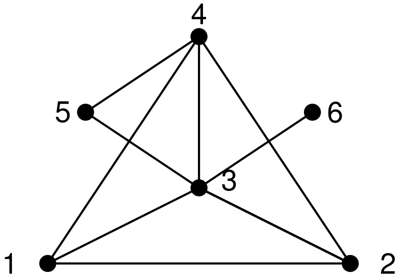
Section 2.3: Degree Sequences

Definition

A *degree sequence* of a graph is a list of the degrees of the vertices in the graph.



- According to labels: 3, 3, 5, 4, 2, 1
- Non-increasing: 5, 4, 3, 3, 2, 1
- Non-decreasing: 1, 2, 3, 3, 4, 5

An interesting question

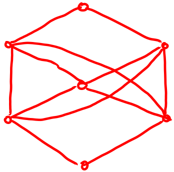
- Suppose I write down a sequence of numbers. Under what conditions will that sequence be a degree sequence for a graph?
- If it is a degree sequence for at least one graph, then the sequence is *graphical*.
- Is 1, 2, 3, 4, 4, 5 graphical?

No: There are 3 odd numbers

Handshake Theorem says no such graph

- Is 4, 4, 4, 4, 4, 2, 2 graphical?

Yes:



Necessary conditions, revisited

Recall: A *necessary condition* for a sequence to be graphical is a condition of the form, "If a sequence s does not satisfy property X, then s is not graphical."

Restated: The necessary condition must be satisfied in order for us to even have a *chance* of getting what we want.

What are some necessary conditions for a sequence to be graphical?

- Sum of terms must be even

5, ______

- Largest number ~~must be~~ not too big relative to # of terms

↓

\# of nonzero terms

- Can't have relatively prime entries (retracted)

Sufficient conditions, revisited

Recall: A *sufficient condition* for a sequence to be graphical is a condition of the form, "If a sequence s satisfies property X, then s is graphical."

Restated: A sufficient condition *guarantees* what we want - but it may stronger than what we need.

What are some sufficient conditions for a sequence to be graphical?

n copies of $n-1$ $\rightarrow K_n$

If s_1 and s_2 are graphical, ~~then~~ $s_1 \cup s_2$ is graphical.

n copies of any even #

sequence of all 0's

A necessary AND sufficient condition

Havel-Hakimi Theorem:

Let $s : d_1, d_2, \ldots, d_n$ be a non-increasing sequence of nonnegative integers, where $d_1 \geq 1$ and $n \geq 2$. The sequence s is graphical if and only if the sequence

$$s_1 = \underbrace{d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1}_{\text{modify first } d_1 \text{ entries that remain}}, \overbrace{d_{d_1+2}, \ldots, d_n}^{\text{the rest stay the same}}$$

is graphical.

- Notice that the sequence s_1 is created by deleting the largest element of the sequence (d_1), and subtracting 1 from the next d_1 members of the sequence.
- Iteratively applying this theorem leads us eventually to a sequence that is either obviously graphic or obviously not graphic.

A necessary AND sufficient condition

Havel-Hakimi Theorem:

Let $s : d_1, d_2, \ldots, d_n$ be a non-increasing sequence of nonnegative integers, where $d_1 \geq 1$ and $n \geq 2$. The sequence s is graphical if and only if the sequence

$$s_1 = d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is graphical.

- To prove "If s_1 is graphical, then s is graphical": Not too hard, we can just add a new vertex and connect it to the vertices with the "missing" degree.
- To prove, "If s is graphical, then s_1 is graphical": This direction is difficult, because we would like to peel of a vertex of max degree, but it may not be adjacent to the next-highest degree vertices.

Use the Havel-Hakimi Theorem to determine if the following sequences are graphic:

1. 2, 7, 5, 5, 3, 4, 7, 5
2. 8, 7, 6, 6, 5, 3, 2, 2, 2, 1

Then prove that the sequence $x, 1, 2, 3, 5, 5$ is not graphical for any $1 \leq x \leq 5$.

2, 7, 5, 5, 3, 4, 7, 5

6 odd numbers: OK

Reorder: 7, 7, 5, 5, 5, 4, 3, 2

delete ↓ subtract 1

S_1: 6, 4, 4, 4, 3, 2, 1 → ordered

delete ↓ subtract

S_2: 3, 3, 3, 2, 1, 0 → ordered

delete ↓ subtract

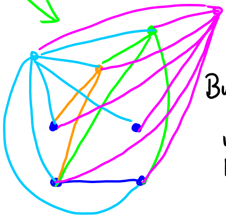
S_3: 2, 2, 1, 1, 0 → ordered

delete ↓ subtract

S_4: 1, 0, 1, 0 → reorder

1, 1, 0, 0

Graphical!



Build graph working backwards

8, 7, 6, 6, 5, 3, 2, 2, 2, 1   already ordered, 4 odd #'s

S_1: 6, 5, 5, 4, 2, 1, 1, 1, 1

S_2: 4, 4, 3, 1, 0, 0, 1, 1 $\rightarrow$ reorder

4, 4, 3, 1, 1, 1, 0, 0

S_3: 3, 2, 0, 0, 1, 0, 0 $\rightarrow$ reorder

3, 2, 1, 0, 0, 0, 0

S_4: problem! Can't subtract 1 from 0.

S_3 isn't graphical, so the original isn't, either

Prove: The sequence $x, 1, 2, 3, 5, 5$ is not graphical for any $1 \leq x \leq 5$.

Observations: 4 odd numbers, so can't add one more

1, 3, 5 are out!

Just look at 5, 5, 4, 3, 2, 1 and 5, 5, 3, 2, 2, 1

First two rounds of H-H will require subtracting from 1, which is a problem

<u>Proof</u>: Observe that $x, 1, 2, 3, 5, 5$ cannot be graphical when x is odd by the Handshake Theorem. Hence $x = 1, 3,$ and 5 all fail to lead to a graphical sequence.

When $x = 2$ or $x = 4$, the first two terms in non-increasing order are 5, and the last is <u>1</u>.

By Havel-Hakimi, $5, 5, y_1, y_2, y_3, 1$ is graphical if and only if $4, y_1-1, y_2-1, y_3-1, 0$ is graphical. However, this sequence is not graphical, because the maximum degree vertex needs 4 neighbors but only 3 are available.

Therefore $x, 1, 2, 3, 5, 5$ is not graphical for any $1 \leq x \leq 5$.