2.7. The Poisson Distribution

1. *Definition.* (Poisson Distribution)

   Let $\lambda > 0$ be a fixed parameter. If X is a random variable with pmf

   $$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, 3, \cdots ,$$

   then we say X has the Poisson(λ) distribution.

2. *Formulas from Calculus.*

$$e = \lim_{h \to 0} (1+h)^{1/h}.$$

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \cdots\cdots .$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots\cdots .$$

3. *Formula.* (mean and variance of Poisson distribution)

If X has the Poisson(λ) distribution, then

$$M(t) = \exp(\lambda(e^t - 1)),$$

$$E(X) = \lambda,$$

$$Var(X) = \lambda.$$

4. *Example of a counting process.*

   Suppose we are interested in the number of customers arriving at a gas station and also the patterns in their arrival times.

   Anytime a customer arrives, we say a change occurs.

   The process of counting customers (recording changes) is a counting process.

5. *Definition.* (Poisson Process)

   The approximate Poisson process is a special type of counting process.

   Let the number of changes that occur in a given interval be counted.

   Then we have the Poisson process with parameter $\lambda > 0$ if the following conditions are satisfies:

   (a) The numbers of changes occurring on non-overlapping intervals are independent.

   (b) The probability of exactly one change occurring in a sufficiently short interval of length h is approximately λh.

   (c) The probability of two or more changes occurring in a sufficiently short interval is essentially zero.

6. Conditions (b) and (c) imply that: If X is number of changes during a

a time interval of length h and h is small, then

$$P(X = 1) = \lambda h, \qquad P(X = 0) = 1 - \lambda h.$$

Please keep in mind that

$$E(X) = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + \cdots .$$

7. In an approximate Poisson process with parameter $\lambda > 0$, the number of occurrences in a time interval of length T has the Poisson(λT) distribution.

8. *Proof.*

   Consider an approximate Poisson process with parameter $\lambda > 0$.

   Let X be the number of occurrences in a time interval of length T.

   Divide the time interval of length T into n subintervals, call them I_1,

$I_2, \cdots, I_n$, each of which has length T/n.

For each $i = 1, 2, 3 \cdots, n$, let X_i be the number of changes (occurrences) in the interval I_i.

It is clear that $X = X_1 + \cdots + X_n$.

Then $X_1, \cdots, X_n$ are independent and identically distributed.

For each $i = 1, 2, \cdots, n$, we have

$$P(X_i = 0) \approx 1 - \frac{\lambda T}{n}, \quad P(X_i = 1) \approx \frac{\lambda T}{n}.$$

So, approximately X has the $b(n, \lambda T/n)$ distribution, that is

$$P(X = x) \approx \binom{n}{x} \left(1 - \frac{\lambda T}{n}\right)^{n-x} \left(\frac{\lambda T}{n}\right)^{x}, \quad x = 0, 1, 2, 3 \cdots, n.$$

As n increases, the approximation gets more and more accurate, so

$$P(X=x)=\lim_{n\to\infty}\binom{n}{x}\left(1-\frac{\lambda T}{n}\right)^{n-x}\left(\frac{\lambda T}{n}\right)^{x}=\frac{e^{-\lambda T}\,(\lambda T)^{x}}{x!}.$$

In fact, for each $x=0,1,2,3,\cdots$, we have

$$\begin{aligned}
P(X=x) &= \lim_{n\to\infty}\binom{n}{x}\left(1-\frac{\lambda T}{n}\right)^{n-x}\left(\frac{\lambda T}{n}\right)^{x}\\
&= \lim_{n\to\infty}\frac{n!}{x!(n-x)!}\left(1-\frac{\lambda T}{n}\right)^{n}\left(1-\frac{\lambda T}{n}\right)^{-x}\left(\frac{\lambda T}{n}\right)^{x}\\
&= \frac{(\lambda T)^{x}}{x!}\lim_{n\to\infty}\frac{n!}{(n-x)!}\left(1-\frac{\lambda T}{n}\right)^{n}\left(\frac{1}{n}\right)^{x}\\
&= \frac{(\lambda T)^{x}}{x!}\lim_{n\to\infty}\frac{n!}{(n-x)!}\left(\frac{1}{n}\right)^{x}\left(1-\frac{\lambda T}{n}\right)^{n}
\end{aligned}$$

9. *R Code:*

```
barplot(dpois(0:10,4),ylab="Probability",xlab="n",
space=2,ylim=c(0,0.2), names.arg=0:10)
```