Solutions and Initial Value Problems

Explicit Solution: A function $\phi(x)$ that when substituted for y satisfies the equation for all x in the interval I is called the **explicit solution** to the equation I.

Implicit Solution: A relation $G(x,y)=0$ is said to be an **implicit solution** on the interval I if it defines one or more explixit solutions on I.

The general for of an n^{th}-order equation with x independent, y dependent, can be expressed as

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$$

Where F is a function that depends on x, y, and the other derivatives of y up to order n. We assume that the equation holds for all x in an open Interval $I(a<x<b)$. In many cases we can isolate the highest-order term $d^n y/dx^n$ and write the equation as

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1} y}{dx^{n-1}}\right)$$

Which is often preferable for theoretical and computational purposes.

Initial Value Problem: By an **initial value problem** for an n^{th}-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$\begin{aligned} y(x_0) &= y_0, \\ \frac{dy}{dx}(x_0) &= y_1, \\ &\vdots \\ \frac{d^{n-1}y}{dx^{n-1}}(x_0) &= y_{n-1} \end{aligned}$$

where $x_0 \in I$ and $y_0, y_1, \dots, y_{n-1}$ are given constants.

Theorem: Existence and Uniqueness of Solution

Consider the inital value problem

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0$$

If f and $\partial f/\partial y$ are continuous in some rectangle

$$R = \{(x,y) : a<x<b,\ c<y<d\}$$

that contains the point (x_0, y_0), then the initial value problem has aunique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$ is a positive number.