**Theorem: Bounded Monotonic Sequences**

If a sequence $(a_n)$ is bounded and monotonic, then it converges.

**Proof:**

Assume that the sequence $(a_n)$ is non-decreasing, and each term is positive. Because the sequence is bounded, there must exists an upper bound $M$ such that

$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots \leq M$$

. From the completeness axiom, there is a least upper bound $L$ such that

$$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots \leq L$$

For $\varepsilon > 0$, it follows that $L - \varepsilon < L$, and therefore $L - \varepsilon$ cannot be an upper bound for the sequence. Consequently, at least one term of $(a_n)$ is greater than $L - \varepsilon$. That is, $L - \varepsilon < a_N$ for some positive $N$. Hence, $a_N \leq a_n$ for $n > N$. Finally, for $n > N$ $L - \varepsilon < a_N \leq a_n \leq L < L + \varepsilon \Rightarrow |a_n - L| < \varepsilon$ for $n > N$.