1.7. Bayes' Theorem.

1. *Bayes' Theorem.* Let S be the sample space. If

    $B_1, B_2, \cdots, B_m$ form a partition of S,

    $P(B_i) > 0$, for all $i = 1, 2, \cdots, m$,

    $A \subset S$ is such that $P(A) > 0$,

    Then,

    $A = (A \cap B_1) \cup \cdots \cup (A \cap B_m)$,

    $P(A) = \sum_{i=1}^{m} P(A \cap B_i) = \sum_{i=1}^{m} P(B_i) P(A|B_i)$, and for each $k = 1, 2, \cdots, m$, we have

$$P(B_k|A) = \frac{P(B_k \cap A)}{P(A)} = \frac{P(B_k)P(A|B_k)}{\sum_{i=1}^{m} P(B_i)P(A|B_i)}.$$

2. In the Bayes' formua, the probabilities $P(B_k)$ are called prior probabilities, the probabilities $P(B_k|A)$ are called posterior probabilities.

3. We now look at the proof of Bayes' formula in the special case of $n = 3$. So we assume that B_1, B_2, B_3 form a partition of S, $P(B_1) > 0$, $P(B_2) > 0$, $P(B_3) > 0$, and A is an event such that $P(A) > 0$.

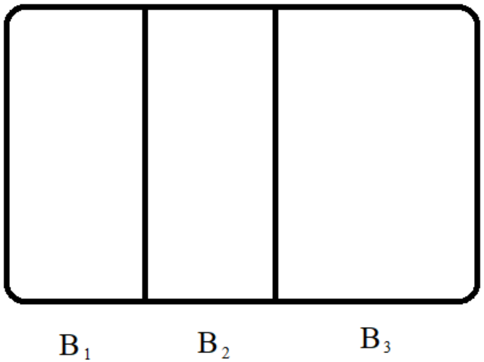
   Since B_1, B_2, B_3 form a partition of S, the events

$$B_1 \cap A, \quad B_2 \cap A, \quad B_3 \cap A$$

   form a partition of the event A. It follows that

$$P(A) = P(B_1 \cap A) + P(B_2 \cap A) + P(B_3 \cap A). \tag{1}$$

   The relations between the events are illustrated by Venn diagrams on the next page.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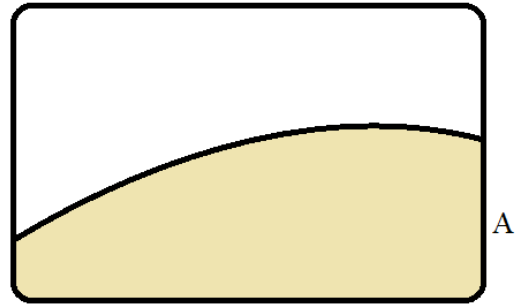
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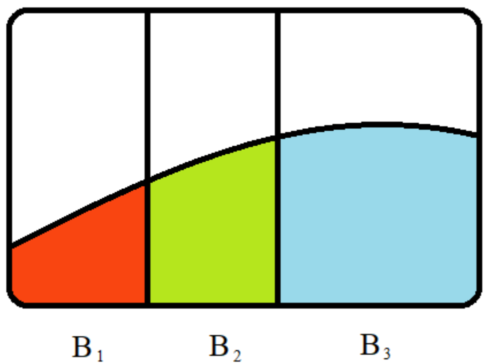
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We have

$$P(B_1 \cap A) = P(A|B_1)P(B_1), \quad P(B_2 \cap A) = P(A|B_2)P(B_2), \tag{2}$$

$$P(B_3 \cap A) = P(A|B_3)P(B_3). \tag{3}$$

If we combine Equations (1), (2) and (3), we get

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3).$$

To summarize, we have

$$\begin{aligned} P(B_1|A) &= \frac{P(B_1 \cap A)}{P(A)} \\ &= \frac{P(A|B_1)P(B_1)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)}. \end{aligned}$$

We have similar results for $P(B_2|A)$ and $P(B_3|A)$.

4. *Example.* A patient goes to see a doctor. The doctor performs a test with 99 percent reliability — that is, 99 percent of people who are sick test positive and 99 percent of the healthy people test negative. The doctor knows that only 1 percent of the people in the country are sick. Now the question is: if the patient tests positive, what are the chances the patient is sick? (This problem was posed by Professor Chris Wiggins.)

— *Solution.* For easy reference, call the patient Mr. X.

Let B_1 be the event that Mr. X is healthy, and let B_2 be the event that Mr. X is sick. Also, let A be the event the test on Mr. X shows positive. Now we translate the information in the example into mathematical language:

$$P(B_1) = 0.99, \quad P(B_2) = 0.01,$$

$$P(A|B_1) = 0.01, \quad P(A|B_2) = 0.99.$$

By Bayes' formula, we have

$$\begin{aligned} P(B_2|A) &= \frac{P(A|B_2)P(B_2)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2)} \\ &= \frac{0.99 \times 0.01}{0.01 \times 0.99 + 0.99 \times 0.01} = \frac{1}{2}. \end{aligned}$$

Here, $P(B_2) = 0.01$ is a prior probability, it reflects our knowledge about the conditions of Mr. X before the test is done. $P(B_2|A) = 0.5$ is a posterior probability, it reflects our knowledge about the conditions of Mr. X after the test is done. The test results help us re-evaluate the conditions of Mr. X.