Limit of a Function

$$\lim_{x \to c} f(x)$$

Introduction

We are interested in finding the (two-sided) limit, if it exists, of a function $f(x)$ as x approaches the value c (in short, $x \to c$), where c is *Real* (and thus finite.) Stating this more informally, we would like to know whether $f(x)$ approaches the same finite value as x approaches some value c on both sides of c. That finite value of $f(x)$ is called the "limit". It is very important to always remember that this is NOT the same as asking what is $f(c)$ (the function does not even have to be defined at c!) since in determining the limit, x never equals c; x only gets arbitrarily close to c from either side of c.

The Rigorous Definition of a Limit: (ε, δ)

When the foundations of calculus were firmed up in the 19th century, the rigorous definition of the limit was developed by Bernard Bolzano (1781-1848), restated by Augustin-Louis Cauchy (1789-1857), and formalized in modern form by Karl Weierstrass (1815-1897). This rigorous definition is known as the (ε, δ)-*Definition*:

(ε, δ)-Definition of a Limit

Let $f(x)$ be a function defined on an open interval I containing c (except $f(c)$ may be undefined) and let L be a *Real* number. The statement

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta, \text{then } |f(x) - L| < \varepsilon$$

This rigorous but confusing definition is very difficult to apply for anything other than the simplest functions. Furthermore, this definition does not analytically determine the limit L if it exists, but rather only proves if a particular value of L is the limit.

However, we need not despair since the (ε, δ)-*Definition* is used to prove a number of theorems giving us important "tools" to determine the limits of nearly all functions encountered in mathematics, science, engineering, etc., without the need to directly apply the (ε, δ)-*Definition.* Our first set of tools is the *Properties of Limits.*

The Properties of Limits

The (ε, δ)-*Definition of a Limit* may be used to prove the following important *Properties of Limits*.

Properties of Limits

Let b and c be *Real* numbers, let n be a positive integer, and let $f(x)$ and $g(x)$ be functions with the following *Real* number limits:

$$\lim_{x\to c} f(x) = L$$

$$\lim_{x\to c} g(x) = M$$

Scalar Multiple	$\lim_{x\to c}[b \cdot f(x)] = bL$
Sum/Difference	$\lim_{x\to c}[f(x) \pm g(x)] = L \pm M$
Product	$\lim_{x\to c}[f(x) \cdot g(x)] = LM$
Quotient	$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}$, $M \neq 0$ (*see note below*)
Power	$\lim_{x\to c}[f(x)]^n = L^n$
Composite	$\lim_{x\to c} f\big(g(x)\big) = f\left(\lim_{x\to c} g(x)\right) = f(M)$ if $f(M)$ defined

Important Notes:

1. These limit properties *do* **not** require that either function be defined at $x = c$. Furthermore, if either or both functions are defined at $x = c$, these limit properties do **not** require that $L = f(c)$ or that $M = g(c)$.

2. If the *Quotient Property* limit results in non-zero divided by zero, the limit does not exist (it is an "infinite" condition.) If the *Quotient Property* limit results in the *indeterminate* $0/0$, then the limit may or may not exist and further analysis is required to resolve the limit.

Basic "Building Block" Functions $f(x)$ *where* $\lim_{x\to c} f(x) = f(c)$

The (ε, δ)-*Definition of a Limit* may be used, with great difficulty in some cases, to prove that the limit as $x \to c$ for each of the following basic "Building Block" functions may be determined by *direct substitution* provided that c is in the ***interior*** of the function's domain.

Basic "Building Block" Functions $\boldsymbol{f(x)}$ ***where*** $\boldsymbol{\lim_{x\to c} f(x) = f(c)}$

c is a *Real* number in the ***interior*** of the function's domain

$f(x) =$	Constant
	x^r where r is *Real*. Examples: $x, x^2, x^{-1} = 1/x$, $x^{1/2} = \sqrt{x}$, etc.
	$\lvert x\rvert$ (absolute value)
	$\ln x\,,\, e^x$
	$\sin x\,,\, \cos x\,,\, \tan x\,,\, \csc x\,,\, \sec x\,,\, \cot x$
	$\sin^{-1} x\,,\, \cos^{-1} x\,,\, \tan^{-1} x\,,\, \csc^{-1} x\,,\, \sec^{-1} x\,,\, \cot^{-1} x$
	All hyperbolic and inverse hyperbolic functions

Direct Substitution Rule

Putting together the results of the prior two sections, it is obvious that we may use direct substitution to find the limit of any function which is an arithmetic and/or compositional combination of the basic "Building Block" functions, provided the limit value c is in the ***interior*** of the function's domain (this is important!)

This pretty much includes nearly all functions studied in mathematics, science, engineering, business, etc. For example, the "student's nightmare" $\lim_{x\to c} f(x)$, where $f(x) = \ln[\sin(x^2 - 5x + \frac{1}{x} + e^{\sec x})]$, is simply $f(c)$ provided c is in the *interior* of this function's domain. The *Direct Substitution Rule* formalizes this:

Direct Substitution Rule for Determining the Limit of a Function:

$$\lim_{x \to c} f(x) = f(c)$$

Provided:

1. $f(x)$ is an arithmetic and/or compositional combination (per the *Properties of Limits*) of one or more "Building Block" functions.
2. The limit value c is an ***interior*** point in the domain of $f(x)$.

Note: An *interior point* c in the domain of $f(x)$ is a point in some finite *open* interval (a, b) where *all* points in that open interval are in the domain of $f(x)$.

An Unusual Example: $\lim_{x \to 0} \sqrt{-\sqrt{x}}$

The *Direct Substitution Rule* is very specific about the requirements for its use in determining the limit for some function. The first requirement, that the function is an arithmetic and/or compositional combination of the "Building Block" functions, is very easy to determine, and pretty much applies to almost every function studied by Calculus students. The second requirement is also very specific: it requires that there be some finite interval around c in which the function is defined (that is, all points in that finite interval around c, and c itself, are in the domain of the function.)

The following example fulfills the first requirement, but fails the second requirement, providing a valuable lesson about the need for a finite interval around c where the function is defined:

$$\lim_{x \to 0} \sqrt{-\sqrt{x}}$$

It is obvious that the function $f(x) = \sqrt{-\sqrt{x}}$ is an arithmetic and compositional combination of the "Basic Functions." But does this function fulfill the second requirement so we may use direct substitution to evaluate the limit? If we could use *direct substitution*, the limit would be zero since $\sqrt{-\sqrt{0}} = 0$.

It is easy to see that the domain of this function has only one element: $\{0\}$. For all other values of x the function is undefined in the *Real* numbers. So why cannot we use the *Direct Substitution Rule*? The reason is simple: zero is not in the ***interior*** of the domain. This is because there is no *interior* in the domain! In fact, the limit does not even exist since x may not be any value other than zero itself.