Example 4: Solve

$$(-3x + y + 6)dx + (x + y + 2)dy = 0$$

Solution

Since $a_1b_2 = (-3)(1) \neq (1)(1) = a_2b_1$, we will use the translation of axes $x = u + h, y = v + k$, where h and k satisfy the system

$$\begin{aligned} -3h + k + 6 &= 0 \\ h + k + 2 &= 0 \end{aligned}$$

Solving the above system for h and k gives $h = 1, k = -3$. Hence, we let $x = u + 1$ and $y = v - 3$. Because $dy = dv$ and $dx = du$, substituting in $(-3x + y + 6)dx + (x + y + 2)dy = 0$ for x and y yields

$$\begin{aligned} (-3u + v)du + (u + v)dv &= 0 \\ \frac{dv}{du}\frac{= 3 - \left(\frac{u}{v}\right)}{1 + \left(\frac{v}{u}\right)} \end{aligned}$$

The last equation is homogeneous, so we let $z = \frac{v}{u}$. Then $\frac{\frac{dv}{du} = z + u(dz}{du)}$, and substituting for $\frac{v}{u}$, we obtain

$$z + u\frac{dz}{du} = \frac{3 - z}{1 + z}$$

Separating variables gives

$$\begin{aligned} \int \frac{z + 1}{z^2 + 2z - 3}dz &= -\int \frac{1}{u}du \\ \frac{1}{2}\ln|z^2 + 2z - 3| &= -\ln|u| + C_1 \end{aligned}$$

for which it follows that

$$z^2 + 2z - 3 = Cu^{-2}$$

When we substitute back in for z, u, and v, we find

$$\begin{aligned} \left(\frac{v}{u}\right)^2 + 2\left(\frac{v}{u}\right) - 3 &= Cu^{-2} \\ v^2 + 2uv - 3u^2 &= C \\ (y + 3)^2 + 2(x - 1)(y + 3) - 3(x - 1)^2 &= C \end{aligned}$$

This last equation gives an implicit solution to $(-3x + y + 6)dx + (x + y + 2)dy = 0$