Graph Theory, Homework 1
Due Fri. Jan. 24

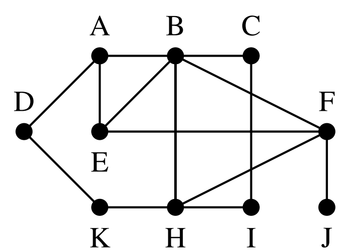
For this homework, you have TWO options:

(a) Complete the assignment as described here (ignoring the challenge questions at the end) - this is what I expect most students will do.

(b) Complete questions 1 and 2, plus the challenge questions at the end - this is for those of you that already feel very confident with your logic and basic proof-writing skills, because I want you to be challenged!

Basic skills

Complete all of the basic skills questions (question 1-5).



Question 1. Use the graph G above to answer the following questions.

(a) What is the order of G? 10 What is the size of G? 15

(b) What is $V(G)$? $\{A, B, C, D, E, F, K, H, I, J\}$

(c) What is $deg(F)$? 4

(d) Find $N(A)$. $\{D, E, B\}$

(e) If $S = \{C, D\}$, find $N[S]$. $\{C, D, B, I, A, K\}$

(f) Is the graph connected? yes

(g) Are vertices D and F adjacent? no

(h) List all the neighbors of vertex F. B, E, H, J

(i) Is $AEFBC$ a walk in this graph? yes

(j) Is $AEFBHFJ$ a path is this graph? no

(k) Give a cycle of length four in the graph. B, C, I, H, B

(l) Give a circuit of length six that is NOT a cycle. B, F, H, B, E, A, B

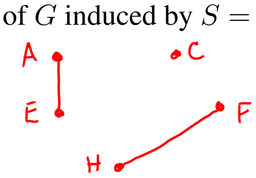
(m) Is C_3 a subgraph of G? yes

(n) What is the diameter of G? 4

(o) Find an $A - J$ geodesic. A, B, F, J

(p) Find an odd cycle in G. B, F, H, B

(q) Draw the subgraph of G induced by $S = \{A, E, H, F, C\}$. Is this graph connected?



not connected

Please complete the remainder of the assignment on a separate piece of paper (or two or three).

Question 2. Answer the following questions based on the course syllabus. (Just making sure we are all on the same page!)

(a) What do you need to do if you want to be excused from class? *Email me and let me know you are missing for an excused reason. Before the next class meeting, look at the notes from the day you missed and email me a question about them.*

(b) How many participation points can you lose before your your grade is affected? *Five, which means two unexcused absences and one time late to class, or one unexcused absence and three times late to class.*

(c) Are you allowed to use the internet to help with your homework? *NO!*

(d) How many warnings will I give you on academic integrity violations before I report you for misconduct? *NONE!*

(e) Do you have any concerns about this class that you would like to share with me? *Please feel free to reach out to me with concerns at any point during the semester!*

Question 3. Rewrite each statement formally, and write its negation. (You don't need to prove or disprove them, so don't worry about what the words mean!)

(a) Every tree has at most one perfect matching.
Rewrite: $\forall$ *trees* T*,* T *has at most one perfect matching.*
Negate: $\exists$ *tree* T *such that* T *has more than one perfect matching.*

(b) If G is a graph with more edges than vertices, then G contains a cycle.
Rewrite: $\forall$ *graphs* G*, if* G *has more edges than vertices, then* G *contains a cycle.*
Negate: $\exists$ *graph* G *such that* G *has more edges than vertices, and* G *does not contain a cycle.*

(c) There are graphs that are Eulerian but not Hamiltonian.
Rewrite: $\exists$ *graph* G *such that* G *is Eulerian and* G *is not Hamiltonian.*
Negate: $\forall$ *graphs* G*,* G *is not Eulerian or* G *is Hamiltonian.*

(d) If G is a connected, r-regular graph and $\overline{G}$ is also connected, then G is Eulerian or $\overline{G}$ is Eulerian. *Rewrite:* $\forall$ *graphs* G*, if* G *is connected and* r*-regular, and* $\overline{G}$ *is connected, then* G *is Eulerian or* $\overline{G}$ *is Eulerian.*
Negate: $\exists$ *graph* G *such that* G *is connected and* r*-regular, and* $\overline{G}$ *is connected, and* G *is not Eulerian and* $\overline{G}$ *is not Eulerian.*

Question 4. Consider the statement, "For any n-vertex graph G, if G is connected, then G is a tree or G has at least n edges.

(a) Write the negation of the statement.
There exists an n*-vertex graph* G *such that* G *is connected, and* G *is not a tree and* G *has less than* n *edges.*

(b) Write the converse of the statement.
For any n*-vertex graph* G*, if* G *is a tree or* G *has at least* n *edges, then* G *is connected.*

(c) Write the contrapositive of the statement.
For any n*-vertex graph* G*, if* G *is not a tree and* G *has less than* n *edges, then* G *is not connected.*

(d) Repeat (a), (b), and (c) with the statement, "If every vertex in a graph G has even degree and G is connected, then G has an Euler circuit."

- *Negation: There exists a graph G such that every vertex in G has even degree and G is connected, and G does not have an Euler circuit.*
- *Converse: For all graphs G, if G has an Euler circuit, then every vertex in G has even degree and G is connected.*
- *Contrapositive: For all graphs G, if G does not have an Euler circuit, then some vertex in G has odd degree or G is not connected.*

Question 5. Describe the BASIC IDEA behind a proof of the following types. (Imagine you are explaining to your classmates how you would write each kind of proof.)

(a) A direct proof of a UNIVERSAL statement
Since we can't directly check all elements of the domain to see if they satisfy the statement (since the domain is usually infinite), we verify the statement for an arbitrary member of the domain, using only the fact that we know the element comes from the domain. If we are proving a universal conditional statement, we also assume the arbitrary element satisfies the hypothesis, since otherwise the conditional statement would be vacuously true.

(b) A proof of an EXISTENTIAL statement
We just need to verify that a witness exists - that the given statement is true for one substitution from the domain.

(c) A proof by contradiction of a UNIVERSAL statement
We assume the given statement is false, that is, that its negation is true. We than use this and other known facts from mathematics to show that something bad happens, like $0 = 1$, or there is something smaller than the smallest thing we picked, or an integer is equal to a non-integer (i.e. we make the universe explode). Since assuming the statement is false gives rise to this impossibility, we conclude that the statement can't be false, and therefore it has to be true.

(d) A proof by contrapositive.
Since a statement is logically equivalent to its contrapositive, we can just complete a direct proof of the contrapositive of the statement instead of proving the given statement.

Question 6. Suppose I wish to prove the statement: "A graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half the vertices." Without worrying about what α and β mean, write (formally) what TWO statements would need to be proven in order to successfully prove this statement. Then write the contrapositive of each statement.
To prove:

- *For all graphs G, if G is bipartite, then every subgraph H of G has an independent set consisting of at least half the vertices.*
  Contrapositive: For all graphs G, if there exists a subgraph H of G in which every independent set consists of less than half the vertices, then G is not bipartite.
- *For all graphs G, if every subgraph H of G has an independent set consisting of at least half the vertices, then G is bipartite.*
  Contraositive: For all graphs G, if G is not bipartite, then there exists a subgraph H of G in which every independent set consists of less than half the vertices.

The fun problems

Question 7. *If H is a subgraph of a graph G containing distinct vertices u and v, then H is a subgraph of $G + uv$.*

Proof. Suppose G is an arbitrary graph containing distinct vertices u and v, and let H be a subgraph of G. Since H is a subgraph of G, H is a graph satisfying $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Since $V(G) = V(G + uv)$, it follows that $V(H) \subseteq V(G + uv)$. Since $E(G) \subseteq E(G + uv)$ and the subset relation is transitive, $E(H) \subseteq E(G + uv)$. Therefore H is a subgraph of $G + uv$ by definition. $\square$

Question 8. *There exists a graph G with an induced subgraph H such that H is not an induced subgraph of $G + uv$.*

Proof. Let G be the graph below. Note that the empty graph with two vertices is an induced subgraph of G, because the subgraph induced by $\{u, v\}$ is an empty graph. However, the empty graph with two vertices is not an induced subgraph of $G + uv$, because any two vertices in $G + uv$ have an edge between them, and the induced subgraph must contain that edge. $\square$

Question 9. *There is a graph with order at least 6 that has diameter 1.*

Proof. K_6 is an example of such a graph. Since there is an edge between every pair of vertices, the distance between any two vertices is 1. Therefore the maximum distance between any two vertices is 1. (Note that K_6 is, in fact, the ONLY example with 6 vertices. In general, K_n is the only graph with diameter 1 for all $n \geq 2$.) $\square$

Question 10. *If every component of G is bipartite, then G is bipartite.*

Proof. Let G be a graph such that every component of G is bipartite. Let $G_1, \ldots, G_t$ be the components of G. By definition, for each G_i, the vertex set $V(G_i)$ can be partitioned into two sets X_i and Y_i such that all edges in G_i have one endpoint in X_i and one endpoint in Y_i. Let $X = \cup_{i=1}^{t} X_i$ and $Y = \cup_{i=1}^{t} Y_i$. This is a partition of $V(G)$, since every $v \in V(G)$ is in some $V(G_i)$. It remains to verify that for every edge $uv \in E(G)$, $u \in X$ and $v \in Y$ (or vice versa). Let uv be an arbitrary edge in G. Assume by symmetry that $u \in X$. Then $u \in X_i$ for some i. Since u is only adjacent to vertices in its component, $v \in V(G_i)$. Since G_i is bipartite, $v \in Y_i$, and hence $v \in Y$. Since uv was arbitrary, G is bipartite by definition. $\square$

Question 11. *Suppose G is a connected graph with $u, v \in V(G)$. Then $diam(G + uv) \leq diam(G)$.*

Proof. Let G be a connected graph with $u, v \in V(G)$. Since the diameter is the maximum distance between any pair of vertices in a graph, it suffices to show that for any $x, y \in V(G + uv)$, $d_{G+uv}(x, y) \leq d_G(x, y)$. Let x and y be arbitrary vertices in $G + uv$. Since $V(G) = V(G + uv)$, x and y are also vertices in $V(G)$. Since G is connected, there is an $x - y$ walk in G. Let W be a shortest $x - y$ walk in G. By the definition of distance, W has length $d_G(x, y)$. Since $E(G) \subseteq E(G + uv)$, W is also an $x - y$ walk in $G + uv$. Hence $d_{G+uv}(x, y)$ is at most the length of W by definition of distance. Thus $d_{G+uv}(x, y) \leq d_G(x, y)$. Since x and y were arbitrary, the distance between any pair of vertices in $G + uv$ is at most the distance in G. Therefore $diam(G + uv) \leq diam(G)$. $\square$