A Property of Invertible Matrices

A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

Proof

To prove the theorem in one direction, assume A is invertible. We know that the system of linear equations represented by $Ax = 0$ has only the trivial solution. But this implies that the augmented matrix $\begin{bmatrix} A & 0 \end{bmatrix}$ can be rewritten in the form $\begin{bmatrix} I & 0 \end{bmatrix}$(using elementary row operations corresponding to $E_1, E_2, \ldots, E_k$). So, $E_k \cdots E_2 E_1 A = I$ and it follows that $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$. A can be rewritten as the product of elementary matrices.

To prove the theorem in the other direction, assume A is the product of elementary matrices. Every elementary matrix is invertible and the product of invertible matrices is invertible, so it follows that A is invertible. This proof is complete.