2.6. The Moment-Generating Function

1. *Definition.* If X is a random variable, then the function

$$M(t) = E(e^{tX})$$

is called the moment-generating function (mgf) of X.

2. *Formula.* If X is random variable, the range of X is $\{x_1, x_2, \cdots, x_m\}$, and the pmf of X is $f(x)$, then

$$M(t) = e^{tx_1} f(x_1) + \cdots + e^{tx_m} f(x_m).$$

3. *Example.* Suppose that X is a discrete random variable with pmf:

x	1	3	5
$f(x)$	0.2	0.5	0.3

Find the mgf $M(t)$ of X.

— *Solution.* Note that e^{tX} is also a random variable. Then

x	1	3	5
e^{tx}	e^t	e^{3t}	e^{5t}
$f(x)$	0.2	0.5	0.3

It follows that

$$M(t) = E(e^{tX}) = \frac{1}{5}e^t + \frac{1}{2}e^{3t} + \frac{3}{10}e^{5t}.$$

4. *Properties of moment-generating functions.* Moment generating functions are positive and log-convex, with $M(0) = 1$.

5. *Theorem.* If $-1 < x < 1$, then

$$x + x^2 + x^3 + x^4 + x^5 + \cdots\cdots = \frac{x}{1-x}.$$

— *Proof.* If we let $T = x + x^2 + x^3 + x^4 + x^5 + \cdots\cdots$, then

$$\begin{aligned} T &= x + \left(x^2 + x^3 + x^4 + x^5 + \cdots\cdots\right) \\ &= x + x\left(x + x^2 + x^3 + x^4 + x^5 + \cdots\cdots\right) = x + xT. \end{aligned}$$

In summary, $T = x(1+T)$. Solving for T, we get

$$T = \frac{x}{1-x}.$$

6. *Example.* Let X be a discrete random variable. If the range of X is $\{1, 2, 3, 4, \cdots\}$, and the pmf of X is

$$f(x) = \frac{1}{2^x}, \quad x = 1, 2, 3, \cdots ,$$

then the mgf of X is

$$M(t) = \frac{e^t}{2 - e^t}.$$

— *Proof.* The proof is an easy application of Theorem 4: The mgf of X is

$$M(t) = e^t \frac{1}{2} + e^{2t} \frac{1}{2^2} + e^{3t} \frac{1}{2^3} + e^{4t} \frac{1}{2^4} + \cdots\cdots .$$

Let $\alpha = e^t/2$, then

$$M(t) = \alpha^2 + \alpha^3 + \alpha^4 + \cdots = \frac{\alpha}{1 - \alpha} = \frac{e^t/2}{1 - e^t/2} = \frac{e^t}{2 - e^t}.$$

7. *Theorem.* If X is a random variable with mgf $M(t)$, then

$$M'(0) = E(X), \quad M''(0) = E(X^2), \quad \cdots\cdots .$$

In general, for each $r \geq 0$,

$$M^{(r)}(0) = E(X^r).$$

8. *Proof.* Suppose that X is random variable, the range of X is $\{x_1, x_2, \cdots, x_m\}$, and the pmf of X is $f(x)$, then

$$M(t) = e^{tx_1} f(x_1) + \cdots + e^{tx_m} f(x_m).$$

It follows that

$$M'(t) = x_1 e^{tx_1} f(x_1) + \cdots + x_m e^{tx_m} f(x_m),$$

$$M'(0) = x_1 0 f(x_1) + \cdots + x_m f(x_m) = E(X).$$

9. *Mean and variance of binomial distribution.*

Let X be a random variable with Binomial(n, p) distribution. Then, X has pmf

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, 3, \cdots, n;$$

and,

$$M(t) = (1 - p + pe^t)^n,$$

$$\mu = np, \quad \sigma^2 = np(1-p).$$

Here, $\mu = E(X)$, $\sigma^2 = Var(X)$, and $M(t)$ is the mgf of X.

10. *Definition.* (geometric distribution)

Let $p \in [0, 1]$ be a fixed parameter.

In a sequence of independent Bernoulli(p) trials, let X be the ordinal number of the trial at which the first success occurs.

Then X is a random variable, and the pmf of X is

$$f(x) = P(X = x) = p(1-p)^{x-1}, \quad x = 1, 2, 3, \cdots .$$

We say X has the geometric distribution with parameter p.

11. *Mean and variance of geometric distribution.*

    Let $p \in [0, 1]$ be a fixed parameter.

    If X is a random variable with the geometric(p) distribution, then

$$M(t) = \frac{pe^t}{1 - (1-p)e^t},$$

$$E(X) = \frac{1}{p}, \quad Var(X) = \frac{1-p}{p^2}.$$

12. The geometric distribution is a special case of the negative binomial distribution.

13. *Example.* We perform a sequence of Bernoulli(p) trials until two successes are observed. (The two success are not necessarily consecutive.) Let X be the number of trials (needed to observe the two successes.)

It is clear that the sample space is

$$S = \{ \; 11, \quad 011, 101, \quad 0011, 0101, 1001, \quad 00011, 00101, 01001, 10001, \quad \cdots\cdots\cdots \}.$$

The pmf of X is

x	2	3	4	5	$\cdots$
$f(x)$	p^2	$2(1-p)p^2$	$3(1-p)^2p^2$	$4(1-p)^3p^2$	$\cdots$

Or, equivalently,

$$f(x) = P(X = x) = \binom{x-1}{1} p^2 (1-p)^{x-2}, \quad x = 2, 3, 4, 5, 6, \cdots .$$

14. *Definition.* (negative binomial distribution)

Let $p \in [0, 1]$ be a fixed parameter and r a fixed positive integer.

In a sequence of independent Bernoulli(p) trials, let X be the ordinal number of the trial at which the r-th success occurs.

Then X is a random variable, and the pmf of X is

$$f(x) = P(X = x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, r+2, \cdots .$$

We say X has the negative binomial distribution with parameters (r, p).

15. The negative binomial distribution is related to the following negative binomial expansion:

16. *Formula.* If $-1 < x < 1$ and r is a positive integer, then

$$\frac{1}{(1-x)^r} = 1+\binom{r}{1}x+\binom{r+1}{2}x^2+\binom{r+2}{3}x^3+\binom{r+3}{4}x^4+\cdots\cdots.$$

17. The binomial distribution is related to the following binomial formula:

18. *Formula.* If r is a positive integer, then

$$(1+x)^r = \sum_{i=0}^{r}\binom{r}{i}x^i.$$

19. *Mean and variance of negative binomial distribution.*

Let X be a random variable with the negative binomial distribution with parameters (r, p). Then

$$M(t) = \frac{(pe^t)^r}{(1-(1-p)e^t)^r},$$

$$E(X) = \frac{r}{p}, \quad Var(X) = \frac{r(1-p)}{p^2}.$$