Example 1: Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0$$

but $\psi(x) = x^3$ is not.

Solution

The function $\phi(x) = x^2 - x^{-1}, \phi'(x) = 2x + x^{-2}, and \phi''(x) = 2 - 2x^{-3}$ are defined for all $x \neq 0$. Substitution of $\phi(x)$ for y in the equation gives

$$(2 - 2x^{-3}) - \frac{2}{x^2}(x^2 - x^{-1}) = (2 - 2x^{-3}) - (2 - 2x^{-3}) = 0$$

Since this is valid for any $x \neq 0$, the function $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the equation on $(-\infty, 0)$ and also on $(0, \infty)$.

For $\psi(x) = x^3$ we have $\phi'(x) = 3x^2, \psi''(x) = 6x$ and substitution into the equation gives

$$6x - \frac{2}{x^2}x^3 = 4x = 0$$

which is valid only at the point $x = 0$ and not on an interval. Hence $\psi(x)$ is not a solution.