Differential Equation Homework Pg. 99-100: 9, 14, 23, 24, 25

Section 3.2: Compartmental Analysis

9) In 1990 the Department of Natural Resources released 1000 splake (a crossbreed of fish) into a lake. In 1997 the population in the lake was estimated to be 3000. Using the Malthusian law for population growth, estimate the population of splake in the lake in the year 2020.

Solution: Let $y(t)$ be the population of splake into the lake at any time t. Then, **the population is proportional to the speed to which the population grows** (Malthusian growth model). This can be represented as a differential equation

$$y'(t) = ky(t)$$

where k is the constant of proportionality.

Initially, 1000 splake was released in the lake.ting time from 1990, we obtain the initial condition

$$y(0) = 1000$$

In 1997, which is 7 years later, the population was estimated to be 3000. That gives as the second initial condition

$$y(7) = 3000$$

Thus, the mathematical model for this problem is

$$y'(t) = ky(t), \quad y(0) = 1000 \quad y(7) = 3000$$

This is **separable ODE**. Rearranging terms in the equation gives

$$\frac{dy}{dt} = ky \Rightarrow \frac{1}{y}dy = kdt$$

Integration on both sides gives

$$\int \frac{1}{y}dy = \int kdt$$
$$\ln|y| = kt + C$$

where C is the constant of integration. By taking exponents, we obtain

$$|y| = e^{kt+C}e^{C}$$

Hence

$$y = Ce^{kt}$$

Where $C = e^C$. Now, we can use the first initial condition to determine the numeric value of the constant C. Substitute 0 for t and 1000 for y in the equation.

$$1000 = y(0) = Ce^{k\cdot 0}$$
$$1000 = Ce^{0}$$
$$C = 1000$$

To determine the numeric value of the constant of proportionality k, we use the second initial condition and the obtained result. Subtitute 7 for t and 3000 for y in the equation.

$$
\begin{aligned}
3000 = y(7) = Ce^{k\cdot 7} &= 1000e^{7k}\\
e^{7k} &= 3\\
k &= \frac{\ln 3}{7}
\end{aligned}
$$

Therefore, the population of splake in the lake at any moment t is

$$y = 1000e^{\frac{\ln 3}{7}t}$$

To estimate the population of splake in the lake in 2020 using the obtained model, we need to calculate the value $y(30)$, since the initial observation of the population was in 1990, which is 30 years prior to 2020. Therefore, we obtain

$$y(30) = 1000e^{\frac{\ln 3}{7}\cdot 30} = 100,868$$

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14) In 1980 the population of alligators on the Kennedy Space Center grounds was estimated to be 1500. In 2006 the population had grown to an estimated 6000. Using the Malthusian law for population growth, estimate the alligator population on the Kennedy Space Center grounds in the year 2020

Solution: Let $y(t)$ be the population of alligators on Kennedy Space Center at any time t. Then, **the population is proportional to the speed to which the population grows** (Malthusian growth model). This can be represented as a differential equation

$$y'(t) = ky(t)$$

where k is the constant of proportionality.

In 1980, the number of aligators was 1500. Counting time from 1980, we obtain the initial condition

$$y(0) = 1500$$

In 2006, which is 26 years later, the population was estimated to be 6000. That gives as the second initial condition

$$y(26) = 6000$$

Thus, the mathematical model for this problem is

$$y'(t) = ky(t) \quad y(0) = 1500 \quad y(26) = 6000$$

This is **separable ODE**. Rearranging terms in the equation gives

$$\frac{dy}{dt} = ky \Rightarrow \frac{1}{y}dy = kdt$$

Integration on both sides gives

$$
\begin{aligned}
\int \frac{1}{y}dy &= \int kdt\\
\ln|y| &= kt + C
\end{aligned}
$$

where C is the constant of integration. By taking exponents, we obtain

$$|y| = e^{kt+C}e^{C}$$

Hence

$$y = Ce^{kt}$$

Where $C = e^C$. Now, we can use the first initial condition to determine the numeric value of the constant C. Substitute 0 for t and 1500 for y in the equation.

$$\begin{aligned}1500 &= y(0) = Ce^{k\cdot 0}\\ 1500 &= Ce^0\\ C &= 1500\end{aligned}$$

To determine the numeric value of the constant of proportionality k, we use the second initial condition and the obtained result. Substitute 26 for t and 6000 for y in the equation.

$$\begin{aligned}6000 &= y(26) = 1500e^{26k}\\ e^{26l} &= 4\\ k &= \frac{\ln 4}{26}\end{aligned}$$

Therefore, the population of alligators on the Kennedey Space Center at any moment t is

$$y = 1500e^{\frac{\ln 4}{26}t}$$

To estimate the population of alligators on the Kennedy Space Center using the obtained model , we need to calculate the value $y(40)$, since the initial observation of the population was in 1980, which is 40 years prior to 2020. Therefore, we obtain

$$y(40) = 1500e^{\frac{\ln 4}{26}\cdot 40} = 12,657$$

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23) If initially there are $50g$ of radioactive substance and after 3 days there are only $10g$ remaining, what percentage of the original amount remains after 4 days?

Solution: Let $m(t)$ be the mass of the radioactive substance, where t represents time measured in days. By **the law of decay**, we obtain the differential equation

$$\frac{dm}{dt} = km(t)$$

where k is the decay constant, which depends on the substance. This is **separable ODE**. Rearranging terms in the equations gives

$$\frac{1}{m}dm = kdt$$

Integration on both sides gives

$$\begin{aligned}\int \frac{1}{m}dm &= \int kdt\\ \ln|m| &= kt + C\end{aligned}$$

By taking the exponents we obtain

$$\begin{aligned}|m| &= e^{kt+C}\\ m &= e^{kt}e^C\end{aligned}$$

Therefore

$$m = Ce^{kt}$$

Where $C + e^C$. If the initial amount of the substance was m_0, we obtain the initial condition

$$m(0) = m_0$$

Substituting 0 for t and m_0 for m in the general solution gives

$$\begin{aligned} m_0 &= m(0) = Ce^0 \\ C &= m_0 \end{aligned}$$

Therefore, we obtain that

$$m = m_0 e^{kt}$$

Here we have that $m_0 = 50g$ and $m(3) = 10g$. We ca use the second condition to dtermine the numeric value of the decay constant k. Substituting 50 for m_0, 3 for t and 10 for m in the equation gives

$$\begin{aligned} 10 = 50e^{3k} &\Rightarrow \frac{1}{5} = e^{3k} \\ &\Rightarrow \ln\frac{1}{5} = 3k \\ &\Rightarrow k = \frac{\ln\frac{1}{5}}{3} \end{aligned}$$

Therefore, the decay is modeled by the equation

$$\begin{aligned} m(t) &= 50e^{\frac{\ln\frac{1}{5}}{3}\cdot t} \\ &= 50\left(\frac{1}{5}\right)^{\frac{t}{3}} \end{aligned}$$

After 4 days, the amount of the substance is given by

$$m(4) = 50\left(\frac{1}{5}\right)^{\frac{4}{3}} \approx 5.848g$$

and that is

$$\frac{m_4}{m_0}\cdot 100 = \frac{5.848}{50}\cdot 100 = 11.63\%$$

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24) If initially there are $300g$ of a radioactive substance and after 5 years there are $200g$ remaininf, how much time must elapse before $10g$ remain?

Solution: Let $m(t)$ be the mass of the radioactive substance, where t represents time measured in years. By **the law of decay**. we obtain the differential equation

$$\frac{dm}{dt} = km(t)$$

where k is the decay constant, which depends on the substance. This is **separable ODE**. Rearranging terms in the equations gives

$$\frac{1}{m}dm = kdt$$

Integration on both sides gives

$$\int \frac{1}{m}dm = \int kdt$$
$$\ln|m| = kt + C$$

By taking the exponents we obtain

$$|m| = e^{kt+C}$$
$$m = e^{kt}e^{C}$$

Therefore

$$m = Ce^{kt}$$

Where $C + e^{C}$. If the initial amount of the substance was m_0, we obtain the initial condition

$$m(0) = m_0$$

Substituting 0 for t and m_0 for m in the general solution gives

$$m_0 = m(0) = Ce^0$$
$$C = m_0$$

Therefore, we obtain that

$$m = m_0 e^{kt}$$

Here we have that $m_0 = 300g$ and $m(5) = 200g$. we can use the second condition to determine the numeric value of the decay constant k. Substituting 300 for m_0, 5 for t and 200 for m in the equation gives

$$200 = 300e^{5k} \Rightarrow \frac{2}{3} = e^{5k}$$
$$\Rightarrow \ln\frac{2}{3} = 5k$$
$$\Rightarrow k = \frac{\ln\frac{2}{3}}{5}$$

Therefore, the decay is modeled by the equation

$$m(t) = 300e^{\frac{\ln\frac{2}{3}}{5}}$$
$$= 300\left(\frac{2}{3}\right)^{0.2t}$$

Let $\hat{t}$ be the moment when only $10g$ of the substance has left. To determine $\hat{t}$. we need to solve the equation

$$m(\hat{t}) = 10$$

for $\hat{t}$

$$
\begin{aligned}
10 = m(\hat{t}) &= 300\left(\frac{2}{3}\right)^{0.2\hat{t}} \\
\Rightarrow t &= \frac{5\ln 30}{\ln 1.5} \approx 41.94
\end{aligned}
$$

Which means that will happen in about 42 years.

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25) Carbon dating is often used to determine the age of a fossil. For example, a humanoid skull was found in a cave in South Africa along with the remains of a campfire. Archaeologists believe the age of the skull to be the same age as the campefire. It is determined that only 2% of the original amout of carbon-14 remainds in the burnt wood of the campfire. Estimate the age of the skull if the half-life of carbon-14 is about 5600 years.

Solution: Let the substance present at any time t is A(t).As it is given that rate of decay is proportional to amount present at that time so

$$
\frac{d(A(t))}{dt} = -kA(t)
$$

where k is the decay constant, which depends on the substance. This is **separable ODE**. Rearranging terms in the equations gives

$$
\frac{d(A(t))}{A(t)} = -kdt
$$

Integration on both sides assuming at $t = 0$ amount present is A_0

$$
\begin{aligned}
\int_{A_0}^{A(t)} \frac{d(A(t))}{A(t)} dA &= \int_0^t -kdt \\
\ln A(t) - \ln A_0 &= -kt \\
\ln \frac{A(t)}{A_0} &= -kt \\
\frac{A(t)}{A_0} &= e^{-kt} \\
A(t) &= A_0 e^{-kt}
\end{aligned}
$$

It is given that time when $A(t) = 0.5 \times A_0$ is 5600

$$
\begin{aligned}
A(5600) &= A_0 e^{-5600k} \\
0.5 \times A_0 &= A_0 e^{-5600k} \\
k &= \frac{\ln 2}{5600} = 1.2377 \times 10^{-4}
\end{aligned}
$$

Time t is to be calculated when $A(t) = 0.02 \times A_0$

$$
\begin{aligned}
A(t) &= A_0 e^{-1.2377 \times 10^{-4} \times t} \\
0.02 \times A_0 &= A_0 e^{-1.2377 \times 10^{-4} \times t} \\
\ln .02 &= -1.2377 \times 10^{-4} \times t \\
t &\approx 31607
\end{aligned}
$$