Section 5.1: Cut-vertices

In Chapter 4, we talked about bridges – edges whose deletion increases the number of components.

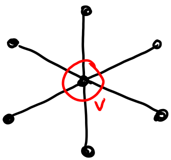
The analogous idea for vertices:

Definition

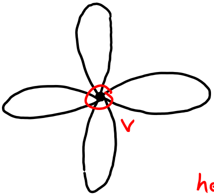
A vertex $v \in V(G)$ is a *cut-vertex* if $G - v$ has more components than G.

Question: If G is connected, is there a limit to the number of components $G - v$ can contain? No!

$K_{1,n}$



Delete v,
n components



More generally: v can have any graphs attached to it, here we would have 4 "blobs" if we delete v

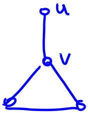
Relating brides and cut-vertices

True or False: If uv is a bridge in G, then u and v are both cut-vertices of G.

Super false:

neither u nor v are cut-vertices

Less false:



v is a cut-vertex, but u is not

In general: If $d(u)=1$, then deleting it won't increase the number of components, even though deleting its incident edge would.

Theorem

Let uv be a bridge in a connected graph G. Then u is a cut-vertex of G if and only if $\deg(u) \geq 2$.

(Proof is pretty much the previous slide)

Corollary

Every connected graph with at least 3 vertices that contains a bridge also contains a cut-vertex.

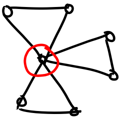
prevents K_2 counterexample from previous slide

The other direction

If G has a bridge (and G isn't lame), then one of the bridge endpoints is a cut-vertex.

The converse: If v is a cut-vertex, must v be the endpoint of a bridge?

No :



v is a cut vertex, but it is not the endpoint of a cut edge.

Generally: The existence of a cut edge implies a cut vertex (aside from lame cases), but the existence of a cut vertex does not necessarily lead to a cut edge.

A characterization

Theorem

A vertex v in a connected graph G is a cut-vertex if and only if there exist vertices u and w distinct from v such that v lies in every $u - w$ path.

To prove:

Let G be a connected graph.

($\Rightarrow$) If $v \in V(G)$ is a cut-vertex, then $\exists$ vertices $u, w \neq v$ s.t. $\forall$ u-w paths, v is contained inside.

contrapositive: If $\forall$ vertices $u, w \neq v$, $\exists$ u-w path s.t. v is not contained inside, then v is not a cut-vertex.

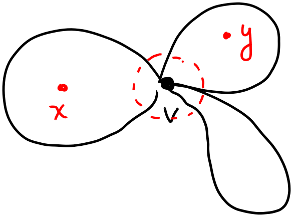
($\Leftarrow$) If $\exists$ vertices $u, w \neq v$ s.t. v lies in every u-w path, then v is a cut-vertex.

contrapositive: If v is not a cut-vertex, then $\forall$ vertices $u, w \neq v$, $\exists$ u-w path s.t. v is not contained inside.

Proof: Let G be a connected graph.

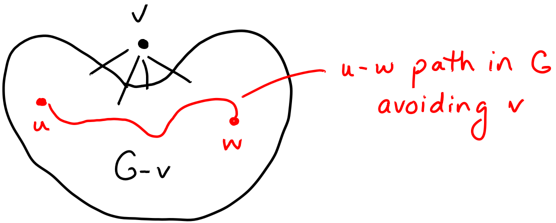
($\Rightarrow$) Suppose $v \in V(G)$ is a cut vertex in G. Then $G-v$ is not connected by definition. Let x and y be in different components of $G-v$. Note that G is connected, so there exist x-y paths in G; let P be an arbitrary such path. Since x and y are not connected in $G-v$, P cannot be a path in $G-v$. Hence P contains v. Since P was arbitrary, all x-y paths contain v.

Idea:



No x-y path in $G-v$, so all x-y paths in G must go through v

($\Leftarrow$) Now suppose v is not a cut-vertex. Hence $G-v$ is connected. Let u and w be arbitrary vertices in G distinct from v. Since $G-v$ is connected, there is a $u-w$ path in $G-v$, say Q. Since $G-v$ is a subgraph of G, Q is also a $u-w$ path in G, and it avoids v as needed.



Note: In class, we also saw that we could, in effect, "reverse" this proof to prove the contrapositive in the other direction.