First Fundamental Theorem of Calculus: Generalized Upper/Lower Limits

The *First Fundamental Theorem of Calculus* (some textbooks call it the *Second*) states the following amazing result:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

This theorem establishes that integration (as in finding *Riemann Sums*) and differentiation (as in finding the change of rate of a function) are essentially inverse operations. The above theorem is also used to prove the *Second Fundamental Theorem of Calculus* (which some textbooks call the *First*), which makes it easy to **exactly** calculate the value of a definite integral using the antiderivative of the integrand.

Consult a reputable calculus textbook (e.g., James Stewart) for geometric and analytic explanations, and rigorous proofs, of the two-part *Fundamental Theorem of Calculus*. All calculus students should thoroughly study, understand, and apply the *Fundamental Theorem of Calculus*.

Sometimes students are asked to determine the differential of a definite integral like above but with one or both limits being more complicated functions of x. The usual approach is to apply the *Chain Rule* and the properties of integrals, and sometimes it gets to be a tad confusing. What will be given, without proof, is the general case where each limit is some function of x. The student need only do simple substitution into the general case "formula" to find the differential of an integral of the above form.

First Fundamental Theorem of Calculus (Generalized)

Given the following function in which $u(x)$ and $v(x)$ are differentiable at x, and $f(t)$ is continuous in the *closed* interval between $u(x)$ and $v(x)$:

$$F(x) = \int_{u(x)}^{v(x)} f(t)dt$$

The derivative $F'(x)$ is given by:

$$F'(x) = \frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f\big(v(x)\big)\cdot v'(x) - f\big(u(x)\big)\cdot u'(x)$$

If we set $u(x) = a$ and $v(x) = x$, the above equation reduces to the *First Fundamental Theorem of Calculus!*

Example 1

Simplify:

$$\frac{d}{dx}\int_{x^2}^{x^3}(2t^2-4)\,dt$$

$$\frac{d}{dx}\int_{x^2}^{x^3}(2t^2-4)\,dt = [2(x^3)^2-4]\cdot\frac{d}{dx}x^3 - [2(x^2)^2-4]\cdot\frac{d}{dx}x^2$$

$$\frac{d}{dx}\int_{x^2}^{x^3}(2t^2-4)\,dt = (2x^6-4)\cdot 3x^2 - (2x^4-4)\cdot 2x$$

$$\frac{d}{dx}\int_{x^2}^{x^3}(2t^2-4)\,dt = 6x^8-4x^5-12x^2+8x \quad \blacksquare$$

As a check, if we integrate the indefinite integral, substitute the lower and upper limits for t, and then differentiate the resulting function in x, we will obtain the same result:

$$F(x) = \int_{x^2}^{x^3}(2t^2-4)\,dt$$

$$F(x) = \left[\frac{2}{3}t^3-4t\right]_{x^2}^{x^3} = \left[\frac{2}{3}x^9-4x^3\right] - \left[\frac{2}{3}x^6-4x^2\right]$$

$$F(x) = \frac{2}{3}x^9-\frac{2}{3}x^6-4x^3+4x^2$$

$$F'(x) = \frac{d}{dx}\int_{x^2}^{x^3}(2t^2-4)\,dt = 6x^8-4x^5-12x^2+8x \quad \blacksquare$$