Theorem 8.3

Let G be a *bipartite graph with partite sets* U *and* W *such that* $r = |U| \leq |W|$. *Then* G *contains a matching of cardinality* r *if and only if* G *statisfies Hall's condition.*

Proof:

If Hall's condition is not satisfied, then there is some subset S of U such that $|S| > |N(S)|$. Since S cannot be matched to a subset of W, it follows that U cannot be matched to a subset of W.

The converse is verified by the Strong Principle of Mathematical Induction. We proceed by induction on the cardinality of U. Suppose first that Hall's condition is satisfied and $|U| = 1$. Since $|N(U)| \geq |U| = 1$, there is a vertex in W adjacent to the vertex in U and so U can be matched to a subset of W. Assume, for an integer $k \geq 2$, that if G_1 is any bipartite graph with partite sets U_1 and W_1 where

$$|U_1| \leq |W_1| \, and \, 1 \leq |U_1| < k$$

that satisfies Hall's condition, then U_1 can be matched to a subset of W_1. Let G be a bipartite graph with partite sets U and W, where $k = |U| \leq |W|$, such that Hall's condition is satisfied. We show that U can be matched to a subset of W. We consider two cases.

case 1. F*or every subset* S *of* U *such that* $1 \leq |S| < |U|$, it follows that $|N(S) > |S|$. Let $u \in U$. By assumption, u is adjacent to two or more vertices of W. Let w be a vertex adjacent to u. Now let H be the bipartite subgraph of G with partite sets $U - \{u\}$ and $W - \{w\}$. For each subset S of $U - \{u\}$, $|N(S) \geq |S|$ in H. By the induction hypotesis, $U - \{u\}$ can be matched to a subset of $W - \{w\}$. This matching together with the edge uw shows that U can be matched to a subset of W.

case 2. *There exists a proper subset* X *of* U *such that* $|N(X) = |X|$. Let F be the bipartite subgraph of G with partite sets X and $N(X)$. Since Hall's condition is satisfied in F, it follows by the induction hypothesis that X can be matched to a subset of $N(X)$. Indeed, since $|N(X) = |X|$, the set X can be matched to $N(X)$. Let M' be such a matching.

Next, consider the bipartite subgraph H of G with partite sets $U - X$ and $W - N(X)$. Let S be a subset of $U - X$ and let

$$S' = N(S) \cap (W - N(X))$$

We show that $|S| \leq |S'|$. By assumption, $|N(X \cup S| \geq |X \cup S|$. Hence

$$|N(X) + |S'| = |N(X \cup S)| \geq |X| + |S|$$

Since $|N(X)| = |X|$, it follows that $|S'| \geq |S'|$. Thus Hall's condition is satisfied in H and so there is a matching M'' from $U - x$ to $W - N(X)$. Therefore, $M' \cup M''$ is a matching from U to W in G.