Theorem 8.20

For every integer $k \geq 1$, the complete graph K_{2k} can be factored into $k-1$ Hamiltonian cycles and a 1-factor.

Proof:

Since the result is true for $k=1$ and $k=2$, we assume that $k \geq 3$. Let $G = K_{2k}$, where $V(G) = \{v_0, v_1, \ldots, v_{2k-1}\}$. Let $v_1, v_2, \ldots, v_{2k-1}$ be the vertices of a regular $(2k-1)$-gon and place v_0 in the center of the $(2k-1)$-gon. Join each two vertices of G by a straight line segment. Let G_1 be the spanning subgraph of G whose edges consist of (1) v_0v_1 and v_0v_{k+1}, (2) all edges parallel to v_0v_1 and (3) all edges parallel to v_0v_{k+1}. Then $G_1 = C_{2k}$. For $1 \leq i \leq k-1$, let G_i be the spanning subgraph of G whose edges consist of (1) v_0v_i and v_0v_{k+i}, (2) all edges parallel to v_0v_i and (3) all edges parallel to v_0v_{k+i}. Then $G_i = C_{2k}$ for each $i(1 \leq i \leq k-1)$ and every edge of G belongs to some subgraph $G_i(1 \leq i \leq k-1)$ except for the edges $v_1v_{2k-1}, v_2v_{2k-2}, \ldots, v_{k-1}v_{k+1}$ and v_ov_k, which are the edges of a 1-factor G_k of G. Thus G can be factored into $G_1, G_2, \ldots, G_k$.