Theorem 8.16

A graph G is 2-factorable *if and only if* G is r-regular for some positive even integer r.

Proof:

We have already observed that every 2-factorable graph r-regular for some positive even integer r. Therefore, we need only establish the converse. Let G be an r-regular graph, where $r = 2k$ and $k \geq 1$. Without loss of generality, we may assume that G is connected. By Theorem 6.1, G is Eulerian and therefore contains an Eulerian circuit C. (Of course, a vertex of G can appear more than once in C. In fact, each vertex of G appears exactly k times in C.)

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. We construct a bipartite graph G with partite sets

$$U = \{u_1, u_2, \ldots, u_n\}\, and\, W = \{w_1, w_2, \ldots, w_n\}$$

where the vertices u_i and $w_j (1 \leq i, j \leq n)$ are adjacent in H if v_j immediately follows v_i on C. Since every vertex of G appears exactly k times in C, the graph H is k-regular. By Theorem 8.15, H is 1-factorable and so H can be factored into k 1-factors $F_1', F_2', \ldots, F_k'$.

Next, we show each 1-factor $F_i' (1 \leq i \leq k)$ of H corresponds to a 2-factor F_i of G. Consider the 1-factor F_1', for example. Since F_1' is a perfect matching of H, it follows that $E(F_1')$ is an independt set of k edges of H, say

$$E(F_1') = \{u_1 w_{i1}, u_2 w_{i2}, \ldots u_n w_{in}\}$$

where the integers $i_1, i_2, \ldots i_n$ are the integers $1, 2, \ldots, n$ in some order and $i_j \neq j$ for each $j(1 \leq j \leq n)$. Suppose that $i_t = 1$. Then the 1-factor F_1' gives rise to a cycle $C^{(1)} = (v_1, v_{i1}, \ldots, v_t, v_{i_t} = 1)$. If $C^{(1)}$ has length n, then the Hamiltonian cycle $C^{(1)}$ of G is a 2-factor of G. If the length of $C^{(1)}$ is less than n, then there is a vertex of v_ℓ of G that is not o $C^{(1)}$. Suppose that $i_s = \ell$. This gives rise to a second cycle $C^{(2)} = (v_\ell, v_{i\ell}, \ldots, v_{i_s} = v_\ell)$. Continuing in this manner, we obtain a collection of pairwise vertex-disjoint cycles that contain each vertex of G once, producing a 2-factor F_1 of G. In general then, the 1-factorization of H into 1-factors $F_1', F_2', \ldots, F_k'$ produces a 2-factorization of G into 2-factors $F_1, F_2, \ldots, F_k$ as desired.