Suppose f and g are real-valued functions of a variable x.

1. If, for sufficiently large values of x, the values of $|f|$ are less than those of a multiple of $|g|$, then f is of ordee at most g, or $f(x)$ is $O(g(x))$.

2. If, for sufficiently large values of x, the values of $|f|$ are greater than those of a multiple of $|g|$, then f is of order at least g, or $f(x)$ is $\Omega(g(x))$.

3. If, for sufficiently large values of x, the values of $|f|$ are bounded above and below by those of multiples of $|g|$, then f is of order g, or $f(x)$ is $\Theta(g(x))$.

Definition

Let f and g be real-valued functions defined on the same set of nonnegative real numbers. Then

1. f **is of order at least** g, written $f(x)$ **is** $\Omega(g(x))$, if and only if, if there exist a positive real number A and a nonnegative real number a such that

$$A|g(x)| \leq |f(x)| \qquad \text{for all real numbers } x > a$$

2. f **is of order at most** g, written $f(x)$ **is** $O(g(x))$, if and only if, there exists a positive real number B and a nonnegative real number b such that

$$|f(x)| \leq B|g(x)| \qquad \text{for all real numbers } x > b$$

3. f **is of order** g, written $f(x)$ **is** $\Theta(g(x))$, if and only if, there exists a positive real number A, B, and a non-negative real number k such that

$$A|g(x)| \leq |f(x)| \leq B|g(x)| \qquad \text{for all real numbers } x > k$$

Example 1: Translating to Θ-Notation

Use Θ-Notation

$$10|x^6| \leq |17x^6 - 45x^3 + 2x + 8| \leq 30|x^6| \qquad \text{for all real numbers } x > 2$$

Solution

Let $A = 10, B = 30$, and $k = 2$. Then the statement translates to

$$A|x^6| \leq |17x^6 - 45x^3 + 2x| \leq B|x^6| \qquad \text{for all real numbers } x > k$$

so by definition of Θ-notation

$$17x^6 - 45x^3 + 2x + 8 \text{ is } \Theta(x^6)$$

Example 2: Translating to O- and Ω- Notations

a. Use Ω and O notations to express the statements

$$\text{(i)} \quad 15|\sqrt{x}| \leq \left|\frac{14\sqrt{x}(2x+9)}{x+1}\right| \qquad \text{for all real numbers } x > 0$$

$$\text{(ii)} \quad \left|\frac{15\sqrt{x}(2x+9)}{x+1}\right| \leq 45|\sqrt{x}| \qquad \text{for all real numbers } x > 7$$

b. Justify the statement:

$$\frac{15\sqrt{x}(2x+9)}{x+1} \text{ is } \Theta(\sqrt{x})$$

Solution

a. (i) Let $A = 15$ and $a = 0$. The given statement translates to

$$A|\sqrt{x} \leq \left|\frac{15\sqrt{x}(2x+9)}{x+1}\right| \qquad \text{for all real numbers } x > a$$

so by definition of Ω-notation

$$\frac{15\sqrt{x}(2x+9)}{x+1} \text{ is } \Omega(\sqrt{x})$$

(ii) Let $B = 45$ and $b = 7$. The given statement translates to

$$\left|\frac{15\sqrt{x}(2x+9)}{x+1}\right| \text{ is } O(\sqrt{x})$$

b. Let $A = 15, B = 45$ and let k be the larger of 0 and 7. Then when $x > k$, both inequalities in (i) and (ii) are satisfied, and so

$$A|\sqrt{x}| \leq \left|\frac{15\sqrt{x}(2x+9)}{x+1}\right| \leq B|\sqrt{x}| \qquad \text{for all real numbers } x > k$$

hence by definition of Θ-notation

$$\frac{15\sqrt{x}(2x+9)}{x+1} \text{ is } \Theta(\sqrt{x}$$

Orders of Power Functions

Observe that if

$$1 < x$$

then

$$x < x^2 \qquad \textbf{multiply both sides by } x$$

and so

$$x^2 < x^3 \qquad \textbf{multiply both sides by } x$$

For any rational number r and s

$$\text{if } x > 1 \text{ and } r < s, \text{ then } x^r < x^s$$

This property has the following consequence for orders
For any rational numbers r and s,

$$\text{if } r < s, \text{ then } x^r \text{ is } O(x^s)$$

Orders of Polynomial Functions

Example 3: A Polynomial Inequality

Show that for any real number x,

$$\text{if} x > 1 \quad \text{then} \quad 3x^3 + 2x + 7 \leq 12x^3$$

Solution

Suppose x is a real number and $x > 1$. Then by the orders of powers property ,

$$x < x^3 \quad \text{and} \quad 1 < x^3$$

Multiply the left-hand inquality by 2 and the right-hand inequality by 7 to get

$$2x < 2x^3 \quad \text{and} \quad 7 < 7x^3$$

Now add $3x^3 \leq 3x^3, 2x < 2x^3$, and $7 < 7x^3$ to obtain

$$3x^3 + 2x + 7 \leq 3x^3 + 2x^3 + 7x^3 = 12x^3$$

Example 4: Using the Definitions to Show That a Polynomial Function with Positive Coefficients Has a Cetain Order

Use the definitions of big-Omega, big-O, and big-Theta to show that $2x^4 + 3x^3 + 5$ is $\Theta(x^4)$

Solutions

Define functions f and g as follows. For all real numbers x

$$f(x) = 2x^4 + 3x^3 + 5$$
$$g(x) = x^4$$

Observe that for all real numbers $x > 0$

$$2x^4 \leq 2x^4 + 3x^3 + 5 \qquad \textbf{because } 3x^3 + 5 > 0 \textbf{ for } x > 0$$

and so

$$2|x^4| \leq |2x^4 + 3x^3 + 5| \qquad \textbf{because all terms on both sides of the inequality are positive}$$

Let $A = 2$ and $a = 0$. Then

$$A|x^4| \leq |2x^4 + 3x^3 + 5 \qquad \text{for all } x > a$$

and so by definition of Ω-notation, $2x^4 + 3x^3 + 5$ is $\Omega(x^4)$.
Also for $x > 1$

$$2x^4 + 3x^3 + h \leq 2x^4 + 3x^4 + 5x^4$$
$$2x^4 + 2x^3 + 5 \leq 10x^4$$
$$|2x^4 + 3x^3 + 5| \leq 10|x^4|$$

Let $B = 10$ and $b = 1$. Then

$$|2x^4 + 3x^3 + 5| \leq B|x^4| \qquad \text{for all } x > b$$

and so, by definition of O-notation, $2x^4 + 3x^3 + 5$ is $O(x^4)$.
Since $2x^4 + 3x^3 + 5$ is both $\Omega(x^4)$ and $O(x^4)$ By the Properties of O-, Ω-, and Θ-Notations

Example 5: A Big-O Approximation for a polynomial with Some Negative Coefficients

a. Use the definition of O-notation to show that $3x^3 - 1000x - 200$ is $O(x^3)$

b. Show that $3x^3 - 1000x - 200$ is $O(x^s)$ for all integers $s > 3$.

Solution

a. According to the triangle inequality for absolute value

$$|a + b| \leq |a| + |b| \qquad \text{for all real numbers } a \text{ and } b$$

If $-b$ is substituted in place of b, the result is

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$$
$$|a - b| \leq |a| + |b|$$

It follows that for all real numbers $x > 1$

$$|3x^3 - 1000x - 200| \leq |3x^3| + |1000x| + |200|$$
$$|3x^3 - 1000x - 200| \leq 3x^3 + 1000x + 200$$
$$|3x^3 - 1000x - 200| \leq 1203x^3$$
$$|3x^3 - 1000x - 200| \leq 1203|x^3|$$

Let $b = 1$ and $B = 1203$. Then

$$|3x^3 - 1000x - 200| \leq B|x^3| \qquad \text{for all real numbers } x > b$$

So, by definition of O-notation, $3x^3 - 1000x - 200$ is $O(x^3)$.

b. Suppose s is an integer with $s > 3$. By the order of polynomial property, $x^3 < x^s$ for all real numbers $x > 1$. So $B|x^3| < B|x^s|$ for all real numbers $x > b$ (because $b = 1$), and thus by part (a)

$$|3x^3 - 1000x - 200| \leq B|x^s| \qquad \text{for all real numbers } x > b$$

Hence, by definition of O-notation, $3x^3 - 1000x - 200$ is $O(x^s)$ for all integers $s > 3$.

Example 6: A Big-Omega Approximation for a Polynomial with Some Negative Coefficients

a. Use the definition of Ω-notation to show that $3x^3 - 1000x - 200$ is $\Omega(x^3)$.

b. Show that $3x^3 - 1000x - 200$ is $\Omega(x^r)$ for all integers $r < 3$.

Solution

a. To show that $3x^3 - 1000x - 200$ is $\Omega(x^3)$, you need to find numbers a and A so that $A|x^3| \leq |3x^3 - 1000x - 200|$ for all real numbers $x > a$

Choose a as follows: Add up the absolute values of the coeeficiencts of the lower order terms of $3x^3 - 1000x - 200$, divide by the absolute value of the highest-power term, and multiply the result by 2. The result is $a = 2(1000 + 200)/3$, which equals 800. A can be taken to be one-half of the absolute value of the highest power of the polynomial.

$$\begin{aligned}
x &> 800 \\
x &> 2\left(\frac{1000+200}{3}\right) \\
x &> \left(\frac{2\cdot 1000}{3}\right) + \left(\frac{2\cdot 200}{3}\right) \\
x &> \left(\frac{2\cdot 1000}{3}\cdot\frac{1}{x}\right) + \left(\frac{2\cdot 200}{3}\cdot\frac{1}{x^2}\right) \\
\frac{3}{2}x^3 &> 1000x + 200 \\
3x^3 - \frac{3}{2}x^3 &> 1000x + 200 \\
3x^3 - 1000x - 200 &> \frac{3}{2}x^3 \\
|3x^3 - 1000x - 200| &> \frac{3}{2}|x^3|
\end{aligned}$$

Let $A = \frac{3}{2}$ and let $a = 800$. Then

$$A|x^3| \leq |3x^3 - 1000x - 200| \qquad \text{for all real numbers } x > a$$

So, by definition of Ω-notations, $3x^3 - 100x - 200$ is $\Omega(x^3)$

b. Suppose r is an integer with $r < 3$. By the order ofpolynomials property , $x^r < x^3$ for all real numbers $x > 1$. So, since $a = 800 > 1$, $A|x^r| < A|x^3|$ for all real numbers $x > a$. Thus, by part (a)

$$A|x^r| \leq |3x^3 - 1000x - 200| \qquad \text{for all real numbers } x > a$$

Hence, by definition of Ω-notations, $3x^3 - 1000x - 200$ is $\Omega(x^r)$ for all integers $r < 3$.

Example 7: Calculating Polynomial Orders Using the Theorem on Polynomial Orders
Use the theorem on polynomial orders to find orders for the functions given by the following formulas.

a.

$$f(x) = 7x^5 + 5x^3 - x + 4 \qquad \text{for all real numbers } x$$

b.

$$g(x) = \frac{(x-1)(x+1)}{4} \qquad \text{for all real numbers } x$$

Solution
a. By direct application of the theorem on polynomial orders, $7x^5 + 5x^3 - x + 4$ is $\Theta(x^5)$

b.

$$\begin{aligned} g(x) &= \frac{(x-1)(x+1)}{4} \\ &= \frac{1}{4}(x^2 - 1) \\ &= \frac{1}{4}x^2 - \frac{1}{4} \end{aligned}$$

Thus $g(x)$ is $\Theta(x^2)$ by the theorem on polynomial orders.

Example 8: Showing That Two Power Functions Have Different Orders
Show that x^2 is not $O(x)$, and deduce that x^2 is not $\Theta(x)$.

Solution [argue by contradiction]
Suppose that x^2 is $O(x)$. By the supposition that x^2 is $O(x)$, there exist a positive real number B and a nonnegative real number b such that

$$|x| \le B|x| \qquad \text{for all real numbers } x > b \qquad (1)$$

Let x be a positive real number that is greater than both B and b. Then

$$\begin{aligned} x \cdot x &> B \cdot x \\ |x^2| &> B|x| \end{aligned}$$

Thus there is a real number $x > b$ such that

$$|x^2| > B|x|$$

This contradicts (1). Hence the supposition is false, and x^2 is not $O(x)$. By the Properties of O-,Ω-, and Θ- Notations theorem, if x^2 is $\Theta(x)$, then x^2 is $O(x)$. But x^2 is not $O(x)$, and thus x^2 is not $\Theta(x)$.

Orders for Functions of Integer Variables

It is traditional to use the symbol x to denote a real number variable, whereas n is used to represent an integer variable. Thus, given a statement of the form

$$f(n) \text{ is } \Theta(g(n))$$

we assume that f and g are functions defined on sets of integers. If it is true that

$$f(x) \text{ is } \Theta(g(x))$$

where f and g are functions defined for real numbers, then it is certainly true that $f(n)$ is $\Theta(g(n))$. The reason is that if $f(x)$ is $\Theta(g(x))$, then an inequality

$$A|g(x)| \leq |f(x)| \leq B|g(x)|$$

holds for all real numbers $x > k$. Hence, in particular, the inequality

$$A|g(n)| \leq |f(n)| \leq B|g(n)|$$

holds for all integers $n > k$

Example 9: An Order for the Sum of the First n Integers

Sums of the form $1 + 2 + 3 + \cdots + n$ arise in the analysis of computer algorithms such as selection sort. Show that for a positive integer variable n,

$$1 + 2 + 3 + \cdots + n \text{ is } \Theta(n^2)$$

Solution

By the formula for the sum of the first n integers, for all positive integers n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

But

$$\frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

And, by the theorem on polynomial orders,

$$\frac{1}{2}n^2 + \frac{1}{2}n \text{ is } \Theta(n^2)$$

Hence

$$1 + 2 + 3 + \cdots + n \text{ is } \Theta(n^2)$$

Extension to Functions Composed of Rational Power Functions

Consider a function of the form

$$\frac{(x^{3/2}+3)(x-2)^2}{x^{1/2}(2x^{1/2}+1)} = \frac{x^{7/2}-4x^{5/2}+4x^{3/2}+3x^2-12x+12}{2x+x^{1/2}}$$

When the numerator and denominator are expanded, each is a sum of terms of the form ax^r, where a is a real number and r is a positive rational number. The degree of such a sum can be taken to be the largest exponent of x that occurs in one of its terms. If the difference between the degree of the numerator and that of the denominator is called the degree of the function and denoted d, then it can be shown that $f(x)$ is $\Theta(x^d)$, that $f(x)$ is $O(x^c)$ for all real numbers $c > d$, and that $f(x)$ is not $O(x^c)$ for any real number $c < d$. This means that for ths example $d = 7/2 - 1 = 5/2$ and that

$$\frac{(x^{3/2}+3)(x-2)^2}{x^{1/2}(2x^{1/2}+1)} \text{ is } \Theta(x^{5/2})$$

$$\frac{(x^{3/2}+3)(x-2)^2}{x^{1/2}(2x^{1/2}+1)} \text{ is } O(x^c) \qquad \text{for all real numbers } c > 5/2$$

and

$$\frac{(x^{3/2}+3)(x-2)^2}{x^{1/2}(2x^{1/2}+1)} \text{ is not } O(x^c) \qquad \text{for all real numbers } c > 5/2$$