Example 2: Solve

$$(2xy - \sec^2(x))dx + (x^2 + 2y)dy = 0$$

Solution

Here $M(x, y) = 2xy - \sec^2(x)$ and $N(x, y) = x^2 + 2y$. Because

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$$

$(2xy - \sec^2(x))dx + (x^2 + 2y)dy = 0$ is exact. To find $F(x, y)$, we begin by intergrating M with respect to x:

$$\begin{aligned} F(x, y) &= \int (2xy - \sec^2(x))dx + g(y) \\ &= x^2 y - \tan(x) + g(y) \end{aligned}$$

Next we take the partial derivative of $F(x, y) = \int (2xy - \sec^2(x))dx + g(y)$ with respect to y and substitute $x^2 + 2y$ for N:

$$\begin{aligned} \frac{\partial F}{\partial y}(x, y) &= N(x, y) \\ x^2 + g'(y) &= x^2 + 2y \end{aligned}$$

Thus $g'(y) = 2y$, and since the choice of the constant of integration is not important, we can take $g'(y) = y^2$. Hence from $F(x, y) = \int (2xy - \sec^2(x))dx + g(y)$, we have $x^2 y - \tan(x) + y^2$, and the solution to $(2xy - \sec^2(x))dx + (x^2 + 2y)dy = 0$ is given implicitly by $x^2 y - \tan(x) + y^2 = C$