# Section 4.6 Variation of Parameters

**Problem** Solve: $a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x)$ $(*)$

**Solution** Let $y_1(x)$ and $y_2(x)$ be two L.I. solutions of the homogeneous equation:

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

Seek a solution to equation $(*)$ of the form

$$y = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where

$$u_1'y_1 + u_2'y_2 = 0 \quad \text{——— } (1)$$

Then, $y' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2$

$= u_1y_1' + u_2y_2'$

and, $y'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'$

Substitute into eqn $(*)$:

$$a_0[u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'] + a_1[u_1y_1' + u_2y_2'] + a_2[u_1y_1 + u_2y_2] = F$$

$$\therefore u_1[a_0y_1'' + a_1y_1' + a_2y_1] + u_2[a_0y_2'' + a_1y_2' + a_2y_2] + a_0[u_1'y_1' + u_2'y_2'] = F$$

$$\therefore u_1'y_1' + u_2'y_2' = \frac{F}{a_0} \quad \text{——— } (2)$$

Solve eqns ① + ② for $u_1'$ and $u_2'$:

$$\left.\begin{aligned} ①\quad & u_1' y_1 + u_2' y_2 = 0 \\ ②\quad & u_1' y_1' + u_2' y_2' = \frac{E}{a_0} \end{aligned}\right\}$$

Suppose that $u_1' = \varpi_1$ and $u_2' = \varpi_2$.

Then, $u_1 = A + \int \varpi_1\, dx$ and $u_2 = B + \int \varpi_2\, dx$

<u>Gen. Soln.</u> $y = u_1 y_1 + u_2 y_2$

$$= Ay_1 + y_1 \int \varpi_1\, dx + By_2 + y_2 \int \varpi_2\, dx$$

$$= \underbrace{Ay_1 + By_2}_{y_c} + \underbrace{y_1 \int \varpi_1\, dx + y_2 \int \varpi_2\, dx}_{y_p}$$

<u>Example1</u> Solve: $y'' + y = \tan x$

<u>Solution</u> $y'' + y = 0$

$$y_1 = \cos x \,, \quad y_2 = \sin x$$

Seek a solution $y = u_1 \cos x + u_2 \sin x$

where $u_1' \cos x + u_2' \sin x = 0$ ——— ①

Substitution yields: $u_1' (\cos x)' + u_2' (\sin x)' = \dfrac{\tan x}{1}$

$\therefore\; -u_1' \sin x + u_2' \cos x = \tan x$ ——— ②

$$(\sin x)\ ①\ :\ u_1' \sin x\cos x + u_2' \sin^2 x = 0$$
$$(\cos x)\ ②\ :\ -u_1' \sin x\cos x + u_2' \cos^2 x = \tan x\cos x$$
Add

$$u_2'(\sin^2 x + \cos^2 x) = \sin x$$

$$\therefore\ u_2' = \sin x$$

From ① : $\quad u_1' \cos x + \sin^2 x = 0$

$$\therefore\ u_1' = -\frac{\sin^2 x}{\cos x}$$

Thus, $u_1 = A - \int \frac{\sin^2 x}{\cos x}\,dx = A - \int \frac{1-\cos^2 x}{\cos x}\,dx$

$$= A - \int \sec x - \cos x\,dx$$

$$= A - \ln(\sec x + \tan x) + \sin x$$

and, $\quad u_2 = B + \int \sin x\,dx = B - \cos x.$

<u>Gen. Soln.</u> $\quad y = u_1 y_1 + u_2 y_2$

$$= A\cos x - (\cos x)\ln(\sec x + \tan x) + \sin x\cos x + B\sin x - \sin x\cos x$$

$$= A\cos x + B\sin x - (\cos x)\ln(\sec x + \tan x)$$

<u>Example 2</u> Solve : $\quad y'' - 2y' + y = xe^x \ln x \quad , \quad x > 0$

<u>Solution</u> $\quad y'' - 2y' + y = 0$

Aux. Eqn. $$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1$$

$$y_1 = e^x \quad , \quad y_2 = xe^x$$

Seek a solution $y = u_1 e^x + u_2 x e^x$

where, $$u_1' e^x + u_2' x e^x = 0 \quad \text{——— } ①$$

Substitution yields : $$u_1'(e^x)' + u_2'(xe^x)' = \frac{xe^x \ln x}{1}$$

$$\therefore u_1' e^x + u_2'(x+1)e^x = xe^x \ln x \quad \text{——— } ②$$

$$\left.\begin{array}{l} ① \Rightarrow \quad u_1' + x u_2' = 0 \\ ② \Rightarrow \quad u_1' + (x+1) u_2' = x\ln x \end{array}\right\}$$

$$u_2' = x \ln x \quad \text{and} \quad u_1' = -x u_2' = -x^2 \ln x .$$

Thus, $$u_1 = A - \int x^2 \ln x \, dx$$ $\quad u = \ln x \quad dv = x^2 dx$, $\quad du = \frac{1}{x} dx \quad v = \frac{1}{3} x^3$

$$= A - \left[ \tfrac{1}{3} x^3 \ln x - \int \tfrac{1}{3} x^2 \, dx \right]$$

$$= A - \left[ \tfrac{1}{3} x^3 \ln x - \tfrac{1}{9} x^3 \right]$$

$$= A - \tfrac{1}{9} x^3 \left[ 3 \ln x - 1 \right]$$

and, $$u_2 = B + \int x \ln x \, dx$$ $\quad u = \ln x \quad dv = x\,dx$, $\quad du = \frac{1}{x} dx \quad v = \frac{x^2}{2}$

$$u_2 = B + \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x \, dx$$

$$= B + \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$$

$$= B + \frac{1}{4}x^2\left[2\ln x - 1\right]$$

<u>Gen. Soln.</u> $y = u_1 y_1 + u_2 y_2$

$$= Ae^x - \frac{1}{9}x^3 e^x\left[3\ln x - 1\right]$$
$$+ Bxe^x + \frac{1}{4}x^3 e^x\left[2\ln x - 1\right]$$

$$= Ae^x + Bxe^x + x^3 e^x\left[-\frac{1}{3}\ln x + \frac{1}{9} + \frac{1}{2}\ln x - \frac{1}{4}\right]$$

$$= (A + Bx)e^x + x^3 e^x\left[\frac{1}{6}\ln x - \frac{5}{36}\right]$$

$$= (A + Bx)e^x + \frac{1}{36}x^3 e^x\left(6\ln x - 5\right)$$

<u>H.W.</u> Page 193 #'s 1, 3, 5, 7, 11, 16, 18