Integration on Curves

Definition of a Smooth Curve

If P is a curve of the form

$$P(t) = \Big(x(t), y(t), z(t)\Big)$$

for $a \le t \le b$ then we say that P is a **smooth curve** in R^3 if its velocity P' is continuous at every number t in $[a, b]$. The same sort of definition can be given for smooth curves in the plane R^2.

Integrals of the Type $\int_P f dx, \int_P f dy$, and $\int_P f dz$

We suppose that P is a smooth curve of the form

$$P(t) = \Big(x(t), y(t), z(t)\Big)$$

for $a \le t \le b$ and this curve runs in a region of space on which a given function f is continuous. We define the integral $\int_P f dx$ by the equation

$$\int_P f dx = \int_a^b f\Big(x(t), y(t), z(t)\Big)x'(t)dt$$

Thus

$$\int_P f dx = \int_a^b f\Big(x(t), y(t), z(t)\Big)x'(t)dt = \int_a^b f\Big(P(t)\Big)x'(t)dt$$

and, in the same way, we define

$$\int_P f dy = \int_a^b f\Big(x(t), y(t), z(t)\Big)y'(t)dt = \int_a^b f\Big(P(t)\Big)y'(t)dt$$

and

$$\int_P f dz = \int_a^b f\Big(x(t), y(t), z(t)\Big)z'(t)dt = \int_a^b f\Big(P(t)\Big)z'(t)dt$$

Example

We take $f(x, y, z) = xy + (xy + 4z)^2$ and

$$P(t) = (t, t^2, t^3)$$

for $1 \le t \le 7$

$$\int_P f dx = \int_1^7 \Big(tt^2(tt^2 + 4t^3)^2\Big)1dt = \frac{20,592,750}{7}$$
$$\int_P f dy = \int_1^7 \Big(tt^2(tt^2 + 4t^3)^2\Big)2dt = \frac{180,183,612}{5}$$
$$\int_P f dz = \int_1^7 \Big(tt^2(tt^2 + 4t^3)^2\Big)3dt$$

Integrals of the Type $\int_P F \cdot dP = \int_P F \cdot (dx, dy, dz) = \int_P f dx + g dy + h dz$

If P is a smooth curve of the form

$$P(t) = \Big(x(t), y(t), z(t)\Big)$$

for $a \leq t \leq b$ and if this curve runs in a region of sace on which a given vector field $F = (f, g, h)$ is continuous, then we define

$$\begin{aligned}
\int_P \cdot dP &= \int_P F \cdot (dx, dy, dz) \\
&= \int_P (f, g, h) \cdot (dx, dy, dz) \\
&= \int_P f dx + g dy + h dz \\
&= \int_a^b f\Big(P(t)\Big)x'(t)dt + \int_a^b g\Big(P(t)\Big)y'(t)dt + \int_a^b h\Big(P(t)\Big)z'(t)dt \\
&= \int_a^b \Bigg(f\Big(P(t)\Big)x'(t) + g\Big(P(t)\Big)y'(t) + h\Big(P(t)\Big)z'(t)\Bigg)dt \\
&= \int_a^b F\Big(P(t)\Big) \cdot P'(t)dt
\end{aligned}$$

Application to Work Done by a Force

Integrals of the type

$$\int_P F \cdot dP$$

are tailor-made for the description of work done by a force as it pushe a particle along a curve.

We suppose that F is a force.

If the domain of P runs from $t = a$ to $t = b$ then

$$\begin{aligned}
\int_P F \cdot dP &= \int_1^b F\Big(P(t)\Big) \cdot P'(t)dt \\
&= \int_a^b \Big|\Big|F\Big(P(t)\Big)\Big|\Big|\Big|\Big|P'(t)\Big|\Big| \cos\theta(t)dt
\end{aligned}$$

This is the amount of work that F does as it pushes along the entire curve P.

Examples of Integrals on Smooth Curves

Example 1

In this example we consider the curve P defined as

$$P(t) = (t, 3t, t^2)$$

for $0 \leq t \leq 1$

So in this example we have

$$\begin{aligned} x(t) &= t \\ y(t) &= 3t \\ z(t) &= t^2 \end{aligned}$$

for each t

If

$$F(x, y, z) = (xy, yz, zx)$$

then

$$\begin{aligned} \int_P F \cdot dP &= \int_0^1 (t3t, 3tt^2, t^2t) \cdot (1, 3, 2t) dt \\ &= \int_0^1 \left(3t^2 1 + 3t^3 3 + t^3 2t\right) dt \\ &= \frac{73}{20} \end{aligned}$$

Example 2
In this example, we want to work out the integral

$$\int_P F \cdot dP$$

given that

$$F(x, y, z) = \Big(\frac{-y}{x^2 + y^2}, \frac{-x}{x^2 + y^2}, 0\Big)$$

for each point (x, y, z) that is not on the z-axis and given that

$$P(t) = \Big(\cos(t), \sin(t), 3\Big)$$

for $0 \leq t \leq 2\pi$

$$\begin{aligned}
\int_P F \cdot dP &= \int_0^{2\pi} \left(\frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2(t)}, 0\right)\Big(-\sin(t), \cos(t), -3\sin(3t)\Big)dt \\
&= \int_0^{2\pi} \left(\frac{-\sin(t)}{1}, \frac{\cos(t)}{1}, 0\right)\Big(-\sin(t), \cos(t), -3\sin(3t)\Big)dt \\
&= \int_0^{2\pi} (-\sin(t), \cos(t), -3\sin(3t)dt \\
&= 2\pi
\end{aligned}$$

Example 3
In this example we shall work out each of the two integrals $\int_P F \cdot dP$ and $\int_Q F \cdot dQ$ given that

$$F(x, y, z) = (xy, yz, zx)$$

for each point (x, y, z) and that

$$P(t) = (t, t^2, t^3)$$

and

$$Q(t) = (t, t, t)$$

for $0 \leq t \leq 1$
The message of this example is that an integral of the type $\int_P F \cdot dP$ depends on the actual formula for $P(t)$ and that it is not good enough to know only where the curve P begins and where it ends.

$$\begin{aligned}
\int_P F \cdot dP &= \int_0^1 (tt^2, t^2t^3, t^3t) \cdot (1, 2t, 3t^2)dt \\
&= \int_0^1 (5t^6 + t^3)dt = \frac{27}{28}
\end{aligned}$$

$$\begin{aligned}
\int_Q F \cdot dQ &= \int_0^1 (tt, tt, tt) \cdot (1, 1, 1)dt \\
&= \int_0^1 3t^2 dt = 1
\end{aligned}$$

The Fundamental Theorem of Integrals of the Type $\int_P F \cdot dP$

We suppose that F is a conservative vector fied of the form

$$F(x, y, z) = \Big(f(x, y, z), g(x, y, z), h(x, y, z)\Big)$$

on a set of points (x, y, z) in a region in R^3 and we suppose that P is a curve in this regionthat has the form

$$P(t) = \Big(x(t), y(t), z(t)\Big)$$

for $a \leq t \leq b$

We are assuming that there is a real function v whose gradient is F

We are assuming that there is a function v such that

$$\begin{aligned} \Big(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\Big) &= F \\ \Big(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial z}\Big) &= (f, g, h) \end{aligned}$$

and this means that

$$\begin{aligned} \frac{\partial v}{\partial x} &= f \\ \frac{\partial v}{\partial y} &= g \\ \frac{\partial v}{\partial z} &= h \end{aligned}$$

The fundamental theorem now tells us that, if we choose a potential v for F, then

$$\int_P F \cdot dP = v\Big(P(b)\Big) - v\Big(P(a)\Big)$$

Proof

We define

$$\varphi(t) = v\Big(P(t)\Big) = v\Big((x(t), y(t), z(t)\Big)$$

for $a \leq t \leq b$ and we see from the chain rule that

$$\varphi'(t) = \frac{\partial v}{\partial x}x'(t) + \frac{\partial v}{\partial y}y'(t) + \frac{\partial v}{\partial z}z'(t)$$

In other words, $\varphi(t)$ is an antiderivative of

$$\frac{\partial v}{\partial x}x'(t) + \frac{\partial v}{\partial y}y'(t) + \frac{\partial v}{\partial z}z'(t)$$

and so

$$\begin{aligned} \int_P F \cdot dP &= \int_a^b \Big(\frac{\partial v}{\partial x}x'(t) + \frac{\partial v}{\partial y}y'(t) + \frac{\partial v}{\partial z}z'(t)\Big)dt \\ &= \Big[\varphi(t)\Big]_a^b \\ &= \varphi(b) - \varphi(a) \\ &= v\Big(P(b)\Big) - v\Big(P(a)\Big) \end{aligned}$$

Path Independence of Integrals of Conservative Fields

Path Independence

We suppose that F is conservative vector field on a region Ω in sace and that P and Q are both curves in Ω both running from a point A to a point B. Choose a potential v for F.

$$\int_P F \cdot dP = v(B) - v(A) = \int_Q F \cdot dQ$$

What if the Curve is Closed?

We call a curve P **closed** if it begins and ends at the same point.

We suupose that F is conservative vector field on q region Ω in space and that P is a **closed** curve in Ω

Integration of a Function of Two Variables

Iterated Integrals with Constant Limits of Integration

Suppose that we are given $a \leq b$ and $c \leq d$ and that $f(x, y)$ is defined whenever $a \leq x \leq b$ and $c \leq y \leq d$

Integrals of the Type $\int_a^b \int_c^d f(x, y)dydx$ and $\int_c^d \int_a^b f(x, y)dxdy$

What should we mean by $\int_a^b \int_c^d f(x, y)dydx$
we mean

$$\int_a^b \left(\int_c^d f(x, y)dy \right) dx$$

What should we mean by

$$\int_c^d \int_a^b f(x, y)dxdy$$

We mean

$$\int_c^d \left(\int_a^b f(x, y)dx \right) dy$$

More General Iterated Integrals

We could have an iterated integral of the form

$$\int_c^d \int_{a(y)}^{b(y)} f(x,y)dxdy$$

We could also have an integral of the form

$$\int_a^b \int_{c(y)}^{d(y)} f(x,y)dxdy$$

The expression

$$\int_{c(y)}^{d(y)} \int_a^b f(x,y)dxdy$$

is meaningless.

The Fichtenholz Theorem

The Fichtenholz Theorem tells us that the equation

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dy \right) dx$$

will always hold when the limits of integration are constant and the integrals are ordinary Riemann integrals.

In the event that the integrals may be improper, the equation

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dy \right) dx$$

still holds if the function f is nonnegative and also holds if the integrals are absolutely convergent.

Integrals over Regions in R^1

Integrating on an Interval

We assume that a and b are real numbers and that $a \le b$

We suppose that f is a function defined on R and that $[a.b]$ is a given closed interval and we look at

$$f_{[a,b]}(x) = \begin{cases} f(x), & \text{if } a \le x \le b \\ 0 & \text{if } x < a \text{ or } x > b \end{cases}$$

Note that

$$\int_{-\infty}^{\infty} f_{[a,b]}(x)dx = \int_{-\infty}^{a} 0dx + \int_a^b f(x)dx + \int_b^{\infty} 0dx = \int_a^b f(x)dx$$

The General Case of a Region in R^1

If S is a region of numbers x in R^1 and f is a given function defined on S then we introduce a new fuction f_S, called **truncate to S of f** and which we define by the equation

$$f_S = \begin{cases} f(x) & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

We now define

$$\int_S f(x)dx = \int_{-\infty}^{\infty} f_S(x)dx$$

This integral looks improper but, as you will see from the examples that follow, it doesn't have to be improper.

When $S = [a, b]$ we are saying that

$$\int_{[a,b]} f(x)dx = \int_a^b f(x)dx = \int_{-\infty}^{\infty} f_S(x)dx$$

Another Example

We take

$$S = [1, 3] \cup [4, 7]$$

$$\begin{aligned} \int_S f(x)dx &= \int_{-\infty}^{1} 0dx + \int_1^3 f(x)dx + \int_3^4 0dx + \int_4^7 f(x)dx + \int_7^{\infty} 0dx \\ &= \int_1^3 f(x)dx + \int_4^7 f(x)dx \end{aligned}$$

Integrals over Regions in R^2

If S is a region of numbers x in R^2

and f is a given function defined on S then we introduce a new function f_S, called **truncate to S of f** and which we define by the formula

$$f_S(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in S \\ 0 & \text{if } x \notin S \end{cases}$$

We now define the **double integral** $\int\int_S f(x, y)d(x, y)$ by the equation

$$\int\int_S f(x, y)d(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_S(x, y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_S(x, y)dydx$$

This integral looks improper but, as you will see from the exercises that follow, it doesn't have to be improper.

Approximating Double Integrals by Sums

The ability to approximate double integrals with sums that we are about to discuss plays an important role in the applications of these integrals in mathematics, science, economics and technology. The principla theorem that tells how to approximate a double integral with sums is known as **Darboux's theorem** and this theorem is an analogue for double integrals of the Darboux theorem that appears in the optional Section 6.4.

We shall not attempt to state Darboux's theorem precisely. We shall be content to say that, if f is a continuous function on a region S which is partitioned into non-overlapping subregions $S_1, S_2, S_3, \ldots, S_n$ and if we choose a point (x_j, y_j) in each subregion S_j

then we can make the sum

$$\sum_{j=1}^{n} f(x_j, y_j) area(S_j)$$

as close as we like to the integral

$$\int\int_S f(x,y)d(x,y)$$

by taking a small enough number r and requiring that every one of the sets $S_1, S_2, S_3, \ldots, S_n$ is small enough to fit inside a disk with radius r.

The Gamma and Beta Functions

Introducing the Gamma Function

Definition of the Gamma Function

If a is any positive number then we define the value $\Gamma(a)$ of the **gamma function** at a to be improper integral

$$\int_0^\infty a^{a-1}e^{-x}dx$$

Some Examples to Illustrate the Gamma Function

We work out $\Gamma(1)$

$$\Gamma(1) = \int_0^\infty 1e^{-x}dx = \Big[-e^x\Big]_0^\infty = -(-1) = 1$$

We work out $\Gamma(2)$

$$\begin{aligned}\Gamma(2) &= \int_0^\infty x^{2-1}e^{-x}dx \\ &= \int_0^\infty xe^{-x}dx \\ &= \int_0^\infty (x)\Big(\frac{d}{dx}(-e^{-x})\Big)dx \\ &= \Big[(x)(-e^{-x})\Big]_0^\infty - \int_0^\infty \Big(\frac{d}{dx}x\Big)\Big((-e^{-x})\Big)dx \\ &= 0 - 0 - \int_0^\infty 1\Big((-e^{-x})\Big)dx = 1\end{aligned}$$

We work out $\Gamma(3)$

$$\begin{aligned}
\Gamma(3) &= \int_0^\infty x^{3-1}e^{-x}dx = \int_0^\infty x^2e^{-x}dx \\
&= \int_0^\infty (x^2)\Big(\frac{d}{dx}(-e^{-x})\Big)dx \\
&= \Big[(x^2)(-e^{-x})\Big]_0^\infty - \int_0^\infty \Big(\frac{d}{dx}x^2\Big)\Big((-e^{-x})\Big)dx \\
&= 0 - 0 + 2\int_0^\infty xe^{-x}dx = 1 \\
&= (2)\Gamma(2) = (2)(1)
\end{aligned}$$

We work out $\Gamma(4)$

$$\begin{aligned}
\Gamma(4) &= \int_0^\infty x^{4-1}e^{-x}dx = \int_0^\infty x^3e^{-x}dx \\
&= \int_0^\infty (x^3)\Big(\frac{d}{dx}(-e^{-x})\Big)dx \\
&= \Big[(x^3)(-e^{-x})\Big]_0^\infty - \int_0^\infty \Big(\frac{d}{dx}x^3\Big)\Big((-e^{-x})\Big)dx \\
&= 0 - 0 + 3\int_0^\infty x^2e^{-x}dx = 1 \\
&= (3)\Gamma(3) = (3)(2)(1)
\end{aligned}$$

If n is any positive integer, then

$$\Gamma(n) = (n-1)!$$

Some Elementary Facts about the Gamma Function

The Recurrence Formula

If a is any positiv number then

$$\Gamma(a+1) = a\Gamma(a)$$

Proof

$$\begin{aligned}
\Gamma(a+1) &= \int_0^\infty x^{a+1-1}e^{-x}dx \\
&= \int_0^\infty (x^a)\Big(\frac{d}{dx}(-e^{-x})\Big)dx \\
&= \Big[(x^a)(-e^{-x})\Big]_0^\infty - \int_0^\infty \Big(\frac{d}{dx}x^a\Big)(-e^{-x})dx \\
&= 0 - 0 + \int_0^\infty ax^{a-1}e^{-x}dx \\
&= a\Gamma(a)
\end{aligned}$$

The Gamma Function and Factorials

The fact that $\Gamma(1) = 1$ and the recurrence formula $\Gamma(a+1) = a\Gamma(a)$ allows us to see that, if n is any positive integer, then

$$\Gamma(n) = (n-1)!$$

The Substitution $x = t^2$

In the definition

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$$

we make the substitution $x = t^2$ (same as $t = \sqrt{x}$) and get $dx = 2tdt$

$$\begin{aligned}\Gamma(a) &= \int_0^\infty x^{a-1}e^{-x}dx \\ &= \int_0^\infty (t^2)^{a-1}e^{-t^2}2tdt \\ &= 2\int_0^\infty t^{2a-1}e^{-t^2}dt\end{aligned}$$

An Attempt Again to Find the Value of $\Gamma(\frac{1}{2})$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty t^{2\frac{1}{2}-1}e^{-t^2}dt = 2\int_0^\infty e^{-t^2}dt$$

and we are stuck. This integral is hard but very important, especially in statistics.

The Beta Function

Definition of the Beta Function

The beta function is a function of two variables whose domain is the first quadrant. Whenever a and b are positive number, we define

$$\beta(a.b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$$

Some Examples to Illustrate the Beta Function

$$\beta(2,3) = \int_0^1 t^1(1-t)^2dt = \frac{1}{12}$$

Some Elementary Facts About the Beta Function

Symmetry of the Beta Function

We suppose that a and b are positive numbers.

The Substitution $u = ct$

We suppose that a and b are positive numbers.

If q is any positive number, we can make the substitution $u = qt$ in the integral

$$\beta(a,b) = \int_0^1 ta - 1(1-t)^{b-1} dt$$

This change of variable gives us

$$\begin{aligned}\beta(a,b) &= \int_0^q \left(\frac{u}{q}\right)^{a-1} \left(1 - \frac{u}{q}\right)^{b-1} \frac{1}{q} du \\ &= \int_0^q \left(\frac{u}{q}\right)^{a-1} \left(\frac{q-u}{q}\right)^{b-1} \frac{1}{q} du \\ &= \frac{1}{q^{a-1+b-1+1}} \int_0^q u^{a-1}(q-u)^{b-1} du\end{aligned}$$

In other words

$$\frac{1}{q^{a+b-1}} \int_0^q u^{a-1}(q-u)^{b-1} du$$

The Substitution $t = \sin^2(\theta)$

We suppose that a and b are positive numbers.

If we make the substitution $\theta = arc\sin(\sqrt{t})$ which gives us $t = \sin^2(\theta)$ and $dt = 2\sin(\theta)\cos(\theta)d\theta$, then we see that

The Value of $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^{2(\frac{1}{2})-1}(\theta)\cos^{2(\frac{1}{2})-1}(\theta) d\theta = \pi$$

The Relationship $\Gamma(a)\Gamma(b) = \Gamma(a+b)\beta(a,b)$

Using This Relationship to Find $\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}+\frac{1}{2}\right)\beta\left(\frac{1}{2},\frac{1}{2}\right) = 1\pi$$

and so

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and now I can tell you from the equation

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty t^{2\left(\frac{1}{2}\right)-1}e^{-t^2}dt$$

that

$$\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$$

Proving the Equation $\Gamma(a)\Gamma(b) = \Gamma(a+b)\beta(a,b)$

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \Gamma(a)\int_0^\infty y^{b-1}e^{-y}dy \\ &= \int_0^\infty \Big(\Gamma(a)\Big)y^{b-1}e^{-y}dy \\ &= \int_0^\infty \Big(x^{a-1}e^{-x}dx\Big)y^{b-1}e^{-y}dy \\ &= \int_0^\infty \Big(\int_0^\infty x^{a-1}e^{-x}y^{b-1}e^{-y}\Big)dy \\ &= \int_0^\infty \int_0^\infty x^{a-1}e^{-x}y^{b-1}e^{-y}dxdy\end{aligned}$$

and this is a repeated integral. In the inside we substitute $u = x+y$ giving $x = u-y$ and $dx = 1du$ and we notice that, as x runs from 0 to ∞, the new variable u runs from y to ∞.

Therefore

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \int_0^\infty \int_0^\infty (u-y)^{a-1}e^{-u+y}y^{b-1}e^{-y}1dudy \\ &= \int_0^\infty \int_0^\infty (u-y)^{a-1}y^{b-1}e^{-u}dudy\end{aligned}$$

Now we interchange the order of integration and we see that

$$\begin{aligned}\Gamma(a)\Gamma(b) &= \int_0^\infty \int_0^\infty (u-y)^{a-1}e^{-u+y}y^{b-1}e^{-y}1dydu \\ &= \int_0^\infty e^{-u}\int_0^\infty (u-y)^{a-1}y^{b-1}dy\end{aligned}$$

Now recall the equation

$$\beta(a,b) = \frac{1}{q^{a+b-1}}\int_0^q u^{a-1}(q-u)^{b-1}du$$

and so

$$\Gamma(a)\Gamma(b) = \int_0^\infty e^{-u}u^{a+b-1}\beta(a,b) = \beta(a,b)\Gamma(a+b)$$

A Hard Fact About Gamma Functions

Whenever $0 < a < 1$, we have the equation

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

$$\begin{aligned}\Gamma\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big) &= \frac{\pi}{\sin(\pi 2)} = \pi \\ \Gamma\Big(\frac{1}{4}\Big)\Gamma\Big(\frac{1}{4}\Big) &= \frac{\pi}{\sin(\pi 4)} = \pi\sqrt{2} \\ \Gamma\Big(\frac{1}{3}\Big)\Gamma\Big(\frac{1}{3}\Big) &= \frac{\pi}{\sin(\pi 3)} = \frac{2\pi}{\sqrt{3}}\end{aligned}$$

Polar Coordinates

Motivating the Change to Polars

We suppose that S is a region of points (x, y) in R^2 and that we change to polar coordinates

$$x = 4\cos(\theta)$$
$$y = r\sin(\theta)$$

and that (x, y) runs once through the region S as (r, θ) runs through a region S^* in the r, θ plane.

The area of the yellow region is called E is

$$(r_2 - r_1)(\theta_2 - \theta_1)$$

This is r_1 times the area of the yellow rectangle called E^*

$$\int_S \int f(x,y)d(x,y) = \int \int_{S^*} f\Big(r\cos(\theta), r\sin(\theta)\Big) r d(r,\theta)$$

Integration of a Function of Three Variables

Iterated Integrals with Constant Limits

Suppose that we are given $a_1 \le a_2$ and $b_1 \le b_2$ and $c_1 \le c_2$, and that $f(x, y, z)$ is defined whenever $a_1 \le x \le a_2$ and $b_1 \le y \le b_2$ and $c_1 \le z \le c_2$

The iterated integral

$$\int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x,y,z)dzdydx = \int_{a_1}^{a_2} \left(\int_{b_1}^{b_2} \left(\int_{c_1}^{c_2} f(x,y,z)dz \right) dy \right) dx$$

More General Iterated Integrals

An example of a more general iterated integrals is

$$\int_{a_1}^{a_2} \int_{b_1(x)}^{b_2(x)} \int_{a_3(x,y)}^{b_3(x,y)} f(x,y,z)dzdydx = \int_{a_1}^{a_2} \left(\int_{b_2(x)}^{b_2(x)} \left(\int_{a_3(xy)}^{b_3(x,y)} f(x,y,z)dz \right) dy \right) dx$$

Definition of the Integral over a Region in R^3

By analogy with our definition given in Section 12.2.8 of an integral over a region in R^2, we define the **truncate** f_S of a function to a region S in R^3 by the formula

$$f_S(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in S \\ 0 & \text{if } (x.y.z) \notin S \end{cases}$$

and we define the **triple integral** $\int\int\int_S f(x,y,z)d(x,y,z)$ by the equation

$$\int\int\int_S f(x,y,z)d(x,y,z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_S(x,y,z)dxdydz$$

and we can replace the iterated integrals

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_S(x,y,z)dxdydz$$

by any one of the five integrals that can be obtained by changing the order of integration.

Darboux's Theorem

As we said in our discussion of **Darboux's Theorem** for integrals over regions in R^2, we shall not attempt to state Darboux's Theorem precisely. We shall be content to say that, of f is a continuous function on a region S in R^3 which is partitioned into non-overlapping subregions $S_1.S_2.S_3, \ldots, S_n$ and if we choose a point (x_j, y_j, z_j) in each subregion S_j,

then we can make the sum

$$\sum_{j=1}^{n} f(x, y, z) volume(S_j)$$

as close as we like to the integral

$$\int\int\int_S f(x, y, z) d(x, y, z)$$

by taking a small enough number r and requiring that every one of the sets $S_1, S_2.S_3. \ldots, S_n$ is small enough to fit inside a ball with radius r.

Using a Triple Integral to Find Volume

Darboux's theorem allow us to conclude that, if S is a region of points (x, y, z) in R^3 then the colume of the region S can be found by integrating the constant function 1 on S. In other words

$$volume(S) = \int\int\int_S 1 d(x, y, z)$$

Cost of Material

Suppose that $f(x, y, z)$ is the cost in dollars per cubic inch of a region S at each point (x, y, z). The integral

$$\int\int\int_S f(x, y, z) d(x, y, z)$$

is the total value of the solid region S.

Look at a Riemann sum:

$$\int_{j=1}^{n} f(x_j, y_j, z_j) volume(S_j)$$

approximates the total value in dollars of the solid region S.

Introucing the Sysmbols $v_n(r)$

We suppose that r is a positive number.

The Ball in Zero Dimensional Space

All we have is the origin O and ball contains this single point and its zero dimensional "volume" is 1

$$v_0(r) = 1$$

The Ball in One Dimensional Space

This is the interval that runs from $-r$ to r

The one dimensional "volume"' of this one-ball is the length of the interval and this is $2r$

$$v_1(r) = 2r$$

The Ball in Two Dimensional Space

This is the disk inside the circle $x^2 + y^2 = r^2$

And the two dimensional "volume" of this two-ball is πr^2

$$v_2(r) = \pi r^2$$

The Ball in Three Dimensional Space

This is the ball of points (x, y, z) inside the sphere

$$x^2 + y^2 + z^2 = r^2$$

and we have seen the volume of the ball is $\frac{4}{3}\pi r^3$

$$v_3(r) = \frac{4}{4}\pi r^3$$

Spherical Coordinates

Introduction to Spherical Coordinates

In the above figure we are looking at a point $P = (x, y, z)$ that lies above or below the point $Q = (x, y, 0)$ in the x, y plane.

If $(x, y, 0)$ has polar coordinates r and θ then

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

and to make θ unique we shall make $0 \leq \theta < 2\pi$

We now introduce

$$\rho = \sqrt{s^2 + y^2 + z^2}$$

We introduce one more angle φ as shown

$$\rho \geq 0$$
$$0 \leq \theta < 2\pi$$
$$0 \leq \varphi \leq 180^\circ$$

We call ρ and φ and θ **spherical coordinates** for the point P

$$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$$
$$y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$$
$$z = \rho\cos(\varphi)$$

In $\triangle OQP$ we are seeing that

$$\frac{r}{\rho}$$
$$\frac{z}{\rho} = \cos(\theta)$$

$$x = r\cos(\theta) = \rho\sin(\varphi)\cos(\theta)$$
$$y = r\sin(\theta) = \rho\sin(\varphi)\sin(\theta)$$
$$z = \rho\cos(\varphi)$$

Changing Integrals to Spherical Coordinates

The idea of changing an integral of the form

$$\int\int\int_S f(x,y,z)d(x,y,z)$$

to spherical polar coordinates makes sense when we can identify a region that we shall call S^* in ρ, φ, θ space in such a way that the equations

$$\begin{aligned} x &= \rho \sin(\varphi) \cos *\theta) \\ y &= \rho \sin(\varphi) \sin(\theta) \\ z &= \rho \cos(\varphi) \end{aligned}$$

map the region S^* to the region S

The formula for changing to spherical polar coordinates tells us that

$$\int\int\int_S f(x,y,z)d(x,y,z) = \int\int\int_{S^*} f\Big(\rho \sin(\varphi)\cos(\theta), \rho \sin(\varphi)\sin(\theta), \rho\cos(\varphi)\Big)\rho^2 \sin(\varphi) d(\rho,\varphi,\theta)$$

.

Motivating the Change to Spherical Polar Coordinates

The volume of the little box is about

$$(d\rho)(\rho d\varphi)(\rho \sin(\varphi) d\theta) = \rho^2 \sin(\varphi) d\rho d\varphi d\theta$$