Chapter 1. Probability

1.1. Basic Concepts

1. We will study probability theory and statistics in this course.

2. Statistics is a collection of methods for planning experiments, obtaining data, and then organizing, summarizing, presenting, analyzing, interpreting, and drawing conclusions based on the data.

   Roughly speaking, statistics has two parts — descriptive statistics and inferential statistics. Details will be given later in the course.

3. Probability theory is a branch of mathematics that is concerned with random phenomena.

4. Probability theory is the foundation for statistics.

5. *Definition.* A random experiment is an experiment whose result can not be predicted with certainty.

6. *Example.* The toss of a coin is a random experiment. The coin will either turn out heads or turn out tails. But before we actually toss the coin, we don't know what the outcome will be.

7. *Example.* The toss of a die is also a random experiment.

8. *Definition.* The set of all possible outcomes of a random experiment is called the sample space of the experiment and is denoted by S.

The sample space is also called the outcome space or simply the space.

9. *Definition.* A sample point is an outcome that can not be further broken down into simpler components. In other words, a sample point is the most basic outcome of an experiment.

10. We now look at some examples:

11. *Example.* If the experiment consists of the flipping of a coin, then the sample space is

$$S = \{h, t\},$$

where h means that outcome of the toss is a head, and t means that the outcome of the toss is a tail. In this example, the sample space consists of two sample points.

12. *Example.* If the experiment consists of the tossing of a die, then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\} .$$

In this example, the sample space consists of six sample points.

13. *Example.* If the experiment consists of flipping of two coins, then the sample space is

$$S = \{hh, ht, th, tt\}.$$

Here, the entry hh means that both coins turn out heads, the entry ht means that the first coin shows a head but the second coin shows a tail, etc. In this example, the sample space consists of four sample points.

14. *Example.* If the experiment consists of tossing two dice, then the sample space is

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

Each entry in the table is a sample point. For example, the entry $(4,2)$ means that the first die shows a 4 and the second die shows a 2. In this example, the sample space consists of thirty six sample points. That is, the size of the sample space is thirty six.

15. *Example.* If the experiment consists of flipping a coin and tossing a die, then the sample space is

$$S = \left\{ \begin{array}{cccccc} h1 & h2 & h3 & h4 & h5 & h6 \\ t1 & t2 & t3 & t4 & t5 & t6 \end{array} \right\}.$$

The sample space has size 12.

16. *Definition.* An event is a subset of the sample space.

17. *Definition.* A random variable is a function that assigns a numerical value to each sample point.

18. Events can be conveniently described by random variables.

19. *Example.* If the experiment consists of flipping three coins in a row, then the sample space is

$$S = \left\{ \begin{array}{cccc} hhh & hht & hth & thh \\ tth & tht & htt & ttt \end{array} \right\}.$$

Let X be the the number of coins that turn out heads in the experiment. Then X is a random variable, and X has possible values $0, 1, 2, 3$. This random variable X assigns a numerical value to each sample point, see the chart below for details:

	hhh	hht	hth	thh	tth	tht	htt	ttt
X	3	2	2	2	1	1	1	0

For example, if the outcome ttt is observed, then all three coins turn out tails and none of the three coins shows a head — in this case, $X = 0$.

It is clear that the random variable X is a function whose domain is the sample space S and whose codomain is the set of all real numbers, or symbolically, $X : S \to \mathbb{R}$. We can think of this function as a machine — each time we feed a sample point to the machine, it will produce a number. For example,

$$X(hhh) = 3, \quad X(hth) = 2, \cdots, etc.$$

The range of X is defined as the collection of all possible values of X. In this example, the range of X is the set

$$\{0, 1, 2, 3\}.$$

For each $i = 0, 1, 2, 3$, the equation $X = i$ defines an event. For example, if we let A be the event that $X = 1$, then

$$A = \{tth, tht, htt\} \subset S.$$

Hence, each equation in X defines an event. Also, each inequality in X defines an event. For example, if we let B be the event that $X \leq 1$, then

$$B = \{tth, tht, htt, ttt\} \subset S.$$

20. *Example.* If the experiment consists of tossing two dice, then the sample space is

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

We let X be the sum of the outcomes of the two dice. Then X is a random variable.

We can think of X as a function $X : S \to \mathbb{R}$, defined as

$$X((i,j)) = i + j.$$

That is, if the input is the pair (i,j), then the output is the sum $i + j$. For example, $X((3,1)) = 4$. That is, if the first die shows a 3 and the second die shows a 1, then the sum is 4.

If we let A be the event that $X \leq 4$, then

$$A = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}.$$

The size of the event A is six. Symbolically, $|A| = 6$.

21. There are two types of random variables — discrete random variables and continuous random variables. Recall that a random variable X is a function from the sample space to the set of real numbers.

(a) If the set of all possible values of X is a finite set or a countable set, then X is a discrete random variable.

(b) If the set of all possible values of X is an interval of the real numbers, then X is a continuous random variable.

22. *Example.*

(a) Let the experiment be the toss of three coins in a row. Let X be the number of coins that turn out heads. Then X is a discrete random variable. The reason is simple — the range of X is the finite set $\{0, 1, 2, 3\}$.

(b) Let Y be the life of a brand new light bulb (measured in hours). Then Y is a continuous random variable. Here, the range of Y is the interval $[0, \infty)$, which is an infinite set.