1.4. Methods of Enumeration.

1. Enumeration problems are encountered frequently in sciences and real world problems. Let us state two basic rules first.

2. *Addition Principle of Counting.* Suppose that A and B are disjoint events. If $|A| = m$ and $|B| = n$, then

$$|A \cup B| = m + n.$$

3. *Multiplication Principle.* Suppose that the experiment E_1 has n_1 outcomes, and, for each outcome of E_1, an experiment E_2 has n_2 outcomes. Then the composite experiment E_1E_2 that consists of performing E_1 first and then E_2 has n_1n_2 outcomes.

4. *Example.*

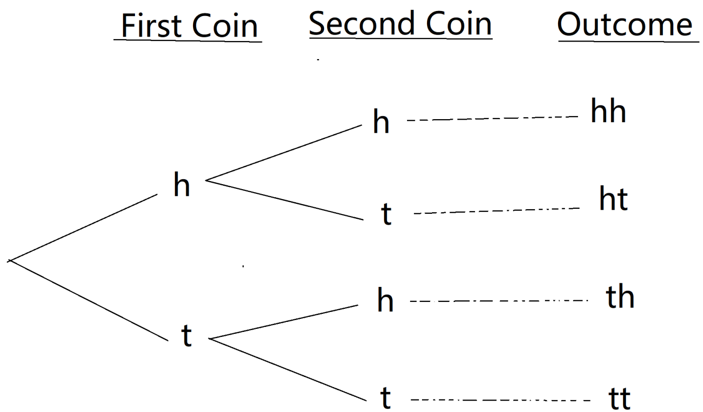
Let E be the experiment that consists of the toss of two coins in a row.

Find the sample space of the experiment.

— Answer. There are two possibilities for the 1st coin, and there are two possibilities for the 2nd coin. If we apply the multiplication principle, the size of the sample space is four. In fact,

$$S = \{hh, ht, th, tt\}.$$

5. Tree diagrams are often useful in counting problems. For the last example, the tree diagram is a simple one.



6. Next, we discuss different ways of sampling. Why sampling? Sampling is done because one usually cannot gather data from the entire population. Suppose, for example, in a study for the purpose of disease control, we are interested in the proportion of the population with some special characteristic, we take a sample.

7. We can take a sample from a population in different ways.

8. Suppose that we are taking a sample of size n from a population of size N, and suppose that the sample consists of the elements

$$a_1, a_2, \cdots, a_n.$$

   (a) If the order of the members of sample matters, then we say the sample is an ordered sample, and we denote this ordered sample by

$$(a_1, a_2, \cdots, a_n).$$

   For example, $(1, 2, 3)$ and $(3, 2, 1)$ are considered as different samples.

(b) If the order of the members of sample does not matter, then we say the sample is an unordered sample, and we denote this unordered sample by

$$[a_1, a_2, \cdots, a_n].$$

For example, $[1, 2, 3]$ and $[3, 2, 1]$ are considered as the same sample. Though the order does not matter, we prefer to sort members of an unordered sample in a certain order, either increasing or decreasing.

(c) If we require that $a_1, a_2, \cdots, a_n$ are all different, we say the sample is a sample without replacement.

(d) If we do not require that $a_1, a_2, \cdots, a_n$ are all different, we say the sample is a sample with replacement.

9. Let us begin with some simple examples.

10. *Example.* In how many ways can we take an ordered sample of size two with replacement from the population $X = \{1, 2, 3, 4\}$?

It is clear that the answer is $4^2 = 16$. We can take an ordered sample of size two with replacement from the population $X = \{1, 2, 3, 4\}$ in 16 ways. The 16 ways ways are:

$$\begin{array}{cccc}
(1,1), & (1,2), & (1,3), & (1,4), \\
(2,1), & (2,2), & (2,3), & (2,4), \\
(3,1), & (3,2), & (3,3), & (3,4), \\
(4,1), & (4,2), & (4,3), & (4,4).
\end{array}$$

11. *Example.* In how many ways can we take an ordered sample of size two without replacement from the population $X = \{1, 2, 3, 4\}$?

It is clear that the answer is $4 \times 3 = 12$. We can take an ordered sample of size two without replacement from the population $X = \{1, 2, 3, 4\}$ in 12 ways. The 12 ways are:

$$(1,2),\quad (1,3),\quad (1,4),\qquad (2,1),\quad (2,3),\quad (2,4),$$

$$(3,1),\quad (3,2),\quad (3,4),\qquad (4,1),\quad (4,2),\quad (4,3).$$

Note that, since we are taking a sample without replacement, members of a sample must be distinct. So the pairs (1,1), (2,2), (3,3), and (4,4) are illegal here.

12. *Example.* In how many ways can we take an unordered sample of size two with replacement from the population $X = \{1, 2, 3, 4\}$.

It is clear that the answer is 10. We can take an unordered sample of size two with replacement from the population $X = \{1, 2, 3, 4\}$ in 10 ways. The 10 ways are:

$$[1, 1], \quad [1, 2], \quad [1, 3], \quad [1, 4],$$

$$[2, 2], \quad [2, 3], \quad [2, 4], \quad [3, 3], \quad [3, 4], \quad [4, 4].$$

This time we are taking a sample with replacement, that means repetition is allowed. For example, if 1 is selected as a member of a sample, then it is put back (replacement) into the the population and is available to be selected again. So the pair $[1, 1]$ is legal this time.

13. *Example.* In how many ways can we take an unordered sample of size two without replacement from the population $X = \{1, 2, 3, 4\}$.

It is clear that the answer is 6. We can take an unordered sample of size two with replacement from the population $X = \{1, 2, 3, 4\}$ in six ways. The six ways are:

$$[1, 2], \quad [1, 3], \quad [1, 4],$$

$$[2, 3], \quad [2, 4], \quad [3, 4].$$

In this example, repetition is not allowed (without replacement), so the pairs $[1, 1]$, $[2, 2]$, $[3, 3]$, $[4, 4]$ are illegal this time.

14. We now prepare some notations before we state the main formulas.

15. *Definition.* If n is a non-negative integer, then we define n factorial by

$$n! = 1 \times 2 \times 3 \times 4 \times \cdots \times (n-1) \times n.$$

In particular, $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, etc.

16. *Notations.* We define the notations P_n^r and C_n^r by

$$P_n^r = \frac{n!}{(n-r)!}, \quad C_n^r = \frac{n!}{r!(n-r)!}.$$

The number C_n^r is also denoted by

$$\binom{n}{r}.$$

17. Now we are ready to state (without proof) the main formulas of this section:

   (a) The number of ways to take an ordered sample of size m with replacement from a population of size n is

$$n^m.$$

   (b) The number of ways to take an ordered sample of size m without replacement from a population of size n is

$$P_n^m = \frac{n!}{(n-m)!}.$$

(c) The number of ways to take an unordered sample of size m without replacement from a population of size n is

$$C_n^m = \binom{n}{m} = \frac{n!}{(n-m)!m!}.$$

(d) The number of ways to take an unordered sample of size m with replacement from a population of size n is

$$\binom{n+m-1}{m} = \binom{n+m-1}{n-1} = \frac{(n+m-1)!}{(n-1)!m!}.$$

18. An ordered sample without replacement is also called a permutation, and an unordered sample without replacement is also called a combination. Detailed definitions are given below:

19. *Definition.* (Combination) By an r-combination of a set S of n elements, we understand an r-subset of S.

20. *Definition.* (Permutation) By an r-permutation of a set S of n elements, we understand an ordered arrangement of r of the n elements.

    An n-permutation of a set S of n elements is simply called a permutation of S or a permutation of the n elements.

21. With these notations and terminology, we can restate two of the mains formulas as follows:

    (a) The number of r-permutations of n elements is P_n^r.

    (b) The number of r-combinations of n elements is C_n^r.

22. Now we look at some more examples:

23. *Example.* In how many ways can we select a four-digit PIN number?

Before we start to solve the problem, let us do some analysis first:

(a) This problem is an example of ordered sampling with replacement.

(b) More accurately, each four-digit PIN number can be considered as an ordered sample of size four with replacement from the population $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Here, the population size is 10, and the sample size is 4. And, by the phrase "with replacement", we mean a digit can be used multiple times. Therefore, the PIN 6557 is legal. By the word "ordered", we mean the order matters. Therefore, the PINs 4567 and 7654 are considered different.

Clearly the answer to the question is 10^4 by a simple application of the multiplication principle.

24. *Example.* In how many ways can we select a four-digit PIN number in which all four digits are distinct?

It is clear that this is an example of ordered sampling without replacement. So the answer to the question is $10 \times 9 \times 8 \times 7 = 5040$ by a simple application of the multiplication principle. Or, by using the formula,

$$P_{10}^{4} = 10 \times 9 \times 8 \times 7 = 5040.$$

25. *Example.* In how many ways can we select a four-digit PIN number in which every digit is greater than the digit on the right?

This is an example of unordered sampling without replacement. The order does not matter. For example, you may have selected the digits 1,6,4,8, in that order; but that order information must be *erased* to satisfy the requirement that every digit is greater than the digit on the right. If you decide to use the digits 1,6,4,8, then you have to rearrange them to form the PIN 8641.

So, in this problem, for example, the PINs 1754 and 6442 are not legal, but the PIN 8641 is.

The answer is, by applying the formula directly,

$$C_{10}^{4} = 10!/(4! \times 6!) = 210.$$

26. *Example.* In how many ways can we select a four-digit PIN number in which every digit is greater than or equal to the digit on the right?

This is an example of unordered sampling with replacement. We can answer the question by applying the formula. Keep in mind that the population size is $n = 10$, and the sample size is $m = 4$. So the answer is

$$\binom{n+m-1}{m} = \binom{13}{4} = 13!/(4! \times 9!) = 715.$$