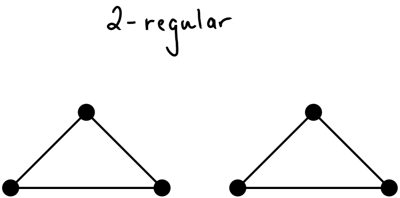
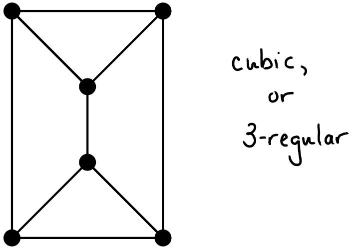
Section 2.2: Regular Graphs

Definition

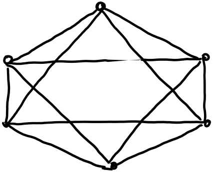
- If every vertex in G has the same degree, then G is *regular.*
- In particular, if every vertex has degree r, then G is r-regular.
- If $r = 3$, then G is a *cubic* graph.





Examples

Draw a 4-regular graph of order 6.



Draw a 3-regular graph of order 7.

not possible: degree sum would be odd, which violates the Handshake Theorem.

Question: What conditions on r and n are needed for there to be an r-regular graph of order n?

A necessary condition: If there exists an r-regular graph of order n, then at least one of $\{r, n\}$ is even.

(so degree sum is even)

Also: We need $0 \leq r \leq n-1$, since $\Delta(G) \leq n-1$ always.

Are these

The next question to ask ourselves: ~~Is this c~~onditions also sufficient? We need to determine whether the *converse* is true, that is, if either r or n is even, then there exists an r-regular graph of order n.

and $0 \leq r \leq n-1$

Existential statement, so we would "just" need to give an example. Except it needs to be proven for all r and n, so we really need an infinite family of examples.

Theorem

Let r and n be integers with $0 \leq r \leq n - 1$. There exists an r-regular graph of order n if and only if at least one of r and n is even.

- This gives a **necessary and sufficient condition** for an r-regular graph of order n to exist.
- Like the theorem about bipartite graphs and odd cycles, this **characterizes** when these regular graphs exist.

What two statements need to be proven?

Assume $0 \leq r \leq n-1$.

1. If there exists an r-regular graph of order n, then at least one of r and n is even. → Contrapositive is true by Handshake Lemma

2. If at least one of r and n is even, then there exists an r-regular graph of order n. → our remaining job

The Harary graph when r is even

The Harary graph $H_{r,n}$ is an example of an n-vertex r-regular graph.

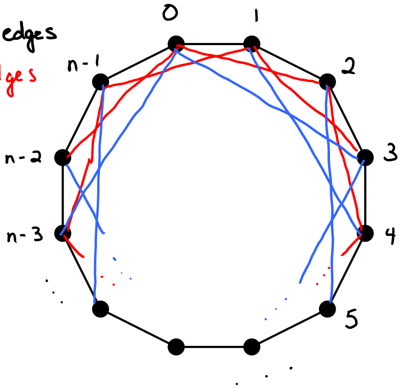
When r is even:

$r=0$: lame

$r=2$: just block edges

$r=4$: add red edges

$r=6$: add blue



In general: Vertex i has $\frac{r}{2}$ "forward" edges and $\frac{r}{2}$ "backwards" edges

$V(H_{r,n}) = \{0,1,2,\dots,n-1\}$

$E(H_{r,n})$: $\{ij : |i-j| \le \frac{r}{2}\}$

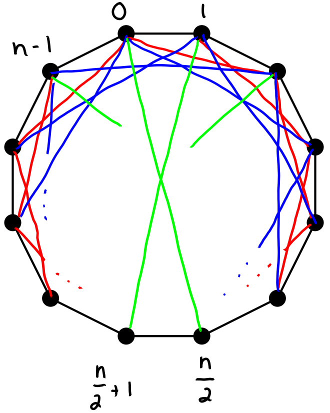
or :

$E(H_{r,n})$ = $\{ij : j = i+k \mod n$ for some $1 \le k \le \frac{r}{2}\}$

The Harary graph when r is odd

Note in this case we know n is even, so $n/2$ is an integer.

new green edges!



$V(H_{r,n}) = \{0,1,2,\ldots,n-1\}$

$E(H_{r,n}) =$

$\{ij : |i-j| \leq \lfloor \frac{n}{2} \rfloor$ or

$i + \frac{n}{2} = j\}$

Conclusion:

Theorem

Let r and n be integers with $0 \leq r \leq n - 1$. There exists an r-regular graph of order n if and only if at least one of r and n is even.

Proof.

Let r and n be integers with $0 \leq r \leq n - 1$.
($\Rightarrow$): Consider the contrapositive. Suppose both r and n are odd. Then no r-regular graph of order n exists by the Handshake Theorem.
($\Leftarrow$): Suppose at least one of n and r is even. Then the Harary graph $H_{r,n}$ is an n-vertex, r-regular graph. $\square$

(Note that the backwards direction is an existential proof of existence, but it isn't just a single example - it is an infinite family of examples.)