Hall's Marriage Theorem

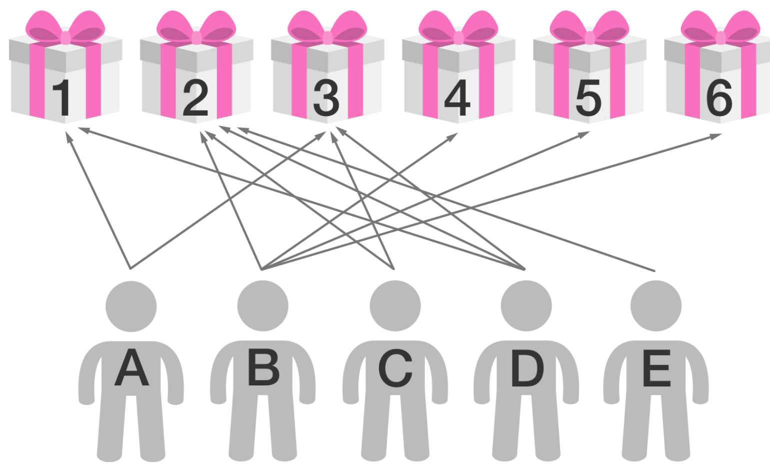
Hall's marriage theorem is a result in combinatorics that specifies when distinct elements can be chosen from a collectio overlapping finite sets. It is equivalent to several beautiful theorems in combinatorics, including Dilworth's theorem. The comes from an application to matchmaking: given a list of potential matches among an equal number of brides and gro the theorem gives a necessary and sufficient condition on the list for everyone to be married to an agreeable match.

EXAMPLE

Suppose I have 6 gifts (labeled $1, 2, 3, 4, 5, 6$) to give at Christmas, to 5 friends (Alice, Bob, Charles, Dot, Edward). Can distribute one gift to each person so that everyone gets something they want?

Certainly this depends on the preferences of my friends. If none of them like any of the gifts, then I am out of luck. But if they all like some of the gifts, I may still not be able to give them out satisfactorily. For instance, if none of them like g or 6, then I will have only 4 gifts to give to my 5 friends. Or suppose that

Alice wants: 1,3
Bob wants: 2,4,5,6
Charles wants: 2,3
Dot wants: 1,2,3
Edward wants: 2



There is still no way to distribute the gifts to make everyone happy. In fact, notice that I have four friends (Alice, Charle Edward) who only want one of the first three gifts, which makes it clear that the problem is impossible.

It turns out, however, that this is the *only* way for the problem to be impossible. As long as there isn't a subset of the fri that collectively likes fewer gifts than there are friends in the subset, there will always be a way to give everyone somet they want. This is the crux of Hall's marriage theorem.

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DEFINITION

Let S be a family of finite sets. Here S is allowed to be infinite, and elements in S can be repeated. A **transversal** T for
set T and a bijection $f: T \to S$ such that $t \in f(t)$ for all $t \in T$.

So a transversal consists of a choice of an element t in each of the sets in S; the point is that the elements chosen must l
unique even if the sets overlap. When S is finite, a transversal for S is often given as an ordered tuple of elements of the
S.

EXAMPLE

Let $S = \{\{1,2,3\},\{2,3,4\},\{1,3\}\}$. Then $(1,4,3)$ is a transversal of S: pick 1 from the first set, 4 from the second s
from the third set. Note that if instead we try to pick 1 from the first set and 3 from the second set, there is no choice o
element from the third set that leads to a transversal.

THEOREM

(Hall's Marriage Theorem) Let S be a finite family of finite sets. Suppose that for every subfamily R of sets in S, the nu
of subsets in R is less than or equal to the total number of elements in those subsets:

$$|R| \le \left| \bigcup_{X \in R} X \right|.$$

Then S has at least one transversal.

Notes:

(1) The condition given in the theorem is clearly necessary for the existence of a transversal, because any transversal mus
include $|R|$ distinct elements from the sets in R, so those sets must together contain at least $|R|$ elements.

(2) The condition given in the theorem is generally called the "marriage condition." So Hall's Marriage Theorem says tha
a transversal if and only if it satisfies the marriage condition.

Application to marriage

Suppose there are n women and n men, all of whom want to get married to someone of the opposite sex. Suppose furt
the women each have a list of the men they would be happy to marry, and that every man would be happy to marry any
who is happy to marry him, and that each person can only have one spouse.

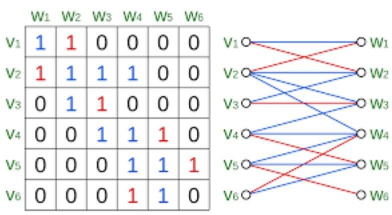
In this case, Hall's marriage theorem says that the men and women can all be paired off in marriage so that everyone is l
and only if the marriage condition holds: if in any group of women, the total number of men who are acceptable to at lea
of the women in the group is greater than or equal to the size of the group.

Bipartite graphs

Hall's marriage theorem can be restated in a graph theory context.

A **bipartite graph** is a graph where the vertices can be divided into two subsets V_1 and V_2 such that all the edges in the connect vertices from V_1 to vertices in V_2. (So none of the vertices in V_1 or in V_2 are connected to each other.)

A **matching** on a graph is a choice of edges with no common vertices. It *covers* a set V of vertices if each vertex in V is a endpoint of one of the edges in the matching.



A matching corresponds to a choice of 1s in the adjacency matrix, with at most one 1 in each row and in each column.

Suppose a bipartite graph has parts V_1 and V_2. Hall's marriage theorem says that there is a matching that covers V_1 if a if, for every subset W of V_1, the number of edges in the graph with endpoints in W is $\geq |W|$.

Other applications

EXAMPLE

Shuffle a deck of cards and deal 13 piles of four cards each. Show that there is always a way to choose a card from eacl in such a way that the 13 cards chosen contain one card of each rank (one ace, one king, etc.)

Consider the family of sets S_i consisting of the ranks of the cards in pile i. A subfamily consists of n subsets, which cor the ranks of $4n$ cards. Since there are only four cards of each rank, there must be at least n distinct ranks. So the marri condition is satisfied, so there is a transversal by Hall's marriage theorem. This transversal is exactly what the problem for. $\square$

EXAMPLE

(Putnam 2012 B3) A round-robin tournament of $2n$ teams lasted for $2n-1$ days, as follows. On each day, every team one game against another team, with one team winning and one team losing in each of the n games. Over the course tournament, each team played every other team exactly once. Can one necessarily choose one winning team from eac without choosing any team more than once?

The answer is yes. On the ith day, let S_i be the set of winning teams. Each set S_i has n elements. To show that a transv exists, we must show that the marriage condition holds.

Suppose it does not--so there are k sets S_i whose union contains less than k elements; that is, suppose there are k da

Other examples are given in this page: Applications of Hall's marriage theorem.

Proof of the theorem

Here is a proof using Dilworth's theorem. Suppose $|S| = n$. Say the sets in S are $X_1, X_2, \ldots, X_n$. Suppose the union c
sets X_i consists of the elements $x_1, x_2, \ldots, x_k$. Now define a partial order on

$$\{x_1, x_2, \ldots, x_k, X_1, X_2, \ldots, X_n\}$$

by $x_i \le X_j$ if and only if $x_i \in X_j$. (And: no two x_i are comparable and no two X_j are comparable.)

The marriage condition implies that the largest antichain in this set is $\{x_1, x_2, \ldots, x_k\}$. This is because if there were a la
antichain consisting of j X's and $> k - j$ x's, those j X's would contain only the x's not in the antichain, of which ther
$< j$, which would violate the marriage condition.

Then by Dilworth's theorem, there is a cover by k chains. Each x_i is in exactly one chain in the cover, and the chains in th
are either $\{x_i\}$ or $\{x_i, X_j\}$. Since every X_j must appear in the chain cover, the elements x_i that appear alongside the
a transversal. $\square$

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