Defining sequences recursively requires both an equation, called a **recurrence relation**, that defines each later term in the sequence by reference to earlier terms and also one or more initial values for the sequence. Defining sequences recursively is similar to proving theorems by mathematical induction. The recurrence relation is like the inductive step and the initial conditions are like the basis step.

Example 1: Computing Terms of a Recursively Defined Sequence

Define a sequence $c_0, c_1, c_2, \ldots$ recursively as follows: For all integers $k \geq 2$

$$\begin{aligned} c_k &= c_{k-1} + kc_{k-2} + 1 \qquad &&\textbf{recurrence relation (1)} \\ c_0 &= 1 \qquad \text{and} \quad c_1 = 2 \qquad &&\textbf{initial conditions (2)} \end{aligned}$$

For c_2, c_3 and c_4

Solution

$$\begin{aligned} c_2 &= c_1 + 2c_0 + 1 \qquad &&\textbf{by substituting } k = 2 \textbf{ into (1)} \\ &= 2 + 2 \cdot 1 + 1 \qquad &&\textbf{since } c_1 = 2 \textbf{ and } c_0 = 1 \textbf{ by (2)} \end{aligned}$$

(3) Thus $c_2 = 5$

$$\begin{aligned} c_3 &= c_2 + 2c_1 + 1 \qquad &&\textbf{by substituting } k = 3 \textbf{ into (1)} \\ &= 5 + 3 \cdot 2 + 1 \qquad &&\textbf{since } c_2 = 5 \textbf{ and } c_1 = 2 \textbf{ by (2)} \end{aligned}$$

(4) Thus $c_3 = 12$

$$\begin{aligned} c_4 &= c_3 + 4c_2 + 1 \qquad &&\textbf{by substituting } k = 4 \textbf{ into (1)} \\ &= 12 + 4 \cdot 5 + 1 \qquad &&\textbf{since } c_3 = 12 \textbf{ by (4) and } c_2 = 5 \textbf{ by (3)} \end{aligned}$$

(5) Thus $c_4 = 33$

Example 2: Writing a Recurrence Relation in More than One Way

Let $s_0, s_1, s_2, \ldots$ be a sequence that satisfies the following recurrence relation:

$$\text{for all integers } k \geq 1, \qquad s_k = 3s_{k-1} - 1$$

Explain why the following statement is true

$$\text{for all integers } k \geq 0, \qquad s_{k+1} = 3s_k - 1$$

Note: Think of the recurrence relation as $s_{\bigcirc} = 3s_{\bigcirc - 1} - 1$, where any positive integer expression may be placed in the circle.

Solution

In informal language, the recurrence relation says that any form of the sequence equals 3 times the previous term minus 1. Now for any integer $k \geq 0$, the term previous to s_{k+1} is s_k. Thus for any integer $k \geq 0, s_{k+1} = 3s_{k-1}$

Example 4: Showing That a Sequence Given by an Explicit Formula Satisfies a Certain Recurrence Relation

For each integer $n \geq 1$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

a. Find C_1, C_2 and C_3

b. Show that this sequence satisfies the recurrence relation $C_k = \frac{4k-2}{k+1}C_{k-1}$ for all integers $k \geq 2$

Solution

a.

$$\begin{aligned} C_1 &= \frac{1}{2}\binom{2}{1} \\ &= \frac{1}{2} \cdot 2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} C_2 &= \frac{1}{3}\binom{4}{2} \\ &= \frac{1}{3} \cdot 6 \\ &= 2 \end{aligned}$$

$$\begin{aligned} C_3 &= \frac{1}{4}\binom{6}{3} \\ &= \frac{1}{4} \cdot 20 \\ &= 5 \end{aligned}$$

b. To obtain the kth and $(k-1)$st terms of the sequence, just substitute k and $k-1$ in place of n in the explicit formula for $C_1, C_2, C_3, \ldots$

$$\begin{aligned} C_k &= \frac{1}{k+1}\binom{2k}{k} \\ C_{k-1} &= \frac{1}{(k-1)+1}\binom{2(k-1)}{k-1} = \frac{1}{k}\binom{2k-2}{k-1} \end{aligned}$$

Then start with the right-hand side of the recurrence relation and transform it into the left-hand side: For each integer $k \geq 2$

$$
\begin{aligned}
\frac{4k-2}{k+1}C_{k-1} &= \frac{4k-2}{k+1}\left[\frac{1}{k}\binom{2k-2}{k-1}\right] && \textbf{by substituting} \\
&= \frac{2(2k-1)}{k+1}\cdot\frac{1}{k}\cdot\frac{(2k-2)!}{(k-1)!(2k-2-(k-1))!} && \textbf{by the formula for } n \textbf{ choose } r \\
&= \frac{1}{k+1}\cdot(2(2k-1))\cdot\frac{(2k-2)!}{(k(k-1)!)(k-1)!} && \textbf{by rearranging the factors} \\
&= \frac{1}{k+1}\cdot(2(2k-1))\cdot\frac{1}{k!(k-1)!\cdot(2k-2)!\cdot\frac{1}{2}\cdot\frac{1}{k}\cdot 2k} && \textbf{because } \tfrac{1}{}\cdot\tfrac{1}{k}\cdot 2k = 1 \\
&= \frac{1}{k+1}\cdot\frac{2}{2}\cdot\frac{1}{k!}\cdot\frac{1}{(k-1)!}\cdot\frac{1}{k}\cdot(2k)\cdot(2k-1)\cdot(2k-2)! && \textbf{by rearranging the factors} \\
&= \frac{1}{k+1}\cdot\frac{(2k)!}{k!k!} && \textbf{because } k(k-1)! = k! \textbf{ and } 2k\cdot(2k-2)! = (2k)! \\
&= \frac{1}{k+1}\binom{2k}{k} && \textbf{by the formula for } n \textbf{ choose } r \\
&= C_k && \textbf{by definition of } C_1, C_2, C_3 \ldots
\end{aligned}
$$