Exact Equations

Exact Differential Form: The differential form $M(x,y)dx + N(x,y)dy$ is said to be **exact** in a rectangle R if there is a function $F(x,y)$ such that

$$\frac{\partial F}{\partial x}(x,y) = M(x,y) \quad \text{and} \quad \frac{\partial F}{\partial y}(x,y) = N(x,y)$$

for all (x,y) in R. That is, the total differential of $F(x,y)$ satisfies

$$dF(x,y) = M(x,ydx + N(x,y)dy$$

If $M(x,y)dx + N(x,y)dy$ is an exact differential form, then the equation

$$M(x,y)dx + N(x,y)dy = 0$$

is called an **exact equation**.

Theorem: Test for Exactness: Suppose the first partial derivatives of $M(x,y)$ and $N(x,y)$ are continuous in a rectangle R. Then

$$M(x,y)dx + N(x,y)dy = 0$$

is an exact equation in R if and only if the compatibility condition

$$\frac{\partial M}{\partial y}(x,y) = \frac{\partial N}{\partial x}(x,y)$$

holds for all (x,y) in R.

Method for Solving Exact Equations

(a) If $M\,dx + N\,dy = 0$ is exact, then $\partial F/\partial x = M$. Integrate this last equation with respect to x to get

$$F(x,y) = \int M(x,y)dx + g(y)$$

(b) To determine $g(y)$, take the partial derivative with respect to y of both sides of equation $F(x,y) = \int M(x,y)dx + g(y)$ and substitute N for $\partial F/\partial y$. We can now solve for $g'(y)$.

(c) Integrate $g'(y)$ to obtain $g(y)$ up to a numerical constant. Substituting $g(y)$ into $F(x,y) = \int M(x,y)dx + g(y)$ gives $F(x,y)$.

(d) The solution to $M\,dx + N\,dy = 0$ is given implicitly by

$$F(x,y) = C$$

(Alternativelt, starting with $\partial F/\partial y = N$, the implicit solution can be found by first integrating with respect to y.