Theorem: Quotient-Remainder Theorem (Existence Part)

Given any integer n and any positive integer d, there exists integers q and r such that

Proof

Let S be the set of all non-negative integers of the form

$$n - dk$$

where k is an integer. This set has at least one element. [For if n is non-negative, then

$$n - 0 \cdot d = n \geq 0$$

and so $n - 0 \cdot d$ is in S. And if n is negative, then

$$n - nd = n(1 - d) \geq 0$$

and so $n - nd$ is in S.] It follows by the well-ordering principle for the integers that S contains a least element r. Then, for some specific integer $k = q$,

$$n - d1 = r$$

[because every integer in S can be written in this form]. Adding dq to both sides gives

$$n = dq + r$$

Furthermore, $r < d$. [For suppose $r \geq d$. Then

$$n - d(q + 1) = n - dq - d = r - d \geq 0$$

and so $n - d(q + 1)$ would be a non-negative integer in S that would be smaller than r. But r is the smallest integer in S. This contradiction shows that the supposition $r \geq d$ must be false.] The preceding arguments prove that there exists integers r and q for which

$$n = dq + r \quad \text{and} \quad 0 \leq r < d$$