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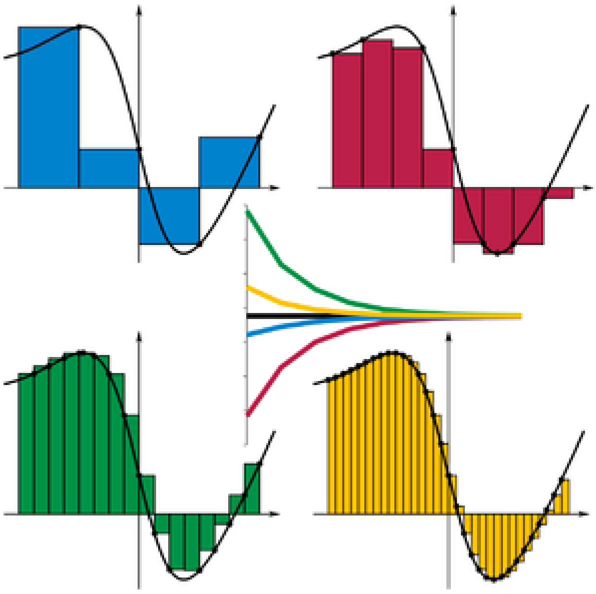
Riemann sum

In mathematics, a **Riemann sum** is a certain kind of approximation of an integral by a finite sum. It is named after nineteenth century German mathematician Bernhard Riemann. One very common application is approximating the area of functions or lines on a graph, but also the length of curves and other approximations.



Four of the Riemann summation methods for approximating the area under curves. **Right** and **left** methods make the approximation using the right and left endpoints of each subinterval, respectively. **Maximum** and **minimum** methods make the approximation using the largest and smallest endpoint values of each subinterval, respectively. The values of the sums converge as the subintervals halve from top-left to bottom-right.

The sum is calculated by dividing the region up into shapes (rectangles, trapezoids, parabolas, or cubics) that together form a region that is similar to the region being measured, then calculating the area for each of these shapes, and finally adding all of these small areas together. This approach can be used to find a numerical approximation for a definite integral even if the fundamental theorem of calculus does not make it easy to find a closed-form solution.

Because the region filled by the small shapes is usually not exactly the same shape as the region being measured, the Riemann sum will differ from the area being measured. This error can be reduced by dividing up the region more finely, using smaller and smaller shapes. As the shapes get smaller and smaller, the sum approaches the Riemann integral.

Contents

Definition

Let $f : [a, b] \to \mathbb{R}$ be a function defined on a closed interval $[a, b]$ of the real numbers, $\mathbb{R}$, and

$$P = \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\},$$

be a partition of *I*, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

A **Riemann sum** S of *f* over *I* with partition *P* is defined as

$$S = \sum_{i=1}^{n} f(x_i^*)\, \Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ and an $x_i^* \in [x_{i-1}, x_i]$.[1] Notice the use of "an" instead of "the" in the previous sentence. Another way of thinking about this asterisk is that you are choosing some random point in this slice, and it does not matter which one; as the difference or width of the slices approaches zero, the only point we can pick is the point our rectangle slice is at. This is due to the fact that the choice of x_i^* in the interval $[x_{i-1}, x_i]$ is arbitrary, so for any given function *f* defined on an interval *I* and a fixed partition *P*, one might produce different Riemann sums depending on which x_i^* is chosen, as long as $x_{i-1} \le x_i^* \le x_i$ holds true.

Some specific types of Riemann sums

Specific choices of x_i^* give us different types of Riemann sums:

- If $x_i^* = x_{i-1}$ for all *i*, then *S* is called a **left rule**[2][3] or **left Riemann sum**
- If $x_i^* = x_i$ for all *i*, then *S* is called a **right rule**[2][3] or **right Riemann sum**
- If $x_i^* = (x_i + x_{i-1})/2$ for all *i*, then *S* is called the **midpoint rule**[2][3] or **middle Riemann sum**
- If $f(x_i^*) = \sup f([x_{i-1}, x_i])$ (that is, the supremum of *f* over $[x_{i-1}, x_i]$), then *S* is defined to be an **upper Riemann sum** or **upper Darboux sum**
- If $f(x_i^*) = \inf f([x_{i-1}, x_i])$ (that is, the infimum of *f* over $[x_{i-1}, x_i]$), then *S* is defined to be an **lower Riemann sum** or **lower Darboux sum**

All these methods are among the most basic ways to accomplish numerical integration. Loosely speaking, a function is Riemann integrable if all Riemann sums converge as the partition "gets finer and finer".

While not technically a Riemann sum, the average of the left and right Riemann sum is the **trapezoidal sum** and is one of the simplest of a very general way of approximating integrals using weighted averages. This is followed in complexity by Simpson's rule and Newton–Cotes formulas.

Any Riemann sum on a given partition (that is, for any choice of x_i^* between x_{i-1} and x_i) is contained between the lower and upper Darboux sums. This forms the basis of the Darboux integral, which is ultimately equivalent to the Riemann integral.

Methods

The four methods of Riemann summation are usually best approached with partitions of equal size. The interval [*a*, *b*] is therefore divided into *n* subintervals, each of length

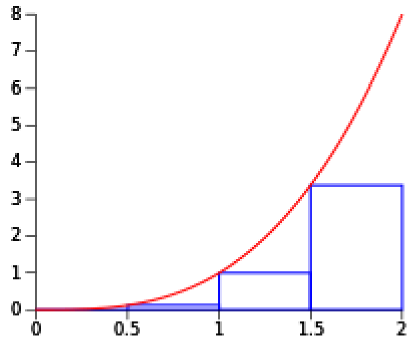
$$\Delta x = \frac{b-a}{n}.$$

The points in the partition will then be

$$a, a + \Delta x, a + 2\,\Delta x, \ldots, a + (n-2)\,\Delta x, a + (n-1)\,\Delta x, b.$$

Left Riemann sum

For the left Riemann sum, approximating the function by its value at the left-end point gives multiple rectangles with base Δ*x* and height *f*(*a* + *i*Δ*x*). Doing this for *i* = 0, 1, ..., *n* − 1, and adding up the resulting areas gives



Left Riemann sum of x^3 over [0,2] using 4 subdivisions

$$\Delta x \left[f(a) + f(a + \Delta x) + f(a + 2\,\Delta x) + \cdots + f(b - \Delta x) \right].$$

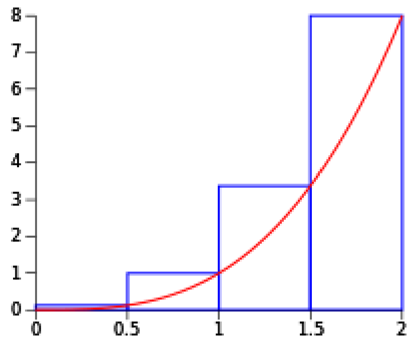
The left Riemann sum amounts to an overestimation if *f* is monotonically decreasing on this interval, and an underestimation if it is monotonically increasing

Right Riemann sum

f is here approximated by the value at the right endpoint. This gives multiple rectangles with base Δ*x* and height *f*(*a* + *i* Δ*x*). Doing this for *i* = 1, ..., *n*, and adding up the resulting areas produces

$$\Delta x \left[f(a + \Delta x) + f(a + 2\,\Delta x) + \cdots + f(b) \right].$$

The right Riemann sum amounts to an underestimation if *f* is monotonically decreasing, and an overestimation if it is monotonically increasing The error of this formula will be



Right Riemann sum of x^3 over [0,2] using 4 subdivisions

$$\left| \int_a^b f(x)\,dx - A_{\mathrm{right}} \right| \le \frac{M_1 (b-a)^2}{2n},$$

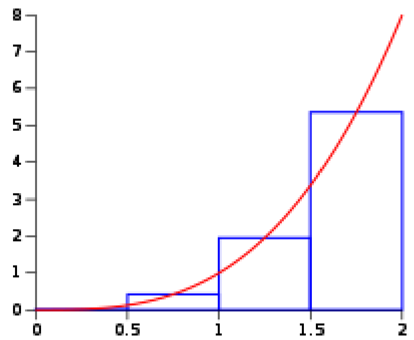
where M_1 is the maximum value of the absolute value of $f'(x)$ on the interval.

Midpoint rule

Approximating *f* at the midpoint of intervals gives *f*(*a* + Δ*x*/2) for the first interval, for the next one *f*(*a* + 3Δ*x*/2), and so on until *f*(*b* − Δ*x*/2). Summing up the areas gives

$$\Delta x \left[f(a + \tfrac{\Delta x}{2}) + f(a + \tfrac{3\,\Delta x}{2}) + \cdots + f(b - \tfrac{\Delta x}{2}) \right].$$

The error of this formula will be

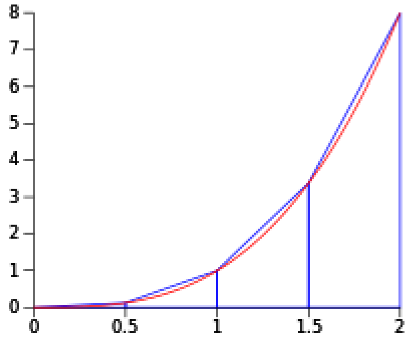


Midpoint Riemann sum of x^3 over [0,2] using 4 subdivisions

$$\left| \int_a^b f(x)\,dx - A_{\mathrm{mid}} \right| \le \frac{M_2 (b-a)^3}{24n^2},$$

where M_2 is the maximum value of the absolute value of $f''(x)$ on the interval.

Trapezoidal rule



Trapezoidal Riemann sum of x^3 over [0,2] using 4 subdivisions

In this case, the values of the function f on an interval are approximated by the average of the values at the left and right endpoints. In the same manner as above, a simple calculation using the area formula

$$A = \tfrac{1}{2}h(b_1 + b_2)$$

for a trapezium with parallel sides b_1, b_2 and height h produces

$$\tfrac{1}{2}\,\Delta x\left[f(a) + 2f(a+\Delta x) + 2f(a+2\,\Delta x) + 2f(a+3\,\Delta x) + \cdots + f(b)\right].$$

The error of this formula will be

$$\left|\int_a^b f(x)\,dx - A_{\mathrm{trap}}\right| \le \frac{M_2(b-a)^3}{12n^2},$$

where M_2 is the maximum value of the absolute value of $f''(x)$.

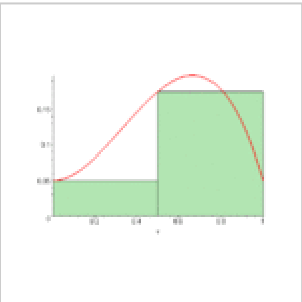
The approximation obtained with the trapezoid rule for a function is the same as the average of the left hand and right hand sums of that function.

Connection with integration

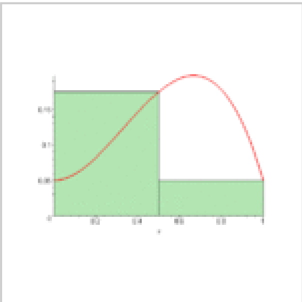
For a one-dimensional Riemann sum over domain $[a, b]$, as the maximum size of a partition element shrinks to zero (that is the limit of the norm of the partition goes to zero), some functions will have all Riemann sums converge to the same value. This limiting value, if it exists, is defined as the definite Riemann integral of the function over the domain,

$$\int_a^b f(x)\,dx = \lim_{\|\Delta x\|\to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

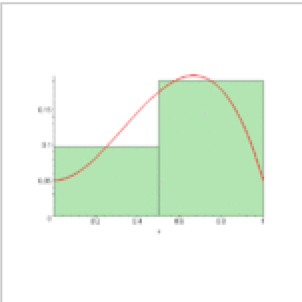
For a finite-sized domain, if the maximum size of a partition element shrinks to zero, this implies the number of partition elements goes to infinity. For finite partitions, Riemann sums are always approximations to the limiting value and this approximation gets better as the partition gets finer. The following animations help demonstrate how increasing the number of partitions (while lowering the maximum partition element size) better approximates the "area" under the curve:



Left sum



Right sum

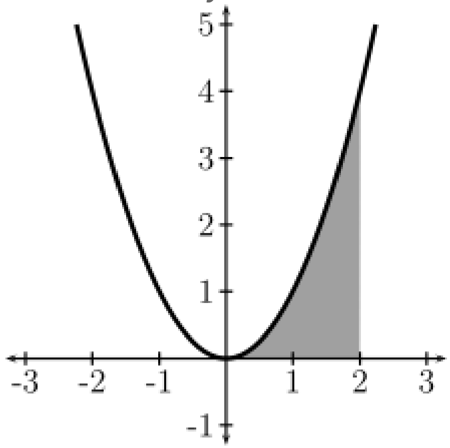


Middle sum

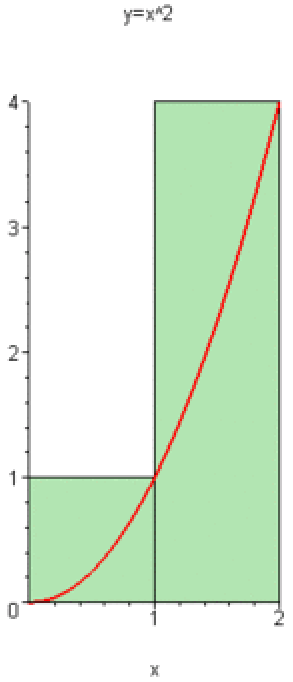
Since the red function here is assumed to be a smooth function, all three Riemann sums will converge to the same value as the number of partitions goes to infinity

Example

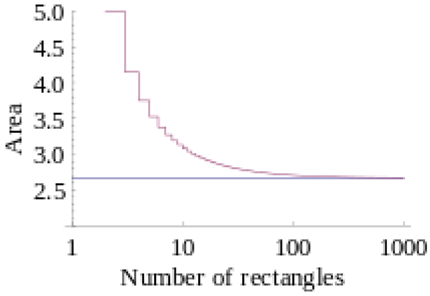
Comparison of right hand sums of the function $y = x^2$ from 0 to 2 with the integral of it from 0 to 2.



A visual representation of the area under the curve $y = x^2$ for the interval from 0 to 2. Using antiderivatives this area is exactly 8/3.



Approximating the area under $y = x^2$ from 0 to 2 using right-rule sums. Notice that because the function is monotonically increasing, right-hand sums will always overestimate the area contributed by each term in the sum (and do so maximally).



The value of the Riemann sum under the curve $y = x^2$ from 0 to 2. As the number of rectangles increases, it approaches the exact area of 8/3.

Taking an example, the area under the curve of $y = x^2$ between 0 and 2 can be procedurally computed using Riemann's method.

The interval [0, 2] is firstly divided into n subintervals, each of which is given a width of $\frac{2}{n}$; these are the widths of the Riemann rectangles (hereafter "boxes"). Because the right Riemann sum is to be used, the sequence of x coordinates for the boxes will be $x_1, x_2, \ldots, x_n$. Therefore, the sequence of the heights of the boxes will be $x_1^2, x_2^2, \ldots, x_n^2$. It is an important fact that $x_i = \frac{2i}{n}$, and $x_n = 2$.

The area of each box will be $\frac{2}{n} \times x_i^2$ and therefore the nth right Riemann sum will be:

$$\begin{aligned}
S &= \frac{2}{n} \times \left(\frac{2}{n}\right)^2 + \cdots + \frac{2}{n} \times \left(\frac{2i}{n}\right)^2 + \cdots + \frac{2}{n} \times \left(\frac{2n}{n}\right)^2 \\
&= \frac{8}{n^3} \left(1 + \cdots + i^2 + \cdots + n^2\right) \\
&= \frac{8}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \frac{8}{n^3} \left(\frac{2n^3 + 3n^2 + n}{6}\right) \\
&= \frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}
\end{aligned}$$

If the limit is viewed as $n \to \infty$, it can be concluded that the approximation approaches the actual value of the area under the curve as the number of boxes increases. Hence:

$$\lim_{n\to\infty} S = \lim_{n\to\infty} \left(\frac{8}{3} + \frac{4}{n} + \frac{4}{3n^2}\right) = \frac{8}{3}$$

This method agrees with the definite integral as calculated in more mechanical ways:

$$\int_0^2 x^2 \, dx = \frac{8}{3}$$

Because the function is continuous and monotonically increasing on the interval, a right Riemann sum overestimates the integral by the largest amount (while a left Riemann sum would underestimate the integral by the largest amount). This fact, which is intuitively clear from the diagrams, shows how the nature of the function determines how accurate the integral is estimated. While simple, right and left Riemann sums are often less accurate than more advanced techniques of estimating an integral such as the Trapezoidal rule or Simpson's rule.

The example function has an easy-to-find anti-derivative so estimating the integral by Riemann sums is mostly an academic exercise; however it must be remembered that not all functions have anti-derivatives so estimating their integrals by summation is practically important.

Higher dimensions

The basic idea behind a Riemann sum is to "break-up" the domain via a partition into pieces, multiply the "size" of each piece by some value the function takes on that piece, and sum all these products. This can be generalized to allow Riemann sums for functions over domains of more than one dimension.

While intuitively, the process of partitioning the domain is easy to grasp, the technical details of how the domain may be partitioned get much more complicated than the one dimensional case and involves aspects of the geometrical shape of the domain.[4]

Two dimensions

In two dimensions, the domain, A may be divided into a number of cells, A_i such that $A = \cup_i A_i$. In two dimensions, each cell then can be interpreted as having an "area" denoted by ΔA_i.[5] The Riemann sum is

$$S = \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta A_i,$$

where $(x_i^*, y_i^*) \in A_i$.

Three dimensions

In three dimensions, it is customary to use the letter V for the domain, such that $V = \cup_i V_i$ under the partition and ΔV_i is the "volume" of the cell indexed by i. The three-dimensional Riemann sum may then be written as[6]

$$S = \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta V_i$$

with $(x_i^*, y_i^*, z_i^*) \in V_i$.

Arbitrary number of dimensions

Higher dimensional Riemann sums follow a similar as from one to two to three dimensions. For an arbitrary dimension, n, a Riemann sum can be written as

$$S = \sum_{i} f(P_i^*)\,\Delta V_i$$

where $P_i^* \in V_i$, that is, it's a point in the n-dimensional cell V_i with n-dimensional volume ΔV_i.

Generalization

In high generality, Riemann sums can be written

$$S = \sum_{i} f(P_i^*)\mu(V_i)$$

where P_i^* stands for any arbitrary point contained in the partition element V_i and μ is a measure on the underlying set. Roughly speaking, a measure is a function that gives a "size" of a set, in this case the size of the set V_i; in one dimension, this can often be interpreted as the length of the interval, in two dimensions, an area, in three dimensions, a volume, and so on.

See also

- Antiderivative
- Euler method and midpoint method, related methods for solving differential equations
- Lebesgue integral
- Riemann integral, limit of Riemann sums as the partition becomes infinitely fine
- Simpson's rule, a powerful numerical method more powerful than basic Riemann sums or even the Trapezoidal rule
- Trapezoidal rule, numerical method based on the average of the left and right Riemann sum

References

1. Hughes-Hallet, Deborah; McCullum, William G.; et al. (2005). *Calculus* (4th ed.). Wiley. p. 252. (Among many equivalent variations on the definition, this reference closely resembles the one given here.)

2. Hughes-Hallet, Deborah; McCullum, William G.; et al. (2005). *Calculus* (4th ed.). Wiley. p. 340. "So far, we have three ways of estimating an integral using a Riemann sum: 1. The **left rule** uses the left endpoint of each subinterval. 2. The **right rule** uses the right endpoint of each subinterval. 3. The **midpoint rule** uses the midpoint of each subinterval."
3. Ostebee, Arnold; Zorn, Paul (2002). *Calculus from Graphical, Numerical, and Symbolic Points of View* (Second ed.). p. M-33. "Left-rule, right-rule, and midpoint-rule approximating sums all fit this definition."
4. Swokowski, Earl W. (1979). *Calculus with Analytic Geometry* (Second ed.). Boston, MA: Prindle, Weber & Schmidt. pp. 821–822. ISBN 0-87150-268-2.
5. Ostebee, Arnold; Zorn, Paul (2002). *Calculus from Graphical, Numerical, and Symbolic Points of View* (Second ed.). p. M-34. "We chop the plane region R into m smaller regions R_1, R_2, R_3, ..., R_m, perhaps of different sizes and shapes. The 'size' of a subregion R_i is now taken to be its *area*, denoted by ΔA_i."
6. Swokowski, Earl W. (1979). *Calculus with Analytic Geometry* (Second ed.). Boston, MA: Prindle, Weber & Schmidt. pp. 857–858. ISBN 0-87150-268-2.

External links

- Weisstein, Eric W. "Riemann Sum". *MathWorld*.
- A simulation showing the convergence of Riemann sums