Theorem 6.1

A non-trivial connected graph G is Eulerian *if and only if* every vertex of G has even degree.

Symbolically: A non-trivial connected graph G is Eulerian $\Leftrightarrow$ every vertex of G has even degree.

Proof:

Assume first that G is Eulerian. Then G contains an Eulerian circuit C. Suppose that C begins at the vertex u (and therefore ends at u). We show that every vertex of G is even. Let v be a vertex of G different from u. Since C neither begins nor ends at v, each time that v is encountered on C, two edges are accounted for (one to enter v and another to exit v). Thus v has even degree. Now to u Since C begins at u, this accounts for one edge. Another edge is accounted for because C ends at u. If u is encounted at other times, two edgesare accounted for. So u is even as well.

For the converse, assume that G is a non-trivial connected graph in which every vertex is even. We show that G contains an Eulerian circuit. Among all trails in G, let T be one of maximum length. Suppose that T is a u—v trail. We claim that $u = v$. If not, then T ends at v. It is possible that v may have been encountered earlier in T. Each such encounter involves two edges of G, one to enter v and another to exit v. Since T ends v, an odd number of edges at v has been encountered. But v has even degree. This means that there is at least one edge at v, say uv, that does not appear on T. But then T can be extended to w, contradicting the assumption that T has maximum length. Thus T is a u—v trail, that is, $C = T$ is a u—u circuit. if C contains all edges of G, then C is an Eulerian circuit and the proof is complete.

Suppose then that C does not contain all edges of G, that is, there are someedges of G that do not lie on C. Since G is connected, some edge $e = xy$ not on C incident with a vertex x that is on C. Let $H = G - E(C)$, that is, H is the spanning subgraph of G obtained by deleting the edges of C. Every of C is incident with an even number of edges on C. Since every vertex of G has even degree, every vertex of H has even degree. It is possible, however that H is disconnected. On the other hand, H has at least one non-trivial component, namely, the component H_1 of H containing the edge xy. This means that H_1 is connected and every vertex of H_1 has even degree. Consider a trail of maximum length in H_1, beginning at x. As we just saw, this trail must also end at x and is an x—x circuit C' of H_1

Now if in the circuit C, we were to attach C' when we arrive at x, we obtain a circuit C'' in G of greater length than C, which is a contradiction. This implies that C contains all edges of G and is an Eulerian circuit.

Corollary

A connected graph G contains an Eurlian trail *if and only if* exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these vertices and ends at the other.

Proof:

Assume first that G contains an Eulerian trail T. Thus T is a u—v trail for some distinct vertices u and v. We now construct a new connected graph H from G by adding a new vertex x of degree 2 to G and joining it to u and v. Then $C = (T, x, u)$ is an Eulerian circuit in H. By Theorem 6.1, every vertex of H is even and so only u and v have odd degrees in $G = H - x$.

For the converse, we proceed in a similiar manner. Let G be a connected graph containing exactly two vertices u and v of odd degree. We show that G contains an Eulerian trail T, where T is either a u—v trail or a v—u trail. Add a new vertex of degree 2 to G and join it to u and v, calling the resulting graph H. Therefore, H is a connected graph all of whose vertices are even. By Theorem 6.1, H is an Eulerian graph containing an Eulerian circuit C. Since it irrelevant which vertex of C is the initial (and terminal) vertex, we assume that C is an x—x circuit. Since x us ubcudebt only with the edges of C Deleting x from C results in an Eulerian trail T of G that begins either at u or v and ends at the other.