Example 2: Solve

$$\frac{dy}{dx} = G(ax + by)$$

then the substitution

$$z = ax + by$$

transforms the equation into a separable one.

Solution

The right-hand side can be expressed as a function of $x - y$, that is,

$$y - x - 1 + (x - y + 2)^{-1} = -(x - y) - y + \big[(x - y) + 2\big]^{-1}$$

so let $z = x - y$. To solve for $\frac{dy}{dx}$, we differentiate $z = x - y$ with respect to x to obtain $\frac{dz}{dx} = 1 - \frac{dy}{dx}$, and so $\frac{dy}{dx} = 1 - \frac{dz}{dx}$. Substituting into $\frac{dy}{dx} = G(ax + by)$ yields

$$1 - \frac{dz}{dx} = -z - 1 + (z + 2)^{-1}$$

or

$$\frac{dz}{dx} = (z + 2) - (z + 2)^{-1}$$

Solving this separable equation, we obtain

$$\int \frac{z + 2}{(z + 2)^2 - 1} dz = \int dx$$
$$\frac{1}{2} \ln\big|(z + 2)^2 - 1\big| = x + C_1$$

from which it follows that

$$(z + 2)^2 = Ce^{2x} + 1$$

Finally, replacing z by $x - y$ yields

$$(x - y + 2)^2 = Ce^{2x} + 1$$

as an implicit solution to $\frac{dy}{dx} = G(ax + by)$