1.6. Independent Events

1. *Definition.* Let A and B be two events.

   (a) If
   $$P(A \cap B) = P(A)P(B),$$
   then we say A and B are independent.

   (b) If
   $$P(A \cap B) \neq P(A)P(B),$$
   then we say A and B are not independent.

2. The following theorem follows immediately from the definition.

3. *Theorem.* If events A and B are independent and $P(B) > 0$, then

$$P(A|B) = P(A).$$

4. Roughly speaking, if A and B are independent, then whether B occurs or not has no influence on the likeliness of A.

5. *Theorem.* $\emptyset$ is independent of any event A. Similarly, S is independent of any event A.

— *Proof.* For example, if A is an event, then

$$P(A)P(\emptyset) = P(A) \times 0 = 0,$$

$$P(A \cap \emptyset) = P(\emptyset) = 0.$$

Hence,

$$P(A)P(\emptyset) = P(A \cap \emptyset).$$

This makes sense because, it does not matter whether A occurs or not, $\emptyset$ will never occur. So $\emptyset$ is independent of any event A.

6. *Example.*

Let the experiment be the toss of two coins in a row.

Let A be the event that the first coin turns up heads.

Let B be the event that the second coin turns up heads.

Then A and B are independent.

— *Proof.* The sample space is

$$S = \{hh, ht, th, tt\} .$$

It is clear that

$$A = \{hh, ht\} , \quad B = \{hh, th\} ,$$

and

$$A \cap B = \{hh\} .$$

It follows that

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{4}.$$

Since

$$P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = P(A \cap B),$$

A and B are independent events.

This conclusion makes sense because A is about the first coin and B is about the second coin, and it is true that the coins behave independently.

7. *Example.*

Let the experiment be the toss of two coins in a row.

Let A be the event that the first coin turns up heads.

Let B be the event that none of the two coins turns up heads.

Then A and B are not independent.

— *Proof.* The sample space is

$$S = \{hh, ht, th, tt\}.$$

It is clear that

$$A = \{hh, ht\}, \quad B = \{tt\},$$

and

$$A \cap B = \emptyset.$$

It follows that

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{4}, \quad P(A \cap B) = 0.$$

Since

$$P(A)P(B) = \frac{1}{8} \neq P(A \cap B),$$

A and B are not independent events.

Again, this conclusion makes sense. Note that $B \cap A = \emptyset$. This means that, if B occurs, then A has no choice but not to occur. So A and B are related that way, and they are not independent.

8. *Theorem.* If A and B are independent events, then A and B' are independent events.

— *Proof.* First, we collect some basic facts:

(a) $P(B') = 1 - P(B)$.

(b) Since A and B are independent, $P(A \cap B) = P(A)P(B)$.

(c) $A = (A \cap B) \cup (A \cap B')$.

(d) $P(A) = P(A \cap B) + P(A \cap B')$.

Now we summarize:

$$\begin{aligned} P(A \cap B') &= P(A) - P(A \cap B) = P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) = P(A)P(B'). \end{aligned}$$

9. More generally, we have the following theorem.

10. *Theorem.* Let A and B be two events. Then

$$\begin{aligned} A \text{ and } B \text{ are independent} &\iff A \text{ and } B' \text{ are independent} \\ &\iff A' \text{ and } B \text{ are independent} \\ &\iff A' \text{ and } B' \text{ are independent.} \end{aligned}$$

11. We now talk about the independence of three events.

12. *Definition.* Let A, B, C be events. If

    (a) A and B are independent, and

    (b) A and C are independent, and

    (c) B and C are independent,

    then we say A, B, and C are pairwise independent.

13. *Definition.* Let A, B, C be events. If

(a) A, B, and C are pairwise independent, and

(b) $P(A \cap B \cap C) = P(A)P(B)P(C)$,

then we say A, B, and C are mutually independent.

14. *Example.* Let the experiment be the toss of two coins in a row.

Let A be the event that the first coin turns up heads.

Let B be the event that the second coin turns up heads.

Let C be the event that both coins turn up the same.

Then A, B, and C are pairwise independent but not mutually independent.

— *Proof.* The sample space is

$$S = \{hh, ht, th, tt\} .$$

It is clear that

$$A = \{hh, ht\}\,, \quad B = \{hh, th\}\,, \quad C = \{hh, tt\}\,.$$

It follows that

$$A \cap B = A \cap C = B \cap C = \{hh\}\,.$$

The probabilities of the events are:

$$P(A) = P(B) = P(C) = \frac{1}{2}$$

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}.$$

It is easily verified that A, B, C are pairwise independent.

On the other hand,

$$A \cap B \cap C = \{hh\}.$$

Hence,

$$P(A \cap B \cap C) = \frac{1}{4}.$$

Since

$$P(A)P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \neq P(A \cap B \cap C),$$

the events A, B, C are not mutually independent.

15. *Example.*

Let the experiment be the toss of three coins in a row.

Let A be the event that the first coin turns up heads.

Let B be the event that the second coin turns up heads.

Let C be the event that the third coin turns up heads.

Then A, B, and C are mutually independent.

— *Proof.* The sample space is

$$S = \{hhh, hht, hth, thh, htt, tht, tth, ttt\} .$$

We have

$$A = \{hhh, hht, hth, htt\}, \quad P(A) = \frac{1}{2}.$$

$$B = \{hhh, hht, thh, tht\}, \quad P(B) = \frac{1}{2}.$$

$$C = \{hhh, hth, thh, tth\}, \quad P(C) = \frac{1}{2}.$$

$$A \cap B = \{hhh, hht\}, \quad P(A \cap B) = \frac{1}{4},$$

$$A \cap C = \{hhh, hth\}, \quad P(A \cap C) = \frac{1}{4},$$

$$B \cap C = \{hhh, thh\}, \quad P(B \cap C) = \frac{1}{4},$$

and,

$$A \cap B \cap C = \{hhh\}, \quad P(A \cap B \cap C) = \frac{1}{8}.$$

With these probabilities, we can easily verify the following

$$P(A)P(B) = P(A \cap B), \quad P(A)P(C) = P(A \cap C),$$

$$P(B)P(C) = P(B \cap C),$$

$$P(A)P(B)P(C) = P(A \cap B \cap C).$$

Hence, A,B,C are not only pairwise independent, but also mutually independent.