Review of Polynomials and Rational Functions

A major focus of Integral Calculus is to integrate (find the antiderivative of) several classes of functions. An important class of functions is *Rational Functions*. A *Rational Function* is a function formed as the quotient of two polynomials, with the denominator being at least a first order (linear) polynomial. Students who have had a comprehensive precalculus/college algebra education should have studied *Polynomial* and *Rational Functions*.

This topic provides a "refresher" overview of the important definitions and properties needed to understand how to integrate *Rational Functions*. Students who did not adequately cover *Polynomial* and *Rational Functions* in precalculus or college algebra should embark on self-study, such as Khan Academy, to prepare for the upcoming lessons on integrating *Rational Functions*.

What is a Polynomial Function?

A *Polynomial Function* of one variable, $P_n(x)$, is a mathematical function that may be expressed in the form:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

where x is the independent variable, n is a *Whole Number* $\{0, 1, 2, 3, 4, \cdots\}$, and each a_i is a constant coefficient with $a_n \neq 0$ (for Calculus we assume all constant coefficients are *Real*).

a_n is called the *Leading Coefficient* and $a_n x^n$ is called the *Leading Term*. Importantly note that while $a_n \neq 0$, any or all of the other constant coefficients may be zero.

Example of a *Polynomial Function*: $P_3(x) = -5x^3 - 2x + 5$

What is the Degree of a Polynomial Function?

The degree of a *Polynomial Function* is n. The degree of the example right above is 3.

A *Polynomial Function* which is a non-zero constant has degree 0 or *zero degree*. From the above definition, a *Polynomial Function* cannot be zero itself, but there are times when it may be convenient to define a zero polynomial—in this case the *Polynomial Function* $P(x) = 0$ has *no degree*, which is NOT the same as *zero degree*.

What is a Linear Function?

A *Linear Function* is a *Polynomial Function* of degree 1, or of the first degree (note that $x = x^1$):

$$P_1(x) = a_1 x + a_0$$

Since *Linear Functions* are often encountered in mathematics, the usual convention is to express the constant coefficients with different notation in what is called the "slope-intercept" form:

$$P_1(x) = mx + b$$

What is a Quadratic Function?

A *Quadratic Function* is a *Polynomial Function* of degree 2:

$$P_2(x) = a_2 x^2 + a_1 x + a_0$$

Since *Quadratic Functions* are often encountered in mathematics, the usual convention is to express the constant coefficients using the following notation:

$$P_2(x) = ax^2 + bx + c$$

Two Types of Quadratic Functions: Reducible and Irreducible

Quadratic Functions with *Real* coefficients may be categorized as either *reducible* or *irreducible*. This is *very* important in our study of integrating *Rational Functions*.

A reducible *Quadratic Function* is a *Quadratic Function* with *Real* coefficients which is factorable into two linear (polynomial) factors with *Real* coefficients. For example:

$$P_2(x) = 2x^2 - 7x - 15 = (2x + 3)(x - 5)$$

An irreducible *Quadratic Function* is a *Quadratic Function* with *Real* coefficients which is not factorable into two linear (polynomial) factors with *Real* coefficients. Example:

$$P_2(x) = x^2 - 3x + 15$$

is not factorable into two linear factors with *Real* coefficients.

The test for reducible vs. irreducible of a *Quadratic Function* with *Real* coefficients is related to its zeros (the zeros will be defined and described later), and this is determined by analyzing the *discriminant* of the *Quadratic Function*. For the *Quadratic Function* $P_2(x) = ax^2 + bx + c$, its *discriminant* is $\Delta = b^2 - 4ac$. When $\Delta \geq 0$, the *Quadratic Function* with *Real* coefficients has two *Real* zeros and is thus *reducible*. When $\Delta < 0$, the *Quadratic Function* with *Real* coefficients has two complex conjugate zeros and is *irreducible*.

It is important to note that an irreducible *Quadratic Function* may be factored into two linear factors with *Complex* constant coefficients, but in Integral Calculus this does not help with integrating *Rational Functions.*

Factoring of Polynomial Functions with Real Coefficients

The following property is very important, and is derived from the *Fundamental Theorem of Algebra.*

All *Polynomial Functions* of degree 3 or higher with *Real* coefficients are completely factorable into linear and/or irreducible quadratic factors with *Real* coefficients. For example:

$$6x^4 + 17x^3 + 6x^2 + 22x - 15 = (2x-1)(x+3)(3x^2+x+5)$$

In general, factoring *Polynomial Functions* of degree 3 and higher is difficult and for degree 5 and higher there is no "closed form" solution for the *Real* zeros which are used to build the factors (see below); for degree 3 and higher the *Real* zeros are usually found by numerical methods, such as using *Newton's Method.* For *Polynomial Functions* with *Integer* or *Rational* coefficients, the *Rational Root Theorem*, which is studied in precalculus/college algebra, may be used to find linear factors with *Integer*/*Rational* coefficients (if any exist). There is a close connection between factoring polynomials and the zeros of polynomials, given next.

The Zeros of a Polynomial Function

The complete set of numbers which solve the n^{th} degree *Polynomial Function Equation:*

$$P_n(x) = 0$$

is referred to as the zeros of the *Polynomial Function.* When the *Polynomial Function* has *Real* coefficients, its zeros may comprise any mix of *Real* numbers and conjugate pairs of *Complex* numbers. For any n^{th} degree *Polynomial Function*, there are a total of n zeros (some of which may be identical.)

The Relationship of the Factors and the Zeros of a Polynomial Function

The following *Polynomial Function*:

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

may be factored as follows:

$$P_n(x) = a_n (x - z_1)(x - z_2)\cdots(x - z_n)$$

where z_i is the i^{th} zero of the n zeros of $P_n(x)$. (As noted previously, some zeros may be identical and yet are counted separately, and the above factorization reveals why this distinction is important!) Note that the zeros of a *Polynomial Function* may be *Real* and/or *Complex*.

Thus, if we can fully factor a *Polynomial Function*, we know its zeros; if we know all the zeros of a *Polynomial Function*, we can factor the function!

This equivalence is a result of the *Zero Product Rule of Algebra*, which states that if $ab = 0$, then either a is zero or b is zero, or both are zero.

Importantly note that when a *Polynomial Function* has *Real* coefficients, any complex zeros come in conjugate pairs (as previously mentioned). When the two factors associated a complex conjugate pair of zeros are multiplied together, the result is an irreducible quadratic with *Real* coefficients. This explains the previous comment that any *Polynomial Function* with *Real* coefficients is factorable into linear and/or irreducible quadratic factors.

Definition of a Rational Function

A *Rational Function* is defined to be the quotient of two polynomials:

$$R(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ is a polynomial of any degree, and $Q(x)$ is a polynomial of degree $n \geq 1$.

(If $Q(x)$ is of zero degree, that is, it is a non-zero constant, then the *Rational Function* is a *Polynomial Function*, which we know how to integrate, so for the purposes of Integral Calculus we are only interested in the denominator of a *Rational Function* being a polynomial of degree $n \geq 1$.)

Example:

$$R(x) = \frac{2x^2 - 3x + 1}{-5x^3 - 2x + 5}$$

Reduced Rational Function

Since a *Rational Function* is a quotient of two polynomials, and each polynomial is fully factorable into linear and/or irreducible quadratic factors (throughout this topic document we are assuming *Real* coefficients), then it is possible that there are common factors between the numerator and denominator of the *Rational Function*.

For purposes of integration, any common factors may be divided out, thereby simplifying the integration (more on this below). However, make sure that the factor removal is done one-for-one. The following *Rational Function*:

$$R(x) = \frac{(4x+7)(x-1)(x-1)(3x-5)}{(2x^2-3)(x-1)(x+2)}$$

may be reduced by removing the $(x-1)$ factor, once from the numerator and once from the denominator, resulting in the *Reduced Rational Function*:

$$R_{red}(x) = \frac{(4x+7)(x-1)(3x-5)}{(2x^2-3)(x+2)}$$

Note that in the above example, the *Reduced Rational Function* still contains a $(x-1)$ factor in the numerator.

For purposes of integration, we should use the *Reduced Rational Function* instead of the original *Rational Function*. The reason is somewhat complicated and is related to “holes” in the graph, the limit of improper integrals, integrability, etc. However, it is important to never forget that the original *Rational Function* is not defined at any x-values where its denominator is zero, even if the common factors, graphically-speaking, create an infinitesimal “hole” in an otherwise smooth graph.

About the Zeros of the Denominator of a Reduced Rational Function

After a *Rational Function* is *reduced* in preparation for integration, it is necessary to factor its denominator, if not already done, or at least determine all the *Real* zeros of the denominator, if any. (Finding all the zeros of any polynomial assists in factoring, and vice versa, as previously mentioned.)

Provided the *Rational Function* is fully reduced, the *Real* zeros of the denominator define vertical asymptotes where the original *Rational Function* tends to positive and/or negative infinity (that is, the original *Rational Function* is undefined at these x-values.) For definite integration of the original *Rational Function*, the open interval of integration *must not* contain any of these x-values; if at least one of the end-points of the interval of integration is one of these x-values, then the integral must be analyzed as an *Improper Integral* (the use of the word “improper” here is not related to the use of the word “improper” in the following section.)

Definition of Improper and Proper Rational Functions

Given the *Rational Function*:

$$R(x) = \frac{P_m(x)}{Q_n(x)}$$

where $P_m(x)$ is a polynomial of degree m and $Q_n(x)$ is a polynomial of degree n, then $R(x)$ is *Improper* if $m \geq n$, and $R(x)$ is *Proper* if $m < n$.

Examples:

$$R(x) = \frac{2x^4 - 3x^2 + x + 7}{3x^2 - 4x + 5} \quad (\textit{Improper Rational Function})$$

$$R(x) = \frac{3x^2 - 4x + 5}{2x^4 - 3x^2 + x + 7} \quad (\textit{Proper Rational Function})$$

A more "catchy", easier-to-remember terminology is to say that an *Improper Rational Function* is "top or even heavy", while a *Proper Rational Function* is "bottom heavy" (where "heavy" refers to the higher polynomial degree.)

Converting an Improper Rational Function into a Proper Rational Function

For the purposes of integrating *Rational Functions*, we must first convert any *Improper Rational Function* into a *Proper Rational Function*. Specifically, an *Improper Rational Function* may always be converted to a polynomial plus a *Proper Rational Function*:

$$R_{improper}(x) = \frac{P_m(x)}{Q_n(x)} = S_{m-n}(x) + R_{proper}(x)$$

where $m \geq n$ and $S_{m-n}(x)$ is a polynomial of degree $m - n$. The *Proper Rational Function* $R_{proper}(x)$ will have the denominator $Q_n(x)$.

This conversion is accomplished using either *Long Division* or, when the denominator is a linear factor with leading coefficient of 1, *Synthetic Division*.

(Note that *reduction*, the removal of common factors between the denominator and numerator as previously described, may be done either before or after conversion of an *Improper Rational Function*.)

Examples (with the *Improper and Proper Rational Functions* highlighted in yellow):

$$\frac{x^3 - 5x^2 + 11x - 10}{x - 3} = x^2 - 2x + 5 + \frac{5}{x - 3}$$

$$\frac{2x^3 - 5x + 4}{x^3 + x^2 + x + 1} = 2 - \frac{2x^2 + 7x - 2}{x^3 + x^2 + x + 1}$$

We are actually ready to integrate (find the antiderivative) of the first *Rational Function* above:

$$\int \frac{x^3 - 5x^2 + 11x - 10}{x - 3}\,dx = \int \left(x^2 - 2x + 5 + \frac{5}{x-3}\right) dx$$

$$\int \frac{x^3 - 5x^2 + 11x - 10}{x - 3}\,dx = \int (x^2 - 2x + 5)\,dx + \int \left(\frac{5}{x-3}\right) dx$$

$$\int \frac{x^3 - 5x^2 + 11x - 10}{x - 3}\,dx = \frac{1}{3}x^3 - x^2 + 5x + 5\ln|x-3| + C$$

Note that the closed interval of integration for any definite integral of this *Rational Function* may not contain $x = 3$.

With regards to the second *Rational Function* above, its denominator is easily factorable, giving us a glimpse as to where we are heading:

$$\frac{2x^3 - 5x + 4}{x^3 + x^2 + x + 1} = 2 - \frac{2x^2 + 7x - 2}{x^3 + x^2 + x + 1} = 2 - \frac{2x^2 + 7x - 2}{(x^2+1)(x+1)}$$

The numerator in the above right *Rational Expression* (a *reducible* quadratic polynomial) can also be factored (although this is unnecessary for integration purposes); the factorization yields irrational coefficients (containing $\sqrt{65}$) for the two linear factors.

Completing the Square

Completing the Square is a technique to convert any quadratic polynomial expression $ax^2 + bx + c$ into an expression of the form $u^2 \pm m^2$ which then allows for easier integration when the quadratic polynomial expression forms the denominator of a *Rational Function*. In general:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

$$ax^2 + bx + c = a\left[\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a}\right] = a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right]$$

$$\boxed{ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}}$$

The Method of Partial Fractions or Partial Fraction Decomposition

The previous topics have provided the necessary definitions, insights, and tools into understanding and using the *Method of Partial Fractions* or *Partial Fraction Decomposition*, which "decomposes" a *Rational Function* into a sum of smaller and directly integrable *Polynomial* and *Rational Functions*.

The details of how to express and determine the decomposition will not be given here as that will be discussed in class and is treated in most precalculus and Calculus textbooks. Khan Academy has a set of videos on this topic.

However, it is important to stress that the *Method of Partial Fractions* can only decompose a *Proper Rational Function*. If one starts with an *Improper Rational Function*, it must first be converted to the sum of a polynomial and a *Proper Rational Function* using *Long Division* or, when applicable, *Synthetic Division*. The *Method of Partial Fractions* is then applied to decompose the *Proper Rational Function* portion of the converted expression.

Reduction, the removal of common factors between the numerator and denominator, if possible, may be done either before or after the conversion of an *Improper Rational Function*. If *reduction* is possible but not done, the *Method of Partial Fractions* will "filter out" the common factors anyway, so best to do the *reduction* before applying the *Method of Partial Fractions* to reduce the overall work.

A Knotty Integral Sometimes Encountered After Partial Fraction Decomposition

When a *Proper Rational Function* contains a single irreducible quadratic factor in its denominator, the integral of the decomposed term associated with the irreducible quadratic factor will be of the following general form (all constant coefficients are *Real* with $a \neq 0$; the constants e or f may either be zero, but not both as that is the trivial case of integrating $\int 0\,dx = C$):

$$\int \frac{ex+f}{ax^2+bx+c}\,dx \quad \text{where } b^2-4ac<0$$

In class, one or more specific integrals of this form will be integrated, to show how it is done, suggesting there is a general form. (Essentially we re-express the denominator into the form u^2+1 using the technique of *Completing the Square* previously described.)

Without proof, the general integral, for $b^2-4ac<0 \;\text{ or }\; 4ac-b^2>0$, is:

$$\boxed{\int \frac{ex+f}{ax^2+bx+c}\,dx = \frac{e}{2a}\ln\left|ax^2+bx+c\right| + \frac{2af-be}{a\sqrt{4ac-b^2}}\tan^{-1}\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right) + C}$$

For the situation where the numerator is a constant ($e = 0$), the above formula reduces to:

$$\int \frac{f}{ax^2 + bx + c}\,dx = \frac{2f}{\sqrt{4ac - b^2}} \tan^{-1}\left(\frac{2ax + b}{\sqrt{4ac - b^2}}\right) + C$$

Example

Integrate the following:

$$\int \frac{4x + 3}{2x^2 + 2x + 1}\,dx$$

The constant coefficients to substitute into the general formula are:

$$a = 2; \quad b = 2; \quad c = 1; \quad e = 4; \quad f = 3$$

Since $4ac - b^2 = 4 \cdot 2 \cdot 1 - 2^2 = 4 > 0$ (which means the denominator is an irreducible quadratic), we may use the general formula previously given.

Substituting:

$$\int \frac{4x + 3}{2x^2 + 2x + 1}\,dx =$$

$$\frac{4}{2 \cdot 2} \ln|2x^2 + 2x + 1| + \frac{2 \cdot 2 \cdot 3 - 2 \cdot 4}{2\sqrt{4 \cdot 2 \cdot 1 - 2^2}} \tan^{-1}\left(\frac{2 \cdot 2 \cdot x + 2}{\sqrt{4 \cdot 2 \cdot 1 - 2^2}}\right) + C$$

Simplifying:

$$\int \frac{4x + 3}{2x^2 + 2x + 1}\,dx = \ln|2x^2 + 2x + 1| + \tan^{-1}(2x + 1) + C \qquad \blacksquare$$