Section 2.3 Linear Equations

$$a_0(x)\frac{dy}{dx} + a_1(x)y = b(x), \quad a_0(x) \neq 0$$

Rewrite in standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Consider the function $\mu(x) = e^{\int P(x)\,dx}$

Differentiate
$$\frac{d\mu}{dx} = \frac{d}{dx} e^{\int P(x)\,dx}$$
$$= e^{\int P(x)\,dx} \frac{d}{dx}\int P(x)\,dx$$
$$= e^{\int P(x)\,dx} \cdot P(x)$$
$$= \mu P.$$

Method of solution for solving a 1ST order linear ODE

1. Write equation in standard form: $\frac{dy}{dx} + P(x)y = Q(x)$
2. Find $\mu(x) = e^{\int P(x)\,dx}$
3. Multiply DE by μ: $\underbrace{\mu\frac{dy}{dx} + \mu P y}_{\frac{d}{dx}(\mu y)} = \mu Q$

$$\frac{d}{dx}(\mu y) = \mu Q$$

(4) Integrate : $\mu y = \int \mu Q \, dx + C$

$$\therefore \; y = \frac{1}{\mu} \int \mu Q \, dx + \frac{1}{\mu} C$$

<u>Example 1</u> Solve : $y' - \frac{2}{x} y = x^2 \sin(3x)$

<u>Solution</u> $\mu = e^{\int -\frac{2}{x} dx} = e^{-2\ln x}$

$$= e^{\ln(x^{-2})} = x^{-2} = \frac{1}{x^2}.$$

Multiply DE by x^{-2} : $\underbrace{x^{-2} y' - y \, 2x^{-3}} = \sin(3x)$

$$\frac{d}{dx}\left(x^{-2} y\right) = \sin(3x)$$

$$\therefore \; x^{-2} y = \int \sin(3x) \, dx$$

$$= -\frac{1}{3}\cos(3x) + C$$

$$\therefore \; y = -\frac{1}{3} x^2 \cos(3x) + C x^2$$

<u>Example 2</u> Solve : $xy' - (x+1) y = x^2 - x^3$

<u>Solution</u> $\frac{dy}{dx} - \left(1 + \frac{1}{x}\right) y = x - x^2$

$$\mu = e^{\int -1 - \frac{1}{x} dx} = e^{-x - \ln x} = e^{-x} e^{-\ln x}$$

$$= e^{-x} e^{\ln(x^{-1})} = e^{-x} (x^{-1}) = \frac{1}{x} e^{-x}$$

Multiply by μ: $\underbrace{\frac{1}{x}e^{-x}\frac{dy}{dx} - \left(\frac{1}{x}+\frac{1}{x^2}\right)e^{-x}y}_{\frac{d}{dx}\left[\frac{1}{x}e^{-x}y\right]} = (1-x)e^{-x}$

$$\frac{d}{dx}\left[\frac{1}{x}e^{-x}y\right] = (1-x)e^{-x}$$

$$\therefore \frac{1}{x}e^{-x}y = \int (1-x)e^{-x}\,dx + C$$

$u = 1-x \quad dv = e^{-x}\,dx$

$du = -\,dx \quad v = -e^{-x}$

$$\therefore \frac{1}{x}e^{-x}y = (1-x)(-e^{-x}) - \int(-e^{-x})(-dx) + C$$

$$= (1-x)(-e^{-x}) - \int e^{-x}\,dx + C$$

$$= (1-x)(-e^{-x}) - (-e^{-x}) + C$$

$$= -e^{-x} + xe^{-x} + e^{-x} + C$$

$$= xe^{-x} + C$$

$$\therefore y = x^2 + Cxe^{x}$$

<u>Example 3</u> Solve: $y' + (\tan x)y = \sin(2x)$, $y(0)=1$

<u>Solution</u> $\mu = e^{\int \tan x\,dx} = e^{\int \frac{\sin x}{\cos x}\,dx}$

$= e^{-\int \frac{-\sin x}{\cos x}\,dx} = e^{-\ln(\cos x)} = e^{\ln(\cos x)^{-1}}$

$= (\cos x)^{-1} = \sec x.$

Multiply $y' + (\tan x)y = \sin(2x)$ by $\sec x$.

$$\underbrace{(\sec x)y' + y(\tan x \sec x)}_{} = \sin(2x)\sec x$$

$$\frac{d}{dx}(\sec x \cdot y) = 2\sin x\cos x \cdot \frac{1}{\cos x}$$

$$= 2\sin x$$

Int

$$(\sec x)y = \int 2\sin x\,dx + C$$

$$\frac{1}{\cos x}y = -2\cos x + C$$

$$y = (C - 2\cos x)\cos x$$

$x=0, y=1$

$$1 = (c - 2(1))(1)$$

$$c = 3$$

$$\therefore y = (3 - 2\cos x)\cos x$$

HW Pg's 51–53

#'s : 1–21 odd, 29, 30