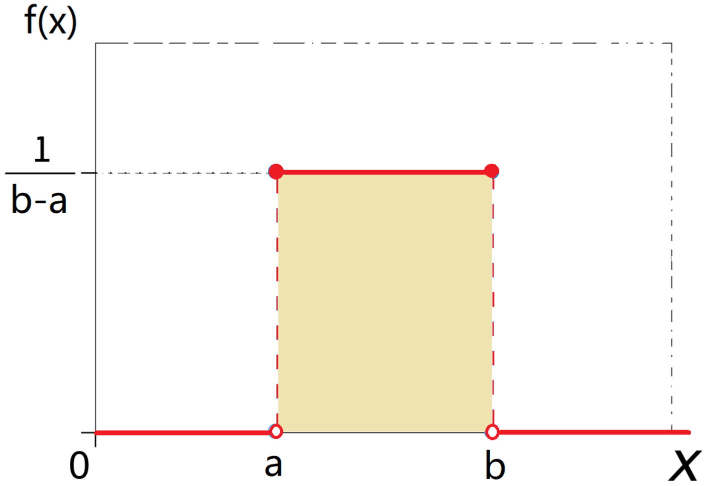
3.3. Uniform and Exponential Distributions

1. *Definition.* Let a and b be real numbers such that $a < b$. If X is a random variable and X has the pdf

$$f(x) = \begin{cases} 0, & x < a, \\ \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & x > b, \end{cases}$$

then we say X has the Continuous Uniform (a, b) distribution. The short notation for the Continuous Uniform (a, b) distribution is $U(a, b)$.

2. In the $U(a, b)$ distribution, the total mass of 1 is evenly distributed over the interval (a, b). Below is the plot of the pdf of $U(a, b)$.



3. *Formula.* If X has the $U(a, b)$ distribution, then

$$M(t) = \frac{e^{bt} - e^{at}}{(b-a)t},$$

$$E(X) = \frac{a+b}{2}, \quad Var(X) = \frac{(b-a)^2}{12}.$$

4. *Example.* Consider the Poisson process with parameter $\lambda > 0$. Let X be the waiting time until the first change (or arrival, or customer). We has shown that X has cdf

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and X has pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

5. *Definition.* Let $\theta > 0$ be a fixed parameter. If the random variable X has the pdf

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

then we say X has the Exponential (θ) distribution.

6. With this definition, we can rephrase the result in Item 4 as:

7. *Example.* Consider the Poisson process with parameter $\lambda > 0$. Let X be the waiting time until the first change (or the first arrival, or the first customer). Then, $X \sim$ Exponential $(1/\lambda)$.

8. *Formula.* (mean and variance of exponential distribution)

If X has the Exponential (θ) distribution, then

$$E(X) = \theta, \quad Var(X) = \theta^2,$$

and X has mgf

$$M(t) = \frac{1}{1 - \theta t}.$$

9. *Convention:* If X has the Exponential (θ) distribution, then X has pdf

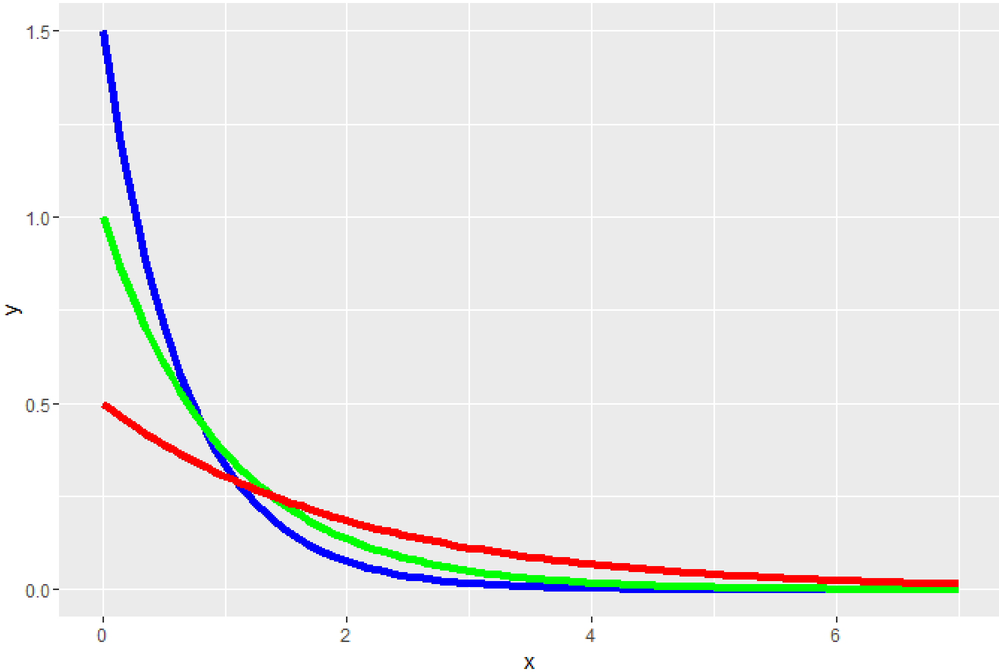
$$f(x) = \begin{cases} \frac{1}{\theta}e^{-x/\theta}, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{1}$$

From this moment on, we will adopt the convention of omitting the trivial part of the distribution. For example, we will simply write the pdf in (1) as

$$f(x) = \frac{1}{\theta}e^{-x/\theta}, \quad x \geq 0.$$

This convention can help us save a lot of writing.

10. The pdf's of Exponential $(2/3)$, Exponential (1), and Exponential (2) distributions are plotted on the next page. The three curves are in blue, green, and red, respectively.



1.5
1.0
0.5
0.0
y
0
2
4
6
x

11. *R Code:*

```
library(ggplot2)
h<-ggplot(data.frame(x=c(0,7)),aes(x=x))
h<-h+stat_function(fun=dexp,geom = "line",size=2,col="blue",args = (mean=1.5))
h<-h+stat_function(fun=dexp,geom = "line",size=2,col="green",args = (mean=1))
h<-h+stat_function(fun=dexp,geom = "line",size=2,col="red",args = (mean=0.5))
h
```