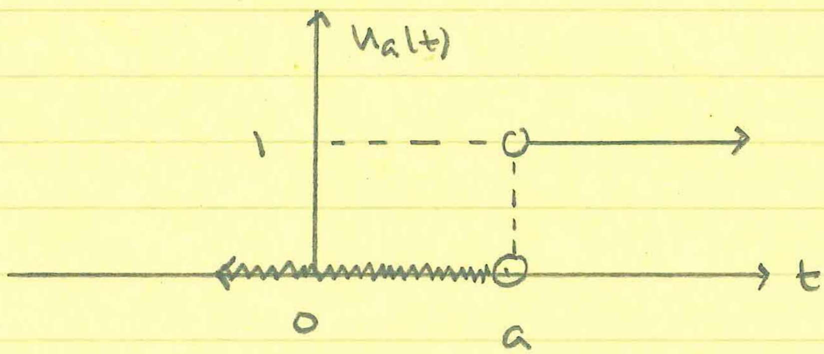
Section 7.6 Transforms of Discontinuous and Periodic Functions

Definition Let $a \geq 0$ then the function

$$u_a(t) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

is called the unit step function or Heaviside function.

Other Notations $u_a(t) = u(t-a) = H(t-a)$



$$\mathcal{L}[u_a(t); t \to s] = \int_0^\infty u_a(t) e^{-st}\, dt = \int_a^\infty e^{-st}\, dt$$

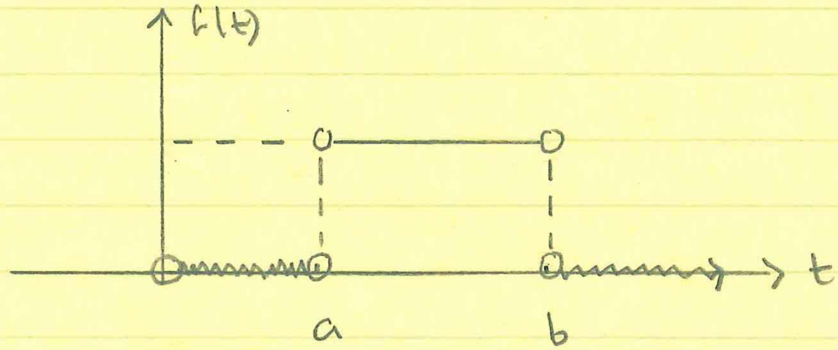
$$= \lim_{N \to \infty} \int_a^N e^{-st}\, dt = \lim_{N \to \infty} \left[-\frac{1}{s} e^{-st} \right]_{t=a}^{t=N}$$

$$= \lim_{N \to \infty} \left(-\frac{1}{s} e^{-sN} + \frac{1}{s} e^{-sa} \right) = \frac{1}{s} e^{-as}, \quad s > 0.$$

$$\therefore \quad \mathcal{L}^{-1}\left[\frac{1}{s} e^{-as} ; s > 0 \right] = u_a(t)$$

Example For $f(t) = \begin{cases} 0, & 0 < t < a \\ 1, & a < t < b \\ 0, & b < t < \infty \end{cases}$ find $F(s)$

<u>Solution</u>



$$f(t) = \begin{cases} 0 &, 0 < t < a \\ 1 &, a < t < b \\ 1 &, b < t < \infty \end{cases} - \begin{cases} 0 &, 0 < t < a \\ 0 &, a < t < b \\ 1 &, b < t < \infty \end{cases}$$

$$= u_a(t) - u_b(t)$$

$$\therefore \bar{f}(s) = \frac{1}{s} e^{-as} - \frac{1}{s} e^{-bs} \quad , s > 0$$

<u>The Second Shifting Theorem</u>

$$\mathcal{L}[u_a(t) f(t-a) ; t \to s]$$

$$= \int_0^\infty u_a(t) f(t-a) e^{-st} \, dt$$

$$= \int_a^\infty 1 . f(t-a) e^{-st} \, dt \qquad \leftarrow \text{Let } x = t - a, \; dx = dt$$

$$= \int_0^\infty f(x) e^{-s(a+x)} \, dx$$

$$= e^{-as} \int_0^\infty f(x) e^{-sx} \, dx$$

$$= e^{-as} \int_0^{\infty} f(t) e^{-st}\, dt$$

$$= e^{-as} F(s)$$

$$\therefore \mathcal{L}^{-1}\left[e^{-as} F(s) ; s \to t \right] = u_a(t) f(t-a)$$

or

$$\mathcal{L}^{-1}\left[e^{-as} F(s) ; s \to t \right] = u_a(t) \mathcal{L}^{-1}\left[F(s) ; s \to t-a \right]$$

Examples ① If $\bar{g}(s) = \frac{s e^{-3s}}{s^2+1}$ find $g(t)$.

$$g(t) = \mathcal{L}^{-1}\left[e^{-3s} \frac{s}{s^2+1} ; s \to t \right]$$

$$= u_3(t) \mathcal{L}^{-1}\left[\frac{s}{s^2+1} ; s \to t-3 \right]$$

$$= u_3(t) \cos(t-3)$$

$$= \cos(t-3) \cdot \begin{cases} 0, & 0 \leq t < 3 \\ 1, & t > 3 \end{cases}$$

$$= \begin{cases} 0, & 0 \leq t < 3 \\ \cos(t-3), & t > 3 \end{cases}$$

Example 2 If $\bar{g}(s) = \frac{5}{s} - \frac{3}{s} e^{-3s} - \frac{2}{s} e^{-7s}$ find $g(t)$.

Solution $g(t) = \mathcal{L}^{-1}\left[\frac{5}{s} - \frac{3}{s} e^{-3s} - \frac{2}{s} e^{-7s} ; s \to t \right]$

$$g(t) = 5 - 3u_3(t)\,\mathcal{L}^{-1}\left[\frac{1}{s};\ s \to t-3\right]$$

$$\qquad - 2u_7(t)\,\mathcal{L}^{-1}\left[\frac{1}{s};\ s \to t-7\right]$$

$$= 5 - 3u_3(t) - 2u_7(t)$$

$$= 5 - 3\begin{Bmatrix} 0, & 0<t<3 \\ 1, & 3<t<\infty \end{Bmatrix} - 2\begin{Bmatrix} 0, & 0<t<7 \\ 1, & 7<t<\infty \end{Bmatrix}$$

$$= 5\begin{Bmatrix} 1, & 0<t<3 \\ 1, & 3<t<7 \\ 1, & 7<t<\infty \end{Bmatrix} - 3\begin{Bmatrix} 0, & 0<t<3 \\ 1, & 3<t<7 \\ 1, & 7<t<\infty \end{Bmatrix} - 2\begin{Bmatrix} 0, & 0<t<3 \\ 0, & 3<t<7 \\ 1, & 7<t<\infty \end{Bmatrix}$$

$$= \begin{Bmatrix} 5, & 0<t<3 \\ 2, & 3<t<7 \\ 0, & 7<t<\infty \end{Bmatrix}$$

<u>Example 3</u> If $\bar{g}(s) = e^{-4s}\left(\frac{2}{s^2} + \frac{5}{s}\right)$ find $g(t)$.

<u>Solution</u> $g(t) = \mathcal{L}^{-1}\left[e^{-4s}\left(\frac{2}{s^2} + \frac{5}{s}\right);\ s \to t\right]$

$$= u_4(t)\,\mathcal{L}^{-1}\left[\frac{2}{s^2} + \frac{5}{s};\ s \to t-4\right]$$

$$= u_4(t)\left(2(t-4) + 5\right)$$

$$= u_4(t) \cdot (2t-3)$$

$$= (2t-3)\begin{cases} 0, & 0 \le t < 4 \\ 1, & 4 < t < \infty \end{cases}$$

$$= \begin{cases} 0, & 0 \le t < 4 \\ 2t-3, & 4 < t < \infty \end{cases}.$$

HW Pages 393–395

#'s 1, 3, 11–19 odd, 29–39 odd