Theorem: Limit Comparison Test

If $a_n > 0$,$b_n > 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L is *finite and positive*, then

$$\sum_{n=1}^{\infty} a_n \;\; and \;\; \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

Proof:

Because $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

there exists $N > 0$ such that

$$0 < \frac{a_n}{b_n} < L + 1, \quad for \;\; n \geq N$$

This implies that

$$0 < a_n < (L+1)b_n$$

So, by the Direct Comparison Test, the convergence of $\sum b_n$ implies the convergence of $\sum a_n$. Similarly, the fact that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{L}$$

can be used to show that convergence of $\sum a_n$ implies the convergence of $\sum b_n$.