Theorem: Properties of O-, Ω-, and Θ- Notations

Let f and g be real-valued functions defined on the same set of nonnegative real numbers.

1. $f(x)$ is $\Omega(g(x))$ and $f(x)$ is $O(g(x))$ if, and only if $f(x)$ is $\Theta(g(x))$.

2. $f(x)$ is $\Omega(g(x))$ if, and only if, $g(x)$ is $O(f(x))$.

3. If $f(x)$ is $O(g(x))$ and $g(x)$ is $O(h(x))$, then $f(x)$ is $O(h(x))$.

Proof

1. We first show that if $f(x)$ is $\Omega(g(x))$, then $g(x)$ is $O(f(x))$. Thus, suppose $f(x)$ is $\Omega(g(x))$. By definition of Ω-notation, there exist a positive real number A and a nonnegative real number a such that

$$A|g(x)| \leq |f(x)| \qquad \text{for all real numbers } x > a$$

Divide both sides by A to obtain

$$|g(x)| \leq \frac{1}{A}|f(x)| \qquad \text{for all real numbers } x > a$$

Let $B = 1/A$ and $b = a$. Then B is a positive real number and b is a nonnegative real number, and

$$|g(x)| \leq B|f(x)| \qquad \text{for all real numbers } x > b$$

and so $g(x)$ is $O(f(x))$.

3. Suppose $f(x)$ is $O(f(x))$ and $g(x)$ is $O(h(x))$. By definition of O-notation, there exist positive real numbers B_1 and B_2, and nonnegative real numbers b_1 and b_2 such that

$$|f(x)| \leq B_1|g(x)| \qquad \text{for all real numbers } x > b_1$$

and

$$|g(x)| \leq B_2|h(x)| \qquad \text{for all real numbers } x > b_2$$

Let $B = B_1B_2$, and let b be the greater of b_1 and b_2. Then if $x > b$,

$$|f(x)| \leq B_1|g(x)| \leq B_1(B_2|h(x)|) \leq B|h(x)|$$

Thus, by definition of O-notation, $f(x)$ is $O(h(x))$.

Theorem: On Polynomial Orders

Suppose $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_n \neq 0$.

1. $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is $O(x^s)$ for all integers $s \geq n$.

2. $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is $\Omega(x^r)$ for all integers $r \leq n$.

3. $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ is $\Theta(x^n)$

Theorem: Limitation on Orders of Polynomial Functions

Let n be a positive integer, and let $a_0, a_1, a_2, \ldots, a_n$ be real numbers with $a_n \neq 0$. If m is any integer with $m < n$, then

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \text{ is not } O(x^m)$$

and

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \text{ is not } \Theta(x^m)$$

Theorem: Orders of Functions Composed of Rational Power Functions

Let m and n be positive integers, and let $r_0, r_1, r_2, \ldots r_n$ and $s_0, s_1, s_2, \ldots, s_m$ be nonnegative rational numbers with $r_0 < r_1 < r_2 < \cdots < r_n$ and $s_0 < s_1 < s_2 < \cdots < s_m$. Let $a_0, a_1, a_2, \ldots, a_n$ and $b_0, b_1, b_2, \ldots, b_m$ be real numbers with $a_n \neq 0$ and $b_m \neq 0$. Then

$$\frac{a_n x^{r_n} + a_{n-1} + \cdots + a_1 x^{r_1} + a_0 x^{r_0}}{b_m x^{s_m} + b_{m-1} x^{s_1} + b_0 x^{s_0}} \text{ is } \Theta(x^{r_n - s_m})$$

$$\frac{a_n x^{r_n} + a_{n-1} + \cdots + a_1 x^{r_1} + a_0 x^{r_0}}{b_m x^{s_m} + b_{m-1} x^{s_1} + b_0 x^{s_0}} \text{ is } O(x^c) \qquad \text{for all real numbers } c > r_n - s_m$$

$$\frac{a_n x^{r_n} + a_{n-1} + \cdots + a_1 x^{r_1} + a_0 x^{r_0}}{b_m x^{s_m} + b_{m-1} x^{s_1} + b_0 x^{s_0}} \text{ is not } O(x^c) \qquad \text{for any real numbers } c < r_n - s_m$$