Theorem 8.7

For every graph G of order n containing no isolated vertices,

$$\alpha'(G) + \beta'(G) = n$$

Proof:

First, suppose that $\alpha'(G) = k$. Then a maximum matching of G consists of k edges, which then cover $2k$ vertices. The remaining $n-2k$ vertices of G can be covered by $n-2k$ edges. Thus $\beta'(G) \leq k+(n-2k) = n$. Hence

$$\alpha'(G) + \beta' \leq k + (n-k) = n$$

It remains only to show that $\alpha'(G) + \beta'(G) \geq n$.

Let X be a minimum edge cover of G. Hence $|X| = \ell = \beta'(G)$.Consider the subgraph $F = G[X]$ induced by X. We begin with an observation: F contains no trail T of length 3. If F did contain a trail T of length 3 and e is the middle edge of T, then $X - \{e\}$ also covers all vertices of G, which is impossible. Therefore, F contains no cycles and no paths of length 3 or more, implying that every component of F is a star.

Since a forest of order n and size $n-k$ contains k components and the size of F is $\ell = n-(n-\ell)$, it follows that F contains $n-\ell$ non-trivial components. Selecting one edge from each of these $n-\ell$ components produces a matching of cardinality $n-\ell$, that is $\alpha'(G) \geq n-\ell$.Therefore,

$$\alpha'(G) + \beta'(G) \geq (n-\ell) + \ell = n$$

Consequently, $\alpha'(G) + \beta'(G) = n$.