

Theorem: Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converge when these two conditions are met.

1. $\lim_{n \rightarrow \infty} a_n = 0$
2. $a_{n+1} \leq a_n$ for all n

Proof:

Consider the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

For this series, the partial sum

$$S_{2n} = (a_1 - a_1) + (a_3 - a_4) + (a_5 - a_6) + \dots + (a_{2n-1} - a_{2n})$$

has all non-negative terms, and therefore (S_{2n}) is a non-decreasing sequence. But you can also write

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \dots - (a_{2n-2} - a_{2n-1}) - a_{2n}$$

which implies that $S_{2n} \leq a_1$ for every integer n . So (S_{2n}) is a bounded, non-decreasing sequence that converges to some value L . Because $S_{2n-1} - a_{2n} = S_{2n}$ and $a_{2n} \rightarrow 0$, you have

$$\begin{aligned} \lim_{n \rightarrow \infty} S_{2n-1} &= \lim_{n \rightarrow \infty} S_{2n} + \lim_{n \rightarrow \infty} a_{2n} \\ &= L + \lim_{n \rightarrow \infty} a_{2n} \\ &= L \end{aligned}$$

Because both S_{2n} and S_{2n-1} converge to the same limit L , it follows that (S_n) also converges to L . Consequently, the given alternating series converges.