Chapter 3. Continuous Distributions

3.1. Random Variables of the Continuous Type

1. *Definition.* The cumulative distribution function (cdf) of a random variable X is

$$F(x) = P(X \le x), \quad -\infty < x < \infty.$$

The domain of a cdf is $\mathbb{R} = (-\infty, +\infty)$, and, for all real numbers x,

$$0 \le F(x) \le 1.$$

2. *Example.* The random variable X has the Binomial$(2, 1/2)$ distribution. Find $F(x)$.

— *Solution.* It is clear that X has pmf

x	0	1	2
f	0.25	0.5	0.25

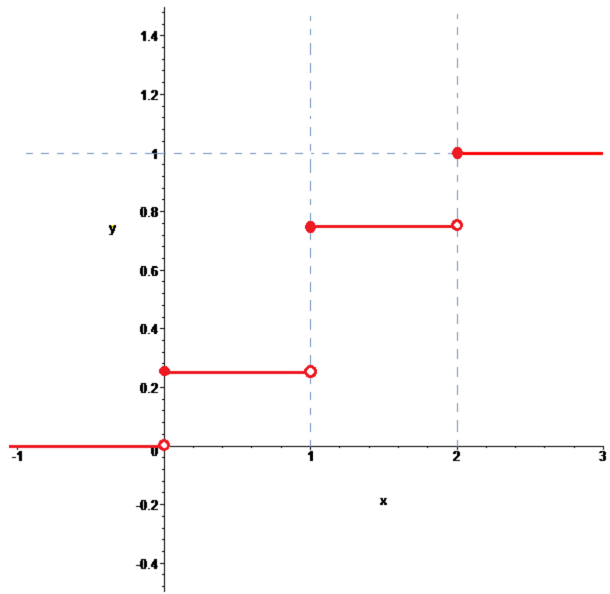
Therefore, X has cdf

$$F(x) = \begin{cases} 0, & x < 0, \\ 0.25, & 0 \le x < 1, \\ 0.75, & 1 \le x < 2, \\ 1, & 2 \le x. \end{cases}$$

For example, by definition,

$$F(1.5) = P(X \leq 1.5) = P(X = 0) + P(X = 1) = 0.25 + 0.5 = 0.75.$$

The graph of the function $F(x)$ is plotted below:



It is clear that the function $F(x)$ is non-decreasing.

The function $F(x)$ has three points of discontinuity: $0, 1, 2, 3$. However, $F(x)$ is continuous from the right at each of these three points. For example,

$$\lim_{x \to 1^+} F(x) = 0.75 = F(1),$$

$$\lim_{x \to 1^-} F(x) = 0.25 \neq F(1).$$

3. *Properties of cdfs.*

   (a) A cdf $F(x)$ is defined over the entire real line.

   (b) $0 \leq F(x) \leq 1$.

   (c) $F(x)$ is non-decreasing in x.

   (d) $F(x) \to 1$ as $x \to +\infty$, and $F(x) \to 0$ as $x \to -\infty$. Or, symbolically
   $$F(+\infty) = 1, \quad F(-\infty) = 0.$$

   (e) $F(x)$ is continuous from the right at each real number x.

4. The cdf is defined in terms of probability. Therefore, probabilities can be calculated using the cdf. (See the next theorem)

5. *Theorem.* Let X be a random variable with cdf $F(x)$. If $a < b$ are real numbers, then

$$P(a < X \leq b) = F(b) - F(a).$$

6. This formula is an easy consequence of additivity axiom of probability. For example, since $(1 < X \leq 3)$ and $(X \leq 1)$ are disjoint, we have

$$P(1 < X \leq 3) + P(X \leq 1) = P((1 < X \leq 3) \cup (X \leq 1)) = P(X \leq 3),$$

that is,

$$P(1 < X \leq 3) + F(1) = F(3),$$

$$P(1 < X \leq 3) = F(3) - F(1).$$

7. Similarly, we have:

8. *Theorem.* Let X be a random variable with cdf $F(x)$.

   (a) If b is a real number, then

   $$P(X > b) = 1 - F(b).$$

   (b) If b is a real number, then

   $$P(X < b) = \lim_{x \to b^-} F(x).$$

   (c) If $a < b$ are real numbers, then

   $$P(a < X < b) = \lim_{x \to b^-} F(x) - F(a).$$

9. *Definition.* Let X be a random variable. If the cdf of X is a continuous function on $(-\infty, \infty)$, then we say X is a continuous random variable.

10. *Example.* Consider the Poisson process with parameter $\lambda > 0$. Let X be the waiting time until the first change, and denote by $F(x)$ the cdf of X. Then X is a continuous random variable.

    — *Proof.* First, if $x < 0$, then $F(x) = 0$. This means that the waiting time until the first customer must be positive — there will be no customer until the store opens.

    If $x \geq 0$, we let Y be the number of customers who arrive during the time interval $[0, x]$. By the definition of the Poisson process,

$$Y \sim \text{Poisson } (\lambda x).$$

This means that

$$P(Y=0)=e^{-\lambda x}.$$

Note that the vent $Y=0$ means that no customer comes during the time interval $[0, x]$, that is, the first customer comes only after time $t=x$, that is, the waiting time until the first customer is longer than x, that is, $X>x$. In summary, if $x \geq 0$, then

$$\begin{aligned}
F_X(x) &= P(X \leq x) \\
&= 1-P(X>x) \\
&= 1-P(Y=0)=1-e^{-\lambda x}.
\end{aligned}$$

Let Y be the number of customers who arrives during the time interval $[0, x]$.

In summary,

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}.$$

Since

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} (1 - e^{-\lambda x}) = 1 - 1 = 0,$$

$F(X)$ is continuous at $x = 0$ and therefore continuous over the real line. Hence, by definition, X is a continuous random variable.

11. *Theorem.* Let X be a continuous random variable. If a is a real number, then

$$P(X = a) = 0.$$

— *Proof.*

$$0 \leq P(X = a) = F(a) - \lim_{s \to a^-} F(x) = 0.$$

12. The implication is, for a continuous random variable, a single point does not matter. This is quite different from the discrete case.

13. *Definition.* Let X be a continuous random variable with cdf $F(x)$. If $F(x)$ is differentiable, then the function

$$f(x) = F'(x), \quad -\infty < x < \infty.$$

is called the probability density function of X.

14. What functions qualify as a pdf?

15. *Properties of pdfs.*

    (a) A pdf $f(x)$ is defined over the entire real line.

    (b) $f(x) \geq 0$;

    (c) $\int_{-\infty}^{\infty} f(x)dx = 1$.

16. Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. Since $f(x) = F'(x)$, we have (by the fundamental theorem of Calculus)

$$\int_a^b f(x)dx = F(b) - F(a). \tag{1}$$

Here, a and b are real numbers such that $a < b$. By Theorem 5, we have

$$F(b) - F(a) = F(a < X \leq b). \tag{2}$$

Combining,

$$F(a < X \leq b) = \int_a^b f(x)dx. \tag{3}$$

Note that

$$\int_a^b f(x)dx$$

is geometrically interpreted as an area in Calculus II.

17. Since a single point does not matter for a continuous random variable, we have extend equation (3) as follows:

18. *Theorem.* If X is a continuous random variable with pdf $f(x)$, then

$$\begin{aligned} \int_a^b f(x)dx &= P(a < X \leq b) = P(a \leq X \leq b) \\ &= P(a \leq X < b) = P(a < X < b). \end{aligned}$$

19. Similarly, we have:

20. *Theorem.* If X is a continuous random variable with pdf $f(x)$, then

$$P(X \leq a) = P(X < a) = \int_{-\infty}^{a} f(x)dx,$$

$$P(X > a) = P(X \geq a) = \int_{a}^{+\infty} f(x)dx.$$

Here, a is a fixed real number.

21. *Example.* Consider the Poisson process with parameter $\lambda > 0$. Let X be the waiting time until the first change. Then X has cdf

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Find the pdf of X.

— *Solution.* We have

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

22. *Definition.* Let X be a continuous random variable with pdf $f(x)$. Then,

(a) the expectation of X is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx,$$

(b) the variance of X is

$$\sigma^2 = Var(X) = E((X - E(X))^2),$$

(c) then standard deviation of X is $\sigma = \sqrt{\sigma^2}$,

(d) the moment generating function of X is

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx.$$