Example 2: Solve the initial value problem

$$\frac{dy}{dx} = \frac{y-1}{x+3}, \quad y(-1) = 0$$

Solution

Separating the variables and integrating gives

$$\begin{aligned}
\frac{dy}{y-1} &= \frac{dx}{x+3} \\
\int \frac{dy}{y-1} &= \int \frac{dx}{x+3} \\
\ln|y-1| &= \ln|x+3| + C
\end{aligned}$$

At this point, we can either solve for y explicitly (retaining the constant C) or use the initial condition to determine C and then solve explicitly for y. Let's try the first approach.

Exponentiating $\ln|y-1| = \ln|x+3| + C$, we have

$$\begin{aligned}
e^{\ln|y-1|} &= e^{\ln|x+3|+C} = e^C e^{\ln|x+3|} \\
|y-1| &= e^C|x+3| = C_1|x+3|
\end{aligned}$$

where $C_1 := e^C$. Now depending on the values of y, we have $|y-1| = \pm(y-1)$; and similarly, $|x+3| = \pm(x+3)$. Thus $|y-1| = e^C|x+3| = C_1|x+3|$ can be written as

$$y - 1 = \pm C_1(x+3) \quad \text{or} \quad y = 1 \pm C_1(x+3)$$

where the choice of sign depends on the values of x and y. because C_1 is a it positive constant (recall that $C_1 = e^C > 0$), we can replace $\pm C_1$ by C, where C now represents an *arbitrary* nonzero constant. We then obtain

$$y = 1 + C(x+3)$$

Finally, we determine C such that the initial condition $y(-1) = 0$ is satisfied. Putting $x = -1$ and $y = 0$ in equation $y = 1 + C(x+3)$ gives

$$0 = 1 + C(-1+3) = 1 + 2C$$

and so $C = -\frac{1}{2}$. Thus the solution to the initial value problem is

$$y = 1 - \frac{1}{2}(x+3) = -\frac{1}{2}(x+1)$$

Alternative Approach

The second appraoch is to first set $x = -1$ and $y = 0$ in $\ln|y-1| = \ln|x+3| + C$ and solve for C. In this case, we obtain

$$\begin{aligned}\ln|0-1| &= \ln|-1+3| + C\\ 0 = \ln(1) &= \ln(2) + C\end{aligned}$$

and so $C = -\ln(2)$. Thus from $\ln|y-1| = \ln|x+3| + C$, the solution is given implicitly by

$$\ln(1-y) = \ln(x+3) - \ln(2)$$

Here we have replaced $|y-1|$ by $1-y$ and $|x+3|$ by $x+3$, since we are interested in x and y near the initial values $x = -1$, $y = 0$ (for such values, $y - 1 < 0$ and $x + 3 > 0$). Solving for y, we find

$$\begin{aligned}\ln(1-y) = \ln(x+3) - \ln(2) &= \ln\left(\frac{x+3}{2}\right)\\ 1 - y &= \frac{x+3}{2}\\ y &= 1 - \frac{1}{2}(x+3) = -\frac{1}{2}(x+1)\end{aligned}$$

which agrees with the solution $y = 1 - \frac{1}{2}(x+3) = -\frac{1}{2}(x+1)$ found by the first method.