MATH 3332, Chapter 5 Exercise — Part 2

Problems:

1. Suppose that $X \sim N(0,1)$. Name the distribution of X^2 with all parameters of the distribution specified.

   Formula. If $X \sim N(0,1)$, then $X^2 \sim \chi^2(1)$.

   Hint. Apply the formula directly.

   Answer. $\chi^2(1)$

2. $X_1, X_2, \cdots, X_9$ are iid $N(0,1)$ random variables. ("iid" is the short for "independent and identically distributed.") Name the distribution of $T = X_1^2 + \cdots + X_9^2$ with all parameters of the distribution specified.

Formula.

(a) If $X \sim N(0,1)$, then $X^2 \sim \chi^2(1)$.

(b) If $X \sim \chi^2(m)$, $Y \sim \chi^2(n)$, and X, Y are independent, then $X + Y \sim \chi^2(m+n)$.

Hint. Apply the formula (a) first, and then apply formula (b).

Answer. $\chi^2(9)$

3. X is a random variable with $\chi^2(20)$ distribution.

   Y is a random variable with $N(0,1)$ distribution.

   X and Y are independent.

   Name the distribution of $T = \frac{Y}{\sqrt{X/20}}$ with all parameters specified.

   Formula. If $X \sim N(0,1)$, $Y \sim \chi^2(n)$, and X, Y are independent, then $X/\sqrt{Y/n}$ has Students's t-distribution of n degrees of freedom.

   Hint. Apply the formula directly.

   Answer. t-distribution with 20 degrees of freedom

4. X is a random variable with $N(2,3)$ distribution.

   Name the distribution of $2X+3$ with all parameters specified.

   Formula.

   (a) If $X \sim N(\mu, \sigma^2)$, then $cX \sim N(c\mu, c^2\sigma^2)$.

   (b) If $X \sim N(\mu, \sigma^2)$, then $X + c \sim N(\mu + c, \sigma^2)$.

   Hint. Apply formula (a) first to get the distribution of $2X$, then apply formula (b).

   Answer. $N(7, 12)$

5. X is a random variable with $N(3,1)$ distribution. Y is a random variable with $N(4,2)$ distribution. X and Y are independent.

   Name the distribution of $T = X + 3Y$ with all parameters specified.

   Formula.

   (a) If $X \sim N(\mu, \sigma^2)$, then $cX \sim N(c\mu, c^2\sigma^2)$.

   (b) If $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, and X, Y are independent, then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

   Hint. Apply formula (a) to get the distribution of $3Y$, then apply formula (b).

   Answer. $N(15, 19)$

6. $X_1, X_2, \cdots, X_{15}$ are iid $N(6, 2)$ distributions.

$$S^2 = \frac{1}{14} \sum_{i=1}^{15} (X_i - \bar{X})^2.$$

Name the distribution of $T = 7S^2$ with all parameters specified.

Formula. If the population $X \sim N(\mu, \sigma^2)$, and $X_1, \cdots, X_n$ are a random sample of size n from the population X, then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

Here, S^2 is the sample variance.

Hint. Collect the information $n = 15$ and $\sigma^2 = 2$ from the problem, then apply the formula directly.

Answer. $\chi^2(14)$

7. X is a random variable with $\chi^2(7)$ distribution. Y is a random variable with $\chi^2(8)$ distribution. X and Y are independent.

   Name the distribution of $T = X + Y$ with all parameters specified.

   Formula. If $X \sim \chi^2(m)$, $Y \sim \chi^2(n)$, and X, Y are independent, then $X + Y \sim \chi^2(m+n)$.

   Hint. Apply the formula directly.

   Answer. $\chi^2(15)$

8. X is a random variable with $\chi^2(6)$ distribution.

   Y is a random variable with $\chi^2(8)$ distribution.

   X and Y are independent.

   Name the distribution of $T = \frac{X/6}{Y/8}$ with all parameters specified.

   Formula. If $X \sim \chi^2(m)$, $Y \sim \chi^2(n)$, and X, Y are independent, then

$$\frac{X/m}{Y/n}$$

   has the F-distribution with m and n degrees of freedom.

   Hint. Apply the formula directly.

   Answer. F distribution with 6 and 8 degrees of freedom

9. X_1, X_2, X_3 are iid geometric$(1/3)$ distributions.

Name the distribution of $T = X_1 + X_2 + X_3$ with all parameters specified.

Formula.

(a) The Geometric(p) distribution is the Negative Binomial distribution with parameters $(1, p)$.

(b) If $X \sim$ Negative Binomial(m, p), $Y \sim$ Negative Binomial(n, p), and X, Y are independent, then $X + Y \sim$ Negative Binomial$(m + n, p)$.

Hint. Apply the formula (a), then (b).

Answer. Negative Binomial$(3, 1/3)$

10. Suppose that $X \sim Binomial(4, 1/3)$, $Y \sim Binomial(5, 1/3)$, X and Y are independent. Name the distribution of $X + Y$ with all parameters specified.

    Formula. If $X \sim Binomial(m, p)$ $Y \sim Binomial(n, p)$, and X, Y are independent, then $X + Y \sim Binomial(m + n, p)$.

    Hint. Apply the formula directly.

    Answer. Binomial$(9, 1/3)$

11. Suppose that $X \sim Negative\ Binomial(4, 1/3)$,

    $Y \sim Negative\ Binomial(5, 1/3)$, X and Y are independent. Name the distribution of $X + Y$ with all parameters specified.

    Formula. If $X \sim$ Negative Binomial(m, p), $Y \sim$ Negative Binomial(n, p), and X, Y are independent, then $X + Y \sim$ Negative Binomial$(m + n, p)$.

    Hint. Apply the formula directly.

    Answer. Negative Binomial $(9, 1/3)$

12. Suppose that $X \sim$ Student's t-distribution with 5 degrees of freedom. Name the distribution of X^2 with all parameters specified.

Formula. If $X \sim Student's$ t-distribution with n degrees of freedom, then X^2 has the F-distribution with 1 and n degrees of freedom.

Hint. Apply the formula directly.

Answer. F-distribution with 1 and 5 degrees of freedom

13. Suppose that X, Y, Z are mutually independent, and

$$E(X) = 3, \quad E(Y) = -4, \quad E(Z) = 11.$$

Find the expectation of $2X + 5Y + Z$.

Formula. $E(aX + bY) = aE(X) + bE(Y)$.

Hint. Apply the formula directly.

14. Suppose that X, Y, Z are mutually independent, and

$$Var(X) = 3, \quad Var(Y) = 4, \quad Var(Z) = 11.$$

Find the variance of $2X + 5Y + Z$.

Formula. If X, Y are independent, then

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y).$$

Hint. Apply the formula directly.

15. Suppose that X, Y, Z are iid $Gamma(2, 3)$ distributions. Name the distribution of $X + Y + Z$ with all parameters specified.

    Formula. If $X \sim$ Gamma(α_1, β), $Y \sim$ Gamma(α_2, β), and X, Y are independent, than $X + Y \sim$ Gamma$(\alpha_1 + \alpha_2, \beta)$.

    Hint. Apply the formula directly.

    Answer. Gamma$(6, 3)$

16. Suppose that X, Y, Z are iid $Exponential(3)$ distributions. Name the distribution of $X + Y + Z$ with all parameters specified.

Formula.

(a) If X ~Exponential(β) distribution, then X ~Gamma$(1, \beta)$.

(b) If $X \sim$ Gamma(α_1, β), $Y \sim$ Gamma(α_2, β), and X, Y are independent, than $X + Y \sim$ Gamma$(\alpha_1 + \alpha_2, \beta)$.

Hint. Apply formula (a) first, then (b).

Answer. Gamma$(3, 3)$

17. Suppose that X and Y are independent random variables, and suppose that $X \sim Poisson(4)$, $Y \sim Poisson(6)$. Name the distribution of $X+Y$ with all parameters specified.

    Formula. If $X \sim$Poisson(λ_1), $Y \sim$ Poisson(λ_2), and X, Y are independent, then $X + Y \sim$Poisson$(\lambda_1 + \lambda_2)$.

    Hint. Apply the formula directly.

    Answer. Poisson(10)