2.2. Mathematical Expectation

1. *Definition.* Let S be the sample space of a random experiment, and let X be a discrete random variable. Suppose that the range of X is $\{x_1, x_2, \cdots, x_m\}$.

   Let $f(x)$ be the pmf of X.

   Then the mathematical expectation of X is

$$E(X) = \sum_{i=1}^{m} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + \cdots + x_m f(x_m).$$

   Each term is a possible value of X times its corresponding probability.

   The mathematical expectation of X is also called the expected value of X, or the expectation of X, or the mean of X.

2. *Example.*

Let the experiment be the toss of two coins. Let X be the number of coins that turn up heads. Then X is a discrete random variable. Find the expectation of X.

— *Solution.* X has possible values $0, 1, 2$. The pmf of X is

x	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Therefore,

$$E(X) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1.$$

The result makes sense. If we toss two coins, then we expect one of them turns out heads.

3. Properties of expectation.

(a) $E(c) = c$

(b) $E(cX) = cE(X)$.

(c) $E(X + Y) = E(X) + E(Y)$.

(d) $E(\alpha X + \beta Y) = \alpha E(X) + \beta E(Y)$.

4. *Example.* Let the experiment be the toss of one die. Let X be the outcome of the die. Then X is a discrete random variable. Find $E(X)$.

— *Solution.* The pmf of X is given in the table form:

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

It follows that

$$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5.$$

5. *Formula.* (The expectation of the hyper-geometric distribution)

A box contains N_1 red chips and N_2 blue chips. n objects are selected from the box at random without replacement. Here, it must be true that $n \leq N_1 + N_2$. Let X be the number of red chips in the n objects selected.

Then X is a random variable with p.m.f.

$$f(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N_1+N_2}{n}}, \quad x = 0, 1, 2, 3, \cdots, n$$

We say X has a hyper-geometric distribution with parameters N_1, N_2, n.

We have

$$E(X) = n\frac{N_1}{N_1 + N_2}. \tag{1}$$

6. The proof of the formula on the last page is elementary but time-consuming. We will just point out that the formula does make sense. It is clear that

$$p = \frac{N_1}{N_1 + N_2}$$

is the proportion of red chips in the population. When we draw a sample of size of n from the population, we expect that

$$np = n\frac{N_1}{N_1 + N_2}$$

objects in the sample are red.

7. *Example.* A box contains 5 red chips and 9 blue chips. Seven objects are selected from the box at random without replacement.

Let X be the number of red chips in the seven objects that are selected. Then X is a discrete random variable. In fact, X has the hyper-geometric distribution with parameters 5,9,7.

The expectation of X is

$$E(X) = 7 \times \frac{5}{5+9} = 2.5.$$

8. A function of a random variable is also a random variable.

9. Let X be a discrete random variable. Suppose that the range of X is $\{x_1, x_2, \cdots, x_m\}$. Let $f(x)$ be the pmf of X.

   If $g(x)$ is a function of X, then $Y = g(X)$ is also a random variable, Y has possible values

$$y_1 = g(x_1),\ y_2 = g(x_2),\ \cdots,\ y_m = g(x_m).$$

   And, the mathematical expectation of Y is

$$E(Y) = \sum_{i=1}^{m} g(x_i) f(x_i) = g(x_1)f(x_1) + \cdots + g(x_m)f(x_m).$$

   Each term in the sum is a possible value of Y times the corresponding probability.

10. *Example.*

Let the experiment be the toss of three coins. Let X be the number of coins that turn up heads. Then X is a discrete random variable. Find the expectation of $Y = X^2$.

— *Solution.* X has possible values $0, 1, 2, 3$. The pmf of X is

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The pmf of $Y = X^2$ is

y	0	1	4	9
$h(y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

. Therefore,

$$E(Y) = 0^2 \times \frac{1}{8} + 1^2 \times \frac{3}{8} + 2^2 \times \frac{3}{8} + 3^2 \times \frac{1}{8} = 3.$$