Number of Solutions of a Linear System

For a system of linear equations, precisely one of the statements below is true.

(1) The system has exactly one solution.

(2) The system has infinitely many solutions.

(3) The system has no solution

Proof

Represent the system by the matrix operation $Ax = b$. If the system has exactly one solution or no solution, then there is nothing to prove. So, assume that the system has at least two distinct solutions x_1 and x_2. If you show that this assumption implies that the system has infinitely man solutions, then the proof is complete. When x_1 and x_2 are solutions, you have $Ax_1 = Ax_2 = b$ and $A(x_1 - x_2) = 0$. This implies that the (nonzero) column matrix $x_h = x_1 - x_2$ is a solution of the homogeneous system of linear equations $Ax = 0$. So, for any scalar c,

$$A(x_1 + cx_h) = Ax_1 + A(cx_h) = b + c(Ax_h) = b + c0 = b$$

Then $x_1 + cx_h$ is a solution of $Ax = b$ for any scalar at c. There are infineitely many possible values of c and each va;ue produces a different solution, so the system has infinitely many solutions.