Reviewing Induction

Principle of Mathematical Induction

Suppose $P(k)$ is a property about integers. If:

1. $P(1)$ is true, and
2. The statement, "For all integers $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is true," is true;

then $P(k)$ is true for all $k \geq 1$.

- Two parts to a proof by induction: The *base case* and the *inductive step*.
- PMI also works if we start at $k = 0$, or $k = 5$, etc.
- Notice: Proving the inductive step involves proving a universal conditional statement! Choose an arbitrary element of the domain satisfying the hypothesis.

Our statement:

Theorem

For all $k \geq 1$, if a graph G contains a walk of length k connecting u and v for any distinct $u, v \in V(G)$, then G also contains a path connecting u and v.

Important: Avoid the induction trap

- In the inductive step, we started with a walk of length $k+1$, and we went and found a walk of length k inside it.
- You CANNOT start with the walk of length k, and build up the walk of length $k+1$ from there!
- Why not? Because we have to verify the result for an *arbitrary* walk of length $k+1$, not a walk of length $k+1$ that was built by adding an edge to a walk of length k.
- In this case, it is true that all walks of length $k+1$ can be built by adding an edge to a walk of length k, but that won't always be true!

Always START from the larger object, and FIND the smaller object inside it.

Have you met Strong Induction?

Strong Mathematical Induction

Suppose $P(k)$ is a property about integers. If:

1. $P(1)$ is true, and again, $P(2)$ or $P(0)$ is OK
2. The statement, "For all $k \geq 1$, if $P(i)$ is true for all $i \leq k$, then $P(k+1)$ is true," is true

stronger assumption

then $P(k)$ is true for all $k \geq 1$.

- This is very similar to PMI, but instead of just using the kth case to prove the $(k+1)$st case, we are allowed to use ANY previous case to prove the $(k+1)$st case.
- In other words, in our attempt to prove that $P(k+1)$ holds, we can use the fact the $P(k)$ holds, that $P(k-1)$ holds, etc.

Example

Theorem Every closed odd walk contains an odd cycle.

~~Every circuit of odd length at least 3 contains a cycle.~~

Start by rewriting it as a statement about integers:

$\forall\, k \geq 3$, if G contains a closed walk of length k and k is odd, then that walk contains a cycle.

Alternative: $\forall$ odd $k \geq 3$, if G contains a closed walk of length k, then that walk contains a cycle.

One more: $\forall\, k \geq 1$, if G contains a closed walk of length $2k+1$, then the walk contains a cycle.

What will the *inductive hypothesis* be?

Suppose for some odd $k \geq 3$, closed odd walk of length i, $3 \leq i \leq k$, contains a cycle.

To show: Closed odd walks of length $k+2$ contain cycles.

Theorem

implicit universal quantifier

For all odd $k \geq 3$, if G contains a ~~circuit~~ closed walk of length k, then that ~~circuit~~ walk contains a cycle.

Proof (by induction): For the base case, consider a walk of length 3 in G, say x, y, z, x. Since G does not contain loops, x, y, and z are all distinct. Thus the walk is itself a cycle, and the base case holds.

For the inductive step, suppose for some odd $k \geq 3$ that if a graph G contains a closed walk of odd length i, $3 \leq i \leq k$, then that walk contains a cycle.

Let G be a graph with a closed walk of length $k+2$, say $W = x_1, x_2, \ldots, x_k, x_{k+1}, x_{k+2}, x_1$. We must show that W contains a cycle.

If W does not repeat vertices, then W is itself a cycle, and the result holds. Otherwise, for some $f < g$, $x_f = x_g$. Consider two new closed walks:

$$C_1 = x_1, x_2, \ldots, x_f, x_{g+1}, \ldots, x_{k+2}, x_1$$

$$C_2 = x_f, x_{f+1}, \ldots, x_{g-1}, x_g$$

Note that every edge in W appears in exactly one of $\{C_1, C_2\}$. Let ℓ_1 and ℓ_2 be the length of C_1 and C_2, respectively. Since G has no loops, we know $\ell_1 \geq 2$ and $\ell_2 \geq 2$ Since $\ell_1 + \ell_2$ is odd, either ℓ_1 or ℓ_2 is odd. By the inductive hypothesis, whichever closed walk is odd must contain a cycle C_i and this cycle C is also contained in W.

(Now by the Principle of Strong Mathematical Induction, the result holds.)

Notice we most likely are not using the case where the smaller walk has length k! STRONG induction.

How you might see this proof in a textbook

Theorem

Every closed walk ~~*circuit*~~ *of odd length contains a cycle.*

Proof. Since every closed walk of length 3 is a cycle, the statement holds for closed walks of length 3. Now suppose

$$C = v_1, v_2, \ldots, v_k, v_1$$

is a closed walk of odd length greater than 3. If C is a cycle, we are done. Otherwise, there must be some vertex repeated in C (other than v_1); say $v_i = v_j$ for $i < j$. Consider the two closed walks:

$$C_1 = v_1, v_2, \ldots, v_i = v_j, v_{j+1}, \ldots, v_k, v_1 \text{ and } C_2 = v_i, v_{i+1}, \ldots, v_j = v_i$$

Since their length combines to the length of C, one of them must be odd. By the inductive hypothesis, the odd closed walk contains a cycle, and that cycle is also contained in C.