Theorem 6.6

Let G be a graph of order $n \geq 3$. If

$$deg(u) + deg(v) \geq n$$

for each pair u, v of non-adjacent vertices of G, then G is *Hamiltonian*.

Proof:

Assume, to the contrary, that there exists a non-Hamiltonian graph G of order $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for each pair u, v of non-adjacent vertices of G. It may be the case that if we add certain edges to G, we obtain K_n, which is obviously Hamiltonian. Add as many edges as possible to G so that the resulting graph H is not Hamiltonian. Therefore, adding any edge to H results in a Hamiltonian graph. Also, $deg_H(u) + deg_H(v) \geq n$ for every pair of u, v of non-adjacent vertices of H.

Since H is not complete, H contains pairs of non-adjacent vertices. Let x and y be two non-adjacent vertices of H. Thus $H + xy$ is Hamiltonian. Furthermore, every Hamiltonian cycle of $H + xy$ must contain the edge xy. This means that H contains a Hamiltonian x—y path $P = (x = x_1, x_2, \ldots, x_n = y)$. We claim that whenever $x_1 x_i$ is an edge of H, where $2 \leq i \leq n$, then $x_{i-1} x_n$ is not an edge of H, for otherwise,

$$(x_1, x_i, x_{i+1}, \ldots, x_n, x_{i-1}, x_{i-2}, \ldots, x_1)$$

is a Hamiltonian cycle of H is impossible. Hence for each vertex in $\{x_2, x_3, \ldots, x_n\}$ that is adjacent to x_1, there is a vertex in $\{x_1, x_2, \ldots, x_{n-1}\}$ that is not adjacent to x_n. Howeve, this means that $deg(x_n) \leq (n-1) - deg(x_1)$ and so

$$deg_H(x) + deg_H(y) \leq n - 1$$

This is a contradiction.

Corollary 6.7

Let G be a graph of order $n \geq 3$. *If* $deg(v) \geq \frac{n}{2}$ for each vertex u of G, *then* G is *Hamiltonian.*

Proof:

Certainly, if $G = K_n$, then G is Hamiltonian. We may therefore assume that G is not complete. Let u and v be two non-adjacent vertices of G. Thus

$$deg(u) + deg(v) \geq \frac{n}{2} + \frac{n}{2} = n$$

By Theorem 6.6, G is Hamiltonian.