Section 1.2 Solutions and Initial Value Problems

Definition

A function $y = f(x)$ which satisfies an n'th order o.d.e. identically is called an <u>explicit solution</u>.

Example 1 $y = 2e^x + e^{2x}$ is an explicit solution of $y'' - 3y' + 2y = 0$.

Solution

$$y = 2e^x + e^{2x}$$
$$y' = 2e^x + 2e^{2x}$$
$$y'' = 2e^x + 4e^{2x}$$

$$\begin{aligned} LHS &= y'' - 3y' + 2y \\ &= (2e^x + 4e^{2x}) - 3(2e^x + 2e^{2x}) + 2(2e^x + e^{2x}) \\ &= e^x(2 - 6 + 4) + e^{2x}(4 - 6 + 2) \\ &= 0 \\ &= RHS \end{aligned}$$

$\therefore y = 2e^x + e^{2x}$ is an explicit solution.

Example 2 $A = 20 + ce^{-t/5}$ is an explicit solution of $A' + \frac{1}{5}A = 4$, $c =$ constant.

Solution $A = A(t) = 20 + ce^{-\frac{1}{5}t}$

$$A' = \frac{dA}{dt} = -\frac{1}{5}ce^{-\frac{1}{5}t}$$

$$LHS = A' + \frac{1}{5}A = -\frac{1}{5}ce^{-\frac{1}{5}t} + 4 + \frac{1}{5}ce^{-\frac{1}{5}t}$$

$= 4 = RHS$

$\therefore A = 20 + Ce^{-t/5}$ is an explicit solution.

<u>Example 3</u> Find an explicit solution of $\frac{dy}{dx} = 2x$.

<u>Solution</u>

$dy = 2x\,dx$

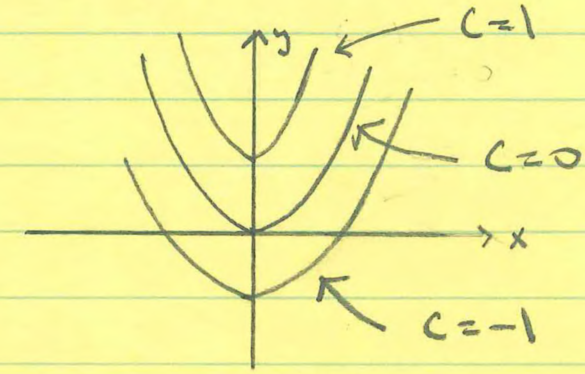
$\therefore \int dy = \int 2x\,dx$

$y + c_1 = x^2 + c_2$

$y = x^2 + (c_2 - c_1)$

$y = x^2 + C$, $C = c_2 - c_1$

There is a <u>one-parameter</u> family of solutions



Graphs of solutions: <u>integral curves</u>

<u>Definition</u>

A relation $G(x,y) = 0$ is said to be an <u>implicit solution</u> to an n'th order ode if it defines one or more explicit solutions.

<u>Example</u> Show that $x^2 + y^2 = 4$ is an implicit solution to $\frac{dy}{dx} = -\frac{x}{y}$.

<u>Solution 1</u> $x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y_1 = +\sqrt{4-x^2}$, $y_2 = -\sqrt{4-x^2}$

$$\frac{dy_1}{dx} = \frac{1}{2}(4-x^2)^{-1/2}(-2x) = (-x)\frac{1}{(4-x^2)^{1/2}} = -\frac{x}{y_1}$$

$\therefore y_1 = \sqrt{4-x^2}$ is an explicit solution of
$\frac{dy}{dx} = -\frac{x}{y}$.

$\therefore x^2+y^2 = 4$ is an implicit solution of
$\frac{dy}{dx} = -\frac{x}{y}$.

[Check that $y_2 = -\sqrt{4-x^2}$ is also an explicit solution of $\frac{dy}{dx} = -\frac{x}{y}$.]

<u>Solution 2</u> — Use implicit differentiation

$$x^2 + y^2 = 4$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}(y(x))^2 = \frac{d}{dx}4$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

$\uparrow$

Given ode — Hence, $x^2+y^2=4$ is a solution

In general a solution (explicit or implicit) to an n'th order o.d.e. involves n constants of integration

& thus takes the form

$$y = G(x, c_1, c_2, c_3, \ldots, c_n) \qquad (1)$$

That is, the solution is not unique. (1) is called an <u>n-parameter family of solutions</u> or the <u>general solution</u>.

Solutions obtained from (1) by giving the c's particular values are called <u>particular solutions</u> or <u>particular integrals</u>.

<u>Definition</u> A problem of the type:

<u>Solve</u> $y^{(n)} = G(x, y, y', y'', \ldots, y^{(n-1)})$

<u>Subject to</u>

$$\left.\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ y''(x_0) &= y_2 \\ &\vdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}\right\} \text{Initial Conditions}$$

is called an <u>initial value problem</u> (IVP).

In contrast a problem such as:

<u>Solve</u> $\dfrac{d^2y}{dx^2} = 0$

<u>Subject to</u> $y(0) = 0 \;,\; y(1) = 2$

in which the conditions are given at different values of x is called a boundary value problem (b.v.p.).

<u>Example 1</u> Solve the IVP : $y''=1$, $y(0)=0$, $y'(0)=1$

<u>Solution</u> $y''=1 \Rightarrow y' = x + A$

$$y'(0)=1 \Rightarrow 1 = 0 + A \Rightarrow A = 1$$

$$\therefore y' = x + 1$$

$$y = \frac{1}{2}x^2 + x + B$$

$$y(0)=0 \Rightarrow 0 = 0 + 0 + B \Rightarrow B = 0$$

$$\therefore y = \frac{1}{2}x^2 + x$$

<u>Example 2</u> Solve the BVP : $y''=0$, $y(0)=0$, $y(1)=2$

<u>Solution</u> $y''=0 \Rightarrow y' = A$

$$\Rightarrow y = Ax + B$$

$$y(0)=0 \Rightarrow 0 = A(0) + B \Rightarrow B = 0$$

$$y(1) = 2 \Rightarrow 2 = A(1) + B \Rightarrow A = 2$$

$$\therefore y = 2x$$

<u>HW</u> pp. 13-14 : 1, 3, 5, 7, 9, 11, 20, 21, 22