1.5. Conditional Probability

1. *Definition.* (conditional probability) Let A, B be two events. Suppose that $P(B) > 0$. The conditional probability of A, given that B has occurred, is denoted by $P(A|B)$ and is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

provided that $P(B) > 0$.

2. It follows from this definition that

$$P(A \cap B) = P(A|B)P(B),$$

$$P(A \cap B) = P(B|A)P(A).$$

3. *Example.* Let the experiment be the toss of two coins in a row.

Let A be the event that the first coin turns up heads. Let B be the event that both coins turn up heads.

Find $P(A)$, $P(B)$, $P(A|B)$, and $P(B|A)$.

— *Solution.* The sample space is

$$S = \{hh, ht, th, tt\}.$$

It is clear that

$$A = \{hh, ht\}, \quad B = \{hh\}, \quad A \cap B = \{hh\}.$$

Hence,

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{4}, \quad P(A \cap B) = \frac{1}{4}.$$

By definition,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/4} = 1,$$

and

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{1/2} = \frac{1}{2}.$$

4. *Properties of conditional probability:* In all the following formulas, we assume that $P(B) > 0$.

   (a) $P(A|B) \geq 0$.

   (b) $P(B|B) = 1$.

   (c) If A_1 and A_2 are disjoint, then

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B).$$

   (d) If $A_1, A_2, \cdots, A_k$ are pairwise disjoint, then

$$P(A_1 \cup \cdots \cup A_k|B) = P(A_1|B) + \cdots + P(A_k|B).$$

5. *Proof of (c).* First, we have

$$(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B),$$

which can be easily verified by drawing a Venn diagram.

Since A_1 and A_2 are disjoint, $A_1 \cap B$ and $A_2 \cap B$ are disjoint. By the additivity of the probability, we have

$$P((A_1 \cup A_2) \cap B) = P((A_1 \cap B) \cup (A_2 \cap B)) = P(A_1 \cap B) + P(A_2 \cap B).$$

In short,

$$P((A_1 \cup A_2) \cap B) = P(A_1 \cap B) + P(A_2 \cap B).$$

Divide each side by $P(B)$, we get

$$\frac{P((A_1 \cup A_2) \cap B)}{P(B)} = \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}.$$

That is,

$$P(A_1 \cup A_2|B) = P(A_1|B) + P(A_2|B).$$

6. *Example.* Let the experiment be the toss of two dice. Let X be the outcome of the first die, and let Y be the outcome of the second die.

Let A be the event that $X+Y \geq 9$. Let B be the event that $Y \geq 5$.

Find $P(A|B)$ and $P(B|A)$.

— *Solution.* The sample space is

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}.$$

$$S = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix} \right\}.$$

It is clear that

$$P(A) = \frac{10}{36}, \quad P(B) = \frac{12}{36}, \quad P(A \cap B) = \frac{7}{36}.$$

It follows that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{7}{12}, \quad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{7}{10}.$$

7. *Example.*

A box contains 4 red blocks, 5 blue blocks, 6 red balls, and 7 blue balls.

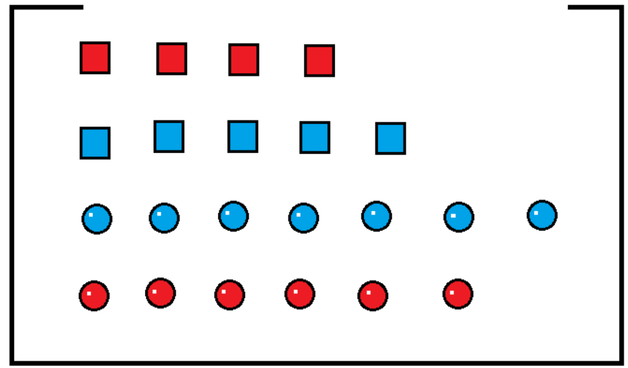
One object is selected at random from the box.

Let A be the event that the object selected is blue.

Let B be the event that the object selected is a block.

Find $P(A)$, $P(B)$, $P(A|B)$, and $P(B|A)$.

— *Solution.* Here is the sample space (the content of the box, where we can take a sample):



It is clear that

$$P(A) = \frac{12}{22}, \quad P(B) = \frac{9}{22}, \quad P(A \cap B) = \frac{5}{22}.$$

It follows that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{5}{9}, \quad P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{5}{12}.$$