Half-Angle Substitution for Integrals of the Form $\int f(\sin x\,,\cos x)\,dx$

If the integrand of an integral can be expressed in the form $f(\sin x\,,\cos x)$, then the following rational substitutions will convert it into a non-trigonometric form that *may* be easier to integrate. Although this powerful technique is attributed to German mathematician Karl Weierstrass (1815–1897), and thus is also known as *Weierstrass Substitution*, it was used much earlier by Leonhard Euler (1707–1783).

Integral	Forward Substitution
$\int f(\sin x\,,\cos x)\,dx$	The underlying transformation of x into u uses the following half-angle "back" substitution: $u = \tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$ Interval: $-\pi < x < \pi$ Thus, in transforming the integral from x to u, make the following rational substitutions: $\sin x \;\Rightarrow\; \frac{2u}{1+u^2}$ $\cos x \;\Rightarrow\; \frac{1-u^2}{1+u^2}$ $dx \;\Rightarrow\; \frac{2}{1+u^2}\,du$

Once the antiderivative (or indefinite integral) has been determined as a function of u (if possible), the antiderivative may be transformed back into the variable x by simple back-substitution of u by $\tan(x/2)$, or by its equivalent from the tangent half-angle identity given in the table above:

$$u \;\Rightarrow\; \tan\left(\frac{x}{2}\right) = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

Example 1

Find the indefinite integral:

$$\int \frac{1}{3\sin x - 4\cos x}\,dx$$

Making the substitution given in the table above:

$$\int \frac{1}{3\sin x - 4\cos x}\,dx \;\Rightarrow\; \int \frac{\dfrac{2}{1+u^2}}{3\,\dfrac{2u}{1+u^2} - 4\,\dfrac{1-u^2}{1+u^2}}\,du = \int \frac{1}{2u^2+3u-2}\,du$$

Since the quadratic denominator on the right integral above is factorable in the integers, we can use partial fraction expansion and then integration of the parts to readily obtain the integral expression in terms of u:

$$\int \frac{1}{2u^2+3u-2}\,du = \frac{1}{5}\int \left(\frac{2}{2u-1} - \frac{1}{u+2}\right)du$$

$$\int \frac{1}{2u^2+3u-2}\,du = \frac{1}{5}\,\left[\,\ln|2u-1| - \ln|u+2|\,\right] + C$$

$$\int \frac{1}{2u^2+3u-2}\,du = \frac{1}{5}\,\ln\left|\frac{2u-1}{u+2}\right| + C$$

Back-substituting for u, and using one of the identities for $\tan(\frac{x}{2})$ to convert it back into an expression containing $\sin x$ and $\cos x$ (like the original integrand), we obtain our final result:

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \frac{1}{5}\,\ln\left|\frac{2\tan(\frac{x}{2}) - 1}{\tan(\frac{x}{2}) + 2}\right| + C$$

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \frac{1}{5}\,\ln\left|\frac{2\left(\dfrac{1-\cos x}{\sin x}\right) - 1}{\left(\dfrac{1-\cos x}{\sin x}\right) + 2}\right| + C$$

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \frac{1}{5}\,\ln\left|\frac{2 - 2\cos x - \sin x}{1 - \cos x + 2\sin x}\right| + C \quad \blacksquare$$