Exponential and Logarithmic Functions: Graphs and Orders

Graphs of Exponential Functions The exponential function with base $b > 0$ is the function that send each real number x to b^x. The graph of any exponential function with base $b > 1$ has a shape that is similar to the graph of the exponential function with base 2. If $0 < b < 1$, then $\frac{1}{b} > 0$ and the graph of the exponential function with base b is the reflection across the vertical axis of the exponential function with base $\frac{1}{b}$.

Graphs of Logarithmic Functions

Definition

If b is a positive real number not equal to 1, then the **logarithmic function with base** b**,** $\log_b : R^+ \to R$, is the function that sends each positive real number x to the number $\log_b x$, which is the exponent to which b must be raised to obtain x.

The logarithmic function with base b is the inverse of the exponential function with base b. It follows that the graphs of the two functions are symmetric with respect to the line $y = x$.

If its base b is greater than 1, the logarithmic function is increasing. Analytically, this means that

If $b > 1$, then all positive numbers x_1 and x_2,

$$\text{if } x_1 < x_2, \text{ then } \log_b(x_1) < \log_b(x_2)$$

Example 1: Base 2 Logarithms of Numbers between Two Consecutive Powers of 2

Provde the following Property:
If k is an integer and x is a real number with

$$2^k \leq x < 2^{k+1}, \text{then } \lfloor \log_2 x \rfloor = k$$

Solution
Proof:
Suppose that k is an integer and x is a real number with

$$2^k \leq x < 2^{k+1}$$

Because the logarithmic function with base 2 is increasing, this implies that

$$\log_2(2^k) \leq \log_2 x < \log_2(2^{k+1})$$

But $\log_2(2^k) = k$ and $\log_2(2^{k+1}) = k + 1$ Hence

$$k \leq \log_2 x < k + 1$$

By definition of the floor function, then,

$$\lfloor \log_2 x \rfloor = k$$

The previouus property can be rewritten as

If x is a positive number that lies between two consecutive integer powers of 2, the floor of the logarithm with base 2 of x is the exponent of the smaller power of 2.

Example 2: When $\lfloor \log_2(n-1) \rfloor = \lfloor \log_2 n \rfloor$

Prove the following property:
For any odd integer $n > 1$, $\lfloor \log_2(n-1) \rfloor = \lfloor \log_2 n \rfloor$

Solution
If n is an odd integer that is greater than 1, then n lies strictly between two successive powers of 2:

$$2^k < n < 2^{k+1} \qquad \text{for sime integer } k > 0$$

It follows that $2^k \leq n - 1$ because $2^k < n$ and both 2^k and n are integers. Consequently,

$$2^k \leq n - 1 < 2^{k+a}$$

Applying the property from example 1 gives

$$\lfloor \log_2 n \rfloor = k \qquad \text{and also} \quad \lfloor \log_2(n-1) \rfloor = k$$

Hence $\lfloor \log_2 n \rfloor = \lfloor \log_2(n-1) \rfloor$

Application: Number of Bits Needed to Represent an Integer in Binary Notation

Given a positive integer n, how many binary digits are needed to represent n? To answer recall that any positive integer n can be written in a unique way as

$$n = 2^k + c_{k-1} \cdot 2^{k-1} + \cdots c_2 \cdot 2^2 + c_1 \cdot 2 + c_0$$

where k is a nonnegative integer and each $c_0, c_1, c_2 \ldots c_{k-1}$ is either 0 or 1. Then the binary representation of n is

$$1c_{k-1}c_{k-2} \cdots c_2c_1c_0$$

and so the number of binary digits needed to represent n is $k+1$

What is $k+1$ as a function of n? Observe that since each $c_1 \leq 1$,

$$n = 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0 \leq 2^k + 2^{k-1} + \cdots 2^2 + 2 + 1$$

But by the formula for the sum of a geometric sequence

$$2^k + 2^{k-1} + \cdots + 2^2 + 2 + 1 = \frac{2^{k+1} - 1}{2 - 1} = 2^{k+1} - 1$$

Hence, by transitivity of order

$$n \leq 2^{k+1} - 1 < 2^{k+1}$$

In addition, because each $c_i \geq 0$

$$2^k \leq 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0 = n$$

Putting inequalities and together gives the double inequality

$$2^k \leq n < 2^{k+1}$$

.

But then, by the property in example 1

$$k = \lfloor \log_2 n \rfloor$$

Thus the number of binary bits needed to represent n is $\lfloor \log_2 n \rfloor + 1$

Example 4: A Recurrence Relation with a Logarithmic Solution

Define a sequence $a_1, a_2, a_3 \ldots$ recursively as follows:

$$\begin{aligned} a_1 &= 1 \\ a_k &= 2a_{\lfloor k/2 \rfloor} \qquad \text{for all integers } k \geq 2 \end{aligned}$$

a. Use iteration to guess an explicit formula for this sequence.

b. Use strong mathematical induction to confirm the correctness of the formula obtained in part (a).

Solution

a. Begin by iterating to find the values of the first few terms of the sequence.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2a_{\lfloor 2/2 \rfloor} = 2a_1 = 2 \cdot 1 = 2 \\ a_3 &= 2a_{\lfloor 3/2 \rfloor} = 2a_1 = 2 \cdot 1 = 2 \\ a_4 &= 2a_{\lfloor 4/2 \rfloor} = 2a_2 = 2 \cdot 2 = 4 \\ a_5 &= 2a_{\lfloor 5/2 \rfloor} = 2a_2 = 2 \cdot 2 = 4 \\ a_6 &= 2a_{\lfloor 6/2 \rfloor} = 2a_3 = 2 \cdot 2 = 4 \\ a_7 &= 2a_{\lfloor 7/2 \rfloor} = 2a_3 = 2 \cdot 2 = 4 \\ a_8 &= 2a_{\lfloor 8/2 \rfloor} = 2a_4 = 2 \cdot 4 = 8 \\ a_9 &= 2a_{\lfloor 9/2 \rfloor} = 2a_4 = 2 \cdot 4 = 8 \\ &\vdots \\ a_{15} &= 2a_{\lfloor 15/2 \rfloor} = 2a_7 = 2 \cdot 4 = 8 \\ a_{16} &= 2a_{\lfloor 16/2 \rfloor} = 2a_8 = 2 \cdot 8 = 16 \\ &\vdots \end{aligned}$$

Note that in each case when the subscript n between two powers of 2, a_n equals the smaller power of 2. More precisely:

$$\text{If } 2^i \leq n < 2^{i+1}, \text{ then } a_n = 2^i.$$

But since n satisfies the inequality

$$2^i \leq n < 2^{i+1}$$

then by the property in example 1

$$i = \lfloor \log_2 n \rfloor$$

Substituting into If $2^i \leq n < 2^{i+1}$, then $a_n = 2^i$ gives

$$a_n = 2^{\lfloor \log_2 n \rfloor}$$

b. The following proof shows that if $a_1, a_2, a_3, \ldots$ is a sequence of numbers that satisfies

$$a_1 = 1 \quad a_k = 2a_{\lfloor k/2 \rfloor} \qquad \text{for all integers } k \geq 2$$

then the sequence satisfies the formula

$$a_n = 2^{\lfloor \log_2 n \rfloor} \qquad \text{for all integers } n \geq 1$$

Proof

Let $a_1, a_2, a_3, \ldots$ be the sequence defined by specifying that $a_1 = 1$ and $a_k = 2_{\lfloor a_{k/2} \rfloor}$ for all integers $k \geq 2$, and let the property $P(n)$ be the equation

$$a_n = 2^{\lfloor \log_2 n \rfloor} \qquad \leftarrow P(n)$$

We will use strong mathematical induction to prove that for all integers $n \geq 1$, $P(n)$ s true

Show that $P(1)$ is true: By definition of $a_1, a_2, a_3, \ldots$, we have that $a_1 = 1$. But it is also the case that $2^{\lfloor \log_2 1 \rfloor} = 2^0 = 1$. Thus $a_1 = 1 = 2^{\lfloor \log_2 1 \rfloor}$ and $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(i)$ is true for all integers i from 1 through k, then $P(k+1)$ is also true: Let k be any integers with $k \geq 1$, and suppose that

$$a_i = 2^{\lfloor \log_2 i \rfloor} \qquad \text{for all integers } i \text{ with } 1 \geq i \geq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$a_{k+1} = 2^{\lfloor \log_2 (k+1) \rfloor} \qquad \leftarrow P(k+1)$$

Consider the two cases: k is even and k is odd

Case 1:

$$\begin{aligned}
a_{k+1} &= 2a_{\lfloor (k+1)/2 \rfloor} \\
&= 2a_{\lfloor k/2 \rfloor} \\
&= 2 \cdot 2^{\lfloor k/2 \rfloor} \\
&= 2^{\lfloor (k/2)+1 \rfloor} \\
\\
&= 2^{\lfloor \log_2 (k/2) \rfloor + 1} \\
&= 2^{\lfloor \log_2 k - \log_2 2 \rfloor + 1} \\
\\
&= 2^{\lfloor \log_2 k - 1 \rfloor + 1} \\
&= 2^{\lfloor \log_2 k \rfloor - 1 + 1} \\
&= 2^{\lfloor \log_2 k \rfloor} \\
&= 2^{\lfloor \log_2 (k+1) \rfloor + 1}
\end{aligned}$$

Case 2 (k is odd)

Thus in either case, $a_n = 2^{\lfloor \log_2 (k+1) \rfloor}$

Exponential and Logarithmic Orders

How do graphs of logarithmic and exponential functions compare with graphs of power functions? It turns out that for large values of x, the graph of the logarithmic function with any base $b > 1$ lies below the graph of any positive power function, and the graph of the exponential function with any base $b > 1$ lies above the graph of any positive power function. In analytic terms, this says the following:

For all real numbers b and r with $b > 1$ and $r > 0$

$$\log_b x \leq x^r \qquad \text{for all sufficiently large real numbers } x$$

and

$$x^r \leq b^x \qquad \text{for all sufficiently large real numbers of } x$$

These statements have the following implications for O-notation.

For all real numbers b and r with $b > 1$ and $r > 0$

$$\log_b x \qquad \text{is} \quad O(x^r)$$

and

$$x^r \qquad \text{is} O(b^x)$$

Another important function in the analysis of algorithms is the function f defined by the formula

$$f(x) = x \log_b x \qquad \text{for all real numbers } x > 0$$

For large values of x, the graph of this function fit in between the graph of the identity function and the graph of the squaring function. More precisely:

For all real numbers b with $b > 1$ and for all sufficiently large real numbers x,

$$x \leq x \log_b x \leq x^2$$

.

The O-notation versions of these facts are as follows:

For all real numbers $b > 1$,

$$x \quad \text{is} \quad O(x \log_b x) \quad \text{and} \quad x \log_b x \quad \text{is} \quad O(x^2)$$

Example 5: Deriving an Order from Logarithmic Inequalities

Show that $x + x\log_2 x$ is $\Theta(x\log_2 x)$.

First observe that $x + x\log_2 x$ is $\Omega(x\log_2 x)$ because for all real numbers $x > 1$,

$$x\log_2 x \le x + x\log_2 x$$

and since all quantities are positive,

$$|x\log_2 x| \le |x + x\log_2 x|$$

Let $A = 1$ and $a = 1$. Then

$$A|x\log_2 x| \le |x + x\log_2 x| \qquad \text{for all } x > a$$

Hence, by definition of Ω-notation

$$x + x\log_2 x \quad \text{is} \quad \Omega(x\log_2 x)$$

To show that $x + x\log_2 x$ is $O(x\log_2 x)$, note that according to the property for all real numbers b with $b > 1$ with $b = 2$, there is a number b such that for all $x > b$

$$x < x\log_2 x$$
$$\Rightarrow x + x\log_2 x < 2x\log_2 x$$

Thus, if b is taken to be greater than 2, then

$$|x + x\log_2 x| < 2|x\log_2 x|$$

Let $B = 2$. Then

$$|x + x\log_2 x| \le B|x\log_2 x| \qquad \text{for all } x > b$$

Hence, by definition of O-notation

$$x + x\log_2 x \quad \text{x} \quad O(x\log_2 x)$$

Therefore, since $x + x\log_2 x$ is $\Omega(x\log_2 x)$ and $x + x\log_2 x$ is $O(x\log_2 x)$, by Theorem: Properties of O-,Ω-, and Θ- notations

$$x + x\log_2 x \quad \text{is} \quad \Theta(x\log_2 x)$$

Example 6: Logarithm Base b Is Big-Theta of Logarithm with Base c

Show that if b and c are real numbers such that $b > 1$ and $c > 1$, then $\log_b x$ is $\Theta(\log_c x)$.

Solution

Suppose b and c are real numbers and $b > 1$ and $c > 1$. To show that $\log_b x$ is $\Theta(\log_c x)$, positive real numbers A, B, and k must be found such that

$$A|\log_c x| \le |\log_b x| \le B|\log_c x| \qquad \text{for all real numbers } x > k.$$

by part (d) of the Properties of Logarithms

$$\log_b x = \frac{\log_c x}{\log_c b} = \left(\frac{1}{\log_c b}\right)\log_c x$$

Since $b > 1$ and the logarithmic function with base c is strictly increasing, then $\log_c b > \log_c 1 = 0$, and so $\frac{1}{\log_c b} > 0$ also. Futhermore, if $x > 1$, then $\log_b x > 0$ and $\log_c x > 0$.

It follows from the equation for $\log_b x$, therefore that

$$\left(\frac{1}{\log_c b}\right)\log_c x \le \log_b x \le \left(\frac{1}{\log_c b}\right)\log_c x$$

for all real numbers $x > 1$. Accordingly, let $A = \frac{1}{\log_c b}$, $B = \frac{1}{\log_c b}$, and $k = 1$. Then, since all quantities in the above inequality are positive,

$$A|\log_c x| \le |\log_b x| \le B|\log_c x| \qquad \text{for all real numbers } x > k$$

Hence, by definition of Θ-notation,

$$\log_b x \qquad \text{is} \quad \Theta(\log_c x)$$

Example 7 shows how a logarithmic order can arise from the computation of a certain kind of sum. It requires thw following fact from calculus:

The area underneath the graph of $y = 1/x$ betwen $x = 1$ and $x = n$ equals $\ln n$, where $\ln n = \log_c n$.

Example 7: Order of a Harmonic Sum

Sums of the form $1+\frac{1}{2}+\cdots+\frac{1}{n}$ are called harmonic sums. They ocur in the analysis of various computer algorithms such as quick sort. Show that $1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n}$ is $\Omega(\ln n)$ by performing the steps:

Show that

a.

$$\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n} \leq \ln n$$

and

$$\ln n \leq 1+\frac{1}{2}+\frac{1}{3}\cdots+\frac{1}{n}$$

b. Show that if n is an integer that is at least 3, then $1 \leq \ln n$.

c. Deduce from (a) and (b) that if the inteer n is greater than or equal to 3, then

$$\ln n \leq 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \leq 2\ln n$$

d. Deduce from (c) that

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \quad \text{is} \quad \Theta(\ln n)$$

Solution

To find the area of a curve we can take the left and right riemann sum. The rectangle whose bases are the intervals between each pair of integers from 1 to n and whose heights of the graph of $y = 1/x$ above the right-hand endpoints of the intervals is the Right Hand Riemann Summ. Rectangles with the same bases but whose heights are the heights of the graph above the left-hand endpoints of the intervals is Left Hand Riemann Summ.

Now the area of each rectangle is its base times its height. Since all the rectangles have base 1, the area of each rectangle equals its height. Thus

$$\text{the area of the rectangle from 1 to 2 is } \frac{1}{2}$$

$$\text{the area of the rectangle from 2 to 3 is } \frac{1}{3}$$

$$\vdots$$

$$\text{the area of the rectangle from } n-1 \text{ to } n \text{ is } \frac{1}{n}$$

So the sum of the areas of all rectangles is $\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$. This sum is less than the area underneath the graph of f between $x = 1$ and $x = n$, which is known to equal n. Hence

$$\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} \leq n$$

A similiar analysis of the areas of the combined Left and Right Summ shows that

$$\ln n \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

b. Suppose n is an integer and $n \geq 3$. Since $e \approx 2.718$, then $n \geq e$. Now the logarithmic function with base e is strictly increasing. Thus since $e \leq n$, then $1 = \ln e \leq \ln n$

c. By part (a)

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq \ln n$$

and by part (b)

$$1 \leq \ln n$$

Adding these two inequalities together gives

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 2 \ln n \qquad \text{for any integer } n \geq 3$$

d. Putting together the results of parts (a) and (c) leads to the conclusion that for all integers $n \geq 3$,

$$\ln n \leq 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 2 \ln n$$

And because all the quantities are positive for $n \geq 3$,

$$|\ln n| \leq \left| 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right| \leq 2 |\ln n|$$

Let $A = 1$, $B = 2$, and $k = 3$. Then

$$A|\ln n| \leq \left| 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \leq 2 \ln n \right| \leq B|\ln n| \qquad \text{for all } n > k.$$

Hence by definition of Θ-notation,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \quad \text{is} \quad \Theta(\ln n)$$