Section 2.2 Separable ODE's

A 1^{st} order ODE is said to be separable if it can be written in the form:

$$p(x)\,dx + q(y)\,dy = 0$$

To such an equation there is a one-parameter family of solutions given by:

$$\int p(x)\,dx + \int q(y)\,dy = c$$

Example 1 Solve: $\frac{dy}{dx} = x^{-3}y^3$

Solution $y^{-3}\,dy = x^{-3}\,dx$

$$\therefore \; x^{-3}\,dx - y^{-3}\,dy = 0 \qquad \text{(separable)}$$

$$\therefore \int x^{-3}\,dx - \int y^{-3}\,dy = c_0$$

$$\frac{1}{-2}x^{-2} - \frac{1}{-2}y^{-2} = c_0$$

$$\therefore \; x^{-2} - y^{-2} = -2c_0$$

$$\therefore \frac{1}{x^2} - \frac{1}{y^2} = c \qquad (c = -2c_0)$$

Example 2 Solve: $8\cos^2 y\,dx + \csc^2 x\,dy = 0$, $y\left(\frac{\pi}{12}\right) = \frac{\pi}{4}$

Solution $8\cos^2 y\,dx + \frac{1}{\sin^2 x}\,dy = 0$

$$\therefore\ 8\sin^2 x\, dx + \frac{1}{\cos^2 y}\, dy = 0 \qquad \text{(separable)}$$

Int : $\displaystyle\int 8\sin^2 x\, dx + \int \sec^2 y\, dy = C$

$$\int 8\left(\tfrac{1}{2}\right)(1-\cos 2x)\, dx + \int \sec^2 y\, dy = C$$

$$4\left(x - \tfrac{1}{2}\sin 2x\right) + \tan y = C$$

$$4x - 2\sin 2x + \tan y = C$$

Let $x = \frac{\pi}{12},\ y = \frac{\pi}{4}$: $\quad \frac{\pi}{3} - 2\sin\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{4}\right) = C$

$$\frac{\pi}{3} - 2\left(\tfrac{1}{2}\right) + 1 = C$$

$$\therefore\ C = \frac{\pi}{3}$$

Part. Soln. $\quad 4x - 2\sin(2x) + \tan y = \frac{\pi}{3}$

<u>Example 3</u> Solve : $\displaystyle\frac{dy}{dx} = \frac{3x + xy^2}{y + x^2y}, \quad y(1) = 3$

<u>Solution</u> $\quad (y + x^2y)\, dy = (3x + xy^2)\, dx$

$$y(1+x^2)\, dy = x(3+y^2)\, dx$$

$$\therefore\ \frac{y}{3+y^2}\, dy = \frac{x}{1+x^2}\, dx$$

$$\therefore\ \frac{x}{1+x^2}\, dx - \frac{y}{3+y^2}\, dy = 0 \qquad \text{(separable)}$$

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Int $\qquad \int \frac{x}{1+x^2}\,dx - \int \frac{y}{3+y^2}\,dy = c$

$$\frac{1}{2}\int \frac{2x}{1+x^2}\,dx - \frac{1}{2}\int \frac{2y}{3+y^2}\,dy = c$$

$$\frac{1}{2}\ln|1+x^2| - \frac{1}{2}\ln|3+y^2| = c$$

$$\ln(1+x^2) - \ln(3+y^2) = 2c$$

$$\ln\left(\frac{1+x^2}{3+y^2}\right) = 2c$$

$$\frac{1+x^2}{3+y^2} = e^{2c} = \text{constant} = c_0$$

$$1+x^2 = c_0(3+y^2)$$

$x=1, y=3 \qquad 1+1 = c_0(3+9)$

$$2 = c_0(12)$$

$$\therefore c_0 = \frac{1}{6}$$

Part. Soln. $\qquad 1+x^2 = \frac{1}{6}(3+y^2)$

$$6+6x^2 = 3+y^2$$

$$\therefore y^2 - 6x^2 = 3$$

HW Pg 43, #'s: 1-25 odd