Theorem 5.17

A non-trivial graph H is k-connected for some integer $k \geq 2$ *if and only if* for each pair of u, v of distinct vertices of G there are at least k internally disjoint u—v paths in G.

Proof:

Since the result holds if G is complete, we may assume that G is not complete. Assume that G is k-connected graph, where $k \geq 2$. Let u and v be two distinct vertices of G. Suppose first that u and v are not adjacent and let U be a minimum u—v separating set. Then $|U| \geq \kappa(G) \geq k$. By the Theorem 5.16, G contains at least k internally disjoint u—v paths. Next, suppose that u and v are adjacent, where $e = uv$. Then $G - e$ is $(k-1)$-connected. Let W be a minimum u—v separating set in $G - e$. This

$$|W| \geq \kappa(G - e) \geq k - 1$$

By Theorem 5.16, $G - e$ contains at least $k - 1$ internally disjoint u—v paths.

For the converse, assume that G is a graph containing at least k internally disjoint u—v paths for every pair u, v of distinct vertices of G. Let U be a minimum vertex-cut of G. Then $|U| = \kappa(G)$. Let x and y be vertices in distinct components of $G - U$, Thus U is an x—y separating set of G. Since there are at least k internally disjoint x—y paths in G, it follows by Theorem 5.16 that $\kappa(G) = |U| \geq k$. Therefore G is k-connected.