In the sequence

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n$$

each individual element a_k is called a **term**. The k in a_k is called a **subscript** or **index**, m is the subscript of the **initial term**, and n is the subscript of the **final term**. The notation

$$a_m, a_{m+1}, a_{m+2}, \ldots$$

denotes an **infinite sequence**. An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

Example 1 Finding the Terms of Sequences Given by Explicit Formulas

Define sequences $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ by following wxplicit formulas:

$$a_k = \frac{k}{j+1} \qquad \text{for all integers } k \geq 1$$

$$b_i = \frac{i-1}{i} \qquad \text{for all integers } i \geq 2$$

Solution

Compute the first five terms of both sequences.

$a_1 = \frac{1}{1+1} = \frac{1}{2}$	$b_2 = \frac{2-1}{2} = \frac{1}{2}$
$a_2 = \frac{2}{2+1} = \frac{2}{3}$	$b_3 = \frac{3-1}{3} = \frac{2}{3}$
$a_3 = \frac{3}{3+1} = \frac{3}{4}$	$b_4 = \frac{4-1}{4} = \frac{3}{4}$
$a_4 = \frac{4}{4+1} = \frac{4}{5}$	$b_5 = \frac{5-1}{5} = \frac{4}{5}$
$a_5 = \frac{5}{5+1} = \frac{5}{6}$	$b_6 = \frac{6-1}{6} = \frac{5}{6}$

Example 2 An Alternating Sequence

Compute the first six terms of the sequence $c_0, c_1, \ldots$ defined as follows

$$c_j = (-1)^j \qquad \text{for all integers } j \geq 0$$

Solution

$$\begin{aligned}
c_0 &= (-1)^0 = 1 \\
c_1 &= (-1)^1 = -1 \\
c_2 &= (-1)^2 = 1 \\
c_3 &= (-1)^3 = -1 \\
c_4 &= (-1)^4 = 1 \\
c_5 &= (-1)^5 = -1
\end{aligned}$$

Even powers of -1 equal 1 and odd powers of -1 equal -1. The sequence oscilates between 1 and -1.

Example 6 Changing from Summation Notation to Expanded Form

Write the following summation in expanded form

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$$

Solution

$$\begin{aligned}
\sum_{i=0}^{n} &= \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1} \\
&= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1} \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}
\end{aligned}$$

Example 7 Changing from Expanded form to Summation Notation

Express the following using summation notation

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

Solution

The general term of this summation can be expressed as $\frac{k+1}{n+k}$ for integers k from 0 to n.

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=0}^{n} \frac{k+1}{n+k}$$

Example 8 Evaulating $a_1, a_2, a_3, \ldots, a_n$ for Small n
What is the value of the expression

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}$$

when $n = 1$? $n = 2$? $n = 3$?
Solution

$$\begin{aligned}
&\text{When } n=1, \text{the expression equals } \frac{1}{1\cdot 2}=\frac{1}{2}\\
&\text{When } n=2, \text{the expression equals } \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}\\
&\text{When } n=3, \text{the expression equals } \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}
\end{aligned}$$

A more mathematically precise definition of summation, called a **recursive definition** is the following: If m is any integer, then

$$\sum_{k=m}^{m} a_k = a_m \qquad \text{and} \qquad \sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n \qquad \text{for all integers } n > m$$

When solving , it is often useful to rewrite a summation using the recursive form of the definition, either by separating off the final term of a summation or by adding a final term to a summation.

Example 9 Separating Off a Final Term and Adding On a Final Term

(a) Rewrite $\sum_{i=1}^{n+1} \frac{1}{i^2}$ by separating off the final term.

(b) Write $\sum_{k=0}^{n} 2^k + 2^{n+1}$ as a single summation.

Solution

$$\text{(a)} \ \sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

$$\text{(b)} \ \sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

Example 10 A Telescoping Sum

Some sums can be transformed into telescoping sums, which then can be rewritten as a simple expression. Observe that

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}$$

Use the identity to find a simple expression for $\sum\limits_{k=1}^{n} \frac{1}{k(k+1)}$

Solution

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k(k+1)} &= \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= 1 - \frac{1}{n+1}
\end{aligned}$$

Example 13 Transforming a Sum by a Change of Variable

Transform the following summation by making the specific change of variable

$$\text{summation: } \sum_{k=0}^{6} \frac{1}{k+1} \qquad \text{change of variable: } j = k+1$$

Solution

First calculate the lower and upper limits of the new summation:

$$\begin{aligned}
&\text{When } k = 0, \qquad j = k+1 = 0+1 = 1 \\
&\text{When } k = 6, \qquad j = k+1 = 6+1 = 7
\end{aligned}$$

Thus the new sum goes from $j = 1$ to $j = 7$.

Next calculate the general term of the new summation. You will need to replace each occurrence of k by an expression in j:

$$\text{Since } j = k+1, \qquad \text{then } k = j-1$$

$$\text{Hence } \frac{1}{k+1} = \frac{1}{(j-1)+1} = \frac{1}{j}$$

Finally, put the steps together to obtain

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j}$$

it is legal to write

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{k=1}^{7} \frac{1}{k}$$

Example 14 When the Upper Limit Appears in the Expression to Be Summed

(a) Transform the following summation by making the specified change of variable.

$$\text{summation: } \sum_{k=1}^{n+1}\left(\frac{k}{n+k}\right) \qquad \text{change of variable: } j = k-1$$

(b) Transform the summation obtained in part (a) by changing all j's to k's

Solution

(a) When $k = 1$, then $j = k - 1 = 1 - 1 = 0$. (The new lower limit is 0.) When $k = n + 1$, then $j = k - 1 = (n + 1) - 1 = n$. (The new upper limit is n)

Since $j = k - 1$, then $k = j + 1$. Also note that n is a constant as far as the terms of the sum are concerned.

$$\frac{k}{n+k} = \frac{j+1}{n+(j+1)}$$

and so the general term of the new summation is

$$\frac{j+1}{n+(j+1)}$$

Therefore,

$$\sum_{k=1}^{n+1}\frac{k}{n+k} = \sum_{j=0}^{n}\frac{j+1}{n+(j+1)}$$

(b) Changing all the j's to k's in the right-hand side of equations gives

$$\sum_{j=0}^{n}\frac{j+1}{n+(j+1)} = \sum_{k=0}^{n}\frac{k+1}{n+(k+1)}$$

Combining equations and results

$$\sum_{k=1}^{n+1}\frac{k}{n+k} = \sum_{k=0}^{n}\frac{k+1}{n+(k+1)}$$

Decimal to Binary Conversion Using Repeated Division by 2

The input is a non-negative integer a. The aim of the algorithm is to produce a sequence of binary digits $r[0], r[1], r[2], \ldots, r[k]$ so that the binary representation of a is

$$(r[k]r[k-1]\cdots r[2]r[1]r[0])_2$$

That is

$$a = 2^k \cdot r[k] + 2^{k-1} \cdot r[k-1] + \cdots + 2^2 \cdot r[2] + 2^1 \cdot r[1] + 2^0 \cdot r[0]$$

Input: a [a non-negative integer]

Output: $r[0], r[1], r[2], \ldots, r[i-1]$—a sequence of integers

Algorithm Body:

```
q := a,i := 0
while (i = 0 or q ≠ 0)
      r[i] := q mod 2
      q := q div 2
      i := i + 1
end while
```

where `div` is the Division Algorithm on pg 219.