Example 1: Find the general solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos(x), \quad x > 0$$

Solution

To put this linear equation in standard form, we multiply by x to obtain

$$\frac{dy}{dx} - \frac{2}{x}y = x^2\cos(x)$$

Here $P(x) = \frac{-2}{x}$, so

$$\int P(x)dx = \int \frac{-2}{x}dx = -2\ln|x|$$

Thus, an integrating factor is

$$\mu(x) = e^{-2\ln|x|} = e^{\ln(x^{-2})} = x^{-2}$$

Multiplying

$$\frac{dy}{dx} - \frac{2}{x}y = x^2\cos(x)$$

by $\mu(x)$ yields

$$\underbrace{x^{-2}\frac{dy}{dx} - 2x^{-3}y}_{\frac{d}{dx}(x^{-2}y)} = \cos(x)$$

$$\frac{d}{dx}(x^{-2}y) = \cos(x)$$

We now integrate both sides and solve for y to find

$$x^{-2}y = \int \cos(x)dx = \sin(x) + C$$

$$y = x^2\sin(x) + Cx^2$$

It is easily checked yhat this solution is valid for all $x > 0$