Example 1: Solve

$$(xy + y^2 + x^2)dx - x^2dy = 0$$

A check will show that $(xy + y^2 + x^2)dx - x^2dy = 0$ is not separable, exact or linear. If we express $(xy + y^2 + x^2)dx - x^2dy = 0$ in the derivative form

$$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$$

then we see that the right-hand side of $\frac{dy}{dx} = \frac{xy+y^2+x^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$ is a function of just $\left(\frac{y}{x}\right)$. Thus, $(xy + y^2 + x^2)dx - x^2dy = 0$ is homogeneous.

Now let $v = \frac{y}{x}$ and recall that $\frac{dy}{dx} = v + x(\frac{dv}{dx})$. With these substitutions, $\frac{dy}{dx} = \frac{xy+y^2+x^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$ becomes

$$v + x\frac{dv}{dx} = v + v^2 + 1$$

The above equation is separable, and, on separating the variables and integrating, we obtain

$$\int \frac{1}{v^2+1}dv = \int \frac{1}{x}dx$$
$$\arctan(x) = \ln|x| + C$$

Hence,

$$v = \tan(\ln|x| + C)$$

Finally, we substitute $\frac{y}{x}$ for v and solve for y to get

$$y = x\tan(\ln|x| + C)$$