}^n$ as a set is the quotient of the set

$\mathbf{A}^{n+1} - \{(0,\ldots,0)\}$ under the equivalence relation which identifies points lying on the same line through the origin.

An element of $\mathbf{P}^n$ is called a point. If P is a point, then any $(n+1)$-tuple $(a_0,\ldots,a_n)$ in the equivalence class P is called a *set of homogeneous coordinates for P.*

Let S be the polynomial ring $k[x_0,\ldots,x_n]$. We want to regard S as a graded ring, so we recall briefly the notion of a graded ring.

A *graded ring* is a ring S, together with a decomposition $S = \bigoplus_{d\geqslant 0} S_d$ of S into a direct sum of abelian groups S_d, such that for any $d,e \geqslant 0$, $S_d \cdot S_e \subseteq S_{d+e}$. An element of S_d is called a *homogeneous element* of *degree d*. Thus any element of S can be written uniquely as a (finite) sum of homogeneous elements. An ideal $\mathfrak{a} \subseteq S$ is a *homogeneous ideal* if $\mathfrak{a} = \bigoplus_{d\geqslant 0}(\mathfrak{a} \cap S_d)$. We will need a few basic facts about homogeneous ideals (see, for example, Matsumura [2, §10] or Zariski–Samuel [1, vol. 2, Ch. VII, §2]). An ideal is homogeneous if and only if it can be generated by homogeneous elements. The sum, product, intersection, and radical of homogeneous ideals are homogeneous. To test whether a homogeneous ideal is prime, it is sufficient to show for any two *homogeneous* elements f,g, that $fg \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$.

We make the polynomial ring $S = k[x_0,\ldots,x_n]$ into a graded ring by taking S_d to be the set of all linear combinations of monomials of total weight d in $x_0,\ldots,x_n$. If $f \in S$ is a polynomial, we cannot use it to define a function on $\mathbf{P}^n$, because of the nonuniqueness of the homogeneous coordinates. However, if f is a homogeneous polynomial of degree d, then $f(\lambda a_0,\ldots,\lambda a_n) = \lambda^d f(a_0,\ldots,a_n)$, so that the property of f being zero or not depends only on the equivalence class of $(a_0,\ldots,a_n)$. Thus f gives a function from $\mathbf{P}^n$ to $\{0,1\}$ by $f(P) = 0$ if $f(a_0,\ldots,a_n) = 0$, and $f(P) = 1$ if $f(a_0,\ldots,a_n) \neq 0$.

Thus we can talk about the *zeros* of a homogeneous polynomial, namely $Z(f) = \{P \in \mathbf{P}^n | f(P) = 0\}$. If T is any set of homogeneous elements of S, we define the *zero set* of T to be

$$Z(T) = \{P \in \mathbf{P}^n | f(P) = 0 \text{ for all } f \in T\}.$$

If $\mathfrak{a}$ is a homogeneous ideal of S, we define $Z(\mathfrak{a}) = Z(T)$, where T is the set of all homogeneous elements in $\mathfrak{a}$. Since S is a noetherian ring, any set of homogeneous elements T has a finite subset $f_1,\ldots,f_r$ such that $Z(T) = Z(f_1,\ldots,f_r)$.

Definition. A subset Y of $\mathbf{P}^n$ is an *algebraic set* if there exists a set T of homogeneous elements of S such that $Y = Z(T)$.

Proposition 2.1. *The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.*

Proof. Left to reader (it is similar to the proof of (1.1) above).

Definition. We define the *Zariski topology* on $\mathbf{P}^n$ by taking the open sets to be the complements of algebraic sets.

Once we have a topological space, the notions of irreducible subset and the dimension of a subset, which were defined in §1, will apply.

Definition. A *projective algebraic variety* (or simply *projective variety*) is an irreducible algebraic set in $\mathbf{P}^n$, with the induced topology. An open subset of a projective variety is a *quasi-projective variety*. The *dimension* of a projective or quasi-projective variety is its dimension as a topological space.

If Y is any subset of $\mathbf{P}^n$, we define the *homogeneous ideal* of Y in S, denoted $I(Y)$, to be the ideal generated by $\{f \in S | f \text{ is homogeneous and } f(P) = 0 \text{ for all } P \in Y\}$. If Y is an algebraic set, we define the *homogeneous coordinate ring* of Y to be $S(Y) = S/I(Y)$. We refer to (Ex. 2.1–2.7) below for various properties of algebraic sets in projective space and their homogeneous ideals.

Our next objective is to show that projective n-space has an open covering by affine n-spaces, and hence that every projective (respectively, quasi-projective) variety has an open covering by affine (respectively, quasi-affine) varieties. First we introduce some notation.

If $f \in S$ is a linear homogeneous polynomial, then the zero set of f is called a *hyperplane*. In particular we denote the zero set of x_i by H_i, for $i = 0, \ldots, n$. Let U_i be the open set $\mathbf{P}^n - H_i$. Then $\mathbf{P}^n$ is covered by the open sets U_i, because if $P = (a_0, \ldots, a_n)$ is a point, then at least one $a_i \neq 0$, hence $P \in U_i$. We define a mapping $\varphi_i : U_i \to \mathbf{A}^n$ as follows: if $P = (a_0, \ldots, a_n) \in U_i$, then $\varphi_i(P) = Q$, where Q is the point with affine coordinates

$$\left(\frac{a_0}{a_i}, \ldots, \frac{a_n}{a_i}\right),$$

with a_i/a_i omitted. Note that φ_i is well-defined since the ratios a_j/a_i are independent of the choice of homogeneous coordinates.

Proposition 2.2. *The map φ_i is a homeomorphism of U_i with its induced topology to $\mathbf{A}^n$ with its Zariski topology.*

PROOF. φ_i is clearly bijective, so it will be sufficient to show that the closed sets of U_i are identified with the closed sets of $\mathbf{A}^n$ by φ_i. We may assume $i = 0$, and we write simply U for U_0 and $\varphi : U \to \mathbf{A}^n$ for φ_0.

Let $A = k[y_1, \ldots, y_n]$. We define a map α from the set S^h of homogeneous elements of S to A, and a map β from A to S^h. Given $f \in S^h$, we set $\alpha(f) = f(1, y_1, \ldots, y_n)$. On the other hand, given $g \in A$ of degree e, then

$x_0^e g(x_1/x_0, \ldots, x_n/x_0)$ is a homogeneous polynomial of degree e in the x_i, which we call $\beta(g)$.

Now let $Y \subseteq U$ be a closed subset. Let $\bar{Y}$ be its closure in $\mathbf{P}^n$. This is an algebraic set, so $\bar{Y} = Z(T)$ for some subset $T \subseteq S^h$. Let $T' = \alpha(T)$. Then straightforward checking shows that $\varphi(Y) = Z(T')$. Conversely, let W be a closed subset of $\mathbf{A}^n$. Then $W = Z(T')$ for some subset T' of A, and one checks easily that $\varphi^{-1}(W) = Z(\beta(T')) \cap U$. Thus φ and φ^{-1} are both closed maps, so φ is a homeomorphism.

Corollary 2.3. *If Y is a projective (respectively, quasi-projective) variety, then Y is covered by the open sets $Y \cap U_i$, $i = 0, \ldots, n$, which are homeomorphic to affine (respectively, quasi-affine) varieties via the mapping φ_i defined above.*

EXERCISES

2.1. Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbf{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$. [*Hint*: Interpret the problem in terms of the affine $(n + 1)$-space whose affine coordinate ring is S, and use the usual Nullstellensatz, (1.3A).]

2.2. For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(i) $Z(\mathfrak{a}) = \varnothing$ (the empty set);
(ii) $\sqrt{\mathfrak{a}}$ = either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
(iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

2.3. (a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.
(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.
(c) For any two subsets Y_1, Y_2 of $\mathbf{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.
(d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \varnothing$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.
(e) For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \bar{Y}$.

2.4. (a) There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbf{P}^n$, and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note*: Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S.
(b) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.
(c) Show that $\mathbf{P}^n$ itself is irreducible.

2.5. (a) $\mathbf{P}^n$ is a noetherian topological space.
(b) Every algebraic set in $\mathbf{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

2.6. If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint*: Let $\varphi_i: U_i \to \mathbf{A}^n$ be the homeomorphism of (2.2), let Y_i be the affine variety $\varphi_i(Y \cap U_i)$, and let $A(Y_i)$ be its affine coordinate ring.

Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.]

2.7. (a) $\dim \mathbf{P}^n = n$.
(b) If $Y \subseteq \mathbf{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \bar{Y}$. [*Hint*: Use (Ex. 2.6) to reduce to (1.10).]

2.8. A projective variety $Y \subseteq \mathbf{P}^n$ has dimension $n - 1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbf{P}^n$.

2.9. *Projective Closure of an Affine Variety.* If $Y \subseteq \mathbf{A}^n$ is an affine variety, we identify $\mathbf{A}^n$ with an open set $U_0 \subseteq \mathbf{P}^n$ by the homeomorphism φ_0. Then we can speak of $\bar{Y}$, the closure of Y in $\mathbf{P}^n$, which is called the *projective closure* of Y.
(a) Show that $I(\bar{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).
(b) Let $Y \subseteq \mathbf{A}^3$ be the twisted cubic of (Ex. 1.2). Its projective closure $\bar{Y} \subseteq \mathbf{P}^3$ is called the *twisted cubic curve* in $\mathbf{P}^3$. Find generators for $I(Y)$ and $I(\bar{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\bar{Y})$.

2.10. *The Cone Over a Projective Variety* (Fig. 1). Let $Y \subseteq \mathbf{P}^n$ be a nonempty algebraic set, and let $\theta: \mathbf{A}^{n+1} - \{(0, \ldots, 0)\} \to \mathbf{P}^n$ be the map which sends the point with affine coordinates $(a_0, \ldots, a_n)$ to the point with homogeneous coordinates $(a_0, \ldots, a_n)$. We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0, \ldots, 0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbf{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \ldots, x_n]$.
(b) $C(Y)$ is irreducible if and only if Y is.
(c) $\dim C(Y) = \dim Y + 1$.
Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbf{P}^{n+1}$. This is called the *projective cone* over Y.

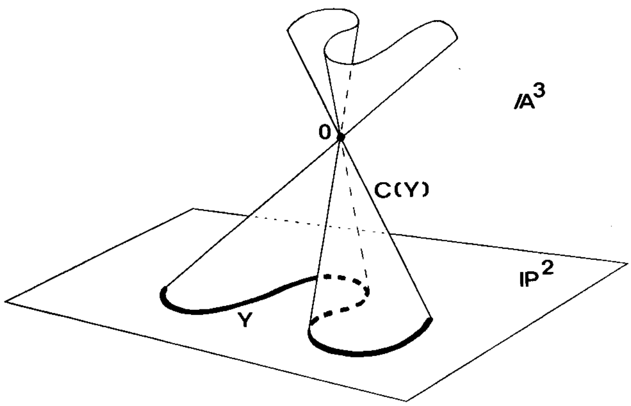

Figure 1. The cone over a curve in $\mathbf{P}^2$.

2.11. *Linear Varieties in* $\mathbf{P}^n$. A hypersurface defined by a linear polynomial is called a *hyperplane.*

(a) Show that the following two conditions are equivalent for a variety Y in $\mathbf{P}^n$:

(i) $I(Y)$ can be generated by linear polynomials.

(ii) Y can be written as an intersection of hyperplanes.

In this case we say that Y is a *linear variety* in $\mathbf{P}^n$.

(b) If Y is a linear variety of dimension r in $\mathbf{P}^n$, show that $I(Y)$ is minimally generated by $n - r$ linear polynomials.

(c) Let Y,Z be linear varieties in $\mathbf{P}^n$, with $\dim Y = r, \dim Z = s$. If $r + s - n \geqslant 0$, then $Y \cap Z \neq \varnothing$. Furthermore, if $Y \cap Z \neq \varnothing$, then $Y \cap Z$ is a linear variety of dimension $\geqslant r + s - n$. (Think of $\mathbf{A}^{n+1}$ as a vector space over k, and work with its subspaces.)

2.12. *The d-Uple Embedding.* For given $n,d > 0$, let $M_0,M_1,\ldots,M_N$ be all the monomials of degree d in the $n + 1$ variables $x_0,\ldots,x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d : \mathbf{P}^n \to \mathbf{P}^N$ by sending the point $P = (a_0,\ldots,a_n)$ to the point $\rho_d(P) = (M_0(a),\ldots,M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbf{P}^n$ in $\mathbf{P}^N$. For example, if $n = 1, d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ is a conic.

(a) Let $\theta : k[y_0,\ldots,y_N] \to k[x_0,\ldots,x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbf{P}^N$.

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

(c) Now show that ρ_d is a homeomorphism of $\mathbf{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbf{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

2.13. Let Y be the image of the 2-uple embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$. This is the *Veronese surface.* If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbf{P}^5$ such that $V \cap Y = Z$.

2.14. *The Segre Embedding.* Let $\psi : \mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^N$ be the map defined by sending the ordered pair $(a_0,\ldots,a_r) \times (b_0,\ldots,b_s)$ to $(\ldots,a_ib_j,\ldots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding.* Show that the image of ψ is a *subvariety* of $\mathbf{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbf{P}^N$ be $\{z_{ij} | i = 0,\ldots,r, j = 0,\ldots,s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0,\ldots,x_r,y_0,\ldots,y_s]$ which sends z_{ij} to x_iy_j. Then show that $\operatorname{Im}\psi = Z(\mathfrak{a})$.]

2.15. *The Quadric Surface in* $\mathbf{P}^3$ (Fig. 2). Consider the surface Q (a *surface* is a variety of dimension 2) in $\mathbf{P}^3$ defined by the equation $xy - zw = 0$.

(a) Show that Q is equal to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

(b) Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\},\{M_t\}$, each parametrized by $t \in \mathbf{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \varnothing$; if $M_t \neq M_u$, $M_t \cap M_u = \varnothing$, and for all t,u, $L_t \cap M_u =$ one point.

(c) Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbf{P}^1 \times \mathbf{P}^1$ (where each $\mathbf{P}^1$ has its Zariski topology).

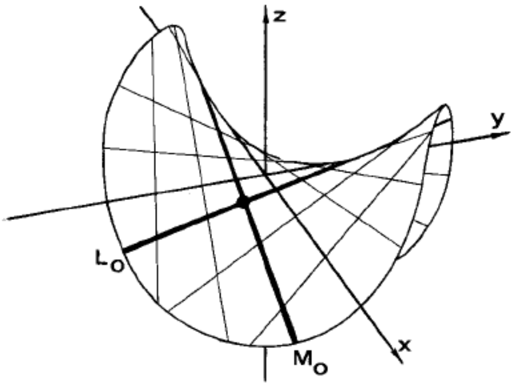

Figure 2. The quadric surface in $\mathbf{P}^3$.

2.16. (a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbf{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.

(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbf{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

2.17. *Complete intersections.* A variety Y of dimension r in $\mathbf{P}^n$ is a (*strict*) *complete intersection* if $I(Y)$ can be generated by $n - r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.

(a) Let Y be a variety in $\mathbf{P}^n$, let $Y = Z(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geqslant n - q$.

(b) Show that a strict complete intersection is a set-theoretic complete intersection.

*(c) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbf{P}^3$ (Ex. 2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees 2,3 respectively, such that $Y = H_1 \cap H_2$.

**(d) It is an unsolved problem whether every closed irreducible curve in $\mathbf{P}^3$ is a set-theoretic intersection of two surfaces. See Hartshorne [1] and Hartshorne [5, III, §5] for commentary.