Evaluating Definite Integrals Using Substitution: Handling the Limits of Integration

When a student is tasked with analytically evaluating a definite integral, and transforming the variable of integration (i.e., "substitution") is required, the student must decide between two approaches:

1. First find the antiderivative, in the *original variable of integration*, of the indefinite integral associated with the definite integral. Then evaluate that antiderivative between the original limits of integration using the *Fundamental Theorem of Calculus.*
2. Transform the definite integral, *including the limits of integration,* and evaluate the antiderivative of the transformed integrand between the transformed limits of integration using the *Fundamental Theorem of Calculus.*

A recommendation will be given as to which approach to use. But first it is best to demonstrate both techniques by analytically evaluating the following definite integral:

$$\int_2^3 x\sqrt{x^2+1}\,dx$$

Approach 1: Evaluating Using the Original Variable of Integration

Given:

$$\int_2^3 x\sqrt{x^2+1}\,dx$$

Write the associated indefinite integral for which we will find its antiderivative in the original variable x.

$$\int x\sqrt{x^2+1}\,dx$$

Use "u-Substitution":

$$u = x^2 + 1$$

$$du = 2xdx \quad \rightarrow \quad \frac{1}{2}du = xdx$$

Substitute and integrate:

$$\int x\sqrt{x^2+1}\,dx \quad \rightarrow \quad \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = \frac{1}{3}u^{3/2} + C$$

Transform the antiderivative in u back into the original variable x:

$$\frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2+1)^{3/2} + C$$

Now we can evaluate the original definite integral using the *Fundamental Theorem of Calculus*. Of course, we pick $C = 0$. Note that we use the original limits of integration.

$$\int_2^3 x\sqrt{x^2+1}\,dx = \left[\frac{1}{3}(x^2+1)^{3/2}\right]_2^3 = \left[\frac{1}{3}10^{3/2}\right] - \left[\frac{1}{3}5^{3/2}\right] = \boxed{\frac{1}{3}[10\sqrt{10} - 5\sqrt{5}]}$$

Approach 2: Evaluating Using the Transformed Variable of Integration

Given:

$$\int_2^3 x\sqrt{x^2+1}\,dx$$

Use "u-Substitution":

$$u = x^2 + 1$$

$$du = 2xdx \quad \rightarrow \quad \frac{1}{2}du = xdx$$

In addition, note that when $x = 2$ (the lower limit of integration), $u = 5$; when $x = 3$ (the upper limit of integration), $u = 10$.

$$\int_2^3 x\sqrt{x^2+1}\,dx = \int_5^{10} \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\int_5^{10} u^{1/2}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_5^{10} = \left[\frac{1}{3}u^{3/2}\right]_5^{10}$$

$$= \left[\frac{1}{3}10^{3/2}\right] - \left[\frac{1}{3}5^{3/2}\right] = \boxed{\frac{1}{3}[10\sqrt{10} - 5\sqrt{5}]}$$

Note that if we had not transformed the limits of integration, we would have gotten an incorrect answer. When transforming a definite integral, ALL parts of the definite integral must be transformed into the new variable of integration, <u>including the</u>

limits of integration. Not transforming the limits of integration is the most common mistake made by Calculus II students who use this approach.

Recommendation

Either approach may be used to analytically evaluate a definite integral when a substitution technique is used to find the antiderivative of an integrand. However, you should consider using Approach 1 (that is, evaluate the antiderivative in the original variable of integration) for the following situations:

1. You will evaluate the original definite integral for more than one pair of integration limits.
2. You are using multiple techniques to solve the integral, such as employing the method of *Parts* in combination with substitution.
3. You are using *Trigonometric Substitution*.
4. You often forget to transform the limits of integration when using Approach 2. (As previously noted, this is a *very* common mistake.)