Example 6: Show that $\phi(x) = \sin(x) - \cos(x)$ is a solution to the initial value problem

$$\frac{d^2y}{dx^2} + y = 0; \quad y(0) = -1, \quad \frac{dy}{dx}(0) = 1$$

Solution

Observe that $\phi(x) = \sin(x) - \cos(x)$, $\frac{d\phi}{dx} = \cos(x) + \sin(x)$, and $\frac{d^2\phi}{dx^2} = -\sin(x) + \cos(x)$ are all defined on $(-\infty, \infty)$. Substituting into the differential equation gives

$$\Big(-\sin(x) + \cos(x)\Big) + \Big(\sin(x) - \cos(x)\Big) = 0$$

which holds for all $x \in (-\infty, \infty)$. Hence, $\phi(x)$ is a solution to the differential equation in $\frac{d^2y}{dx^2} + y = 0; \quad y(0) = -1, \quad \frac{dy}{dx}(0) = 1$ on $(-\infty, \infty)$. When we check the initial conditions, we find

$$\begin{aligned} \phi(0) &= \sin(0) - \cos(0) = -1 \\ \frac{d\phi}{dx}(0) &= \cos(0) + \sin(0) = 1 \end{aligned}$$

Which meets the requirements of $\frac{d^2y}{dx^2} + y = 0; \quad y(0) = -1, \quad \frac{dy}{dx}(0) = 1$. Therefore, $\phi(x)$ is a solution to the given initial value problem.