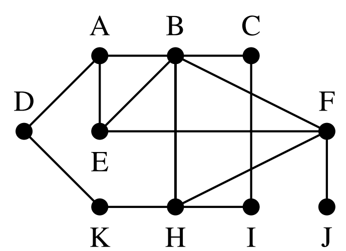
For this homework, you have TWO options:

(a) Complete the assignment as described here (ignoring the challenge questions at the end) - this is what I expect most students will do.

(b) Complete questions 1 and 2, plus the challenge questions at the end - this is for those of you that already feel very confident with your logic and basic proof-writing skills, because I want you to be challenged!

Basic skills

Complete all of the basic skills questions (question 1-5).



Question 1. Use the graph G above to answer the following questions.

(a) What is the order of G?______________ What is the size of G?______________

(b) What is $V(G)$?______________

(c) What is $deg(F)$?______________

(d) Find $N(A)$.______________

(e) If $S = \{C, D\}$, find $N[S]$.______________

(f) Is the graph connected?______________

(g) Are vertices D and F adjacent?______________

(h) List all the neighbors of vertex F. ______________

(i) Is $AEFBC$ a walk in this graph?______________

(j) Is $AEFBHFJ$ a path is this graph?______________

(k) Give a cycle of length four in the graph.______________

(l) Give a circuit of length six that is NOT a cycle.______________

(m) Is C_3 a subgraph of G?______________

(n) What is the diameter of G? ______________

(o) Find an $A - J$ geodesic. ______________

(p) Find an odd cycle in G. ______________

(q) Draw the subgraph of G induced by $S = \{A, E, H, F, C\}$. Is this graph connected?

Please complete the remainder of the assignment on a separate piece of paper (or two or three).

Question 2. Answer the following questions based on the course syllabus. (Just making sure we are all on the same page!)

(a) What do you need to do if you want to be excused from class?

(b) How many participation points can you lose before your your grade is affected?

(c) Are you allowed to use the internet to help with your homework?

(d) How many warnings will I give you on academic integrity violations before I report you for misconduct?

(e) Do you have any concerns about this class that you would like to share with me?

Question 3. Rewrite each statement formally, and write its negation. (You don't need to prove or disprove them, so don't worry about what the words mean!)

(a) Every tree has at most one perfect matching.

(b) If G is a graph with more edges than vertices, then G contains a cycle.

(c) There are graphs that are Eulerian but not Hamiltonian.

(d) If G is a connected, r-regular graph and $\overline{G}$ is also connected, then G is Eulerian or $\overline{G}$ is Eulerian.

Question 4. Consider the statement, "For any n-vertex graph G, if G is connected, then G is a tree or G has at least n edges.

(a) Write the negation of the statement.

(b) Write the converse of the statement.

(c) Write the contrapositive of the statement.

(d) Repeat (a), (b), and (c) with the statement, "If every vertex in a graph G has even degree and G is connected, then G has an Euler circuit."

Question 5. Describe the BASIC IDEA behind a proof of the following types. (Imagine you are explaining to your classmates how you would write each kind of proof.)

(a) A direct proof of a UNIVERSAL statement

(b) A proof of an EXISTENTIAL statement

(c) A proof by contradiction of a UNIVERSAL statement

(d) A proof by contrapositive.

Question 6. Suppose I wish to prove the statement: "A graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half the vertices." Write (formally) what TWO statements would need to be proven in order to successfully prove this statement. Then write the contrapositive of each statement. You don't need to prove anything.

The fun problems - Remember to turn in your scrap work!

Prove each statement below using either an *example* or a *direct proof*, whichever is appropriate. What to do:

(a) Rewrite each statement formally, so you know what kind of statement it is.

(b) If it is an existential statement, look for an example that proves it. If you don't think of something right away, then write your preliminary attempts on scrap paper. You'll probably need to include some explanation with your example (but it's fine if your example is a drawing of a graph).

(c) If it is universal statement, you have more work to do! For each statement, on your scrap paper, carefully write the first and the last sentence of a proof.

(d) Follow that by writing any relevant definitions (working from both the front and the back).

(e) Make the two ends meet. (Note that these are very straightforward, so writing the definitions will already be almost enough to make the ends meet. Don't be surprised if the proofs are short!)

(f) Once you have written a proof, see if you can make any edits that would improve it - make it more clear, make it less wordy, make the notation better, etc.

(g) Copy the proof over on your "official" paper, but save the scrap work to turn in.

REMEMBER: NO GOOGLING FOR SOLUTIONS!

Question 7. If H is a subgraph of a graph G and G contains distinct nonadjacent vertices u and v, then H is a subgraph of $G + uv$.

Question 8. There exists a graph G with an induced subgraph H such that H is not an induced subgraph of $G + uv$.

Question 9. There is a graph with order at least 6 that has diameter 1.

Question 10. If every component of G is bipartite, then G is bipartite. (Hint: Notice the definition of bipartite has both an existential and a universal quantifier. When verifying the definition, for the existential part, you'll have to describe an example. For the universal part, you'll have to do what you always do when you are proving a universal statement: show it is true for an arbitrary selection from the domain.)

Question 11. Suppose G is a connected graph with $u, v \in V(G)$. Then diam$(G + uv) \leq$diam(G).

The following challenge questions are intended only for the people who feel REALLY comfortable with their logic and proofs. Instead of the assignment above, you can complete questions 1 and 2 plus these three challenge questions. Write formal proofs; you can use any method. Remember to turn in your scrap paper. If you have two of them (mostly) correct and some scrap paper ideas for the third one, that will still earn you full credit.

Challenge Question A. Prove that for all connected graphs G, if P and Q are two longest paths in G, then there is some vertex v that is in both P and Q.

Challenge Question B. Suppose that G is a connected graph that is not a complete graph. Prove that every vertex of G belongs to an induced P_3 subgraph of G (that is, an induced subgraph that is isomorphic to P_3).

Challenge Question C. Suppose G is a graph with vertices x and y where $x \neq y$. What is the minimum order of a connected subgraph of G containing both x and y? (Note that the answer isn't something like "7". Instead, the answer is in terms of some quantity in G, like the degree of x, or $\frac{1}{2}|V(G)|$, or something like that.) Once you have decided what the minimum order is, *prove* your result.