4.2. Distributions of Two Continuous Random Variables

1. We will go over the definition of a single continuous distribution.

2. *Definition.* Let X be a random variable. Define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = P(X \leq x).$$

This function $F(x)$ is called the cdf of X. If $F(x)$ is a continuous function, then we say X is a continuous random variable, and

$$f(x) = F'(x)$$

is called the pdf of X. And, if $a < b$ are real numbers, then

$$P(a < X \leq b) = F(b) - F(a) = \int_a^b f(x)dx.$$

3. *Definition.* Suppose that X and Y are continuous random variables. For each $(x, y) \in \mathbb{R}^2$, let

$$F(x, y) = P(X \leq x, Y \leq y).$$

Then F is called the joint cumulative distribution function of X and Y.

Let

$$f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y),$$

then f is called the joint probability density function of X and Y.

And, if $A \subset \mathbb{R}^2$, then

$$P((X,Y) \in A) = \iint_A f(x,y)dxdy.$$

Note that, here in the case of the joint pdf of two random variables, every probability is a volume whose base is a plane region in the xy-plane.

4. While in the uni-variate case, for example,

$$P(a < X \leq b) = F(b) - F(a) = \int_a^b f(x)dx$$

is geometrically an area whose base is an interval of the x-axis.

5. Some properties of the joint pdf f are listed below:

6. Suppose that $f(x, y)$ is the joint pdf of X and Y as defined in Item 3, then

   (a) $f(x, y) \geq 0$.

   (b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$.

7. *Example.* Suppose that X, Y are continuous random variables, and their joint cdf is

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-2y}), & x>0, y>0, \\ 0, & \text{otherwise}, \end{cases}$$

then, the joint pdf of X and Y is

$$f(x,y) = \begin{cases} 2e^{-x-2y}, & x>0, y>0, \\ 0, & \text{otherwise}, \end{cases}$$

and, for example,

$$P(X>1, Y>3) = \int_1^\infty \int_3^\infty f(x,y)dxdy = e^{-7}.$$

8. *Definition.* Suppose that X and Y are continuous random variables, Let f be the joint probability density function of X and Y.

   (a) For each $x \in \mathbb{R}$, let

   $$f_1(x) = \int_{-\infty}^{\infty} f(x, y)dy.$$

   Then f_1 is called the marginal pdf of X.

   (b) For each $y \in \mathbb{R}$, let

   $$f_2(y) = \int_{-\infty}^{\infty} f(x, y)dx.$$

   Then f_2 is called the marginal pdf of Y.

If, for each $(x, y) \in \mathbb{R}^2$,

$$f(x, y) = f_1(x) f_2(y),$$

then we say X and Y are independent.

9. *Example.* Suppose that X and Y are continuous random variables, and their joint pdf is

$$f(x, y) = x + y, \quad 0 < x < 1, 0 < y < 1.$$

(Note that, following our convention, the trivial parts of distribution are omitted in this definition of $f(x, y)$.)

(a) Find the marginal distributions.

(b) Are X and Y independent?

(c) Find $P(Y > X)$.

— *Solution.* (a) We will now find the marginal pdf $f_1(x)$. First of all, if either $x < 0$ or $x > 1$, we have $f(x, y) = 0$, which implies that

$$f_1(x) = 0.$$

If $0 < x < 1$, then

$$\begin{aligned} f_1(x) &= \int_{-\infty}^{\infty} f(x, y)dy \\ &= \int_0^1 f(x, y)dy \\ &= \int_0^1 (x + y)dy = x + \tfrac{1}{2}. \end{aligned}$$

In summary, X has marginal pdf $f_1(x)$:

$$f_1(x) = x + \frac{1}{2}, \quad 0 < x < 1.$$

In a similar way, Y has marginal pdf $f_2(y)$:

$$f_2(y) = y + \frac{1}{2}, \quad 0 < y < 1.$$

(b) For $0 < x < 1, 0 < y < 1$, we have

$$f(x, y) = x + y,$$

$$f_1(x) f_2(y) = (x + \tfrac{1}{2})(y + \tfrac{1}{2}).$$

It is clear that

$$f(x, y) \neq f_1(x) f_2(y).$$

By definition, X and Y are not independent.

(c) Define a plane region:

$$R = \left\{(x, y) \in \mathbb{R}^2 : 0 < x < y < 1\right\}.$$

It is clear that

$$\begin{aligned} P(Y > X) &= P((X, Y) \in R) \\ &= \iint_R f(x, y)dxdy \\ &= \int_0^1 \int_0^y (x + y)dxdy = \frac{1}{2}. \end{aligned}$$

10. We continue to study Example 9:

    (a) Figure 1 (on the next page) shows the plot of the joint pdf

    $$f(x, y) = x + y, \quad 0 < x < 1, 0 < y < 1.$$

    (b) Figure 2: plot of the base (or support) of the distribution, which is the unit rectangle $(0 < x < 1, 0 < y < 1)$ in the xy-plane.

    (c) Figure 3: plot of the mass that represents $P(Y > X)$.

    (d) Figure 4: plot of the base (or support) of the mass that represents $P(Y > X)$. This base is the triangle $0 < x < y < 1$ in the xy-plane.

Figure 1:

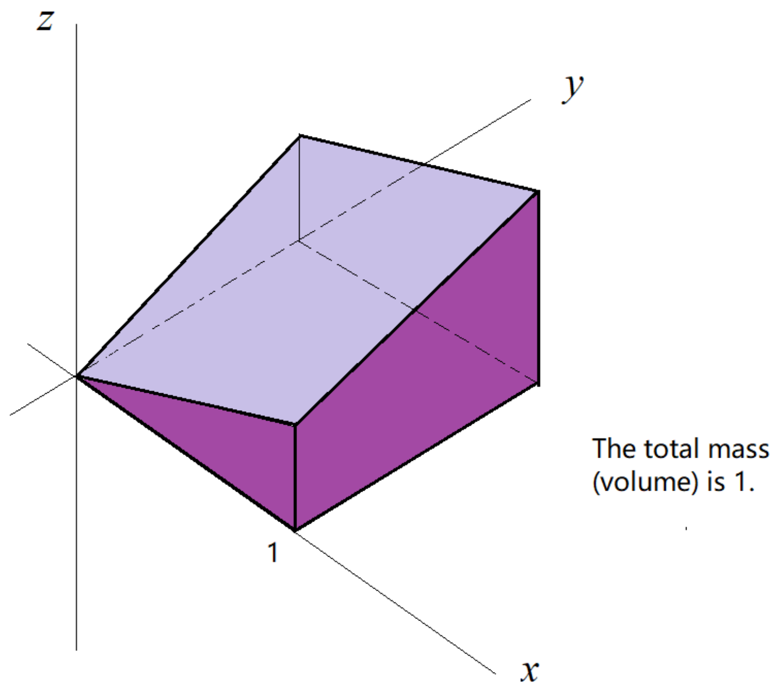


Figure 2:

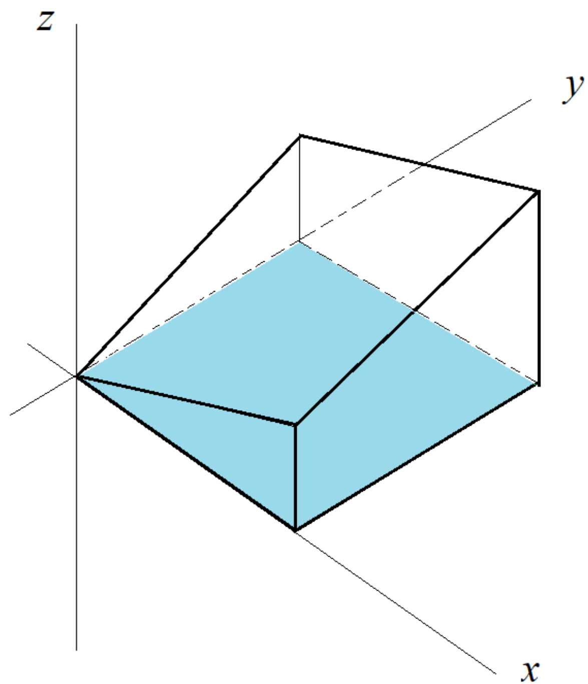


Figure 3:

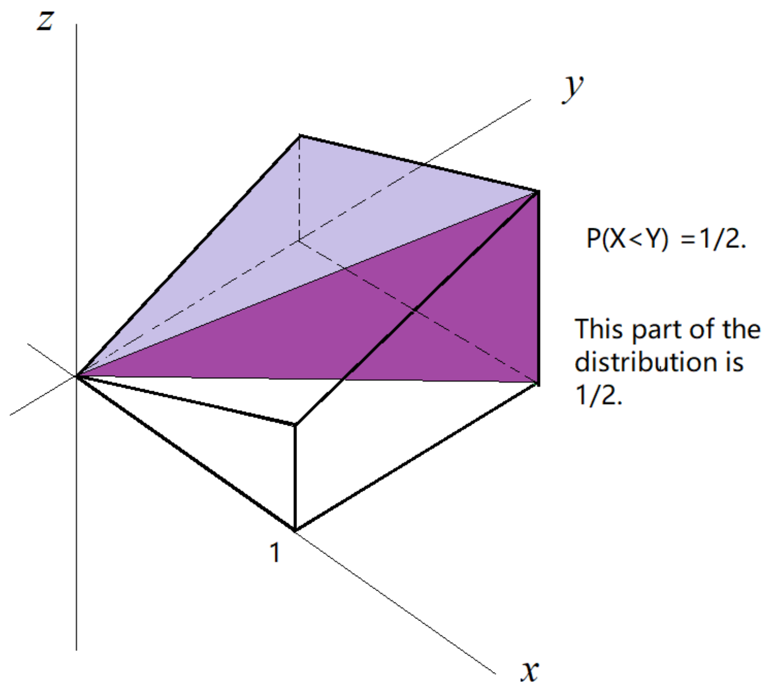
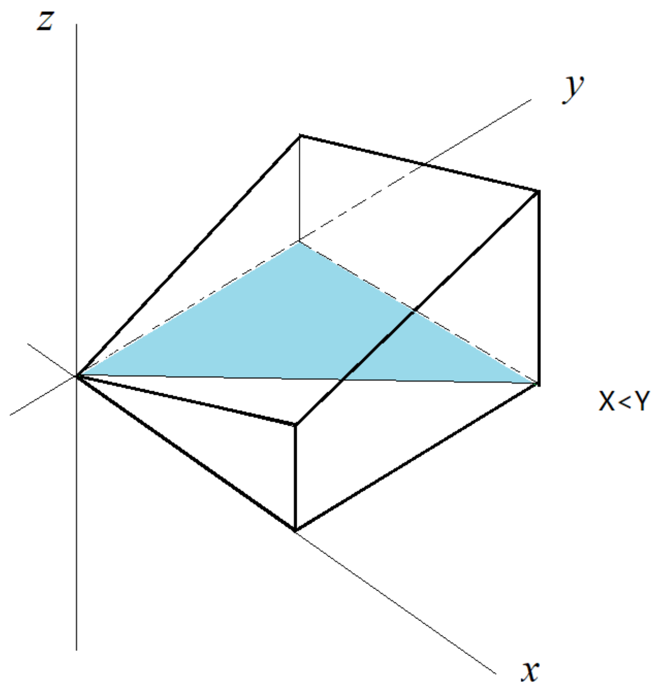


Figure 4:

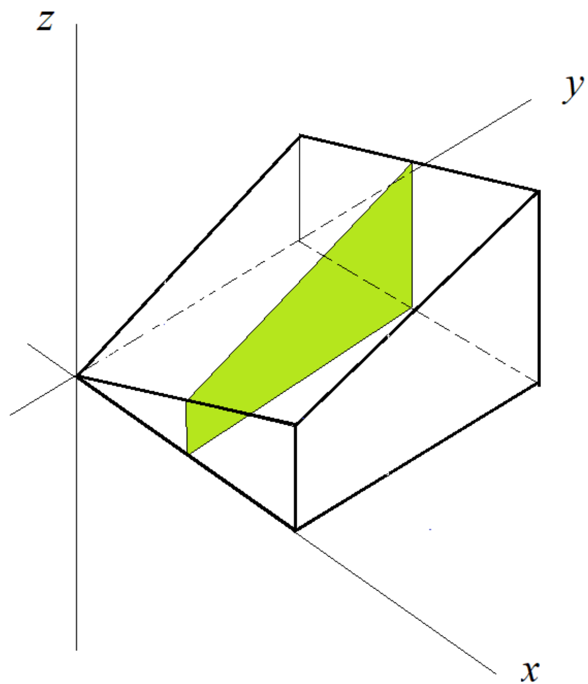


11. We continue to study Example 9: For each $0 < a < 1$, we have

$$f_1(a) = \int_0^1 f(a, y)dy,$$

which is an area. For example, $f_1(\frac{1}{2})$ is an area, and we plot this area in the figure on the next page. This area is the cross section of the joint distribution at $x = \frac{1}{2}$. (Recall that $f_1(x)$ is the marginal pdf of X.)

Figure 5:



12. We continue to study Example 9: For each $0 < b < 1$, we have

$$f_2(b) = \int_0^1 f(x, b)dx,$$

which is also an area. For example, $f_2(\frac{1}{3})$ is an area, and we plot this area in the figure on the next page. This area is the cross section of the joint distribution at $y = \frac{1}{3}$. (Recall that $f_2(y)$ is the marginal pdf of Y.)

Figure 6: