Theorem: Ration Test

Let $\sum a_n$ be a series with non-zero terms.

1. The series $\sum a_n$ converges absolutely when

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$$

2. The series $\sum a_n$ diverges when

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1 \ \ or \ \ \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$$

3. The Ratio Test is inconclusive when

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$$

Proof:

Assume that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$$

and choose R such that $0 \leq r < R < 1$. By the definition of the limit of a sequence, there exists some $N > 0$ such that $|\frac{a_{n+1}}{a_n}| < R$ for all $n > N$. Therefore, you can write the following inequalities.

$$|a_{N+1}| < |a_N|R$$

$$|a_{N+2}| < |a_{N+1}|R < |a_N|R^2$$

$$|a_{N+3}| < |a_{N+2}|R < |a_{N+1}|R^2 < |a_N|R^3$$

$$\vdots$$

The geometric series $\sum_{n=1}^{\infty} |a_N|R^n = |a_N|R + |a_N|R^2 + \ldots + |a_N|R^n + \ldots$ converges, and so, by the Direct Comparison Test, the series

$$\sum_{n=1}^{\infty} |a_{N+n}| = |a_{N+1}| + |a_{N+2}| + \ldots + |a_{N+n}| + \ldots$$

also converges. This in turn implies that the series $\sum |a_n|$ converges, because discarding a finite number of terms ($n = N - 1$) does not affect convergence. Consequently, by Theorem 9.16, the series $\sum a_n$ converges absolutely.