6.2. Confidence Intervals for Means

1. *Motivating Example.* Assume that $X_1, X_2, \cdots, X_{60}$ are a random sampleof size 60 from the population $X \sim N(\mu, 16)$. Here, μ is unknown. We have shown (in the last section) that the MLE for μ is $\hat{\mu} = \bar{X}$.

   Now, suppose that someone made a claim that $H_0 : \mu = 10.0$, and we wish to find a way to test this claim H_0. For example, if the observed value of $\bar{X}$ is $\bar{x} = 12$, do we accept the claim H_0 or reject the claim H_0?

   A second related question is: can we find some kind of estimate for the difference between this observed value $\bar{x} = 12$ and the true value of μ?

2. *Definition.*

   For each $\alpha \in (0,1)$, let z_α be the real number at which the value of the cdf of a standard normal distribution equals $1-\alpha$.

   In other words, if $X \sim N(0,1)$, then

$$P(X > z_\alpha) = \alpha.$$

3. Some commonly used values of z_α are listed below:

$$z_{0.1} \approx 1.28.$$

$$z_{0.05} \approx 1.645,$$

$$z_{0.025} \approx 1.96.$$

4. *Example.* Let $X_1, \cdots, X_n$ be iid $N(\mu, \sigma^2)$ distributions, where μ is unknown and σ^2 is known. If

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

then $\bar{X} \sim N(\mu, \sigma^2/n)$. It follows that

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1).$$

For each $\alpha \in (0, 1)$, we have

$$P\left(-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1 - \alpha.$$

In other words, for each $\alpha \in (0, 1)$, we have

$$P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

And, once $\bar{x}$ is observed, the interval

$$\left[\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right] \tag{1}$$

is called the confidence interval for μ with confidence coefficient $1 - \alpha$.

5. *Example.* Let $X_1, \cdots, X_{50}$ be iid $N(\mu, 16)$ distributions, where μ is unknown. Suppose that the observed value of $\bar{X}$ is $\bar{x} = 11$. Construct a confidence interval for μ with a confidence level of 90%.

— *Solution.* First, we collect the necessary information:

(a) Since $1 - \alpha = 0.9$, $\alpha/2 = 0.05$, and $z_{\alpha/2} = 1.645$.

(b) $\bar{x} = 10$, $n = 50$, $\sigma = \sqrt{16} = 4$.

Substitute these number into Formula (1), we get: the interval

$$\left[11 - 1.645 \cdot \frac{4}{\sqrt{50}}, 11 + 1.645 \cdot \frac{4}{\sqrt{50}}\right],$$

that is, the interval

$$[10.069, 11.930]$$

is the confidence interval for μ with a confidence coefficient of 90%.

6. *Definition.*

   For each $\alpha \in (0,1)$, let $t_\alpha(n-1)$ be the real number at which the value of the cdf of a t_{n-1} distribution equals $1-\alpha$.

   If other words, if $X \sim t_{(n-1)}$, then

$$P(X > t_\alpha(n-1)) = \alpha.$$

7. Some commonly used $T_\alpha(n-1)$ values are:

$$t_{0.1}(10) \approx 1.372,$$

$$t_{0.05}(20) \approx 1.725.$$

8. *Example.* Let $X_1, \cdots, X_n$ be iid $N(\mu, \sigma^2)$ distributions, where both μ and σ^2 are unknown. In this case, the sample statistic

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}},$$

has the t_{n-1} distribution.

For each $\alpha \in (0,1)$, we have

$$P\left(-t_{\alpha/2}(n-1) < \frac{\bar{X} - \mu}{S/\sqrt{n}} < t_{\alpha/2}(n-1)\right) = 1 - \alpha.$$

In other words, for each $\alpha \in (0,1)$, we have

$$P\left(\bar{X} - t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2}(n-1)\frac{S}{\sqrt{n}}\right) = 1 - \alpha.$$

And, once $\bar{x}$ and s^2 are observed, the interval

$$\left[\bar{x} - t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}, \bar{x} + t_{\alpha/2}(n-1)\frac{s}{\sqrt{n}}\right]$$

is called the confidence interval for μ with confidence coefficient $1 - \alpha$.

9. *Suggested Reading:*

   One-sided confidence intervals.