8.1: Matchings

One of my favorite topics!

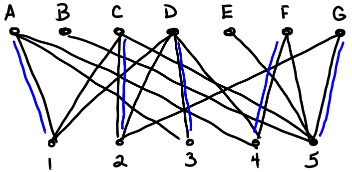
Question. Delta has five departments (we'll number them 1 through 5) within their company that have open positions. Seven applicants (which we will call A, B, C, D, E, F, and G to preserve their anonymity) apply for promotions to those five positions. Not all applicants are qualified for all positions; the table below lists which applicants are qualified for the five departments. Can Delta fill all of the positions from these seven applicants?

- Department 1: A, C, D
- Department 2: C, D, G
- Department 3: A, D
- Department 4: A, D, F
- Department 5: B, C, E, F, G

Hard to make sense of this as a list...

Modeling this type of problem

Form a bipartite graph



Department 1: A, C, D

Department 2: C, D, G

Department 3: A, D

Department 4: A, D, F

Department 5: B, C, E, F, G

We seek a set of edges so that:

- No vertex is the endpoint of more than one edge
- As many vertices as possible are contained in an edge

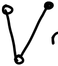
certainly at most 5 edges

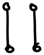
Other examples of where matchings like this may be useful?

- Matching up roommates (edge = compatible)
- Matching med students with hospitals

lots of others!

Definitions


not independent


independent

Definition

- A set of edges is *independent* if they have no common endpoints.
- A set of independent edges is called a *matching*.
- A matching *covers* (or *saturates*) a set of vertices S if each vertex in S is the endpoint of an edge in the matching.
- The size of a maximum matching (i.e. a matching of maximum cardinality) in a graph G is denoted $\alpha'(G)$.

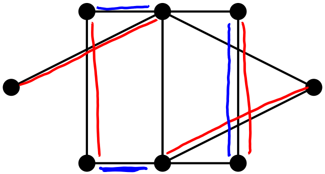
Book has more examples: Examples 8.1 and 8.2

Maximal versus maximum

biggest possible

Find a maximum matching in the graph below. Then find a maximal matching that is not maximum.

not contained in a larger one

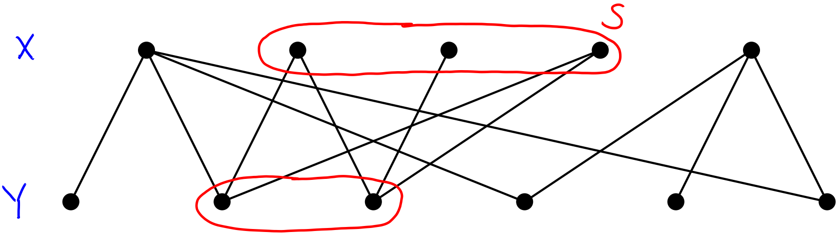


Red edges must be maximum, because all the vertices are matched up

Blue edges are ~~maximum~~ maximal because no additional edges can be added

Shifting to Bipartite Graphs

Does the graph below have a matching that covers the set X?
(Restated: Is $\alpha'(G) = |X|$?)



Problem: The 3 vertices in S have only two vertices in their neighborhood, so we can't cover all 3 with a matching.

A general observation

Let G be a bipartite graph with bipartite sets X and Y. If G contains a set $S \subseteq X$ such that $\underbrace{|N(S)| < |S|}_{\text{our red set}}$, then G does not contain a matching covering X.

Restated: $|N(S)| \geq |S|$ for all $S \subseteq X$ is a *necessary condition* for the existence of a matching covering X.

My favorite acronym in mathematics: TONCAS!

The **O**bvious **N**ecessary **C**ondition is **A**lso **S**ufficient

One of my favorite theorems

* Fundamental result!

Theorem (Hall's Theorem.)

Let G be a bipartite graph with bipartite sets X and Y. G has a matching covering X if and only if for every $S \subseteq X$, $|N(S)| \geq |S|$.

- The condition that for every $S \subseteq X$, $|N(S)| \geq |S|$ is called *Hall's Condition*.
- We already showed the forward direction.
- The reverse direction is not as "obvious" - proof is in text, but we will skip it.
- The book has a nice bio about Philip Hall, who first proved the theorem - read it!

Perfect matchings

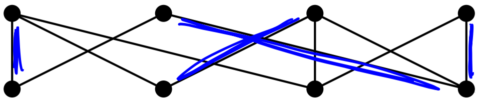
each edge covers two vertices

In any graph of order n, $\alpha'(G) \leq n/2$ (for bipartite graphs, this requires the partite sets have the same size).

Definition

definition works for all graphs

A *perfect matching* in a (not necessarily bipartite) graph G is a matching covering all the vertices of G.



perfect matching

Rephrasing Hall's Theorem when the bipartite sets are balanced:

Theorem (Marriage Theorem.)

obviously necessary

Let G be a bipartite graph with partite sets X and Y with $|X| = |Y|$. G has a perfect matching if and only if G satisfies Hall's Condition (i.e. for every $S \subseteq X$, $|N(S)| \geq |S|$).

An application

Theorem

Every r-regular bipartite graph has a perfect matching.

* Think Hall's Theorem!

Verify $|N(S)| \geq |S|$ for all $S \subseteq X$.

choose arbitrary!

Proof: Let G be an r-regular bipartite graph with bipartition X and Y. We verify that G satisfies Hall's condition.

Let S be an arbitrary nonempty subset of X. Since G is r-regular, each vertex of S has r neighbors in Y. Hence there are $r \cdot |S|$ edges connecting S to $N(S)$.

Each vertex in $N(S)$ has degree r, so each vertex in $N(S)$ can be the endpoint of at most r edges incident on vertices in S. Hence $|N(S)| \geq \frac{r|S|}{r} = |S|$.

Since S was arbitrary, $|N(S)| \geq |S|$ for all $S \subseteq X$.

Therefore G has a matching saturating X, and since $|X| = |Y|$, this is a perfect matching.