Vectors in Space

What Do We Mean by R^n?

R^n for a given integer n stands for the collection of all possible strings

$$(a_1, a_2, a_3, \cdots, a_n)$$

Such a string is called n-tuple of nummbes.

When $n = 1$ all we have is a real number.

Addition, Subtraction, and Scalar Multiplication in R^n

We suppose that n is a given positive integer and that $A = (a_1, a_2, a_3, \cdots, a_n)$ and $B = (b_1, b_2, b_3, \cdots, b_n)$ are points in R^n.

We define

$$A + B = (a_1, a_2, a_3, \cdots, a_n) + (b_1, b_2, b_3, \cdots, b_n) = (a_1 + b_1, a_2 + b_2, \cdots a_n + b_n)$$

and

$$A - B = (a_1, a_2, a_3, \cdots, a_n) - (b_1, b_2, b_3, \cdots, b_n) = (a_1 - b_1, a_2 - b_2, \cdots a_n - b_n)$$

We suppose that n is a given integer, that $A = (a_1, a_2, a_3, \cdots, a_n)$, and that t is any given number. The product tA is defined by the equation

$$tA = t(a_1, a_2, a_3, \cdots, a_n) = (ta_1, ta_2, ta_3, \cdots, ta_n)$$

The Norm $||A||$ of a point A in R^n

If $A = (a_1, a_2, a_3, \cdots, a_n)$ then we define

$$||A|| = \sqrt{a_1^2, a_2^2, a_3^2, \cdots, a_n^2}$$

Special Case: The Norm $||A||$ of a point A in R^n

The norm $||(x, y)||$ of a point (x, y) in R^2 is $\sqrt{x^2 + y^2}$

The Norm of a Point P in R^3

In R^3 the number $||P||$ is the distance from P to O.

Applying the theorem of Pythagoras to $\triangle OQP$ we see that

$$\begin{aligned}(OP)^2 &= (OQ)^2 + (QP)^2 \\ &= (\sqrt{x^2 + y^2})^2 + z^2 \\ &= x^2 + y^2 + z^2 \\ OP &= \sqrt{x^2 + y^2 + z^2} = ||P||\end{aligned}$$

The norm of a point is its distance to the point $O = (0, 0, 0)$.

Some Properties of the Arithmetic in R^n

Assume that $A = (a_1, a_2, a_3, \cdots, a_n)$ and $B = (b_1, b_2, b_3, \cdots, b_n)$ and $C = (c_1, c_2, c_3, \cdots, c_n)$ and that s and t are numbers.

The Commutative Law $A + B = B + A$

Proof

$$\begin{aligned} A + B &= (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_2) \\ &= (b_1 + a_1, b_2 + a_2, \cdots, b_n + a_n) \\ &= B + A \end{aligned}$$

The Associative Law $(A + B) + C = A + (B + C)$

Proof

The left side is

$$\begin{aligned} a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n + (c_1, c_2, c_3, \cdots, c_n) &= ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2, \cdots, (a_n + b_n) + c_n) \\ &= (a_1 + b_1 + c_1, a_2 + b_2 + c_2, \cdots, a_n + b_n + c_2) \end{aligned}$$

The Point O

This is the point $(0, 0, 0)$

The Equation $A + O = A$

The Equation $A - A = 0$

The Equation $1A = A$

The Equation $0A = O$

The Equation $(s + t)A = sA + tA$

The Equation $t(A + B) = tA + tB$

The Equation $||tA|| = |t|\,||A||$

Proof

I want to point out that, if t is any real number, then $\sqrt{t^2} = |t|$

$$\begin{aligned} ||tA|| &= ||t(a_1, a_2, a_3, \cdots, a_n)|| \\ &= (ta_1, ta_2, ta_3, \cdots, ta_n)|| \\ &= \sqrt{(ta_1)^2 + (ta_2)^2 + (ta_3)^2 + \cdots + (ta_n)^2} \\ &= \sqrt{t^2(a_1^2 + a_2^2 + \cdots a_n^2)} \\ &= |t| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2} \\ &= |t|\,||A|| \end{aligned}$$

The Standard Basis in R^n

The **standard basis** in R^3 is the set $\{(1,0,0),(0,1,0),(0,0,1)\}$ and we can extend this to R^n

We write $(1,0,0) = I$ and $(0,1,0) = J$ and $(0,0,1) = K$

In R^n we define

$$\begin{aligned} I_1 &= (1,0,0,\cdots,0) \\ I_2 &= (0,1,0,\cdots,0) \\ I_3 &= (0,0,1,0,\cdots,0) \\ &\vdots \\ I_n &= (0,0,0,\cdots,1) \end{aligned}$$

and the set $\{I_1, I_2, I_3, \cdots, I_n\}$ is called the **standad basis** in R^n.

Linear Combinations

A **linear combination** of points A and B and C is any point of the form

$$rA + sB + tC$$

where r and s and t can be any numbers. In the same way, we can have a linear combination of many points.

We can extend this idea to larger sets. For example, since

$$(3,-2,5,0,7) = 3(1,0,0,0,0) - 2(0,1,0,0,0) + (0,0,1,0,0) + 0(0,0,0,1,0) + 7(0,0,0,0,1)$$

we see that $(3,-2,5,0,7)$ is a linear combination of the set $\{I_1, I_2, I_3, I_4, I_5\}$

Linear Independence

A set of points in R^n is said to be **linearly independent** if it is impossible to make linear combination of these points equal to O unless all the coeffiencients are zero.

For example, since

$$a(1,0,0,0) + b(0,1,0,0) + c(0,0,1,0) + d(0,0,0,1) = (a,b,c,d)$$

the only way we can make a linear combination

$$a(1,0,0,0) + b(0,1,0,0) + c(0,0,1,0) + d(0,0,0,1)$$

equal to O is by making $a = b = c = d$

An Important Principle Studied in a First Course in Linear Algebra

If a set $\{A, B, C\}$ of points in R^3 is linearly independent, then *every* point in R^3 can be expressed as a linear combination of A, B, and C.

In other words, if W is any given point in R^3, then the equation

$$xA + yB + zC = W$$

can be solved for x, y, and z

So, for example, if $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ and $C = (c_1, c_2, c_3)$ and $W = (w_1, w_2, w_3)$, then the equation says that

$$x(a_1, a_2, a_3) + y(b_1, b_2, b_3) + z(c_1, c_2, c_3) = (w_1, w_2, w_3)$$

and this says that

$$\begin{aligned} a_1x + b_1y + c_1z &= w_1 \\ a_2x + b_2y + c_2z &= w_2 \\ a_3x + b_3y + c_3z &= w_3 \end{aligned}$$

and we are saying that this system of three equations in the three unknowns x, y, and z can be solved. We can make a similiar statement for R^n for each n.

Geometric Interpretation of the Arithmetic in R^2 and R^3

Line Segments with the Same Length and Direction

We suppose that we have four points

$$\begin{aligned} A &= (x_1, y_1, z_1) \\ B &= (x_2, y_2, z_2) \\ C &= (x_3, y_3, z_3) \\ D &= (x_4, y_4, z_4) \end{aligned}$$

in R^3 and we look at the two line segments AB and CD.

The conditions for these two line segments to have the same length and also point in the same direction is that the equations

$$\begin{aligned} x_2 - x_1 &= x_4 - x_3 \\ y_2 - y_1 &= 7y_4 - y_3 \\ z_2 - z_1 &= z_4 - z_3 \end{aligned}$$

hold. Thus the two line segments AB and CD will have the same length and direction if and only if

$$\begin{gathered} (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (x_4 - x_3, y_4 - y_3, z_4 - z_3) \\ B - A = D - C \end{gathered}$$

because this says that

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1) = (x_4 - x_3, y_4 - y_3, z_4 - z_3)$$

We can, of course, make the same observation about points in R^2.

The Notation $\vec{AB}$

Given any two points A and B in R^n we define

$$\vec{AB} = B - A$$

Given points A and B and C and D in space, the condition

$$\vec{AB} = \vec{CD}$$

says that

$$B - A = D - C$$

and this is the condition AB and CD are line segments with the same length and the same direction.

We need a name for $\vec{AB}$

The symbol $\vec{AB}$ is given the name AB

Note that, if A and B are points in space then the symbol AB stands for the line segment between A and B and, sometimes, we use the symbol AB for the *length* of the line segment AB.

On the other hand, vector AB, which is written as $\vec{AB}$, means a **point** in space. In fact, $\vec{AB}$ is the point $B - A$.

So, for example, if $A = (2, 4, -3)$ and $B = (5, 1, -8)$, then

$$\begin{aligned}\vec{AB} &= B - 1\\ &= (5, 1, -8) - (2, 4, -3)\\ &= (3, -3, -5)\end{aligned}$$

Using the Norm to Find the Length of a Line Segment in R^3

If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ are any points in R^3, then the distance formula tells us that the distance from A to B is

$$\begin{aligned}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} &= ||(x_2 - x_1, y_2 - y_1, z_2 - z_1)||\\ &= ||B - A||\\ &= ||\vec{AB}||\end{aligned}$$

Another way of looking at this topic is to define $P = B - A$. From the equation

$$P - O = P = B - A$$

we see that the line segments OP and AB have the same length (and direction) and so

$$dis(AB) = dist(OP) = ||P|| = ||B - A|| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Line Segments with the Same Length

Two line segments AB and CD have the same length if and only if $||\vec{AB}|| = ||\vec{CD}||$

$$||B - A|| = ||D - C||$$

Two Line Segments with the Same Direction

We begin by looking at a special case

Suppose that AB and CD have the same direction but that AB is twice as long as CD and that

$$\begin{aligned} A &= (x_1, y_1, z_1) \\ B &= (x_2, y_2, z_2) \\ C &= (x_3, y_3, z_3) \\ D &= (x_4, y_4, z_4) \end{aligned}$$

In this case

$$\begin{aligned} x_2 - x_1 &= 2(x_4 - x_3) \\ y_2 - y_1 &= 2(y_4 - y_3) \\ z_2 - z_1 &= 2(z_4 - z_3) \end{aligned}$$

Un this case we see that

$$B - A = 2(D - C)$$

and this says that

$$\vec{AB} = 2\vec{CD}$$

Now we extend the idea

Suppose that t is a positive number. The condition

$$B - A = t(D - C)$$

which is

$$\vec{AB} = t\vec{CD}$$

says that the line segments AB and CD have the same direction but that AB is t times as long as CD

Line Segments with Opposite Directions

1. Given points A and B we can see that

$$\vec{BA} = A - B = -(B - A) = -\vec{AB}$$

and we can see, the directions of the line segments AB and BA are opposite to one another.

2. Now suppose that A, B, C, and D are points in space and that $\vec{AB} = -\vec{CD}$. This tells us that

$$\vec{AB} = B - A = -(D - C) = C - D = \vec{DC}$$

and so the line segments AB and CD have the same length but opposite directions.

3. Now suppose that A, B, C, and D are point in space ad that t is a negative number and that

$$\vec{AB} = t(\vec{CD})$$

This gives us

$$\begin{aligned} \vec{AB} &= t(D - C) \\ &= -t(C_D) \\ &= (-t)(\vec{DC} \end{aligned}$$

The number it is positive. The line segment AB is $-t$ times as long as the line segment CD and the directions of the line segments AB and CD are opposite to one another.

The Inner Product (Dot Product)

Definition of the Inner Product (Good in R^n for any positive integer n)

Given two points $A = (a_1, a_2, a_3, \cdots, a_n)$ and $B = (b_1, b_2, b_3, \cdots, b_n)$ in R^n, we define

$$A \cdot B = a_1 b_1 + a_2 b_2 + a_3 b_3 + \cdots a_n b_n$$

Note that $A \cdot B$ is a number.

When We Are in R^3

If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ then

$$A \cdot B = x_1 x_2 + y_1 y_2 + z_1 z_2$$

Simple Facts About the Dot Product

We suppose that $A = (a_1, a_2, a_3, \cdots, a_n)$ and $B = (b_1, b_2, b_3, \cdots, b_n)$ and $C = (c_1, c_2, c_3, \cdots, c_n)$ and that s and t are numbers

The Equation $A \cdot A = ||A||^2$

The reason for this is

$$\begin{aligned} A \cdot A &= (a_1, a_2, a_3, \cdots, a_n) \cdot (a_1, a_2, a_3, \cdots, a_n) \\ &= a_1^2 + a_2^2 + a_3^2 + \cdots a_n^2 \\ &= ||A||^2 \end{aligned}$$

The Commutative Law $A \cdot B = B \cdot A$

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

The Distributive Law $A \cdot (B + C) = A \cdot B + A \cdot C$

The equation $t(A \cdot B) = (tA) \cdot B = A \cdot (tB)$

$$\begin{aligned} t(A \cdot B) &= t(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \\ &= ta_1 b_1 + ta_2 b_2 + \cdots + ta_n b_n \\ &= a_1 tb_1 + a_2 tb_2 + \cdots + a_n tb_n \\ &= A \cdot (tB) \end{aligned}$$

The Cauchy-Schwarz Inequality $|A \cdot B| \leq ||A||||B||$
If $\angle AOB = \theta$, then

$$A \cdot B = (OA)(OB)\cos(\theta) = ||A||\,||B||\cos(\theta)$$

and so

$$\begin{aligned} |A \cdot B| &= ||A||\,||B|||\cos(\theta)| \\ &\leq ||A||\,||B||1 \end{aligned}$$

The Minkoski Inequality $||A + B|| \leq ||A|| + ||B||$

$$||A + B|| = ||C|$$

which is the distance from O to C and this can't be more than the distance from O to A plus the distance from O to B.

The Triangle Inequality $||A - C|| \leq ||A - B|| + ||B - C||$
Given the points A and B and C we have

$$||A - C|| \leq ||A - B|| + ||B - C||$$

Review of the Law of Cosines

Explain why

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos\angle A \\
b^2 &= a^2 + c^2 - 2ac\cos\angle B \\
c^2 &= a^2 + b^2 - 2ab\cos\angle C
\end{aligned}
$$

We suppose that $\triangle ABC$ is any triangle and we move $\triangle ABC$ into a coordinate system as follows.

1. we slide $\triangle ABC$ until A is at O.
2. we roate $\triangle ABC$ until B is on the right side of the x-axis.
3. If necessary, we flip $\triangle ABC$ to make C sit above the x-axis.

$$
\begin{aligned}
A &= (0,0) \\
B &= (c,0)
\end{aligned}
$$

The point C is a bit harder.

The angle $\angle A$ is drawn in standard position. The angle winds to the terminal line $OC = AC$ whose length is b and so, if we write C as (x,y) just for the moment, then we have

$$
\begin{aligned}
\cos\angle A &= \frac{x}{n} \\
\sin\angle A &= \frac{y}{b}
\end{aligned}
$$

and so

$$
\begin{aligned}
C &= (b\cos\angle A, b\sin\angle A) \\
A &= (0,0) \\
B &= (c,0) \\
C &= (b\cos\angle A, b\sin\angle A)
\end{aligned}
$$

From the distance formula we can see that

$$
(BC)^2 = (b\cos\angle A - c)^2 + (b\sin\angle A - 0)^2
$$

and so

$$
\begin{aligned}
a^2 &= b^2\cos^2\angle A - 2bc\cos\angle A + c^2 + b^2\sin^2\angle A \\
a^2 &= b^2\cos^2\angle A + b^2\sin^2\angle - 2bc\cos\angle A + c^2 \\
a^2 &= b^2(\cos^2\angle A + \sin^2\angle A) - 2bc\cos\angle A + c^2 \\
a^2 &= b^2(1) - 2bc\cos\angle A + c^2
\end{aligned}
$$

A Geometric Interpretation of the Inner Product in R^2 and R^3

If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ are points in R^3 (or R^2) that are unequal to the origin O and $\theta = \angle AOB$. Then

$$A \cdot B = ||A||\,||B|| \cos(\theta)$$

Proof

We apply the law of cosines to $\triangle AOB$ and get

$$\begin{aligned}
(AB)^2 &= (OA)^2 + (OB)^2 - 2(OA)(OB)\cos(\theta) \\
(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 &= x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2||A||\,||B|| \cos(\theta) \\
x_1^2 - 2x_1x_2 + x_2^2 + y_1^2 - 2y_1y_2 + y_2^2 + z_1^2 - 2z_1z_2 + z_2^2 &= x_1^2 + y_1^2 + z_1^2 + x_2^2 + y_2^2 + z_2^2 - 2||A||\,||B|| \cos(\theta) \\
-2x_1x_2 - 2y_1y_2 - 2z_1z_2 &= -2||A||\,||B|| \cos(\theta) \\
x_1x_2 + y_1y_2 + z_1z_2 &= ||A||\,||B|| \cos(\theta) \\
A \cdot B &= ||A||\,||B|| \cos(\theta)
\end{aligned}$$

Points Orthogonal to One Another

We say that A and B are **orthogonal** to each other when $A \cdot B = 0$

Orthogonality in R^2 or R^3

Saying that $A \cdot B = 0$ is saying that

$$||A||\,||B|| \cos \angle AOB = 0$$

and this gives us

$$\cos \angle AOB = 0$$

and this holds when $\angle AOB = 90°$ and this says that $OA \perp OB$

A condition for Two Line Segments to be Perpendicular to One Another

We define $P = B - A = \vec{AB}$ and $Q = D - C = \vec{CD}$.

The line segments AB and OP have length and the same direction.

The line segments CD and OQ have the same length and the samwe direction.

The condition for CD to be perpendicular to AB is that $OP \perp OQ$ and this says that $P \cdot Q = 0$ and this says that

$$(\vec{AB} \cdot (\vec{CD}) = 0$$

The Points $A+B$ and $A-B$ are Orthogonal to Each other if and only if $||A|| = ||B||$

To say that $A+B$ iw orthogonal to $A-B$ is to say that

$$(A+B)\cdot(A-B) = 0$$

and this says that

$$\begin{aligned} A\cdot A + B\cdot A - A\cdot B - B\cdot B = 0 \\ A\cdot A + A\cdot B - A\cdot B - B\cdot B = 0 \end{aligned}$$

and this says that

$$||A||^2 - ||B||^2 = 0$$

and says that

$$||A|| = ||B||$$

The Cross Product

The Definition of the $\times$ Product in R^3

Given $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ we define

$$\begin{aligned} A\times B &= (x_1, y_1, z_1)\times(x_2, y_2, z_2) \\ &= (y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2) \end{aligned}$$

Another way to get this is to work out

$$det \begin{bmatrix} I & J & K \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{aligned} &= I(y_1z_2 - y_2z_1) + J(x_2z_1 - x_1z_2) + K(x_1y_2 - x_2y_1) \\ &= (y_1z_2 - y_2z_1, x_2z_1 - x_1z_2.x_1z_2, x_1y_2 - x_2y_1) \end{aligned}$$

The Product $A\times A$

Given $A = (x, y, z)$ we have

$$\begin{aligned} A\times A &= (x, y, z)\times(x, y, z) \\ &= (yz - zy, zx - xz, xy - yx) \\ &= (0, 0, 0) \\ &= O \end{aligned}$$

Some Facts About the $\times$ Product

The Equation $B \times A = -A \times B$

We assume that $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$

$$\begin{aligned} A \times B &= (x_1, y_1, z_1) \times (x_2, y_2, z_2) \\ &= (y_1 z_2 - z_1 y_2, z_1 x_2 - x_1 z_2, x_1 y_2 - y_1 x_2) \end{aligned}$$

On the other hand

$$\begin{aligned} B \times A = (x_2, y_2, z_2) \times (x_1, y_1, z_1) \\ &= (y_2 z_1 - z_2 y_1, z_2 x_1 - x_2 z_1, x_2 y_1 - y_2 x_1) \end{aligned}$$

$$B \times A = -A \times B$$

The Equation $A \times (B + C) = A \times B + A \times C$

We assume that $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ and $C = (x_3, y_3, z_3)$ and we see that

$$\begin{aligned} A \times (B + C) &= (x_1, y_1, z_1) \times (x_2 + x_3, y_2 + y_3, z_2 + z_3) \\ &= \Big(y_1(z_2 + z_3) - z_1(y_2 + y_3), z_1(x_2 + x_3) - x_1(z_2 + z_3), x_1(y_2 + y_3) - y_1(x_2 + x_3)\Big) \end{aligned}$$

The Equation $(tA) \times B = t(A \times B)$

Failure of the Law $A \times (B \times C) = (A \times B) \times C$

The Norm of $A \times B$

Given A and B we have the following equation

$$||A \times B||^2 = ||A||^2||B||^2 - (A \cdot B)^2$$

If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ then

$$\begin{aligned}||A \times B||^2 &= ||y_1z_2 - z_1y_2, z_1x_2 - x_1z_2, x_1y_2 - y_1x_2||^2 \\ &= (y_1z_2 - z_1y_2)^2 + (z_1x_2 - x_1z_2)^2 + (x_1y_2 - y_1x_2)^2\end{aligned}$$

We can make an important conclusion

If $\theta = \angle AOB$ then we know that

$$A \cdot B = ||A||\,||B|| \cos(\theta)$$

Now we use the equation

$$||A \times B||^2 = ||A||^2||B||^2 - (A \cdot B)^2$$

and we can see that

$$\begin{aligned}||A \times B||^2 &= ||A||^2||B||^2 - ||A||^2||B||^2\cos^2(\theta) \\ &= ||A||^2||B||^2(1 - \cos^2(\theta) \\ &= ||A||^2||B||^2\sin^2(\theta)\end{aligned}$$

and so

$$||A \times B|| = ||A||\,||B||\,|\sin(\theta)|$$

Our understanding of θ is that $0 \leq \theta \leq 180°$ and so $\sin(\theta)$ can't be negative and so we can drop the absolute value sign

$$||A \times B|| = ||A||\,||B|| \sin(\theta)$$

The Direction of $A \times B$

We write $P = A \times B$ and I want to explain that OP is perpendicular both to OA and to OB and so OP is perpendicular to $\triangle OAB$

The Vector Triple Product

Given A and B and C

$$A \times (B \times C) = (A \times C)B - (A \cdot B)C$$

Lines and Planes in R^3

Equation of a Plane

We can specify a plane in R^3 by giving a point $A = (x_1, y_1, z_1)$ in R^3 that will be in the plane and giving a vector $U = (a, b, c) \neq O$ that is perpendicular to the plane.

When we say U is perpendicular to the green plane we mean that the line segment OU is perpendicular to the plane.

Given any point $P = (x, y, z)$, the condition for P to be in the green plane is that $AP \perp OU$ and this says that

$$\vec{AP} \cdot \vec{OU} = 0$$

and this says that

$$(x - x_1, y - y_1, z - z_1) \cdot (a, b, c) = 0$$

and we get

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Parametric Equations of a Line

We can specify a line in R^3 by giving a point $A = (x_1, y_1, z_1)$ in the line and giving a vector $U = (a, b, c) \neq O$ that is parallel to the line.

The condition for a point $P - (x, y, z)$ to be in the given line is that AP is parallel to OU.

This says that for some number t we have

$$\vec{AP} = t\vec{OU}$$

and this says that

$$(x - x_1, y - y_1, z - z_1) = t(abc)$$

Maybe we might like to write this equation in this form

$$\begin{aligned} x - x_1 &= ta \\ y - y_1 &= tb \\ z - z_1 &= tc \end{aligned}$$

and this says that

$$\begin{aligned} x &= x_1 + at \\ y &= y_1 + bt \\ z &= z_1 + ct \end{aligned}$$