Example 2: A rock contains two radioactive isotopes, RA_1 and RA_2, that belong to the same radioactive series; that is, RA_1 decays into RA_2, which then decays into stable atoms. Assume that the rate at which RA_1 decays into RA_2 is $50e^{-10t}$kg/sec. Because the rate of decay of RA_2 is proportional to the mass $y(t)$ of RA_2 present, the rate of change in RA_2 is

$$\frac{dy}{dt} = \quad \text{rate of creation} - \text{rate of decay}$$
$$\frac{dy}{dt} = 50e^{-10t} - ky$$

where $k > 0$ is the decay constant. If $k = 2$/sec and initially $y(0) = 40$kg, find the mass $y(t)$ of RA_2 for $t \geq 0$

Solution

$\frac{dy}{dt} = 50e^{-10t} - ky$ is linear, so we begin by writing it in standard form

$$\frac{dt}{dt} + 2y = 50e^{-10t}, \quad y(0) = 40$$

where we have substituted $k = 2$ and displayed the initial condition. We now see that $P(t) = 2$, so $\int P(t)dt = \int 2\,dt = 2t$. Thus, an integrating factor is $\mu(t) = e^{2t}$. multiplying $\frac{dt}{dt} + 2y = 50e^{-10t}$ by $\mu(t)$ yields

$$\underbrace{e^{2t}\frac{dy}{dx} + 2e^{2t}y} = 50e^{-10t+2t} = 50e^{-8t}$$
$$\frac{d}{dt}(e^{2t}y) = 50e^{-8t}$$

Integrating both sides and solving for y, we find

$$e^{2t}y = -\frac{25}{4}e^{-8t} + C$$
$$y = \frac{25}{4}e^{-10t} + Ce^{-2t}$$

Substituting $t = 0$ and $y(0) = 40$ gives

$$40 = -\frac{25}{4}e^{0} + Ce^{0} = -\frac{25}{4} + C$$

so $C = 40 + \frac{25}{4} = \frac{185}{4}$. Thus, the mass $y(t)$ of RA_2 at time t is given by

$$y(t) = \left(\frac{185}{4}\right)e^{-2t} - \left(\frac{25}{4}\right)e^{-10t}, \quad t \geq 0$$