Theorem: The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \textit{ and } \int_1^{\infty} f(x)dx$$

either both converge or both diverge.

Proof:

Partition the interval $[1, n]$ into $(n-1)$ unit intervals. The total areas of the inscribed rectangles and the circumscribed rectangles are

$$\sum_{i=2}^{n} f(i) = f(2) + f(3) + \ldots + f(n) \quad \textbf{Inscribed area}$$

and

$$\sum_{i=1}^{n-1} f(i) = f(1) + f(2) + \ldots + f(n-1) \quad \textbf{Circumscribed area}$$

The exact area under the graph of f from $x = 1$ to $x = n$ lies between the inscribed and circumscribed areas.

$$\sum_{i=2}^{n} f(i) \leq \int_1^n f(x)dx \leq \sum_{i=1}^{n-1} f(i)$$

Using the n^{th} partial sum, $S_n = f(1) + f(2) + \ldots + f(n)$, you can write the inequality as

$$S_n - f(1) \leq \int_1^n f(x)dx \leq S_{n-1}$$

Now, assuming that $\int_1^{\infty} f(x)dx$ converges to L, it follows that for $n \geq 1$,

$$S_n - f(1) \leq L \Longrightarrow S_n \leq L + f(1)$$

Consequently, (S_n) is bounded and monotonic, and by Theorem 9.5 it converges. So, $\sum a_n$ converges. For the other direction of the proof, assume that the improper integral diverges. Then $\int_1^n f(x)dx$ approaches infinity as $n \to \infty$, and the inequality $S_{n-1} \geq \int_1^n f(x)dx$ implies that (S_n) diverges. So, $\sum a_n$ diverges.