Differential Equation Homework Pg. 99: 1, 3, 4, 6

Section 3.2: Compartmental Analysis

1) A brine solution of salt flows at a constant rate of 8 L/min into a large tank that initially held $100L$ of brine solution in which was dissolved 0.5 kg of salt. The solution inside the tank is ket well stirred and flows out of the tank at the same rate. If the concentration of salt in the brine entering the tank is 0.05 kg/L, determine the mass of salt in the tank after t minutes. When will the concentration of salt in the tank reach 0.02 kg/L?

Solution:

rate of change=input rate-output rate

$$\frac{dx}{dt} = \left(\frac{8L}{min}\right)\left(\frac{0.05\ kg}{L}\right) - \left(\frac{8L}{min}\right)\left(\frac{x(t)}{100}\cdot\frac{kg}{L}\right)$$
$$\frac{dx}{dt} = \frac{0.4\ kg}{min} - \frac{0.08x\ kg}{min}$$

so now we have,

$$\frac{dx}{dt} = 0.4 - 0.08x$$

$$\frac{dx}{dt} + 0.08x = 0.4$$

this is a linear equation, it has an integrating factor, where,

$$\begin{aligned} P(t) &= 0.08 \\ &= \int P(t)dt \\ &= \int 0.08\ dt \\ &= 0.08t \end{aligned}$$

and so,

$$\begin{aligned} \mu(t) &= e^{\int P(t)\ dt} \\ &= e^{\int 0.08t} \\ &= e^{0.08t} \end{aligned}$$

multiplying $\frac{dx}{dt} + 0.08x = 0.4$ by $\mu(t)$

$$\begin{aligned} e^{0.08t}\frac{dx}{dt} + 0.08e^{0.08t} &= 0.4e^{0.08t} \\ \frac{d}{dt[e^{0.08t}]} &= 0.4e^{0.08t} \end{aligned}$$

integrating both sides,

$$\int \frac{d}{dt}[e^{0.08t}x]dt = \int 0.4e^{0.08t}dt$$
$$e^{0.08t}x = 5e^{0.08t} + C$$
$$x(t) = 5 + Ce^{-0.08t}$$

applying the initial value condition, $x(0) = 0.5$, to obtain the value of C

$$0.5 = 5Ce^0$$
$$0.5 - 5 = C$$
$$-4.5 = C$$

and so

$$x(t) = 5 - 4.5e^{-0.08t}$$

To determine when will the concentration of salt in the tank reach 0.02 kg/L, we first need to obtain the mass of the salt, where

$$\text{mass=concentration} \times \text{ volume}$$

$$= \left(0.02\frac{kg}{L}\right)(100L) = 2\ kg$$

and so

$$2 = 5 - 4.5e^{-0.08t}$$
$$-3 = -4.5e^{-0.08t}$$
$$\frac{2}{3} = e^{-0.08t}$$
$$\ln\frac{2}{3} = -0.08t$$
$$t = \frac{\ln\frac{2}{3}}{-0.08}$$
$$t \approx 5.07 \text{ min}$$

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3) A nitric acid solution flows at a constant rate of 6 L/min into a large tank that initially held 200 L of 0.5% nitric acid solution. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 L/min. If the solution entering the tank is 20% nitric acid, determine the volume of nitric acid in the tank after t minutes. When will the percentage of nitric acid in the tan reach 10%?

Solution: Let $y(t)$ be the volume of nitric acid in the container after t minutes. Then, **its time rate of change, y', is the difference between its inflow and outflow (Balance law)**. Since $6L$ of nitric acid solution runs in the container per minute, containing 20% of nitric acid, the needed inflow rate is

$$6 \cdot 0.2 = 1.2$$

The amount of the solution in the container at any moment t is

$$200 + (6 - 8)t = 200 - 2t$$

since the tank initially contained $200L$ of the solution and nitric solution is pumped into the top of the tank at the rate 6 l/min, while well-mixed solution leaves the tank at the rate 8 L/min. Now, the outflow is 8 L of the mix in a minute. That is $\frac{8}{200-2t}$ of the total mix content in the container, hence $\frac{4}{100-t}$ of the nitric acid content $y(t)$, that is

$$\frac{4y(t)}{100-t}$$

Initially the $200L$ tank contained solution with 0.5% of nitric acid, therefore, we obtain the initial condition

$$y(0) = 200 \cdot \frac{0.5}{100} = 1$$

Thus, the mathematical model for the problem is

$$y'(t) = 1.2 - \frac{4y(t)}{100-t}, \quad y(0) = 1$$

This is **linear ODE**. In this case

$$P(t) = \frac{4}{100-t}, \quad Q(t) = 1.2$$

Hence,

$$\begin{aligned} h &= \int P\,dt \\ &= 4\int \frac{dt}{100-t} \\ &= \ln(100-t)^{-4} \end{aligned}$$

So, an integrating factor is

$$\begin{aligned} e^h &= e^{\ln(100-t)^{-4}} \\ &= \frac{1}{(100-t)^4} \end{aligned}$$

and the general solution is

$$\begin{aligned} y(t) = e^{-h}\left(c + \int Qe^h\,dt\right) \\ &= (100-t)^4\left(c + \int 1.2 \cdot \frac{1}{(100-t)^4}\right)dt \\ &= c(100-t)^4 + \frac{1.2}{3}(100-t)^4 \cdot (100-t)^{-3} \\ &= c(100-t)^4 + 0.4(100-t) \end{aligned}$$

Now, we can use the initial condition to determine the numeric value of c. Substitute 0 for t and 1 for y in the equation.

$$\begin{aligned} 1 = y(0) = c(100-0)^4 + 0.4(100-0) &\Rightarrow c \cdot 100^4 + 40 \\ &\Rightarrow 1 - 40 = c \cdot 10^8 \\ &\Rightarrow c = -39 \cdot 10^{-8} \end{aligned}$$

Hence,

$$y(t) = -39 \cdot 10^{-8}(100-t)^4 + 0.4(100-t)$$

is the volume of the nitric acid in the tank after t minutes.

We want to find the moment when the container hold 10% of nitric acid. The amount of the solution in the tank at any minute t is $200-2t$ and $y(t)$ represents the amount of the nitric acid in the solution. Hence, we need to solve

$$y(t) = (200-2t) \cdot \frac{10}{100} = 0.2(100-t)$$

for t

On the other hand, we know that the general solution $y(t)$ is equal to $-39 \cdot 10^{-8}(100-t)^4 + 0.4(100-t)$. Hence, we obtain

$$\begin{aligned}
-39 \cdot 10^{-8}(100-t)^4 + 0.4(100-t) &= 0.2(100-t) \\
\frac{-39 \cdot 10^{-8}(100-t)^4 + 0.4(100-t)}{0.2(100-t)} &= 1 \\
0.2 - 19.5 \cdot 10^{-8}(100-t^3) &= 0.1 \\
(100-t)^3 &= 51280.513
\end{aligned}$$

Therefore,

$$t = 19.95 \text{ min}$$

which means that the container will hold 10% of nitric acid in about 20 minutes.

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4) A brine solution of salt flows at a constant rate of 4L/min into a large tank that initially held $100L$ of pure water. The solution inside the tank is kept well stirred and flows out of the tan at a rate 3L/min. If the concentration of salt in the brine entering the tank is 0.2 kg/L, determine the mass of salt in the tank after t min. When will the concentration of salt in the tank reach 0.1 kg/L?

Solution: Let $y(t)$ be the amount (the mass) of salt in the container after t minutes. Then, **its time of rate of change, y', is the difference between its inflow and outflow (Balance Law)**.

Since $4L$ of saktwater runs in the container per minute, containing 0.2 kg of salt per liter, the needed inflow is

$$4 \cdot 0.2 = 0.8$$

The volume of the solution in the container at any moment t is

$$v(t) = 100 + (4-3)t = 100 + t$$

the tank initially contained 100 L of water and saltwater is pumped into the top of the tank at the rate 4 L/min, while well-mixed solution leaves the tank at the rate 3L/min.

We can determine the concentration of the salt in the tank by dividing the amount of salt $y(t)$ with the volume of the solution $v(t)$ at any time t. Hence,

$$c(t) = \frac{y(t)}{v(t) = \frac{y(t)}{100+t}}$$

Now, the outflow is 3L of the mix in a minute. That is $\frac{3}{100+t}$ of the total mix content in the container, hence $\frac{3}{100+t}$ of the salt content $y(t)$, that is

$$\frac{3y(t)}{100+t}$$

Initially, the tank contained 100 L of clear water. Therefore, the concentraion of salt at the beginning was 0 and we obtain the initial condition

$$y(0) = 0$$

Thus, the mathematical model for this problem is

$$y'(t) = 0.8 - \frac{3y(t)}{100+t}, \quad y(0)$$

This is **linear ODE**. In this case

$$P(t) = \frac{3}{100+t}, \quad Q(t) = 0.8$$

Hence,

$$\begin{aligned} h &= \int P\,dt \\ &= 3\int \frac{dt}{100+t} \\ &= \ln(100+t)^3 \end{aligned}$$

So, an integrating factor is

$$\begin{aligned} e^h &= e^{\ln(100+t)^3} \\ &= (100+t)^3 \end{aligned}$$

and the general solution is

$$\begin{aligned} y(t) &= e^{-h}\left(c + \int Qe^h\,dt\right) \\ &= (100+t)^{-3}\left(c + \int = 0.8 \cdot (100+t^3)\,dt\right) \\ &= c(100+t)^{-3} + 0.8(100+t)^{-3} \cdot \frac{(100+t)^{3+1}}{3+1} \\ &= \frac{c}{(100+t)^3} + \frac{0.8}{4} \cdot (100+t)^4 \\ &= \frac{c}{(100+t)^3} + 0.2(100+t) \end{aligned}$$

Now, we can use the initial condition to determine the numeric value of c. Substritute 0 for t and 0 for y in the equation

$$\begin{aligned} 0 = y(0) = \frac{c}{(100+0)^3} + 0.2(100+0) \Rightarrow 0 &= \frac{c}{100^3} + 20 \\ &\Longrightarrow -20 = c \cdot 10^{-6} \\ &\Longrightarrow c = -20 \cdot 10^6 \\ &\Longrightarrow c = -2 \cdot 10^7 \end{aligned}$$

Hence,

$$y(t) = \frac{-2 \cdot 10^7}{(100+t)^3 + 0.2(100+t)}$$

is the mass of salt in the tank after t minutes.

We want to find the moment when the concentration of salt in the tank reaches 0.1kg/L. Hence, we need to solve

$$c(t) = 0.1$$

for t.

Therefore, we need to solve

$$\frac{y(t)}{v(t)} = \frac{0.2(100+t) - \frac{2\cdot 10^7}{(100+t)^3}}{100+t} = 0.1$$

for t.

$$\begin{aligned}
0.2 - \frac{2 \cdot 10^7}{(100+t)^4} &= 0.1 \\
\frac{2 \cdot 10^7}{(100+t)^4} &= 0.2 - 0.1 \\
\frac{2 \cdot 10^7}{(100+t)^4} &= 0.1 \\
(100+t)^4 &= \frac{2 \cdot 10^7}{0.1} \\
t + 100 &= 118.92
\end{aligned}$$

Therefore,

$$t = 18.92 \text{ min}$$

which means that the solution will reach the given concentration in about 19 minutes.

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6) The air in a small room 12ft by 8ft by 8ft is 3% carbon monoxide. Starting at $t = 0$, fresh air containing no carbon monoxide is blown into the room at a rate of $100ft^3/min$. If air in the room flows out through a vent at the same rate, when will the air in the room be 0.01% carbon monoxide?

Solution: First, we need to calculate the volume of the small room.

$$V = 12 \cdot 8 \cdot 8 = 768\text{ft}^3$$

Now, let $y(t)$ be the amount of carbon monoxide in the room at time t. Then, **its time rate of change, y', is the difference between its inflow and outflow (Balance law)**.

The needed inflow is 0, because fresh air is inserted in the room, with no carbon monoxide.

Now, the outflow is 100ft^3 of the air in a minute. That is $\frac{100}{V} = \frac{100}{768}$ of the total content in the room, hene $\frac{25}{192}$ of the carbon monoxide content $y(t)$, that is

$$\frac{25y(t)}{192}$$

Initially, at $t = 0$, the air in the room contained 0.3% of carbon monoxide. Therefore, the amount of carbon monoxide at the beginning was

$$768 \cdot \frac{0.3}{100} = 23.04$$

and we obtain the initial condition

$$y(0) = 23.04$$

Thus, the mathematical model for this problem is

$$y'(t) = -\frac{25y(t)}{192}, \quad y(0) = 23.04$$

This is **separable ODE**. Rearranging the terms in the equation gives

$$\begin{aligned} \frac{dy}{dt} &= -\frac{25y}{192} \\ \frac{1}{y}dy &= -0.13dt \end{aligned}$$

Integration on both sides gives

$$\begin{aligned} \int \frac{1}{y}dy &= -\int 0.13dt \\ \ln|y| &= -0.13t + C \end{aligned}$$

By taking exponents, we obtain

$$\begin{aligned} |y| &= e^{-0.13t+C} \\ &= e^{-0.13t}e^{C} \end{aligned}$$

Hence

$$y = Ce^{-0.13t}$$

where $C = e^C$.

Now, we can use the initial condition to determine the numeric value of the constant C. Substitute 0 for t and 23.04 for y in the equation.

$$\begin{aligned} 23.04 &= Ce^{-0.13 \cdot 0} \\ \Rightarrow C &= 23.04 \end{aligned}$$

Therefore, the amount of carbon monoxide at any moment t is given by

$$y = 23.04e^{-0.13t}$$

To find the moment when the air in the room will be 0.01% carbon monoxide, we need to solve

$$y(t) = \frac{0.01}{100}V$$

for t. Hence,

$$23.04e^{-0.13t} = 768 \cdot 10^{-3}$$

and we obtain that

$$t \approx 43.8$$

which means that the air in the room will be 0.01% carbon monoxide in about 44 minutes.