Chapter 2. Discrete Distributions

2.1. Random variables of the discrete type

1. *Definition.* Let S be the sample space of an experiment, and let X be a random variable.

   The range of X is defined as the set of all possible values of X.

2. *Definition.* Let S be the sample space of an experiment, and let X be a discrete random variable.

   Suppose that the range of X is $\{x_1, x_2, \cdots, x_m\}$. For each $i = 1, 2, \cdots, m$, we let A_i be the event $X = x_i$. Then, the events $A_1, A_2, \cdots, A_m$ form a partition of the sample space S.

For each $i = 1, 2, \cdots, m$, we let

$$f(x_i) = P(X = x_i).$$

Then, f is called the probability mass function (pmf) of X.

It is easy to see that

$$f(x_i) \geq 0, \quad i = 1, 2, \cdots, m$$

$$\sum_{i=1}^{m} f(x_i) = 1.$$

3. *Example.* Let the experiment be the toss of three coins.

Let X be the number of coins that turn up heads.

Then X is a discrete random variable. The pmf of X is

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

4. *Example.* Let the experiment be the toss of two dice. Let X be the sum of the outcomes of two dice.

Then X is a discrete random variable. The pmf of X

x	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

5. *Definition.* (uniform distribution) Let m be a positive integer.

If the random variable X has range $\{1, 2, 3, \cdots, m\}$, and

$$P(X = i) = \frac{1}{m}, \quad i = 1, 2, 3, \cdots, m,$$

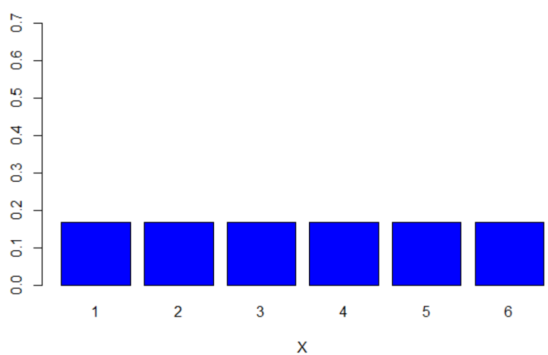
then we say X has a uniform distribution — the discrete uniform distribution over the numbers $1, 2, 3, \cdots, m$. In this case, the pmf of X is

x	1	2	3	$\cdots$	m
$f(x)$	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{1}{m}$	$\cdots$	$\frac{1}{m}$

6. *Example.* Let the experiment be the toss of a die. Let X be the outcome of the die. Then X has the discrete uniform distribution over the numbers $1, 2, 3, 4, 5, 6$. The pmf is, in the table form:

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

or, represented graphically,



7. *Definition.* Suppose that a box contains N_1 red chips and N_2 blue chips. Let the experiment be the random selection of n objects from the box without replacement. Here, it must be true that $n \leq N_1 + N_2$.

Let X be the number of red chips in the n objects selected. Then X is a random variable, and

$$f(x) = P(X = x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N_1+N_2}{n}}, \qquad x = 0, 1, 2, 3, \cdots, n.$$

Note that $P(X = x) = 0$ if $x < N_2 - n$, and $P(X = x) = 0$ if $x > N_1$.

We say the random variable X has a hyper-geometric distribution with parameters N_1, N_2, n.

8. Note that, in the formula

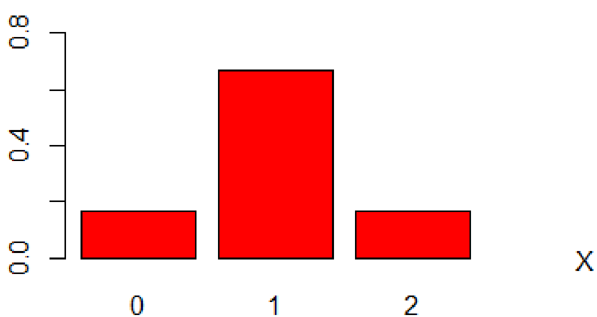
$$P(X = x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N_1+N_2}{n}},$$

the denominator $\binom{N_1+N_2}{n}$ is the size of the sample sample, which is also the number of ways to take an unordered sample of size n without replacement from a population of size $N_1 + N_2$; the numerator $\binom{N_1}{x}\binom{N_2}{n-x}$ is the size of the event that $X = x$. To make $X = x$ happen, we need select x red chips from the available N_1 red chips and then select $n - x$ blue chips from the available N_2 blue chips.

9. *Example.* A box contains 2 red chips and 2 blue chips. Two objects are selected from the box at random without replacement. Let X be the number of red chips in the 2 objects selected. Then X is a random variable, X $\sim$Hyper-geometric$(2, 2, 2)$, and X has pmf

x	0	1	2
$f(x)$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

or, represented graphically,



10. *Exercise.* A box contains 3 red chips and 4 blue chips. Five objects are selected from the box at random without replacement. Let X be the number of red chips in the 5 objects selected.

Then X is a discrete random variable.

In fact, $X \sim$Hyper-geometric$(3, 4, 5)$. Find the pmf of X.