Particular Integrals by the Method of Undetermined Coefficients

The general solution of the 2nd order, linear, non-homogeneous ODE:

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = F(x) \qquad (*)$$

is given by:

$$y = y_c + y_p$$

where y_p is any particular solution of $(*)$ + y_c is the general solution of the corresponding homogeneous equation (the complimentary equation):

$$a_0(x)\,y'' + a_1(x)\,y' + a_2(x)\,y = 0$$

Notes ① $y_c(x)$ is called the complimentary function.

② Similar results hold for n'th order linear equations.

③ If equation $(*)$ has constant coefficients and $F(x)$ is a sum of polynomials, exponentials + sines or cosines we can often determine a particular integral by the <u>method of undetermined coefficients</u> which is essentially a technique in which we guess the form of the particular integral. This is best illustrated by example:

Example 1 Solve : $y'' - 3y' + 2y = x+1$

C.F. $y'' - 3y' + 2y = 0$

Aux. Eqn. $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$\therefore m = 1, 2$

$\therefore y_c = c_1 e^{1.x} + c_2 e^{2.x}$

P.I. Try $y_p = ax + b$

Then $y_p' = a$ & $y_p'' = 0$.

Substitute : $y_p'' - 3y_p' + 2y_p = x+1$

$0 - 3a + 2(ax+b) = x+1$

$\therefore 2ax + (2b-3a) = x+1$

Match like terms : $2a = 1$ $\quad$ $2b - 3a = 1$

$a = 1/2$ $\quad$ $2b = 1 + 3a = 5/2$

$\therefore b = 5/4$

$\therefore y_p = \frac{1}{2}x + \frac{5}{4}$.

Gen. Soln. $y = y_c + y_p = c_1 e^x + c_2 e^{2x} + \frac{1}{2}x + \frac{5}{4}$

Example 2 Solve : $y'' - 3y' + 2y = e^{3x}$

C.F. $y_c = c_1 e^x + c_2 e^{2x}$

P.I. Try $y_p = a e^{3x}$

Then $\quad y_p' = 3ae^{3x} \;\&\; y_p'' = 9ae^{3x}$

Substitute : $\quad y_p'' - 3y_p' + 2y_p = e^{3x}$

$$9ae^{3x} - 9ae^{3x} + 2ae^{3x} = e^{3x}$$
$$2ae^{3x} = e^{3x}$$
$$\therefore\; 2a = 1$$
$$a = 1/2$$

$$\therefore\; y_p = \tfrac{1}{2}e^{3x}$$

<u>Gen. Soln.</u> $\quad y = y_c + y_p = c_1e^x + c_2e^{2x} + \frac{1}{2}e^{3x}$

<u>Example 3</u> Solve : $y'' - 3y' + 2y = e^x$

<u>C.F.</u> $\quad y_c = c_1e^x + c_2e^{2x}$

<u>P.I.</u> An expression of the form $y_p = ae^x$ will not work as e^x is a solution to the homogeneous eqn. $y'' - 3y' + 2y = 0$

Try instead, $\quad y_p = axe^x$

Then, $\quad y_p' = a(x+1)e^x \;\&\; y_p'' = a(x+2)e^x$

Substitute : $\quad y_p'' - 3y_p' + 2y_p = e^x$

$$a(x+2)e^x - 3a(x+1)e^x + 2axe^x = e^x$$
$$ax + 2a - 3ax - 3a + 2ax = 1$$
$$2a - 3a = 1$$
$$a = -1$$

$$\therefore \quad y_p = -xe^x$$

Gen Soln. $\quad y = c_1 e^x + c_2 e^{2x} - xe^x$

Example 4 $\quad$ Solve : $y'' + 9y = \sin 2x$

C.F. $\quad y'' + 9y = 0$

Aux Eqn. $\quad m^2 + 9 = 0$

$$m = \pm 3i$$

$$y_c = c_1 \cos 3x + c_2 \sin 3x$$

P.I. $\quad$ Try $\quad y_p = a\cos 2x + b\sin 2x$

Then $\quad y_p' = -2a\sin 2x + 2b\cos 2x$

$+ \quad y_p'' = -4a\cos 2x - 4b\sin 2x$

Substitute : $\quad y_p'' + 9y_p = \sin 2x$

$$(-4a\cos 2x - 4b\sin 2x) + 9(a\cos 2x + b\sin 2x) = \sin 2x$$

$$\therefore 5a\cos 2x + 5b\sin 2x = \sin 2x$$

$$= 0 \cdot \cos 2x + 1 \cdot \sin 2x$$

Match like terms : $\quad 5a = 0 \qquad 5b = 1$

$$a = 0 \qquad b = 1/5$$

$$\therefore \quad y_p = \frac{1}{5}\sin 2x$$

Gen. Soln. $\quad y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{5}\sin 2x$

Example 5 Solve : $y'' + 9y = \cos 3x$

C.F. $y_c = c_1 \cos 3x + c_2 \sin 3x$

P.I. Try $y_p = x(a\cos 3x + b\sin 3x)$

Then $y_p' = x(-3a\sin 3x + 3b\cos 3x) + (a\cos 3x + b\sin 3x).1$

and $y_p'' = x(-9a\cos 3x - 9b\sin 3x) + (-3a\sin 3x + 3b\cos 3x)$
$+ (-3a\sin 3x + 3b\cos 3x)$

Substitute $y_p'' + 9y_p = \cos 3x$

$-9ax\cos 3x - 9bx\sin 3x + 9ax\cos 3x + 9bx\sin 3x = \cos 3x$
$-6a\sin 3x + 6b\cos 3x$

$\therefore \quad -6a\sin 3x + 6b\cos 3x = 0.\sin 3x + 1.\cos 3x$

$\therefore \quad -6a = 0 \quad \Rightarrow \quad a = 0, \quad b = \frac{1}{6}$
$6b = 1$

$\therefore \quad y_p = \frac{1}{6} x \sin 3x$

Gen. Soln. $y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{6} x \sin 3x$

Example 6 Solve : $y'' - 3y' + 2y = \cos x + e^{2x}$

C.F. $y_c = c_1 e^{x} + c_2 e^{2x}$

P.I. Try $y_p = a\cos x + b\sin x + cxe^{2x}$

Show that $a = \frac{1}{10}$, $b = -\frac{3}{10}$, $c = 1$

Gen. Soln. $y = c_1 e^x + c_2 e^{2x} + \frac{1}{10}(\cos x - 3\sin x) + xe^{2x}$

Example 7 Solve : $y'' + 2y' + y = 4e^{-x}$

$y(0) = 1, \quad y'(0) = 0$

C.F.

$$y'' + 2y' + y = 0$$

$$m^2 + 2m + 1 = 0$$

$$m = -1, -1$$

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

P.I. Try $y_p = a x^2 e^{-x}$

Show that $a = 2$

Gen. Soln. $y = c_1 e^{-x} + c_2 x e^{-x} + 2x^2 e^{-x}$

Part Soln. $y = (1 + x + 2x^2) e^{-x}$

Example 8 Solve : $y^{(6)} - y^{(4)} = \cos x$

C.F. Aux Eqn : $m^6 - m^4 = 0$

$$m^4(m^2 - 1) = 0$$

$$m = 0, 0, 0, 0, -1, 1$$

$$y_c = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^x + c_6 e^{-x}$$

P.I. Try $y_p = a\cos x + b\sin x$

Show that $a = -\frac{1}{2}$, $b = 0 \Rightarrow y_p = -\frac{1}{2}\cos x$

Gen. Soln. $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{x} + c_6 e^{-x} - \frac{1}{2}\cos x$

Example 9 Solve: $y^{(6)} - y^{(4)} = x + 1$

C.F. $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{x} + c_6 e^{-x}$

P.I. Try $y_p = x^4(ax+b) = ax^5 + bx^4$

Show that $a = -\frac{1}{120}$, $b = -\frac{1}{24} \Rightarrow y_p = -\frac{1}{120}x^4(x+5)$

Gen. Soln. $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{x} + c_6 e^{-x} - \frac{x^4}{120}(x+5)$

Example 10 Solve: $y''' - y'' + y' - y = x + 2e^{x}$

C.F. $y_c = c_1 e^{x} + c_2\cos x + c_3 \sin x$

P.I. Try $y_p = ax + b + cxe^{x}$

Show that $a = -1$, $b = -1$, $c = 1 \Rightarrow y_p = -x - 1 + xe^{x}$

Gen. Soln. $y = c_1 e^{x} + c_2\cos x + c_3\sin x - x - 1 + xe^{x}$

HW Page 182 #'s 9, 13, 15, 17, 23, 33, 34, 35
Page 188 #'s 23, 25, 27, 29, 37, 39