Section 2.2: More Predicates and Quantified Statements

Recall: The *negation* of a statement has the opposite truth value as the original statement.

If p is true, then $\sim p$ is false, and vice versa.

What do you think is the negation of the following statements?

1. Every math class meets at 9am.
2. Some students take Discrete Math.

Every math class meets at 9am

(lots of suggestions; correct answer in two slides)

Some students take Discrete Math.

Theorem

Negating Quantified Statements:

- $\sim(\forall x \in D, Q(x)) \equiv$ $\exists$ $x \in D$ s.t. $\sim Q(x)$
- $\sim(\exists x \in D, Q(x)) \equiv$ $\forall$ $x \in D$, $\sim Q(x)$

Example. Negate the following. Start by writing them formally.

1. Every math class meets at 9am.

   Rewrite: $\forall$ math classes x, x meets at 9am.

   Negate: $\exists$ math class x s.t. x does not meet at 9am.

   Restated: There exists a math class that does not meet at 9am.

2. Some students take Discrete Math.

   Rewrite: $\exists$ student x s.t. x takes DM.

   Negate: $\forall$ students x, x does not take DM.

   Restated: All students do not take DM.
   No students take DM.

Notes:

1. When we say two *quantified* statements are logically equivalent, we mean they have the same truth value for **every possible substitution** of **predicate, predicate variables, and choice of domain.**
2. An important summary:
   - The negation of a universal statement is

     existential
   - The negation of an existential statement is

     universal
   - The negation of a conditional statement is

     an and statement
   - The negation of an and statement is

     an or statement

More examples: Negate the following. First rewrite them formally, then negate them formally, then rewrite the negation informally.

1. Every integer has a multiplicative inverse.

Rewrite: $\forall\ x \in \mathbb{Z}$, x has a mult. inverse

Negate: $\exists\ x \in \mathbb{Z}$ s.t. x doesn't have a mult. inverse

Informal: There exists an integer without a mult. inverse.

2. There is a prime number greater than 1 million.

Rewrite: $\exists$ prime n s.t. $n > 1{,}000{,}000$

Negate: $\forall$ primes n, $n \leq 1{,}000{,}000$

Informal: All primes are at most 1,000,000.

Negating Universal Conditional Statements

How can we rewrite $\sim(\forall x \in D, P(x) \rightarrow Q(x))$?

$\exists\, x \in D$ s.t. $\sim(P(x) \rightarrow Q(x))$

$\exists\, x \in D$ s.t. $P(x) \wedge \sim Q(x)$

* Negate one layer at a time!

Theorem

$\sim(\forall x \in D, P(x) \rightarrow Q(x)) \equiv \exists\, x \in D$ s.t. $P(x) \wedge \sim Q(x)$

Example. Negate the following. Write them formally.

1. For every integer n, if n is even, then $n/2$ is an integer.

   $\forall\, n \in \mathbb{Z},\ (n \text{ even}) \rightarrow \left(\frac{n}{2} \in \mathbb{Z}\right)$.

   Negate: $\exists\, n \in \mathbb{Z}$ s.t. $(n \text{ even}) \wedge \left(\frac{n}{2} \notin \mathbb{Z}\right)$

2. For every real number x, if x is not rational, then $2x$ is not rational.

   $\forall\, x \in \mathbb{R},\ (x \notin \mathbb{Q}) \rightarrow (2x \notin \mathbb{Q})$

   Negate: $\exists\, x \in \mathbb{R}$ s.t. $(x \notin \mathbb{Q}) \wedge (2x \in \mathbb{Q})$

Revisiting vacuous truth AGAIN

Rewrite formally, as a conditional and then negate: "All positive integers less than -5 are divisible by 3."

$\forall n \in \mathbb{Z}^+$, if $n < -5$ then n is divisible by 3.

Negate: $\exists n \in \mathbb{Z}^+$ s.t. $n < -5$ and n is not divisible by 3.

Recall that either a statement OR its negation must be true. Which of the statements above is true?

Certainly not the second! There doesn't exist a positive integer that is less than -5.

So the original must be. Vacuously.

Inverse, Converse, Contrapositive revisited

The inverse, converse, and contrapositive of a universal conditional statement yields a new universal conditional statement.

Given the statement, $\forall x \in D, P(x) \rightarrow Q(x)$, we can find the:

- inverse: $\forall\ x \in D,\ \sim P(x) \rightarrow \sim Q(x)$
- converse: $\forall\ x \in D,\ Q(x) \rightarrow P(x)$
- contrapositive: $\forall\ x \in D,\ \sim Q(x) \rightarrow \sim P(x)$

note these are contrapositives of each other

Still universal! (unlike the negation)

Again:

- the contrapositive has the same truth value as the original statement, but
- there is no guaranteed relationship between the truth value of the original and its converse or inverse

Example. Consider the statement: $\forall n \in \mathbb{Z}^+$, if n is divisible by 6, then n is divisible by 3. Find the:

false

1. inverse:

   $\forall\, n \in \mathbb{Z}^+$, if n is not divisible by 6, then n is not divisible by 3.

2. converse:

   $\forall\, n \in \mathbb{Z}^+$, if n is divisible by 3, then n is divisible by 6.

3. contrapositive:

   $\forall\, n \in \mathbb{Z}^+$, if n is not divisible by 6, then n is not divisible by 3.

Which of these has the same truth value as the original?

contrapositive: both are true!

When you have finished the homework for 2.2, you should be able to:

1. negate quantified statements really, really really, really, really, really well.
2. find the inverse, converse, and contrapositive of a universal conditional statement.