Section 2.4 Exact Equations

Definition An expression $M(x,y)\,dx + N(x,y)\,dy$ is said to be an exact differential if there is a function $F(x,y)$ such that

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = M\,dx + N\,dy$$

Proposition Let $M(x,y)$ & $N(x,y)$ & their first partial derivatives be continuous functions of (x,y) in a rectangular region R. Then $M(x,y)\,dx + N(x,y)\,dy$ is exact in R iff $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ in R.

Proof $(\Rightarrow)$ Suppose that $M\,dx + N\,dy$ is exact.

Then there is a function $F(x,y)$ such that

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = M\,dx + N\,dy \quad , \text{ for any } dx, dy$$

$$\therefore \quad M = \frac{\partial F}{\partial x} \;,\; N = \frac{\partial F}{\partial y}$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$(\Leftarrow)$ Suppose that $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

(Reqd to produce a function $F(x,y)$ s.t. $dF = M\,dx + N\,dy$)

Define $F(x,y) = \int M(x,y)\,dx + \phi(y)$

Clearly, $\frac{\partial F}{\partial x} = M$

Furthermore, $\frac{\partial F}{\partial y} = \int \frac{\partial M}{\partial y}\, dx + \phi'(y)$

$$= N \text{ , if } \quad \phi'(y) = N - \int \frac{\partial M}{\partial y}\, dx$$

This last equation is meaningful only if the RHS is independent of x.

Let us show that this is the case.

$$\frac{\partial}{\partial x}\left[N - \int \frac{\partial M}{\partial y}\, dx \right] = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\therefore \phi(y) = \int \left[N - \int \frac{\partial M}{\partial y}\, dx \right] dy + c$$

$$\therefore F(x,y) = \int M(x,y)\, dx + \int \left[N - \int \frac{\partial M}{\partial y}\, dx \right] dy + C$$

is such that $dF = M dx + N dy$.

<u>Definition</u>

The 1^{st} order ODE $M(x,y)\, dx + N(x,y)\, dy = 0$ is said to be <u>exact</u> if it's LHS is an exact differential.

In that case the equation may be written as

$$dF(x,y) = 0$$

which has the implicit solution

$$F(x,y) = C$$

Example 1 Show that $x\,dx + y\,dy = 0$ is exact & hence solve it.

Solution
$$x\,dx + y\,dy = 0$$

$$M(x,y) = x \qquad N(x,y) = y$$

$$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 0$$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ Equation is exact.

$\therefore \exists\, F(x,y)$ such that $dF = \dfrac{\partial F}{\partial x}dx + \dfrac{\partial F}{\partial y}dy = M\,dx + N\,dy = 0$

$\therefore \dfrac{\partial F}{\partial x} = M = x \qquad \Rightarrow F(x,y) = \int x\,dx + \phi(y)$

$$= \tfrac{1}{2}x^2 + \phi(y)$$

and

$\dfrac{\partial F}{\partial y} = N = y$

$$\frac{\partial F}{\partial y} = 0 + \phi'(y)$$

$$\therefore \phi'(y) = y$$

$$\phi(y) = \tfrac{1}{2}y^2 + c_0$$

$$F(x,y) = \tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 + c_0$$

A one-parameter family of solutions is given by:

$$F(x,y) = \text{constant} = c_1$$

$$\therefore \frac{1}{2}x^2 + \frac{1}{2}y^2 + c_0 = c_1$$

$$\therefore x^2 + y^2 = c \quad , \quad (c = 2(c_1 - c_0))$$

<u>Example 2</u> Show that $2xyy' = x^2 - y^2$ is exact & hence solve it.

<u>Solution</u> Rewrite equation in the form $M\,dx + N\,dy = 0$.

$$2xy \frac{dy}{dx} = x^2 - y^2$$

$$2xy\,dy = (x^2 - y^2)\,dx$$

$$(y^2 - x^2)\,dx + 2xy\,dy = 0$$

$$M(x,y) = y^2 - x^2 \quad , \quad N(x,y) = 2xy$$

$$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = 2y$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ Equation is exact.

$\therefore \exists\, F(x,y)$ s.t. $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = M\,dx + N\,dy = 0$

$\frac{\partial F}{\partial x} = M = y^2 - x^2 \quad \Rightarrow F(x,y) = \int y^2 - x^2\,dx + \phi(y)$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad = xy^2 - \frac{1}{3}x^3 + \phi(y)$

and

$\frac{\partial F}{\partial y} = N = 2xy$

$$\frac{\partial F}{\partial y} = 2xy + \phi'(y)$$

$$\therefore \quad 2xy + \phi'(y) = 2xy$$

$$\phi'(y) = 0$$

$$\phi(y) = c_0$$

$$\therefore \quad F(x,y) = xy^2 - \frac{1}{3}x^3 + c_0$$

There is a one-parameter family of solutions $F(x,y) = \text{constant} = c_1$

$$\therefore \quad xy^2 - \frac{1}{3}x^3 + c_0 = c_1$$

$$xy^2 - \frac{1}{3}x^3 = c \qquad (c = c_1 - c_0)$$

<u>Example 3</u> Solve $y' = \dfrac{xy^2 - 1}{1 - x^2y}$, $y(0) = 1$

<u>Solution</u> Rewrite eqn in the form $M\,dx + N\,dy = 0$.

$$\frac{dy}{dx} = \frac{xy^2 - 1}{1 - x^2y}$$

$$(1 - x^2y)\,dy = (xy^2 - 1)\,dx$$

$$(xy^2 - 1)\,dx + (x^2y - 1)\,dy = 0$$

$$M = xy^2 - 1 \qquad N = x^2y - 1$$

$$\frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = 2xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow \text{Eqn is exact.}$$

$$\therefore \exists F(x,y) \text{ s.t. } dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = M\,dx + N\,dy = 0$$

$$\therefore \frac{\partial F}{\partial x} = M = xy^2 - 1 \qquad \Rightarrow F(x,y) = \int xy^2 - 1 \, dx + \phi(y)$$

$$= \tfrac{1}{2}x^2y^2 - x + \phi(y)$$

and

$$\frac{\partial F}{\partial y} = N = x^2y - 1$$

$$\frac{\partial F}{\partial y} = x^2y + \phi'(y)$$

$$\therefore x^2y - 1 = x^2y + \phi'(y)$$

$$\phi'(y) = -1$$

$$\phi(y) = -y + C_0$$

$$F(x,y) = \tfrac{1}{2}x^2y^2 - x - y + C_0$$

There is a one-parameter family of solutions:

$$F(x,y) = \tfrac{1}{2}x^2y^2 - x - y + C_0 = \text{constant} = C_1$$

$$\tfrac{1}{2}x^2y^2 - x - y = C \qquad (C = C_1 - C_0)$$

Let $x=0, y=1$: $\quad 0 - 0 - 1 = C \quad \Rightarrow C = -1$

$$\therefore \tfrac{1}{2}x^2y^2 - x - y = -1$$

$$\therefore x + y - \tfrac{1}{2}x^2y^2 = 1$$

Integrating Factors

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact but $\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0$ is exact, then $\mu(x,y)$ is called an <u>integrating factor</u>.

<u>Example</u> Show that $(xy^2+2y)\,dx + (3x^2y+2x)\,dy = 0$ is not exact but has an integrating factor $\mu = (xy)^{-2}$ & hence solve it.

<u>Solution</u> $(xy^2+2y)\,dx + (3x^2y+2x)\,dy = 0 \qquad (*)$

$M(x,y) = xy^2 + 2y \qquad N(x,y) = 3x^2y + 2x$

$\dfrac{\partial M}{\partial y} = 2xy + 2 \qquad \dfrac{\partial N}{\partial x} = 6xy + 2$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ Eqn $(*)$ is not exact.

Multiply through $(*)$ by $\mu = (xy)^{-2} = \dfrac{1}{x^2y^2}$:

$$\left(\frac{1}{x} + \frac{2}{x^2y}\right)dx + \left(\frac{3}{y} + \frac{2}{xy^2}\right)dy = 0 \qquad (**)$$

$M = x^{-1} + 2x^{-2}y^{-1} \qquad N = 3y^{-1} + 2x^{-1}y^{-2}$

$\dfrac{\partial M}{\partial y} = -2x^{-2}y^{-2} \qquad \dfrac{\partial N}{\partial x} = -2x^{-2}y^{-2}$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow$ Eqn $(**)$ is exact.

$\exists\ F(x,y)$ s.t. $dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = M\,dx + N\,dy = 0$

$\therefore \frac{\partial F}{\partial x} = M = x^{-1} + 2x^{-2}y^{-1} \Rightarrow F(x,y) = \int \frac{1}{x} + \frac{2}{y}x^{-2}\,dx + \phi(y)$

and

$\frac{\partial F}{\partial y} = N = 3y^{-1} + 2x^{-1}y^{-2}$

$$= \ln x + \frac{2}{y}\frac{x^{-1}}{-1} + \phi(y)$$

$$= \ln x - \frac{2}{xy} + \phi(y)$$

$$\frac{\partial F}{\partial y} = \frac{2}{xy^2} + \phi'(y)$$

$$\therefore \frac{3}{y} + \frac{2}{xy^2} = \frac{2}{xy^2} + \phi'(y)$$

$$\phi'(y) = \frac{3}{y}$$

$$\phi(y) = 3\ln y + c_0$$

$$\therefore F(x,y) = \ln x - \frac{2}{xy} + 3\ln y + c_0$$

There is a one-parameter family of solutions:

$$F(x,y) = \text{constant} = c_1$$

$$\ln(xy^3) = c_1 + 2/xy$$

$$xy^3 = e^{c_1}e^{2/xy}$$

$$\therefore xy^3 = c\,e^{2/xy} \qquad (c = e^{c_1})$$

HW Pgs 61 – 62, #'s: 9 – 25 odd, 30