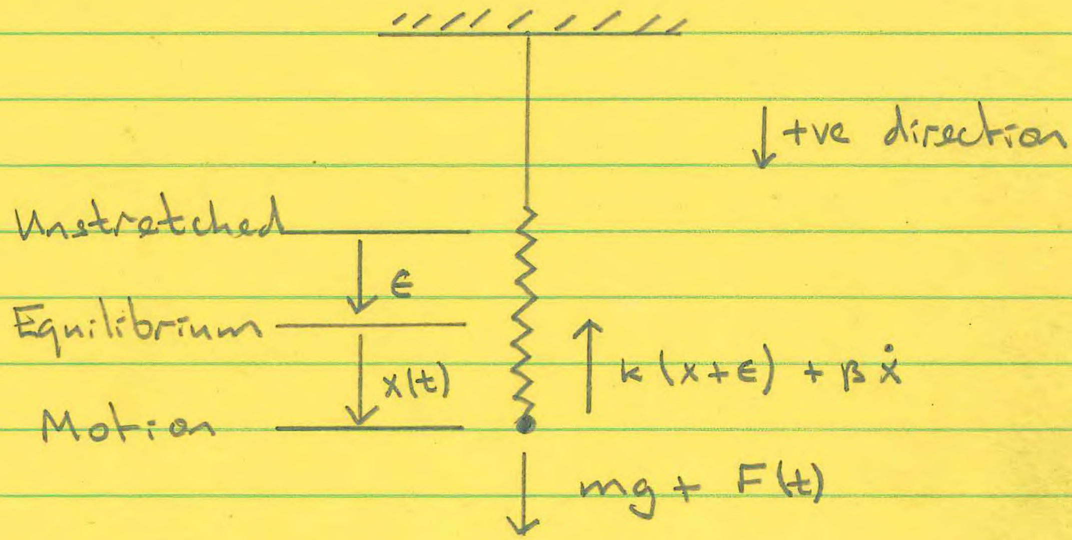
Section 4.10 A Closer Look at Forced Mechanical Vibrations

Problem A mass m is attached to the free end of a linear spring of constant k which hangs vertically in a medium with viscous damping β. Find the d.e. which describes the system if the mass is acted upon by an external force $F(t)$.

Solution



$$\text{N2} \Rightarrow \frac{d}{dt}(m\dot{x}) = mg + F(t) - k(x+\epsilon) - \beta\dot{x}$$

$$m\frac{d}{dt}\dot{x} = \cancel{mg} + F(t) - kx - \cancel{k\epsilon} - \beta\dot{x}$$

$$\therefore \quad m\ddot{x} + \beta\dot{x} + kx = F(t)$$

General Solution : $x = \underbrace{x_c(t)}_{\text{Transient Response}} + \underbrace{x_p(t)}_{\text{Steady-State Response}}$

Apply I.C.'s :

Example A 16-lb weight is placed upon the lower end of a coil spring suspended from a rigid beam. The spring constant is known to be 10 lb/ft. The weight comes to rest in it's equilibrium position. Beginning at $t=0$, an external force given by $F(t) = 5\cos 2t$ is applied to the system. The medium offers a resistance in pounds numerically equal to $2\dot{x}$, where $\dot{x}$ is the instantaneous velocity in ft/sec.

(a) Determine the displacement of the weight as a function of the time.

(b) Determine the amplitude, period, and frequency of the steady-state terms.

Solution $$m\ddot{x} + \beta\dot{x} + kx = F(t)$$

$$W = mg = 16 \Rightarrow m = \frac{16}{32} = \frac{1}{2}$$

$$k = 10 \quad , \quad \beta = 2 \quad , \quad F(t) = 5\cos 2t$$

$$\tfrac{1}{2}\ddot{x} + 2\dot{x} + 10x = 5\cos 2t$$

$$\therefore \quad \ddot{x} + 4\dot{x} + 20x = 10\cos 2t$$

C.F. $$\ddot{x} + 4\dot{x} + 20x = 0$$

A.E. $$m^2 + 4m + 20 = 0$$
$$(m+2)^2 + 16 = 0$$

$$(m+2)^2 = -16$$
$$m+2 = \pm 4i$$
$$m = -2 \pm 4i$$

$$\therefore \quad x_c(t) = e^{-2t}\left[c_1 \cos 4t + c_2 \sin 4t \right]$$

P.I. Try $x_p = a\cos 2t + b\sin 2t$

$$\dot{x}_p = -2a \sin 2t + 2b\cos 2t$$

$$\ddot{x}_p = -4a\cos 2t - 4b\sin 2t$$

Substitute $\ddot{x}_p + 4\dot{x}_p + 20x_p = 10\cos 2t$

$$(\cos 2t)(-4a + 8b + 20a) + (\sin 2t)(-4b - 8a + 20b) = 10\cos 2t.$$

Match: $$\left.\begin{aligned} 16a + 8b &= 10 \\ -8a + 16b &= 0 \end{aligned}\right\}$$

$$40b = 10 \qquad a = 2b$$
$$b = \tfrac{1}{4} \qquad a = \tfrac{1}{2}$$

$$\therefore \quad x_p = \tfrac{1}{2}\cos 2t + \tfrac{1}{4}\sin 2t$$

Gen. Soln. $$x = e^{-2t}\left[c_1\cos 4t + c_2 \sin 4t\right] + \tfrac{1}{2}\cos 2t + \tfrac{1}{4}\sin 2t$$

$$\dot{x} = e^{-2t}\left[-4c_1\sin 4t + 4c_2\cos 4t\right] - 2e^{-2t}\left[c_1\cos 4t + c_2\sin 4t\right] - \sin 2t + \tfrac{1}{2}\cos 2t$$

$\underline{t=0,\ x=0}$ $\quad 0 = 1[c_1] + \frac{1}{2} \Rightarrow c_1 = -\frac{1}{2}$

$\underline{t=0,\ \dot{x}=0}$ $\quad 0 = 1[4c_2] - 2[c_1] + \frac{1}{2}$

$$0 = 4c_2 + \frac{3}{2}$$

$$\Rightarrow c_2 = -3/8$$

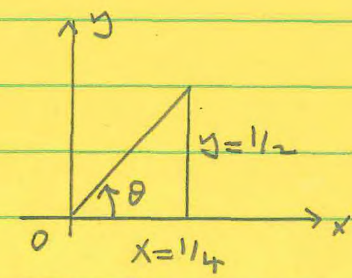
$$\therefore\ x = \underbrace{e^{-2t}\left[-\frac{1}{2}\cos 4t - \frac{3}{8}\sin 4t\right]}_{\text{Transient Response}} + \underbrace{\frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t}_{\text{Steady-State Response}}$$

(b) $x_p = \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$

Let $A = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{\sqrt{5}}{4}$

and $\sin\theta = \frac{1/2}{A} = \frac{1/2}{\sqrt{5}/4} = \frac{y}{r}$

$\cos\theta = \frac{1/4}{A} = \frac{1/4}{\sqrt{5}/4} = \frac{x}{r}$



$\frac{1}{2} = \frac{\sqrt{5}}{4}\sin\theta$

$\tan\theta = \frac{y}{x} = 2$

$\frac{1}{4} = \frac{\sqrt{5}}{4}\cos\theta$

$\theta = \tan^{-1} 2$.

$$\therefore\ x = \frac{1}{2}\cos(2t) + \frac{1}{4}\sin(2t)$$

$$= \frac{\sqrt{5}}{4}\sin\theta\cos 2t + \frac{\sqrt{5}}{4}\cos\theta\sin 2t$$

$$= \frac{\sqrt{5}}{4}\left[\sin 2t\cos\theta + \cos 2t\sin\theta\right]$$

$$= \frac{\sqrt{5}}{4}\sin(2t+\theta).$$

Amplitude $= \frac{\sqrt{5}}{4}$

Period $= \frac{2\pi}{2} = \pi$

Frequency $= \frac{1}{\pi}$.

Undamped Forced Motion

Example

Solve: $\ddot{x} + \omega^2 x = F_0 \sin \gamma t$

where: $x(0) = 0$, $\dot{x}(0) = 0$, F_0 = constant and $\gamma \neq \omega$.

Gen. Soln: $x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \dfrac{F_0}{\omega^2 - \gamma^2} \sin \gamma t$

Part. soln: $x = \dfrac{F_0}{\omega(\omega^2 - \gamma^2)} \left(-\gamma \sin \omega t + \omega \sin \gamma t \right)$, $\gamma \neq \omega$.

↑

Eqn is not defined for $\gamma = \omega$.
What happens as $\gamma \to \omega$ (that is "tune in" the frequency of the driving force ($\gamma / 2\pi$) to the frequency of the free vibrations ($\omega / 2\pi$)).

$$x(t) = \lim_{\gamma \to \omega} F_0 \frac{-\gamma \sin \omega t + \omega \sin \gamma t}{\omega(\omega^2 - \gamma^2)}$$

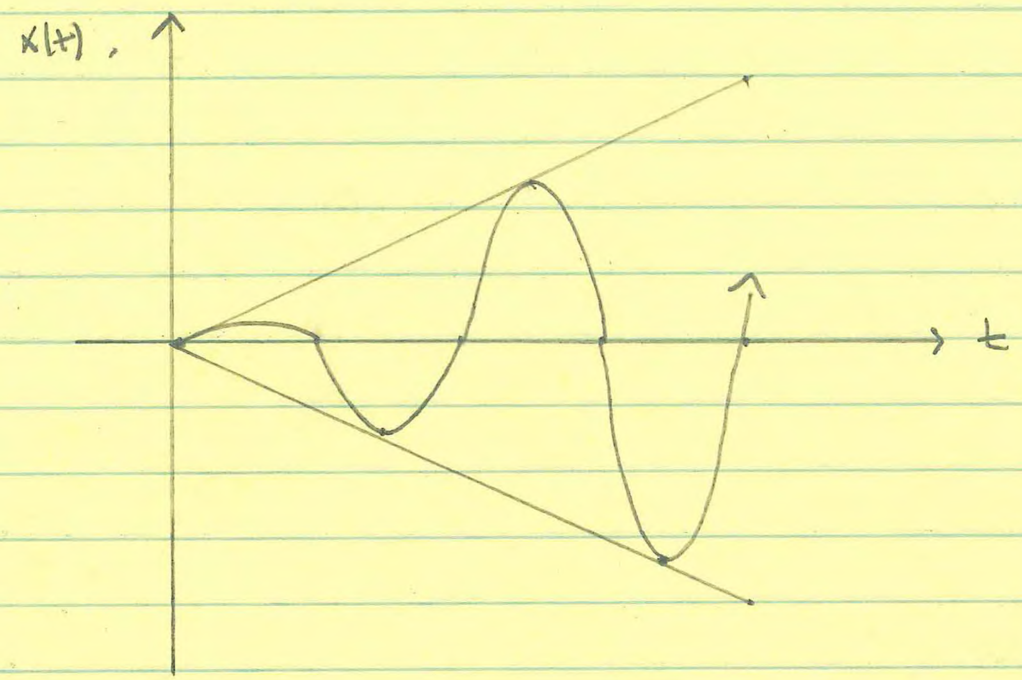
$$= F_0 \lim_{\gamma \to \omega} \frac{\frac{d}{d\gamma}\left(-\gamma \sin \omega t + \omega \sin \gamma t\right)}{\frac{d}{d\gamma}\, \omega(\omega^2 - \gamma^2)}$$

$$= F_0 \lim_{\gamma \to \omega} \frac{-\sin \omega t + \omega t \cos \gamma t}{-2\omega \gamma}$$

$$= F_0 \frac{-\sin \omega t + \omega t \cos \omega t}{-2\omega^2}$$

$$= \frac{F_0}{2\omega^2} \sin\omega t - \frac{F_0}{2\omega} t \cos\omega t$$

As $t \to \infty$ displacements will become large.



The above phenomenon is called <u>resonance</u>.

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The tendency of a system to oscillate with greater amplitude at some frequencies than at others.

<u>HW</u> Pages 230–231, #'s 3, 9, 11, 13