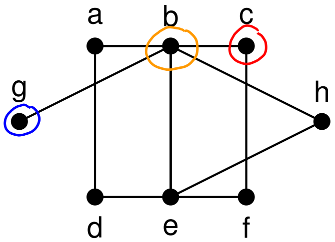
Section 2.1: The Degree of a Vertex

Some definitions:

- The *degree* of a vertex, denoted $\deg(v)$ or $d(v)$ (or sometimes $\deg_G(v)$, when the graph in question is unclear), is the number of neighbors v has.
- The *minimum degree* of a graph G, denoted $\delta(G)$, is the minimum degree among the vertices of G.
- The *maximum degree*, denoted $\Delta(G)$, is the maximum degree among the vertices of G.



$d(c) = 2$

$\delta(G) = 1$

$\Delta(G) = 5$

A theorem for you to prove

The Handshake Theorem.

If G is a graph with size m, then

Total degree of G

$$\underbrace{\sum_{v \in V(G)} \deg(v)}_{\text{Total degree of } G} = 2m.$$

* For practice:
Try proving this by induction on m!

Proof: Let G be a graph with m edges. Note that each edge in G is incident on exactly two vertices, so each edge contributes one to the degree of two distinct vertices. Hence each edge contributes two to $\sum_{v \in V(G)} \deg(v)$. Since each edge also contributes two to $2m$,

$$\sum_{v \in V(G)} \deg(v) = 2m.$$

An easy consequence

Corollary. No graph can have an odd number of vertices of odd degree.

Proof: The sum of the vertex degrees is $2m$, so it is even. Hence the number of odd numbers in the sum must be even.

An application of graphs!

Say I have 15 students in one of my classes (with the delightful names A, B, C, and so on), and I want everyone to pair up and conduct interviews with exactly three other students. (So pairs interview each other.) Use a graph to model how I could assign those interviews to the students.

15 students $\longrightarrow$ 15 vertices

edges $\longrightarrow$ interview between a pair

degree of each vertex = 3

Problem! 15 vertices with odd degree is impossible!

Relating degree to connectivity

What degree condition could we put on a graph G to guarantee that G is connected?

Conjectures:

1. If $\delta(G)=0$ and $G \neq K_1$, then G is not connected. True!

2. If total degree $=3k+1$ for some k, then G is connected.
   False: Let $G = K_2 \cup K_2$

3. If total degree $\geq \frac{(n-1)(n-2)}{2}+1$, then G is connected.
   Also false: $K_6 \cup K_6$

4. If $\delta(G)=n-1$, then G is connected. True: G is K_n

5. If $|E(G)|<n-1$, then G is not connected. True!

6. If $\Delta(G)=n-1$, then G is connected. True!