1.2. Events

1. The set of all possible outcomes of a random experiment is called the sample space of the experiment and is denoted by S.

   The sample space is also called the outcome space or simply the space.

2. A sample point is an outcome that can not be further broken down into simpler components.

3. If $A \subset S$, then we say A is an event.

4. Let A be an event. If the random experiment is performed and the observed outcome is an element of A, then we say that A has occurred.

5. *Example.* Consider the experiment of tossing three coins in a row. The sample space of the experiment is

$$S = \{hhh, hht, hth, thh, tth, tht, htt, ttt\}.$$

Let A be the event that exactly one of three coins shows a head. That is,

$$A = \{tth, tht, htt\} \subset S.$$

If any of the three sample points tth, tht, htt is observed, then A has occuurred. For example, if the observed outcome is tht, then the event A has occurred.

6. We make a review of set algebra (and related concepts in probability theory).

   (a) The universal set S: In each mathematical problem there is a universal set, it includes every thing that we are interested and everything related.

       In a probability problem, the sample space of the experiment is the universal set, because it includes all possible outcomes. The sample space S is also called the sure event.

   (b) The empty set $\emptyset$.

       In probability, $\emptyset$ is called the impossible event. Note that, since $\emptyset \subset S$, the set $\emptyset$ is an event.

(c) Subset: Let A, B be sets. If very element of A is an element of B, then we say A is a subset of B, and we write $A \subset B$.

In a probability problem, if two events A, B satisfy the relation $A \subset B$, then it simply means that if A occurs then B occurs.

As an example, consider the experiment of tossing three coins in a row. In this experiment, w define random variable X to be the number of coins that show a head. Then the event $X = 1$ is a subset of the event $X \leq 1$. In fact,

$$S = \{hhh, hht, hth, thh, htt, tht, tth, ttt\},$$

$$(X = 1) = \{htt, tht, tth\}, \quad (X \leq 1) = \{htt, tht, tth, ttt\}.$$

It is clear that if $X = 1$ occurs, then $X \leq 1$ occurs. In other words, if $X = 1$, then $X \leq 1$.

(d) Union $A \cup B$: If A, B are sets, then

$$A \cup B = \{x : x \in A \text{ or } x \in B\}\,.$$

In probability, the union of two events is also an event. And, the event $A \cup B$ occurs if and only if either A or B occurs. (An example will be given shortly.)

(e) Intersection $A \cap B$: If A, B are sets, then

$$A \cap B = \{x : x \in A \text{ and } x \in B\}\,.$$

In probability, the intersection of two events is also an event. And, the event $A \cap B$ occurs if and only if both A and B occur. (An example will be given shortly.)

(f) Complement of A: If A is a set, then its complement is

$$A' = \{x \in S : x \notin A\}.$$

In probability theory, if A is an event, then A' is also a event. And, A occurs if and only if A' does not occur. (An example will be given shortly.)

7. *Example.* Let the experiment be the toss of a die, and let X be the number shown by the die. Then X is a random variable. It is clear that the sample space is

$$S = \{1, 2, 3, 4, 5, 6\} .$$

Let A be the event X is even, and let B be the event $X \geq 5$. That is,

$$A = \{2, 4, 6\} \subset S, \qquad B = \{5, 6\} \subset S.$$

It is clear that $A \cap B = \{6\}$. In other words, $A \cap B$ is the event that $X = 6$. It is clear that

$$A \cap B \text{ occurs} \iff X = 6 \iff \text{both } A \text{ and } B \text{ occur.}$$

8. *Example.* Let the experiment be the toss of a die, and let X be the number shown by the die. Then X is a random variable. The sample space is

$$S = \{1, 2, 3, 4, 5, 6\} .$$

If we let A be the event that $X \leq 3$. Then A' is the event that $X \geq 4$.

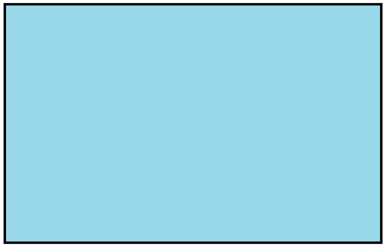
In other words, if $A = \{1, 2, 3\}$, then $A' = \{4, 5, 6\}$.

It is clear that

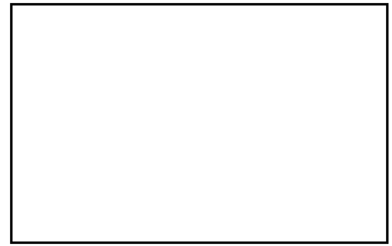
$$A \text{ occurs} \iff A' \text{ does not occur.}$$

For example, if the outcome $X = 1$ is observed, then A occurs and A' does not occur.

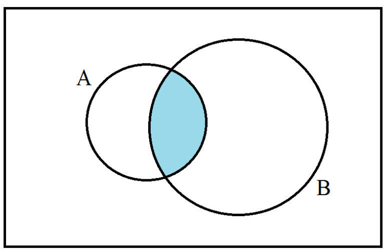
9. In set theory, we use Venn diagrams frequently to visualize relations between sets and operations on sets. In each of the following diagrams, we use the rectangular box to reprensent the sample space S.



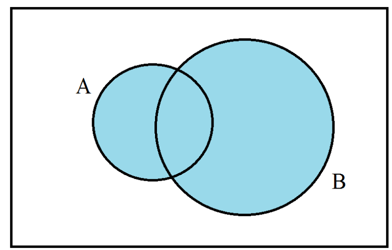
S



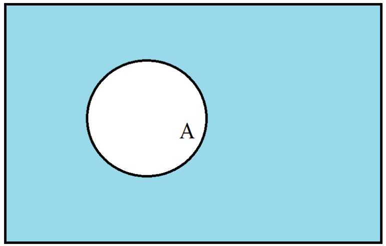
$\emptyset$



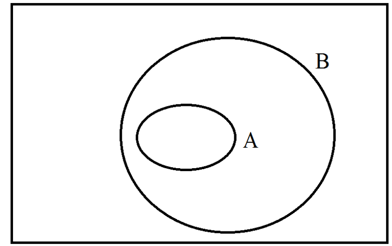
$A \cap B$



$A \cup B$



A'



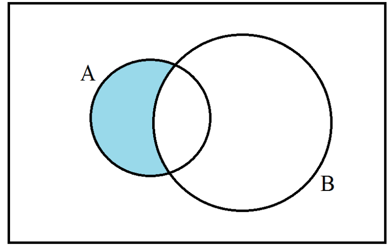
$A \subset B$

10. For convenience, sometimes we denote

$$A \backslash B = A \cap B'.$$

It is clear that $A \backslash B$ occurs if and only if

$$A \text{ occurs but } B \text{ does not occur.}$$



11. *Example.* If we toss two dice in a row, then each die shows a number. Let X be the sum of the two numbers.

Let A be the event that $X \geq 9$. Let B be the event that the second die turns out a number greater than 4.

Question: What is the size of $A \cap B$?

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \right\}.$$

It is clear that

$$A \cap B = \{(3,6), (4,5), (4,6), (5,5), (5,6), (6,5), (6,6)\}.$$

So $|A \cap B| = 9$.

12. *Definition.* Let $A_1, A_2, \cdots, A_k$ be events.

(a) If $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then we say the events $A_1, A_2, \cdots, A_k$ are pairwise disjoint.

(b) If

$$A_1 \cup A_2 \cup \cdots \cup A_k = S,$$

then we say the events $A_1, A_2, \cdots, A_k$ are exhaustive.

(c) If events $A_1, A_2, \cdots, A_k$ are pairwise disjoint and also exhaustive, then we say they form a partition of the sample space S.

13. *Example.* Let the experiment be the toss of three coins in a row. Let X be the number of coins that turn out heads. We will show that the events $X = 0$, $X = 1$, $X = 2$, and $X = 3$ form a partition of the sample space.

In fact, the sample space is

$$S = \{hhh, hht, hth, thh, tth, tht, htt, ttt\} .$$

It is clear that the events

$$(X = 3) = \{hhh\} , \qquad (X = 2) = \{hht, hth, thh\}$$

$$(X = 1) = \{tth, tht, htt\} \qquad (X = 0) = \{ttt\}$$

form a partition of the sample space.