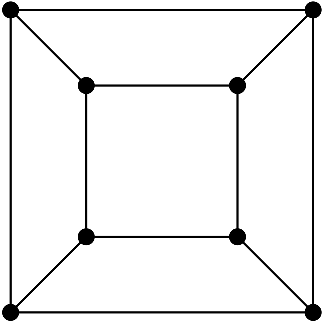
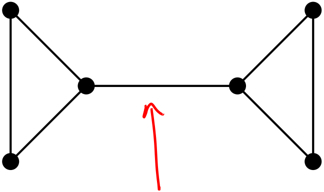
4.1: Bridges

- We know what it means for a graph to be connected.
- But some graphs are "more connected" than others



Taking out any single edge keeps the graph connected.



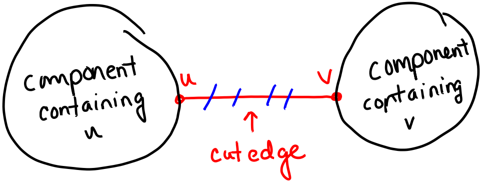
Deleting one edge disconnects the graph

Definition

An edge $e = uv$ of a graph G is a *bridge* of G (also called a *cutedge* of G) if $G - e$ contains one more component than G.

If G is connected, then G-e is not.

Question. If G is connected, is there a limit to the number of components $G - e$ can contain?



only two!

Theorem

An edge e of a connected graph G is a bridge if and only if e does not lie on a cycle in G.

What two statements need to be proved? Write both the statements and their contrapositive.

Rewrite: $\forall$ connected graphs G, ($e \in E(G)$ is a bridge) $\Leftrightarrow$ (e does not lie on a cycle in G)

define!

define!

define!

Let G be a connected graph.

($\Rightarrow$) If $e \in E(G)$ is a bridge, then e does not lie on a cycle in G.

or: If e lies in a cycle in G, then e is not a bridge.

($\Leftarrow$) If e does not lie on a cycle in G, then e is a bridge.

or: If e is not a bridge, then e lies on a cycle in G.

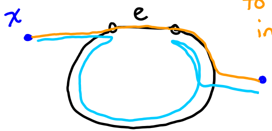
The plan

Let G be a connected graph.

- If $e = uv$ lies on a cycle, then uv is not a bridge.

"not a bridge" means $G-e$ is connected, i.e. there is a walk connecting any pair of vertices.
We will choose an arbitrary pair of vertices!

Picture:

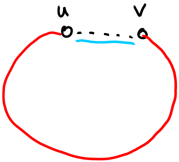


If we go through e to get from x to y in G, then we will reroute around the cycle to get there in G-e.

- If $e = uv$ is not a bridge, then uv lies on a cycle.

so G-e is connected

just need to find one!



must be a path in G-e from u to v

Adding the edge uv to that path makes a cycle in G.

Theorem

An edge e of a connected graph G is a bridge if and only if e does not lie on a cycle in G.

Proof.

Let G be a connected graph, and $e = uv$ an edge in G.

($\Leftarrow$) Suppose first that uv is not a bridge in G. Then $G - uv$ is connected. Let P be a path between u and v in $G - uv$. Now P together with uv forms a cycle in G, so uv lies on a cycle.

($\Rightarrow$) Now suppose uv lies on a cycle C. Thus in C, there is a path P joining u to v that avoids uv. It suffices to show that $G - uv$ is connected. Let x and y be arbitrary vertices in G. Since G is connected, there is a path Q from x to y. If Q does not contain uv, then Q is also a path in $G - uv$, and x and y are connected. If Q does contain uv, then replace uv in Q with the path P from u to v. Now this is a walk in $G - uv$ from x to y, and again x and y are connected. Since x and y were arbitrary, $G - uv$ is connected, and uv is not a bridge in G. $\square$