Section 7.2 Definition of the Laplace Transform

Improper Integrals

$$\int_0^{\infty} f(t)\,dt = \lim_{N\to\infty} \int_0^N f(t)\,dt$$

If the limit exists then the integral is said to converge. If the limit does not exist then the integral is said to diverge.

Examples

1. $$\int_0^{\infty} t^2 e^{-t^3}\,dt = \lim_{N\to\infty} \int_0^N t^2 e^{-t^3}\,dt$$

$$= \lim_{N\to\infty} \left[-\tfrac{1}{3} e^{-t^3} \right]_{t=0}^{t=N}$$

$$= \lim_{N\to\infty} \left\{ -\tfrac{1}{3} e^{-N^3} + \tfrac{1}{3} \right\}$$

$$= \tfrac{1}{3}$$

Integral converges.

2. $$\int_0^{\infty} e^{2t}\,dt = \lim_{N\to\infty} \int_0^N e^{2t}\,dt$$

$$= \lim_{N\to\infty} \left[\tfrac{1}{2} e^{2t} \right]_{t=0}^{t=N}$$

$$= \lim_{N\to\infty} \left\{ \tfrac{1}{2} e^{2N} - \tfrac{1}{2} \right\}$$

$$= +\infty.$$

Integral diverges.

<u>Exercise</u> Determine whether the given integral converges or diverges. If it converges, find it's value.

1. $\int_0^\infty e^{-t}\,dt$ — 1
2. $\int_0^\infty t e^{-t^2}\,dt$ — $1/2$
3. $\int_0^\infty e^{3t}\,dt$ — Diverges
4. $\int_0^\infty t e^{-t}\,dt$ — 1
5. $\int_0^\infty \frac{1}{1+t^2}\,dt$ — $\pi/2$

<u>Definition</u> Let $f(t)$ be a real valued function defined for $t>0$, then the <u>Laplace Transform</u> $\bar{F}(s)$ of $f(t)$ is defined by

$$\bar{F}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

<u>provided the integral exists</u>.

<u>Notation</u> $\bar{F}(s) = F(s) = \mathcal{L}[\,f(t)\,;\, t \to s\,]$

<u>Example 1</u> Find the Laplace Transform of

$$f(t) = \begin{cases} t, & 0 < t < 2 \\ 3, & t > 2 \end{cases}$$

<u>Solution</u> $F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$

$$= \underbrace{\int_0^2 t\, e^{-st}\, dt}_{I_1} + \underbrace{\int_2^\infty 3\, e^{-st}\, dt}_{I_2}$$

<u>For I_1:</u> Let $u = t \quad dv = e^{-st}\, dt$

$du = dt \quad v = -\frac{1}{s} e^{-st}$

$$I_1 = \left[-\frac{t}{s} e^{-st} \right]_{t=0}^{t=2} - \int_0^2 -\frac{1}{s} e^{-st}\, dt$$

$$= -\frac{2}{s} e^{-2s} - 0 + \left[-\frac{1}{s^2} e^{-st} \right]_{t=0}^{t=2}$$

$$= -\frac{2}{s} e^{-2s} + \left\{ -\frac{1}{s^2} e^{-2s} + \frac{1}{s^2} \right\}$$

$$= \frac{1}{s^2} - \frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s}$$

<u>For I_2:</u> $I_2 = \lim_{N \to \infty} \int_2^N 3\, e^{-st}\, dt$

$$= \lim_{N \to \infty} \left[\frac{3}{-s} e^{-st} \right]_{t=2}^{t=N}$$

$$= \lim_{N\to\infty} \left\{ -\frac{3}{s} e^{-sN} + \frac{3}{s} e^{-2s} \right\}$$

$$= -\frac{3}{s}(0) + \frac{3}{s} e^{-2s} \qquad (\text{provided } s>0)$$

$$= \frac{3}{s} e^{-2s}$$

$$\therefore \bar{f}(s) = I_1 + I_2 = \frac{1}{s^2} + \frac{1}{s} e^{-2s} - \frac{1}{s^2} e^{-2s}, \quad s>0$$

<u>Example 2</u> (a) Find the Laplace Transform of $f(t) = 1$.

<u>Solution</u> $\mathcal{L}[1;\, t\to s] = \int_0^\infty 1.e^{-st}\,dt$

$$= \lim_{N\to\infty} \int_0^N e^{-st}\,dt = \lim_{N\to\infty} \left[-\frac{1}{s} e^{-st} \right]_{t=0}^{t=N}$$

$$= \lim_{N\to\infty} \left\{ -\frac{1}{s} e^{-sN} + \frac{1}{s} \right\} = -\frac{1}{s}(0) + \frac{1}{s}, \quad s>0$$

$$= \frac{1}{s}, \quad s>0$$

<u>Example 2</u> (b) Find the Laplace Transform of $f(t) = e^{at}$.

<u>Solution</u> $\mathcal{L}[e^{at};\, t\to s] = \int_0^\infty e^{at} e^{-st}\,dt$

$$= \lim_{N\to\infty} \int_0^N e^{t(a-s)}\,dt = \lim_{N\to\infty} \left[\frac{1}{a-s} e^{t(a-s)} \right]_{t=0}^{t=N}$$

$$= \frac{1}{a-s} \lim_{N\to\infty} \left\{ e^{N(a-s)} - 1 \right\}$$

$$= \frac{1}{a-s}\left\{ 0 - 1 \right\}, \text{ provided that } a-s<0$$

$$= \frac{1}{s-a}, \quad s>a$$

<u>Example 2</u> (c) Find the Laplace Transform of $f(t) = \sin(bt)$.

<u>Solution</u> $\mathcal{L}[\sin bt\,;\, t\to s] = \int_0^\infty \sin bt \,.\, e^{-st}\,dt$

$$= \lim_{N\to\infty} \int_0^N \sin bt \,.\, e^{-st}\,dt \qquad \leftarrow \text{use parts.}$$

$$u = \sin bt \qquad dv = e^{-st}\,dt$$
$$du = b\cos bt\,dt \qquad v = -\tfrac{1}{s}e^{-st}$$

$$\therefore F(s) = \lim_{N\to\infty}\left\{ \left[-\tfrac{1}{s}e^{-st}\sin bt\right]_{t=0}^{t=N} - \int_0^N -\tfrac{b}{s}e^{-st}\cos bt\,dt \right\}$$

$$= \lim_{N\to\infty}\left\{ -\tfrac{1}{s}e^{-sN}\sin bN - 0 + \tfrac{b}{s}\int_0^N e^{-st}\cos bt\,dt \right\}$$

$$= \underset{(s>0)}{0} + \frac{b}{s}\lim_{N\to\infty}\int_0^N e^{-st}\cos bt\,dt$$

parts

$$u = \cos bt \qquad dv = e^{-st}\,dt$$
$$du = -b\sin bt\,dt \qquad v = -\tfrac{1}{s}e^{-st}$$

$$\therefore F(s) = \frac{b}{s}\lim_{N\to\infty}\left\{ \left[-\tfrac{1}{s}e^{-st}\cos bt\right]_{t=0}^{t=N} - \int_0^N \tfrac{b}{s}e^{-st}\sin bt\,dt \right\}$$

$$= \frac{b}{s}\lim_{N\to\infty}\left\{ -\tfrac{1}{s}e^{-Ns}\cos bN + \tfrac{1}{s} - \tfrac{b}{s}\int_0^N e^{-st}\sin bt\,dt \right\}$$

$$= \frac{b}{s}\left\{ 0 + \frac{1}{s} - \frac{b}{s}\int_0^\infty e^{-st} \sin bt \, dt \right\}$$

$$= \frac{b}{s^2} - \frac{b^2}{s^2} \bar{f}(s)$$

$$\therefore \left(1 + \frac{b^2}{s^2}\right) \bar{f}(s) = \frac{b}{s^2}$$

$$\therefore \bar{f}(s) = \frac{b/s^2}{1 + b^2/s^2} \cdot \frac{s^2}{s^2} = \frac{b}{s^2+b^2}, \quad s > 0$$

Example 2 (d) Show that $\mathcal{L}[\cos bt \,;\, t \to s]$

$$= \frac{s}{s^2+b^2}, \quad s > 0$$

Example 2 (e) Find the Laplace Transform of $f(t) = t^n$ where n is a positive integer.

Solution $\quad \frac{d}{ds} e^{-st} = (-t) e^{-st}$

$$\frac{d^2}{ds^2} e^{-st} = (-t)^2 e^{-st}$$

$$\frac{d^3}{ds^3} e^{-st} = (-t)^3 e^{-st}$$

In general, $\frac{d^n}{ds^n} e^{-st} = (-t)^n e^{-st} = (-1)^n t^n e^{-st}$

Hence, $\mathcal{L}[t^n \,;\, t \to s] = \int_0^\infty t^n e^{-st} \, dt$

$$= \int_0^\infty \frac{1}{(-1)^n} \frac{d^n}{ds^n} e^{-st}\, dt = \frac{1}{(-1)^n} \frac{d^n}{ds^n} \int_0^\infty e^{-st}\, dt$$

$$= \frac{1}{(-1)^n} \frac{d^n}{ds^n} \mathcal{L}[1; t \to s] = \frac{1}{(-1)^n} \frac{d^n}{ds^n} s^{-1}, \quad s > 0$$

$$\left.\begin{aligned} \frac{d}{ds} s^{-1} &= (-1) s^{-2} \\ \frac{d^2}{ds^2} s^{-1} &= (-1)(-2) s^{-3} \\ \frac{d^3}{ds^3} s^{-1} &= (-1)(-2)(-3) s^{-4} \end{aligned}\right\} \quad \text{Generalize}$$

$$\frac{d^n}{ds^n} s^{-1} = (-1)(-2)(-3)\cdots(-n)\, s^{-(n+1)} = (-1)^n n! \frac{1}{s^{n+1}}$$

Thus, $\mathcal{L}[t^n; t \to s] = \frac{1}{(-1)^n} \frac{d^n}{ds^n} s^{-1}$

$$= \frac{1}{(-1)^n} (-1)^n n! \frac{1}{s^{n+1}}$$

$$= \frac{n!}{s^{n+1}}, \quad s > 0$$

See Table 7-1 (Brief Table of Laplace Transforms) on page 359.

Example 3 Show that the Laplace Transform is a linear transformation. That is,

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t); t \to s] = c_1 \mathcal{L}[f_1(t); t \to s] + c_2 \mathcal{L}[f_2(t); t \to s]$$

<u>Solution</u> $\mathcal{L}[c_1 f_1(t) + c_2 f_2(t) \,;\, t \to s]$

$$= \int_0^\infty \left(c_1 f_1(t) + c_2 f_2(t)\right) e^{-ts}\, dt$$

$$= \int_0^\infty c_1 f_1(t) e^{-ts} + c_2 f_2(t) e^{-ts}\, dt$$

$$= c_1 \int_0^\infty f_1(t) e^{-ts}\, dt + c_2 \int_0^\infty f_2(t) e^{-ts}\, dt$$

$$= c_1 \mathcal{L}[f_1(t) \,;\, t \to s] + c_2 \mathcal{L}[f_2(t) \,;\, t \to s]$$

<u>Example</u> Use the results of examples 2 and 3 to compute the Laplace Transforms of the following:

(a) $f(t) = t^2 + 4 - e^{-3t}$

$$F(s) = \mathcal{L}[t^2 + 4 - e^{-3t} \,;\, t \to s]$$

$$= \mathcal{L}[t^2 \,;\, t \to s] + 4\mathcal{L}[1 \,;\, t \to s] - \mathcal{L}[e^{-3t} \,;\, t \to s]$$

$$= \frac{2!}{s^3} + 4\,\frac{1}{s} - \frac{1}{s-(-3)}$$

$$= \frac{2}{s^3} + \frac{4}{s} - \frac{1}{s+3}$$

(b) $f(t) = (1 + \sin 3t)^2$

$$F(s) = \mathcal{L}[(1+\sin 3t)^2 \,;\, t \to s]$$

$$= \mathcal{L}[\,1 + 2\sin 3t + \sin^2 3t\ ;\ t \to s\,]$$

$$= \mathcal{L}[\,1 + 2\sin 3t + \tfrac{1}{2}(1 - \cos 6t)\ ;\ t \to s\,]$$

$$= \mathcal{L}[\,\tfrac{3}{2} + 2\sin 3t - \tfrac{1}{2}\cos 6t\ ;\ t \to s\,]$$

$$= \frac{3}{2}\cdot\frac{1}{s} + 2\cdot\frac{3}{s^2+3^2} - \frac{1}{2}\,\frac{s}{s^2+6^2}$$

$$= \frac{3}{2s} + \frac{6}{s^2+9} - \frac{s}{2(s^2+36)}$$

<u>HW</u> Page 360

#'s 1, 3, 9–12, 13–19 odd