5.5. Random Functions Associated with Normal Distributions

1. *Recall that:* If $X \sim N(\mu, \sigma^2)$, then X has mgf

$$M(t) = \exp(\mu t + \sigma^2 t^2/2).$$

2. The next theorem tells us that the sum of two independent normal distributions is also normal.

3. *Theorem.*

If X and Y are independent random variables,

$X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, then

$$X + Y \sim N\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right).$$

* Proof. One way to prove the theorem is to use the transformation technique that we learned in Section 5.2. Here, however, we will use the moment-generating function technique from the last section.

X has mgf $M_X(t) = \exp(\mu_1 t + \sigma_1^2 t^2/2)$,

Y has mgf $M_Y(t) = \exp(\mu_2 t + \sigma_2^2 t^2/2)$.

Since X and Y are independent,

$$M_{X+Y}(t) = M_X(t) M_Y(t) = \exp((\mu_1 + \mu_2)t + (\sigma_1^2 + \sigma_2^2)t^2/2),$$

which is precisely the mgf of the normal distribution

$$N\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right).$$

4. In a very similar way, we can prove the following result:

5. *Theorem.*

   If $X \sim N(\mu, \sigma^2)$, then $cX \sim N(c\mu, c^2\sigma^2)$.

6. More generally, we have the following theorem, which shows that a linear combination of independent normal distributions is also normal.

7. *Theorem.* Suppose that $X_1, X_2, \cdots, X_n$ are independent random variables, and suppose that each $X_i \sim N(\mu_i, \sigma_i^2)$, $1 \le i \le n$. If $Y = \sum_{i=1}^{n} c_i X_i$, then

$$Y \sim N\left(\sum_{i=1}^{n} c_i\mu_i, \sum_{i=1}^{n} c_i^2\sigma_i^2\right).$$

8. *Exercise.* Suppose that $X \sim N(2,1)$, $Y \sim N(4,3)$, and suppose that X, Y are independent. Find the distribution of $U = X + 2Y$.

9. *Exercise.* Suppose that $X \sim N(2,1)$, $Y \sim N(-3,5)$, and suppose that X, Y are independent. Find the distribution of $V = 3X - 2Y$.

10. *Recall that:* If X and Y have joint pdf $f(x,y)$ and they have marginal pdfs $f_1(x)$ and $f_2(y)$, respectively. Then X and Y are independent if and only if

$$f(x,y) \equiv f_1(x) f_2(y).$$

11. *Theorem.* Suppose that X_1, X_2 are a random sample of size 2 from a population X, and X has standard normal distribution $N(0,1)$. Define

$$U = \frac{1}{2}(X_1 + X_2), \quad V = \frac{1}{2}(X_1 - X_2),$$

$$S^2 = 2V^2.$$

Then

(a) $U \sim N(0, 1/2)$, $V \sim N(0, 1/2)$, $S^2 \sim \chi^2(1)$,

(b) U and V are independent; U and S^2 are independent.

* If we apply Theorem 7 of this section, it is an easy matter to prove that $U \sim N(0, 1/2)$, $V \sim N(0, 1/2)$.

Since $V \sim N(0, 1/2)$, we have $\sqrt{2}V \sim N(0, 1)$. It follows that

$$(\sqrt{2}V)^2 \sim \chi^2(1).$$

In other words,

$$S^2 = (\sqrt{2}V)^2 \sim \chi^2(1).$$

The proof of the independence of U and S^2 is by the fact in Item 10 of this section. Since the proof of this independence is out of the scope of this course, it is omitted here.

12. More generally, we have the following result, which will be stated without proof.

13. *Theorem.* Suppose that $X_1, \cdots, X_n$ are a random sample of size n from a population X and X has normal distribution $N(\mu, \sigma^2)$. Define the sample mean

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i,$$

and define the sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

Then, $\bar{X}$ and S^2 are independent,

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right), \qquad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

And, as a consequence,

$$E(S^2) = \sigma^2.$$

14. *Definition.* Suppose $Z \sim N(0,1)$, $U \sim \chi^2(r)$, and, Z and U are independent. It is clear that U has mean r and variance $2r$. Let

$$T = \frac{Z}{\sqrt{U/r}}.$$

Then the distribution of the random variable T is called the student's t-distribution with parameter r, or the student's t-distribution with r degrees of freedom.

15. *Theorem.* If T has the student's t-distribution with parameter r, then the pdf of T is

$$f(t) = \frac{\Gamma((r+1)/2)}{\sqrt{\pi r}\,\Gamma(r/2)\,(1+t^2/r)^{(r+1)/2}}, \qquad -\infty < t < \infty.$$

* The proof is by the transformation technique that we learned in Sections 5.1 and 5.2. Since this is our first reading of the subject, the proof is omitted here, for now.

16. *An approximation.* If r is large positive, then $t_r \approx N(0,1)$.

* Again, the proof will be omitted here, for now. What I can tell you now is: For this proof, we need the formulas

$$\frac{\Gamma((r+1)/2)}{\Gamma(r/2)} \approx \sqrt{\frac{r}{2}}, \quad r \to \infty,$$

$$(1+1/r)^r \approx e, \quad r \to \infty.$$

More generally, we have, for each fixed complex number z, it holds that

$$\lim_{n\to\infty} \frac{\Gamma(n+z)}{\Gamma(n)\, n^z} = 1.$$

17. The t-distributions appear naturally in the study of random samples. See the example below.

18. *Example.* Suppose that $X_1, \cdots, X_n$ is a random sample of size n from the normal distribution $N(\mu, \sigma^2)$. Define the sample mean $\bar{X}$ and the sample variance S^2 as before. By Theorem 13 of this section, $\bar{X}$ and S^2 are independent,

$$\bar{X} \sim N(\mu, \sigma^2/n), \tag{1}$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1), \tag{2}$$

and, $E(S^2) = \sigma^2$.

Note that $\bar{X}$ has a normal distribution, it can be standardized into $N(0,1)$. That is,

$$\frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \sim N(0,1). \tag{3}$$

For easy reference, we denote

$$Z = \frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}}, \quad K = \frac{(n-1)S^2}{\sigma^2}.$$

Now equations (3) and (2) read $Z \sim N(0,1)$ and $K \sim \chi^2_{n-1}$. By Theorem 13 of this section, Z and K are independent.

By the definition of the t-distribution, we have

$$\frac{Z}{\sqrt{K/(n-1)}} \sim t_{n-1}.$$

That is,

$$\frac{\frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}}}{\sqrt{\left(\frac{(n-1)S^2}{\sigma^2}\right)/(n-1)}} \sim t_{n-1}, \tag{4}$$

which, after some algebra, simplifies into

$$\frac{\bar{X}-\mu}{\sqrt{S^2/n}} \sim t_{n-1}.$$

Since $E(S^2) = \sigma^2$, S^2 can be used as an estimator for σ^2. For this reason, we sometimes denote S^2 by $\hat{\sigma}^2$, and with this new notation we can rewrite the last equation as

$$\frac{\bar{X} - \mu}{\sqrt{\hat{\sigma}^2/n}} \sim t_{n-1}. \tag{5}$$

Note that this last formula (Equation (5)) looks very similar to the following formula:

$$\frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \sim N(0, 1).$$

19. *A Relation.* The t-distributions are related to the F-distributions in the following manner: If $X \sim t_r$, then $X^2 \sim F_{1,r}$. Symbolically,

$$F_{1,r} = t_r^2.$$

* *Proof.* The formula can be proved easily by comparing the definitions for the t-distribution and the F-distribution.