Linear Equations

Method for Solving Linear Equations:

(a) Write the equation in the standard form

$$\frac{dy}{dx}P(x)y = Q(x)$$

(b) Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)dx\right]$$

(c) Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{d}{dx}[\mu(x)y]$, obtain

$$\underbrace{\mu(x)\frac{dy}{dx} + P(x)\mu(x)y}_{} = \mu(x)Q(x)$$

$$\frac{d}{dx}\{\mu(x)y\} = \mu(x)Q(x)$$

(d) Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$$

Existence and Uniquenes of Solution

Suppose $P(x)$ and $Q(x)$ are continuous on an interval (a, b) that contains the point x_0. Then for any choice of initial value y_0, there exists a unique solution $y(x)$ on (a, b) to the initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \quad y(x_0) = y_0$$

In fact the solution is given by $y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)dx + C\right]$ for a suitable value of C.