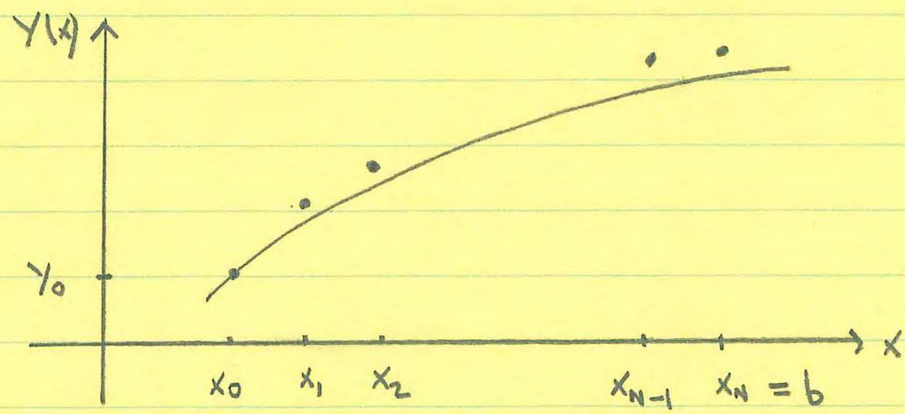
Section 1.4 The Approximation Method of Euler

Let $Y(x)$ be the true solution of the IVP with initial value Y_0:

$$Y'(x) = f(x, Y(x)) \quad , \quad x_0 \le x \le b$$
$$Y(x_0) = Y_0$$

We shall obtain an approximate solution at a discrete set of nodes:

$$x_0 < x_1 < x_2 < x_3 < \dots < x_N = b$$



We shall assume that the nodes are evenly spaced:

$$x_n = x_0 + nh \quad , \quad n = 0, 1, 2, \dots, N$$

The approximate solution at the node point is denoted $y(x_n)$ or y_n.

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Derivation of Eulers Method

$$Y'(x) = \lim_{h \to 0} \frac{Y(x+h) - Y(x)}{h}$$

For h small : $Y'(x) \doteq \frac{1}{h}[Y(x+h) - Y(x)]$

$$\therefore f(x, Y(x)) \doteq \frac{1}{h}[Y(x+h) - Y(x)]$$

Let $x = x_n$: $f(x_n, Y(x_n)) \doteq \frac{1}{h}[Y(x_n+h) - Y(x_n)]$

$$\therefore \underbrace{Y(x_n+h)}_{\doteq y_{n+1}} \doteq \underbrace{Y(x_n)}_{\doteq y_n} + h\, \underbrace{f(x_n, Y(x_n))}_{\doteq f(x_n, y_n)}$$

$$\therefore y_{n+1} \doteq y_n + h f(x_n, y_n)$$

Eulers Method is defined by taking this to be exact.

Eulers Method : $y_{n+1} = y_n + h f(x_n, y_n)$, $n = 0, 1, 2, \ldots, N-1$

Usually choose $y_0 = Y_0$.

<u>Example 1</u> Use Euler's method with step size $h = 0.1$ to approximate the solution to the initial value problem

$$\frac{dy}{dx} = x\sqrt{y}, \quad y(1) = 4$$

at the points $x = 1.1, 1.2, 1.3, 1.4, 1.5$

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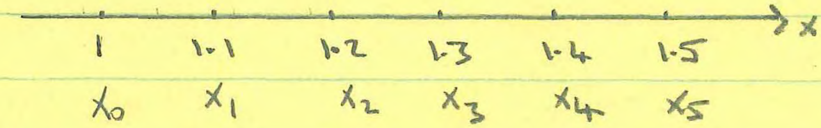
Solution $h = 0.1, \ x_0 = 1$



Let $y_0 = Y_0 = 4$

$y_{n+1} = y_n + h f(x_n, y_n) = y_n + (0.1)\, x_n \sqrt{y_n}$

$\underline{n=0}$ $\quad y_1 = y_0 + (0.1)\, x_0 \sqrt{y_0}$

$= 4 + (0.1)(1)\sqrt{4}$

$= 4.2$

$\underline{n=1}$ $\quad y_2 = y_1 + (0.1)\, x_1 \sqrt{y_1}$

$= 4.2 + (0.1)(1.1)\sqrt{4.2}$

$\doteq 4.42543 \longrightarrow a$

$\underline{n=2}$ $\quad y_3 = y_2 + (0.1)\, x_2 \sqrt{y_2}$

$= a + (0.1)\, 1.2 \sqrt{a}$

$\doteq 4.67787 \longrightarrow b$

$\underline{n=3}$ $\quad y_4 = y_3 + (0.1)\, x_3 \sqrt{y_3}$

$= b + (0.1)\, 1.3 \sqrt{b}$

$\doteq 4.95904 \longrightarrow c$

$\underline{n=4}$ $\quad y_5 = y_4 + (0.1)\, x_4 \sqrt{y_4}$

$= c + (0.1)\, 1.4 \sqrt{c}$

$\doteq 5.27081 \longrightarrow d$

See data of Table 1.1 on page 25.

n	x_n	y_n	Y_n	Error $= Y_n - y_n$
0	1	4	4	0
1	1.1	4.2	4.21276	.01276
2	1.2	4.42543	4.45210	.02667
3	1.3	4.67787	4.71976	.04189
4	1.4	4.95904	5.01760	.05856
5	1.5	5.27081	5.34766	.07685

<u>HW</u> Pg 28, #'s : 1, 3, 5, 7