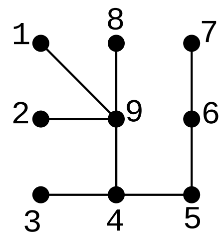
Note: This assignment can't be submitted late. I will be posting solutions right away so that you can use them to prepare for the exam on the 18th.

Basic skills

Complete all of the basic skills questions.

Question 1. Construct the Prufer code corresponding to the labeled tree below. Show me the steps of your algorithm.

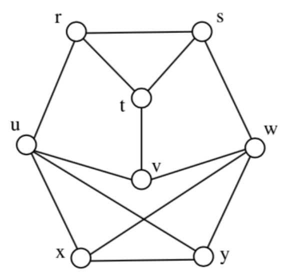


Question 2. Construct the tree corresponding to the Prufer code $(2, 3, 2, 6, 1, 3)$. Show me the steps of your algorithm.

Question 3. In the graph below, find:

(a) $\kappa(H)$

(b) $\lambda(H)$



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Question 4. In the graph above, for each of the following, either find one or explain why it doesn't exist:

(a) A minimum vertex cut that is not minimal

(b) A minimum vertex cut that is minimal

(c) A minimal vertex cut that isn't minimum

Question 5. Give an example of a graph G with the following properties, or explain why no such graph exists.

(a) 2-connected but not 1-connected

(b) 3-connected but not 4-connected

(c) 3-connected with $\kappa(G) = 3$

(d) 2-connected with $\kappa(G) = 3$

(e) $\kappa(G) = 1$, $\lambda(G) = 3$, $\delta(G) = 5$

(f) $\kappa(G) = 3$, $\lambda(G) = 2$, $\delta(G) = 1$

Question 6. Draw a graph G with vertices x and y satisfying the following conditions, or state why no such graph exists:

(a) Contains 3 internally disjoint $x - y$ paths, and contains an $x - y$-separating set of size 4.

(b) Contains 3 internally disjoint $x - y$ paths, and contains an $x - y$-separating set of size 2.

(c) Contains 3 internally disjoint $x - y$ paths, and contains a vertex cut of size 2.

Question 7. Read the proof below, taken from your textbook, and answer the questions about it that follow.

Corollary 4.6 *Every forest of order n with k components has size $n - k$.*

Proof. Suppose that the size of a forest F is m. Let $G_1, G_2, \ldots, G_k$ be the components of F, where $k \geq 1$. Furthermore, suppose that G_i has order n_i and size m_i for $1 \leq i \leq k$. Then $n = \sum_{i=1}^{k} n_i$ and $m = \sum_{i=1}^{k} m_i$. 1 Since each component G_i $(1 \leq i \leq k)$ is a tree, it follows by Theorem 4.4 that $m_i = n_i - 1$. Therefore,

$$m = \sum_{i=1}^{k} m_i \overset{2}{=} \sum_{i=1}^{k} (n_i - 1) \overset{3}{=} n - k. \quad \blacksquare$$

(a) What kind of proof is this? (Direct? Contradiction? Induction?) How do you know?

(b) Explain what the notation in the sentence ending with a red "1" means in words.

(c) Why must each G_i be a tree?

(d) What is Theorem 4.4, and how is it used?

(e) Explain why the equation with a "2" above it holds.

(f) Explain why the equation with a "3" above it holds.

Question 8. Read the proof below, taken from your textbook, and answer the questions about it that follow.

Theorem 5.10 *For every positive integer n, $\lambda(K_n) = n - 1$.*

Proof. By definition, $\lambda(K_1) = 0$. Let $G = K_n$ for $n \geq 2$. Since every vertex of G has degree $n - 1$, if we remove the $n - 1$ edges incident with a vertex, then a disconnected graph results. Thus $\lambda(G) \leq n - 1$. 1 Now let X be a minimum edge-cut of G. So $|X| = \lambda(G)$. Then $G - X$ has exactly two components G_1 and G_2, where G_1 has order k, say, and G_2 has order $n - k$. Since (1) X consists of all edges joining G_1 and G_2 and (2) G is complete, it follows that $|X| = k(n-k)$. Because $k \geq 1$ and $n - k \geq 1$, we have $(k-1)(n-k-1) \geq 0$ and so

$$(k-1)(n-k-1) \overset{2}{=} k(n-k) - n + 1 \geq 0.$$

Hence $\lambda(G) = |X| = k(n-k) \overset{3}{\geq} n - 1$. Therefore, $\lambda(K_n) = n - 1$. $\blacksquare$

(a) Explain how the sentence labeled 1 follows from the previous sentence.

(b) Why is $|X| = \lambda(G)$?

(c) If G_1 has order k, why does G_2 have order $n-k$?

(d) Explain how (1) and (2) in the proof imply that $|X| = k(n-k)$.

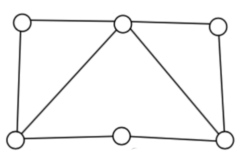
(e) Why are $k \geq 1$ and $n-k \geq 1$?

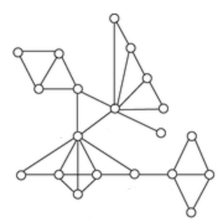
(f) Explain the equation with a "2" over it.

(g) Explain the inequality with a "3" over it.

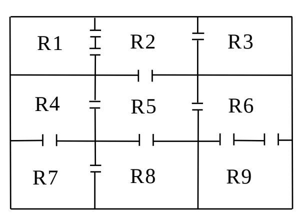
(h) Explain the general idea behind this proof. What is the method taken by the proof to show that $\lambda(K_n) = n-1$?

Question 9. A **block** in a graph G is a maximal (there's that word!) 2-connected subgraph of G. (Your book has some information about blocks in section 5.2, if you would like to read about them. Your book uses the term nonseparable, which is a synonym of 2-connected.) For the two graphs below, find all of the bridges, cut-vertices, and blocks.





Question 10. The picture below indicates the layout of nine rooms, with doorways between them. Is it possible to start in some room and take a walk through the rooms, going through each doorway exactly once? (Note that rooms *can* be revisited, and you can end in any room.) Defend your answer.



The fun problems

These should all be proven formally, unless otherwise indicated. You may use any method, and you may use any results we proved in class. Remember to keep and submit your scrap work for each problem. There are three problems required. **You may complete a fourth, beyond what is required, for one extra point.**

Complete either Question 11 or Question 12:

Question 11. No formal proof required, just a convincing explanation: Suppose T is a tree, and it has exactly $t-1$ vertices that are not leaves, its maximum degree is t, and no non-leaves have the same degree. Determine $|V(T)|$ (in terms of t). Hint: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Question 12. Let $n \geq 3$. Use the Prufer correspondence to determine (and then prove) how many trees with vertex set $[n]$ have exactly $n-2$ leaves. Then determine (with proof) how many have exactly 2 leaves. (Hint: What do leaves look like in the Prufer code?)

Complete either question 13 or question 14:

Question 13. Prove that a connected, 3-regular graph G has a cut-vertex if and only if G has a cut-edge. (Hint: For the forward direction, draw a picture!)

Question 14. Let G be a connected graph with more than one edge. Prove that G is 2-connected if and only if any two edges in G with a common endpoint lie together in some cycle of G.

Complete one of the following questions (more might be added):

Question 15. Let G be a graph with $\delta(G) \geq m$. Then for every tree T with m edges, T is a subgraph of G. (Hint: Use induction on m.)

Question 16. Let G be a connected regular graph which does not contain an Eulerian circuit. Prove that if $\overline{G}$ is connected, then $\overline{G}$ has an Eulerian circuit.