Section 4.4: Counting Labeled Trees

How many distinct **labeled** trees with n vertices are there?

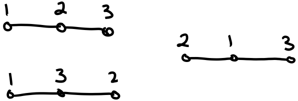
$n = 2$

(Note 

is the same graph, not distinct.)

only 1

$n = 3$



These are all isomorphic, but not equal.

3 total

$n = 4$?

Possible labels: choose 2 labels for the degree 2 vertices, then 2 ways to finish the leaves for each, so $\binom{4}{2} \cdot 2 = 12$ possible

4 ways to label the degree 3 vertex, the rest don't matter

$12 + 4 = 16$ total

Cayley's Formula

Theorem

There are n^{n-2} distinct labeled trees on n vertices.

- Proof idea: Give a bijection between the set of labeled trees on n vertices and the sequences of length $n-2$ with elements chosen from $[n] = \{1, 2, \ldots, n\}$.
- If a bijection exists between the two sets, then they must contain the same number of elements.
- There are n^{n-2} sequences of length $n-2$ with elements chosen from $[n]$, so ta-da!

$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \cdots\ \underline{\ \ }$
$n\ \ n\ \ n\ \ \ \ \ \ \ n$

n choices for each, n-2 positions. $\longrightarrow n^{n-2}$

Getting a sequence from a tree

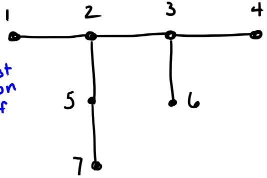
Algorithm. Given a tree T with vertex set $[n]$:

1. Iteratively delete the smallest-labeled remaining leaf, and record its neighbor.
2. Stop when K_2 remains.

Will this work?

- Note that the algorithm will have $n - 2$ steps, so $n - 2$ entries will be recorded.
- Certainly the entries in the sequence will be chosen from $[n]$.
- The resulting sequence is called the *Prufer code* for the tree.

Example



smallest label on a leaf

Sequence: (2, 3, 3, 2, 5)

↑

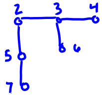
Prüfer code

Notice: $n = 7$, so

$7 - 2 = 5$ entries

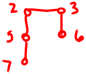
all entries in $[7]$

Step 1: Delete 1, record 2



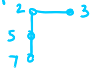
← smallest

Step 2: Delete 4, record 3



← smallest

Step 3: Delete 6, record 3



← smallest

Step 4: Delete 3, record 2

← smallest

Step 5: Delete 2, record 5

K_2, so stop.

What about the other direction?

*We need a bijection; how to get the inverse?

How do we recover the tree from the Prufer code?

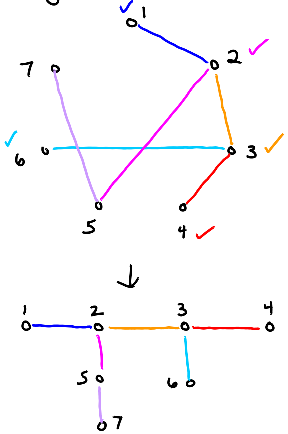
given a sequence of length $n-2$

1. Start with n isolated vertices.
2. Find the smallest vertex not appearing in the sequence, and join it to the first entry in the sequence; mark it, and remove the first entry in the sequence.
3. In each step thereafter, find the smallest unmarked vertex not appearing in what remains of the sequence, join it to the current first entry of the sequence, mark it, and delete the first entry in the sequence.
4. Once the sequence is empty, connect the last two unmarked vertices.

Example

Build a tree from the Prufer code (2, 3, 3, 2, 5)

Length is 5, so $n = 7$



Step 1: 1 doesn't appear, join it to 2, delete 2, mark 1

(3, 3, 2, 5)

Step 2: 4 doesn't appear, join it to 3, delete 3, mark 4

(3, 2, 5)

Step 3: 6 doesn't appear, join it to 3, delete 3, mark 6

(2, 5)

Step 4: 3 doesn't appear, join it to 2, delete 2, mark 3

(5)

Step 5: 2 doesn't appear, join it to 5, delete 5, mark 2

Step 6: Join the two unmarked vertices

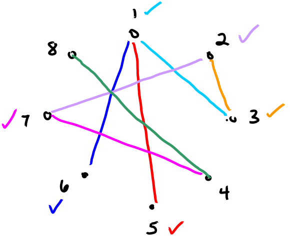
Question: Have we established that there are n^{n-2} labeled trees on n vertices?

- Our algorithm is a bijection, because it has an inverse! (Hand wavy...)
- Given a tree, we know how to build the Prufer code.
- Given a sequence (any of which can be interpreted as Prufer code), we can recover the tree.
- Since the two sets are finite and there is a bijection between them, they have the same number of elements!

Illustrating that the maps are inverses

Go from code to tree and back to code: $(1, 1, 3, 2, 7, 4)$

$n = 8$



Tree to code: (1, 1, 3, 2, 7, 4)

① Delete 5, record 1

② Delete 6, record 1

③ Delete 1, record 3

④ Delete 3, record 2

⑤ Delete 2, record 7

⑥ Delete 7, record 4

They match!

Step 1: Join 5 to 1, mark 5
(1, 3, 2, 7, 4)

Step 2: Join 6 to 1, mark 6
(3, 2, 7, 4)

Step 3: Join 1 to 3, mark 1
(2, 7, 4)

Step 4: Join 3 to 2, mark 3
(7, 4)

Step 5: Join 2 to 7, mark 2 (4)

Step 6: Join 7 to 4, mark 7

Last step: Join 4 to 8