Theorem 5.11

For every graph G

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Proof:

If G is disconnected or trivial, then $\kappa(G) = \lambda(G) = 0$ and the inequalities hold; while ig $G = K_n$ for some integer $n \geq 2$, then $\kappa(G) = \lambda(G) = \delta(G = n - 1$. Thus we may assume that G is a connected graph of order $n \geq 3$ that is not complete. Hence $\delta(G) \leq n - 2$.

First, we show that $\lambda(G) \leq \delta(G)$. Let v be a vertex of G with $deg(v) = \delta(G)$. Since the set of the $\delta(G)$ edges incident with v in G is an edge-cut, it follows that

$$\lambda(G) \leq \delta(G) \leq n - 2$$

It remains to show that $\kappa(G) \leq \lambda(G)$. Let X be a minimum edge-cut of G. Then $|X| = \lambda(G) \geq n - 2$ Necessarily, $G - X$ contains exactly two components G_1 and G_2. Suppose that the order of G_1 is k. Thus the order of G is $n - k$, where $k \geq 1$ and $n - k \geq 1$. Consequently, every edge in X joins a vertex of G_1 and a vertex of G_2. We consider two cases.

Case 1. *Every vertex of G_1 is adjacent in G to every vertex of G_2.* Thus $|X| = k(n - k)$. Since $(k - 1)(n - k - 1) \geq 0$, it follows that

$$(k - 1)(n - k - 1) = k(n - k) - n + 1 \geq 0$$

and so $\lambda(G) = |X| = k(n - k) \geq n - 1$. However, $\lambda(G) \leq n - 2$; so this case cannot occur.

Case 2. *There exists vertices in u in G_1 and v in G_2 such that u and v are not adjacent in G.* We now define a set U of vertices of G. For each $e \in X$, we select a vertex for U in the following way. If u is incident with e, then choose the other vertex in G_2 that is incident with e as an element of U; otherwise, select the vertex that is incident with e and belongs to G_1 as an element of U. it follows that $G - U$ is disconnected and so U is a vertex-cut. Hence

$$\kappa(G) \leq |U| \leq |X| = \lambda(G)$$

as desired.