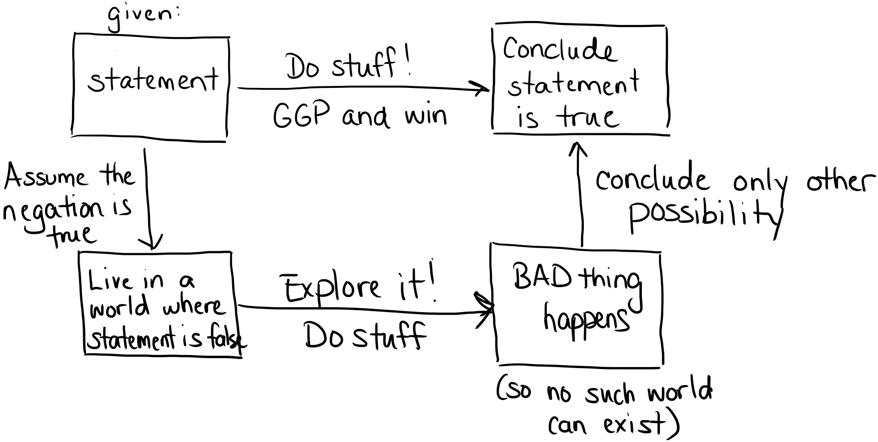
Section 3.6
Indirect Argument

Another method of proof: **Proof by Contradiction**



Proof by Contradiction

General steps:

1. Rewrite statement formally, and write its negation.
(unless I say not to)

2. Write "Proof (by contradiction):"

3. Assume that the statement is false, i.e. the negation is true.

4. Apply definitions.

Example: Prove there is no smallest integer.

① Hard to write formally and negate, so we'll skip it.

② Proof (by contradiction):

③ Assume otherwise, that is, assume there is a smallest integer.

④ (None here)

5. Work toward a contradiction, i.e. make the universe explode.

6. Conclude that the original statement is true.

5 Let n be the smallest integer. Note that $n-1$ is an integer, and $n-1<n$. This contradicts our assumption that n is the smallest integer.

6 Therefore there is no smallest integer.

Important note: Proving a statement by contradiction is VERY VERY different from disproving a statement by proving its negation!

- When we are *proving a statement by contradiction*, we are trying to show the negation is FALSE. We assume that the negation is true in the hopes that the assumption makes the universe explode.
- When we are *disproving* a statement, we are trying to show with one of our methods that the negation is TRUE. **When PROVING the negation of a statement, we never ever assume the negation is true!**

Example 2. Prove that no integer is both even and odd.

Proof (by contradiction): Suppose otherwise, that some integer is both even and odd. Let n be such an integer.
By the definition of even, $n = 2k$ for some integer k.
By the definition of odd, $n = 2\ell + 1$ for some integer ℓ.
Substituting, $2k = 2\ell + 1$, so $2k - 2\ell = 1$, and $k - \ell = \frac{1}{2}$.
However, k and ℓ are integers, so $k - \ell$ is an integer, a contradiction.
Therefore no integer is both even and odd.