Theorem: The Inverse of a Matrix Using its Adjoint

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

Proof

Begin by proving that the product of A and its adjoint is equal to the product of the determinant of A and I_n. Consider the product

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The entry in the i^{th} row and j^{th} column of this product is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

If $i = j$, then this sum is simply the cofactor expansion of A in its i^{th} row, which means that the sum is the determinant of A. On the other hand, if $i \neq j$, then the sum is zero.

$$A\big[adj(A)\big] = \begin{bmatrix} det(A) & 0 & \cdots & 0 \\ 0 & det(A) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & det(A) \end{bmatrix} = det(A)I$$

The matrix A is invertible, so $det(A) \neq 0$ and you can write

$$\frac{1}{det(A)} A\big[adj(A)\big] = I \quad \text{or} \quad A\big[\frac{1}{det(A)} adj(A)\big] = I$$

by Theorem Uniqueness of an Inverse Matrix and the definition of the inverse matrix, it follows that

$$\frac{1}{det(A)} adj(A) = A^{-1}$$