Section 5.7 Electrical Systems

Notation $Q = Q(t)$ = instantaneous charge (Coulombs)

$I = I(t) = \dfrac{dQ}{dt}$ = current (Amps)

R = Resistance (Ohms)

L = Inductance (Henries)

C = Capacitance (Farads)

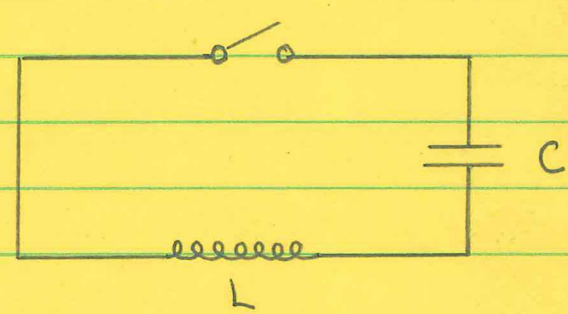
E or V = Emf or Potential (Volts)

Ohms Law $V_R = RI$

Law of Inductance $V_L = L\dfrac{dI}{dt}$

Law of Capacitance $V_c = \dfrac{Q}{C}$

Example 1



At time $t=0$ the key in the circuit shown opposite is closed. If the initial charge on the capacitor is Q_0 describe the state of the circuit at time $t>0$.

Solution Let $Q(t)$ be the charge on the capacitor at time t.

$$V_L + V_C = 0$$

$$L\frac{dI}{dt} + \frac{Q}{C} = 0$$

$$\therefore L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

$$\therefore \ddot{Q} + \omega^2 Q = 0 \quad , \quad \omega = \frac{1}{\sqrt{LC}}$$

Gen Soln $\quad Q(t) = A\cos\omega t + B\sin\omega t$

$t=0,\ Q=Q_0 \qquad Q_0 = A(1) + B(0) \Rightarrow A = Q_0$

$$\dot{Q}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$$

$t=0,\ \dot{Q} = I = 0 \qquad 0 = -A\omega(0) + B\omega(1) \Rightarrow B = 0$

$$\therefore Q(t) = Q_0\cos\omega t = Q_0\cos\left(\frac{t}{\sqrt{LC}}\right)$$

$$I(t) = \dot{Q}(t) = -\frac{Q_0}{\sqrt{LC}}\sin\left(\frac{t}{\sqrt{LC}}\right)$$

Example 2

A circuit consists of an inductor of 0.05 H, a resistance of 20 Ω, a capacitor of 100×10^{-6} F, and an emf of 100 Volts. Describe the state of the circuit at time $t>0$ if $Q=0$ and $I=0$ at $t=0$.

Solution $\quad V_L + V_R + V_C = E$

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E \qquad \leftarrow \text{Replace } I \text{ by } \dot{Q}$$

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = E$$

$$\frac{1}{20}\ddot{Q} + 20\dot{Q} + \frac{1}{10^{-4}}Q = 100$$

$$\therefore \quad \ddot{Q} + 400\dot{Q} + 200,000\,Q = 2,000$$

C.F. $\quad \ddot{Q} + 400\dot{Q} + 200,000\,Q = 0$

A.E. $\quad m^2 + 400m + 200,000 = 0$

$$m = \frac{-400 \pm \sqrt{400^2 - 4(1)200,000}}{2(1)}$$

$$= \frac{-400 \pm \sqrt{-640,000}}{2}$$

$$= -200 \pm 400i$$

$$Q_c(t) = Ae^{-200t}\cos(400t) + Be^{-200t}\sin(400t)$$

P.I. $\quad$ Try $Q_p = a$

$$\dot{Q}_p = 0$$

$$\ddot{Q}_p = 0$$

Substitute $\quad \ddot{Q}_p + 400\dot{Q}_p + 200,000\,Q_p = 2000$

$$0 + 0 + 200,000a = 2,000$$

$$\therefore \quad a = \frac{2{,}000}{200{,}000} = \frac{1}{100}$$

$$\therefore \quad Q_p = \frac{1}{100}$$

<u>Gen. Soln.</u> $\quad Q(t) = e^{-200t}\left(A\cos 400t + B\sin 400t\right) + \frac{1}{100}$

<u>$t=0,\ Q=0$</u> $\quad 0 = 1(A) + \frac{1}{100} \Rightarrow A = \frac{-1}{100}$

$$\dot{Q}(t) = e^{-200t}\left(-400A\sin 400t + 400B\cos 400t\right) - 200e^{-200t}\left(A\cos 400t + B\sin 400t\right)$$

<u>$t=0,\ \dot{Q}=0$</u> $\quad 0 = 1(400B) - 200(A)$

$$400B = 200A = -2$$

$$\therefore B = \frac{-1}{200}$$

<u>Part. Soln.</u> $\quad Q(t) = \frac{-1}{200}e^{-200t}\left(2\cos 400t + \sin 400t\right) + \frac{1}{100}$

$$I(t) = \dot{Q}(t) = e^{-200t}\left(-400A\sin 400t + 400B\cos 400t - 200A\cos 400t - 200B\sin 400t\right)$$

$$= e^{-200t}\left(4\sin 400t - 2\cos 400t + 2\cos 200t + \sin 400t\right)$$

$$= 5e^{-200t}\sin 400t$$

<u>Example 3</u> Repeat Example 2 assuming that there is now a variable emf of $E(t) = 100\cos(200t)$.

Solution $\frac{1}{20}\ddot{Q} + 20\dot{Q} + \frac{1}{10^{-4}}Q = 100\cos(200t)$

$$\ddot{Q} + 400\dot{Q} + 200{,}000\,Q = 2{,}000\cos(200t)$$

C.F. $Q_c(t) = e^{-200t}\left(A\cos 400t + B\sin 400t\right)$

P.I. Try $Q_p = a\cos(200t) + b\sin(200t)$

$$\dot{Q}_p = -200a\sin(200t) + 200b\cos(200t)$$

$$\ddot{Q}_p = -40{,}000a\cos(200t) - 40{,}000b\sin(200t)$$

Substitute $\ddot{Q}_p + 400\dot{Q}_p + 200{,}000\,Q_p = 2{,}000\cos(200t)$

$$(\cos 200t)\left[-40{,}000a + 80{,}000b + 200{,}000a\right] = 2000\cos 200t$$
$$+ (\sin 200t)\left[-40{,}000b - 80{,}000a + 200{,}000b\right]$$

$$\left.\begin{aligned} \cos 200t &: \; 160{,}000a + 80{,}000b = 2000 \\ \sin 200t &: \; -80{,}000a + 160{,}000b = 0 \end{aligned}\right\} \rightarrow a = 2b$$

$$160{,}000a + 40{,}000a = 2{,}000$$

$$a = \frac{2{,}000}{200{,}000} = \frac{1}{100}$$

$$b = \frac{1}{200}$$

$$\therefore Q_p(t) = \frac{1}{100}\cos(200t) + \frac{1}{200}\sin(200t)$$

Gen Soln. $Q = e^{-200t}\left(A\cos 400t + B\sin 400t\right)$

$$+ \frac{1}{100}\cos 200t + \frac{1}{200}\sin 200t$$

$\underline{t=0,\ Q=0}$ $\quad 0 = 1(A) + \frac{1}{100} \Rightarrow A = -\frac{1}{100}$

$$\dot{Q}(t) = e^{-200t}\left(-400A\sin 400t + 400B\cos 400t\right)$$
$$-200e^{-200t}\left(A\cos 400t + B\sin 400t\right)$$
$$-2\sin 200t + \cos 200t$$

$\underline{t=0,\ \dot{Q}=0}$ $\quad 0 = 1(400B) - 200(A) + 1$

$$0 = 400B + 3$$
$$B = -\frac{3}{400}$$

Part soln $\quad Q = -\frac{1}{400}e^{-200t}\left(4\cos 400t + 3\sin 400t\right)$

$$+\frac{1}{100}\cos 200t + \frac{1}{200}\sin 200t$$

$$I(t) = \dot{Q}(t) = e^{-200t}\left(4\sin 400t - 3\cos 400t\right)$$
$$+ e^{-200t}\left(2\cos 400t + \frac{3}{2}\sin 400t\right)$$
$$-2\sin 200t + \cos 200t$$

$$= e^{-200t}\left(\frac{4}{2}\sin 400t - \cos 400t\right)$$
$$-2\sin 200t + \cos 200t$$

H.W. Pages 296-297, #'s 1, 3, 11, 13