The Method of Reduction of Order

Proposition

Let $y_1(x)$ be a non-trivial solution of the homogeneous 2'nd order linear ode $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ then the change of variables $y = y_1(x)v(x)$ transforms the differential equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = g(x)$ into a 1^{ST} order linear ode.

Example 1 Solve $x^2y'' - xy' + y = x^3$, $y_1(x) = x$

Solution Corresponding Homogeneous Eqn : $x^2y'' - xy' + y = 0$.

If $y_1(x) = x$ then, $y_1' = 1$ and $y_1'' = 0$.

Thus, $x^2y_1'' - xy_1' + y_1 = x^2(0) - x(1) + x = 0$.

So $y_1(x)$ is indeed a solution to the corresponding homogeneous eqn.

In the given non-homogeneous equation we must substitute $y = y_1(x)v(x) = x.v$.

For which, $y' = \frac{d}{dx}(xv) = xv' + v.1 = xv' + v$

and, $y'' = \frac{d}{dx}(xv' + v) = xv'' + v'.1 + v' = xv'' + 2v'$

Substitute $y = xv$ into the given equation :

$$x^2\left(xv'' + 2v'\right) - x\left(xv' + v\right) + xv = x^3$$

$$x^3v'' + 2x^2v' - x^2v' - xv + xv = x^3$$

$$\therefore \quad x^3v'' + x^2v' = x^3$$

or $\quad \dfrac{d}{dx}v' + \dfrac{1}{x}v' = 1 \quad \leftarrow$ 1st order linear in v'.

$$\mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\therefore \quad x\frac{dv'}{dx} + 1\cdot v' = x$$

$$\frac{d}{dx}\left[xv'\right] = x$$

Int $\quad xv' = \frac{1}{2}x^2 + B$

$$v' = \frac{1}{2}x + B\frac{1}{x}$$

Int $\quad v = \frac{1}{4}x^2 + B\ln x + A$

Gen. Soln. $\quad y = xv$

$$y = \frac{1}{4}x^3 + Ax + Bx\ln x$$

$$= \frac{1}{4}x^3 + x\left(A + B\ln x\right)$$

Example 2 Show that $y_1(x) = x^2$ is a solution of $x^2y'' - 4xy' + 6y = 0$ & hence find the general solution.

Solution $\qquad x^2y'' - 4xy' + 6y = 0$

$\uparrow$

Note that the given equation is homogeneous. Must show that $y_1(x) = x^2$ is a solution.

$y_1(x) = x^2$

$y_1' = 2x$

$y_1'' = 2$

$x^2y_1'' - 4xy_1' + 6y_1$

$= x^2(2) - 4x(2x) + 6(x^2)$

$= 0$ ✓

In the given equation now make the substitution $y = y_1(x)v(x) = x^2v$.

$y = x^2v$

$y' = \frac{d}{dx}(x^2v) = x^2v' + v2x$

$y'' = \frac{d}{dx}(x^2v' + 2xv) = x^2v'' + v'2x + 2x.v' + v.2$

$= x^2v'' + 4xv' + 2v$

Substitute $\quad x^2(x^2v'' + 4xv' + 2v) - 4x(x^2v' + 2xv) + 6x^2v = 0$

$x^4v'' + 4x^3v' + 2x^2v - 4x^3v' - 8x^2v + 6x^2v = 0$

$\therefore \; x^4v'' = 0$

$\therefore \; v'' = 0$

$\therefore \; v = A + Bx$

Gen. Soln. $\quad y = x^2v = Ax^2 + Bx^3$

Example 3 Show that $y_1(x) = e^x$ is a solution of $xy'' - (x+2)y' + 2y = 0$ & hence find the general solution.

Solution $y_1 = e^x$, $y_1' = e^x$, $y_1'' = e^x$

$$\therefore \; xy_1'' - (x+2)y_1' + 2y_1$$
$$= xe^x - (x+2)e^x + 2e^x$$
$$= xe^x - xe^x - 2e^x + 2e^x$$
$$= 0.$$

$\therefore$ $y_1(x) = e^x$ is a solution to the given homogeneous equation.

One must now substitute $y = e^x v$.

For which, $y' = \frac{d}{dx}(e^x v) = e^x v' + ve^x = e^x(v'+v)$

and, $y'' = \frac{d}{dx}[e^x(v'+v)] = e^x(v''+v') + (v'+v)e^x$
$= e^x(v'' + 2v' + v)$

Substitute $xe^x(v''+2v'+v) - (x+2)e^x(v'+v) + 2e^x v = 0$

$$\therefore \; x(v'' + 2v' + v) - (x+2)(v'+v) + 2v = 0$$

$$xv'' + 2xv' + \cancel{xv} - xv' - 2v' - \cancel{xv} - \cancel{2v} + \cancel{2v} = 0$$

$$xv'' + xv' - 2v' = 0$$

$$xv'' + (x-2)v' = 0$$

$$\therefore \frac{d}{dx}v' + \left(1 - \frac{2}{x}\right)v' = 0$$

I.F. $\mu = e^{\int 1 - \frac{2}{x}\,dx} = e^{x - 2\ln x} = e^x e^{-2\ln x}$

$$= e^x e^{\ln x^{-2}} = e^x \cdot x^{-2} = \frac{1}{x^2}e^x.$$

$$\therefore \underbrace{\left(\frac{1}{x^2}e^x\right)\frac{d}{dx}v' + v'\left(\frac{1}{x^2} - \frac{2}{x^3}\right)e^x}_{} = 0$$

$$\frac{d}{dx}\left[\frac{1}{x^2}e^x \cdot v'\right] = 0$$

Int $\quad \frac{1}{x^2}e^x v' = B$

$$\therefore v' = B x^2 e^{-x}$$

Int $\quad v = B\int x^2 e^{-x}\,dx + A$

$$u = x^2, \quad dw = e^{-x}\,dx$$
$$du = 2x\,dx, \quad w = -e^{-x}$$

$$v = B\left\{-x^2e^{-x} + 2\int x e^{-x}\,dx\right\} + A$$

$$u = x, \quad dw = e^{-x}\,dx$$
$$du = dx, \quad w = -e^{-x}$$

$$v = B\left\{-x^2e^{-x} + 2\left[-xe^{-x} + \int e^{-x}\,dx\right]\right\} + A$$

$$= B\left\{-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right\} + A$$

$$= -Be^{-x}\left(x^2 + 2x + 2\right) + A$$

Gen Soln $\quad y = e^x v$

$$y = -B(x^2+2x+2) + Ae^x$$

or $\quad y = c_1(x^2+2x+2) + c_2 e^x.$

H.W. $\quad$ Page 202, #'s 45–48, 51

↑

Do not solve by using formula (13) on page 198. Instead, demonstrate knowledge of solution technique.