5.3. Several Random Variables

1. Most of the formulas in this section are straightforward and easy to prove. We will prove only one or two of the formulas in this section. But you are encouraged to verify them by yourself if you have any doubt.

2. *Formulas.* If X and Y are random variables and a, b are real numbers, then

$$E(X+Y) = E(X) + E(Y),$$

$$E(aX) = aE(X),$$

$$E(aX+bY) = aE(X) + bE(Y).$$

Simply put, expectation is a linear operation.

3. *Formula.* Suppose that $X_1, X_2, \cdots, X_n$ are random variables, and $a_1, a_2, \cdots, a_n$ are scalars. If

$$Y = a_1 X_1 + \cdots + a_n X_n,$$

then

$$E(Y) = a_1 E(X_1) + \cdots + a_n E(X_n).$$

That is, if

$$Y = \sum_{i=1}^{n} a_i X_i,$$

then

$$E(Y) = \sum_{i=1}^{n} a_i E(X_i).$$

In words, the expectation of a linear combination is the linear combination of the expectations.

4. *Formula.* If X and Y are independent random variables, then

$$E(XY) = E(X)E(Y).$$

* *Proof of the formula in the continuous case.* Suppose that X is a continuous random variable with pdf $f_1(x)$, and Y is a continuous random variable with pdf $f_2(y)$. In this case, by definition,

$$E(X) = \int x f_1(x) dx, \quad E(Y) = \int y f_2(y) dy.$$

Since X and Y are independent, their joint pdf is the product of the marginal pdfs. That is, the joint pdf of X and Y is

$$f(x, y) = f_1(x) f_2(y).$$

Here, $f(x, y)$ denotes the joint pdf of X and Y.

Again, the key is, in the expression for $f(x, y)$, the variables x and y can be separated. If follows that

$$\begin{aligned} E(XY) &= \iint xyf(x,y)dxdy \\ &= \iint xyf_1(x)f_2(y)dxdy. \end{aligned}$$

Since the variables x and y are separated, we have

$$\begin{aligned} E(XY) &= \left(\int xf_1(x)dx\right)\left(\int yf_2(y)dy\right) \\ &= E(X)\cdot E(Y). \end{aligned}$$

The proof is complete. Note that this proof requires a background of Calculus III.

5. The contrapositive of the last theorem if also true (of course):

6. If $E(XY) \neq E(X)E(Y)$, then X and Y are not independent.

7. On the other hand, it is possible that X and Y are not independent, but still $E(XY) = E(X)E(Y)$. See the example below.

8. *Example.* Suppose that X and Y are discrete random variables, whose joint pmf is given by the following table

	$X=-1$	$X=0$	$X=1$
$Y=-1$	0.25	0	0.25
$Y=1$	0	0.5	0

We leave it to the audience to show that, in this example, X and Y are not independent, but still $E(XY) = E(X)E(Y)$.

9. More generally, we have the following result.

10. *Theorem.* If $X_1, X_2, \cdots, X_n$ are mutually independent random variables, then

$$E(X_1 X_2 \cdots X_n) = E(X_1)E(X_2) \cdots E(X_n),$$

11. Next, we develop some formulas regarding the variance.

12. *Formula.* If X is a random variable, and c is a scalar, then

$$Var(c\,X) = c^2 Var(X).$$

13. *Exercise.* We leave it to the audience to prove this last formula.

14. *Caution:* In general,

$$Var(X + Y) \neq Var(X) + Var(Y),$$

$$Var(c\,X) \neq cVar(X).$$

Simply put, variance is not a linear operation.

15. *Example.* Let X be a random variable such that $Var(X) \neq 0$. Then, we have

$$Var(2X) = 2^2 Var(X) = 4Var(X) \neq 2Var(X),$$

and

$$Var(X + X) = 4Var(X) \neq Var(X) + Var(X).$$

Again,

$$Var(2X) \neq 2Var(X),$$

$$Var(X + X) \neq Var(X) + Var(X).$$

16. *Formula.* If X, Y are random variables, then

$$Var(X+Y) = Var(X) + 2Cov(X,Y) + Var(Y).$$

17. *Exercise.* Prove the last formula.

18. *Formula.* If X and Y are independent random variables, then

$$Cov(X,Y) = 0.$$

* *Proof.* Since X and Y are independent, $E(XY) = E(X)E(Y)$. It follows that

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0.$$

19. If we combine the last two formulas, we get:

20. *Formula.* If X and Y are independent random variables, then

$$Var(X+Y) = Var(X) + Var(Y).$$

21. More generally, we have the following result.

22. *Theorem.* If $X_1, X_2, \cdots, X_n$ are mutually independent random variables, then

$$Var(X_1 + X_2 + \cdots + X_n) = Var(X_1) + Var(X_2) + \cdots + Var(X_n).$$

23. Even more generally, we have the following result.

24. *Theorem.* If $X_1, X_2, \cdots, X_n$ are independent random variables, and

$$Y = \sum_{i=1}^{n} a_i X_i = a_1 X_1 + \cdots + a_n X_n,$$

then

$$Var(Y) = \sum_{i=1}^{n} a_i^2 Var(X_i) = a_1^2 Var(X_1) + \cdots + a_n^2 Var(X_n).$$

25. We now go over the definition for a sample:

26. *Definition.* Let X be a random variable with pmf/pdf $f(x)$.

    If $X_1, X_2, \cdots, X_n$ are random variables such that

    (a) $X_1, X_2, \cdots, X_n$ are mutually independent;

    (b) Each X_i has the pmf/pdf $f(x)$,

    then we say $X_1, X_2, \cdots, X_n$ is a random sample of size n from the population X.

27. *Remark.* In statistics, the word *population* is an abstract term, it is simply defined as the common distribution of the members in a sample.

For example, when we toss five coins in a row, we are taking a sample of size five from the entire population — the population of all fair coins in the world. But this understanding of the population does not help much, because the only information we need about this population is the population distribution — if we toss all the coins at one time, then half of them will turn up heads, and the other half of them will turn up tails. For us, the population is nothing but the distribution

heads	tails
$\frac{1}{2}$	$\frac{1}{2}$

28. If we apply the formulas (regarding the expectation and variance) that we learned in the first half of the section to the case of a sample, we have the following theorem:

29. *Formula.* Suppose that $X_1, X_2, \cdots, X_n$ is random sample of size n from a population X, and suppose that the population X has mean μ and variance σ^2. We define the sample mean by

$$\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n).$$

Then

$$E(\bar{X}) = \mu, \qquad Var(\bar{X}) = \frac{\sigma^2}{n}.$$

* In words, the expectation of the sample mean is the population mean; the variance of the sample mean is the population variance divided by the sample size.

30. *Example.* Let the population $X \sim N(7, 8)$, and let $X_1, X_2, \cdots, X_9$ be a sample of size 9 from this population. Let $\bar{X}$ denote the sample mean, that is

$$\bar{X} = \tfrac{1}{9}(X_1 + X_2 + \cdots + X_9).$$

Find $E(\bar{X})$ and $Var(\bar{X})$.

* It is clear that the population mean is 7, and the population variance is 8. So the answer is

$$E(\bar{X}) = 7,$$

$$Var(\bar{X}) = \frac{8}{9}.$$