Example 2: Use Euler's method to find approximation to the solution of the initial value problem

$$y' = y, \quad y(0) = 1$$

at $x = 1$, taking $1, 2, 4, 8$, and 16 steps.

Remark. Observe that the solution to $y' = y, \quad y(0) = 1$ is just $\phi(x) = e^x$, so Euler's method will generate algebraic approximations to the transcendental number e.

Solution

Here $f(x, y) = y, x_0$, and $y_0 = 1$. The recursive formula for Euler's method is

$$y_{n+1} = y_n + hy_n = (1 + h)y_n$$

To obtain approximations at $x = 1$ with N steps, we take the step size $h = \frac{1}{N}$. For $N = 1$, we have

$$\phi(1) \approx y_1 = (1 + 1)(1) = 2$$

For $N = 2, \phi(x_2) = \phi(1) \approx y_2$. In this case we get

$$\begin{aligned} y_1 &= (1 + 0.5)(1) = 1.5 \\ \phi(1) \approx y_2 &= (1 + 0.5)(1.5) = 2.25 \end{aligned}$$

For $N = 4, \phi(x_4) = \phi(1) \approx y_4$. In this case we get

$$\begin{aligned} y_1 &= (1 + 0.25)(1) = 1.25 \\ y_2 &= (1 + 0.25)(1.25) = 1.5625 \\ y_3 &= (1 + 0.25)(1.5625) = 1.95313 \\ y_4 &= (1 + 0.25)(1.95313) = 2.44141 \end{aligned}$$

Notice smaller steps sizes give better approximations. (But suffer the drawback of more computations and round-off error)