Section 5.2 Differential Operators and the Elimination Method for Systems

Problem Solve for $x(t)$ and $y(t)$ if:

$$a_1 \frac{dx}{dt} + a_2 \frac{dy}{dt} + a_3 x + a_4 y = F_1(t)$$
$$b_1 \frac{dx}{dt} + b_2 \frac{dy}{dt} + b_3 x + b_4 y = F_2(t)$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are constants.

Example 1 Solve:
$$\left.\begin{aligned} 2x' - 2y' - 3x &= t \\ 2x' + 2y' + 3x + 8y &= 2 \end{aligned}\right\} (*)$$

Solution $\quad ' = \frac{d}{dt} = D \leftarrow$ differential operator

$$2x' = 2\frac{dx}{dt} = 2\frac{d}{dt}x = 2Dx$$

Rewrite system:
$$2Dx - 2Dy - 3x = t$$
$$2Dx + 2Dy + 3x + 8y = 2$$

$$\therefore \left.\begin{aligned} (2D-3)x - 2Dy &= t \\ (2D+3)x + (2D+8)y &= 2 \end{aligned}\right\} \leftarrow \text{Eliminate } y.$$

$$\left.\begin{aligned} (2D+8)(2D-3)x - (2D+8)2Dy &= (2D+8)t \\ 2D(2D+3)x + 2D(2D+8)y &= 2D2 \end{aligned}\right\}$$

$$\left.\begin{aligned} (4D^2+10D-24)x - (4D^2+16D)y &= 2(1)+8t \\ (4D^2+6D)x + (4D^2+16D)y &= 2(0) \end{aligned}\right\} \text{ Add}$$

$$(8D^2 + 16D - 24)x = 2 + 8t$$

$$\therefore \ (D^2 + 2D - 3)x = t + \tfrac{1}{4}$$

or $$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = t + \frac{1}{4}$$

<u>C.F.</u> $$\ddot{x} + 2\dot{x} - 3x = 0$$

Let $x = e^{mt}$

Aux. Eqn: $m^2 + 2m - 3 = 0$

$$(m+3)(m-1) = 0$$

$$m = -3, 1$$

$$x_1 = e^{-3t}, \quad x_2 = e^{t}$$

$$\therefore \ x_c = c_1 e^{-3t} + c_2 e^{t}$$

<u>P.I.</u> Try $x_p = at + b$

$$\dot{x}_p = a$$

$$\ddot{x}_p = 0$$

<u>Substitute</u> $$\ddot{x}_p + 2\dot{x}_p - 3x_p = t + \frac{1}{4}$$

$$0 + 2a - 3(at + b) = t + 1/4$$

$$-3at + (2a - 3b) = t + 1/4$$

$-3a = 1 \Rightarrow a = -1/3$

$2a - 3b = 1/4 \Rightarrow 3b = 2a - \frac{1}{4} = -\frac{2}{3} - \frac{1}{4} = -\frac{11}{12}$

$$\therefore \ b = -\frac{11}{36}$$

Thus, $x_p = -\frac{1}{3}t - \frac{11}{36}$

and, $x = x_c + x_p = c_1 e^{-3t} + c_2 e^{t} - \frac{1}{3}t - \frac{11}{36}$

To obtain $y(t)$ return to the original system (*) and add appropriate multiples of the two equations together so as to eliminate y':

$$\left.\begin{aligned} 2x' - 2y' - 3x &= t \\ 2x' + 2y' + 3x + 8y &= 2 \end{aligned}\right\} \text{ Add}$$

$$4x' + 8y = t + 2$$

$$y = -\tfrac{1}{2}x' + \tfrac{1}{8}t + \tfrac{1}{4}$$

$$= -\tfrac{1}{2}\left(-3c_1 e^{-3t} + c_2 e^{t} - \tfrac{1}{3}\right) + \tfrac{1}{8}t + \tfrac{1}{4}$$

$$= \tfrac{3}{2}c_1 e^{-3t} - \tfrac{1}{2}c_2 e^{t} + \tfrac{1}{8}t + \tfrac{5}{12}$$

<u>Gen. Soln</u>
$$x = c_1 e^{-3t} + c_2 e^{t} - \tfrac{1}{3}t - \tfrac{11}{36}$$
$$y = \tfrac{3}{2}c_1 e^{-3t} - \tfrac{1}{2}c_2 e^{t} + \tfrac{1}{8}t + \tfrac{5}{12}$$

should involve at most 2 arbitrary constants.

<u>Example 2</u> Solve:
$$x' + y' - x = -2t$$
$$x' + y' - 3x - y = t^2$$

<u>Solution</u> Replace $'$ by D.

$$\left.\begin{aligned} Dx + Dy - x &= -2t \\ Dx + Dy - 3x - y &= t^2 \end{aligned}\right\}$$

$$\left.\begin{array}{l}(D-1)x + Dy = -2t \\ (D-3)x + (D-1)y = t^2\end{array}\right\}$$ Eliminate y.

$$\left.\begin{array}{l}(D-1)(D-1)x + (D-1)Dy = (D-1)(-2t) \\ D(D-3)x + D(D-1)y = D(t^2)\end{array}\right\}$$

$$\left.\begin{array}{l}(D^2-2D+1)x + (D^2-D)y = (-2)(1) + 2t \\ (D^2-3D)x + (D^2-D)y = 2t\end{array}\right\}$$ Subtract

$$\left[(D^2-2D+1)-(D^2-3D)\right]x = -2$$

$$\therefore \ [D+1]x = -2$$

$$\frac{dx}{dt} + x = -2$$ ← 1ST order linear

$$\mu = e^{\int 1\,dt} = e^t$$

$$\therefore \ \frac{d}{dt}(e^t x) = -2e^t$$

$$\therefore \ e^t x = -2e^t + C$$

$$\therefore \ x = -2 + Ce^{-t}$$

Return to the original system:

$$\left.\begin{array}{l}x' + y' - x = -2t \\ x' + y' - 3x - y = t^2\end{array}\right\}$$ Subtract (to eliminate y')

$$2x + y = -2t - t^2$$

$$\therefore\ y = -2x - 2t - t^2 = -2(-2 + ce^{-t}) - 2t - t^2$$
$$= 4 - 2t - t^2 - 2ce^{-t}$$

Gen. Soln.
$$x = -2 + ce^{-t}$$
$$y = 4 - 2t - t^2 - 2ce^{-t}$$

Example 3 Solve :
$$3x' + 2y' - x + y = t - 1$$
$$x' + y' - x = t + 2$$

Solution Replace $' = \frac{d}{dt}$ by D.

$$\left.\begin{array}{l}(3D-1)x + (2D+1)y = t-1 \\ (D-1)x + Dy = t+2\end{array}\right\} \text{ Eliminate } y.$$

$$\left.\begin{array}{l}D(3D-1)x + D(2D+1)y = D(t-1) \\ (2D+1)(D-1)x + (2D+1)Dy = (2D+1)(t+2)\end{array}\right\} \text{ Subtract.}$$

$$[(3D^2 - D) - (2D^2 - D - 1)]x = [1] - [2 + t + 2]$$

$$\therefore [D^2 + 1]x = -t - 3$$

or $\ddot{x} + x = -t - 3$

C.F. $\ddot{x} + x = 0$

$x = e^{mt}$

A.E. $m^2 + 1 = 0$

$m = \pm i$

$x_1 = \cos t,\ y_2 = \sin t$

$\therefore x_c = c_1 \cos t + c_2 \sin t$

P.I. Try $x_p = at + b$

$\dot{x}_p = a$

$\ddot{x}_p = 0$

Sub $\ddot{x}_p + x_p = -t - 3$

$0 + at + b = -t - 3$

$a = -1,\ b = -3$

$\therefore\ x_p = -t - 3$

$x = x_c + x_p = c_1 \cos t + c_2 \sin t - t - 3$

Return to the original system:

$$\left.\begin{aligned} 3x' + 2y' - x + y &= t - 1 \\ 2x' + 2y' - 2x &= 2t + 4 \end{aligned}\right\} \quad \text{Eliminate } y'$$

$x' + x + y = -t - 5$

$\therefore\ y = -x' - x - t - 5$

$$= -(-c_1 \sin t + c_2 \cos t - 1) - (c_1 \cos t + c_2 \sin t - t - 3) - t - 5$$

$$= (\cos t)(-c_1 - c_2) + (\sin t)(c_1 - c_2) - 1$$

$$= -(c_1 + c_2)(\cos t) + (c_1 - c_2)(\sin t) - 1$$

Gen. Soln. $x = c_1 \cos t + c_2 \sin t - t - 3$

$y = -(c_1 + c_2)(\cos t) + (c_1 - c_2)(\sin t) - 1$

H.W. Pages 250-251 #'s 7-21 odd