5.2. Transformations of Two Random Variables

1. This section is a long section. Some key words are: transformation, inverse transformation, Jacobian matrix, Jacobian determinant, exponential distribution, Gamma distribution, χ^2 distribution, F distribution, $Gamma$ function, $Beta$ function.

2. Let us begin with a motivating example.

3. *Example.* Suppose that X_1, X_2 are continuous random variables with joint probability density function

$$f(x_1, x_2) = 1, \quad 0 < x_1 < 1, 0 < x_2 < 1.$$

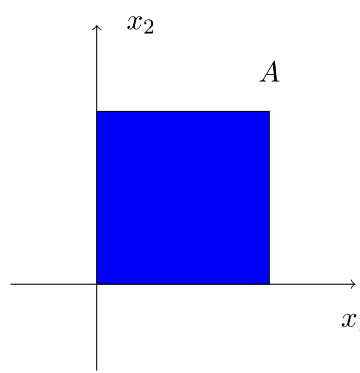
Let $Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$, then Y_1, Y_2 are also random variables. Find the joint pdf of Y_1 and Y_2.

* We will be able to solve this problem after we develop a formula. But let us do some preliminary analysis. If we define $A \subset \mathbb{R}^2$ as

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1, 0 < x_2 < 1\},$$

then A is the support of the random vector (X_1, X_2).

Geometrically, A is a square in the x_1x_2-plane.

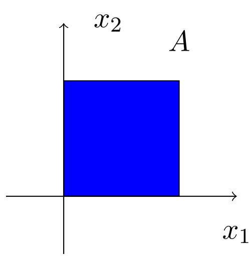


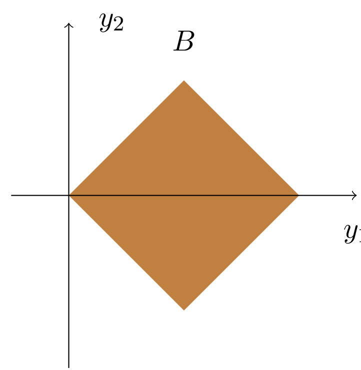
If we define

$$u_1(x_1, x_2) = x_1 + x_2, \quad u_2(x_1, x_2) = x_1 - x_2,$$

$$u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2)) = (x_1 + x_2, x_1 - x_2),$$

then the vector function u transforms (X_1, X_2) into (Y_1, Y_2). And, the function u transforms the square A in the x_1x_2-plane into another square B in the y_1y_2-plane.





If we solve the equations

$$y_1 = x_1 + x_2, \quad y_2 = x_1 - x_2$$

for x_1 and x_2, we get

$$x_1 = \frac{y_1}{2} + \frac{y_2}{2}, \quad x_2 = \frac{y_1}{2} - \frac{y_2}{2}.$$

So now it is clear that the inverse function of $u(x_1, x_2)$ is

$$v(y_1, y_2) = (v_1(y_1, y_2), v_2(y_1, y_2)) = (\tfrac{1}{2}(y_1 + y_2), \tfrac{1}{2}(y_1 - y_2)).$$

The function u transforms region A to region B, and the function v transforms region B back to A.

4. In summary, the purpose of this section is to develop a general procedure for solving the following general problem.

5. *General problem of this section:* Let A be a region in the x_1x_2 plane. Let B be a region in the y_1y_2-plane. Let $u : A \to B$ be a bijection. Denote by v the inverse function of u. Then $v : B \to A$ is a bijection.

   X_1, X_2 are continuous random variables with joint pdf

   $$f(x_1, x_2), \qquad (x_1, x_2) \in A.$$

   Here, A is the support of the random vector (X_1, X_2).

   Now the function u transforms the random vector (X_1, X_2) to the random vector (Y_1, Y_2). That is, $(Y_1, Y_2) = u(X_1, X_2)$. Then, the support of the random vector (Y_1, Y_2) is B. The question is — What is the joint pdf of Y_1 and Y_2?

6. Now we state the general formula.

7. *Formula.*

Let $A, B \subset \mathbb{R}^2$. Suppose that $u : A \to B$ is a bijection. Denote by v the inverse function of u, then $v : B \to A$ is also a bijection. Denote

$$u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2)),$$

and denote

$$v(y_1, y_2) = (v_1(y_1, y_2), v_2(y_1, y_2)).$$

Here $u_1, u_2 : A \to \mathbb{R}$, and $v_1, v_2 : B \to \mathbb{R}$.

Suppose that X_1, X_2 are random variables, the random vector (X_1, X_2) has support A, and (X_1, X_2) has joint pdf

$$f(x_1, x_2), \quad (x_1, x_2) \in A.$$

The function u transforms the random vector (X_1, X_2) to another random vector (Y_1, Y_2). That is,

$$(Y_1, Y_2) = u(X_1, X_2),$$

or, equivalently,

$$Y_1 = u_1(X_1, X_2), \quad Y_2 = u_2(X_1, X_2),$$

Then, the function v transforms the random vector (Y_1, Y_2) back to the random vector (X_1, X_2). That is,

$$(X_1, X_2) = v(Y_1, Y_2),$$

or, equivalently,

$$X_1 = v_1(Y_1, Y_2), \quad X_2 = v_2(Y_1, Y_2),$$

Under these conditions, we have the conclusion that

Y_1, Y_2 are continuous random variables, (Y_1, Y_2) has support B, and (Y_1, Y_2) has joint pdf

$$g(y_1, y_2) = |J| f(v_1(y_1, y_2), v_2(y_1, y_2)), \quad (y_1, y_2) \in B,$$

where

$$J = J(x_1, x_2; y_1, y_2) = \det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix}.$$

8. *Definition.* The matrix

$$\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix}$$

is usually called the Jacobian matrix of x_1, x_2 with respect to y_1, y_2. And, the expression

$$J = J(x_1, x_2; y_1, y_2) = \det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{pmatrix}$$

is usually called the Jacobian determinant of x_1, x_2 with respect to y_1, y_2.

9. *Formula.* If

$$r = r(x, y), \quad s = s(x, y),$$

then

$$x = x(r, s), \quad y = y(r, s),$$

and

$$J(x, y; r, s) \cdot J(r, s; x, y) = 1.$$

10. *Suggested Reading:* The reader is encouraged to learn more about Jocobian matrix and Jocobian determinant from any standard textbook on Calculus III.

11. *Example.*

Suppose that $X_1 \sim$Exponential(1), $X_2 \sim$Exponential(1), and X_1 and X_2 are independent.

Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Find the joint pdf of Y_1 and Y_2.

* Let us state the answer first — Y_1 and Y_2 have the joint pdf

$$g(y_1, y_2) = \frac{1}{2}e^{-y_1}, \quad y_1 > 0, -y_1 < y_2 < y_1.$$

* And here is the proof. By definition, the pdf of X_1 is

$$f_1(x_1) = e^{-x_1}, \quad x_1 > 0;$$

the pdf of X_2 is

$$f_2(x_2) = e^{-x_2}, \quad x_2 > 0.$$

Since X_1 and X_2 are independent, their joint pdf is

$$f(x_1, x_2) = f_1(x_1)f_2(x_2) = e^{-x_1}e^{-x_2} = e^{-x_1-x_2}, \qquad x_1 > 0, x_2 > 0.$$

So, the random vector (X_1, X_2) has support

$$A = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}.$$

Geometrically, A is the first quadrant of x_1, x_2-plane. The function u that transforms (X_1, X_2) to (Y_1, Y_2) is

$$u(x_1, x_2) = (x_1 + x_2, x_1 - x_2).$$

This function u transforms the region A to the region

$$B = \{(y_1, y_2) : y_1 > 0, -y_1 < y_2 < y_1\}.$$

For example, the lower edge of A is the line $x_2 = 0$, since

$$u(x_1, 0) = (x_1, x_1)\,,$$

the lower edge of A is transformed to the line $y_2 = y_1$, which is the upper edge of B. Both A and B are plotted below.

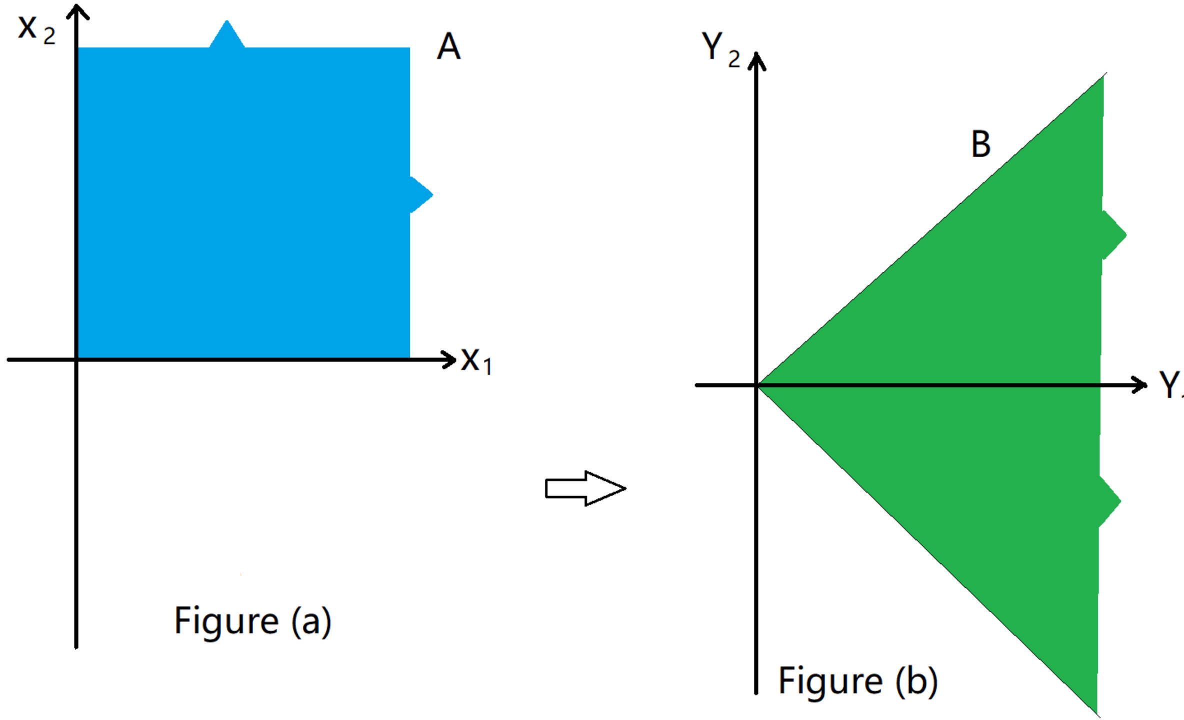


Figure (a)

Figure (b)

If we solve the equations

$$y_1 = x_1 + x_2, \quad y_2 = x_1 - x_2$$

for x_1 and x_2, we get

$$x_1 = \frac{y_1}{2} + \frac{y_2}{2}, \quad x_2 = \frac{y_1}{2} - \frac{y_2}{2}.$$

It follows that

$$J(x_1, x_2; y_1, y_2) = \det \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = -\frac{1}{2}.$$

Now, we put all this information into the formula

$$g(y_1, y_2) = |J(x_1, x_2; y_1, y_2)| f(v_1(y_1, y_2), v_2(y_1, y_2)),$$

we get that the joint pdf of Y_1 and Y_2 is

$$\begin{aligned} g(y_1, y_2) &= |J(x_1, x_2; y_1, y_2)| f(v_1(y_1, y_2), v_2(y_1, y_2)) \\ &= |(-1/2)| f(y_1/2 + y_2/2, y_1/2 - y_2/2) \\ &= \tfrac{1}{2} e^{-(y_1/2+y_2/2)-(y_1/2-y_2/2)} \\ &= \tfrac{1}{2} e^{-y_1}, \quad (y_1, y_2) \in B. \end{aligned}$$

The proof is now complete.

* Before we move on, let us find the marginal distribution of Y_1. To find this marginal distribution of Y_1, we simply integrate $g(y_1, y_2)$ with respect to y_2. Note that the support of (Y_1, Y_2) is the region

$$B : y_1 > 0, -y_1 < y_2 < y_1.$$

Outside of this region, the joint density of (Y_1, Y_2) is zero!

So, the marginal pdf of Y_1 is, for $y_1 > 0$,

$$\begin{aligned} g_1(y_1) &= \int_{-y_1}^{y_1} g(y_1, y_2) dy_2 \\ &= \int_{-y_1}^{y_1} \tfrac{1}{2} e^{-y_1} dy_2 \\ &= y_1 e^{-y_1}, \end{aligned}$$

which is precisely the Gamma(2,1) distribution.

* In a very similar way, we can show that the marginal pdf of Y_2 is

$$g_2(y_2) = \tfrac{1}{2} \exp(-|y_2|), \quad -\infty < y_2 < \infty.$$

12. We now state these results formally as two theorems:

13. *Theorem.* Suppose that $X_1 \sim Exponential(1)$, $X_2 \sim Exponential(1)$, and X_1 and X_2 are independent. If $Y_2 = X_1 - X_2$, then Y_2 has pdf

$$g(t) = \frac{1}{2}\exp(-|t|), \quad -\infty < t < \infty.$$

The distribution is called a double exponential distribution.

14. *Theorem.* Suppose that $X_1 \sim Exponential(1)$, $X_2 \sim Exponential(1)$, and X_1 and X_2 are independent. If $Y_1 = X_1 + X_2$, then Y_1 has the $Gamma(2,1)$ distribution.

15. Since $Exponential(1)$ is the $Gamma(1, 1)$ distribution, we can restate the last results as:

16. *Theorem.* Suppose that $X_1 \sim Gamma(1, 1)$, $X_2 \sim Gamma(1, 1)$, and X_1 and X_2 are independent. If $Y = X_1 + X_2$, then Y has the $Gamma(2, 1)$ distribution.

17. In fact, we have the following result which is more general:

18. *Theorem.* Let $X_1 \sim Gamma(\alpha, \theta)$, $X_2 \sim Gamma(\beta, \theta)$. If X_1 and X_2 are independent, and

$$Y = X_1 + X_2$$

has the $Gamma(\alpha + \beta, \theta)$ distribution.

19. Note that χ^2 distributions are a special case of the Gamma family, we have the following special case of the last theorem:

20. *Theorem.* Suppose that $X_1 \sim \chi^2_m$, $X_2 \sim \chi^2_n$. If X_1 and X_2 are independent, then

$$X_1 + X_2 \sim \chi^2_{m+n}.$$

21. *Definition.* The Beta function is defined as

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x > 0, y > 0.$$

22. *Example.* For example,

$$B(3,5) = \frac{\Gamma(3)\Gamma(5)}{\Gamma(8)} = \frac{2! \cdot 4!}{7!}.$$

23. *Definition.* Let $U \sim \chi^2(r_1)$, $V \sim \chi^2(r_2)$. If U and V are independent, and

$$W = \frac{U/r_1}{V/r_2},$$

then we say W has the F distribution with r_1 and r_2 degrees of freedom.

24. *Formula.* If $X \sim F(r_1, r_2)$, then its pdf is given by

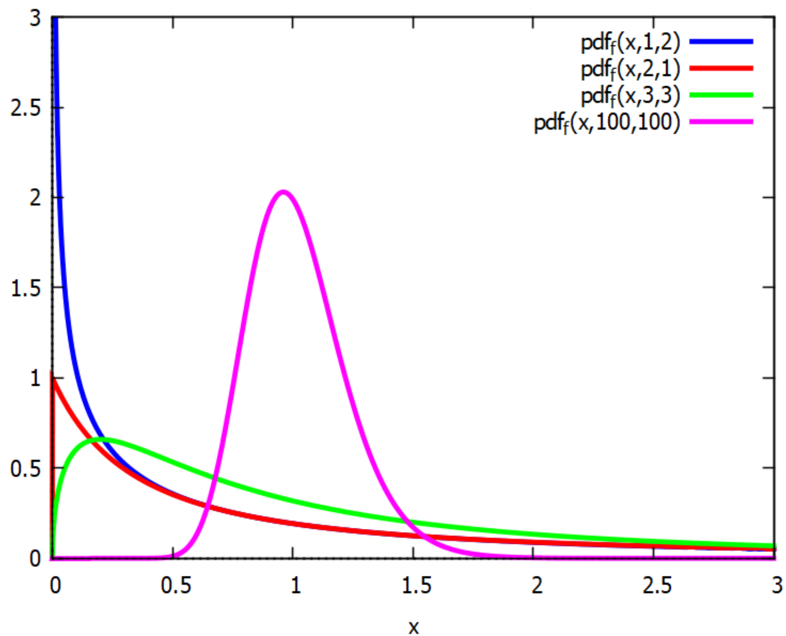
$$f(x) = \frac{(r_1/r_2)^{r_1/2}}{B(r_1/2, r_2/2)} x^{r_1/2-1}(1 + r_1 x/r_2)^{-(r_1+r_2)/2}, \quad x > 0.$$

Here, again, $B(\cdot, \cdot)$ is the Beta function.

25. *Formula.* If $X \sim F(r_1, r_2)$ and $r_2 > 2$, then

$$E(X) = \frac{r_2}{r_2 - 2}.$$

26. Four members in the F-family are plotted below:



In particular, the pink curve in the plot is the pdf of the $F_{100,100}$.

27. *MAXIMA Code for generating the plot:*

```
(%i1) load(distrib);
(%i7) plot2d([pdf_f(x,1,2),pdf_f(x,2,1),pdf_f(x,3,3),pdf_f(x,100,100)],
      [x,0,3],[y,0,3],
      [style, [lines,3],[lines,3],[lines,3],[lines,3]] );
```

28. *Theorem.* Let $X_1 \sim Gamma(\alpha, \theta)$, $X_2 \sim Gamma(\beta, \theta)$. If X_1 and X_2 are independent, and

$$Y = \frac{X_1}{X_1 + X_2},$$

then Y has pdf

$$g(y) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < y < 1. \tag{1}$$

Here, $B(\cdot, \cdot)$ is the Beta function. This distribution in (1) is called the $Beta(\alpha, \beta)$ distribution.

29. The next example is left to the reader as an exercise:

30. *Example.* Suppose that X, Y have joint pdf

$$f(x,y) = \frac{4}{\pi}, \quad x^2 + y^2 < 1,\ x > 0,\ y > 0.$$

Let $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1,\ x > 0, y > 0\}$.

Let $B = \{(r,t) \in \mathbb{R}^2 : 0 < r < 1,\ 0 < t < \pi/2\}$.

Let $v(r,t) = (r\cos t, r\sin t)$. That is, the function v masp the vector (r,t) to the vector $(r\cos t, r\sin t)$. Note that the vector function $u(r,t)$ has two components

$$v_1(r,t) = r\cos t, \quad v_2(r,t) = r\sin t.$$

Let $u(x,y) = (\sqrt{x^2+y^2}, \arctan(y/x))$. That is, the function u maps the vector (x,y) to the vector $(\sqrt{x^2+y^2}, \arctan(y/x))$.

The vector function $v(r,t)$ has two components

$$u_1(x,y)=\sqrt{x^2+y^2},\quad u_2(x,y)=\arctan(y/x).$$

Note that the function $u:A\to B$ is a bijection,

and $v:B\to A$ is also a bijection.

Let $(R,T)=u(X,Y)$, that is, let

$$R=\sqrt{X^2+Y^2},\quad T=\arctan(Y/X).$$

Then R,T are also random variables. Find the joint pdf of R and T.

* **Answer.** The random vector (R,T) have joint pdf

$$g(r,t)=\frac{4r}{\pi},\quad 0<r<1,\ 0\le t\le \pi/2.$$