The validity of proof by mathematical induction is generally taken as an axiom. That is why it is referred to as the principle of mathematical induction rather than as a theorem. It is equivalent to the following property of the integers, which is easy to accept.

Suupose S is any set of integers satisfying the statement $P(a)$ is true. Represented in set notation as $a \in S$ and for all integers $k \geq a$, if $k \in S$ then $k+1 \in S$. Then S must contain every integer greater than or equal to a.

Proving a statement by mathematical induction is a two-step process. The first step is called the **basis step**, and the second step is called the **inductive step**.

Theorem: Sum of the First n Integers
For all integers $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Proof (by mathematical induction):
Let the property $P(n)$ be the equation

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Show that $P(1)$ is true:
To establish $P(1)$, we must show that

$$1 = \frac{1(1+1)}{2}$$

But the left-hand side of this equation is 1 and the right-hand side is

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

also. Hence $P(1)$ is true.

Show that for all integers $k \geq 1$ if $P(k)$ is true then $P(k+1)$ is also true:
Suppose that k is any integer with $k \geq 1$ such that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \qquad \leftarrow P(k) \textbf{ inductive hypothesis}$$

We must show that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)[(k+1)+1]}{2}$$

or equivalently, that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
1 + 2 + 3 + \cdots + (k+1) &= 1 + 2 + 3 + \cdots + k + (k+1) && \leftarrow \textbf{make the next-to-last term explicit} \\
&= \frac{k(k+1)}{2} + (k+1) && \leftarrow \textbf{substitution from the inductive hypothesis} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k^2+k}{2} + \frac{2(k+1)}{2} \\
&= \frac{k^2+k}{2} + \frac{2k+2}{2} \\
&= \frac{k^2+3k+2}{2}
\end{aligned}$$

and the right-hand side of $P(k+1)$ is

$$\frac{(k+1)(k+2)}{2} = \frac{k^2+3k+2}{2}$$

Thus the two sides of $P(k+1)$ are equal to the same quantity and so they are equal to each other. Therefore the equation $P(k+1)$ is true.

Theorem: Sum of a Geometric Sequence

For any real unmber r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

Proof (by mathematical induction):

Suppose r is a particular but arbitrarily chosen real number that is not equal to 1, and let the property $P(n)$ be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1} \qquad \leftarrow P(0)$$

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

also because $r^1 = r$ and $r \neq 1$. Hence $P(0)$ is true.

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is also true:

Let k be any integer $k \geq 0$, and suppose that

$$\sum_{i=0}^{n} r^i \frac{r^{k+1}-1}{r-1} \qquad \leftarrow P(k) \textbf{ inductive hypothesis}$$

We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1}-1}{r-1}$$

or, equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1} \qquad \leftarrow P(k+1)$$

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=0}^{k+1} r^i &= \sum_{i=0}^{k+1} r^i + r^{k+1} && \leftarrow \textbf{write the } (k+1)\textbf{st term separetly from the first } k \textbf{ terms} \\
&= \frac{r^{k+1}-1}{r-1} + r^{k+1} - 1 && \leftarrow \textbf{substitute from the inductive hypothesis} \\
&= \frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1} && \leftarrow \textbf{find common denominator} \\
&= \frac{(r^{k+1}-1) + r^{k+1} - (r-1)}{r-1} && \leftarrow \textbf{add factors} \\
&= \frac{(r^{k+1}-1) + r^{k+2} - r^{k+1}}{r-1} \\
&= \frac{r^{k+2}-1}{r-1}
\end{aligned}$$

which is the right-hand side of $P(k+1)$.