Theorem: Sum of the First n Integers

For all intgers $n \geq 1$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Proof (by mathematical induction)

Let the property $P(n)$ be the equation

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad \longleftarrow P(n)$$

Show that $P(1)$ is true:

To establish $P(1)$, we must show that

$$1 = \frac{1(1+1)}{2} \qquad \longleftarrow P(1)$$

But the left-hand side of the equation is 1 and the right-hand side is

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

also. Hence $P(1)$ is true.

Show that for all integers $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is also true:

[Suppose that $P(k)$ is true for a particular but arbitrarily chosen integer $k \geq 1$. That is:] Suppose that k is any integer with $k \geq 1$ such that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} \qquad \longleftarrow P(k) \text{ inductive hypothesis}$$

[We must show that $P(k+1)$ is true, That is:] We must show that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)[(k+1)+1]}{2}$$

or, equivalently, that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)[(k+2)]}{2}$$

[We will show that the left-hand side and the right-hand side of $P(k+1)$ are equal to the same quantity and thus are equal to each other.]

The left-hand side of $P(k+1)$ is

$$\begin{aligned}
1 + 2 + 3 + \cdots + (k+1) &= 1 + 2 + 3 + \cdots + k + (k+1) \qquad \text{making the next-to-last term explict} \\
&= \frac{k + (k+1)}{2} + (k+1) \\
&= \frac{k + (k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k^2 + k}{2} + \frac{2(k+1)}{2} \\
&= \frac{k^2 + 3k + 2}{2}
\end{aligned}$$

And the right-hand side of $P(k+1)$ is

$$\frac{(k+1)(k+2)}{2} = \frac{k^2 + 3k + 2}{2}$$