Area of a Triangle in the xy-Plane

The area of a triangle with vertices

$$(x_1), (y_1), (x_2), (y_2), \text{ and } (x_3), (y_3)$$

$$Area = \pm\frac{1}{2}det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

where the sign ($\pm$) is chosen to give a positive area.

Proof

Prove the case for $y_i > 0$. Assume that $x_1 \leq x_3 \leq x_2$ and that $\big(x_3, y_3\big)$ lies above the line segment connecting $\big(x_1, y_1\big)$ and $\big(x_2, y_2\big)$. Consider the three trapezoids whose vertices are

$$\text{Trapezoid 1: } \big(x_1, 0\big), \big(x_1, y_1\big), \big(x_3, y_3\big), \big(x_3, 0\big)$$

$$\text{Trapezoid 2: } \big(x_3, 0\big), \big(x_3, y_3\big), \big(x_2, y_2\big), \big(x_2, 0\big)$$

$$\text{Trapezoid 3: } \big(x_1, 0\big), \big(x_1, y_1\big), \big(x_2, y_2\big), \big(x_2, 0\big)$$

The area of the triangle is equal to the sum of the areas of the first two trapezoids minus the area of the third trapezoid. So,

$$\begin{aligned} Area &= \frac{1}{2}\big(y_1 + y_2\big)\big(x_3 - x_1\big) + \frac{1}{2}\big(y_3 + y_2\big)\big(x_2 - x_3\big) - \frac{1}{2}\big(y_1 + y_2\big)\big(x_2 - x_1\big) \\ &= \frac{1}{2}\big(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2\big) \\ &= \frac{1}{2}\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \end{aligned}$$

If the vertices do not occur in the order $x_1 \leq x_3 \leq x_2$ and that $\big(x_3, y_3\big)$ or if the vertex $\big(x_3, y_3\big)$ is not above the line segment connecting the other two vertices, then the formula above may yield the negative of the area. So, use $\pm$ and choose the correct sign to give a positive area.