Theorem 5.20

If G is a k-connected graph, $k \geq 2$, then every k vertices of G lie on a common cycle of G.

Proof:

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of k vertices of G. We show that there exists a cycle in G containing every vertex of S. Among all cycles in G, let C be one containing a maximum number ℓ of vertices of S We claim that $\ell = k$. Assume, to the contrary, that $\ell < k$. Since G is k-connected, $k \geq 2$ it follows that G is 2-connected and so $2 \leq \ell < k$ by Theorem 5.7. We may assume that C contains the vertices $v_1, v-2, \ldots v_\ell$ of S and that the vertices of S on C appear in the order $v_1, v-2, \ldots v_\ell$ as we proceed cyclically about C.

Since $\ell < k$, there is a vertex $u \in S$ that does not belong to C. Furthermore, since $2 \leq \ell < k$, the graph G is ℓ-connected as well. Suppose first that the order of C is ℓ. Applying Corollary 5.19 to the vertices $u, v_1, v_2, \ldots, v_\ell$, we see that G contains internally disjoint u—v_i paths $P_i (1 \leq i \leq \ell)$. Replacing the edge v_1, v_2 by P_1 and P_2 produces a cycle containing the vertices $u, v_1, v_2, \ldots, v_\ell$, which gives a contradiction.

Hence we may assume that C contains a vertex $v_0 \notin S$. Since $2 \leq \ell + 1 \leq k$, the graph G is $(\ell + 1$-connected. Applying Corollary 5.19 to the vertices $u, v_0, v_1, v_2, \ldots, v_\ell$, we see that G contains internally disjoint u—v_i paths P_i $(0 \leq i \leq \ell)$. Let $v_i'(0 \leq i \leq \ell)$ be the first vertex of P_i that belongs to C (possibly $v_i' = v_i$ abd ket P_i' be the u—v_i' subpath of P_i. Since there are $\ell + 1$ paths P_i' and ℓ vertices C that belong to S, there are distinct vertices v_r' and vertices belonging to S. Deleting the interior vertices of P' from C and adding the paths P_r' and P_t' produces a cycle containing the vertices $u, v_1, v_2, \ldots, v_\ell$, which is a contradiction.