3.3. The Gamma and Chi-Square Distributions

1. Consider the Poisson process with parameter $\lambda > 0$. Let X be the waiting time until the first change. We have shown that X has cdf

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

and X has pdf:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0. \tag{1}$$

2. Consider the Poisson process with parameter $\lambda > 0$. Let X_2 be the waiting time until the second change. We will find the cdf and pdf of X_2. Denote by $F_2(x)$ the cdf of X_2 and denote by $f_2(x)$ the pdf of X_2.

   If $x < 0$, then $F_2(x) = 0$. For each fixed $x > 0$, denote by Y the number of customers who arrive during the time interval $[0, x]$. Then

$$Y \sim \text{Poisson}(\lambda x).$$

   It follows that

$$P(Y = 0) = e^{-\lambda x}, \quad P(Y = 1) = e^{-\lambda x}\lambda x.$$

Note that the event $X_2 > x$ means that the second customer does not come during the time interval $[0, x]$, so

$$(X_2 > x) = (Y = 0) \cup (Y = 1).$$

By definition,

$$\begin{aligned} F_2(x) &= P(X \leq x) \\ &= 1 - P(X_2 > x) \\ &= 1 - P(Y = 0 \text{ or } Y = 1) \\ &= 1 - P(Y = 0) - P(Y = 1) \\ &= 1 - e^{-\lambda x} - e^{-\lambda x}\lambda x. \end{aligned}$$

In summary, the cdf of X_2 is

$$F_2(x) = \begin{cases} 1 - e^{-\lambda x} - e^{-\lambda x}\lambda x, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The pdf of X is

$$f_2(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

3. Consider the Poisson process with parameter $\lambda > 0$. Let X_α be the waiting time until the α-th change occurs. Then the pdf of X is

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1}}{(\alpha-1)!} e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

4. *Definition.* The gamma function is

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx, \quad t > 0.$$

5. *Formula.*

$$\Gamma(1) = 1.$$

$$\Gamma(2) = 1.$$

$$\Gamma(3) = 2.$$

$$\Gamma(n) = (n-1)!.$$

$$\Gamma(t) = (t-1)\Gamma(t-1), \quad t > 1. \tag{1}$$

6. *Definition.* If random variable X has pdf

$$f(x) = \frac{x^{\alpha-1}e^{-x/\theta}}{\Gamma(\alpha)\theta^{\alpha}}, \quad x > 0,$$

then we say X has the Gamma distribution with parameters (α, θ).

Here, only the nontrivial part of the distribution is given. We will adopt this convention throughout the rest of the book.

7. In the Poisson process with parameter $\lambda > 0$, the waiting time X until the α-th change has the Gamma $(\alpha, 1/\lambda)$ distribution.

8. *Formula.*

If X has the Gamma (α, θ) distribution, then

$$\mu = \alpha\theta,$$

$$\sigma^2 = \alpha\theta^2,$$

and X has mgf

$$M(t) = \frac{1}{(1-\theta t)^\alpha}.$$

9. *Definition.* Let r be a positive integer. Then the Gamma $(r/2, 2)$ distribution is also called the chi-square distribution with r degrees of freedom. The short notation of the chi-square distribution with r degrees of freedom is $\chi^2(r)$.

10. *Formula.* If the random variable X has chi-square distribution with r degrees of freedom, then the pdf of X is

$$f(x) = \frac{x^{r/2-1}e^{-x/2}}{\Gamma(r/2)2^{r/2}}, \quad x > 0.$$

11. *Formula.* If X has the chi-square distribution with r degrees of freedom, then

$$M(t) = (1 - 2t)^{-r/2},$$

$$\mu = r,$$

$$\sigma^2 = 2r.$$

12. *R code:*

```
library(ggplot2);
h <- ggplot(data.frame(x = c(0, 4)), aes(x = x)) ;
h<-h+stat_function(fun=dgamma, geom = "line",size=2,col="red",args=list(shape=2, rate=1));
h<-h+stat_function(fun=dgamma, geom = "line",size=2,col="blue",args=list(shape=1, rate=1));
h<-h+stat_function(fun=dgamma, geom = "line",size=2,col="green",args=list(shape=1.5, rate=1));
h<-h+stat_function(fun=dgamma, geom = "line",size=2,col="black",args=list(shape=3, rate=1));
h
```

```
library(ggplot2);
h <- ggplot(data.frame(x = c(0, 6)), aes(x = x)) ;
h<-h+stat_function(fun=dgamma, geom = "line",size=1,col="red",args=list(shape=2, scale=1));
h<-h+stat_function(fun=dgamma, geom = "line",size=2,col="blue",args=list(shape=2, scale=2));
h <- h+ lims(y = c(0, 1));
h
```