Theorem: Direct Comparison Test

Let $0 < a_n \leq b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$

Proof:

Let $L = \sum_{n=1}^{\infty} b_n$ and let

$$S_n = a_1 + a_2 + \ldots + a_n$$

Because $0 < a_n \leq b_n$, the sequence $S_1, S_2, S_3, \ldots$ is non-decreasing and bounced above by L. So, it must converge. Because

$$\lim_{n \to \infty} S_N = \sum_{n=1}^{\infty} a_n$$

it follows that $\sum_{n=1}^{\infty} a_n$ converges. The second property is logically equivalent to the first.