Note: This homework can NOT be submitted late, since I will be posting solutions on the 5th in anticipation of the exam on the 7th. Remember, searching the internet is not allowed!

Basic skills

Complete all of the basic skills questions (questions 1-7).

Question 1. Read the definition of the Cartesian product of graphs on page 24, and look at the examples. Then:

(a) Draw $K_4 \times K_3$.

(b) How many vertices would $C_6 \times K_5$ have? How many edges?

(c) Draw the complement of $P_2 \times K_4$.

Question 2. Explain the idea behind a proof by induction. (Pretend you are teaching someone why induction works.)

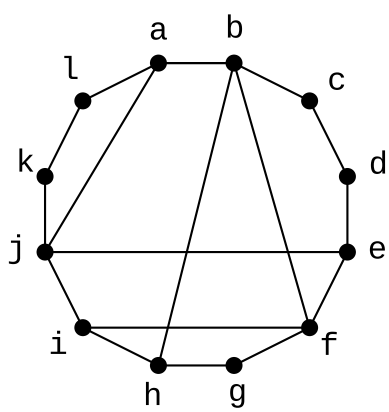
Question 3. Basics about bipartite graphs.

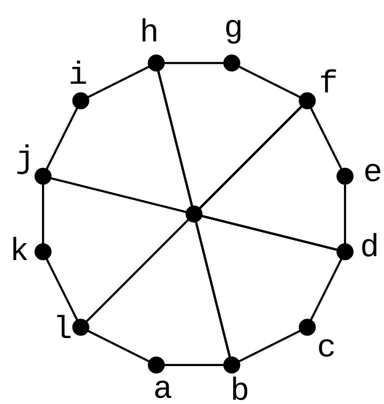
(a) Draw $K_{2,5}$ and $\overline{K_{2,5}}$.

(b) How many edges are there in $K_{n,m}$? Defend your answer.

(c) Suppose G is a bipartite graph with bipartition X and Y, $|X| = 540$, $|Y| = 108$, and every vertex in X has degree 3. If every vertex in Y has the same degree, what is that degree? Defend your answer.

(d) Determine if the graphs below are bipartite. If they are, redraw them to clearly indicate the bipartition. If not, give a convincing reason why not.





Question 4. For each of the graphs above:

(a) What is $d(a, e)$?

(b) Find their diameter, and explain how you know your answer is correct.

(c) Find a $c - h$ geodesic.

Question 5. Determine if the statements are true or false, and briefly defend your answer. No formal proof needed.

(a) For any graph G, if G is bipartite, then $\overline{G}$ is bipartite.

(b) For any graph G, if G is bipartite, then $\overline{G}$ is not bipartite.

Question 6. Read the proof below, and answer the questions that follow. The numbers in brackets are not part of the proof; they relate to the questions that follow. Note that once you understand the proof, the questions should be quite easy - don't second-guess yourself. Also, feel free to use a picture to help explain any of your answers.

Theorem. Suppose G is an n-vertex graph that does not contain any triangles. Then the number of edges in G is at most $\lfloor n^2/4 \rfloor$, where $\lfloor x \rfloor$ is the function that rounds x down to the nearest integer. (If x is already an integer, then $\lfloor x \rfloor = x$.)

Proof. Suppose G is an n-vertex triangle-free graph. Let $k = \Delta(G)$, and let x be a vertex of maximum degree in G. Since G has no triangles, there are no edges between neighbors of x [1]. Hence summing the degrees of x and its non-neighbors counts at least one endpoint of every edge [2], so this sum is at least $|E(G)|$. The sum is taken over $n - k$ vertices, each having degree at most k, so $|E(G)| \leq (n - k)k$ [3].

Observe that for fixed n, $(n - k)k$ is a concave-down function of k with a maximum value when $k = n/2$ [4]. Hence $n(n - k) \leq n(n/2)$ [5]. Therefore $|E(G)| \leq \lfloor n^2/4 \rfloor$. □

(a) What kid of proof is this? (Direct, contradiction, induction?) How do you know?

(b) Explain why the sentence ending with [1] is true.

(c) Explain why the sentence ending with [2] is true.

(d) Explain the inequality just before [3].

(e) Explain why the sentence ending with [4] is correct.

(f) The inequality in [5] doesn't have a floor function, but the final inequality does. Why?

(g) Construct a 10-vertex graph and an 11-vertex graph that achieve this edge bound (that is, that have exactly $\lfloor n^2/4 \rfloor$ edges). The proof should help you.

Question 7. No formal proofs needed for these.

(a) Draw a graph G with exactly two components, whose complement also has two components, or state why no such graph exists.

(b) Draw a disconnected graph G with 17 vertices, where every vertex has degree 8, or state why no such graph exists.

The fun problems

Prove one of the two following questions **using induction.**

Question 8. The complete graph on n vertices has $n(n - 1)/2$ edges. (Note that induction is NOT the easiest way to prove this! But it is definitely the most fun.)

Question 9. (More challenging) Let G_k be a graph with vertex set $V(G) = \{x_1, y_1, x_2, y_2, \ldots, x_k, y_k\}$. Suppose $x_i y_i \notin E(G_k)$ for any $1 \leq i \leq k$, and no pair of vertices in $V(G_k) \setminus \{x_1\}$ have the same degree. Prove that for any graph satisfying these properties, $d(x_1) = k - 1$.

Choose **three** of the following problems to complete. They should all be proven formally, unless otherwise indicated. You may use any method. **Remember to keep and submit your scrap work for each problem.**

Question 10. Suppose for a graph G, an edge $e \in E(G)$ appears an odd number of times in a closed walk W. Prove that W contains a cycle in which e appears.

Question 11. Let n be an arbitrary positive integer. Let G_n be the graph whose vertices consist of all of the subsets of $\{1, 2, \ldots, n\}$, and whose edge set consists of all pairs of vertices $\{A, B\}$ such that $|A| = |B| - 1$ and $A \subset B$, or $|B| = |A| - 1$ and $B \subset A$. Prove or disprove: For all $n \geq 1$, G_n is bipartite.

Question 12. A set of vertices I is an *independent set* if $G[I]$ (the subgraph induced by I) is an empty graph. Prove that G is a bipartite graph if and only if every subgraph of G has an independent set containing at least half of the vertices.

Question 13. For any connected graph G, if the diameter of G is at least 3, then the diameter of $\overline{G}$ is at most 3. (Hint: If a graph has diameter at least 3, then for any two vertices x and y, the neighborhoods of x and y do not overlap.)

Question 14. Use induction on n to prove that every n-vertex graph G with at least one edge has a bipartite subgraph whose size is greater than $|E(G)|/2$.