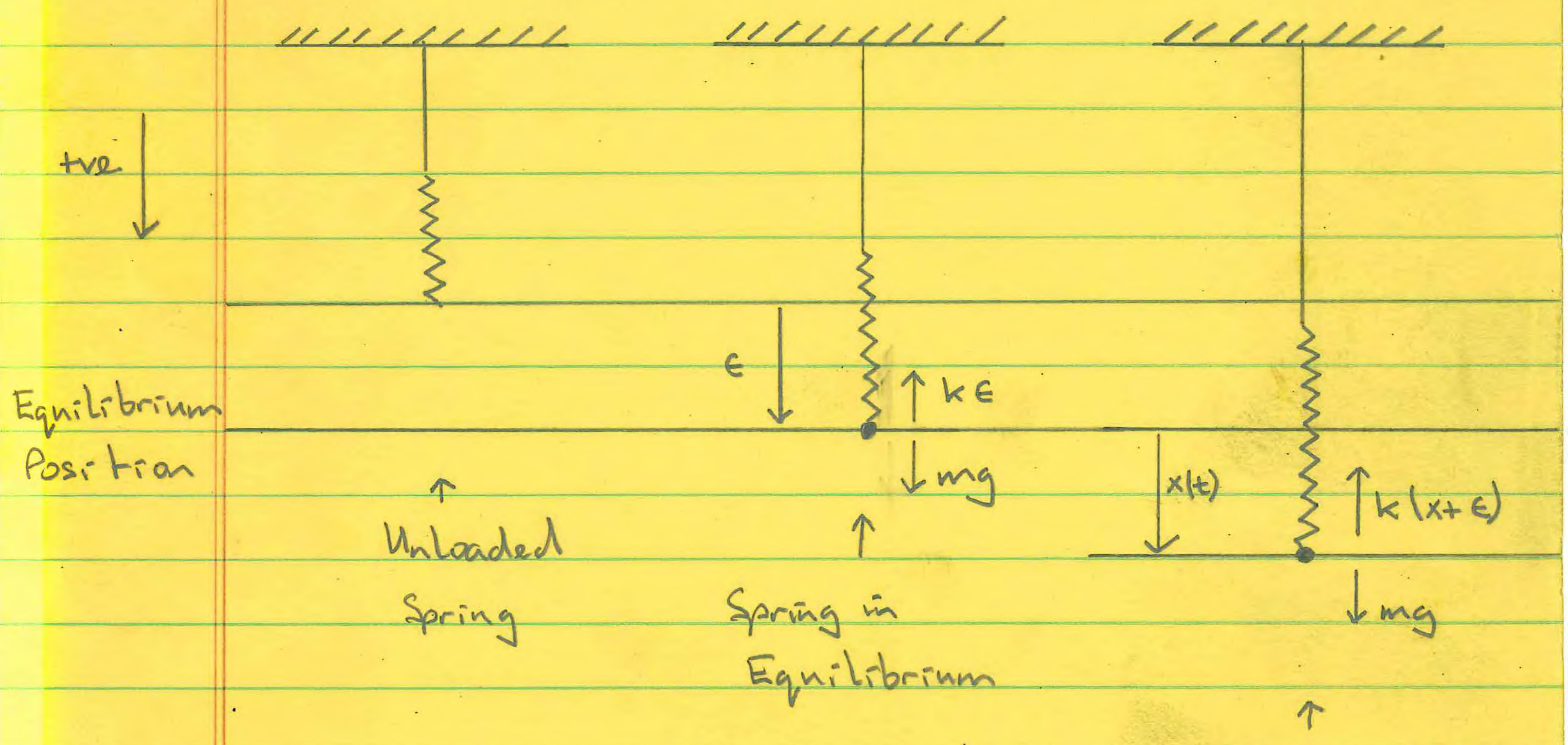
Section 4.9 A Closer Look at Free Mechanical Vibrations

Problem A mass m is attached to the free end of a linear spring of constant k which hangs vertically & the system is then allowed to come to rest. If the mass is given an initial displacement x_0 from the equilibrium position & released from rest discuss the resulting motion.

Solution



$$mg - k\epsilon = 0$$
$$\Rightarrow mg = k\epsilon$$

A linear spring of constant k is one for which:
applied force = k . extension.

Let the displacement of the mass from the equilibrium position at time t be $x(t)$.

$$\text{N2} \Rightarrow \quad \frac{d}{dt}(m\dot{x}) = mg - k(x+\epsilon)$$

$$\therefore m\ddot{x} = mg - kx - k\epsilon$$

$$\therefore m\ddot{x} = -kx$$

$$\therefore m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}x = 0$$

Let $\omega = \sqrt{\frac{k}{m}}$ ← angular frequency (radians/unit time)

Then, $\ddot{x} + \omega^2 x = 0$ ← Simple Harmonic Motion.

<u>Gen. Soln.</u> $x(t) = A\cos\omega t + B\sin\omega t$.

<u>$t=0, x=x_0$</u>

$$x_0 = A\cos(0) + B\sin(0)$$
$$x_0 = A(1) + B(0)$$
$$\Rightarrow A = x_0$$

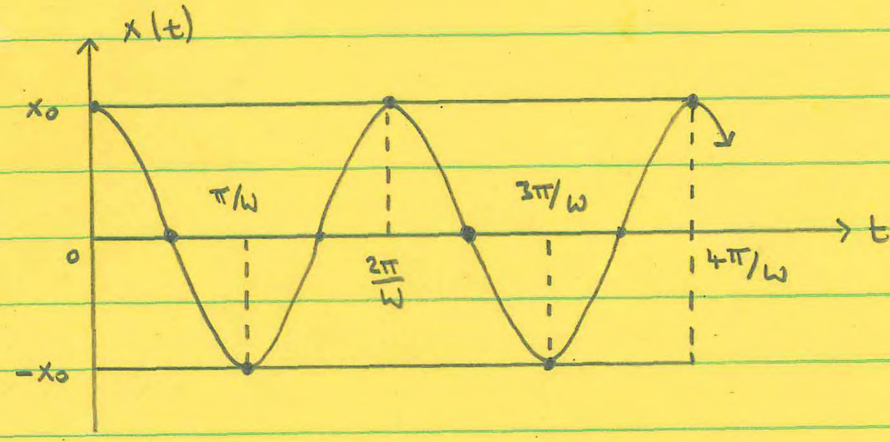
$$\text{velocity} = \dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t$$

<u>$t=0, \dot{x}=0$</u>

$$0 = -A\omega\sin(0) + B\omega\cos(0)$$
$$0 = -A\omega(0) + B\omega(1)$$
$$\Rightarrow B = 0$$

<u>Part. Soln.</u> $x = x_0 \cos \omega t$

Assume x_0 is positive then amplitude $= x_0$ and period $= 2\pi/\omega$.



<u>Notes</u>

1. Angular Frequency $= \omega = \sqrt{\frac{k}{m}}$

2. Period $= T = 2\pi/\omega$

3. Frequency $= f = 1/T = \omega/2\pi$ (cycles/unit time)

4. Particle is x_0 units beneath the equilibrium position at $t = 0, \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \frac{6\pi}{\omega}, \ldots$

5. Particle is x_0 units above the equilibrium position at $t = \frac{\pi}{\omega}, \frac{3\pi}{\omega}, \frac{5\pi}{\omega}, \ldots$

6. Particle passes through the equilibrium position at $t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \frac{5\pi}{2\omega}, \frac{7\pi}{2\omega}, \ldots$

7. Amplitude remains constant.

Example

A mass weighing 16 lb stretches a spring 2 ft. At $t=0$ the mass is released from a point 1 ft below the equilibrium position with an upward velocity of 2 ft/sec. Determine the amplitude and period of the subsequent motion.

Solution $\quad m\ddot{x} + kx = 0$

$$W = mg = 16 \Rightarrow m = \frac{16}{32} = \frac{1}{2}$$

$$F = k\Delta L \Rightarrow 16 = k\,2 \Rightarrow k = 8$$

$$\therefore \quad \frac{1}{2}\ddot{x} + 8x = 0$$

$$\ddot{x} + 16x = 0$$

$$\therefore \quad x = A\cos 4t + B\sin 4t$$

$t=0,\ x=1$ $\qquad 1 = A(1) + B(0) \Rightarrow A = 1$

$$\dot{x} = -4A\sin 4t + 4B\cos 4t$$

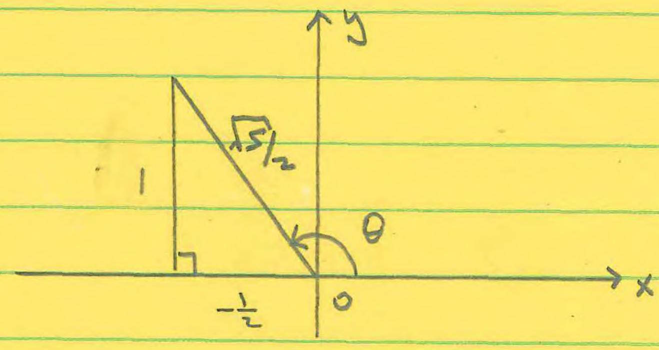
$t=0,\ \dot{x}=-2$ $\qquad -2 = -4A(0) + 4B(1) \Rightarrow B = -\frac{1}{2}$

Thus, $\quad x = 1.\cos(4t) + \left(-\frac{1}{2}\right)\sin(4t)$

Let $A = \sqrt{(1)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$

and, $\sin\theta = \dfrac{1}{A} = \dfrac{1}{\sqrt{5}/2} = \dfrac{y}{r}$

$$\cos\theta = \frac{-1/2}{A} = \frac{-1/2}{\sqrt{5}/2} = \frac{x}{r}$$



$$x = 1 \cdot \cos(4t) + \left(-\tfrac{1}{2}\right) \sin(4t)$$

$$= \frac{\sqrt{5}}{2} \sin\theta \cos(4t) + \frac{\sqrt{5}}{2} \cos\theta \sin(4t)$$

$$= \frac{\sqrt{5}}{2} \left[\sin 4t \cos\theta + \cos 4t \sin\theta \right]$$

$$= \frac{\sqrt{5}}{2} \sin(4t + \theta)$$

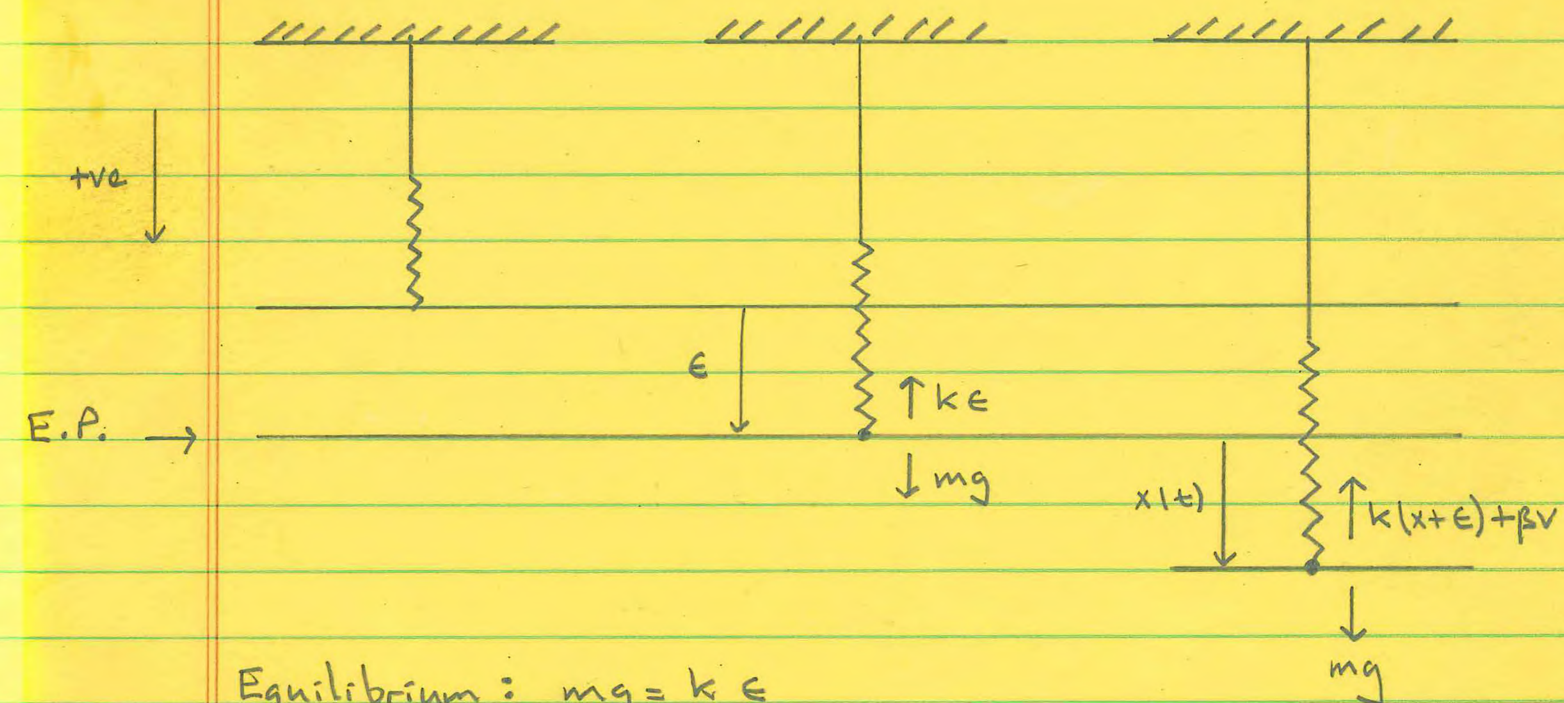
Amplitude $= \dfrac{\sqrt{5}}{2}$, Period $= \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

Phase Angle $= \theta = \pi - \tan^{-1}(2)$

Damped Free Vibrations of a Spring-Mass System

Problem A mass m is attached to the free end of a linear spring of constant k which hangs vertically in a medium with linear velocity damping β. The system is then allowed to come to rest. Find the d.e. which describes the motion when the mass is given an initial displacement x_0 from the equilibrium position + released from rest.

Solution



Equilibrium : $mg = k\epsilon$

Motion : $\frac{d}{dt}(m\dot{x}) = mg - k(x+\epsilon) - \beta v$

$$\therefore \; m\ddot{x} + \beta\dot{x} + kx = 0$$

Initial Conditions $x(0) = x_0$, $\dot{x}(0) = 0$

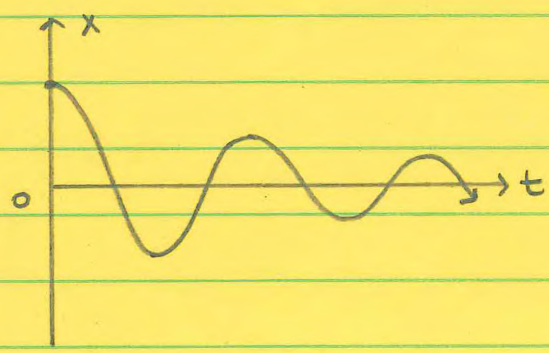
$$m\ddot{x} + \beta\dot{x} + kx = 0$$

Let $x = e^{\alpha t}$

Aux Eqn : $m\alpha^2 + \beta\alpha + k = 0$

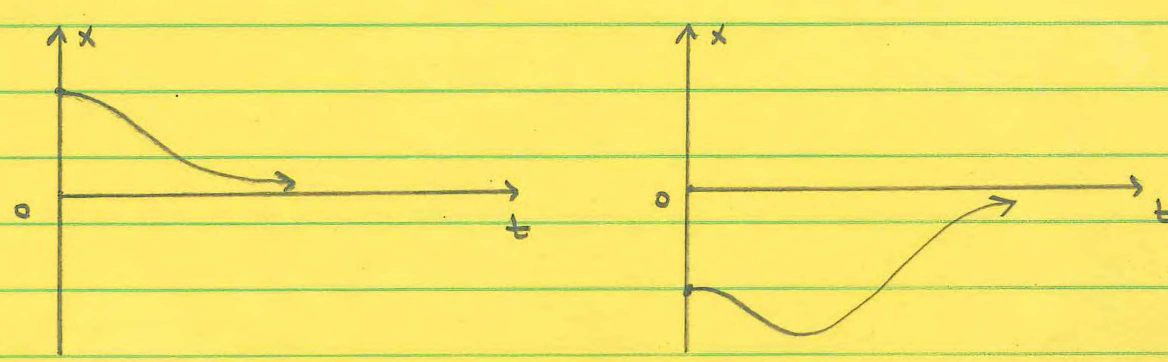
$$\therefore \alpha = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$$

Case I ($\beta^2 < 4mk$ – Underdamping)



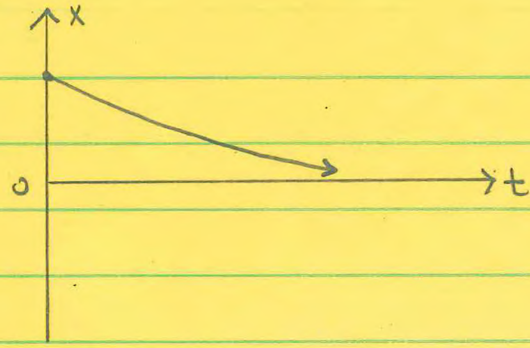
Case II ($\beta^2 = 4mk$ – Critical Damping)

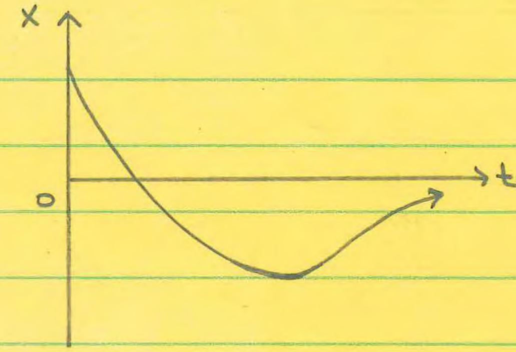
Possible graphs for $x(t)$.



Case III ($\beta^2 > 4mk$ – Overdamping)

Possible graphs for $x(t)$





Example An 8 lb weight is attached to the lower end of a coil spring suspended from the ceiling and stretches the spring 0.4 ft. The weight is pulled down 6" below it's equilibrium position and released at $t=0$. The resistance of the medium in pounds is numerically equal to twice the instantaneous velocity.

(a) Determine the displacement of the weight as a function of time.

(b) Find the time when the weight first passes through the equilibrium position.

Solution

$$W = mg = 8 \Rightarrow m = \frac{8}{32} = \frac{1}{4}$$

$$mg = k\epsilon \Rightarrow 8 = k(0.4) \Rightarrow k = 8/_{2/5} = 20$$

$$R = \beta v = 2v \Rightarrow \beta = 2$$

$$m\ddot{x} + \beta\dot{x} + kx = 0$$

$$\tfrac{1}{4}\ddot{x} + 2\dot{x} + 20x = 0$$

$$\ddot{x} + 8\dot{x} + 80x = 0$$

A.E. $\quad \alpha^2 + 8\alpha + 80 = 0$

$$(\alpha+4)^2 = -64$$

$$\alpha = -4 \pm 8i$$

$$x = e^{-4t}\left[c_1 \cos 8t + c_2 \sin 8t\right]$$

$$v = \dot{x} = e^{-4t}\left[-8c_1 \sin 8t + 8c_2 \cos 8t\right] - 4e^{-4t}\left[c_1 \cos 8t + c_2 \sin 8t\right]$$

<u>$t=0,\ x=\tfrac{1}{2}$</u> $\qquad \tfrac{1}{2} = 1\left[c_1 + 0\right] \Rightarrow c_1 = \tfrac{1}{2}$

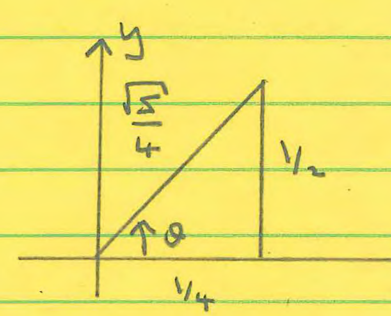
<u>$t=0,\ \dot{x}=0$</u> $\qquad 0 = 1\left[8c_2\right] - 4\left[c_1\right] \Rightarrow c_2 = \tfrac{1}{4}$

$$\therefore\ x(t) = e^{-4t}\left[\tfrac{1}{2}\cos 8t + \tfrac{1}{4}\sin 8t\right]$$

Let $A = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{4}\right)^2} = \dfrac{\sqrt{5}}{4}$

$$\sin\theta = \frac{1/2}{\sqrt{5}/4} = \frac{y}{r}$$

$$\cos\theta = \frac{1/4}{\sqrt{5}/4} = \frac{x}{r}$$



$$x = e^{-4t}\left[\tfrac{1}{2}\cdot \cos 8t + \tfrac{1}{4}\cdot \sin 8t\right]$$

$$= e^{-4t}\left[\frac{\sqrt{5}}{4}\sin\theta \cdot \cos 8t + \frac{\sqrt{5}}{4}\cos\theta \cdot \sin 8t\right]$$

$$= \frac{\sqrt{5}}{4} e^{-4t} \left[\sin 8t \cos\theta + \cos 8t \sin\theta \right]$$

$$= \frac{\sqrt{5}}{4} e^{-4t} \sin(8t + \theta)$$

Amplitude $= \frac{\sqrt{5}}{4} e^{-4t}$, Quasi Period $= \frac{2\pi}{8} = \frac{\pi}{4}$

Phase Angle $= \theta = \tan^{-1}(2)$

Particle passes through equilibrium position when $x(t) = 0$

$$\Rightarrow \quad \sin(8t + \theta) = 0$$

$$\Rightarrow \quad 8t + \theta = 0, \ \pm\pi, \ \pm 2\pi, \ \pm 3\pi, \ldots$$

$$\Rightarrow \quad 8t = -\theta, \ -\theta \pm \pi, \ -\theta \pm 2\pi, \ -\theta \pm 3\pi, \ldots$$

$$\Rightarrow \quad t = \ldots, \frac{-\pi-\theta}{8}, \ \underset{\substack{\uparrow \\ \text{negative}}}{\frac{-\theta}{8}}, \ \frac{\pi-\theta}{8}, \ \frac{2\pi-\theta}{8}, \ldots$$

1st pass through E.P. at $t = \frac{1}{8}(\pi - \theta)$

$$= \frac{1}{8}(\pi - \tan^{-1} 2)$$

<u>HW</u> Page 222 #'s 1,3,5,7,8,9,11