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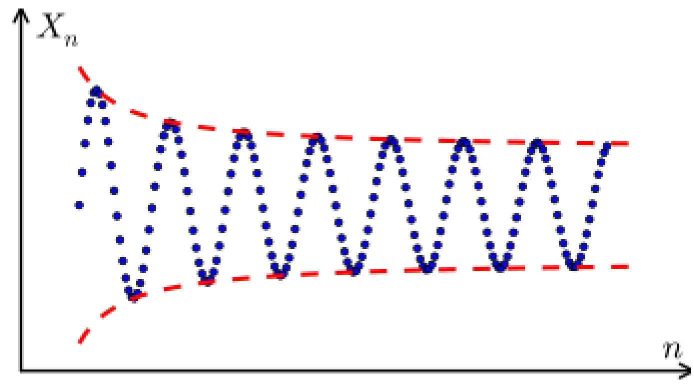
# Sequence

In mathematics, a **sequence** is an enumerated collection of objects in which repetitions are allowed. Like a set, it contains members (also called *elements*, or *terms*). The number of elements (possibly infinite) is called the *length* of the sequence. Unlike a set, the same elements can appear multiple times at different positions in a sequence, and order matters. Formally, a sequence can be defined as a function whose domain is either the set of the natural numbers (for infinite sequences) or the set of the first $n$ natural numbers (for a sequence of finite length $n$). The position of an element in a sequence is its *rank* or *index*; it is the natural number from which the element is the image. It depends on the context or a specific convention, if the first element has index 0 or 1. When a symbol has been chosen for denoting a sequence, the $n$th element of the sequence is denoted by this symbol with $n$ as subscript; for example, the $n$th element of the Fibonacci sequence is generally denoted $F_n$.

For example, (M, A, R, Y) is a sequence of letters with the letter 'M' first and 'Y' last. This sequence differs from (A, R, M, Y). Also, the sequence (1, 1, 2, 3, 5, 8), which contains the number 1 at two different positions, is a valid sequence. Sequences can be *finite*, as in these examples, or *infinite*, such as the sequence of all even positive integers (2, 4, 6, ...). In computing and computer science, finite sequences are sometimes called strings, words or lists, the different names commonly corresponding to different ways to represent them in computer memory; infinite sequences are called streams. The empty sequence ( ) is included in most notions of sequence, but may be excluded depending on the context.




An infinite sequence of real numbers (in blue). This sequence is neither increasing, decreasing, convergent, nor Cauchy. It is, however, bounded.

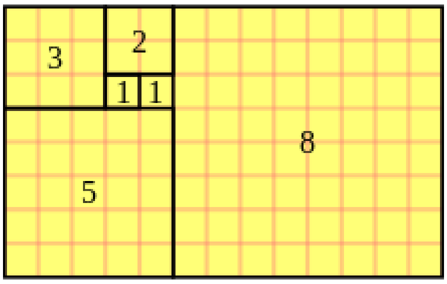
## Contents

# Examples and notation

A sequence can be thought of as a list of elements with a particular order. Sequences are useful in a number of mathematical disciplines for studying functions, spaces, and other mathematical structures using the convergence properties of sequences. In particular, sequences are the basis for series, which are important in differential equations and analysis. Sequences are also of interest in their own right and can be studied as patterns or puzzles, such as in the study of prime numbers.

There are a number of ways to denote a sequence, some of which are more useful for specific types of sequences. One way to specify a sequence is to list the elements. For example, the first four odd numbers form the sequence (1, 3, 5, 7). This notation can be used for infinite sequences as well. For instance, the infinite sequence of positive odd integers can be written (1, 3, 5, 7, ...). Listing is most useful for infinite sequences with a pattern that can be easily discerned from the first few elements. Other ways to denote a sequence are discussed after the examples.

## Examples



A tiling with squares whose sides are successive Fibonacci numbers in length.

The prime numbers are the natural numbers bigger than 1 that have no divisors but 1 and themselves. Taking these in their natural order gives the sequence (2, 3, 5, 7, 11, 13, 17, ...). The prime numbers are widely used in mathematics and specifically in number theory.

The Fibonacci numbers are the integer sequence whose elements are the sum of the previous two elements. The first two elements are either 0 and 1 or 1 and 1 so that the sequence is (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...).

For a large list of examples of integer sequences, see On-Line Encyclopedia of Integer Sequences.

Other examples of sequences include ones made up of rational numbers, real numbers, and complex numbers. The sequence (.9, .99, .999, .9999, ...) approaches the number 1. In fact, every real number can be written as the limit of a sequence of rational numbers, e.g. via its decimal expansion. For instance, π is the limit of the sequence (3, 3.1, 3.14, 3.141, 3.1415, ...). A related sequence is the sequence of decimal digits of π, i.e. (3, 1, 4, 1, 5, 9, ...). This sequence does not have any pattern that is easily discernible by eye, unlike the preceding sequence, which is increasing.

## Indexing

Other notations can be useful for sequences whose pattern cannot be easily guessed, or for sequences that do not have a pattern such as the digits of π. One such notation is to write down a general formula for computing the *n*th term as a function of *n*, enclose it in parentheses, and include a subscript indicating the range of values that *n* can take. For example, in this notation the sequence of even numbers could be written as $(2n)_{n\in\mathbb{N}}$. The sequence of squares could be written as $(n^2)_{n\in\mathbb{N}}$. The variable *n* is called an index, and the set of values that it can take is called the index set.

It is often useful to combine this notation with the technique of treating the elements of a sequence as variables. This yields expressions like $(a_n)_{n\in\mathbb{N}}$, which denotes a sequence whose *n*th element is given by the variable $a_n$. For example:

$$
\begin{aligned}
a_1 &= \text{1st element of } (a_n)_{n\in\mathbb{N}} \\
a_2 &= \text{2nd element} \\
a_3 &= \text{3rd element} \\
&\vdots \\
a_{n-1} &= (n-1)\text{th element} \\
a_n &= n\text{th element} \\
a_{n+1} &= (n+1)\text{th element} \\
&\vdots
\end{aligned}
$$

Note that we can consider multiple sequences at the same time by using different variables; e.g. $(b_n)_{n\in\mathbb{N}}$ could be a different sequence than $(a_n)_{n\in\mathbb{N}}$. We can even consider a sequence of sequences: $((a_{m,n})_{n\in\mathbb{N}})_{m\in\mathbb{N}}$ denotes a sequence whose $m$th term is the sequence $(a_{m,n})_{n\in\mathbb{N}}$.

An alternative to writing the domain of a sequence in the subscript is to indicate the range of values that the index can take by listing its highest and lowest legal values. For example, the notation $(k^2)_{k=1}^{10}$ denotes the ten-term sequence of squares $(1, 4, 9, \ldots, 100)$. The limits $\infty$ and $-\infty$ are allowed, but they do not represent valid values for the index, only the supremum or infimum of such values, respectively. For example, the sequence $(a_n)_{n=1}^{\infty}$ is the same as the sequence $(a_n)_{n\in\mathbb{N}}$, and does not contain an additional term "at infinity". The sequence $(a_n)_{n=-\infty}^{\infty}$ is a **bi-infinite sequence**, and can also be written as $(\ldots, a_{-1}, a_0, a_1, a_2, \ldots)$.

In cases where the set of indexing numbers is understood, the subscripts and superscripts are often left off. That is, one simply writes $(a_k)$ for an arbitrary sequence. Often, the index $k$ is understood to run from 1 to ∞. However, sequences are frequently indexed starting from zero, as in

$$(a_k)_{k=0}^{\infty} = (a_0, a_1, a_2, \ldots).$$

In some cases the elements of the sequence are related naturally to a sequence of integers whose pattern can be easily inferred. In these cases the index set may be implied by a listing of the first few abstract elements. For instance, the sequence of squares of odd numbers could be denoted in any of the following ways.

- $(1, 9, 25, \ldots)$
- $(a_1, a_3, a_5, \ldots), \qquad a_k = k^2$
- $(a_{2k-1})_{k=1}^{\infty}, \qquad a_k = k^2$
- $(a_k)_{k=1}^{\infty}, \qquad a_k = (2k-1)^2$
- $((2k-1)^2)_{k=1}^{\infty}$

Moreover, the subscripts and superscripts could have been left off in the third, fourth, and fifth notations, if the indexing set was understood to be the natural numbers. Note that in the second and third bullets, there is a well-defined sequence $(a_k)_{k=1}^{\infty}$, but it is not the same as the sequence denoted by the expression.

## Defining a sequence by recursion

Sequences whose elements are related to the previous elements in a straightforward way are often defined using recursion. This is in contrast to the definition of sequences of elements as functions of their positions.

To define a sequence by recursion, one needs a rule to construct each element in terms of the ones before it. In addition, enough initial elements must be provided so that all subsequent elements of the sequence can be computed by the rule. The principle of mathematical induction can be used to prove that in this case, there is exactly one sequence that satisfies both the recursion rule and the initial conditions. Induction can also be used to prove properties about a sequence, especially for sequences whose most natural description is recursive.

The Fibonacci sequence can be defined using a recursive rule along with two initial elements. The rule is that each element is the sum of the previous two elements, and the first two elements are 0 and 1.

$$a_n = a_{n-1} + a_{n-2}, \quad \text{with} \quad a_0 = 0 \text{ and } a_1 = 1.$$

The first ten terms of this sequence are 0, 1, 1, 2, 3, 5, 8, 13, 21, and 34. A more complicated example of a sequence that is defined recursively is Recaman's sequence.[1] We can define Recaman's sequence by

$a_0 = 0$ and $a_n = a_{n-1} - n$, if the result is positive and not already in the list. Otherwise, $a_n = a_{n-1} + n$.

Not all sequences can be specified by a rule in the form of an equation, recursive or not, and some can be quite complicated. For example, the sequence of prime numbers is the set of prime numbers in their natural order, i.e. (2, 3, 5, 7, 11, 13, 17, ...).

Many sequences have the property that each element of a sequence can be computed from the previous element. In this case, there is some function *f* such that for all *n*, $a_{n+1} = f(a_n)$.

# Formal definition and basic properties

There are many different notions of sequences in mathematics, some of which (*e.g.*, exact sequence) are not covered by the definitions and notations introduced below.

## Formal definition

For the purposes of this article, we define a sequence to be a function whose domain is an interval of integers. This definition covers several different uses of the word "sequence", including one-sided infinite sequences, bi-infinite sequences, and finite sequences (see below for definitions). However, many authors use a narrower definition by requiring the domain of a sequence to be the set of natural numbers. The narrower definition has the disadvantage that it rules out finite sequences and bi-infinite sequences, both of which are usually called sequences in standard mathematical practice. In some contexts, to shorten exposition, the codomain of the sequence is fixed by context, for example by requiring it to be the set **R** of real numbers,[2] the set **C** of complex numbers,[3] or a topological space.[4]

Although sequences are a type of function, they are usually distinguished notationally from functions in that the input is written as a subscript rather than in parentheses, i.e. $a_n$ rather than *f*(*n*). There are terminological differences as well: the value of a sequence at the input 1 is called the "first element" of the sequence, the value at 2 is called the "second element", etc. Also, while a function abstracted from its input is usually denoted by a single letter, e.g. *f*, a sequence abstracted from its input is usually written by a notation such as $(a_n)_{n\in A}$, or just as $(a_n)$. Here *A* is the domain, or index set, of the sequence.

Sequences and their limits (see below) are important concepts for studying topological spaces. An important generalization of sequences is the concept of nets. A **net** is a function from a (possibly uncountable) directed set to a topological space. The notational conventions for sequences normally apply to nets as well.

## Finite and infinite

The **length** of a sequence is defined as the number of terms in the sequence.

A sequence of a finite length *n* is also called an *n*-tuple. Finite sequences include the **empty sequence** ( ) that has no elements.

Normally, the term *infinite sequence* refers to a sequence that is infinite in one direction, and finite in the other—the sequence has a first element, but no final element. Such a sequence is called a **singly infinite sequence** or a **one-sided infinite sequence** when disambiguation is necessary. In contrast, a sequence that is infinite in both directions—i.e. that has neither a first nor a final element

—is called a **bi-infinite sequence**, **two-way infinite sequence**, or **doubly infinite sequence**. A function from the set **Z** of *all* integers into a set, such as for instance the sequence of all even integers ( …, −4, −2, 0, 2, 4, 6, 8… ), is bi-infinite. This sequence could be denoted $(2n)_{n=-\infty}^{\infty}$.

## Increasing and decreasing

A sequence is said to be *monotonically increasing*, if each term is greater than or equal to the one before it. For example, the sequence $(a_n)_{n=1}^{\infty}$ is monotonically increasing if and only if $a_{n+1} \geq a_n$ for all $n \in \mathbf{N}$. If each consecutive term is strictly greater than (>) the previous term then the sequence is called **strictly monotonically increasing**. A sequence is **monotonically decreasing**, if each consecutive term is less than or equal to the previous one, and **strictly monotonically decreasing**, if each is strictly less than the previous. If a sequence is either increasing or decreasing it is called a **monotone** sequence. This is a special case of the more general notion of a monotonic function.

The terms **nondecreasing** and **nonincreasing** are often used in place of *increasing* and *decreasing* in order to avoid any possible confusion with *strictly increasing* and *strictly decreasing*, respectively.

## Bounded

If the sequence of real numbers ($a_n$) is such that all the terms are less than some real number $M$, then the sequence is said to be **bounded from above**. In other words, this means that there exists $M$ such that for all $n$, $a_n \leq M$. Any such $M$ is called an *upper bound*. Likewise, if, for some real $m$, $a_n \geq m$ for all $n$ greater than some $N$, then the sequence is **bounded from below** and any such $m$ is called a *lower bound*. If a sequence is both bounded from above and bounded from below, then the sequence is said to be **bounded**.

## Subsequences

A **subsequence** of a given sequence is a sequence formed from the given sequence by deleting some of the elements without disturbing the relative positions of the remaining elements. For instance, the sequence of positive even integers (2, 4, 6, ...) is a subsequence of the positive integers (1, 2, 3, ...). The positions of some elements change when other elements are deleted. However, the relative positions are preserved.

Formally, a subsequence of the sequence $(a_n)_{n\in\mathbb{N}}$ is any sequence of the form $(a_{n_k})_{k\in\mathbb{N}}$, where $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive integers.

## Other types of sequences

Some other types of sequences that are easy to define include:

- An **integer sequence** is a sequence whose terms are integers.
- A **polynomial sequence** is a sequence whose terms are polynomials.
- A positive integer sequence is sometimes called **multiplicative**, if $a_{nm} = a_n a_m$ for all pairs $n$, $m$ such that $n$ and $m$ are coprime.[5] In other instances, sequences are often called *multiplicative*, if $a_n = na_1$ for all $n$. Moreover, a *multiplicative* Fibonacci sequence[6] satisfies the recursion relation $a_n = a_{n-1} a_{n-2}$.
- A binary sequence is a sequence whose terms have one of two discrete values, e.g. base 2 values (0,1,1,0, ...) , a series of coin tosses (Heads/Tails) H,T,H,H,T, ..., the answers to a set of True or False questions (T, F, T, T, ...), and so on.

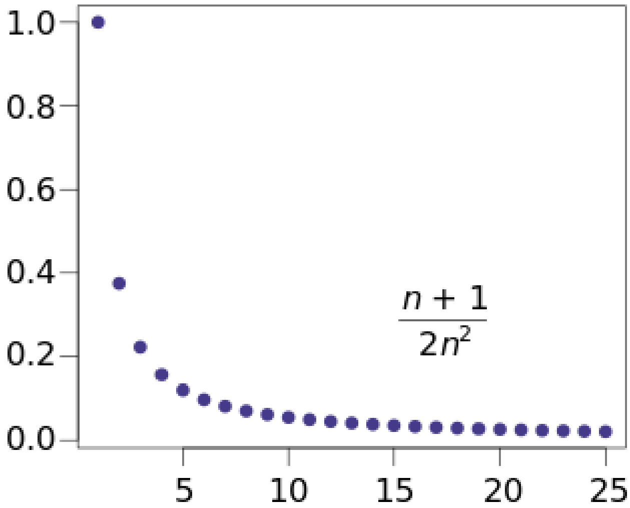
# Limits and convergence

An important property of a sequence is *convergence*. If a sequence converges, it converges to a particular value known as the *limit*. If a sequence converges to some limit, then it is **convergent**. A sequence that does not converge is **divergent**.

Informally, a sequence has a limit if the elements of the sequence become closer and closer to some value $L$ (called the limit of the sequence), and they become and remain *arbitrarily* close to $L$, meaning that given a real number $d$ greater than zero, all but a finite number of the elements of the sequence have a distance from $L$ less than $d$.

For example, the sequence $a_n = \frac{n+1}{2n^2}$ shown to the right converges to the value 0. On the other hand, the sequences $b_n = n^3$ (which begins 1, 8, 27, …) and $c_n = (-1)^n$ (which begins -1, 1, -1, 1, …) are both divergent.

If a sequence converges, then the value it converges to is unique. This value is called the **limit** of the sequence. The limit of a convergent sequence $(a_n)$ is normally denoted $\lim_{n\to\infty} a_n$. If $(a_n)$ is a divergent sequence, then the expression $\lim_{n\to\infty} a_n$ is meaningless.



The plot of a convergent sequence ($a_n$) is shown in blue. From the graph we can see that the sequence is converging to the limit zero as $n$ increases.

## Formal definition of convergence

A sequence of real numbers $(a_n)$ **converges to** a real number $L$ if, for all $\varepsilon > 0$, there exists a natural number $N$ such that for all $n \geq N$ we have[2]

$$|a_n - L| < \varepsilon.$$

If $(a_n)$ is a sequence of complex numbers rather than a sequence of real numbers, this last formula can still be used to define convergence, with the provision that $|\cdot|$ denotes the complex modulus, i.e. $|z| = \sqrt{z^* z}$. If $(a_n)$ is a sequence of points in a metric space, then the formula can be used to define convergence, if the expression $|a_n - L|$ is replaced by the expression $\operatorname{dist}(a_n, L)$, which denotes the distance between $a_n$ and $L$.

## Applications and important results

If $(a_n)$ and $(b_n)$ are convergent sequences, then the following limits exist, and can be computed as follows:[2][7]

- $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$ for all $c$
- $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided that $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = \left(\lim_{n\to\infty} a_n\right)^p$ for all $p > 0$

Moreover:

- If $a_n \leq b_n$ for all $n$ greater than some $N$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.[a]
- (Squeeze Theorem)
  If $(c_n)$ is a sequence such that $a_n \leq c_n \leq b_n$ for all $n > N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $(c_n)$ is convergent, and $\lim_{n\to\infty} c_n = L$.
- If a sequence is bounded and monotonic then it is convergent.

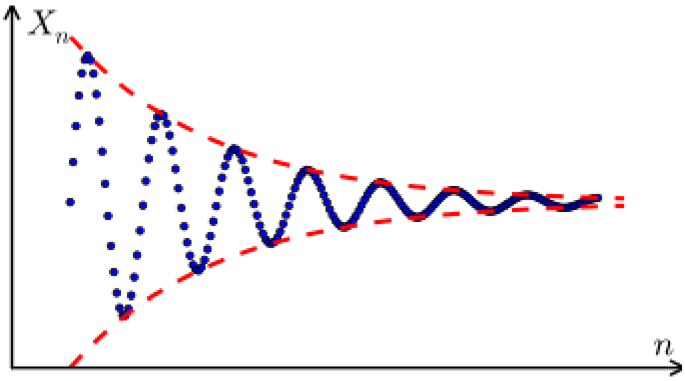
- A sequence is convergent if and only if all of its subsequences are convergent.

## Cauchy sequences

A Cauchy sequence is a sequence whose terms become arbitrarily close together as n gets very large. The notion of a Cauchy sequence is important in the study of sequences in metric spaces, and, in particular, in real analysis. One particularly important result in real analysis is *Cauchy characterization of convergence for sequences*:

> A sequence of real numbers is convergent (in the reals) if and only if it is Cauchy.

In contrast, there are Cauchy sequences of rational numbers that are not convergent in the rationals, e.g. the sequence defined by $x_1 = 1$ and $x_{n+1} = \frac{x_n + \frac{2}{x_n}}{2}$ is Cauchy, but has no rational limit, cf. here. More generally, any sequence of rational numbers that converges to an irrational number is Cauchy, but not convergent when interpreted as a sequence in the set of rational numbers.




The plot of a Cauchy sequence ($X_n$), shown in blue, as $X_n$ versus $n$. In the graph the sequence appears to be converging to a limit as the distance between consecutive terms in the sequence gets smaller as $n$ increases. In the real numbers every Cauchy sequence converges to some limit.

Metric spaces that satisfy the Cauchy characterization of convergence for sequences are called complete metric spaces and are particularly nice for analysis.

## Infinite limits

In calculus, it is common to define notation for sequences which do not converge in the sense discussed above, but which instead become and remain arbitrarily large, or become and remain arbitrarily negative. If $a_n$ becomes arbitrarily large as $n \to \infty$, we write

$$\lim_{n\to\infty} a_n = \infty.$$

In this case we say that the sequence **diverges**, or that it **converges to infinity**. An example of such a sequence is $a_n = n$.

If $a_n$ becomes arbitrarily negative (i.e. negative and large in magnitude) as $n \to \infty$, we write

$$\lim_{n\to\infty} a_n = -\infty$$

and say that the sequence **diverges** or **converges to negative infinity**.

# Series

A **series** is, informally speaking, the sum of the terms of a sequence. That is, it is an expression of the form $\sum_{n=1}^{\infty} a_n$ or $a_1 + a_2 + \cdots$, where $(a_n)$ is a sequence of real or complex numbers. The **partial sums** of a series are the expressions resulting from replacing the infinity symbol with a finite number, i.e. the $N$th partial sum of the series $\sum_{n=1}^{\infty} a_n$ is the number

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N.$$

The partial sums themselves form a sequence $(S_N)_{N\in\mathbb{N}}$, which is called the **sequence of partial sums** of the series $\sum_{n=1}^{\infty} a_n$. If the sequence of partial sums converges, then we say that the series $\sum_{n=1}^{\infty} a_n$ is **convergent**, and the limit $\lim_{N\to\infty} S_N$ is called the **value** of the series. The same notation is used to denote a series and its value, i.e. we write $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$.

# Use in other fields of mathematics

## Topology

Sequences play an important role in topology, especially in the study of metric spaces. For instance:

- A metric space is compact exactly when it is sequentially compact.
- A function from a metric space to another metric space is continuous exactly when it takes convergent sequences to convergent sequences.
- A metric space is a connected space if and only if, whenever the space is partitioned into two sets, one of the two sets contains a sequence converging to a point in the other set.
- A topological space is separable exactly when there is a dense sequence of points.

Sequences can be generalized to nets or filters. These generalizations allow one to extend some of the above theorems to spaces without metrics.

### Product topology

The topological product of a sequence of topological spaces is the cartesian product of those spaces, equipped with a natural topology called the product topology.

More formally, given a sequence of spaces $(X_i)_{i\in\mathbb{N}}$, the product space

$$X := \prod_{i\in\mathbb{N}} X_i,$$

is defined as the set of all sequences $(x_i)_{i\in\mathbb{N}}$ such that for each $i$, $x_i$ is an element of $X_i$. The **canonical projections** are the maps $p_i : X \to X_i$ defined by the equation $p_i((x_j)_{j\in\mathbb{N}}) = x_i$. Then the **product topology** on $X$ is defined to be the coarsest topology (i.e. the topology with the fewest open sets) for which all the projections $p_i$ are continuous. The product topology is sometimes called the **Tychonoff topology**.

## Analysis

In analysis, when talking about sequences, one will generally consider sequences of the form

$$(x_1, x_2, x_3, \ldots) \text{ or } (x_0, x_1, x_2, \ldots)$$

which is to say, infinite sequences of elements indexed by natural numbers.

It may be convenient to have the sequence start with an index different from 1 or 0. For example, the sequence defined by $x_n = 1/\log(n)$ would be defined only for $n \geq 2$. When talking about such infinite sequences, it is usually sufficient (and does not change much for most considerations) to assume that the members of the sequence are defined at least for all indices large enough, that is, greater than some given $N$.

The most elementary type of sequences are numerical ones, that is, sequences of real or complex numbers. This type can be generalized to sequences of elements of some vector space. In analysis, the vector spaces considered are often function spaces. Even more generally, one can study sequences with elements in some topological space.

### Sequence spaces

A sequence space is a vector space whose elements are infinite sequences of real or complex numbers. Equivalently, it is a function space whose elements are functions from the natural numbers to the field **K**, where **K** is either the field of real numbers or the field of complex numbers. The set of all such functions is naturally identified with the set of all possible infinite sequences with elements in **K**, and can be turned into a vector space under the operations of pointwise addition of functions and pointwise scalar multiplication. All sequence spaces are linear subspaces of this space. Sequence spaces are typically equipped with a norm, or at least the structure of a topological vector space.

The most important sequences spaces in analysis are the $\ell^p$ spaces, consisting of the $p$-power summable sequences, with the $p$-norm. These are special cases of $L^p$ spaces for the counting measure on the set of natural numbers. Other important classes of sequences like convergent sequences or null sequences form sequence spaces, respectively denoted $c$ and $c_0$, with the sup norm. Any sequence space can also be equipped with the topology of pointwise convergence, under which it becomes a special kind of Fréchet space called an FK-space.

## Linear algebra

Sequences over a field may also be viewed as vectors in a vector space. Specifically, the set of $F$-valued sequences (where $F$ is a field) is a function space (in fact, a product space) of $F$-valued functions over the set of natural numbers.

## Abstract algebra

Abstract algebra employs several types of sequences, including sequences of mathematical objects such as groups or rings.

### Free monoid

If $A$ is a set, the free monoid over $A$ (denoted $A^*$, also called Kleene star of $A$) is a monoid containing all the finite sequences (or strings) of zero or more elements of $A$, with the binary operation of concatenation. The free semigroup $A^+$ is the subsemigroup of $A^*$ containing all elements except the empty sequence.

### Exact sequences

In the context of group theory, a sequence

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} G_n$$

of groups and group homomorphisms is called **exact**, if the image (or range) of each homomorphism is equal to the kernel of the next:

$$\mathrm{im}(f_k) = \ker(f_{k+1})$$

Note that the sequence of groups and homomorphisms may be either finite or infinite.

A similar definition can be made for certain other algebraic structures. For example, one could have an exact sequence of vector spaces and linear maps, or of modules and module homomorphisms

### Spectral sequences

In homological algebra and algebraic topology, a **spectral sequence** is a means of computing homology groups by taking successive approximations. Spectral sequences are a generalization of exact sequences, and since their introduction by Jean Leray (1946), they have become an important research tool, particularly in homotopy theory.

## Set theory

An ordinal-indexed sequence is a generalization of a sequence. If α is a limit ordinal and $X$ is a set, an α-indexed sequence of elements of $X$ is a function from α to $X$. In this terminology an ω-indexed sequence is an ordinary sequence.

## Computing

Automata or finite state machines can typically be thought of as directed graphs, with edges labeled using some specific alphabet, Σ. Most familiar types of automata transition from state to state by reading input letters from Σ, following edges with matching labels; the ordered input for such an automaton forms a sequence called a *word* (or input word). The sequence of states encountered by the automaton when processing a word is called a *run*. A nondeterministic automaton may have unlabeled or duplicate out-edges for any state, giving more than one successor for some input letter. This is typically thought of as producing multiple possible runs for a given word, each being a sequence of single states, rather than producing a single run that is a sequence of sets of states; however, 'run' is occasionally used to mean the latter.

## Streams

Infinite sequences of digits (or characters) drawn from a finite alphabet are of particular interest in theoretical computer science. They are often referred to simply as *sequences* or *streams*, as opposed to finite *strings*. Infinite binary sequences, for instance, are infinite sequences of bits (characters drawn from the alphabet {0, 1}). The set $C = \{0, 1\}^\infty$ of all infinite binary sequences is sometimes called the Cantor space.

An infinite binary sequence can represent a formal language (a set of strings) by setting the $n$ th bit of the sequence to 1 if and only if the $n$ th string (in shortlex order) is in the language. This representation is useful in the diagonalization method for proofs.[8]

# See also

- Enumeration
- On-Line Encyclopedia of Integer Sequences
- Recurrence relation
- Sequence space

## Operations

- Cauchy product

## Examples

- Discrete-time signal
- Farey sequence
- Fibonacci sequence
- Look-and-say sequence
- Thue–Morse sequence

## Types

- ±1-sequence
- Arithmetic progression
- Automatic sequence

- Cauchy sequence
- Constant-recursive sequence
- Geometric progression
- Holonomic sequence
- Regular sequence

### Related concepts

- List (computing)
- Net (topology) (a generalization of sequences)
- Ordinal-indexed sequence
- Recursion (computer science)
- Set (mathematics)
- Tuple

# Notes

a. Note that if the inequalities are replaced by strict inequalities then this is false: There are sequences such that $a_n < b_n$ for all $n$, but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

# References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# External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Sequence", *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- The On-Line Encyclopedia of Integer Sequences
- Journal of Integer Sequences (free)
- "Sequence". *PlanetMath*.

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