Homogeneous Linear Equations With Constant Coefficients

2'nd order: $$a_0 y'' + a_1 y' + a_2 y = 0 \qquad (1)$$
a_0, a_1, a_2 constants ; $a_0 \neq 0$

Seek a solution to equation (1) of the form $y = e^{mx}$.

If $y = e^{mx}$ then $y' = me^{mx}$ and $y'' = m^2 e^{mx}$.

Substitute $y = e^{mx}$ into (1) : $a_0 m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = 0$

$$\Rightarrow e^{mx}\left(a_0 m^2 + a_1 m + a_2\right) = 0$$

$$\Rightarrow \underbrace{a_0 m^2 + a_1 m + a_2 = 0}_{\text{Auxiliary Equation.}} \qquad (2)$$

$$m = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_0}$$

Eqn (2) can have 3 different types of solution depending upon the value of the discriminant $a_1^2 - 4a_0a_2$.

<u>Case 1</u> ($a_1^2 - 4a_0a_2 > 0$; real + distinct roots)

If $a_1^2 - 4a_0a_2 > 0$ then eqn (2) has two real + distinct roots $m = m_1$ and $m = m_2$.

Eqn ① has two linearly independent solutions $y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$ and general solution $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ where c_1 + c_2 are arbitrary constants.

<u>Case 2</u> ($a_1^2 - 4a_0a_2 = 0$; real + equal roots)

If $a_1^2 - 4a_0a_2 = 0$ then eqn ② has two real + equal roots $m = m_1$ and $m = m_1$.

Eqn ① has two linearly independent solutions $y_1 = e^{m_1 x}$, $y_2 = x e^{m_1 x}$ and general solution $y = c_1 e^{m_1 x} + c_2 x e^{m_1 x} = e^{m_1 x}(c_1 + c_2 x)$ where c_1 + c_2 are arbitrary constants.

<u>Case 3</u> ($a_1^2 - 4a_0a_2 < 0$; complex roots)

If $a_1^2 - 4a_0a_2 < 0$ then eqn ② has two complex roots $m = \alpha \pm \beta i$.

Eqn ① has two linearly independent solutions $y_1 = e^{\alpha x}\cos\beta x$, $y_2 = e^{\alpha x}\sin\beta x$ and general solution

$$y = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$
$$= e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$$

where c_1 + c_2 are arbitrary constants.

<u>Note</u> Similar results hold for n'th order equations.

<u>Example 1</u> Find the general solution of the following:

(a) $y'' - y' - 2y = 0.$

Aux.Eqn. $m^2 - m - 2 = 0$

$(m-2)(m+1) = 0$

$m = 2, \ m = -1$

$y_1 = e^{2x}, \ y_2 = e^{-x}$

$y = c_1 e^{2x} + c_2 e^{-x}$

(b) $y'' + 2y' + y = 0$

Aux Eqn. $m^2 + 2m + 1 = 0$

$(m+1)(m+1) = 0$

$m = -1, \ m = -1$

$y_1 = e^{-x}, \ y_2 = xe^{-x}$

$y = c_1 e^{-x} + c_2 x e^{-x}$

$= (c_1 + c_2 x) e^{-x}$

(c) $y'' + 4y = 0$

Aux.Eqn. $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$

$m = \pm 2i$

or, $m = 0 \pm 2i$

$y_1 = e^{0 \cdot x} \cos 2x = \cos 2x$

$y_2 = e^{0 \cdot x} \sin 2x = \sin 2x$

$y = c_1 \cos 2x + c_2 \sin 2x$

(d) $y'' + y' + y = 0$

Aux Eqn. $m^2 + m + 1 = 0$

$m^2 + m = -1$

$m^2 + m + \frac{1}{4} = -1 + \frac{1}{4}$

$(m + \frac{1}{2})^2 = -\frac{3}{4}$

$m + \frac{1}{2} = \pm\sqrt{-\frac{3}{4}} = \pm\frac{\sqrt{3}}{2} i$

$\therefore m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$y_1 = e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2} x\right)$

$y_2 = e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2} x\right)$

OVER

$$y = c_1 y_1 + c_2 y_2 = e^{-\frac{1}{2}x}\left[c_1 \cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\tfrac{\sqrt{3}}{2}x\right)\right]$$

Example 2 Solve : $y^{(4)} - 16y = 0$

Solution

Aux. Eqn. $m^4 - 16 = 0$

$(m^2-4)(m^2+4) = 0$

$m^2 = 4,\ m^2 = -4$

$m = \pm 2,\ m = \pm 2i$

$y_1 = e^{2x}$

$y_2 = e^{-2x}$

$y_3 = \cos 2x$

$y_4 = \sin 2x$

$$\therefore y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

Example 3 Solve : $y''' + y'' - y' - y = 0$

Solution

Aux. Eqn. $m^3 + m^2 - m - 1 = 0$

$m^2(m+1) - (m+1) = 0$

$(m+1)(m^2-1) = 0$

$(m+1)(m+1)(m-1) = 0$

$m = -1, -1, 1$

$y_1 = e^{-x}$

$y_2 = xe^{-x}$

$y_3 = e^{x}$

$$\therefore y = c_1 e^{-x} + c_2 x e^{-x} + c_3 e^{x} = (c_1 + c_2 x)e^{-x} + c_3 e^{x}$$

Example 4 Solve : $y^{(4)} - 2y^{(3)} + y^{(2)} = 0$

Solution

Aux. Eqn. $m^4 - 2m^3 + m^2 = 0$ $\qquad y_1 = e^{0 \cdot x} = 1$

$m^2(m^2 - 2m + 1) = 0$ $\qquad y_2 = x \cdot 1 = x$

$m^2(m-1)^2 = 0$ $\qquad y_3 = e^{1 \cdot x} = e^x$

$m = 0, 0, 1, 1$ $\qquad y_4 = xe^x$

Gen. soln. $y = c_1 \cdot 1 + c_2 \cdot x + c_3 e^x + c_4 x e^x$

$= c_1 + c_2 x + (c_3 + c_4 x) e^x.$

<u>Example 5</u> Given that the polynomial

$$m^9 + 3m^6 + 3m^3 + 1$$

has roots $m = -1, -1, -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

obtain the general solution to the o.d.e.

$$y^{(9)} + 3y^{(6)} + 3y^{(3)} + y = 0$$

<u>Solution</u> $y = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$

$+ c_4 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_5 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$

$+ c_6 x e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_7 x e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$

$+ c_8 x^2 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_9 x^2 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$

$= (c_1 + c_2 x + c_3 x^2) e^{-x}$

$+ (c_4 + c_6 x + c_8 x^2) e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right)$

$+ (c_5 + c_7 x + c_9 x^2) e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$

<u>H.W.</u> Pages 165–166 #'s 1, 5, 9, 13, 17, 37, 39, 41, 43

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