Example 3: Solve

$$(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0$$

Solution

Here $M = 1 + e^x y + xe^x y$ and $N = xe^x + 2$. Because

$$\frac{\partial M}{\partial y} = e^x + xe^x y = \frac{\partial N}{\partial x}$$

$(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0$ is exact. If we now integrate $N(x, y)$ with respect to y, we obtain

$$F(x, y) = \int (xe^x + 2)dy + h(x) = xe^x y + 2y + h(x)$$

When we take the partial derivative with respect to x and substitute for M, we get

$$\frac{\partial F}{\partial x}(x, y) = M(x, y)$$
$$e^x y + xe^x y + h'(x) = 1 + e^x y + xe^x y$$

Thus, $h'(x) = 1$, so we take $h(x) = x$. Hence, $F(x, y) = xe^x y + 2y + x$, and the solution to $(1 + e^x y + xe^x y)dx + (xe^x + 2)dy = 0$ is given implicitly by $xe^x y + 2y + x = C$. In this case we can solve explicitly for y to obtain $y = \frac{(C-x)}{2+xe^x}$