Reviewing Induction

Principle of Mathematical Induction

Suppose $P(k)$ is a property about integers. If:

1. $P(1)$ is true, and
2. The statement, "For all integers $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is true," is true;

then $P(k)$ is true for all $k \geq 1$.

- Two parts to a proof by induction: The *base case* and the *inductive step*.
- PMI also works if we start at $k = 0$, or $k = 5$, etc.
- Notice: Proving the inductive step involves proving a universal conditional statement! Choose an arbitrary element of the domain satisfying the hypothesis. (Choose arbitrary $k \geq 1$ s.t. $P(k)$ is true, show $P(k+1)$ must also be true.)

Rewriting the statement so it is a statement about integers:

Theorem (Already proved by contradiction)

For all $k \geq 1$, if a graph G contains a walk of length k connecting u and v for any distinct $u, v \in V(G)$, then G also contains a path connecting u and v.

P(k)

base

Proof (by induction): For the base case, consider $k=1$. Let G be an arbitrary walk of length 1 connecting u and v for some distinct $u, v \in V(G)$. Since a walk of length 1 cannot repeat vertices, this walk is a path. Thus G contains a u-v path.

For the inductive step, let $k \geq 1$ be an integer, and assume that if G contains a walk of length k connecting u and v for any distinct u and v in $V(G)$, then G also contains a u-v path. (Inductive hypothesis)

For arbitrary vertices $x, y \in V(G)$, suppose W is an x-y walk of length $k+1$ in G. We must show that G contains an x-y path.

Let xa be the first edge of W. If $a = y$, then xa is an x-y path. Otherwise, removing xa from W leaves an a-y walk of length k. By the inductive hypothesis, there is an a-y path in G, P. We consider two cases:

Case 1: P does not contain x. Then adding xa to P results in an x-y path in G.

Case 2: P does contain x. Then the subpath of P connecting x and y is an x-y path in G.

Therefore G contains an x-y path, and the inductive step is complete.

(Therefore by the Principle of Mathematical Induction, the theorem is true.)

Important: Avoid the induction trap

- In the inductive step, we started with a walk of length $k + 1$, and we went and found a walk of length k inside it.
- You CANNOT start with the walk of length k, and build up the walk of length $k + 1$ from there!
- Why not? Because we have to verify the result for an *arbitrary* walk of length $k + 1$, not a walk of length $k + 1$ that was built by adding an edge to a walk of length k.
- In this case, it is true that all walks of length $k + 1$ can be built by adding an edge to a walk of length k, but that won't always be true!

Always START from the larger object, and FIND the smaller object inside it.