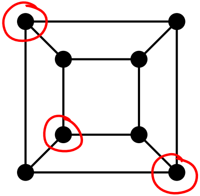
5.3: Connectivity

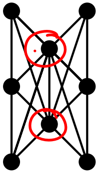
- We already discussed how graphs with cut-edges and cut-vertices are somehow “less connected” than other graphs.

- We can generalize this idea: Which of the graphs below is “more connected”?

Both need 3 edges deleted to disconnect



This needs 3 vertices deleted to disconnect it



This only needs 2

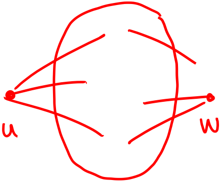
Vertex cuts

Definition

A *vertex cut* in a connected graph G is a set of vertices $U \subseteq V(G)$ such that $G - U$ is disconnected. A vertex cut in G containing the fewest vertices is called a *minimum vertex cut*.

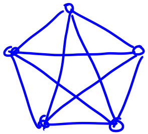
Question: Does every graph contain a vertex cut?

Yes!



Take 2 nonadjacent vertices and delete everything else

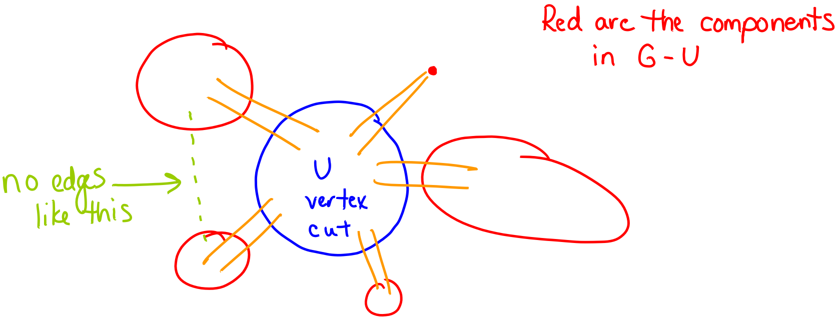
Well... almost.



In a complete graph, we can't disconnect the graph by deleting vertices.

An important idea when proving things about cuts

What is the general structure of a graph with a vertex cut U?



All edges are within a single red blob, or between red and blue, or within blue.

Important new graph parameter

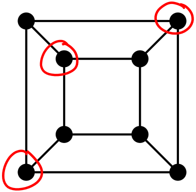
read: "kappa of G"

Definition

interesting cases

The *connectivity* of a connected, non-complete graph G, denoted $\kappa(G)$, is the cardinality of a minimum vertex cut of G. If $G = K_n$, then $\kappa(G) = n - 1$. If G is disconnected, then $\kappa(G) = 0$. Lame cases

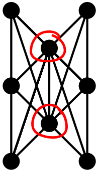
Find $\kappa(G)$ for the graphs below:



$\kappa(G) = 3$

There is a cut-set of size 3

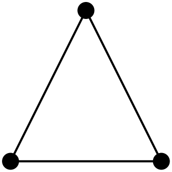
There isn't one of size 2



$\kappa(G) = 2$

There is a cutset of size 2

There isn't a cut-vertex.

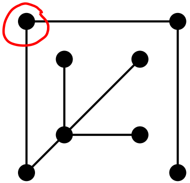


$\kappa(G) = 2$

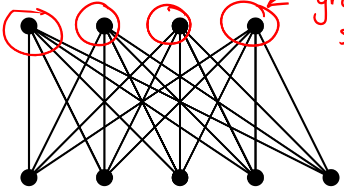
By definition

Find $\kappa(G)$ for the graphs below. What theorems are suggested by the examples?

Leaving any of these behind keeps the graph connected, so $\kappa(G) \geq 4$



$\kappa(G) = 1$



$\kappa(G) = 4$

Conjecture: $\kappa(T) \leq 1$ for any tree T.

(In fact: $\kappa(T) = 1$ unless $T = K_1$.)

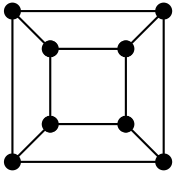
Conjecture: $\kappa(K_{n,m}) = \min\{n, m\}$.

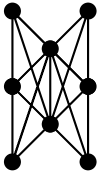
Another definition

Definition

A graph G is *k-connected* if $\kappa(G) \geq k$.

Restated: You have to delete at least k vertices to disconnect the graph.





3-connected
2-connected
1-connected
same as connected
NOT 4-connected

2-connected
1-connected
NOT 3-connected

Another definition

Definition

A graph G is *k-connected* if $\kappa(G) \geq k$.

Notes:

- If G is k-connected, then it is also t-connected for every nonnegative $t < k$.
- Another way to think of k-connected: You couldn't disconnect G by deleting *fewer* than k vertices, but you *might* disconnect G by deleting k vertices.
- Saying G is k-connected is LESS information than saying $\kappa(G) = k$. If $\kappa(G) = k$, then G DOES have a vertex cut of size k. If G is k-connected, it may or it may not.
- Connected graphs are all 1-connected. Graphs without cut-vertices are 2-connected.

Connectivity: Why do we care?

What are some applications of graphs for which we may care about the connectivity?

computer networks

flight networks

<u>any</u> networks...

roads

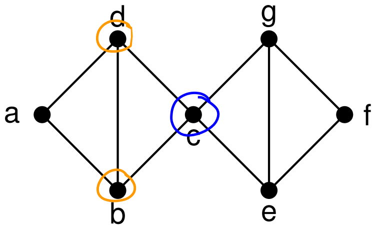
power grids (nodes = transformers)

MUM versus MAL

An important idea in graph theory: The difference between **minimum** versus **minimal**.

- **MiniMUM** means the very smallest, period.
- **MimiMAL** means it doesn't CONTAIN a smaller one.

Example:



minimum: $\{c\}$

minimal but not minimum:
$\{b, d\}$

Edge connectivity

We have the analogous idea for edges.

Definition

- A set of edges $X \subseteq E(G)$ in a connected graph G is an *edge-cut* in G if $G - X$ is disconnected.
- If X is an edge cut in G with the fewest number of edges, X is a *minimum edge-cut*.
- The *edge connectivity* of G, denoted $\lambda(G)$, is the size of a minimum edge cut in G.
- G is *k-edge-connected* if $\lambda(G) \geq k$.

No exception needed for complete graphs: $\lambda(K_n) = n - 1$. (Proven in text.)

A key relationship between graph parameters

Theorem

For every graph G,

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

Can equality hold?

Are there graphs where $\kappa(G) = \lambda(G) = \delta(G)$?

Can strict inequality hold?

Are there graphs where $\kappa(G) < \lambda(G) < \delta(G)$?

Sparse graphs can't be too connected

Theorem

If G is a graph of order n and size m, then

$$\kappa(G) \le \left\lfloor \frac{2m}{n} \right\rfloor .$$