Using the Limit of a Riemann Sum to Determine the Value of a Definite Integral (with Examples)

Many problems in mathematics, science, and engineering (such as finding the area under a plane curve) are solved by constructing a *Riemann Sum* of a function $f(x)$ on the closed interval $[a, b]$:

$$\sum_{i=1}^{n} f(u_i)\Delta x_i$$

and then taking the limit of this sum as the partition norm goes to zero:

$$\lim_{max \Delta x_i \to 0} \sum_{i=1}^{n} f(u_i)\Delta x_i$$

(This document assumes the student has already studied *Riemann Sums* and understands the terminology/nomenclature.)

When the function $f(x)$ is continuous on the closed interval $[a, b]$, the above limit amazingly (and fortunately!) has the same value regardless of how the x interval is partitioned and which value of x within each i^{th} subinterval (that is, u_i) is selected. Thus, for $f(x)$ continuous on $[a, b]$, we represent the above complicated limit expression using the much simpler and more useful definite integral notation:

$$\lim_{max \Delta x_i \to 0} \sum_{i=1}^{n} f(u_i)\Delta x_i = \int_a^b f(x)\, dx$$

Prior to learning the two-part *Fundamental Theorem of Calculus*, students are usually given problems to determine the definite integral of a function using the basic definition of taking the limit of the associated *Riemann Sum*. This turns out to involve a **lot** of work even for low order, simple polynomial functions (constant, linear, and quadratic), and the student quickly concludes that it may be nigh impossible to use this technique for more complicated functions such as certain transcendental functions. Fortunately, the *Fundamental Theorem of Calculus* comes to the rescue!

Nevertheless, this document provides two example integrands whose definite integrals with given limits will be exactly determined by applying the limit of *Riemann Sums*: $f(x) = x^2 - 2x + 2$, and $f(x) = \ln x$. The second integrand function, $\ln x$, rarely appears in this context in calculus problem sets, but is included to give the student a glimpse into the difficulty of determining, in general, the exact value of a definite integral using the limit of *Riemann Sums*.

Before presenting and solving the example problems, we will save some work and "pre-construct" the *Riemann Sum* for *any* function $f(x)$ (continuous on $[a, b]$) by specifying that all n sub-intervals of the partition have equal width and that for each sub-interval select the right-side x value (in some calculus books this is called the "right-side sum".) The student may find the resulting distilled "template" below to be a useful starting-point for solving any *Riemann Sum* limit problem as it eliminates the usually confusing and complicated first step of constructing the *Riemann Sum* for the problem.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{b-a}{n}i\right)\right],\ f(x)\ \textit{continuous on}\ [a,b]$$

Exact Value of $\int_1^4 (x^2-2x+2)dx$ *Using Riemann Sums*

The *template* substitutions are: $f(x)=x^2-2x+2$, $a=1$, and $b=4$:

$$a+\frac{b-a}{n}i = 1+\frac{3i}{n}$$

$$f\left(a+\frac{b-a}{n}i\right) = f\left(1+\frac{3i}{n}\right) = \left(1+\frac{3i}{n}\right)^2 - 2\left(1+\frac{3i}{n}\right)+2$$

The right side, after multiplying and simplifying, we obtain:

$$f\left(a+\frac{b-a}{n}i\right) = \frac{9i^2}{n^2}+1$$

Therefore, substituting into our template:

$$\int_1^4 (x^2-2x+2)dx = \lim_{n\to\infty}\left[\frac{3}{n}\sum_{i=1}^{n}\left(\frac{9i^2}{n^2}+1\right)\right]$$

Since $\sum(a+b)=\sum a+\sum b$

$$\int_1^4 (x^2-2x+2)dx = \lim_{n\to\infty}\left[\frac{3}{n}\left(\sum_{i=1}^{n}\frac{9i^2}{n^2}+\sum_{i=1}^{n}1\right)\right]$$

Since

$$\sum_{i=1}^{n} c = nc \quad \text{where } c \text{ is a constant}$$

$$\int_1^4 (x^2-2x+2)dx = \lim_{n\to\infty}\left[\left(\frac{3}{n}\sum_{i=1}^{n}\frac{9i^2}{n^2}\right)+3\right]$$

Within the sigma summation, $9/n^2$ is a constant, therefore:

$$\int_1^4 (x^2-2x+2)dx = \lim_{n\to\infty}\left[\left(\frac{27}{n^3}\sum_{i=1}^{n}i^2\right)+3\right]$$

From the special topic document *The First Seven Faulhaber Formulas*:

$$S_p(n) = \sum_{i=1}^{n} i^p \quad (p \textit{ Natural})$$

we find:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6}$$

Substituting to remove the sigma substitution:

$$\int_1^4 (x^2-2x+2)dx = \lim_{n\to\infty}\left[\frac{27}{n^3}\left(\frac{2n^3+3n^2+n}{6}\right)+3\right]$$

$$\int_1^4 (x^2-2x+2)dx = \lim_{n\to\infty}\left(\frac{54n^3+81n^2+27n}{6n^3}+3\right)$$

$$\int_1^4 (x^2-2x+2)dx = \lim_{n\to\infty}\left(\frac{54n^3}{6n^3}+\frac{81n^2}{6n^3}+\frac{27n}{6n^3}+3\right)$$

$$\int_1^4 (x^2-2x+2)dx = \lim_{n\to\infty}\left(9+\frac{27}{2n}+\frac{9}{2n^2}+3\right)$$

$$\int_1^4 (x^2-2x+2)dx = \lim_{n\to\infty}\left(12+\frac{27}{2n}+\frac{9}{2n^2}\right)$$

As $n \to \infty$, the terms $27/2n$ and $9/2n^2$ limit to zero. Thus:

$$\int_1^4 (x^2-2x+2)dx = 12 \quad \blacksquare$$

Exact Value of $\int_1^2 \ln x\, dx$ Using Riemann Sums

The *template* substitutions are $f(x) = \ln x$, $a = 1$, and $b = 2$. Substituting, with minor simplification:

$$\int_1^2 \ln x\, dx = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n}\ln\left(1+\frac{i}{n}\right)\right]$$

Combining the two terms inside the natural logarithm:

$$\int_1^2 \ln x\, dx = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=1}^{n}\ln\left(\frac{n+i}{n}\right)\right\}$$

Using the logarithmic property $\ln a - \ln b = \ln(a/b)$:

$$\int_1^2 \ln x\,dx = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=1}^{n}[\ln(n+i)-\ln n]\right\}$$

Since $\sum(a-b) = \sum a - \sum b$:

$$\int_1^2 \ln x\,dx = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=1}^{n}[\ln(n+i)] - \frac{1}{n}\sum_{i=1}^{n}\ln n\right\}$$

Since

$$\sum_{i=1}^{n} c = nc \quad \text{where } c \text{ is a constant}$$

The summation on the right above simplifies:

$$\int_1^2 \ln x\,dx = \lim_{n\to\infty}\left\{\frac{1}{n}\sum_{i=1}^{n}[\ln(n+i)] - \ln n\right\}$$

The other summation is complicated. If one writes it out for certain specific values of n, and after some thought and trial-error, and recalling the logarithmic property that $\ln a + \ln b = \ln(a \cdot b)$, we can generalize it as follows:

$$\sum_{i=1}^{n}[\ln(n+i)] = \ln\frac{(2n)!}{n!} = \ln(2n)! - \ln n!$$

At this point the student may despair, but there is this result of mathematics (a variant form of what is known as *Stirling's Formula*):

$$\ln n! \to \frac{1}{2}\ln(2\pi n) + n\ln\left(\frac{n}{e}\right) \quad \text{as} \quad n\to\infty$$

and thus

$$\ln(2n)! \to \frac{1}{2}\ln(4\pi n) + 2n\ln\left(\frac{2n}{e}\right) \quad \text{as} \quad n\to\infty$$

Therefore, after substitution and simplification

$$\sum_{i=1}^{n}[\ln(n+i)] \to \frac{1}{2}\ln 2 + n\ln\frac{4n}{e} \quad \text{as} \quad n\to\infty$$

Since we are taking the limit as $n\to\infty$, we can substitute the above into our last equation for the definite integral, and then algebraically distill it:

$$\int_1^2 \ln x\,dx = \lim_{n\to\infty}\left\{\frac{1}{n}\left(\frac{1}{2}\ln 2 + n\ln\frac{4n}{e}\right) - \ln n\right\}$$

$$\int_1^2 \ln x\,dx = \lim_{n\to\infty}\left\{\frac{1}{2n}\ln 2 + \ln\frac{4n}{e} - \ln n\right\}$$

$$\int_1^2 \ln x\,dx = \lim_{n\to\infty}\left\{\frac{1}{2n}\ln 2 + \ln\frac{4}{e}\right\}$$

$$\int_1^2 \ln x\,dx = \lim_{n\to\infty}\left\{\frac{1}{2n}\ln 2 + \ln 4 - \ln e\right\}$$

Now, as $n \to \infty$, $1/2n \to 0$, and since $\ln e = 1$, we have our *exact* answer:

$$\int_1^2 \ln x\,dx = \ln 4 - 1 \quad \blacksquare$$

The student should conclude that, in general, directly evaluating a definite integral using the limit of its *Riemann Sum* is *very* difficult and for some functions may be nigh impossible! As noted previously, we are very thankful for the *Fundamental Theorem of Calculus* to more easily and more universally evaluate definite integrals with continuous integrand functions!