Determinant of a Matrix Product

If A and B are square matrices of order n, then $det(AB) = det(A)\, det(B)$

Proof

To begin, observe that if E is an elementary matrix, then by the Theorem of Elementary Row Operations and Determinants, the next three statements are true. If you obtain E from I by interchanging two rows, then $|E| = -1$. If you obtain E by multiplying a row of I by a nonzero constant c, then $|E| = c$. If you obtain E by adding a multiple of one row of I to another row of I, the $|E| = 1$. Additionaly by Theorem of Representing Elementary Row Operations, if E results from performing an elementary row operation on I and the same elementary row operations is performed on B, then the matrix EB results. It follows that $|EB| = |E||B|$.

This can be generalized to conclude that $|E_k \cdots E_2 E_1 B| = |E_k| \cdots |E_2||E_1||B|$, where E_i is an elementary matrix. Now consider the matrix AB If A is *nonsingular*, then, by Theorem Property of Invertible Matrices, it can be written as the product $A = E_k \cdots E_2 E_1$, and

$$\begin{aligned} |AB| &= |E_k \cdots E_2 E_1 B| \\ &= |E_k| \cdots |E_2||E_1||B| \\ &= |E_k \cdots E_2 E_1||B| \\ &= |A||B| \end{aligned}$$

If A is *singular*, then A is row-equivalent to a matrix with an entire row of zeros. From Theorem Conditions That Yield a Zero Determinant, $|A| = 0$. Moreover, it follows that AB is also singular. (If AB were nonsingular, then $A\big[B(AB)^{-1}\big] = I$ would imply that A is nonsingular.) So, $|AB| = 0$, and you can conclude that $|AB| = |A||B|$.