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Proof by exhaustion

Proof by exhaustion, also known as **proof by cases**, **proof by case analysis**, **complete induction**, or the **brute force method**, is a method of mathematical proof in which the statement to be proved is split into a finite number of cases or sets of equivalent cases and each type of case is checked to see if the proposition in question holds.[1] This is a method of direct proof. A proof by exhaustion contains two stages:

1. A proof that the set of cases is exhaustive; i.e., that each instance of the statement to be proved matches the conditions of (at least) one of the cases.
2. A proof of each of the cases.

The prevalence of digital computers has greatly increased the convenience of using the method of exhaustion. Computer expert systems can be used to arrive at answers to many of the questions posed to them. In theory, the proof by exhaustion method can be used whenever the number of cases is finite. However, because most mathematical sets are infinite, this method is rarely used to derive general mathematical results.[2]

In the Curry–Howard isomorphism, proof by exhaustion and case analysis are related to ML-style pattern matching.

Contents

Example

To prove that every integer that is a perfect cube is a multiple of 9, or is 1 more than a multiple of 9, or is 1 less than a multiple of 9.

Proof:
Each cube number is the cube of some integer n. Every integer n is either a multiple of 3, or 1 more or 1 less than a multiple of 3. So these 3 cases are exhaustive:

- Case 1: If $n = 3p$, then $n^3 = 27p^3$, which is a multiple of 9.
- Case 2: If $n = 3p + 1$, then $n^3 = 27p^3 + 27p^2 + 9p + 1$, which is 1 more than a multiple of 9. For instance, if $n = 4$ then $n^3 = 64 = 9x7 + 1$.
- Case 3: If $n = 3p - 1$, then $n^3 = 27p^3 - 27p^2 + 9p - 1$, which is 1 less than a multiple of 9. For instance, if $n = 5$ then $n^3 = 125 = 9×14 - 1$.∎

Elegance

Mathematicians prefer to avoid proofs by exhaustion with large numbers of cases, which are viewed as inelegant. An illustration of how such proofs might be inelegant is to prove that every year in which the modern Summer Olympic Games is held is divisible by 4.

Proof: the first modern Summer Olympics were held in 1896, and then every 4 years thereafter (neglecting years in which the games were not held due to World War I and World War II). Since 1896 = 474 × 4 is divisible by 4, the next Olympics would be in year 474 × 4 + 4 = (474 + 1) × 4, which is also divisible by four, and so on (this is a proof by mathematical induction). Therefore the statement is proved.

The statement can also be proved by exhaustion by listing out every year in which the Summer Olympics were held, and checking that every one of them can be divided by four. With 28 total Summer Olympics as of 2016, this is a proof by exhaustion with 28 cases. In addition to being more elegant, the proof by mathematical induction also proves the statement indefinitely into the future, while after each new Summer Olympics the proof by exhaustion will require an extra case.

Number of cases

There is no upper limit to the number of cases allowed in a proof by exhaustion. Sometimes there are only two or three cases. Sometimes there may be thousands or even millions. For example, rigorously solving an endgame puzzle in chess might involve considering a very large number of possible positions in the game tree of that problem.

The first proof of the four colour theorem was a proof by exhaustion with 1,936 cases. This proof was controversial because the majority of the cases were checked by a computer program, not by hand. The shortest known proof of the four colour theorem today still has over 600 cases.

In general the probability of an error in the whole proof increases with the number of cases. A proof with a large number of cases leaves an impression that the theorem is only true by coincidence, and not because of some underlying principle or connection. Other types of proofs—such as proof by induction (mathematical induction)—are considered more elegant. However, there are some important theorems for which no other method of proof has been found, such as

- The proof that there is no finite projective plane of order 10.
- The classification of finite simple groups.
- The Kepler conjecture.
- The Boolean Pythagorean triples problem.

See also

- British Museum algorithm
- Computer-assisted proof
- Enumerative induction
- Mathematical induction

Notes

1. Reid, D. A; Knipping, C (2010), *Proof in Mathematics Education: Research, Learning, and Teaching*, Sense Publishers, p. 133, ISBN 978-9460912443.
2. S., Epp, Susanna (2011-01-01). *Discrete mathematics with applications* (https://www.worldcat.org/oclc/970542319). Brooks/Cole. ISBN 0495391328. OCLC 970542319 (https://www.worldcat.org/oclc/970542319).