Example 3: For the initial value problem

$$y' + y = \sqrt{1+\cos^2(x)}, \quad y(1) = 4$$

find the value of $y(2)$

Solution

The intergrating factor for the differential equation is, from $\mu(x) = e^{\int P(x)\,dx}$

$$\mu(x) = e^{\int 1dx} = e^x$$

The general solution from $y(x) = \frac{1}{\mu(x)\left[\int \mu(x)Q(x)dx + C\right]}$ thus reads

$$y(x) = e^{-x}\left(\int e^x\sqrt{1+\cos^2(x)}dx + C\right)$$

and take the definite integral from the initial value $x = 1$ to the desired value $x = 2$:

$$e^x y\Big|_{x=1}^{x=2} = e^2y(2) - e^1y(1) = \int_{x=1}^{x=2} e^x\sqrt{1+\cos^2(x)}dx$$

Inserting the given value of $y(1)$ and solving, we express

$$y(2) = e^{-2+1}(4) + e^{-2}\int_1^2 e^x\sqrt{1+\cos^2(x)}dx$$

Using Simpson's Rule, we find that the definite integral is approximately 4.841, so

$$y(2) \approx 4e^{-1} + 4.841e^{-2} \approx 2.127$$