Chapter 3 Mathematical Models and Numerical Methods Involving First Order Equations

Section 3.2 Compartmental Analysis

Growth and Decay

Example 1 The population of a certain city increases at a rate proportional to the number of inhabitants at any time. If the population doubles in 40 years, in how many years will it triple.

Solution Let P = population after t years.

Rate at which population changes $= \frac{dP}{dt} \propto P$

$$\therefore \frac{dP}{dt} = kP \qquad (k = \text{constant})$$

$$\frac{dP}{P} = k\,dt \qquad (\text{separable})$$

$$\text{Int}: \quad \ln P = kt + c_0$$

$$\therefore P = e^{kt+c_0} = e^{kt}e^{c_0} = Ce^{kt}, \quad C = e^{c_0}$$

Suppose at $t=0$, $P = P_0$ = initial population

$$P_0 = Ce^{k(0)} \Rightarrow C = P_0$$

$$\therefore \ P = P_0 e^{kt}$$

When $t=40$, $P = 2P_0$: $2P_0 = P_0 e^{k40}$

$$\therefore \ 2 = e^{40k}$$

$$e^k = 2^{1/40}$$

$$\therefore \ P = P_0 e^{kt} = P_0 (e^k)^t = P_0 (2^{1/40})^t = P_0 \, 2^{t/40}$$

Let $P = 3P_0$: $3P_0 = P_0 \, 2^{t/40}$

$$3 = 2^{t/40}$$

$$\ln 3 = \ln(2^{t/40}) = \frac{t}{40}\ln 2$$

$$\therefore \ t = \frac{40\ln 3}{\ln 2} \doteqdot 63.40 \text{ yr's}$$

<u>Example 2</u> A mold grows at a rate proportional to the amount present. Initially there is 3oz of this mold, and 10hrs later there is 5oz. How much mold is there at the end of 1 day.

<u>Solution</u> Let P = amount of mold present at time t.

Rate of growth $= \frac{dP}{dt} \propto P$

$$\therefore \ P = Ce^{kt}$$

$\underline{t=0,\ P=3}$ $\rightarrow 3 = Ce^{k(0)} \Rightarrow C=3$

$$P = 3e^{kt}$$

$\underline{t=10,\ P=5}$ $\rightarrow 5 = 3e^{k(10)} \Rightarrow e^k = \left(\frac{5}{3}\right)^{1/10}$

$$P = 3e^{kt} = 3(e^k)^t = 3\left(\frac{5}{3}\right)^{t/10}$$

Hence, $P(24) = 3\left(\frac{5}{3}\right)^{24/10} \doteq 10.22$ oz's

<u>Law of Radioactive Decay</u>

The rate of decay of a radioactive substance is proportional, at any instant, to the amount of substance which is present.

<u>Example</u> At time $t=t_0$, Q_0 units of mass of a radioactive material are present while at time $t=t_1$ Q_1 units are present. Find the amount of material present at time $t>t_0$ & calculate the half-life τ of the material (i.e. the time it takes for half of the material to disintegrate).

<u>Solution</u> Let $Q(t)$ be the amount of material present at time t.

$$\frac{dQ}{dt} \propto Q \Rightarrow \frac{dQ}{dt} = kQ \Rightarrow Q = Ce^{kt}$$

$\underline{Q(t_0)=Q_0}$ $Q_0 = Ce^{kt_0} \rightarrow C = Q_0 e^{-kt_0}$

$$\therefore\ Q = Q_0 e^{-kt_0} e^{kt} = Q_0 e^{k(t-t_0)}$$

<u>$Q(t_1) = Q_1$</u> $\quad Q_1 = Q_0 e^{k(t_1 - t_0)} \Rightarrow e^k = \left(\frac{Q_1}{Q_0}\right)^{1/(t_1 - t_0)}$

Thus, $\quad Q = Q_0 e^{k(t-t_0)}$

$$= Q_0 \left[e^k \right]^{(t-t_0)}$$

$$= Q_0 \left[\left(\frac{Q_1}{Q_0}\right)^{\frac{1}{t_1 - t_0}} \right]^{(t-t_0)}$$

$$= Q_0 \left(\frac{Q_1}{Q_0}\right)^{(t-t_0)/(t_1-t_0)}$$

Half-life $= \tau$

If $Q(t_0) = Q_0$ then $Q(t_0 + \tau) = \frac{1}{2} Q_0$.

<u>Let $t = t_0 + \tau$ and $Q = \frac{1}{2} Q_0$</u> :

$$\frac{1}{2} Q_0 = Q_0 \left(\frac{Q_1}{Q_0}\right)^{(t_0 + \tau - t_0)/(t_1 - t_0)}$$

$$\frac{1}{2} = \left(\frac{Q_1}{Q_0}\right)^{\tau/(t_1 - t_0)}$$

Take ln's : $\quad \ln\left(\frac{1}{2}\right) = \frac{\tau}{(t_1 - t_0)} \ln\left(\frac{Q_1}{Q_0}\right)$

$$\underbrace{\cancel{\ln 1}}_{0} - \ln 2 = \frac{\tau}{(t_1 - t_0)} \ln\left[\left(\frac{Q_0}{Q_1}\right)^{-1}\right] = \frac{-\tau}{(t_1 - t_0)} \ln\left(\frac{Q_0}{Q_1}\right)$$

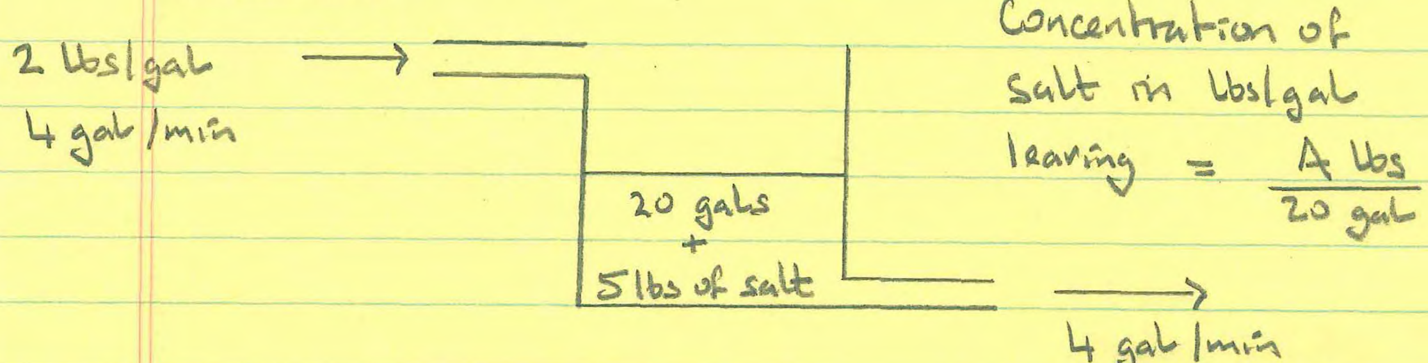
$$\therefore\ \tau = \frac{(t_1 - t_0)\ln 2}{\ln(Q_0/Q_1)}$$

HW Pgs 99 - 101 , #'s 9, 14, 23, 24, 25

Mixture Problems

Example 1 A tank is filled with 20 gals of brine in which is dissolved 5 lbs of salt. Brine containing 2 lbs of salt/gal enters at 4 gal/min & a well stirred mixture leaves at the same rate.
(i) How much salt is in the tank at time $t>0$.
(ii) How much salt is in the tank at time $t = 10$ min.
(iii) How much salt is in the tank after a long time.

Solution



Let $A(t)$ = amount of salt inside the tank at time t.

$$\frac{dA}{dt} = \text{rate at which salt enters} - \text{rate at which salt leaves}$$

$$= \left(2\ \frac{\text{lbs}}{\text{gal}}\right)\left(4\ \frac{\text{gal}}{\text{min}}\right) - \left(\frac{A}{20}\ \frac{\text{lbs}}{\text{gal}}\right)\left(4\ \frac{\text{gal}}{\text{min}}\right)$$

$$= 8 - \frac{1}{5}A$$

$$\therefore \frac{dA}{dt} + \frac{1}{5}A = 8 \quad \text{— 1st order linear}$$

$$P(t) = \frac{1}{5}, \quad \mu(t) = e^{\int \frac{1}{5}dt} = e^{t/5}$$

$$\therefore \underbrace{e^{t/5}\frac{dA}{dt} + \frac{1}{5}e^{t/5}A}_{\frac{d}{dt}\left(e^{t/5}A\right)} = 8e^{t/5}$$

$$\frac{d}{dt}\left(e^{t/5}A\right) = 8e^{t/5}$$

$$e^{t/5}A = \frac{8}{1/5}e^{t/5} + c$$

$$= 40e^{t/5} + c$$

$$\therefore A = 40 + ce^{-t/5}$$

$\underline{t=0,\ A=5}$ $\quad 5 = 40 + c(1) \Rightarrow c = -35$

$$\therefore A = 40 - 35e^{-t/5}$$

(ii) $A(10) = 40 - 35e^{-10/5} \doteq 35.26$ lbs

(iii) $A(t) = 40 - 35e^{-t/5} \xrightarrow{t\to\infty} 40 - 35(0) = 40\text{lbs}$

Example 2 A tank is filled with 20 gals of brine in which is dissolved 5 lbs of salt. Brine containing 2 lbs of salt/gal enters at 4 gal/min + a well stirred mixture leaves at the rate of 5 gal/min.

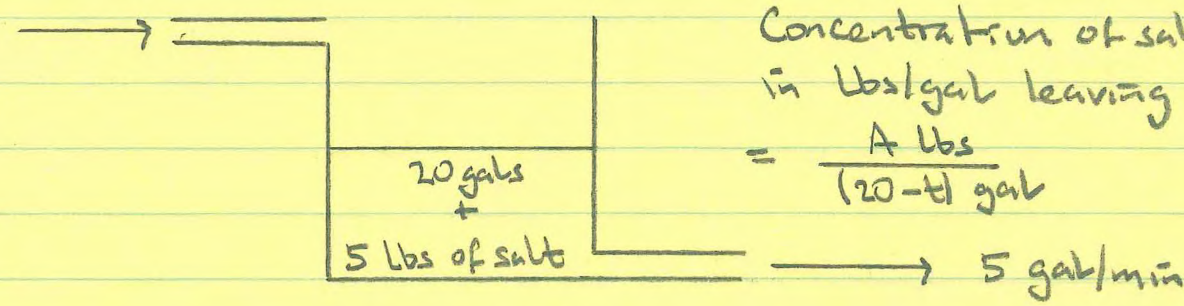
(i) How much salt is in the tank at time $t > 0$.

(ii) How much salt is in the tank at time $t = 10$ min.

(iii) How much salt is in the tank after a long time.

<u>Solution</u>

2 lbs/gal
4 gal/min



Concentration of salt in lbs/gal leaving $= \dfrac{A \text{ lbs}}{(20-t) \text{ gal}}$

Let $A(t)$ = amount of salt in the tank at time t.

Then, $\dfrac{dA}{dt}$ = rate at which salt enters − rate at which salt leaves

$$= \left(2 \frac{\text{lbs}}{\text{gal}}\right)\left(4 \frac{\text{gal}}{\text{min}}\right) - \left(\frac{A}{20-t} \frac{\text{lbs}}{\text{gal}}\right)\left(5 \frac{\text{gal}}{\text{min}}\right)$$

$$= 8 - \frac{5}{20-t} A$$

$$\therefore \frac{dA}{dt} + \frac{5}{20-t} A = 8 \quad - \text{1st Order Linear}$$

$$P(t) = \frac{5}{20-t}, \quad \mu(t) = e^{\int \frac{5}{20-t}\,dt} = e^{-5\int \frac{-1}{20-t}\,dt}$$

$$= e^{-5\ln(20-t)} = (20-t)^{-5}$$

$$\frac{dA}{dt} + 5(20-t)^{-1} A = 8 \quad - \text{Mult. by } \mu$$

$$(20-t)^{-5} \frac{dA}{dt} + 5(20-t)^{-6} A = 8(20-t)^{-5}$$

$$\therefore \quad \frac{d}{dt}\left[(20-t)^{-5} A \right] = 8(20-t)^{-5}$$

Int $\quad (20-t)^{-5} A = \int 8(20-t)^{-5}\,dt + C$

$$= \frac{8(20-t)^{-4}}{(-4)(-1)} + C$$

$$= 2(20-t)^{-4} + C$$

$$\therefore A = 2(20-t) + C(20-t)^5$$

$t=0, A=5$ $\quad 5 = 2(20) + C(20)^5 \Rightarrow C = \frac{-35}{(20)^5}$

$$A = 2(20-t) - \frac{35}{(20)^5}(20-t)^5$$

$$= 2(20-t) - 35\left(\frac{20-t}{20}\right)^5$$

$$= 2(20-t) - 35\left(1-\frac{t}{20}\right)^5$$

(ii) $A(10) = 2(20-10) - 35\left(1-\frac{10}{20}\right)^5 = 18.875$ lbs

(iii) Tank does not hold solution after $t = 20$ min.

HW Page 99, #'s 1,3,4,6