The Two Fundamental Theorems of Calculus

This topic document will demonstrate the important, amazing, and powerful relationship between differentiation and integration, showing that they are essentially inverse operations of each other.

We start by recalling the *Definition of a Definite Integral* based on *Riemann Sums*:

Definition of a Definite Integral

If $f(x)$ is defined on the closed interval $[a,b]$ and the *Riemann Sum* limit

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} f(u_i)\Delta x_i$$

exists, then $f(x)$ is **integrable** on $[a,b]$ and this limit is denoted by

$$\int_a^b f(x)dx$$

$\|P\|$ is the norm of the *Riemann Sum* partition, that is, the maximum partition width among all n partitions. As $\|P\| \to 0$, $n \to \infty$.

The definite integral is graphically understood to be the net *signed* area of the region bounded by the function $f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. From this definition of a definite integral, we can prove a number of easy-to-apply properties of definite integrals—consult any basic calculus text for the proofs of these properties.

This definition does not require the integrand function to be continuous, but only to be "integrable" (i.e., the *Riemann Sum* limit exists.) It can be shown that if the integrand function $f(x)$ is continuous on the interval $[a,b]$, then it is Riemann-integrable; the reverse is not necessary true as it is possible for a function to be Riemann-integrable but not continuous on the interval. Our focus will be on continuous functions.

In addition to using certain basic and well-known properties of definite integrals, our demonstration proofs will employ the following two theorems:

Mean Value Theorem for Integrals

If $f(x)$ is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$ (that is, $a \le c \le b$) such that

$$\int_a^b f(x)dx = f(c)\cdot(b-a)$$

Antiderivatives of a Function Differ Only by a Constant Theorem

If $F(x)$ is *an* antiderivative of $f(x)$ on an interval I, then $G(x)$ is *another* antiderivative of $f(x)$ on the interval I if and only if $G(x)$ is of the form:

$$G(x) = F(x) + C$$

for all x in I and where C is a *Real* constant.

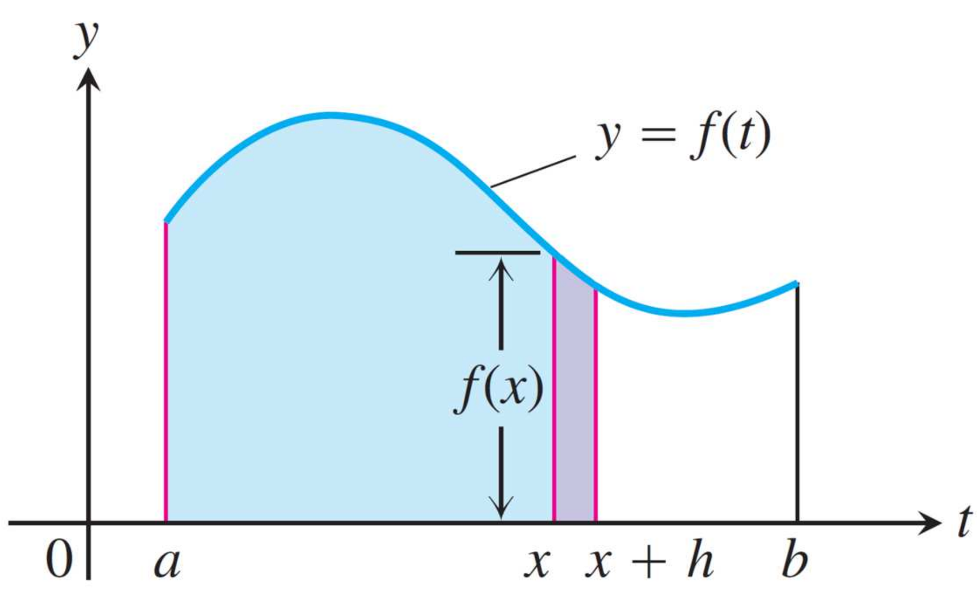
We will begin by proving what is termed the *First Fundamental Theorem of Calculus*.

Demonstration Proof of the First Fundamental Theorem of Calculus

Let us define the following unusual "area function":

$$A(x) = \int_a^x f(t)dt$$

Note that the function's independent variable x is the upper limit of the above definite integral. This definite integral can be graphically interpreted to be the net *signed* area of the region bounded on the left by $t = a$ (where a is a fixed constant), on the right by $t = x$ (we allow x to vary in the closed interval $[a, b]$), the function $f(t)$ (continuous in the closed interval $[a, b]$), and the t-axis. As x varies, the net *signed* area of the bounded region varies. In the following generalized figure where $f(t)$ is positive in the interval (but need not be for the theorem to be valid), $A(x)$ is the area of the light blue region:



In this figure, we are interested in the area of the magenta-colored "ribbon" between the function $f(t)$ and the t-axis, and bounded on the sides between $t = x$ and $t = x + h$ (the ribbon thus has width h.) Using the basic properties of definite integrals and our original definition of $A(x)$, the area of this "ribbon" can be expressed two distinct ways:

$$I.\ \ Area_{ribbon} = \int_x^{x+h} f(t)dt$$

$$II.\ \ Area_{ribbon} = A(x+h) - A(x)$$

Therefore,

$$\int_x^{x+h} f(t)dt = A(x+h) - A(x)$$

Since the function $f(t)$ is assumed continuous, we apply the *Mean Value Theorem for Integrals* presented above; the left side of the above equation can be re-expressed as:

$$f(c) \cdot [(x+h) - (x)] = A(x+h) - A(x)$$

$$where\ x \leq c \leq x + h$$

Simplifying:

$$f(c) \cdot h = A(x+h) - A(x)$$

Dividing both sides of this equation by h:

$$f(c) = \frac{A(x+h) - A(x)}{h}$$

Applying the limit $h \to 0$ to both sides of the equation (we are "narrowing" the ribbon):

$$\lim_{h \to 0} f(c) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

The right side of the above equation should be quite familiar from differential calculus: it is $A'(x)$!

For the left side of the equation, since c is always "sandwiched" between x and $x + h$, then as $h \to 0$, $c \to x$. Therefore, the above equation simplifies to

$$f(x) = A'(x)$$

Thus, $A(x)$ is *an* antiderivative of $f(x)$!!! This result will be used to prove the much more important *Second Fundamental Theorem of Calculus* which follows this proof. We are now close to finalizing our results for the *First Fundamental Theorem of Calculus.*

Since

$$A'(x) = \frac{d}{dx} A(x)$$

and

$$A(x) = \int_a^x f(t)dt$$

we now present the *First Fundamental Theorem of Calculus*:

First Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval I,

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

What does this theorem tell us? First, it shows that differentiation and integration are inverse operations of each other (which will be even clearer once we prove the even more amazing and extremely useful *Second Fundamental Theorem of Calculus*.) Second, it shows that if a function is continuous on some interval, the function has an antiderivative on the same interval; that is, the antiderivative exists.

Besides being a stepping-stone to understanding the "inverse" relationship of integration and differentiation, and proving the much more important *Second Fundamental Theorem of Calculus*, the *First Fundamental Theorem of Calculus* is sometimes of direct use in mathematics, science, and engineering—a separate topic.

Demonstration Proof of the Second Fundamental Theorem of Calculus

One of the most important problems in calculus, and some say the most important, is to exactly determine the value of a definite integral of a continuous function between fixed limits. Definite integrals are ubiquitous in mathematics, science, and engineering, so this is important! Directly using the definition of a *Riemann Sum* "taken to the limit" to find the exact value of a definite integral is difficult and oftentimes impossible; we need a better way! The *First Fundamental Theorem of Calculus* **almost** gives us what we need. The *Second Fundamental Theorem of Calculus*, which will be proven here, finishes the job and provides us with an astounding result and an extremely useful tool.

In our demonstration proof of the *First Fundamental Theorem of Calculus*, we arrived at the amazing conclusion that $A(x)$ is *an* antiderivative of the integrand function $f(x)$ in the following definite integral function:

$$A(x) = \int_a^x f(t)dt$$

If we were to set the upper limit of the definite integral to b, thus giving us an integral with fixed limits, we then have

$$A(b) = \int_a^b f(t)dt$$

This now provides us an equation to exactly evaluate the definite integral of a function between the lower limit a and the upper limit b, *provided* we can find *an* antiderivative of $f(x)$ which also fulfills the requirement that $A(a) = 0$ (since the definite integral between identical limits is zero, one of the basic properties of definite integrals.)

We can cleverly remove this second requirement, and use *any* convenient antiderivative of $f(x)$ by employing the *Antiderivatives of a Function Differ Only by a Constant* theorem presented at the beginning of this topic paper. This theorem states that all antiderivatives of a continuous function differ only by a *Real* constant:

$$A(x) = F(x) + C$$

where $A(x)$ and $F(x)$ are two antiderivatives of $f(x)$, and C is a *Real* constant. Substituting for $A(x)$ into our equation above, we obtain

$$F(x) + C = \int_a^x f(t)dt$$

Setting the upper limit to a, the definite integral will be zero. Thus,

$$F(a) + C = 0 \quad \text{or} \quad C = -F(a)$$

Substituting $-F(a)$ for C in the above definite integral equation, and setting the upper

limit to b, we obtain:

$$\int_a^b f(t)dt = F(b) - F(a)$$

This change from $A(x)$ to $F(x)$ no longer requires the antiderivative of $f(x)$ to be zero at $t = a$, so we can use *any* antiderivative of $f(x)$! Also, since the variable x no longer appears in the above equation, then the variable of integration in the definite integral is truly arbitrary; we will convert the variable of integration from t to x since that is how it is usually presented. This results in the astonishing *Second Fundamental Theorem of Calculus*:

Second Fundamental Theorem of Calculus

If a function $f(x)$ is continuous on the closed interval $[a, b]$ and $F(x)$ is *an* antiderivative of $f(x)$ on $[a, b]$, that is, $F'(x) = f(x)$ on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The simplicity, the beauty, and the power of this theorem cannot be overstated!

As noted by Eric Temple Bell in his famous work *Men of Mathematics* (1937, p. 110), this theorem is "surely one of the most astonishing things a mathematician ever discovered." Martin Gardner, the famous author on "popular" topics in mathematics, wrote: "It is an amazing theorem—one that unites differentiating with integrating. It works like sorcery, almost too good to be true!" (annotation in the 1998 republication of Silvanus P. Thompson's classic *Calculus Made Easy*, first published in 1910, p. 215.)

The Second Fundamental Theorem of Calculus allows us to *exactly* evaluate the definite integral of any continuous function by first finding the antiderivative expression of the integrand function (see note at right), and then calculating the difference between the antiderivative expression evaluated at both end points of the closed interval defined by the lower and upper bounds of the definite integral (also known as the "limits of integration.")

It is important to note that the antiderivatives of many functions cannot be expressed as "finite" formulas of elementary functions!

In essence, we reduce the problem of evaluating a definite integral of a continuous function into finding and evaluating the function's antiderivative. This is <u>*always*</u> easier than the difficult, laborious, and oftentimes impossible task of taking a *Riemann Sum* "to the limit." Integral Calculus is devoted to learning the various and sometimes clever techniques to find the antiderivatives of functions, which we will now restate as finding the *indefinite integral of a function*, given by the following "limit-free" integral notation:

$$\textit{antiderivative of } f(x) \implies \int f(x)\,dx$$

Example 1

Find the exact value of

$$\int_3^6 x^2 dx$$

The integrand is x^2 and it is easy to verify by differentiation that the family of antiderivatives of this integrand function is given by:

$$\int x^2\, dx = \frac{1}{3}x^3 + C$$

Since the integrand function is continuous on the closed interval of integration, we may use the *Second Fundamental Theorem of Calculus*:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is *any* antiderivative of $f(x)$, and $f(x)$ is continuous on $[a, b]$. Substituting our specific quantities:

$$\int_3^6 x^2\, dx = \frac{1}{3}x^3 \Big|_3^6 = \left(\frac{1}{3}6^3\right) - \left(\frac{1}{3}3^3\right) = 63 \quad \blacksquare$$

Since the function $f(x) = x^2$ is everywhere positive in the interval of integration, the value of the definite integral, 63, represents the *exact* area of the graphical region bounded by $f(x) = x^2$, the x-axis, and the vertical lines $x = 3$ and $x = 6$.

We may omit the constant of integration C when evaluating a definite integral using the *Second Fundamental Theorem of Calculus*. The reason is that we may use *any* antiderivative, so set $C = 0$!