Theorem: Convergence of a Power Series

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.

2. There exists a real number $R > 0$ such that the series converges absolutely for

$$|x - c| < R$$

and diverges for

$$|x - c| > R$$

3. The series converges absolutely for all x.

The number R is the **radius of convergence** of the power series. If the series converges only at c, then the radius of convergence is $R = 0$. If the series converges for all x, then the radius of convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

Proof: (for $\sum a_n x^n$ cetered at $x = 0$

By the completeness property: If a non-empty set S of real numbers has an upper bound, then it must have a least upper bound.

It must be shown that if a power series $\sum a_n x^n$ converges at $x = d$,$d \neq 0$, then it converges for all b satisfying $|b| < |d|$. Because $\sum a_n x^n$ converges, $\lim_{n\to\infty} a_n d^n = 0$. So, there exists an integer $N > 0$ such that $|a_n d^n| < 1$ for all $n > N$.

Then for $n > N$,

$$|a_n b^n| = \left|a_n b^n \frac{d^n}{d^n}\right| = \left|a_n d^n\right|\left|\frac{b^n}{d^n}\right| < \left|\frac{b^n}{d^n}\right|$$

So, for $|b| < |d|$, $|\frac{b}{d}| < 1$, which implies that

$$\sum\left|\frac{b^n}{d^n}\right| = \sum\left|\frac{b}{d}\right|^n$$

is a convergent geometric series. By the Comparison Test, the series $\sum a_n b^n$ converges.

Similarly, if the power series $\sum a_n x^n$ diverges at $x = b$, $b \neq 0$ then it diverges for all d satisfying $|d| > |b|$. If $\sum a_n d^n$ converged, then the argument above would imply that $\sum a_n b^n$ converged as well.

To prove the theorem: suppose that neither Case 1 nor Case 3 is true. Then there exists points b and d such that $\sum a_n x^n$ converges at b and diverges at d. Let $S = \{x : \sum a_n x^n \quad \text{converges}\}$.$S$ is non-empty becasue $b \in S$. If $x \in S$, then $|x| \leq |d|$, which shows that $|d|$ is an upper bound for the non-empty set S.

By the completeness property, S has a least upper bound, R. If $|x| > R$, then $x \notin S$, so $\sum a_n x^n$ diverges. If $|x| < R$, then $|x|$ is not and upper bound for S, so there exists b in S satisfying $|b| > |x|$. Because $b \in S$, $\sum a_n b^n$ converges, $\Longrightarrow \sum a_n x^n$ converges.