Theorem 8.10

A graph G contains a 1-factor *if and only if* $k_o(G-S) \leq |S|$ for every proper subset S of $V(G)$.

Proof:

Assume first that G contains a 1-factor F. Let S be a proper subset of $V(G)$. If $G-S$ has odd components, then $k_o(G-S)=0$ and certainly $k_o(G-S) \leq |S|$. Suppose that $k_o(G-S)=k \geq 1$ and let $G_1, G_2, \ldots, G_k$ be the odd components of $G-S$ (There may also be even components of $G-S$.) Since G contains the 1-factor F and the order of each subgraph of $G_i (1 \leq i \leq k)$ is odd, some edge of F must be incident to both a vertex of G_i and a vertex of S and so $k_o(G-s) \leq |S|$.

For the converse, assume that $k_o(G-S) \leq |S|$ for every proper subset S of $V(G)$. In particular, for $S=\emptyset$, er have $k_o(G-S)=k_o(G)=0$, that is, every component of G is even and so G has even order. We now show by induction that every graph G of even order with this property has a 1-factor. There is only one grap of order 2 having only even components, namely K_2, which of course, has a 1-factor. Assume, for an even integer $n \geq 4$, that all graphs H of even order less than n for which $k_o(H-S) \leq |S|$ for every proper subset S of $V(H)$ have a 1-factor. Let G be a graph of order n satisfying $k_o(G-S) \leq |S|$ for every proper subset S of $V(G)$. Thus every component of G has even order.

First, we make an observation. Since every non-trivial component of G contains a vertex that is not a cut-vertex (Corollary 5.6), there are subsets R of $V(G)$ for which $k_o(G-R)=|R|$. (For example, we coulse choose $R=\{v\}$, where v is not a cut-vertex of G.) Among all such sets, let S be one of maximum cardinality and let $G_1, G_2, \ldots, G_k$ be the k odd components of $G-S$. Thus $k=|S| \geq 1$.

Observe that $G_1, G_2, \ldots, G_k$ are the only components of $G-S$, for otherwise $G-S$ has an even component G_0 containing a vertex u_0 that is not a cut-vertex. Then for the set $S_0=S \cup \{u\}$ of cardinality $k+1$,

$$k_o(G-S)=|S_0|=k+1$$

which is impossible. Therefore, as claimed, the odd components $G_1, G_2, \ldots, G_k$ are, in fact, the only components of $G-S$.

Now, for each integer i with $1 \leq i \leq k$, let S_i be the set of vertices of S that are adjacent to at least one vertex in G_i. Since G has only even components, each set S_i is non-empty. We claim next that each integer ℓ with $1 \leq \ell \leq k$, the union of any ℓ of the sets $S_1, S_2, \ldots S_k$ contains at least ℓ vertices. Assume, to the contrary, that there exists an integer j such that the union T of j of the sets $S_1, S_2, \ldots, S_k$ has fewer than j elements. Without loss of generality, we may assume that $T=S_1 \cup S_2 \cup \ldots \cup S_j$ and $|T|<j$. Then

$$k_o(G-T \geq j > |T|$$

which is impossible. Thus as claimed, for each integer ℓ with $1 \leq \ell \leq k$, the union of any ℓ of the sets $S_1, S_2, \ldots, S_k$ contains at least ℓ vertices.

By Theorem 8.4, there exists a set $\{v_1, v_2, \ldots, v_k\}$ of k distinct vertices such that $v_i \in S_i$ for $1 \leq i \leq k$. Since every graph $G_i (1 \leq i \leq k)$ contains a vertex u_i for which $u_i v_i \in E(G)$, it follows that $\{u_i v_i : 1 \leq i \leq k\}$ is a matching of G.

Next, we show that if $G_i (1 \leq i \leq k$ is non-trivial, then $G_i - u_i$ has a 1-factor. Let W be a proper subset of $V(G_i - u_i)$. We claim that

$$k_o(G_i - u_i - W) \leq |W|$$

Assume, to the contrary that $k_o(G_i - u_i - W) > |W|$. Since $G_i - u_i$ has even order, $k_o(G_i - u_i - W)$ and $|W|$ are either both even or both odd. Hence $k_o(G_i - u_i - W) \geq |W| + 2$. Let $S' = S \cup W \cup \{u_i\}$. Then.

$$|S'| \geq k_o(G-S') = k_o(G-S) + k_o(G_i - u_i - W) - 1 \geq |S| + (|W|+2) - 1 = |S| + |W| + 1 = |S'|$$

which implies that $k_o(G-S')=|S'|$, contradicting our choice of S. Therefore, $k_o(G_i - u_i - W) \leq |W|$, as claimed.

By the induction hypothesis, if $G_i (1 \leq i \leq k)$ is non-trivial, then $G_i - u_i$ has a 1-factor. The collection of 1-factors of $G_i - u_i$ for all non-trivial graphs $G_i (1 \leq i \leq k)$ and the edges in $\{u_i v_i : 1 \leq i \leq k\}$ produce a 1-factor of G.