Properties of Real Numbers

For any real numbers a, b, c:

Communicative Properties:

Addition

$$a + b = b + a$$

Multiplication

$$ab = ba$$

Associative Properties:

Addition

$$a + (b + c) = (a + b) + c$$

Multiplication

$$a(bc) = (ab)c$$

Additive identity Property:

$$a + 0 = 0 + a = a$$

Additive inverse Property:

$$-a + a = a + (-a) = 0$$

Multiplicative identity Property:

$$a \cdot 1 = 1 \cdot a = a$$

Multiplicative inverse Property:

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1\ (a \neq 0)$$

Distributive Property:

$$a(b + c) = ab + ac$$

Absolute Value

For any real number a,

$$|a| = \begin{cases} a, & if\ a \geq \mathbf{0} \\ -a, & if\ a < \mathbf{0} \end{cases}$$

When a is non-negative, the absolute value of a is a. When a is negative, the absolute value of a is the opposite, or additive inverse, of a.

$\therefore$ $|a|$ is never negative; that is, for any real number a, $|a| \geq \mathbf{0}$.

Distance Between two Points on the Number line

For any real numbers a and b, the distance between a and b is:

$$|a - b|$$

or

$$|b - a|$$

Integers as Exponents

For any positive integer n

$$a^n = a \cdot a \cdot a \cdots a$$

Where a, is the base and n is the exponent.

Zero and negative-integer exponents

For any non-zero real number a and any integer m,

$$a^0 = 1$$

and

$$a^{-m} = \frac{1}{a^m}$$

For any non-zero numbers a and b and any integers m and n,

$$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m}$$

Properties of Exponents

For any real numbers **a** and **b** and any integers **m** and **n**, assuming **0** is not raised to a non-positive number:

Product Rule:

$$a^m \cdot a^n = a^{m+n}$$

Quotient Rule:

$$\frac{a^m}{a^n} = a^{m-n} \ (a \neq 0)$$

Power Rule:

$$(a^m)^n = a^{mn}$$

Raising a product to a power

$$(ab)^m = a^m b^m$$

Raising a quotient to a power

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m} \ (b \neq 0)$$

Scientific notation

Scientific notation for a number is an expression of the type.

$$N \times 10^m,$$

Where $\mathbf{1 \leq N < 10}$, $\boldsymbol{N}$ is in decimal notation, and **m** is an integer.

Rules For order of Operations

1. Do all calculations within grouping symbols before operations outside. When nested grouping symbols are present, work from the inside out.

2. Evacuate all exponential expressions.

3. Do all multiplication and divisions in order from left-to-right.

4. Do all additions and subtractions in order from left-to-right.

Polynomials In one variable

A **polynomial in one variable** is any expression of the type:

$$a^n x^n + a_{n-1}x^{n-1} + \cdots a_2 x^2 + a_1 x + a_0$$

Where **n** is a non-negative integer ad $a_n, \cdots a_0$ are real numbers, called **coefficients**. The parts of a polynomial separated by plus signs are called **terms.** The **leading coefficient** is a_m, and the **constant term** is a_0. If $a_0 \neq 0$, the **degree** of the polynomial is **n**. The polynomial is said to be written in **descending order**, because the exponents decrease from left-to-right.

Special Products of Binomials

Square of a Sum

$$(A+B)^2 = A^2 + 2AB + B^2$$

Square of a Difference

$$(A-B)^2 = A^2 - 2AB + B^2$$

Product of a sum and a Difference

$$(A+B)(A-B) = A^2 - B^2$$

Difference of Squares

$$A^2 - B^2 = (A+B)(A-B)$$

Sum of Cubes

$$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$$

Difference of Cubes

$$A^3 - A^3 = (A-B)(A^2 + AB + B^2)$$

Linear Equations

A **linear equation in one variable** is an equation that is equivalent to one of the form $ax + b = 0$, where **a** and **b** are real numbers and $a \neq \mathbf{0}$.

Quadratic Equations

A quadratic equation **is an equation that is equivalent to one of the form**
$ax^2 + bx + c = 0$ **where** a, b, **and** c **are real numbers and** $a \neq 0$

Equation solving Principles

For any real numbers **a**, **b**, **c**,

The Addition Principle

If $a = b$ is true, then $a + c = b + c$ is true

The Multiplication Principle

If $a = b$ is true, then $ac = bc$ is true.

The Principle of Zero Products

If $ab = 0$ is true, then $a = 0$ or $= 0$, and if $a = 0$ or $b = 0$, then $ab = 0$.

The Principle of Square Roots

If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$

Nth Root

A number **c** is said to be an **nth root** of **a** if $c^n = a$**.**

Properties of Radicals

Let **a** and **b** be any real number or expressions for which the given roots exist. For any natural number **m** and **n** $(n \neq 1)$:

1. If n is even, $\sqrt[n]{a^n} = |a|$
2. If n is odd, $\sqrt[n]{a^n} = a$
3. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
4. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ $(b \neq 0)$
5. $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

The Pythagorean Theorem

The sum of the squares of the lengths of the legs of a right triangle is equal to the square of the length of the hypotenuse"

$$a^2 + b^2 = c^2$$

Rational Exponents

For any real number **a** and any natural numbers **m** and **n**, $n \neq 1$, for which $\sqrt[n]{a}$ exists:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

$$a^{-\frac{m}{n}} = \frac{1}{a^{\frac{m}{n}}}$$

X-Intercept

An **x-intercept** is a point $(a, 0)$. To find **a**, let $y = 0$ and solve for **x.**

Y-Intercept

A **y-intercept** is a point $(0, b)$. To find **b**, let $x = 0$ and solve for **y**.

Distance Formula

The **distance d** between any two point (x_1, y_1) and (x_2, y_2) is give by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula

If the endpoints of a segment are (x_1, y_1) and (x_2, y_2), then the coordinates of the **midpoint** of the segment are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Equation of a Circle

The standard form of the equation of a circle with center (h, k) and radius **r:**

$$(x - h)^2 + (y - k)^2 = r^2$$

Function

A **function** is a correspondence between a first set, called the **domain,** and a second set called the **range**, such that each member of the domain corresponds to exactly one member of the range.

Relation

A **relation** is a correspondence between a first set, called the **domain**, and a second set, called the **range**, such that each member of the domain corresponds to *at least one* member of the range.

Vertical Line Test

If it is possible for a vertical line to cross a graph more than once, then the graph is *not* the graph of a function.

Visualizing Domain and Range

Domain = the set of a function's input, found on the horizontal **x-axis**.

Range = the set of a function's output on the vertical **y-axis**.

Linear Function

A function is a **linear function** if it can be written as

$$f(x) = mx + b$$

Where **m** and **b** are constants.

If $m = 0$, the function is a **constant function** $f(x) = b$, if $m = 1$ and $b = 0$, the function is the **identity function** $f(x) = x$.

Horizontal Lines and Vertical Lines

Horizontal lines are given by equations of the type $y = b$ or $f(x) = b$. (They are functions)

Vertical Lines

Vertical lines are given by equations of the type $x = a$. (They are *not* functions)

Slope

The **slope m** of a line containing points (x_1, y_1) and (x_2, y_2) is give by

$$m = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2}$$

Horizontal Lines and Vertical Lines

If a line is horizontal, the change is **y** for any two points is **0** and the change in x is non-zero. Thus a horizontal line has slope 0. If a line is vertical, the change in x for any two points is **0**. Thus the slope is *not defined* because we cannot divide by 0.

Average Rate of Change

Slope can also be considered as an **average rate of change**. To find the average rate of change between any two data points on a graph, we determine the slope of the line that passes through the two points.

Slope-Intercept Equation

The linear function $\boldsymbol{f}$ given by

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{mx} + \boldsymbol{b}$$

is written in slope-intercept form. The graph of an equation in this form is a straight line parallel to $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{mx}$. The constant **m** is called the slope, and the **y-intercept** is $(\mathbf{0}, \boldsymbol{b})$.

Point-Slope Equation

The **point-slope equation** of the line with slope **m** passing though $(\boldsymbol{x_1}, \boldsymbol{y_1})$ is

$$\boldsymbol{y} - \boldsymbol{y_1} = \boldsymbol{m}(\boldsymbol{x} - \boldsymbol{x_1})$$

Parallel Lines

Vertical lines are **parallel**. Non-vertical lines are parallel ↔ they have the same slope and different **y-intercepts**.

Perpendicular Lines

Two lines with slopes $\boldsymbol{m_1}$and $\boldsymbol{m_2}$ are **perpendicular** ↔ the product of their slope is $-\mathbf{1}$:

$$\boldsymbol{m_1 m_2} = -\mathbf{1}$$

Lines are also **perpendicular** if one is vertical $(\boldsymbol{x} = \boldsymbol{a})$ and the other is horizontal $(\boldsymbol{y} = \boldsymbol{b})$

Linear equation in one variable

A **linear equation in one variable** is in equation that can be expressed in the form $\boldsymbol{mx} + \boldsymbol{b}$, where **m** and **b** are real numbers and $\boldsymbol{m} \neq \mathbf{0}$.

Equation-Solving Principles

For any real numbers **a**, **b**, and **c**:

The Addition Principle

If $\boldsymbol{a} = \boldsymbol{b}$ is true, then $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{c}$ is true

The Multiplication Principle

If $\boldsymbol{a} = \boldsymbol{b}$ is true, then $\boldsymbol{ac} = \boldsymbol{bc}$ is true.

Simple interest rate

$$\boldsymbol{I} = \boldsymbol{Prt}$$

Zeros of Functions

An input **c** of a function $\boldsymbol{f}$is called a **zero** of the function if the output for the function is **0** when the input is **c**. That is **c**, is a zero of $\boldsymbol{f}$ if $(\boldsymbol{c}) = \mathbf{0}$.

Principles for solving Inequalities

For any real numbers **a**, **b**, **c**:

Addition Principle for Inequalities

If $\boldsymbol{a} < \boldsymbol{b}$ is true, then $\boldsymbol{a} + \boldsymbol{c} < \boldsymbol{b}$ is true.

Multiplication Principle for Inequalities

a) If $\boldsymbol{a} < \boldsymbol{b}$ and $\boldsymbol{c} > \mathbf{0}$ are true, then $\boldsymbol{ac} < \boldsymbol{bc}$ is true.
b) If $\boldsymbol{a} < \boldsymbol{b}$ and $\boldsymbol{c} < \mathbf{0}$ are true, then $\boldsymbol{ac} > \boldsymbol{bc}$ is true.

When both sides of an inequality are multiplied by a negative number, the inequality sign must be reversed.

Similar statements hold for $\boldsymbol{a} \leq \boldsymbol{b}$.

Increasing Functions

A function $\boldsymbol{f}$ is said to be **increasing** on an *open interval*, if for all **a** and **b** in that interval, $\boldsymbol{a} < \boldsymbol{b}$ implies $\boldsymbol{f}(\boldsymbol{a}) < \boldsymbol{f}(\boldsymbol{b})$

Decreasing Functions

A function $\boldsymbol{f}$ is said to be **decreasing** on an *open interval*, if for all **a** and **b** in that interval, $\boldsymbol{a} < \boldsymbol{b}$ implies $\boldsymbol{f}(\boldsymbol{a}) > \boldsymbol{f}(\boldsymbol{b})$

Constant Functions

A function $\boldsymbol{f}$ is said to be **constant** on an *open interval*, if for all **a** and **b** in that interval, $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{f}(\boldsymbol{b})$

Relative Maxima

Suppose that $\boldsymbol{f}$ is a function for which $\boldsymbol{f}(\boldsymbol{c})$ exists for some in the domain of $\boldsymbol{f}$. Then

$\boldsymbol{f}(\boldsymbol{c})$ is a **relative maximum** if there exists an *open interval* containing **c** such that $\boldsymbol{f}(\boldsymbol{c}) > \boldsymbol{f}(\boldsymbol{x})$, for all **x** where $\boldsymbol{x} \neq \boldsymbol{c}$.

Relative Minima

Suppose that $\boldsymbol{f}$ is a function for which $\boldsymbol{f}(\boldsymbol{c})$ exists for some in the domain of $\boldsymbol{f}$. Then

$\boldsymbol{f}(\boldsymbol{c})$ is a **relative minimum** if there exists an *open interval* containing **c** such that $\boldsymbol{f}(\boldsymbol{c}) < \boldsymbol{f}(\boldsymbol{x})$, for all **x** where $\boldsymbol{x} \neq \boldsymbol{c}$.

Properties of Functions

Sum Property

$$(\boldsymbol{f} + \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})$$

Difference Property

$$(\boldsymbol{f} - \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{g}(\boldsymbol{x})$$

Products Property

$$(\boldsymbol{fg}) = \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{g}(\boldsymbol{x})$$

Quotients Property

$$\frac{\boldsymbol{f}}{\boldsymbol{g}}(x) = \frac{\boldsymbol{f}(\boldsymbol{x})}{\boldsymbol{g}(\boldsymbol{x})}, provided\ \boldsymbol{g}(\boldsymbol{x}) \neq \boldsymbol{0}$$

Composition of functions

The **composition function** $\boldsymbol{f} \circ \boldsymbol{g}$, the **composition** of **f** and **g,** is defined as

$$(\boldsymbol{f} \circ \boldsymbol{g}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))$$

Algebraic Test of Symmetry

x-axis : If replacing **y** with $-\boldsymbol{y}$ produces an equivalent equation, then the graph is **symmetric** with respect to the **x-axis.**

y-axis : If replacing **x** with $-\boldsymbol{x}$ produces an equivalent equation, then the graph is **symmetric** with respect to the **y-axis.**

Origin : If replacing **x** with $-\boldsymbol{x}$ and **y** with $-\boldsymbol{y}$ produces an equivalent equation, then the graph is **symmetric** with respect to the **origin**.

Even Functions

If the graph of a function $\boldsymbol{f}$ is symmetric with respect to the **y-axis**, we say that it is an **even function**. That is, for each **x** in the domain of $\boldsymbol{f}$

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(-\boldsymbol{x})$$

Odd Functions

If the graph of a function $\boldsymbol{f}$ is symmetric with respect to the **origin**, we say that it is an **odd function**. That is, for each **x** in the domain of $\boldsymbol{f}$

$$f(-x) = -f(x)$$

Vertical Translation

For $b > 0$

the graph of $y = f(x) + b$ is the graph of $y = f(x)$ shifted up **b** units.

the graph of $y = f(x) - b$ is the graph of $y = f(x)$ shifted down **b** units.

Horizontal Translation

For $d > 0$

the graph of $y = f(x - d)$ is the graph of $y = f(x)$ shifted right **d** units.

the graph of $y = f(x + d)$ is the graph of $y = f(x)$ shifted left **d** units.

Reflections

The graph of $y = -f(x)$ is the **reflection** of the graph $y = f(x)$ across the **x-axis**

The graph of $y = f(-x)$ is the **reflection** of the graph $y = f(x)$ across the **y-axis**

If a point (x, y) is on the graph $y = f(x)$**,** then $(x, -y)$ is on the graph of $y = -f(x)$**,** and $(-x, y)$ is on the graph $y = f(-x)$

Vertical Stretching and Shrinking

The graph of $y = af(x)$ can be obtained from the graph $y = f(x)$ by

Stretching vertically for $|a| > 1$, or

Shrinking vertically for $0 < |a| < 1$.

For $a < 0$, the graph is also reflected across the **x-axis**.

(The **y-coordinates** of the graph $y = af(x)$ can be obtained by multiplying the **y-coordinates** of $y = f(x)$ by a.

Horizontal Stretching and Shrinking

The graph of $y = f(cx)$ can be obtained from the graph of $y = f(x)$ by

Shrinking horizontally for $|c| > 1$, or

Stretching horizontally for $0 < |c| < 1$.

For $c < 0$, the graph is also reflected across the **y-axis**.

(The **x-coordinates** of the graph $y = f(cx)$ can be obtained by dividing the **x-coordinates** of the graph of $y = f(x)$ by c.

Direct Variation

If a situation gives rise to a linear function $f(x) = kx$, or $y = kx$, where **k** is a _positive_ constant, we say that we have **direct variation**, or that **y varies directly as x**, or that **y is directly proportional to x**. The number **k** is called the **variation constant**, or the **constant of proportionality**.

Inverse Variation

If a situation gives rise to a function $f(x) = \frac{k}{x}$, or $y = \frac{k}{x}$, where **k** is a _positive_ constant, we sat that we have **inverse variation**, or that **y varies inversely as x**, or that **y is inversely proportional to x**. The number **k** is called the **variation constant**, or the **constant of proportionally**.

Combined Variation

y varies **directly as the nth power of x** if there is some positive constant **k** such that

$$y = kx^n$$

y varies **inversely as the nth power of x** if there is some positive constant **k** such that

$$y = \frac{k}{x^n}$$

y varies **jointly as x and z** if there is some positive constant **k** such that

$$y = kxz$$

The Number i

The number *i* is defined such that

$$i = \sqrt{-1}$$

and

$$i^2 = -1$$

Complex Numbers

A **complex number** is a number of the form $a + bi$, where **a** and **b** are real numbers. The number **a** is said to be the **real part** of $a + bi$, and the number **b** is said to be the **imaginary part** of $a + bi$.

Conjugate of a Complex Number

The **conjugate** of a complex number $a + bi$ is $a - bi$. The number $a + bi$ and $a - bi$ are **complex conjugates**.

Quadratic Equations

A **quadratic equation** is an equation that can be written in the form

$$ax^2 + bx + c = 0, \quad a \neq 0$$

Where **a**, **b**, and **c** are real numbers.

Quadratic Functions

A **quadratic function** f is a function that can be written in the form

$$f(x) = ax^2 + bx + c = 0, \quad a \neq 0$$

Where **a**, **b**, and **c** are real numbers.

Equation-Solving Principles

The Principles of Zero Products: If $ab = 0$ is true, then $a = 0$ or $b = 0$, then $ab = 0$.

The Principle of Square Roots: If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$.

The Quadratic Formula

The solution of $ax^2 + bx + c = 0, \quad a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant

For $ax^2 + bx + c = 0$, where **a**, **b**, and **c** are real numbers:

$$b^2 - 4ac = 0 \rightarrow One\ real\ number\ solution$$

$b^2 - 4ac > 0 \rightarrow$
$two\ different\ \ real\ number\ solutions$

$b^2 - 4ac < 0 \rightarrow$
$Two\ different\ imaginary\ number\ solutions,$
$complex\ conjugates$

Graphing Quadratic Functions

The graph of the function $f(x) = a(x - h)^2 + k$ is a parabola that

- Opens up if $a > 0$ and down if $a < 0$
- Has (h, k) as the vertex
- Has $x = h$ as the axis of symmetry
- Has k as a minimum value (output) if $a > 0$
- Has k as a maximum value if $a < 0$

Vertex of a Parabola

The **vertex** of a graph of $f(x) = ax^2 + bx + c$ is

$$(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right))$$

The Principle of Powers

For any positive integer **n:**

If $a = b$ is true, then $a^n = b^n$ is true

Equations with Absolute Value

For $a > 0$ and an algebraic expression **X**:

$$|X| = a$$

Is equivalent to

$$X = -a \text{ or } X = a$$

Polynomial Function

A **polynomial function** P is given by

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

Where the coefficients $a_n, a_{n-1}, \cdots a_1, a_0$ are real numbers and the exponents are whole numbers.

Domain of a Polynomial Function

The *domain* of a polynomial functions is the set of all real numbers,

$$(-\infty, \infty)$$

Even and Odd Multiplicity

If $(x - c)^k$, $k \geq 1$, is a factor of a polynomial function $P(x)$ and $(x - c)^{k-1}$ is not a factor and:

- k is odd, then the graph crosses the **x-axis** at $(c, 0)$
- k is even, then the graph is tangent to the **x-axis** at $(c, 0)$

Graphing Polynomial Functions

If $P(x)$ is a polynomial function of degree n, then the graph of the function has:

- At most n real zeros, and thus at most n **x-intercepts**
- At most $n - 1$ turning points

(Turning points on a graph, also called **relative maxima** and **minima**, occur when the function changes from decreasing to increasing or from increasing to decreasing.)

To Graph a Polynomial Function

1. Use the leading-term test to determine the end behavior.

2. Find the zeros of the function by solving $f(x) = 0$. Any real zeros are the first coordinates of the **x-intercepts**.

3. Use the **x-intercepts** (zeros) to divide the **x-axis** into intervals, and choose a test point in each interval to determine the sign of all function values in that interval.

4. Find $f(0)$. This gives the **y-intercept** of the function.

5. If necessary, find additional function values o determine the general shape of the graph and then draw the graph.

6. As a partial check, use the facts that the graph has at most n **x-intercepts** and at most $n - 1$ turning points. Multiplicity of zeros can also be considered in order to check where the graph crosses or is tangent to the **x-axis.** We can also check the graph with a graphing calculator.

The Intermediate Value Theorem

For any polynomial function $P(x)$ with real coefficients, suppose that for $a \neq b$, $P(a)$ and $P(b)$ are opposite signs. Then the function has a real zero between a and b.

The intermediate value theorem *cannot* be user to determine whether there is a real zero between a and b when $P(a)$ and $P(b)$ have the *same* sign.

The Remainder Theorem

If a number c is substituted for x in the polynomial $f(x)$, then the result $f(c)$ is the remainder that would be obtained by dividing $f(x)$ by $x - c$. That is, if $f(x) = (x - c) * Q(x) + R$, then $f(c) = R$

The Factor Theorem

For a polynomial $f(x)$, if $f(c) = 0$, then $x - c$ is a factor of $f(x)$.

The Fundamental Theorem of Algebra

Every polynomial function of degree n, with $n \geq 1$, has at least one *zero* in the set of complex numbers.

Polynomial function of degree n

Every polynomial function of degree n, with $n \geq 1$, can be factored into n linear factors (not necessarily unique); that is

$$f(x) = a_n(x - c_1)(x - c_2)\cdots(x - c_n)$$

Non-real Zeros: $a + bi$ AND $a - bi, b \neq 0$

If a complex number $a + bi$, $b \neq 0$, is a zero of a polynomial function $f(x)$ with *real coefficients*, then its conjugate, $a - bi$, is also a zero. For example, if $2 + 7i$ is a zero of a polynomial function $f(x)$, with *real coefficients*, then its conjugate $2 - 7i$, is also a zero. (Non-real zeros occur in conjugate pairs.)

Irrational Zeros: $a + c\sqrt{b}$ AND $a - c\sqrt{b}, b$ is not a Perfect Square

If $a + c\sqrt{b}$, where a, b, and c are rational and b is not a perfect square, is a zero if a polynomial function $f(x)$ with *rational coefficients*, then its conjugate $a - c\sqrt{b}$, is also a zero. For example, if $-3 + 5\sqrt{2}$ is a zero of a polynomial function $f(x)$ with rational coefficients, then its conjugate, $-3 - 5\sqrt{2}$, is also a zero. (Irrational zeros occur in conjugate pairs.)

The Rational zero Theorem

Let

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

Where all the coefficients are integers. Consider a rational number denoted by p/q, where p and q are relatively prime (having no common factor besides -1 and 1). If p/q is zero $P(x)$, then p is a factor of a_0 and q is a factor of a_n

Descartes' Rule of Signs

Let $P(x)$, written in descending order or ascending order, be a polynomial function with real coefficients and a non-zero constant term. The number of positive real zeros of $P(x)$ is either:

1. The same as the number of variations of sign in $P(x)$, or

2. Less than he number of variations of sign in $P(x)$ by a positive even integer.

The number of negative real zeros of $P(x)$ is either:

3. The same as the number of variations of sign in $P(-x)$, or

4. Less than the number of variations of sign in $P(-x)$ by a positive even integer.

A zero of multiplicity m must be counted m times.

Rational Function

A **rational function** is a function that is quotient of two polynomials. That is,

$$f(x) = \frac{p(x)}{q(x)},$$

Where $p(x)$ and $q(x)$ are polynomials and where $q(x)$ is not the zero polynomial. The domain of f consists of all inputs x for which $q(x) \neq 0$

Determining Vertical Asymptotes

For a rational function $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with no common factors other than constants, if a is a zero of the denominator, then the line $x = a$ is a vertical asymptote for the graph of the function.

Determining a Horizontal Asymptote

- When the numerator and the denominator of a rational function have the same degree, the line $\boldsymbol{y} = \boldsymbol{a}/\boldsymbol{b}$ is the horizontal asymptote, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the leading coefficients of the numerator and the denominator, respectively.

- When the degree of the numerator of a rational function id less than the degree of the denominator, the **x-axis**, or $\boldsymbol{y} = \mathbf{0}$, is the horizontal asymptote.

- When the degree of the numerator of a rational function is greater than the degree of the denominator, there is no horizontal asymptote.

Occurrence of Line as Asymptotes of Rational Functions

For a rational function $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{x})/\boldsymbol{q}(\boldsymbol{x})$**,** where $\boldsymbol{p}(\boldsymbol{x})$ and $\boldsymbol{q}(\boldsymbol{x})$ have no common factors other than constants

Vertical asymptotes occur at any **x-values** that make the denominator 0.

The x-axis is the horizontal asymptote occurs when the numerator and the denominator have the same degree.

An oblique asymptote occurs when the degree of the numerator is 1 greater than the degree of the denominator.

There can be only one horizontal asymptote or one oblique asymptote and never both.

An asymptote is *not* part of the graph of the function.

Crossing an Asymptote

- The graph of a rational function *never crosses* a vertical asymptote.
- The graph of a rational function *might cross* a horizontal asymptote but does not necessarily do so.

Graph a rational Function

To graph a rational function $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{p}(\boldsymbol{x})/\boldsymbol{q}(\boldsymbol{x})$, where $\boldsymbol{p}(\boldsymbol{x})$ and $\boldsymbol{q}(\boldsymbol{x})$ have no common factor other than constants:

1. Find any real zeros of the denominator. Determine the domain of the function and sketch any vertical asymptotes.

2. Find the horizontal asymptote or the oblique asymptote, if there is one, and sketch it.

3. Find any zeros of the function. The zeros re found by determining the zeros of the numerator. These are the first coordinates of the **x-intercepts** of the graph.

4. Find $\boldsymbol{f}(\mathbf{0})$. This give the **y-intercept**, $(\mathbf{0}, \boldsymbol{f}(\mathbf{0}))$, of the function.

5. Find other function values to determine the general shape. Then draw the graph.

Solve Polynomial inequality

To solve a polynomial inequality

1. Find an equality with $P(x)$ on one side and 0 on the other.

2. Change the inequality symbol to an equals sign and solve the related equation; that is solve $P(x) = 0$.

3. Use the solutions to divide the **x-axis** into intervals. Then select a test value from each interval and determine the sign of the polynomial on the interval.

4. Determine the intervals for which the inequality is satisfied and write interval notation or set-builder notation for the solution set. Include the endpoints of the intervals in the solution set if the inequality symbol is $\leq$ or $\geq$.

Solve Rational Inequality

To solve a rational inequality:

1. Find an equivalent with 0 on one side.

2. Change the inequality symbol to an equals sign and solve the related equation

3. Find the values of the variable for which the related rational function is not defined.

4. The numbers found in steps (2) and (3) are called **critical values**. Use the critical values to divide the **x-axis** into intervals using an **x-value** from the interval or the graph of the equation.

5. Select the intervals for which the inequality is satisfied and write interval notation or set-builder notation for the solution set. If the inequality symbol is $\leq$ or $\geq$, then the solutions to step (2) should be included in the solution set The **x-values** found in step (3) are never included in the solution set.

Inverse Relation

Interchanging the first and second coordinates of each ordered pair in a relation produces the **inverse relation**.

Inverse Relation

If a relation is defined by an equation, interchanging the variables produces an equation of the **inverse relation**.

One-to-One Functions

A function f is **one-to-one** if different inputs have different outputs—that is ,

If $a \neq b$, then $f(a) \neq f(b)$.

Or, a function f is **one-to-one** if when the outputs are the same, the inputs are the same—that is,

If $f(a) = f(b)$, then $a = b$.

One-to-One Functions AND Inverses

- If a function is one-to-one, then its inverse f^{-1} is a function.

- The domain of a one-to-one function f is the range of the inverse f^{-1}.

- The range of a one-to-one function f is the domain of the inverse function f^{-1}.

- A function that is increasing over its entire domain or is decreasing over its entire domain is a one-to-one function.

Horizontal Line Test

It is possible for a horizontal line to intersect the graph of a function more than once, then the function is *not* one-to-one and its inverse is *not* a function.

Obtaining a Formula for an Inverse

If a function $\boldsymbol{f}$ is one-to-one, a formula for its inverse can generally be found as follows:

1. Replace $\boldsymbol{f(x)}$ with $\boldsymbol{y}$.
2. Interchange $\boldsymbol{x}$ and $\boldsymbol{y}$.
3. Solve for $\boldsymbol{y}$.
4. Replace $\boldsymbol{y}$ with $\boldsymbol{f^{-1}(x)}$.

Graph of f^{-1}

The graph of $\boldsymbol{f^{-1}}$ is a reflection of the graph of $\boldsymbol{f}$ across the line $\boldsymbol{y = x}$.

Inverse Functions and Composition

If a function $\boldsymbol{f}$ is one-to-one, then $\boldsymbol{f^{-1}}$ is the unique function such that each of the following holds:

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$$

For each $\boldsymbol{x}$ in the domain of $\boldsymbol{f}$

$$(f \circ f^{-1})(x) = f\left(f^{-1}(x)\right) = x$$

For each $\boldsymbol{x}$ in the domain of $\boldsymbol{f^{-1}}$

Exponential Function

The function $\boldsymbol{f(x) = a^x}$, where $\boldsymbol{x}$ is a real number, $\boldsymbol{a > 0}$ and $\boldsymbol{a \neq 1}$, is called the **exponential function, base a**

Logarithmic Function, Base 2

"$\boldsymbol{\log_2 x}$," read "the logarithm, base 2, of $\boldsymbol{x}$," means "the power to which we raise 2 to get $\boldsymbol{x}$."

Logarithmic Function, Base a

We define $\boldsymbol{y = \log_a x}$ as that number $\boldsymbol{y}$ such that $\boldsymbol{x = a^y}$, where $\boldsymbol{x > 0}$ and $\boldsymbol{a}$ is a *positive* constant other than $\boldsymbol{1}$.

Common Logs

$$\log_a 1 = 0$$

and

$$\log_a a = 1$$

For any logarithmic base $\boldsymbol{a}$.

Converting Between Exponential Equations and Logarithmic Equations

$$\log_a x = y \leftrightarrow x = a^y$$

Natural Logarithms

$$\ln 1 = 0$$

and

$$\ln e = 1$$

The change-of-Base Formula

For any logarithmic bases $\boldsymbol{a}$ and $\boldsymbol{b}$, and any positive number $\boldsymbol{M}$,

$$\log_b M = \frac{\log_a M}{\log_a b}$$

Logarithmic Product Rule

For any positive numbers M and N and any logarithmic base a,

$$\log_a MN = \log_a M += \log_a N$$

(The logarithm of a product is the sum of the logarithms of the factors.)

Logarithmic Power Rule

For any positive number M, any logarithmic base a, and any real number p.

$$\log_a M^p = p \log_a M$$

(The logarithm of a power of M is the exponent times the logarithm of M.)

Logarithmic Quotient Rule

For any positive numbers M and N and any logarithmic base a,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

(The logarithm of a quotient is the logarithm of the numerator minus the logarithm of the denominator.)

The logarithm of a Base to a Power

For any base a and any real number x,

$$\log_a a^x = x$$

(The logarithm, base a, of a to a power is the power.)

A vase to a Logarithmic Power

For any base a and any real number x,

$$a^{\log_a x} = x$$

Base-Exponent Property

For any $a > 0,\ a \neq 1$,

$$a^x = a^y \leftrightarrow x = y$$

Property of Logarithmic Equality

For any $M > 0$, $a > 0$,and $a \neq 1$,

$$\log_a M = \log_a N \leftrightarrow M = N$$

Growth Rate and Doubling Time

The **growth rate** k and the **doubling time** T are related by

$$kT = \ln 2$$

or

$$k = \frac{\ln 2}{T}$$

or

$$T = \frac{\ln 2}{k}$$

Decay Rate and Half-Life

The decay **rate** k and the half-life T are related by

$$kT = \ln 2$$

or

$$k = \frac{\ln 2}{T}$$

or

$$T = \frac{\ln 2}{k}$$

Converting From Base b to Base e

$$b^x = e^{x(\ln b)}$$

Trigonometric Function Values of n Acute Angle θ

Let $\boldsymbol{\theta}$ be an acute angle of a right triangle. Then the six trigonometric functions of $\boldsymbol{\theta}$ are as follows

$$\sin\theta = \frac{side\ opposite\ \theta}{hypotenuse}$$

$$\cos\theta = \frac{side\ adjecent\ to\ \theta}{hypotenuse}$$

$$\tan\theta = \frac{side\ opposite\ \theta}{side\ adjecent\ to\ \theta}$$

$$\csc\theta = \frac{hypotenuse}{side\ opposite\ \theta}$$

$$\sec\theta = \frac{hypotenuse}{side\ adjecent\ to\ \theta}$$

$$\cot\theta = \frac{side\ adjecent\ to\ \theta}{side\ opposite\ \theta}$$

Reciprocal Functions

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Trigonometric Function Value

The trigonometric function values of $\boldsymbol{\theta}$ depend only on the measure of the angle, *not* the size of the triangle.

Co-function Identities

$$\sin\theta = \cos(90° - \theta)$$

$$\cos\theta = \sin(90° - \theta)$$

$$\tan\theta = \cot(90° - \theta)$$

$$\cot\theta = \tan(90° - \theta)$$

$$\sec\theta = \csc(90° - \theta)$$

$$\csc\theta = \sec(90° - \theta)$$

Trigonometric Functions of Any Angle θ

Suppose that $\boldsymbol{P(x, y)}$ is any point other than the vertex on the terminal side of any angle $\boldsymbol{\theta}$ in standard position, and $\boldsymbol{r}$ is the radius, or distance from the origin to $\boldsymbol{P(x, y)}$. Then the trigonometric functions are defined as follows:

$$\sin\theta = \frac{y - coordinate}{radius} = \frac{y}{r}$$

$$\cos\theta = \frac{x - coordinate}{radius} = \frac{x}{r}$$

$$\tan\theta = \frac{y - coordinate}{x - coordinate} = \frac{y}{x}$$

$$\csc\theta = \frac{radius}{y - coordinate} = \frac{r}{y}$$

$$\sec\theta = \frac{radius}{x - coordinate} = \frac{r}{x}$$

$$\cot\theta = \frac{x - coordinate}{y - coordinate} = \frac{x}{y}$$

Reference Angle

The **reference angle** for an angle is the acute angle formed by the terminal side of the angle and the **x-axis**.

Converting Between Degree Measure and Radian Measure

$$\frac{\pi\ radians}{180°} = \frac{180°}{\pi\ radians} = 1$$

To convert from degree to radian measure, multiply by

$$\frac{\pi\ radians}{180°}$$

To convert from radian to degree measure, multiply by

$$\frac{180°}{\pi\ radians}$$

Radian Measure

The **radian measure** θ of a rotation is the ratio of the distance s traveled by a point at a radius r from the center of rotation, to the length of the radius r.

$$\theta = \frac{r}{s}$$

When we are using the formula $\theta = \frac{r}{s}$, θ must be in radians and s and r must be expressed in the same unit.

Linear Speed in Terms of Angular Speed

The **linear speed** v of a point a distance r from the center of rotation is given by

$$v = r\omega$$

Where ω is the **angular speed** in radians per unit of time.

Linear Speed in Terms of Angular Speed

For the formula $v = r\omega$, the units of distance for v and r must be the same, ω must be in radians per unit of time, and the units of time for v andω must be the same.

Basic Circular Functions

For a real number s that determines a point (x, y) on the unit circle:

$$\sin(s) = second\ coordinate = y$$

$$\cos(s) = first\ coordinate = x$$

$$\tan(s) = \frac{second\ coordinate}{first\ coordinate} = \frac{y}{x}(x \neq 0)$$

$$\csc(s) = \frac{1}{second\ coordinate} = \frac{1}{y}(y \neq 0)$$

$$\sec(s) = \frac{1}{first\ coordinate} = \frac{1}{x}(x \neq 0)$$

$$\cot(s) = \frac{first\ coordinate}{second\ coordinate} = \frac{x}{y}(y \neq 0)$$

Domain and Range of Sine and Cosine Functions

The *domain* of the sine function and the cosine function is $(-\infty, \infty)$

The *range* of the sine function and the cosine function is $[-1,1]$

Periodic Function

A function f is said to be **periodic** if there exists a positive constant p such that

$$f(s + p) = f(s)$$

For all s n the domain of f. The smallest such positive number p is called the period of the function.

Sine Function

1. Continuous
2. Period: $\mathbf{2\pi}$
3. Domain: $\mathbb{R}$
4. Range: $[-\mathbf{1}, \mathbf{1}]$
5. Amplitude: $\mathbf{1}$
6. Odd: $\mathbf{sin}(-\boldsymbol{s}) = -\,\mathbf{sin}(\boldsymbol{s})$

Cosine Function

1. Continuous
2. Period: $\mathbf{2\pi}$
3. Domain: $\mathbb{R}$
4. Range: $[-\mathbf{1}, \mathbf{1}]$
5. Amplitude: $\mathbf{1}$
6. Odd: $\mathbf{cos}(-\boldsymbol{s}) = \mathbf{cos}(\boldsymbol{s})$

Tangent Function

1. Period: $\boldsymbol{\pi}$
2. Domain: $\mathbb{R}$ except $\boldsymbol{\pi}/\mathbf{2} + \boldsymbol{k\pi}$, where $\boldsymbol{k}$ is an integer
3. Range:$\mathbb{R}$

Cotangent Function

1. Period: $\boldsymbol{\pi}$
2. Domain: $\mathbb{R}$ except $\boldsymbol{\pi}/\mathbf{2} + \boldsymbol{k\pi}$, where $\boldsymbol{k}$ is an integer
3. Range:$\mathbb{R}$

Cosecant Function

1. Period: $2\boldsymbol{\pi}$
2. Domain: $\mathbb{R}$ except $\boldsymbol{\pi}/\mathbf{2} + \boldsymbol{k\pi}$, where $\boldsymbol{k}$ is an integer
3. Range:$(-\infty, -\mathbf{1}] \cup [\mathbf{1}, \infty)$

Secant Function

1. Period: $2\boldsymbol{\pi}$
2. Domain: $\mathbb{R}$ except $\boldsymbol{\pi}/\mathbf{2} + \boldsymbol{k\pi}$, where $\boldsymbol{k}$ is an integer
3. Range:$(-\infty, -\mathbf{1}] \cup [\mathbf{1}, \infty)$

Amplitude

The **amplitude** of the graphs of

$\boldsymbol{y = A}\,\mathbf{sin}(\boldsymbol{Bx - C}) + \boldsymbol{D}$

and

$\boldsymbol{y = A}\,\mathbf{cos}(\boldsymbol{Bx - C}) + \boldsymbol{D}$ is $|\boldsymbol{A}|$

Period

The **period** of the graphs of
$\boldsymbol{y = A}\,\mathbf{sin}(\boldsymbol{Bx - C}) + \boldsymbol{D}$ and
$\boldsymbol{y = A}\,\mathbf{cos}(\boldsymbol{Bx - C}) + \boldsymbol{D}$ is $|\mathbf{2}\boldsymbol{\pi}/\boldsymbol{B}|$

Phase Shift

The **phase shift** of the graphs

$$\boldsymbol{y = A}\,\mathbf{sin}(\boldsymbol{Bx - C}) + \boldsymbol{D}$$

$$= \boldsymbol{A}\,\mathbf{sin}\left[\boldsymbol{B}\left(\boldsymbol{x} - \frac{\boldsymbol{C}}{\boldsymbol{B}}\right)\right] + \boldsymbol{D}$$

and

$$\boldsymbol{y = A}\,\mathbf{cos}(\boldsymbol{Bx - C}) + \boldsymbol{D}$$

$$= \boldsymbol{A}\,\mathbf{cos}\left[\boldsymbol{B}\left(\boldsymbol{x} - \frac{\boldsymbol{C}}{\boldsymbol{B}}\right)\right] + \boldsymbol{D}$$

Is

$$\frac{\boldsymbol{C}}{\boldsymbol{B}}$$

Transformation of Sine Functions and Cosine Functions

To graph

$y = A\sin(Bx - C) + D = A\sin\left[B\left(x - \frac{C}{B}\right)\right] + D$ and

$y = A\cos(Bx - C) + D = A\cos\left[B\left(x - \frac{C}{B}\right)\right] + D$

Follow these steps in the order in which they are listed

1. Stretch or shrink the graph horizontally according to B.
   a. $|B| < 1$, stretch horizontally
   b. $|B| > 1$, shrink horizontally
   c. $B < 0$, reflect across **y-axis**
2. Stretch or shrink the graph vertically according to A.
   a. $|A| < 1$, stretch vertically
   b. $|A| > 1$, shrink vertically
   c. $A < 0$, reflect across **x-axis**
3. Translate the graph horizontally according to C/B
   a. $\frac{C}{B} < 0$, $\left|\frac{C}{B}\right|$ units to the left
   b. $\frac{C}{B} > 0$, $\left|\frac{C}{B}\right|$ units to the right

The *phase shift* is C/B

4. Translate the graph vertically according to D
   a. $D < 0$, $|D|$ units down
   b. $D > 0$, $|D|$ units up

Basic Pythagorean Identities

$$\sin(x) = \frac{1}{\csc(x)}$$

$$\cos(x) = \frac{1}{\csc(x)}$$

$$\tan(x) = \frac{1}{\cot(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\cot(x) = \frac{1}{\tan(x)}$$

$$\sin(-x) = -\sin(x)$$

$$\cos(-x) = \cos(x)$$

$$\tan(-x) = -\tan(x)$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Pythagorean Identities

$$\sin^2(x) + \cos^2(x) = 1$$

$$1 + \cot^2(x) = \csc^2(x)$$

$$1 + \tan^2(x) = \sec^2(x)$$

Pythagorean identities Equivalent Forms

$$\sin^2(x) = 1 - \cos^2(x)$$

$$\cos^2(x) = 1 - \sin^2(x)$$

$$1 + \cot^2(x) = \csc^2(x)$$

$$1 = \csc^2(x) - \cot^2(x)$$

$$\cot^2(x) = \csc^2(x) - 1$$

$$1 = \sec^2(x) - \tan^2(x)$$

$$\tan^2(x) = \sec^2(x) - 1$$

Sum and Difference Identities

$$\sin(u \pm v) = \sin(u)\cos(v) \pm \cos(u)\sin(v)$$

$$\cos(u \pm v) = \cos(u)\cos(v) \pm \sin(u)\sin(v)$$

$$\tan(u \pm v) = \frac{\tan(u) \pm \tan(v)}{1 \pm \tan(u)\tan(v)}$$

There are six identities here

Co-function Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot(x)$$

$$\cot\left(\frac{\pi}{2} - x\right) = \tan(x)$$

$$\sec\left(\frac{\pi}{2} - x\right) = \csc(x)$$

$$\csc\left(\frac{\pi}{2} - x\right) = \sec(x)$$

Co-function Identities for Sine and Cosine

$$\sin(x \pm \frac{\pi}{2}) = \pm \cos(x)$$

$$\cos(x \pm \frac{\pi}{2}) = \pm \sin(x)$$

Double-Angle Identities

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

$$= 1 - 2\sin^2(x)$$

$$= 2\cos^2(x) - 1$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

$$\tan^2(x) = \frac{1 - \cos(2x)}{1 + \cos(2x)}$$

Half-Angle Identities

$$\sin(\frac{x}{2}) = \pm\sqrt{\frac{1 - \cos(x)}{2}}$$

$$\cos(\frac{x}{2}) = \pm\sqrt{\frac{1 + \cos(x)}{2}}$$

$$\tan\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos(x)}{1 + \cos(x)}}$$

$$= \frac{\sin(x)}{1 + \cos(x)}$$

$$= \frac{1 - \cos(x)}{\sin(x)}$$

Product-To-Sum Identities

$$\sin(x) \cdot \sin(y) = \frac{\cos(x - y) - \cos(x + y)}{2}$$

$$\cos(x) \cdot \cos(y) = \frac{\cos(x - y) + \cos(x + y)}{2}$$

$$\sin(x) \cdot \cos(y) = \frac{\sin(x + y) + \sin(x - y)}{2}$$

$$\cos(x) \cdot \sin(y) = \frac{\sin(x + y) - \sin(x - y)}{2}$$

Sum-To-Product Identities

$$\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

$$\cos(y) + \cos(x) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos(y) - \cos(x) = \sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Inverse Trigonometric Functions

$y = \sin^{-1}(x)$, where $x = \sin(y)$

Domain:$[-1, 1]$

Range: $[\frac{-\pi}{2}, \frac{\pi}{2}]$

$y = \cos^{-1}(x)$, where $x = \cos(y)$

Domain:$[-1, 1]$

Range: $[0, \pi]$

$y = \tan^{-1}(x)$, where $x = \tan(y)$

Domain:$[-\infty, \infty]$

Range: $[\frac{-\pi}{2}, \frac{\pi}{2}]$

Composition of Trigonometric Functions

$\sin(\sin^{-1}(x)) = x$, for all x in the domain of sin^{-1}

$\cos(\cos^{-1}(x)) = x$, for all x in the domain of cos^{-1}

$\tan(\tan^{-1}(x)) = x$, for all x in the domain of tan^{-1}

Special Cases

$\sin^{-1}(\sin(x)) = x$, for all x in the range of sin^{-1}

$\cos^{-1}(\cos(x)) = x$, for all x in the range of cos^{-1}

$\tan^{-1}(\tan(x)) = x$, for all x in the domain of tan^{-1}

Sine Function

$\sin\theta$

1. Domain: $(-\infty, \infty)$
2. Range: $[-1,1]$

Cosine Function

$\cos\theta$

1. Domain: $(-\infty, \infty)$
2. Range: $[-1,1]$

Tangent Function

$\tan\theta$

1. Domain: $\mathbb{R}$ except $k\pi/2$, k odd
2. Range: $(-\infty, \infty)$

Cosecant Function

$$\csc \boldsymbol{\theta}$$

1. Domain:$\mathbb{R}$ except $\boldsymbol{k\pi}$
2. Range: $(-\infty, -\mathbf{1}] \cup [\mathbf{1}, \infty)$

Secant Function

$$\sec \boldsymbol{\theta}$$

1. Domain: $\mathbb{R}$ except $\boldsymbol{k\pi/2}$, $\boldsymbol{k}$ odd
2. Range: $(-\infty, -\mathbf{1}] \cup [\mathbf{1}, \infty)$

Cotangent Function

$$\cot \boldsymbol{\theta}$$

1. Domain:$\mathbb{R}$ except $\boldsymbol{k\pi}$
2. Range: $(-\infty, \infty)$

Arcsine Function

$$\sin^{-1} \boldsymbol{\theta}$$

1. Domain: $[-\mathbf{1}, \mathbf{1}]$
2. Range: $[\frac{-\pi}{2}, \frac{\pi}{2}]$

Arccosine Function

$$\cos^{-1} \boldsymbol{\theta}$$

1. Domain: $[-\mathbf{1}, \mathbf{1}]$
2. Range: $[\mathbf{0}, \boldsymbol{\pi}]$

Arctangent Function

$$\tan^{-1} \boldsymbol{\theta}$$

1. Domain: $(-\infty, \infty)$
2. Range: $[\frac{-\pi}{2}, \frac{\pi}{2}]$

Arccosecant Function

1. Domain: $(-\infty, -\mathbf{1}) \cup [\mathbf{1}, \infty)$
2. Range: $[\mathbf{0}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \boldsymbol{\pi}]$

Arctangent Function

1. Domain: $(-\infty, \infty)$
2. Range: $(\mathbf{0}, \boldsymbol{\pi}]$

Angle-Angle-Side (AAS)

Two angles of a triangle and a side opposite one of them are known.

Angle-Side-Angle (ASA)

Two angles of a triangle and the included sides are known.

Side-Side-Angle (SSA)

Two sides of a triangle and an angle opposite one of them are known. (In this case, there may be no solution, one solution, or two solutions. The latter is known is the ambiguous case.)

Side-Angle-Side (SAS)

Two sides of a triangle and the included angle are known.

Side-Side-Side (SSS)

All three sides of the triangle are known.

The Law of Sines

In any triangle $\boldsymbol{ABC}$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Area of a Triangle

The area $\boldsymbol{K}$ of any $\triangle \boldsymbol{ABC}$ is one-half of the product of the lengths of two sides and the sine of the included angle:

$$K = \frac{bc\sin A}{2} = \frac{ab\sin C}{2} = \frac{ac\sin B}{2}$$

The Law of Cosines

In any triangle $\boldsymbol{ABC}$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Thus, in any triangle, the square of a side is the sum of the squares of the other two sides, minus twice the product of those sides and the cosine of the included angle. When the included angle is $\mathbf{90}°$, the law of cosines reduces the Pythagorean theorem.

Use The Law of Sines

Use the law of sines for:

AAS

ASA

SSA

Use The Law Of Cosines

SAS

SSS

The law of cosines can also be used for the SSA situation, but since the process involves solving a quadratic equation, it is not included.

Absolute Value of a Complex Number

The **absolute value of a complex number** $a + bi$

Trigonometric Notation For Complex Numbers

$$a + bi = r(\cos\theta + i\sin\theta)$$

Complex Numbers: Multiplication

For any complex numbers $r_1(\cos(\theta_1) + i\sin(\theta_1)$ and $r_2(\cos(\theta_2) + i\sin(\theta_2)$

$$r_1(\cos(\theta_1) + i\sin(\theta_1) \cdot r_2(\cos(\theta_2) + i\sin(\theta_2)$$

$$= r_1 r_2[\cos(\theta_1 + (\theta_1) + i\sin(\theta_1 + \theta_2)]$$

Complex Numbers: Division

For any complex numbers $r_1(\cos(\theta_1) + i\sin(\theta_1)$ and $r_2(\cos(\theta_2) + i\sin(\theta_2) \neq 0$

$$\frac{r_1(\cos(\theta_1) + i\sin(\theta_1)}{r_2(\cos(\theta_2) + i\sin(\theta_2)}$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - (\theta_1) + i\sin(\theta_1 - \theta_2)]$$

Demoivre's Theorem

For any complex number $r(\cos\theta + i\sin\theta)$ and any natural number n,

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos\theta + i\sin n\theta)$$

Roots Of Complex Numbers

The nth roots of a complex number $r(\cos\theta + i\sin\theta)$, $r \neq 0$, are given by

$$r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + k\cdot\frac{360°}{n}\right) + i\sin\left(\frac{\theta}{n} + k\cdot\frac{360°}{n}\right)\right]$$

Where $k = 0,1,2,\cdots,n-1$

Plot points on a polar graph:

To plot points on a polar graph

1. Locate the directed angle θ.
2. Move a directed distance r from the pole. If $r > 0$, move along ray OP. If $r < 0$, move in the opposite direction of ray OP.

Vectors

A **vector** in the plane is a directed line segment. Two vectors are **equivalent** if they have the same *magnitude* and the same *direction*.

Component Form of a Vector

The **component form** of $\overrightarrow{AC}$ with $A = (x_1, y_1)$ and $C = (x_2, y_2)$ is $\overrightarrow{AC} = \langle x_2 - x_1, y_2 - y_1\rangle$

Length Of a Vector

The **length**, o **magnitude**, of a vector $v = \langle v_1, v_2\rangle$ is given by

$$|v| = \sqrt{{v_1}^2 + {v_2}^2}$$

Equivalent Vectors

Let $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$. Then

$$\langle u_1, u_2\rangle = \langle v_1, v_2\rangle$$

If and only if $u_1 = v_1$ and $u_2 = v_2$

Scalar Multiplication

For any real number k and a vector $v = \langle v_1, v_2\rangle$, the **scalar product** of k and v is

$$kv = k\langle v_1, v_2\rangle = \langle kv_1, kv_2\rangle$$

The vector kv is a **scalar multiple** of the vector v

Vector Addition

If $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$. Then

$$u + v = \langle u_1 + v_1, u_2 + v_2\rangle$$

Vector Subtraction

If $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$. Then

$$u - v = \langle u_1 - v_1, u_2 - v_2\rangle$$

Properties of Vector Addition and Scalar Multiplication

For all vectors $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$, and for all scalars $\boldsymbol{b}$, and $\boldsymbol{c}$:

1. $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$
2. $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{v}+\boldsymbol{u})+\boldsymbol{w}$
3. $\boldsymbol{v}+\mathbf{0}=\boldsymbol{v}$
4. $\mathbf{1}\boldsymbol{v}=\boldsymbol{v}; \mathbf{0}\boldsymbol{v}=\mathbf{0}$
5. $\boldsymbol{v}+(-\boldsymbol{v})=\mathbf{0}$
6. $\boldsymbol{b}(\boldsymbol{c}\boldsymbol{v})=(\boldsymbol{b}\boldsymbol{c})\boldsymbol{v}$
7. $(\boldsymbol{b}+\boldsymbol{c})\boldsymbol{v}=\boldsymbol{b}\boldsymbol{v}+\boldsymbol{c}\boldsymbol{v}$
8. $\boldsymbol{b}(\boldsymbol{u}+\boldsymbol{v})=\boldsymbol{b}\boldsymbol{u}+\boldsymbol{b}\boldsymbol{v}$

Dot Product

The **Dot product** of two vectors $\boldsymbol{u}=\langle\boldsymbol{u_1}, \boldsymbol{u_2}\rangle$ and $\boldsymbol{v}=\langle\boldsymbol{v_1}, \boldsymbol{v_2}\rangle$ is

$$\boldsymbol{u}\cdot\boldsymbol{v}=\boldsymbol{u_1}\boldsymbol{v_1}, +\boldsymbol{u_2}\boldsymbol{v_2}$$

(Note that $\boldsymbol{u_1}\boldsymbol{v_1}, +\boldsymbol{u_2}\boldsymbol{v_2}$ is a *scalar*, not a vector.)

Angle Between Two Vectors

If $\boldsymbol{\theta}$ is the angle between two *non-zero* vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then

$$\cos\boldsymbol{\theta}=\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}$$

Solving Systems of Equations in Three Variables

1. Interchange any two equations
2. Multiply on both sides of the equations by a non-zero constant
3. Add a non-zero multiple of one equation to another equation.

Row-Equivalent Operations

1. Interchange any two rows.
2. Multiply each entry in a row by the same non-zero constant
3. Add a non-zero multiple of one to another row.

Row-Echelon Form

1. If a row does not consist entirely of $\mathbf{0}'\boldsymbol{s}$, then the first non-zero element in the row is a $\mathbf{1}$ (called a **leading $\mathbf{1}$**).
2. For any two successive non-zero rows, the leading $\mathbf{1}$ in the lower row is farther to the right than the leading $\mathbf{1}$ in the higher row.
3. All the rows consisting entirely of $\mathbf{0}'\boldsymbol{s}$ are at the bottom of the matrix.

If a fourth property is also satisfied, a matrix is said to be in **reduced row-echelon form**:

4. Each column that contains a leading $\mathbf{1}$ has $\mathbf{0}'\boldsymbol{s}$ everywhere else.

Addition and Subtraction of Matrices

Given two $\boldsymbol{m}\times\boldsymbol{n}$ matrices $\boldsymbol{A}=[\boldsymbol{a_{ij}}]$ and $\boldsymbol{B}=[b_{ij}]$, their sum is

$$\boldsymbol{A}+\boldsymbol{B}=[\boldsymbol{a_{ij}}+\boldsymbol{b_{ij}}]$$

and their difference is

$$\boldsymbol{A}-\boldsymbol{B}=[\boldsymbol{a_{ij}}-\boldsymbol{b_{ij}}]$$

Scalar Product

The **scalar product** of a number $\boldsymbol{k}$ and a matric $\boldsymbol{A}$ is the matrix denoted $\boldsymbol{kA}$, obtained by multiplying each entry of $\boldsymbol{A}$ by the number $\boldsymbol{k}$. The number $\boldsymbol{k}$ is called a **scalar**.

Properties of Matrix Addition and Scalar Multiplication

For any $m \times n$ matrices A, B and C and any scalars k and l:

Communicative Property of Addition

$$A + B = B + A$$

Associative Property of Addition

$$A + (B + C) = (A + B) + C$$

Associative Property Scalar Multiplication

$$(kl)A = k(lA)$$

Distributive Property

$$k(A + B) = kA + kB$$

$$(k + l)A = kA + lA$$

There exists a unique matrix $\mathbf{0}$ such that:

Additive Identity Property

$$A + \mathbf{0} = \mathbf{0} + A = A$$

There exists a unique matrix $-A$ such that:

Additive Inverse Property

$$A + (-A) = -A + A = \mathbf{0}$$

Matrix Multiplication

For any $m \times n$ matrix $A = [a_{ij}]$ and an $n \times p$ matrix $B = [b_{ij}]$ the **product** $AB = [c_{ij}]$ is an $m \times p$ matrix where

$$c_{ij} = a_{i1} \cdot b_j + \cdot\, a_{i2} \cdot b_{2j} + \cdot\, a_{i3} \cdot b_{i3} + \cdots + \cdot\, a_{in} \cdot b_{in}$$

Matrix Multiplication

Note that you can multiply two matrices only when the number of columns in the first matrix is equal to the number of rows in the second matrix.

Properties of Matrix Multiplication

For matrices A, B, and C, assuming that the indicated operations are possible:

Associative Property of Multiplication

$$A(BC) = (AB)C$$

Distributive Property

$$A(B + C) = AB + AC$$

Distributive Property

$$(B + C)A = BA + CA$$

Identity Matrix

For any positive integer n, the $n \times n$ **identity matrix** is an $n \times n$ matrix with $1's$ on the main diagonal and $0's$ elsewhere and is denoted by

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

Then $AI = IA$, for any $n \times n$ matrix A.

Inverse of Matrix

For any $n \times n$ matrix A there is a matrix A^{-1} for which $A^{-1} \cdot A = I = A \cdot A^{-1}$, then A^{-1} is the **inverse** of A.

Matrix Solutions of Systems of Equations

For a system of n linear equations in n variables, $AX = B$, if A is an invertible matrix, then unique solution of the system is given by

$$X = A^{-1}B$$

Determinants of a 2×2 Matrix

The **determinant** of matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is denoted $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$ and is defined as $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$

Minor

For a square matrix $A = [a_{ij}]$,the **minor** M_{ij} of an entry a_{ij} is the determinant of the matrix formed by deleting the ith row and the jth column of A.

Cofactor

For a square matrix $A = [a_{ij}]$, the **cofactor** of A_{ij} of an entry a_{ij} is given by

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Where M_{ij} is the minor of a_{ij}

Minor and Cofactors

Note that minors and cofactors are *numbers*. They are not *matrices*.

Determinant of Any Square Matrix

For any square matrix A of order $n \times n$ $(n > 1)$, we define the **determinant** of A, denoted $|A|$, as follows. Choose any row or column. Multiply each element in that row or column by its cofactor and add the results. The determinant of a 1×1 matrix is simply the elements of the matrix. The value of a determinant will be the same no matter which row or column is chosen.

Cramer's Rule for 2×2 Systems

The solution of the system of equations

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

Is given by

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

Where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

and $D \neq 0$

Cramer's Rule for 3×3 Systems

The solution of the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Were

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad D_2 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

And $D \neq 0$

Linear Inequality In Two Variables

A **Linear inequality in two variables** is an inequality that can be written in the form

$$Ax + By < C$$

Where A, B, and C are real numbers, and A and B are not both zero. The symbol $<$ may be replaced with $\leq, >$, or $\geq$.

Graph A Linear Inequality in Two Variables

To graph a linear inequality in two variables:

1. Replace the inequality with an equals sign and graph this related equation. If the inequality is $<$ or $>$, draw the line dashed. If the inequality is $\leq$ or $\geq$, draw the line solid
2. The graph consist of a half-plane on one side of the line and, if the line is solid, the line as well. To determine which half-plane to shade, test a point no on the line in the original inequality. If that point is a solution, shade the half-plane containing that point. If not, shade the opposite half-plane.

Linear Programming Procedure

To fine the maximum or minimum value of a linear objective function subject to a set of constraints.

1. Graph the region of feasible solutions
2. Determine the coordinates of the vertices of the region.
3. Evaluate the objective function at each vertex. The largest and smallest of those values are the maximum and minimum values of the function, respectively.

Procedure For Decomposing a Rational Expression into Partial Fractions

Consider any rational expression $P(x)/Q(x)$ such that $P(x)$ and $Q(x)$ have no common factor other than 1 or -1,

1. If the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$, divide to express $P(x)/Q(x)$ as a quotient + remainder $/Q(x)$ and follow steps$(2)-(5)$ to decompose the resulting rational expression.

2. If the degree of $P(x)$ is less than the degree of $Q(x)$, factor $Q(x)$ into linear factors of the form $(px+q)^n$ and/or quadratic factors of the form $(ax^2+bx+c)^m$. Any quadratic factor ax^2+bx+c must be irreducible, meaning that it cannot be factored into linear factors with rational coefficients.

3. Assign to each linear factor $(px+q)^n$ the sum of n partial fractions:

$$\frac{A_1}{px+q}+\frac{A_2}{(px+q)^2}+\cdots+\frac{A_n}{(px+q)^n}$$

4. Assign to each quadratic factor $(ax^2+bx+c)^m$ the sum of m partial fractions:

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_mx+C_m}{(ax^2+bx+c)^m}$$

5. Apply algebraic methods, to find the constraints in the numerators of the partial fractions.

Parabola

A **Parabola** is the set of all points in a plane equidistant from a fixed line (the **directrix**) and a fixed pint not on the line (the **focus**).

Standard Equation Of A Parabola with vertex at the origin

The standard equation of a parabola vertex $(\mathbf{0},\mathbf{0})$ and directrix $\boldsymbol{y}=-\boldsymbol{p}$ is

$$x^2 = 4py$$

The focus is $(\mathbf{0},\boldsymbol{p})$ and the **y-axis** is the axis of symmetry.

The standard equation of a parabola with vertex $(\mathbf{0},\mathbf{0})$ and directrix $\boldsymbol{y}=-\boldsymbol{p}$ is

$$y^2 = 4py$$

The focus is $(\boldsymbol{p},\mathbf{0})$ and the **x-axis** is axis of symmetry.

Standard Equation of A parabola with Vertex (h,k) and vertical Axis of Symmetry

The standard equation of a parabola with $(\boldsymbol{h},\boldsymbol{k})$ and vertical axis of symmetry is

$$(x-h)^2 = 4p(y-k)$$

Where the vertex is $(\boldsymbol{h},\boldsymbol{k})$, the focus is $(\boldsymbol{h},\boldsymbol{k}+\boldsymbol{p})$, and the directrix is $\boldsymbol{y}=\boldsymbol{k}-\boldsymbol{p}$

(When $\boldsymbol{p}<\mathbf{0}$, the parabola opens down.)

Standard Equation of A parabola with Vertex (h,k) and Horizontal Axis of Symmetry

The standard equation of a parabola with $(\boldsymbol{h},\boldsymbol{k})$ and Horizontal axis of symmetry is

$$(y-k)^2 = 4p(x-h)$$

Where the vertex is $(\boldsymbol{h},\boldsymbol{k})$, the focus is $(\boldsymbol{h}+\boldsymbol{p},\boldsymbol{k})$, and the directrix is $\boldsymbol{x}=\boldsymbol{h}-\boldsymbol{p}$

(When $\boldsymbol{p}<\mathbf{0}$,the parabola opens to the left.)

Circle

A **circle** is the set of all in a plane that are at a fixed distance from a fixed point (the **center**) in the plane.

Standard Equation Of A Circle

The standard equation of a circle with center $(\boldsymbol{h},\boldsymbol{k})$ and radius $\boldsymbol{r}$ is

$$(x-h)^2 + (y-k)^2 = r^2$$

Ellipse

An **ellipse** is the set of all points in a pane, the sum of whose distances from two fixed points (the **foci**) is constant. The **center** of an ellipse is the midpoint of the segment between the foci.

Standard Equation Of An Ellipse With Center At The Origin

Major Axis Horizontal

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0$$

Vertices:$(-\boldsymbol{a},\mathbf{0}),(\boldsymbol{a},\mathbf{0})$

Y-intercepts:$(\mathbf{0},-\boldsymbol{b}),(\mathbf{0},\boldsymbol{b})$

Foci: $(-\boldsymbol{c},\mathbf{0}),(\boldsymbol{c},\mathbf{0})$, where $c^2 = a^2 - b^2$

Major Axis Vertical

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad a > b > 0$$

Vertices:$(\mathbf{0},-\boldsymbol{a}),(\mathbf{0},\boldsymbol{a})$

Y-intercepts:$(-\boldsymbol{b},\mathbf{0}),(\boldsymbol{b},\mathbf{0})$

Foci: $(\mathbf{0},-\boldsymbol{c}),(\mathbf{0},\boldsymbol{c})$, where $c^2 = a^2 - b^2$

Standard Equation Of An Ellipse With Center At (h, k)

Major Axis Horizontal

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1, \quad a>b>0$$

Vertices: $(h-a, k), (h+a, k)$

Length of minor axis: $2b$

Foci: $(h-c, k), (h+c, k)$, where $c^2=a^2-b^2$

Major Axis Vertical

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1, \quad a>b>0$$

Vertices: $(h, k-a), (h, k+a)$

Length of minor axis: $2b$

Foci: $(h, k-c), (h, k+c)$, where $c^2=a^2-b^2$

Hyperbola

A **hyperbola** is the set of all in lane for which the absolute value of the difference of the distances from two fixed points (the **foci**) is constant. The midpoint of the segment between the foci is the **center** of the hyperbola.

Standard Equation Of Hyperbola With Center At The Origin

Transverse Axis Horizontal

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

Vertices:$(-a, 0), (a, 0)$

Foci: $(-c, 0), (c, 0)$, where $c^2=a^2-b^2$

Transverse Axis Vertical

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

Vertices:$(0, -a), (0, a)$

Foci: $(0, -c), (0, c)$, where $c^2=a^2-b^2$

Standard Equation Of Hyperbola With Center At The Origin (h, k)

Transverse Axis Horizontal

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Vertices: $(h-a, k), (h+a, k)$

Asymptotes: $y-k=\frac{b}{a}(x-h)$,

$$y-k=-\frac{b}{a}(x-h)$$

Foci: $(h-c, k), (h+c, k)$, where $c^2=a^2-b^2$

Transverse Axis Horizontal

$$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$$

Vertices: $(h, k-a), (h, k+a)$

Asymptotes: $y-k=\frac{b}{a}(x-h)$,

$$y-k=-\frac{b}{a}(x-h)$$

Foci: $(h, k-c), (h, k-c)$, where $c^2=a^2-b^2$

Rotation of Axes Formulas

If the **x-** and **y-axes** are rotated about the origin through a positive acute angle θ, then the coordinates $(x, y)$and (x', y') of a point P in the xy and $x'y'$-coordinate systems are related by the following formulas:

$$x' = x\cos\theta + y\sin\theta$$

$$x = x'\cos\theta - y'\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

Eliminating the xy-Term

To eliminate the **xy-term** from the equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0, \qquad B\neq 0$$

Select an angle θ such that

$$\cot 2\theta=\frac{A-C}{B}, 0^\circ < 2\theta < 180^\circ$$

and use the rotation of axes formulas.

Graph of the Equation

The graph of the equation

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

Is, except in degenerate cases,

1. An ellipse or a circle if $B^2-4AC<0$
2. A hyperbola if $B^2-4AC>0$
3. A parabola if $B^2-4AC=0$

An Alternative Definition Of Conics

Let $\boldsymbol{L}$ be a fixed line (the **directrix**), let $\boldsymbol{F}$ be a fixed point (the **focus**) not on $\boldsymbol{L}$, and let $\boldsymbol{e}$ be a positive constant (the **eccentricity**). A **conic** is the set of all points $\boldsymbol{P}$ in the plane such that

$$\frac{PF}{PL} = e$$

Where $\boldsymbol{PF}$ is the distance from $\boldsymbol{P}$ to $\boldsymbol{F}$ and $\boldsymbol{PL}$ is the distance from $\boldsymbol{P}$ *to* $\boldsymbol{L}$. The conic is a parabola if $e = 1$,an ellipse if $0 < e < 1$, and a hyperbola if $e > 1$.

Eccentricity e

For an ellipse and a hyperbola, the **eccentricity** $\boldsymbol{e}$ is given by

$$e = \frac{c}{a}$$

Where c is the distance from the center to a focus and a is the distance from the center to a vertex.

Polar Equations of Conics

A polar equation of any of the four forms

$$r = \frac{ep}{1 \pm e\cos\theta}$$

$$r = \frac{ep}{1 \pm e\sin\theta}$$

Is a conic section. The conic is a parabola if $e = 1$, an ellipse if $0 < e < 1$, and a hyperbola if $e > 1$.

Polar Equations

$$r = \frac{ep}{1 + e\cos\theta}$$

Vertical directrix p unit to the right of the pole (or focus)

$$r = \frac{ep}{1 - e\cos\theta}$$

Vertical directrix p unit to the left of the pole (or focus)

$$r = \frac{ep}{1 + e\sin\theta}$$

Horizontal directrix p units above the pole (or focus)

$$r = \frac{ep}{1 - e\sin\theta}$$

Horizontal directrix p units below the pole (or focus)

Parametric Equations

If f and g are continuous of t on an interval I,then the set of ordered pair (x, y) such that $x = f(t)$ and $y = g(t)$ is a **plane curve** The equation $x = f(t)$ and $y = g(t)$ are **parametric equations** for the curve. The variable t is the **parameter**.

Sequences

An **infinite sequence** is a function having for its domain the set of positive integers, $\{1, 2, 3, 4, 5, \cdots\}$

A **finite sequence** is a function having for its domain a set of positive integers $\{1, 2, 3, 4, 5, \cdots, n\}$, for some positive integer n

Series

Given the infinite sequence

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots,$$

The sum of the terms

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots,$$

Is called an **infinite series**. A **partial sum** is the sum of the first n terms:

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

A partial sum is also called a finite series, or nth **partial sum**, and is denoted S_∞.

Arithmetic Sequence

A sequence is **arithmetic** if there exists a number d, called the **common difference**, such that $a_{n+1} = a_n + d$ for any integer $n \geq 1$

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nth Term of An Arithmetic Sequence

The nth **term** of an arithmetic sequence is given by

$$a_n = a_1 + (n-1)d$$

for any integer $n \geq 1$.

Sum of the First n Terms

The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Geometric Sequence

A sequence is **geometric** if there is a number r, called the **common ratio**, such that

$$\frac{a_{n+1}}{a_n} = r,$$

Or

$$a_{n+1} = a_n\, r,$$

for any integer $n \geq 1$

---

nth Term of An Geometric Sequence

The nth **term** of a geometric sequence is given by

$$a_n = a_1 r^{n-1}$$

for any integer $n \geq 1$.

Sum of the First n Terms

The sum of the first n terms of an geometric sequence is given by

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

for any $r \neq 1$

Limit or Sum of An Infinite Geometric Series

When $|r| < 1$, the limit or sum of an infinite geometric series is given by

$$S_\infty = \frac{a_1}{1-r}$$

The principle of Mathematical Induction

We can prove an infinite sequence of statements S_n by showing the following

1. Basis step: S_1 is true.
2. Induction step: For all natural numbers k, $S_k \rightarrow S_{k+1}$

The Fundamental Counting Principle

Given a combined action, or *event*, in which the first action can be performed n_1 ways, the second action can be performed n_2 ways, and so on, the total number of ways in which the combined action can be performed is the product

$$n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_k$$

Permutation

A **permutation** of a set of n objects is an ordered arrangement of all n objects.

The Total Number of Permutations Of n Objects

The total number of permutations of n objects, denoted as nPn is given by

$$nPn = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

Factorial Notation

For any natural number n

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

For the number 0

$$0! = 1$$

$$nPn = n!$$

Generalizing Factorial

For any natural number $n, n! = n(n-1)!$

Factorial

For any natural numbers k and n, with $k < n$

$$n! = n(n-1)(n-2)\cdots[n-(k-1)] \cdot (n-k)!$$

Permutation of n Objects Taken k At a Time

A **permutation** of a set of n objects take k at a time is an ordered arrangement of k objects from the set.

The Number of Permutations of n Objects Taken k At A Time

The number of permutations of a set of n objects taken k at a time, denoted nPk is given by

$$nPk = (n-1)(n-2)\cdots[n-(k-1)]$$

$$= \frac{n!}{(n-k)!}$$

Number of Distinct Arrangements

The number of distinct arrangements of n objects taken k at a time, allowing repetition is n^k

---

Number of Distinguishable Permutations

For a set of n objects in which n_1 are of one kind n_2 are of another kind, $\cdots$, and n_k are of a kth kind, the number of distinguishable permutations is

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}$$

Combination; Combination Notation

A **combination** containing k objects chosen from a set of n objects $k \le n$, is denoted using **combination notation** nCk

Combinations of n Objects taken at k At A Time

The total number of combinations of n taken at k at a time, denoted nCk isgiven by

$$nCk = \frac{n!}{k!\,(n-k)!}$$

Or

$$nCk = \frac{nPk}{k!}$$

$$= \frac{(n-1)(n-2)\cdots[n-(k-1)]}{k!}$$

Binomial Coefficient Notation

$$\binom{n}{k} = nCk$$

Subsets of Size k and of size $n - k$

$$\binom{n}{k} = \binom{n}{n-k}$$

And

$$nCk = nC_{n-k}$$

The number of subsets of size k of a set with n objects is the same as the number of subsets of size $n-k$. The number of combinations of n objects taken at a time is the same as the number of combinations of n objects taken $n-k$ at a time.

The Binomial Theorem using Pascal's Triangle

For any binomial $a + b$ and any natural number n

$$(a+b)^n = c_0 a^n b^o + c_1 a^{n-1} b^1 + c_2 a^{n-2} b^2 + \cdots c_{n-1} a^1 b^{n-1} + c_n a^0 b^n$$

Where the numbers $c_0, c_1, c_2, \cdots, c_{n-1}, c_n$ are from the $(n+1)st$ row of Pascal's triangle.

Finding The $(k+1)$st Term

The $(k+1)st$ of $(a+b)^n$ is $\binom{n}{k} a^{n-k} b^k$

Total Number of Subsets

The total number of subsets of a set with n elements is 2^n

Principle P (Experimental)

Given an experiment in which n observations are made, if a situation, or event, E occurs m times out of n observations, then we say that the **experimental probability** of the event, $P(E)$, is given by

$$P(E) = \frac{m}{n}$$

Principle P (Theoretical)

If an event E occur m ways out of n possible equally likely outcomes of a sample space S, then the **theoretical probability** of the event $P(E)$, is given by

$$P(E) = \frac{m}{n}$$

Probability Properties

a) If an event E cannot occur, then $P(E) = 0$
b) If an event E is certain to occur, then $P(E) = 1$
c) The probability that an event E will occur is a number from 0 to 1: $0 \leq P(E) \leq 1$

Test for Symmetry

1. The graph of an equation in x and y is symmetric with respect to the **y-axis** when replacing x by $-x$ yields an equivalent equation.

2. The graph of an equation x and y is symmetric with respect to the **x-axis** when replacing y by $-y$ yields an equivalent equation.

3. The graph of an equation x and y is symmetric with respect to the **origin** when replacing x by $-x$ and y by $-y$ yields an equivalent equation.

Definition of the Slope of a line

The **slope** m of the non-vertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad x_1 \neq x_2$$

Slope is not defined for vertical lines.

Slope-Intercept Form of the Equation of a Line

The graph of the linear equation

$$y = mx + b$$

is a line whose slope is m and whose **y-intercept** is $(0, b)$

Summary of Equation of Lines

1. General Form: $Ax + By + C = 0$

2. Vertical Line: $x = a$

3. Horizontal Line: $y = b$

4. Slope-intercept Form: $y = mx + b$

5. Point-Slope Form:
$$y - y_1 = m(x - x_1)$$

Parallel and Perpendicular Lines

1. Two distinct non-vertical lines are **parallel** if and only if their slopes are equal—that is, if and only if
$$m_1 = m_2$$

2. Two non-vertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if
$$m_1 = -\frac{1}{m_2}$$

Definition of a Real-Value Function of a Real Variable

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x**. The **range** of f is a subset of Y and consist of all images of numbers in X.

Basic Types of Transformations ($c > 0$)

Original graph:

$$y = f(x)$$

Horizontal shift up c units to the **right**:

$$y = f(x - c)$$

Horizontal shift up c units to the **left**:

$$y = f(x + c)$$

Vertical shift up c units **downward**:

$$y = f(x) - c$$

Vertical shift up c units **upward**:

$$y = f(x) + c$$

Reflection (above the **x-axis**):

$$y = -f(x)$$

Reflection (above the **y-axis**):

$$y = f(-x)$$

Reflection (about the **origin**):

$$y = -f(-x)$$

Definition of Composite Function

Let f and g be functions. The function $(f \circ g)(x) = f\big(g(x)\big)$ is the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

Test for Even and Odd Functions

The function $y = f(x)$ is **even** when $f(-x) = f(x)$

The function $y = f(x)$ is **odd** when $f(-x) = -f(x)$

Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \frac{\pi}{2}$

$$\sin\theta = \frac{opposite}{hypotenuse}$$

$$\cos\theta = \frac{adjacent}{hypotenuse}$$

$$\tan\theta = \frac{opposite}{adjacent}$$

$$\csc\theta = \frac{hypotenuse}{opposite}$$

$$\sec\theta = \frac{hypotenuse}{adjacent}$$

$$\cot\theta = \frac{adjacent}{opposite}$$

Circular function definitions, where θ is any angle

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \quad y \neq 0$$

$$\sec\theta = \frac{r}{x} \quad x \neq 0$$

$$\cot\theta = \frac{x}{y} \quad y \neq 0$$

Trigonometric Identities

Pythagorean Identities

$$sin^2(x) + cos^2(x) = 1$$

$$1 + cot^2(x) = csc^2(x)$$

$$1 + tan^2(x) = sec^2(x)$$

Sum and Difference Formula

$$\sin(\theta \pm \phi) = \sin(\theta)\cos(\phi) \pm \cos(\theta)\sin(\phi)$$

$$\cos(\theta \pm \phi) = \cos(\theta)\cos(\phi) \pm \sin(\theta)\sin(\phi)$$

$$\tan(\theta \pm \phi) = \frac{\tan(\theta) \pm \tan(\phi)}{1 \pm \tan(\theta)\tan(\phi)}$$

Even/Odd Identities

$$\sin(-\theta) = -\sin(\theta)$$

$$\cos(-\theta) = \cos(\theta)$$

$$\tan(-\theta) = -\tan(\theta)$$

$$\csc(-\theta) = -\csc(\theta)$$

$$\sec(-\theta) = \sec(\theta)$$

$$\cot(-\theta) = -\cot(\theta)$$

Power-Reducing Formulas

$$sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

$$cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$tan^2(\theta) = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Double-Angle Formulas

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(2\theta) = cos^2(\theta) - sin^2(\theta)$$

$$= 1 - 2sin^2(\theta)$$

$$= 2cos^2(\theta) - 1$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - tan^2(\theta)}$$

Quotient Identities

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Slope of Secant Line

$$m_{sec} = \frac{f(c + \Delta x) - f(c)}{c + \Delta x - c} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Limit of $f(x)$ as x approaches c

$$\lim_{x \to c} f(x) = L$$

Common Types of Behavior Associated with Nonexistence of a Limit

1. $f(x)$ approaches a different number from the right side of c that it approaches from the left side.

2. $f(x)$ increases or decreases without bound as x approaches c.

3. $f(x)$ oscillates between two fixed values as x approaches c.

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c), and let L be a real number. The statement

$$\lim_{x \to c} f(x) = L$$

Means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - c| < \delta$$

Then

$$|f(x) - L| < \varepsilon$$

Some Basic Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x \to c} b = b$

2. $\lim_{x \to c} x = c$

3. $\lim_{x \to c} x^n = c^n$

Properties of Limits

Let b and c be real numbers, and let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x \to c} f(x) = L$$

And

$$\lim_{x \to c} g(x) = K$$

1. Scalar multiple:
$$\lim_{x \to c} [bf(x)] = bL$$

2. Sum or difference:
$$\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$$

3. Product:
$$\lim_{x \to c} [f(x)g(x)] = LK$$

4. Quotient :
$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, \quad K \neq 0$$

5. Power:
$$\lim_{x \to c} [f(x)]^n = L^n$$

Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c)$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c) = \frac{p(c)}{q(c)}$$

Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all c when n is odd, and is valid for $c > 0$ when n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Limit of a Composite Function

If f and g are function such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f\big(g(x)\big) = f\left(\lim_{x \to c} g(x)\right) = f(L)$$

Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

$$\lim_{x \to c} \sin(x) = \sin(c)$$

$$\lim_{x \to c} \cos(x) = \cos(c)$$

$$\lim_{x \to c} \tan(x) = \tan(c)$$

$$\lim_{x \to c} \cot(x) = \cot(c)$$

$$\lim_{x \to c} \sec(x) = \sec(c)$$

$$\lim_{x \to c} \csc(x) = \csc(c)$$

Functions That Agree at All but One Point

Let c be a real number $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$$

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution.

2. When the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$ [Choose g such that the limit of $g(x)$ can be evaluated by direct substitution.]

3. Use a *graph* or *table* to reinforce your conclusion.

Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$$

Then $\lim_{x \to c} f(x)$ exists and is equal to L

Two Special Trigonometric Limits

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$$

Definition of Continuity

Continuity at a Point

A function f is **continuous at c** when these three conditions are met.

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

Continuity on an Open Interval

A function is **continuous on an open interval** (a, b) when the function is continuous at each point in the interval. A function that is continuous on the entire real number line $(-\infty, \infty)$ is **everywhere continuous**.

Right Sided Limit

$$\lim_{x \to c^+} f(x) = L$$

Left Sided Limit

$$\lim_{x \to c^-} f(x) = L$$

One Sided Limit Involving Radicals

$$\lim_{x \to c^+} \sqrt[n]{x} = 0$$

Greatest Integer Function

$$[\![x]\!] = greatest\ integer\ \ n\ such\ that\ n \leq x$$

The Existence of a Limit

Let f be a function, and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x \to c^-} f(x) = L$$

And

$$\lim_{x \to c^+} f(x) = L$$

Definition of Continuity on a Closed Interval

A function f is **continuous on the closed interval** $[a, b]$ when f is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a)$$

And

$$\lim_{x \to b^-} f(x) = f(b)$$

The function f is **continuous from the right** at a and **continuous from the left** at b.

Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the functions listed below are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}, \quad g(c) \neq 0$

Continuity of a Composite Function

If g is continuous at c and f is continuous at $g(c)$, then the composite function given by $(f \circ g)(x) = f\big(g(x)\big)$ is continuous at c.

Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that

$$f(c) = k$$

Definition of Infinite Limits

Let f be a function that is defined at every real number in some open interval containing c (except possibly at c itself). The statement

$$\lim_{x \to c^-} f(x) = \infty$$

Means that for each $M > 0$ there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$. Similarly, the statement

$$\lim_{x \to c^-} f(x) = -\infty$$

Means that for each $N < 0$ there exists a $\delta > 0$ such that $f(x) < N$ whenever $0 < |x - c| < \delta$

To define the **infinite limit from the left**, replace $0 < |x - c| < \delta$ by $c - \delta < x < c$. To define the **infinite limit from the right**, replace $0 < |x - c| < \delta$ by $c < x < c + \delta$.

Definition of Vertical Asymptote

If $f(x)$ approaches infinity (or negative infinity) as x approaches c from the right or the left, then the line $x = c$ is a **vertical asymptote** of the graph of f.

Vertical Asymptotes

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interbal containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then graph of the function

$$h(x) = \frac{f(x)}{g(x)}$$

Has a vertical asymptote at $x = c$

Properties of Infinite Limits

Let c and L be real numbers, and let f and g be functions such that$\lim_{x \to c} f(x) = \infty$ and $\lim_{x \to c} f(x) = L$

Sum or difference:

$$\lim_{x \to c} [f(x) \pm g(x)] = \infty$$

Product:

$$\lim_{x \to c} [f(x)g(x)] = \infty, \quad L > 0$$

Quotient:

$$\lim_{x \to c} \frac{g(x)}{f(x)} = 0$$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Definition of Tangent Line with Slope m

If m is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} = \frac{f(c + \Delta x) - f(c)}{\Delta x} = m$$

Exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$

Definition of the Derivative of a Function

The **derivative** of f at x is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Provided the limits exists. For all x for which the limits exists, f' is a function of x.

Alternative Form of Derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

The Constant Rule

The derivative of a constant function is $\mathbf{0}$. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0$$

The Power Rule

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing $\mathbf{0}$.

The Power Rule when $n = 1$

$$\frac{d}{dx}[x] = 1$$

The Constant Multiple Rule

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}[cf(x)] = cf'(x)$$

The Sum and Difference Rules

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.

Sum Rule

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

Difference Rule

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

Derivatives of Sine and Cosine Functions

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

Average Velocity

$$\frac{\textit{Change in distance}}{\textit{Change in time}} = \frac{\Delta s}{\Delta t}$$

Velocity Function

$$v(t) = \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t} = s'(t)$$

Position Function

$$s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$$

The Product Rule

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the first function times the derivative of the second, plus the second function times the derivative of the first.

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

The Quotient Rule

The quotient f/g of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is given by the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2},$$

$$g(x) \neq 0$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\tan(x)] = sec^2(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\cot(x)] - csc^2(x)$$

$$\frac{d}{dx}[\cot(x)] = -\csc(x)\cot(x)$$

The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Or, equivalently

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

Or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1}u'$$

Summary of Differentiation Rules

General Differentiation Rules: Let c be a real number, let n be a rational number, let u and v be differentiable funtions of x, and let f be a differentiable function of u.

Constant Rule

$$\frac{d}{dx}[c] = 0$$

Constant Multiple Rule

$$\frac{d}{dx}[cu] = cu'$$

Product Rule

$$\frac{d}{dx}[uv] = uv' + u'v$$

Chain Rule

$$\frac{d}{dx}[f(u)] = f'(u)u'$$

(Simple) Power Rule

$$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$$

Sum or Difference Rule

$$\frac{d}{dx}[u \pm v] = u' \pm v$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}$$

Guidelines for Implicit Differentiation

1. Differentiate both side of the equation with *respect to* x.
2. Collect all terms involving dy/dx on the left side of the equation and move all other terms to the right side of the equation
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

Guidelines For Solving Related-Rate Problems

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time* t.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

Definition of Extrema

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum of f on I** when $f(c) \le f(x)$ for all x in I

2. $f(c)$ is the **maximum of f on I** when $f(c) \ge f(x)$ for all x in I

The minimum and maximum of a function on an interval are the **extreme values**, or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval. Extrema that occur at the endpoints are called **endpoint extrema**.

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.

Definition of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** of f, or you can say that f has a **relative maximum** at $(c, f(c))$.

2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** of f, or you can say that f has a **relative minimum** at $(c, f(c))$.

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called **local maximum** and **local minimum**, respectively.

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c, then c, is a **critical number** of f.

Relative Extrema Occur Only at Critical Numbers

If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

Guidelines For Finding Extrema On a Closed Interval

To find the extrema of a continuous function f on a closed interval $[a, b]$, use these steps

1. Find the critical numbers of f in (a, b).

2. Evaluate f at each critical number in (a, b).

3. Evaluate f at each endpoint of $[a, b]$.

4. The least of these values is the minimum. The greatest is the maximum.

Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$

The Mean Value Theorem

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f(c) = \frac{f(b) - f(a)}{b - a}$$

Alternative Form of Mean Value Theorem

$$f(b) = f(a) + (b - a)f'(c)$$

Definitions of Increasing and Decreasing Functions

A function f is **increasing** on an interval when, for any two numbers x_1 and x_2 in the interval $x_1 < x_2$ implies $f(x_1) < f(x_2)$

A function f is **decreasing** on an interval when, for any two numbers x_1 and x_2 in the interval $x_1 < x_2$ implies $f(x_1) < f(x_2)$

Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on $[a, b]$.

Guidelines For Finding Intervals on Which A Function is Increasing or Decreasing

Let f be continuous on the interval (a, b). To find the open intervals on which f is increasing or decreasing, use the following steps.

1. Locate the critical numbers of f in (a, b), and use these number to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use the previous test for increasing and decreasing functions to determine whether f is increasing or decreasing on each interval.

The guidelines are also valid when the interval (a, b) is replaced by an interval of the form $(-\infty, b), (a, \infty)$, or $(-\infty, \infty)$.

The First Derivative Test

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows.

1. If $f'(x)$ changes from negative to positive at c, then f has a *relative minimum* at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a *relative maximum* at $(c, f(c))$.
3. If $f'(x)$ changes from positive on both sides of c or negative on both sides of c, then $f(c)$is neither a relative minimum nor a relative maximum.

Definition of Concavity

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I when f' is increasing on the interval and **concave downward** on I when f' is decreasing on the interval.

Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I

2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I

Definition of Point of Inflection

Let f be a function that is continuous on an open interval, and let c be a point in the interval. If the graph f has a tangent line at the point $(c, f(c))$, then this point is a **point of inflection** of the graph of f when the concavity of f changes from upward to downward (or downward to upward) at that point.

Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ does not exist.

Second Derivative Test

Let f be a function such that $f'(c) = 0$and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$

2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$

If $f'' = 0$, then the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the First Derivative Test.

Definition of Limits at Infinity

Let L be a real number.

1. The statement $\lim_{x \to \infty} f(x) = L$ means that for each $\varepsilon > 0$there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$

2. The statement $\lim_{x \to \infty} f(x) = L$ means that for each $\varepsilon > 0$there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$

Definition of a Horizontal Asymptote

The line $y = L$ is a **horizontal asymptote** of the graph of f when

$$\lim_{x \to -\infty} f(x) = L$$

Or

$$\lim_{x \to \infty} f(x) = L$$

Limits at Infinity

If $\boldsymbol{r}$ is a positive rational number and $\boldsymbol{c}$ is any real number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = \mathbf{0}$$

Furthermore, if $\boldsymbol{x^r}$ is defined when $\boldsymbol{x} < \mathbf{0}$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = \mathbf{0}$$

Guidelines for Finding Limits at $\pm\infty$ of Rational Functions

1. If the degree of the numerator is *less than* the degree of the denominator, then the limit of the rational function is $\mathbf{0}$.

2. If the degree of the numerator is *equal to* the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.

3. If the degree of the numerator is *greater than* the degree of the denominator, then the limit of the rational function does not exists.

Definition of Infinite Limits at Infinity

Let $\boldsymbol{f}$ be a function on the interval $(\boldsymbol{a}, \infty)$

1. The $\lim\limits_{x\to-\infty} f(x) = \infty$ means that for each positive number $\boldsymbol{M}$, there is a corresponding number $\boldsymbol{N} > \mathbf{0}$ such that $\boldsymbol{f(x)} > \boldsymbol{M}$ whenever $\boldsymbol{x} > \boldsymbol{N}$.

2. The $\lim\limits_{x\to-\infty} f(x) = -\infty$ means that for each negative number $\boldsymbol{M}$, there is a corresponding number $\boldsymbol{N} > \mathbf{0}$ such that $\boldsymbol{f(x)} < \boldsymbol{M}$ whenever $\boldsymbol{x} > \boldsymbol{N}$.

Similar definitions can be given for the statements

$$\lim_{x\to-\infty} f(x) = \infty$$

And

$$\lim_{x\to-\infty} f(x) = -\infty$$

Guidelines for Analyzing the Graph of a Function

1. Determine the domain and range of the function.

2. Determine the intercepts, asymptotes, and symmetry of the graph.

3. Locate the **x-values** for which $\boldsymbol{f'(x)}$ and $\boldsymbol{f''(x)}$ either are zero or do not exists. Use the results to determine relative extrema and points of inflection.

Guidelines For solving Applied Minimum and Maximum Problems

1. Identify all *given* quantities and all quantities *to be determined*. If possible, make a sketch.

2. Write a **primary equation** for the quantity that is to be maximized or minimized.

3. Reduce the primary equation to one having a *single independent variable*. This may involve the use of **secondary equations** relating the independent variables of the primary equation.

4. Determine the feasible domain of the primary equation. That is, determine the values for which the stated problem makes sense.

5. Determine the desired maximum or minimum value by calculus techniques.

Newton's Method for Approximating the Zeros of a Function

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use these steps.

1. Make an initial estimate x_1 that is close to c

2. Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3. When $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called **iteration**.

Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The **differential of** x (denoted by dx) is any non-zero real number. The **differential of** y (denoted by dy) is

$$dy = f'(x)dx$$

Differential Formulas

Let u and v be differentiable functions of x.

Constant multiple:

$$d[cu] = c\,du$$

Sum or difference:

$$d[u \pm v] = du \pm dv$$

Product:

$$d[uv] = u\,dv + v\,du$$

Quotient:

$$d\left[\frac{u}{v}\right] = \frac{v\,du - udv}{v^2}$$

Definition of Antiderivative

A function F is an **antiderivative** of f on an interval I when $F'(x) = f(x)$ for all x in I

Representation of Antiderivatives

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$ for all x in I, where C is a constant.

Basic Integration Rules

Differentiation Formula

$$\frac{d}{dx}[C] = 0$$

$$\frac{d}{dx}[kx] = k$$

$$\frac{d}{dx}[kf(x)] = kf'(x)$$

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

$$\frac{d}{dx}[\tan(x)] = sec^2(x)$$

$$\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

$$\frac{d}{dx}[\cot(x)] = -csc^2(x)$$

$$\frac{d}{dx}[\csc(x)] = -\csc(x)\cot(x)$$

Integration Formula

$$\int 0\,dx = C$$

$$\int k\ dx = kx + C$$

$$\int kf(x)\ dx = k\int f(x)\ dx$$

$$\int [f(x) \pm g(x)]\ = \int f(x)\ dx\ \pm \int g(x)\ dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \cos(x)\ dx = \sin(x) + C$$

$$\int \sin(x)\ dx = -\cos(x) + C$$

$$\int sec^2(x)\ dx = \tan(x) + C$$

$$\int \sec(x)\tan(x)\ dx = \sec(x) + C$$

$$\int csc^2(x)\ dx = -\cot(x) + C$$

$$\int \csc(x)\cot(x)\ dx = -\csc(x) + C$$

Sigma Notation

The sum of n terms $a_1, a_2, a_3, \cdots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

Where i is the **index of summation**, $a_i$is the ith **term** of the sum, and the **upper and lower bounds of summation** are n and 1.

Summation Formulas

$$\sum_{i=1}^{n} c = cn, \quad c \text{ is a constant}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Limits of the Lower and Upper Sums

Let f be continuous and non-negative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x$$

$$= \lim_{n\to\infty} S(n)$$

Where $\Delta x = (b-a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the ith subinterval.

Definition of the Area of a Region in the Plane

Let f be continuous and non-negative on the interval $[a, b]$. The area of the region bounded by the graph of f, the **x-axis**, and the vertical lines $x = a$ and $x = b$ is

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x$$

Where $x_{i-1} \le c_i \le x_i$ and

$$\Delta x = \frac{(b-a)}{n}$$

Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Where Δx_i is the width of the ith subinterval

$$[x_{i-1}, x_i]$$

If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x, \quad x_{i-1} \le c_i \le x_i$$

Is called a **Riemann sum** of f for the partition Δ.

Regular Partition

$$||\Delta|| = \Delta x = \frac{b-a}{n}$$

General Partition

$$\frac{b-a}{n} = ||\Delta|| \le n$$

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

Exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\ dx$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

Continuity Implies Integrability

If a function f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$. That is $\int_a^b f(x)\ dx$ exists.

The Definite Integral as the Area of a Region

If f is continuous and non-negative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the **x-axis**, and the vertical lines $x = a$ and $x = b$ is

$$Area = \int_a^b f(x)\ dx$$

Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x)\ dx = 0$
2. If f is integrable on $[a, b]$, then $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$

Additive Interval Property

If f is integrable on the three closed intervals determined by a, b, and c, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

Properties of Definite Integrals

If f and g are integrable on $[a, b]$ and k is a constant, then the functions kf and $f \pm g$ are integrable on $[a, b]$, and

$$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$$

$$\int_a^b [f(x) \pm g(x)]\ dx$$

$$= \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$$

Preservation of Inequality

1. If f integrable and non-negative on the closed interval $[a, b]$, then

$$0 \le \int_a^b f(x)\ dx$$

2. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \le g(x)$ for every x in $[a, b]$, then

$$\int_a^b f(x)\ dx \le \int_a^b g(x)\ dx$$

The Fundamental Theorem of Calculus

If a function $\boldsymbol{f}$ is continuous on the closed interval $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{F}$ is an antiderivative on $\boldsymbol{f}$ on the interval $[\boldsymbol{a}, \boldsymbol{b}]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

Guidelines for Using the Fundamental Theorem of Calculus

1. *Provided you can find* an antiderivative of $\boldsymbol{f}$, you now have a way to evaluate a definite integral without having to use the limit of a sum.

2. When applying the Fundamental Theorem of Calculus, the notation shown below is convenient

$$\int_a^b f(x)\ dx = F(x)\Big|_a^b = F(b) - F(a)$$

3. It is not necessary to include a constant of integration $\boldsymbol{C}$ in the antiderivative.

Mean Value Theorem for Integrals

If $\boldsymbol{f}$ is continuous on the closed interval $[\boldsymbol{a}, \boldsymbol{b}]$, then there exists a number $\boldsymbol{c}$ in the closed interval $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\int_a^b f(x)\ dx = f(c)(b-a)$$

Definition of the Average Value of a Function on an Interval

If $\boldsymbol{f}$ is integrable on the closed interval $[\boldsymbol{a}, \boldsymbol{b}]$, then the **average value** of $\boldsymbol{f}$ on the interval

$$\frac{1}{b-a}\int_a^b f(x)\ dx$$

The Second Fundamental Theorem of Calculus

If $\boldsymbol{f}$ is integrable on an open interval $\boldsymbol{I}$ containing $\boldsymbol{a}$, then, for every $\boldsymbol{x}$ in the interval

$$\frac{d}{dx}\left[\int_a^x f(t)\ dt\right] = f(x)$$

The Net Change Theorem

If $\boldsymbol{F'(x)}$ is the rate of change of a quantity $\boldsymbol{F(x)}$, then the definite integral of $\boldsymbol{F'(x)}$ from $\boldsymbol{a}$ to $\boldsymbol{b}$ gives the total change, or **net change**, of $\boldsymbol{F(x)}$ on the interval $[\boldsymbol{a}, \boldsymbol{b}]$

$$\int_a^b F'(x)\ dx = F(b) - F(a)$$

Antidifferentiation of a Composite Function

Let $\boldsymbol{g}$ be a function whose range is an interval $\boldsymbol{I}$, and let $\boldsymbol{f}$ be a function that is continuous on $\boldsymbol{I}$. If $\boldsymbol{g}$ is differentiable on its domain and $\boldsymbol{F}$ is an antiderivative of $\boldsymbol{f}$ on $\boldsymbol{I}$, then

$$\int f\big(g(x)\big)g'(x)\ dx = F\big(g(x)\big) + C$$

Letting $\boldsymbol{u = g(x)}$ gives $\boldsymbol{du = g'(x)dx}$ and

$$\int f(u)\ du = F(u) + C$$

Guidelines for Making a Change of Variables

1. Chose a substitution $u = g(x)$. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $du = g'(x)dx$.
3. Rewrite the integral in terms of the variable u.
4. Find the resulting integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your answer by differentiating.

The General Power Rule for Integration

If g is a differentiable function of x, then

$$\int [g(x)]^n\, g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

Equivalently, if $u = g(x)$, then

$$\int u^n \; du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

Change of Variables for Definite Integrals

If the function $u = g(x)$ has a derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))\, g'^{(x)}\; dx = \int_{g(a)}^{g(b)} f(u)\; du$$

Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$

1. If f is an *even* function, then
$$\int_{-a}^{a} f(x)\; dx = 2\int_0^a f(x)\; dx$$
2. If f is an *odd* function, then
$$\int_{-a}^{a} f(x)\; dx = 0$$

Definition of the Natural Logarithmic Function

The **natural logarithmic function** is defined by

$$\ln(x) = \int_1^x \left(\frac{1}{t}\right)\; dt, \;\; x > 0$$

The domain of the natural logarithmic function is the set of all positive real numbers.

Properties of the Natural Logarithmic Function

The natural logarithmic function has the following properties,

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, increasing, and one-to-one.
3. The graph is concave downward.

Logarithmic Properties

If a and b are positive numbers and n is rational, then the following properties are true.

$$\ln(1) = 0$$

$$\ln(ab) = \ln(a) + \ln(b)$$

$$\ln(a^n) = n\ln(a)$$

$$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$$

Definition of e

The letter e denotes the positive real number such that

$$\ln(e) = \int_1^e \left(\frac{1}{t}\right) dt = 1$$

Derivative of the Natural Logarithmic Function

Let u be a differentiable function of x.

$$\frac{d}{dx}[\ln(x)] = \frac{1}{x}, \quad x > 0$$

$$\frac{d}{dx}[\ln(u)] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}, \quad u > 0$$

Derivative Involving Absolute Value

If u is a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

Log Rule for Integration

Let u be a differentiable function of x.

$$\int \left(\frac{1}{x}\right) dx = \ln|x| + C$$

$$\int \left(\frac{1}{u}\right) du = \ln|u| + C$$

Guidelines for Integration

1. Learn a basic list of integration formulas.

2. Find an integration formula that resembles all or part of the integrand and, by trial and error, find a choice of u that will make the integrand conform to the formula.

3. When you cannot find a u-substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division.

4. If you have access to computer software that will find antiderivatives symbolically use it.

5. Check your result by differentiating to obtain the original integrand.

Integrals of the Six Basic Trigonometric Functions

$$\int \sin(u)\ du = -\cos(u) + C$$

$$\int \cos(u)\ du = \sin(u) + C$$

$$\int \tan(u)\ du = -\ln|\cos(u)| + C$$

$$\int \cot(u)\ du = \ln|\sin(u)| + C$$

$$\int \sec(u)\ du = \ln|\sec(u) + \tan(u)| + C$$

$$\int \csc(u)\ du = -\ln|\csc(u) + \cot(u)| + C$$

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$f(g(x)) = x$ for each x in the domain of g

And

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

Reflective Property of Inverse Functions

The graph of f contains the point (a, b)if and only if the graph of f^{-1} contains the point (b, a).

The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.

2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Guidelines for Finding an Inverse Function

1. Determine whether the function $y = f(x)$ has an inverse function.

2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.

3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

4. Define the domain of f^{-1} as the range of f.

5. Verify that $f\left(f^{-1}(x)\right) = x$ and $f^{-1}(f(x)) = x$

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ontinuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain f^{-1} is continuous on its domain.

2. If f is increasing on its domain f^{-1} is increasing on its domain.

3. If f is decreasing on its domain f^{-1} is decreasing on its domain.

4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0$$

Definition of the Natural Exponential Function

The inverse function of the natural logarithmic function $\boldsymbol{f(x) = \ln(x)}$ is called the **natural exponential function** and is denoted by

$$\boldsymbol{f^{-1}(x) = e^{x}}$$

That is

$$\boldsymbol{y = e^{x}}$$

If and only if $\boldsymbol{x = \ln(y)}$

Inverse relationship between natural logarithmic function and natural exponential

$$\boldsymbol{\ln(e^{x}) = x}$$

And

$$\boldsymbol{e^{\ln(x)} = x}$$

Operations with Exponential Functions

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be any real numbers.

$$\boldsymbol{e^{a}e^{b} = e^{a+b}}$$

$$\boldsymbol{\frac{e^{a}}{e^{b}} = e^{a-b}}$$

Properties of the Natural Exponential Function

1. The domain of $\boldsymbol{f(x) = e^{x}}$ is $(-\infty, \infty)$ and the range is $(\boldsymbol{0}, \infty)$.

2. The function $\boldsymbol{f(x) = e^{x}}$ is continuous, increasing, and one-to-one on its entire domain.

3. The graph of $\boldsymbol{f(x) = e^{x}}$ is concave upward on its entire domain.

4. $\boldsymbol{\lim_{x \to -\infty} (e^{x}) = 0}$

5. $\boldsymbol{\lim_{x \to \infty} (e^{x}) = \infty}$

The natural exponential function is increasing, and its graph is concave upward.

Derivatives of the Natural Exponential Function

Let $\boldsymbol{u}$ be a differentiable function of $\boldsymbol{x}$.

$$\boldsymbol{\frac{d}{dx}[e^{x}] = e^{x}}$$

$$\boldsymbol{\frac{d}{dx}[e^{u}] = e^{u}}$$

Integration Rules for Exponential Functions

Let $\boldsymbol{u}$ be a differentiable function of $\boldsymbol{x}$.

$$\boldsymbol{\int (e^{x})\ dx = e^{x} + C}$$

$$\boldsymbol{\int (e^{u})\ du = e^{u} + C}$$

Definition of Exponential Function to Base a

If a is a positive real number ($a \neq 1$) and x is any real number, then the **exponential function to the base a** is denoted by a^x and is defined by

$$a^x = e^{(\ln(a))x}$$

If $a = 1$, then $y = 1^x = 1$ is constant function.

Definition of Logarithmic Function to Base a

If a is a positive real number ($a \neq 1$) and x is any positive real number. Then the **logarithmic function to the base a** is denoted by $\log_a x$ and is defined as

$$\log_a(x) = \frac{1}{\ln(a)}\ln(x)$$

Properties of Inverse Functions

1. $y = a^x$ if and only if $x = \log_a(y)$
2. $a^{\log_a(x)} = x, \quad x > 0$
3. $\log_a(a^x) = x$, for all x

Derivatives for Bass Other than e

Let a be a positive number ($a \neq 1$), let u be a differentiable function of x.

$$\frac{d}{dx}[a^x] = (\ln(a))a^x$$

$$\frac{d}{dx}[a^u] = (\ln(a))a^u\frac{du}{dx}$$

$$\frac{d}{dx}[\log_a(x)] = \frac{1}{\ln(a)\,x}$$

$$\frac{d}{dx}[\log_a(u)] = \frac{1}{\ln(a)\,u}\frac{du}{dx}$$

The Power Rule for Real Exponents

Let n be any real number, and let u be a differentiable function of x.

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$$

A Limit Involving e

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

Summary of Compound Interest Formulas

Let $P = amount\ of\ deposit$,
$t = number\ of\ years$,
$A = balance\ after\ t\ years$,
$r = annual\ interest\ rate$ (in decimal form), and
$n = number\ of\ compoundings\ per\ year$.

1. Compounded n times per year:
$$A = \left(1 + \frac{r}{n}\right)^{nt}$$
2. Compounded continuously
$$A = Pe^{rt}$$

The Extended Mean Value Theorem

If f and g are differential on an open interval (a, b) and continuous on $[a, b]$ such that $g'(x) \neq 0$ for any x in (a, b), then there exists a point c in (a, b)such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

L'Hôpital's Rule

Let f and g be functions that are differentiable on an open interval (a, b) containing c, except possibly at c itself. Assume that $g'(x) \neq 0$ for all x in (a, b), except possibly at c itself. If the limit of $f(x)/g(x)$ as x approaches c produces the indeterminate form $0/0$, then

$$\lim_{x \to c} \left(\frac{f(x)}{g(x)}\right) = \lim_{x \to c} \left(\frac{f'(x)}{g'(x)}\right)$$

Provided the limit on the right exists (or is infinite). This result also applies when the limit of $f(x)/g(x)$ as x approaches c produces any one of the indeterminate forms ∞/∞, $(-\infty)/\infty$, $\infty/(-\infty)$, or $(-\infty)/(-\infty)$

Definitions of Inverse Trigonometric Functions

$y = sin^{-1}(x)$ if and only if $\sin(y) = x$

Domain: $-1 \leq x \leq 1$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$y = cos^{-1}(x)$ if and only if $\cos(y) = x$

Domain: $-1 \leq x \leq 1$

Range: $0 \leq y \leq \pi$

$y = tan^{-1}(x)$ if and only if $\tan(y) = x$

Domain: $-\infty \leq x \leq \infty$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$y = cot^{-1}(x)$ if and only if $\cot(y) = x$

Domain: $-\infty \leq x \leq \infty$

Range: $0 \leq y \leq \pi$

$y = sec^{-1}(x)$ if and only if $\sec(y) = x$

Domain: $|x| \geq 1$

Range: $0 \leq y \leq \frac{\pi}{2}, \quad y \neq \frac{\pi}{2}$

$y = csc^{-1}(x)$ if and only if $\csc(y) = x$

Domain: $|x| \geq 1$

Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \quad y \neq \frac{\pi}{2}$

Properties of Inverse Trigonometric Functions

If $-\mathbf{1} \le \boldsymbol{x} \le \mathbf{1}$ and $-\boldsymbol{\pi}/\mathbf{2} \le \boldsymbol{y} \le \boldsymbol{\pi}/\mathbf{2}$, then

$$\sin\left(sin^{-1}(x)\right) = x$$

And

$$sin^{-1}(\sin(y)) = y$$

If $-\boldsymbol{\pi}/\mathbf{2} \le \boldsymbol{y} \le \boldsymbol{\pi}/\mathbf{2}$, then

$$\tan\left(tan^{-1}(x)\right)$$

And

$$tan^{-1}(\tan(y)) = y$$

If $|\boldsymbol{x}| \ge \mathbf{1}$ and $\mathbf{0} \le \boldsymbol{y} < \boldsymbol{\pi}/2$or $\boldsymbol{\pi}/\mathbf{2} < \boldsymbol{y} \le \boldsymbol{\pi}$, then

$$\sec\left(sec^{-1}(x)\right) = x$$

And

$$sec^{-1}(\sec(y)) = y$$

Similar properties hold for other inverse trigonometric functions.

Derivatives of Inverse Trigonometric Functions

Let $\boldsymbol{u}$ be a differentiable function of $\boldsymbol{x}$.

$$\frac{d}{dx}\left[sin^{-1}(u)\right] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[cos^{-1}(u)\right] = -\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[tan^{-1}(u)\right] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}\left[cot^{-1}(u)\right] = -\frac{u'}{1+u^2}$$

$$\frac{d}{dx}\left[sec^{-1}(u)\right] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}\left[csc^{-1}(u)\right] = -\frac{u'}{|u|\sqrt{u^2-1}}$$

Basic Differentiation Rules for Elementary Functions

$$\frac{d}{dx}[cu] = cu'$$

$$\frac{d}{dx}[u \pm v] = u' \pm v'$$

$$\frac{d}{dx}[uv] = uv' + u'v$$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}$$

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[u^n] = nu^{n-1}u'$$

$$\frac{d}{dx}[x] = 1$$

$$\frac{d}{dx}[|u|] = \frac{u}{|u|}(u'), \quad u \neq 0$$

$$\frac{d}{dx}[\ln(u)] = \frac{u'}{u}$$

$$\frac{d}{dx}[e^u] = e^u u'$$

$$\frac{d}{dx}[\log_a(u)] = \frac{u'}{(\ln(a))u}$$

$$\frac{d}{dx}[a^u] = (\ln(a))a^u u'$$

$$\frac{d}{dx}[\sin(u)] = (\cos(u))u'$$

$$\frac{d}{dx}[\cos(u)] = -(\sin(u))u'$$

$$\frac{d}{dx}[\tan(u)] = \left(sec^2(u)\right)u'$$

$$\frac{d}{dx}[\cot(u)] = -\left(csc^2(u)\right)u'$$

$$\frac{d}{dx}[\sec(u)] = (\sec(u)\tan(u))u'$$

$$\frac{d}{dx}[\csc(u)] = -(\csc(u)\cot(u))u'$$

$$\frac{d}{dx}\left[sin^{-1}(u)\right] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[cos^{-1}(u)\right] = -\frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[tan^{-1}(u)\right] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}\left[cot^{-1}(u)\right] = -\frac{u'}{1+u^2}$$

$$\frac{d}{dx}\left[sec^{-1}(u)\right] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}\left[csc^{-1}(u)\right] = -\frac{u'}{|u|\sqrt{u^2-1}}$$

Integrals Involving Inverse Trigonometric Functions

Let u be a differentiable function of x, and let $a > 0$

$$\int \frac{du}{\sqrt{a^2-u^2}} = sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a} tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a} sec^{-1}\left(\frac{|u|}{a}\right) + C$$

Basic Integration Rules ($a > 0$)

$$\int kf(u) = k\int f(u)\ du$$

$$\int [f(u) \pm g(u)]\ du = \int f(u)\, du \pm g(u) du$$

$$\int du = u + C$$

$$\int \frac{du}{u} = \ln|u| + C$$

$$\int u^n\ du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int (e^u)\ du = e^u + C$$

$$\int a^u\, du = \frac{1}{\ln(a)} a^u + C$$

$$\int \sin(u)\ du = -\cos(u) + C$$

$$\int \cos(u)\, du = \sin(u) + C$$

$$\int \tan(u)\ du = -\ln|\cos(u)| + C$$

$$\int \cot(u)\ du = \ln|\sin(u)| + C$$

$$\int \sec(u)\ du = \ln|\sec(u) + \tan(u)| + C$$

$$\int \csc(u)\ du = -\ln|\csc(u) + \cot(u)| + C$$

$$\int sec^2(u)\ du = \tan(u) + C$$

$$\int csc^2(u)\ du = -\cot(u) + C$$

$$\int \sec(u) + \tan(u) = \sec(u)$$

$$\int \csc(u) + \cot(u) = -\csc(u) + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} sec^{-1}\left(\frac{|u|}{a}\right) + C$$

Definitions of the Hyperbolic Functions

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \quad x \neq 0$$

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\coth(x) = \frac{1}{\tanh(x)}, \quad x \neq 0$$

Hyperbolic Identities

$$cosh^2(x) - sinh^2(x) = 1$$

$$tanh^2(x) + sech^2(x) = 1$$

$$coth^2(x) - csch^2(x) = 1$$

$$sinh^2(x) = \frac{-1 + cosh^2(2x)}{2}$$

$$sinh^2(2x) = 2\sinh(x)\cosh(x)$$

$$\sinh(x) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

$$\sinh(x - y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$$

$$\cosh(x + y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

$$\cosh(x - y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$$

$$cosh^2(x) = \frac{1 + \cosh(2x)}{2}$$

$$\cosh(2x) = cosh^2(x) + sinh^2(x)$$

Derivatives and Integrals of Hyperbolic Functions

$$\frac{d}{dx}[\sinh(u)] = (\cosh(u))u'$$

$$\frac{d}{dx}[\cosh(u)] = (\sinh(u))u'$$

$$\frac{d}{dx}[\tanh(u)] = \left(sinh^2(u)\right)u'$$

$$\frac{d}{dx}[\coth(u)] = -\left(csch^2(u)\right)u'$$

$$\frac{d}{dx}[\mathrm{sech}(u)] = -(\mathrm{sech}(u)\tanh(u))u'$$

$$\frac{d}{dx}[\mathrm{csch}(u)] = -(\mathrm{csch}(u)\coth(u))u'$$

$$\int[\cosh(u)]\ du = \sinh(u) + C$$

$$\int[\sinh(u)]\ du = \cosh(u) + C$$

$$\int\left[sinh^2(u)\right] du = \tanh(u) + C$$

$$\int\left[csch^2(u)\right]\ du = -\coth(u) + C$$

$$\int[\mathrm{sech}(u)\tanh(u)]\ du = -\mathrm{sech}(u) + C$$

$$\int[\mathrm{csch}(u)\coth(u)] = -\mathrm{csch}(u) + C$$

Inverse Hyperbolic Functions

Domain $(-\infty, \infty)$

$$sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right)$$

Domain $[1, \infty)$

$$cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right)$$

Domain $(-1, 1)$

$$tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

Domain $(0, 1]$

$$sech^{-1}(x) = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

Domain $(-\infty, 0) \cup (0, \infty)$

$$csch^{-1}(x) = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$$

Differentiation and Integration Involving Inverse Hyperbolic Functions

$$\frac{d}{dx}\left[sinh^{-1}(u)\right] = \frac{u'}{\sqrt{u^2+1}}$$

$$\frac{d}{dx}\left[cosh^{-1}(u)\right] = \frac{u'}{\sqrt{u^2-1}}$$

$$\frac{d}{dx}\left[tanh^{-1}(u)\right] = \frac{u'}{1-u^2}$$

$$\frac{d}{dx}\left[coth^{-1}(u)\right] = \frac{u'}{1-u^2}$$

$$\frac{d}{dx}\left[sech^{-1}(u)\right] = -\frac{u'}{u\sqrt{1-u^2}}$$

$$\frac{d}{dx}\left[csch^{-1}(u)\right] = -\frac{u'}{|u|\sqrt{1-u^2}}$$

$$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left(u + \sqrt{u^2 \pm a^2}\right) + C$$

$$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\ln\left(\frac{a^2 \pm u^2}{|u|}\right) + C$$

Exponential Growth Decay

If y is a differentiable function of t such that $y > 0$ and $dy/dt = ky$ for some constant k, then

$$y = Ce^{kt}$$

Where C is the **initial value** of y, and k is the **proportionality constant**. **Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

Definition of First-Order Linear Differential Equation

A **first-order linear differential equation** is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Where P and Q are continuous functions of x. This first-order linear differential equation is said to be in **standard form**.

Solution of a First-Order Linear Differential Equation

An integrating factor for the first-order linear differential equation

$$y' + P(x)y = Q(x)$$

Is $u(x) = e^{\int P(x)\,dx}$. The solution of the differential equation is

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx + C$$

Area of a Region between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \leq f(x)$ for all x on $[a, b]$, then the area of the region bounded by the graphs f and g are the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

The Disk Method

To find the volume of a solid of revolution with the **disk method**, use one of the formulas below

Horizontal Axis of Revolution

$$Volume = V = \pi \int_{a}^{b} [R(x)]^2 \; dx$$

Vertical Axis of Revolution

$$Volume = V = \pi \int_{c}^{d} [R(y)]^2 \; dy$$

The Shell Method

To find the volume of a solid of revolution with the **shell method**, use one of the formulas below

Horizontal Axis of Revolution

$$Volume = V = 2\pi \int_{c}^{d} p(y)h(y) \; dy$$

Vertical Axis of Revolution

$$Volume = V = 2\pi \int_{a}^{b} p(x)h(x) \; dx$$

The Washer Method

$$V = \pi \int_{a}^{b} \left([R(x)]^2 - [r(x)]^2\right) \; dx$$

Volumes of Solids with Known Cross Sections

1. For cross sections of area $A(x)$ taken perpendicular to the **x-axis**,

$$Volume = \int_{a}^{b} A(x) \; dx$$

2. For cross sections of area $A(y)$ taken perpendicular to the **y-axis**,

$$Volume = \int_{c}^{d} A(y) \; dy$$

Definition of Arc Length

Let the function $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The **arc length** of f between a and b is

$$s = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \; dx$$

Similarly, for a smooth curve $x = g(y)$, the **arc length** of g between c and d is

$$s = \int_{c}^{d} \sqrt{1 + [g'(y)]^2} \; dy$$

Definition of Surface of Revolution

When the graph of a continuous function is revolved about a line, the resulting surface is a **surface of revolution**.

Definition of the Area of a Surface Revolution

Let $y = f(x)$ have a continuous derivative on the interval $[a, b]$. The area S of the surface of revolution formed by revolving the graph of f about a horizontal or vertical axis is

$$S = 2\pi \int_a^b r(x)\sqrt{1 + [f'(x)]^2}\ dx$$

Where $r(x)$ is the distance between the graph of f and the axis of revolution. If $x = g(y)$ is on the interval $[c, d]$, then the surface area is

$$S = 2\pi \int_c^d r(y)\sqrt{1 + [g'(y)]^2}\ dy$$

Where $r(y)$ is the distance between the graph of g and the axis of revolution.

Definition of Work Done by a Constant Force

If an object is moved a distance D in the direction of an applied constant force F, then the **work** W done by the force is defined as $W = FD$.

Definition of Work Done by a Variable Force

If an object is moved along a straight line by a continuously varying force $F(x)$, then the **work** W done by the force as the object is moved from

$$x = a$$

To

$$x = b$$

Is given by

$$W = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \Delta W_i = \int_a^b F(x)\ dx$$

Hooke's Law

The force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is

$$F = kd$$

Newton's Law of Universal Gravitation

The force F of attraction between two particles of masses m_1 and m_2 is proportional to the product of the masses and inversely proportional to the square of the distance d between the two particles. That is

$$F = G\frac{m_1 m_2}{d^2}$$

Coulomb's Law

The force F between two charges q_1 and q_2 in a vacuum is proportional to the product of the changes and inversely proportional to the square of the distance d between the two charges. That is,

$$F = k\frac{q_1 q_2}{d^2}$$

When q_1 and q_2 are given electrostatic units and d in centimeters, F will be in dynes for a value of $k = 1$.

Force

$$Force = (mass)(acceleration)$$

Moment

The **moment of m about the point P is**

$$Moment = mx$$

Moment and Center of Mass: One-Dimensional System

Let the point masses $m_1, m_2, \cdots, m_n$ be located at $x_1, x_2, \cdots, x_n$.

1. The **moment about the origin** is

$$M_0 = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n$$

2. The **center of mass** is

$$\bar{x} = \frac{M_0}{m}$$

Where $m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

Moment and Center of Mass: Two-Dimensional System

Let the point masses $m_1, m_2, \cdots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)$

1. The **moment about the y-axis** is

$$M_y = m_1 x_1 + m_2 x_2 + \cdots + m_n x_n$$

2. The **moment about the x-axis**

$$M_x = m_1 y_1 + m_2 y_2 + \cdots + m_n y_n$$

3. The **center of mass** $(\bar{x}, \bar{y})$ (or **center of gravity**) is

$$\bar{x} = \frac{M_y}{m}$$

And

$$y = \frac{M_x}{m}$$

Where

$m = m_1 + m_2 + \cdots + m_n$ is the **total mass** of the system.

Moments and Center of Mass of a Planar Lamina

Let f and g be continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, and consider the planar lamina of uniform density ρ bounded by the graphs of $y = f(x), y = g(x)$, and $a \leq x \leq b$.

1. The **moments about the x- and y-axis** are

$$M_x = \rho \int_a^b \left[\frac{f(x) + g(x)}{2}\right] [f(x) - g(x)]\, dx$$

$$M_y = \rho \int_a^b x[f(x) - g(x)]\ dx$$

2. The **center of mass** $(\bar{x}, \bar{y})$ is given by $\bar{x} = \frac{M_y}{m}$ and $y = \frac{M_x}{m}$, where $m = \rho \int_a^b [f(x) - g(x)]\ dx$ is the mass of the lamina.

The Theorem of Pappus

Let R be in a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R. If r is the distance between the centroid of R and the line, then the volume V of the solid of revolution formed by revolving R about the line is

$$V = 2\pi r A$$

Where A is the area of R. (Note that $2\pi r$ is the distance traveled by the centroid as the region is revolved about the line.)

Definition of Fluid Pressure

The **pressure** P on an object at depth h in a liquid is

$$P = wh$$

Where w is the weight-density of the liquid per unit of volume.

Definition of Force Exerted by a Fluid

The **force F exerted by a fluid** of constant weight-density w (per unit of volume) against a submerged vertical plane region from $y = c$ to $y = d$ is

$$F = w \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} h(y_i) L(y_i) \Delta y$$

$$= w \int_{c}^{d} h(y) L(y) \, dy$$

Where $h(y)$ is the depth of the fluid at y and $L(y)$ is the horizontal length of the region at y.

Procedures for Fitting Integrands to Basic Integration Rules

Technique

1. Expand (numerator)
2. Separate Numerator
3. Complete the Square
4. Divide improper rational function
5. Add and Subtract terms in numerator
6. Use trigonometric identities
7. Multiply and Divide by Pythagorean conjugate

Integration by Parts

If u and v are functions of x and have continuous derivative, then

$$\int u \, dv = uv - \int v \, du$$

Guidelines for Integration by Parts

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.

2. Try letting u be the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand.

Note that dv always includes the dx of the original integrand.

Summary: Common Integrals using Integration By Parts

1. For integrals of the form

$$\int x^n e^{ax} \, dx, \int x^n \sin(ax) \, dx, \int x^n \cos(ax) \, dx$$

Let $u = x^n$ and let
$dv = e^{ax} \, dx, \sin(ax) \, dx, \cos(ax)$.

2. For integrals of the form

$$\int x^n \ln(x) \, dx, \int x^n \sin^{-1}(ax) \, dx, \int x^n \tan^{-1}(ax) \, dx$$

Let $u = \ln(x) \, dx, \ \sin^{-1}(ax) \, dx, \int x^n \tan^{-1}(ax)$, and let $dv = x^n \, dx$

3. For integrals of the form

$$\int e^{ax} \sin(bx) \, dx, \int e^{ax} \cos(bx)$$

Let $u = \sin(bx)$ or $\cos(bx)$ and let $dv = e^{ax} \, dx$

Guidelines for Evaluating Integrals Involving Powers of Sine and Cosine

1. When the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then expand and integrate.

$$\int \sin^{2k+1}(x) x \cos^{n}(x)\ dx$$

$$= \int \sin^2(x)^k \cos^n(x) x \sin(x)\ dx$$

$$= \left(1 - \cos^2(x)\right)^k \cos^n(x) x \sin(x)\ dx$$

2. When the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then expand and integrate.

$$\int \sin^m(x) x \cos^{2k+1}(x)\ dx$$

$$= \left(\sin^m(x)\right)\left(\cos^2\right)^k \cos(x)\ dx$$

$$= \left(\sin^m(x)\right)\left(1 - \sin^2(x)\right)^k \cos(x)\, dx$$

3. When the powers of both the sine and cosine are even and non-negative, make repeated use of the formulas

$$\sin^2(x) = \frac{1 - \cos(2x)}{2}$$

And

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}$$

To convert the integrand to odd powers of the cosine. Then proceed as in the second guideline.

Wallis's Formulas

1. If n is odd $(n \geq 3)$, then

$$\int_0^{\frac{\pi}{2}} \cos^n(x)\ dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{n-1}{n}\right)$$

1. If n is odd $(n \geq 2)$, then

$$\int_0^{\frac{\pi}{2}} \cos^n(x)\ dx = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{n-1}{n}\right)\left(\frac{\pi}{2}\right)$$

The formulas are also valid when $\cos^n(x)$ is replaced by $\sin^n(x)$

Guidelines for Evaluating Integrals Involving Powers of Secant and Tangent

1. When the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

$$\int sec^{2k}(x)\, xtan^{n}(x)\, dx$$

$$= \int \left(sec^2(x)\right)^{k-1} tan^n(x) sec^2(x)\, dx$$

$$= \int \left(1 + tan^2(x)\right)^{k-1} tan^n(x) sec^2(x)\, dx$$

2. When the power of the secant is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.

$$\int sec^{m}(x) tan^{2k+1}(x)\, dx$$

$$= \left(\int sec^{m-1}(x)\right)\left(tan^2(x)\right)^{k} \sec(x)\tan(x)\ dx$$

$$= \left(\int sec^{m-1}(x)\right)\left(sec^2(x-1)\right)^{k} \sec(x)\tan(x)\ dx$$

3. When there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int tan^{n}(x)\ dx$$

$$= \left(tan^{n-2}(x)\right)\left(tan^2(x)\right)\ dx$$

$$= \left(tan^{n-2}(x)\right)\left(sec^2(x-1)\right)\ dx$$

4. When the integral of the form

$$\int sec^{m}(x) dx$$

When m is odd and positive, use integration by parts

5. When the first four guidelines do not apply, try converting to sines and cosines.

Integrals Involving Sine-Cosine Products

$$\sin(mx)\sin(nx)$$

$$= \frac{1}{2}(\cos[(m-n)x] - \cos[(m+n)x])$$

$$\sin(mx)\cos(nx)$$

$$= \frac{1}{2}(\sin[(m-n)x] + \sin[(m+n)x])$$

$$\cos(mx)\cos(nx)$$

$$= \frac{1}{2}(\cos[(m-n)x] + \cos[(m+n)x])$$

Trigonometric Substitution $(a > 0)$

1. For integrals involving $\sqrt{a^2 - u^2}$, let

$$u = a\sin(\theta)$$

Then $\sqrt{a^2 - u^2} = a\cos\theta$, where

$-\pi/2 \le \theta \le \pi/2$

2. For integrals involving $\sqrt{a^2 + u^2}$, let

$$u = a\tan(\theta)$$

Then $\sqrt{a^2 + u^2} = a\sec(\theta)$, where $-\pi/2 < \theta < \pi/2$

3. For integrals involving $\sqrt{u^2 - a^2}$, let

$$u = a\sec(\theta)$$

Then $\sqrt{u^2 - a^2} =$

$$\begin{cases} a\tan(\theta)\ for\ u > a, where\ 0 \le \theta < \pi/2 \\ -a\tan(\theta)\ for\ u < a, where\ \pi/2 < \theta \le \pi \end{cases}$$

Special Integration Formulas ($a > 0$)

$$\int \sqrt{a^2 - u^2}\, du$$

$$= \frac{1}{2}\left(u\sqrt{a^2 - u^2} + a^2 sin^{-1}\left(\frac{u}{a}\right)\right) + C$$

$$\int \sqrt{u^2 - a^2}\, du$$

$$= \frac{1}{2}\left(u\sqrt{u^2 - a^2} - a^2 \ln\left|u + \sqrt{u^2 - a^2}\right| + C\right), u > a$$

$$\int \sqrt{u^2 + a^2}\, du$$

$$= \left(u\sqrt{u^2 + a^2} + a^2 \ln\left|u + \sqrt{u^2 + a^2}\right| + C\right)$$

Decomposition of $N(x)/D(x)$ into Partial Fractions

1. **Divide when improper:** When $N(x)/D(x)$ is an improper fraction (that is, when the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (a\, polynomial) + \frac{N_1(x)}{D(x)}$$

Where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational express $N_1(x)/D(x)$

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \text{ and } \left(ax^2 + bx + c\right)^n$$

Where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions

$$\frac{A_1}{px + q} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $\left(ax^2 + bx + c\right)^n$, the partial fraction decomposition must include the following sum of n fractions

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Guidelines for Solving the Basic Equation

Linear Factors

1. Substitute the roots of the distinct linear factors in the basic equation.

2. For repeated linear factors, use the coefficients determines in the first guideline to rewrite the basic equation. Then substitute other convenient values of x and solve for the remaining coefficients.

Quadratic Factors

1. Expand the basic equation.

2. Collect terms according to powers of x.

3. Equate the coefficients of like power to obtain a system of linear equations involving A, B, C, and so on.

4. Solve the system of linear equations.

The Trapezoidal Rule

Let f be continuous on $[a, b]$. The Trapezoidal Rule for approximating $\int_a^b f(x)\ dx$ is

$$\int_a^b f(x)\ dx \approx \frac{b-a}{2n}[2f(x_0) + 2f(x_1) + 2f(x_2) \cdots 2f(x_{n-1}) + 2f(x_n)]$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\ dx$.

Integration of $p(x) = Ax^2 + Bx + C$

If $p(x) = Ax^2 + Bx + C$, then

$$\int_a^b p(x) = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{b-a}{2}\right) + p(b)\right]$$

Simpson's Rule

Let f be continuous on $[a, b]$ and let n be an even integer. Simpson's Rule for approximating $\int_a^b f(x)\ dx$ is

$$\int_a^b f(x)\ dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Moreover, as $n \to \infty$, the right-hand side approaches $\int_a^b f(x)\ dx$.

Errors in the Trapezoidal Rule and Simpson's Rule

If f has a continuous second derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\ dx$ by the Trapezoidal Rule is

$$|E| \le \frac{(b-a)^3}{12n^2}[max|f''(x)|], \qquad a \le x \le b$$

Moreover, if f as a continuous fourth derivative on $[a, b]$, then the error E in approximating $\int_a^b f(x)\ dx$ by Simpson's Rule is

$$|E| \le \frac{(b-a)^5}{180n^4}\left[max\left|f^{(4)}\right|\right], \qquad a \le x \le b$$

Substitution for Rational Functions of Sine and Cosine

For integrals involving rational functions of sine and cosine, the substitution

$$u = \frac{\sin(x)}{1+\cos(x)} = \tan\left(\frac{x}{2}\right)$$

Yields

$$\cos(x) = \frac{1-u^2}{1+u^2}, \sin(x) = \frac{2u}{1+u^2}, dx = \frac{2\ du}{1+u^2}$$

Definition of Improper Integrals with Integration Limits

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^\infty f(x)\ dx = \lim_{b\to\infty} \int_a^b f(x)\ dx$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\ dx = \lim_{a\to-\infty} \int_a^b f(x)\ dx$$

3. 3 If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^c f(x)\ dx + \int_c^\infty f(x)\ dx$$

Where c is any real number.

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a, b)$ and has infinite discontinuity at b, then

$$\int_a^b f(x)\ dx = \lim_{c\to b^-} \int_a^c f(x)\ dx$$

2. If f is continuous on the interval $(a, b]$ and has infinite discontinuity at a, then

$$\int_a^b f(x)\ dx = \lim_{c\to a^+} \int_c^b f(x)\ dx$$

3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

In the first two cases, the improper integral **converges** when the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

A Special Improper Integral

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ diverges, & p \le 1 \end{cases}$$

Definition of the Limit of a Sequence

Let L be a real number. The **limit** of a sequence $\{a_n\}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

If for each $\varepsilon > 0$, there exists $M > 0$ such that $|a_n - L| < \varepsilon$ whenever $n > M$. If the limit L of a sequence exists, then the sequence **converges** to L. If the limit of a sequence does not exist, then the sequence **diverges**.

Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L$$

If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty}(a_n) = L$$

Properties of Limits of Sequences

Let $\lim_{n\to\infty}(a_n) = L$ and $\lim_{n\to\infty}(b_n) = K$

Scalar multiple: $\lim_{n\to\infty}(ca_n) = cL$, c is any real number

Sum or Difference $\lim_{n\to\infty}(a_n \pm b_n) = L \pm K$

Product: $\lim_{n\to\infty}(a_n b_n) = LK$

Quotient: $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{M}$, $b_n \neq 0$ and $K \neq 0$

Squeeze Theorem for Sequences

If $\lim_{n\to\infty}(a_n) = L = \lim_{n\to\infty}(b_n)$ and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then $\lim_{n\to\infty}(c_n) = L$

Absolute Value Theorem

For the sequence $\{a_n\}$ if

$$\lim_{n\to\infty}|a_n| = 0$$

Then

$$\lim_{n\to\infty}(a_n) = 0$$

Definition of Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** when its terms are non-decreasing

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$$

Or when its terms are non-increasing

$$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots$$

Definition of Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** when there is a real number M such that $a_n \le M$ for all n. The M is called an **upper bound** of the sequence.

2. A sequence $\{a_n\}$ is **bounded below** when there is a real number N such that $N \le a_n$ for all n. The number N is called a **lower bound** of the sequence.

3. A sequence $\{a_n\}$ is **bounded** when it is bounded above and bounded below.

Bounded Monotonic Sequences

If a sequence $\{a_n\}$ is bounded and monotonic, then it converges

---

Infinite Series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$$

Definitions of Convergent and Divergent Series

For the infinite series $\sum_{n\to\infty}^{\infty} a_n$, the **nth partial sum** is

$$S_n = a_1 + a_2 + \cdots + a_n$$

If the sequence of partial sums $\{S_n\}$ converges to S, then the series $\sum_{n\to\infty}^{\infty} a_n$ **converges**. The limit S is called the **sum of the series**.

$$S_n = a_1 + a_2 + \cdots + a_n + \cdots$$

If $\{S_n\}$ diverges, then the series **diverges**.

Telescoping Series

$$(b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + (b_4 - b_5) + \cdots$$

Geometric Series

$$\sum_{n=1}^{\infty} ar^n = a + ar + ar^2 + \cdots ar^n + \cdots, \quad a \ne 0$$

Convergence of a Geometric Series

A geometric series with ratio r diverges when $|r| \ge 1$. If $|r| < 1$, then the series converges to the sum

$$\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1$$

Properties of Infinite Series

Let $\sum a_n$ and $\sum b_n$ be convergent series, and let A, B and c be real numbers. If $\sum a_n = A$ and $\sum b_n = B$, then the following series converge to indicated sums.

$$\sum_{n=1}^{\infty} ca_n = cA$$

$$\sum_{n=1}^{\infty} (a_n + b_n) = A + B$$

$$\sum_{n=1}^{\infty} (a_n - b_n) = A - B$$

Limit of the nth term of a Convergent Series

If $\sum_{n\to\infty}^{\infty} a_n$ converges, then $\lim_{n\to\infty}(a_n) = 0$

---

nth term of a Divergent Series

If $\lim_{n\to\infty}(a_n) \neq 0$ then $\sum_{n\to\infty}^{\infty} a_n$ diverges.

The Integral Test

If f is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n$$

And

$$\int_1^{\infty} f(x)\ dx$$

Either both converge or both diverge.

P-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

Harmonic Series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

Convergence of p-series

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

Converges for $p > 1$ and diverges for $0 < p \leq 1$

Direct Comparison Test

Let $0 < a_n \leq b_n$ for all n

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges

---

Limit Comparison Test

If $a_n > 0, b_n > 0$ an

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

Where L is *finite and positive*, then

$$\sum_{n=1}^{\infty} a_n$$

And

$$\sum_{n=1}^{\infty} b_n$$

Either both converge or both diverge.

Alternating Series Test

Let $a_n > 0$. The alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

And

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

Converge when these two conditions are met

1. $\lim_{n\to\infty} (a_n) = 0$

2. $a_{n+1} \leq a_n$, for all n

Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder $R_N$involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = R_N \leq a_{N+1}$$

Absolute Convergence

If the series $\sum |a_n|$ converges, then the series$\sum a_n$ also converges.

Definitions of Absolute and Conditional Convergence

1. The series $\sum a_n$ is **absolutely convergent** when $\sum |a_n|$ converges.

2. The series $\sum a_n$ is **conditionally convergent** when$\sum a_n$ converges but $\sum |a_n|$ diverges.

Ratio Test

Let $\sum a_n$be a series with non-zero terms

1. The series $\sum a_n$ converges absolutely when $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| < 1$

2. The series $\sum a_n$ diverges when $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| > 1$ or $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| = \infty$

3. The Ratio Test is inconclusive when $\lim_{n\to\infty} \left|\frac{a_n+1}{a_n}\right| = 1$

Root Test

1. The series $\sum a_n$ converges absolutely when $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

2. The series $\sum a_n$ diverges when $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$

3. The Root Test is inconclusive when $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Guidelines for Testing a Series for Convergence or Divergence

1. Does the nth root term approach 0? If not, the series diverges.

2. Is the series one of the special types—geometric, p-series, telescoping, or alternating?

3. Can the Integral Test, the Root Test or the Ratio Test be applied?

4. Can the series be compared favorably to one of the special types?

Definitions of nth Taylor Polynomial and nth Maclaurin Polynomial

If f has n derivatives at c, then the polynomial

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Is called the **nth Taylor polynomial for f** at c. If $c = 0$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \cdots + \frac{f^{(n)}(0)}{n!}(x)^n$$

Is also called the **nth Taylor polynomial for f**.

Taylor Polynomial Remainder

$$f(x) = P_n(x) + R_n(x)$$

Error in Taylor Polynomial

$$Error = |R_n(x)| = |f(x) - P_n(x)|$$

Taylor's Theorem

If a function f is differentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

Where

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots a_n x^n + \cdots$$

Is called a **power series**. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots a_n(x-c)^n + \cdots$$

Is called a **power series centered at c**, where c is a constant.

Convergence of a Power Series

For a power series centered at c, precisely one of the following is true,

1. The series converges only at c.

2. There exists a real number $R > 0$ such that the series converges absolutely for

$$|x-c| < R$$

And diverges for

$$|x-c| > R$$

3. The series converges absolutely for all x

The number R is the **radius of convergence** of the power series. If the series converges only at c, then the radius of convergence $R = 0$. If the series converges at all x, then the radius convergence is $R = \infty$. The set of all values of x for which the power series converges is the **interval of convergence** of the power series.

Properties of Function Defined by Power Series

If the function

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n$$

$$=a_0+a_1(x-c)+a_2(x-c)^2+\cdots a_n(x-c)^n+\cdots$$

Has a radius of convergence of $R>0$, then one the interval

$$(c-R, c+R)$$

f is a differentiable (and therefore continuous). Moreover, the derivative and antiderivative of f are as follows.

$$f'(x)=\sum_{n=1}^{\infty} na_n(x-c)^n$$

$$=a_1+2a_2(x-c)+3a_3(x-c)^3+\cdots$$

$$\int f(x)\ dx=C+\sum_{n=1}^{\infty} a_n\frac{(x-c)^{n+1}}{n+1}$$

$$=C+a_0(x-c)+a_1\frac{(x-c)^2}{2}+a_2\frac{(x-c)^3}{3}+\cdots$$

The *radius of convergence* of the series obtained by differentiating or integrating a power series is the same as that of the original power series. The *interval of convergence*, however, may differ as a result of the behavior at the endpoints.

Operations with Power Series

Let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ and $g(x)=\sum_{n=0}^{\infty} b_n x^n$

$$f(kx)=\sum_{n=0}^{\infty} a_n k^n x^n$$

$$f(x^N)=\sum_{n=0}^{\infty} a_n x^{nN}$$

$$f(x)\pm g(x)=\sum_{n=0}^{\infty}(a_n\pm b_n)x^n$$

The Form of a Convergent Power Series

If f is represented by a power series $f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n$ for all x in an open interval I containing c, then

$$a_n=\frac{f^{(n)}(c)}{n!}$$

And

$$(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\cdots$$
$$+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

Definition of Taylor and Maclaurin Series

If a function $f$has derivative of all orders at $x=c$, then the series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2$$
$$+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

Is called the **Taylor series for f at** $c=0$, then the series is the **Maclaurin series for f**.

Convergence of Taylor Series

If $\lim_{n\to\infty} R_n=0$ for all x in the interval I, then the Taylor series for f converges and equals $f(x)$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$$

Guidelines for Finding a Taylor Series

1. Differentiate $\boldsymbol{f}$ with respect to $\boldsymbol{x}$ several times and evaluate each derivative at $\boldsymbol{c}$.

$$f(c), f'^{(c)}, f''^{(c)}, f'''^{(c)}, \cdots, f^{(n)}(c), \cdots$$

Try to recognize a patter in these numbers

2. Use the sequence developed in the first step to form the Taylor coefficients $a_n = f^{(n)}(c)/n!$ and determine the interval of convergence for the resulting power series

$$f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n + \cdots$$

3. Within this interval of convergence, determine whether the series converges to $\boldsymbol{f}(\boldsymbol{x})$

General Second-Degree Equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Standard Equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2$$

Standard Equation of a Parabola

The **standard form** of the equation of a parabola with vertex $(\boldsymbol{h}, \boldsymbol{k})$ and directrix $\boldsymbol{y} = \boldsymbol{k} - \boldsymbol{p}$ is

Vertical axis

$$(x - h)^2 = 4p(y - k)$$

For directrix $\boldsymbol{x} = \boldsymbol{h} - \boldsymbol{p}$, the equation is

Horizontal axis

$$(y - k)^2 = 4p(x - h)^2$$

The focus lies on the axis $\boldsymbol{p}$ units (*directed distance*) from the vertex. The coordinates of the focus are as follows

Vertical axis

$$(h, k + p)$$

Horizontal axis

$$(h + p, k)$$

Reflective Property of a Parabola

Let $\boldsymbol{P}$ be on a point on a parabola. The tangent line to the parabola at point $\boldsymbol{P}$ makes equal angles with the following two lines.

1. The line passing through $\boldsymbol{P}$ and the focus.

2. The line passing through $\boldsymbol{P}$ parallel to the axis of the parabola.

Standard Equation of an Ellipse

The **standard form** of the equation of an ellipse with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectfully, where $a > b$, is

Major Axis Horizontal

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Or

Major Axis Vertical

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$

Reflective Property of an Ellipse

Let P be a point on an ellipse. The tangent line to the ellipse at point P makes equal angles with the line through P and the foci.

Definition of Eccentricity of an Ellipse

$$e=\frac{c}{a}$$

Standard Equation of a Hyperbola

The **standard form** of the equation of a hyperbola with center at (h, k) is

Transverse Axis Horizontal

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

Or

Transverse Axis Vertical

$$\frac{(y-h)^2}{a^2}-\frac{(x-k)^2}{b^2}=1$$

The vertices are a units from the center, and the foci are c units from the center, where $c^2 = a^2 + b^2$

Asymptotes of a Hyperbola

For a *horizontal* transverse axis, the equation of the asymptotes are

$$y=k+\frac{b}{a}(x-h)$$

And

$$y=k-\frac{b}{a}(x-h)$$

For a *vertical* transverse axis, the equations of the asymptotes are

$$y=k+\frac{a}{b}(x-h)$$

And

$$y=k-\frac{a}{b}(x-h)$$

Definition of Eccentricity of a Hyperbola

The **eccentricity** e of a hyperbola is given the ratio

$$e=\frac{c}{d}$$

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, then the equations

$$x = f(t)$$

And

$$y = g(t)$$

Are **parametric equations** and t is the **parameter**. The set of points (x, y) obtained as t varies on the interval I is the **graph** of the parametric equations. Taken together, the parametric equations and the graph are a **plane curve**, denoted by C.

Definition of a Smooth Curve

A curve C represented by $x = f(t)$ and $y = g(t)$ on an interval I and not simultaneously 0, except possibly at the endpoints of I. The curve C is called **piecewise smooth** when it is smooth on each subinterval of some partition of I.

Parametric Form of the Derivative

If a smooth curve C is given by the equations

$$x = f(t)$$

And

$$y = g(t)$$

Then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0$$

Arc Length in Parametric Form

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Area of a Surface of Revolution

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$, then the area S of the surface of revolution formed by revolving C about the coordinate axes is given by the following.

Revolution about the x-axis: $g(t) \geq 0$

$$S = 2\pi \int_a^b g(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Revolution about the y-axis: $f(t) \geq 0$

$$S = 2\pi \int_a^b f(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Coordinate Conversion

The polar coordinates $(\boldsymbol{r}, \boldsymbol{\theta})$ of a point are related to the rectangular coordinates $(\boldsymbol{x}, \boldsymbol{y})$ of the point as follows

Polar to Rectangular

$$\boldsymbol{x} = \boldsymbol{r}\cos(\boldsymbol{\theta})$$

$$\boldsymbol{y} = \boldsymbol{r}\sin(\boldsymbol{\theta})$$

Rectangular to Polar

$$\tan(\boldsymbol{\theta}) = \frac{\boldsymbol{y}}{\boldsymbol{x}}$$

$$\boldsymbol{r}^2 = \boldsymbol{x}^2 + \boldsymbol{y}^2$$

Slope in Polar Form

If $\boldsymbol{f}$ is a differentiable function of $\boldsymbol{\theta}$, then the *slope* of the tangent line to the graph of $\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{\theta})$ at the point $(\boldsymbol{r}, \boldsymbol{\theta})$ is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta)\cos(\theta) + f'(\theta)\sin(\theta)}{-f(\theta)\sin(\theta) + f(\theta)\cos(\theta)}$$

Provided that $dx/d\theta \neq 0$ at $(\boldsymbol{r}, \boldsymbol{\theta})$

Tangent Lines at the Pole

If $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, then the line $\theta = \alpha$ is tangent at the pole to the graph of $r = f(\theta)$.

Area in Polar Coordinates

If $\boldsymbol{f}$ is continuous and non-negative on the interval $[\alpha, \beta], 0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$$

Arc Length of a Polar Curve

Let $\boldsymbol{f}$ be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The length of the graph $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ is

$$s = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$$

Area of a Surface of Revolution

Let $\boldsymbol{f}$ be a function whose derivative is continuous on an interval $\alpha \leq \theta \leq \beta$. The area of the surface formed by revolving the graph of $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$ about the indicated line is as follows

About the polar axis

$$s = 2\pi \int_{\alpha}^{\beta} f(\theta)\sin(\theta) \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta$$

About the line $\theta = \frac{\pi}{2}$

$$s = 2\pi \int_{\alpha}^{\beta} f(\theta)\cos(\theta) \sqrt{[f(\theta)]^2 + [f'(\theta)]^2} \, d\theta$$

Classification of Conics by Eccentricity

Let $\boldsymbol{F}$ be a fixed point $(focus)$ and let $\boldsymbol{D}$ be a fixed line $(directrix)$ in the plane. Let $\boldsymbol{P}$ be another point in the plane and let $\boldsymbol{e}$ $(eccentricity)$ be the ratio of the distance between $\boldsymbol{P}$ and $\boldsymbol{F}$ to the distance between $\boldsymbol{P}$ and $\boldsymbol{D}$. The collection of all points $\boldsymbol{P}$ with a given eccentricity is a conic.

1. The conic is an ellipse for $0 < e < 1$
2. The conic is a parabola for $e = 1$
3. The conic is a hyperbola for $e > 1$

Polar Equations of Conics

The graph of a polar equation of the form

$$r = \frac{ed}{1 + e\cos(\theta)}$$

Or

$$r = \frac{ed}{1 + e\sin(\theta)}$$

Is a conic, where $e > 0$ is the eccentricity and $|d|$ is the distance between the focus at the pole and its corresponding directrix.

Determining a Conic from its Equation

Ellipse

$$b^2 = a^2 - c^2 = a^2 - (ea)^2 = a^2(1 - e^2)$$

Hyperbola

$$b^2 = a^2 - c^2 = (ea)^2 - a^2 = a^2(e^2 - 1)$$

Definition of Component Form of a Vector in the Plane

If v is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **component form of** v is $v = \langle v_1, v_2\rangle$. The coordinates v_1 and v_2 are called the **components of** v. If both the initial point and the terminal point lie at the origin of v is called the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0\rangle$.

Length of a Vector

$$\|v\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)} = \sqrt{{v_1}^2 + {v_2}^2}$$

Definitions of Vector Addition and Scalar Multiplication

Let $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$ be vectors and let c be a scalar.

1. The **vector sum** of u and v is the vector
$$u + v = \langle u_1 + v_1, u_2 + v_2\rangle$$

2. The **vector multiple** of c and u is the vector
$$cu = \langle cu_1, cu_2\rangle$$

3. The **negative** of v is the vector
$$-v = (-1)v = \langle -v_1, -v_2\rangle$$

4. The **difference** of u and v is
$$u - v = u + (-v) = \langle u_1 - v_1, u_2 - v_2\rangle$$

Properties of Vector Operations

Let u, v and w be vectors in the plane, and let c and d be scalars

Communicative Properties

$$u + v = v = u$$

Associative Property

$$(u + v) + w = u + (v + w)$$

Additive Identity Property

$$u + 0 = u$$

$$c(du) = (cd)u$$

Distributive Property

$$(c + d)u = cu + du$$

$$c(u + v) = cu + cv$$

$$1(u) = u, \quad 0(u) = 0$$

Length of a Scalar Multiple

Let $\boldsymbol{v}$ be a vector and let $\boldsymbol{c}$ be a scalar. Then

$$\|\boldsymbol{cv}\| = |\boldsymbol{c}|\|\boldsymbol{v}\|$$

$|\boldsymbol{c}|$ is the absolute value of $\boldsymbol{c}$.

Unit Vector in the Direction of $\boldsymbol{v}$

If $\boldsymbol{v}$ is a non-zero vector in the plane, then the vector

$$\boldsymbol{u} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{\boldsymbol{1}}{\|\boldsymbol{v}\|}\boldsymbol{v}$$

Has length $\boldsymbol{1}$ and the same direction as $\boldsymbol{v}$.

Standard Unit Vector

$$\boldsymbol{i} = \langle \boldsymbol{1}, \boldsymbol{0} \rangle$$

And

$$\boldsymbol{j} = \langle \boldsymbol{0}, \boldsymbol{1} \rangle$$

Equation of a Sphere

$$(\boldsymbol{x} - \boldsymbol{x_0})^2 + (\boldsymbol{y} - \boldsymbol{y_0})^2 + (\boldsymbol{z} - \boldsymbol{z_0})^2 = \boldsymbol{r}^2$$

Midpoint Formula

$$\left(\frac{\boldsymbol{x_1} + \boldsymbol{x_2}}{\boldsymbol{2}}, \frac{\boldsymbol{y_1} + \boldsymbol{y_2}}{\boldsymbol{2}}, \frac{\boldsymbol{z_1} + \boldsymbol{z_2}}{\boldsymbol{2}}\right)$$

Vectors in Space

Let $\boldsymbol{u} = \langle \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3} \rangle$ and $\boldsymbol{v} = \langle \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} \rangle$ be vectors in space and let $\boldsymbol{c}$ be a scalar.

1. *Equality of Vectors*: $\boldsymbol{u} = \boldsymbol{v}$ if and only if $\boldsymbol{u_1} = \boldsymbol{v_1}\,, \boldsymbol{u_2} = \boldsymbol{v_2}$ and $\boldsymbol{u_3} = \boldsymbol{v_3}$

2. *Component Form*: If $\boldsymbol{u} = \boldsymbol{v}$ is represented by the directed line segment from $\boldsymbol{P}(\boldsymbol{p_1}, \boldsymbol{p_2}, \boldsymbol{p_3})$ to $\boldsymbol{Q}(\boldsymbol{q_1}, \boldsymbol{q_2}, \boldsymbol{q_3})$, then

   $$\boldsymbol{v} = \langle \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} \rangle = \langle \boldsymbol{q_1} - \boldsymbol{p_1}, \boldsymbol{q_2} - \boldsymbol{p_2}, \boldsymbol{q_3} - \boldsymbol{p_3} \rangle$$

3. *Length*: $\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v_1}^2, \boldsymbol{v_2}^2, \boldsymbol{v_3}^2}$

4. *Unit Vector in the Direction of $\boldsymbol{v}$*:

   $$\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \left(\frac{\boldsymbol{1}}{\|\boldsymbol{v}\|}\right)\langle \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} \rangle, \quad \boldsymbol{v} \neq \boldsymbol{0}$$

5. *Vector Addition*:

   $$\boldsymbol{v} + \boldsymbol{u} = \langle \boldsymbol{v_1} + \boldsymbol{u_1}, \boldsymbol{v_2} + \boldsymbol{u_2}, \boldsymbol{v_3} + \boldsymbol{u_3} \rangle$$

6. *Scalar Multiplication*:

   $$\boldsymbol{cv} = \langle \boldsymbol{cv_1}, \boldsymbol{cv_2}, \boldsymbol{cv_3} \rangle$$

Definition of Parallel Vectors

Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** when there is some scalar $\boldsymbol{c}$ such that $\boldsymbol{u} = \boldsymbol{cv}$.

Definition of Dot Product

The **dot product** of $\boldsymbol{u} = \langle \boldsymbol{u_1}, \boldsymbol{u_2} \rangle$ and $\boldsymbol{v} = \langle \boldsymbol{v_1}, \boldsymbol{v_2} \rangle$

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u_1 v_1}, + \boldsymbol{u_2 v_2}$$

The **dot product** of $\boldsymbol{u} = \langle \boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3} \rangle$ and $\boldsymbol{v} = \langle \boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3} \rangle$ is

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u_1 v_1}, + \boldsymbol{u_2 v_2} + \boldsymbol{u_3 v_3}$$

Properties of the Dot Product

Let $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in the plane, and let $\boldsymbol{c}$ be a scalar

Communicative Properties

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$$

Distributive Property

$$\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{w}$$

Associative Property

$$\boldsymbol{0} \cdot \boldsymbol{v} = \boldsymbol{0}$$

$$\boldsymbol{v} \cdot \boldsymbol{v} = \|\boldsymbol{v}\|^2$$

Angle Between Two Vectors

If $\boldsymbol{\theta}$ is the angle between two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, where $\boldsymbol{0} \leq \boldsymbol{\theta} \leq \boldsymbol{\pi}$, then

$$\cos(\boldsymbol{\theta}) = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\| \|\boldsymbol{v}\|}$$

Definition of Orthogonal Vectors

The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal when $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{0}$

Alternative form of Dot Product

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos(\boldsymbol{\theta})$$

Let $\boldsymbol{u}, \boldsymbol{v}$ be non-zero vectors. Moreover, let

$$\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2$$

Where $\boldsymbol{w}_1$ is parallel to $\boldsymbol{v}$ and $\boldsymbol{w}_2$ is orthogonal,

1. $\boldsymbol{w}_1$ is called the **projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$** or the **vector component of $\boldsymbol{u}$ along $\boldsymbol{v}$**, and is denoted by $\boldsymbol{w}_1 = \boldsymbol{proj}_{\boldsymbol{v}}\boldsymbol{u}$

2. $\boldsymbol{w}_2 = \boldsymbol{u} - \boldsymbol{w}_1$ is called the **vector component of $\boldsymbol{u}$ orthogonal to $\boldsymbol{v}$**

Projection Using the Dot Product

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-zero vectors, then the projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is

$$\boldsymbol{proj}_{\boldsymbol{v}}\boldsymbol{u} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{v}\|^2}\right)\boldsymbol{v}$$

Definition of Work

The work $\boldsymbol{W}$ done by a constant force $\boldsymbol{F}$ as its point of application moves along the vector $\overrightarrow{\boldsymbol{PQ}}$ is one of the following

Projection form

$$\boldsymbol{W} = \left\|\boldsymbol{proj}_{\overrightarrow{\boldsymbol{PQ}}}\boldsymbol{F}\right\| \left\|\overrightarrow{\boldsymbol{PQ}}\right\|$$

Dot Product form

$$\boldsymbol{W} = \boldsymbol{F} \cdot \overrightarrow{\boldsymbol{PQ}}$$

Definition of Cross Product of Two Vectors in Space

Let

$$\boldsymbol{u} = \boldsymbol{u}_1\boldsymbol{i} + \boldsymbol{u}_2\boldsymbol{j} + \boldsymbol{u}_3\boldsymbol{k}$$

And

$$\boldsymbol{v} = \boldsymbol{v}_1\boldsymbol{i} + \boldsymbol{v}_2\boldsymbol{j} + \boldsymbol{v}_3\boldsymbol{k}$$

Be vectors in space. The **cross product** of $\boldsymbol{u}$ and $\boldsymbol{v}$ is the vector

$$\boldsymbol{u} \times \boldsymbol{v} = (\boldsymbol{u}_2\boldsymbol{v}_3 - \boldsymbol{u}_3\boldsymbol{v}_2)\boldsymbol{i} - (\boldsymbol{u}_1\boldsymbol{v}_3 - \boldsymbol{u}_3\boldsymbol{v}_1)\boldsymbol{j} + (\boldsymbol{u}_1\boldsymbol{v}_2 - \boldsymbol{u}_2\boldsymbol{v}_1)\boldsymbol{k}$$

Algebraic Properties of the Cross Product

Let $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in space, and let $\boldsymbol{c}$ be a scalar.

1. $\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$
2. $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} \times \boldsymbol{v}) + (\boldsymbol{u} \times \boldsymbol{w})$
3. $\boldsymbol{c}(\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{c}\boldsymbol{u}) \times \boldsymbol{v} = \boldsymbol{u} \times (\boldsymbol{c}\boldsymbol{v})$
4. $\boldsymbol{u} \times \boldsymbol{0} = \boldsymbol{0} \times \boldsymbol{u} = \boldsymbol{0}$
5. $\boldsymbol{u} \times \boldsymbol{u} = \boldsymbol{0}$
6. $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{u}) = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$

Geometric Properties of the Cross Product

Let $\boldsymbol{u}, \boldsymbol{v}$ be non-zero vectors in space, and let θ the angle $\boldsymbol{u}$ and t $\boldsymbol{v}$.

1. $\boldsymbol{u} \times \boldsymbol{v}$ is orthogonal to both t $\boldsymbol{u}$ and $\boldsymbol{v}$.
2. $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin(\boldsymbol{\theta})$
3. $\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0}$ if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are scalar multiples of each other.
4. $\|\boldsymbol{u} \times \boldsymbol{v}\| = \boldsymbol{area\ of\ parallelogram}$ having $\boldsymbol{u}$ and $\boldsymbol{v}$ as adjacent sides.

The Triple Scalar Product

For $\boldsymbol{u} = \boldsymbol{u_1 i} + \boldsymbol{u_2 j} + \boldsymbol{u_3 k}, \boldsymbol{v} = \boldsymbol{v_1 i} + \boldsymbol{v_2 j} + \boldsymbol{v_3 k}$ and $\boldsymbol{w} = \boldsymbol{w_1 i} + \boldsymbol{w_2 j} + \boldsymbol{w_3 k}$, the triple scalar product is

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} \\ \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} \\ \boldsymbol{w_1} & \boldsymbol{w_2} & \boldsymbol{w_3} \end{vmatrix}$$

Geometric Property of the Triple Scalar Product

The volume $\boldsymbol{V}$ of a parallelepiped with vectors $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$ as adjacent edges is

$$\boldsymbol{V} = |\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})|$$

Parametric Equations of a Line in Space

A line $\boldsymbol{L}$ parallel to the vector$\boldsymbol{v} = \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle$ and passing through the point $\boldsymbol{P}(\boldsymbol{x_1}, \boldsymbol{y_1}, \boldsymbol{z_1})$ is represented by the **parametric equations**

$$\boldsymbol{x} = \boldsymbol{x_1} + \boldsymbol{at}, \qquad \boldsymbol{y} = \boldsymbol{y_1} + \boldsymbol{bt}, \quad \boldsymbol{z} = \boldsymbol{z_1} + \boldsymbol{ct}$$

Symmetric Equations

$$\left(\frac{\boldsymbol{x_1} - \boldsymbol{x_2}}{\boldsymbol{a}}, \frac{\boldsymbol{y_1} - \boldsymbol{y_2}}{\boldsymbol{b}}, \frac{\boldsymbol{z_1} - \boldsymbol{z_2}}{\boldsymbol{c}}\right)$$

Standard Equation of a Plane in Space

The plane containing the point $(\boldsymbol{x_1}, \boldsymbol{y_1}, \boldsymbol{z_1})$ and having normal vector $\boldsymbol{n} = \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle$

Can be represented by the **standard form** of the equation of a plane

$$\boldsymbol{a}(\boldsymbol{x_1} - \boldsymbol{x_2}) + \boldsymbol{b}(\boldsymbol{y_1} - \boldsymbol{y_2}) + \boldsymbol{c}(\boldsymbol{z_1} - \boldsymbol{z_2}) = \boldsymbol{0}$$

General form of Equation of plane

$$\boldsymbol{ax} + \boldsymbol{by} + \boldsymbol{cz} + \boldsymbol{d} = \boldsymbol{0}$$

Angle between two planes

$$\cos(\boldsymbol{\theta}) = \frac{|\boldsymbol{n_1} \cdot \boldsymbol{n_2}|}{\|\boldsymbol{n_1}\|\|\boldsymbol{n_2}\|}$$

Distance Between a Point and a Plane

The distance between a plane and a point $\boldsymbol{Q}$ (not in the plane) is

$$\boldsymbol{D} = \|\boldsymbol{proj_n}\overrightarrow{\boldsymbol{PQ}}\| = \frac{|\overrightarrow{\boldsymbol{PQ}} \cdot \boldsymbol{n}|}{\|\boldsymbol{n}\|}$$

Where $\boldsymbol{P}$ is appoint in the plane and is $\boldsymbol{n}$ normal to the plane

Distance Between a Point and a Plane

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance Between a Point and a in Space

The distance between a point Q and a line in space is

$$D = \frac{\|\overrightarrow{PQ} \times \boldsymbol{u}\|}{\|\boldsymbol{u}\|}$$

Where $\boldsymbol{u}$ is a direction vector for the line and P is a point on the line.

Definition of a Cylinder

Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is a **cylinder**. The curve C is the **generating curve** (or **directrix**) of the cylinder, and the parallel lines are **rulings**.

Equations of Cylinders

The equation of a cylinder whose rulings are parallel to one of the coordinate axes contains only the variable corresponding to the other two axes.

Quadric Surface

The equation of a **quadric surface** in space is a second-degree equation in three variables. The **general form** of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

There are six basic types of quadric surfaces: **ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid,** and **hyperbolic paraboloid**.

Surface of Revolution

If the graph of a radius function r is revolved about one of the coordinate axes, then the equation of the resulting surface of revolution has one of the forms.

1. Revolved about the **x-axis**: $y^2 + z^2 = [r(x)]^2$
2. Revolved about the **y-axis**: $x^2 + z^2 = [r(y)]^2$
3. Revolved about the **z-axis**: $x^2 + y^2 = [r(z)]^2$

The Cylindrical Coordinate System

In a **cylindrical coordinate system**, a point P in space is represented by an ordered triple (r, θ, z).

1. (r, θ) is a polar representation of the projection of P in the **xy-plane**.
2. z is the directed distance from (r, θ) to P.

Cylindrical to Rectangular

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

Rectangular to Cylindrical

$$r^2 = x^2 + y^2, \quad \tan(\theta) = \frac{y}{x}, \quad z = z$$

The Spherical Coordinate System

In a **spherical coordinate system**, a point $\boldsymbol{P}$ in space is represented by an ordered triple $(\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{\phi})$, where $\boldsymbol{\rho}$ is the lowercase Greek letter rho and $\boldsymbol{\phi}$ is the lowercase Greek letter phi

1. $\boldsymbol{\rho}$ is the distance between $\boldsymbol{P}$ **and the origin** $\boldsymbol{\rho \geq 0}$.

2. $\boldsymbol{\theta}$ is the same angle used in cylindrical coordinated $r \geq 0$

3. $\boldsymbol{\phi}$ is the angle *between* the positive **z-axis** and the line segment $\overrightarrow{\boldsymbol{OP}}$ $\boldsymbol{0 \leq \phi \leq \pi}$

Note that the first and third coordinates, $\boldsymbol{\rho}$ and $\boldsymbol{\phi}$ are non-negative.

Spherical to Rectangular

$$x = \rho \sin(\phi) \cos(\theta)$$

$$y = \rho \sin(\phi) \sin(\theta),$$

$$z = \rho \cos(\phi)$$

Rectangular to spherical

$$\rho^2 = x^2 + y^2 + z^2,$$

$$\tan(\theta) = \frac{y}{x},$$

$$\phi = cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Spherical to cylindrical $(r \geq 0)$

$$r^2 = \rho^2 sin^2(\phi)$$

$$\theta = \theta$$

$$z = \rho \cos(\phi)$$

Cylindrical to Spherical $(r \geq 0)$

$$\rho = \sqrt{r^2 + z^2}$$

$$\theta = \theta$$

$$\phi = cos^{-1}\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$$

Definition of Vector-Values Function

A function of the form

Plane

$$r(t) = f(t)i + g(t)j$$

Or

Space

$$r(t) = f(t)i + g(t)j + h(t)j$$

Is a **vector-valued function**, where the **component functions** $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$ are real-valued functions of the parameter $\boldsymbol{t}$. Vector-valued functions are sometimes denoted as

Plane

$$r(t) = \langle f(t), g(t) \rangle$$

Or

$$r(t) = \langle f(t), g(t), h(t) \rangle$$

Definition of the Limit of a Vector-Valued Function

1. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{t})\boldsymbol{i} + \boldsymbol{g}(\boldsymbol{t})\boldsymbol{j}$, then

Plane

$$\lim_{t \to a} r(t) = \left[\lim_{t \to a} f(t)\right] i + \left[\lim_{t \to a} g(t)\right] j$$

Provided $\boldsymbol{f}$ and $\boldsymbol{g}$ have limits as $\boldsymbol{t} \to \boldsymbol{a}$

2. If $\boldsymbol{r}$ is a vector-valued function such that $\boldsymbol{r}(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{t})\boldsymbol{i} + \boldsymbol{g}(\boldsymbol{t})\boldsymbol{j} + \boldsymbol{h}(\boldsymbol{t})\boldsymbol{j}$, then

$$\lim_{t \to a} r(t) = \left[\lim_{t \to a} f(t)\right] i + \left[\lim_{t \to a} g(t)\right] j + \left[\lim_{t \to a} h(t)\right] k$$

Provided $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$ have limits as $\boldsymbol{t} \to \boldsymbol{a}$

Definition of Continuity of a Vector-Valued Function

A vector-value d function $\boldsymbol{r}$ is **continuous at the point** given by $\boldsymbol{t} = \boldsymbol{a}$ when the limit $\boldsymbol{r}(\boldsymbol{t})$ exists as $\boldsymbol{t} \to \boldsymbol{a}$ and

$$\lim_{t \to a} r(t) = r(a)$$

A vector-valued function $\boldsymbol{r}$ is **continuous on an interval** $\boldsymbol{I}$ when it is continuous at every point in the interval.

Definition of the Derivative of a Vector-Valued Function

The **derivative of a vector-valued function** $\boldsymbol{r}$ is

$$r'(t) = \lim_{\Delta t \to 0} \frac{r(t + \Delta t)}{\Delta t}$$

For all $\boldsymbol{t}$ for which the limit exists. If $\boldsymbol{r}'(\boldsymbol{t})$ exists, then $\boldsymbol{r}$ is **differentiable at** $\boldsymbol{t}$. If $\boldsymbol{r}'(\boldsymbol{t})$ exists for all $\boldsymbol{t}$ in an interval $\boldsymbol{I}$, then $\boldsymbol{r}$ is **differentiable on the interval** $\boldsymbol{I}$. Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

Differentiation of Vector-Valued Functions

1. If $\boldsymbol{r}(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{t})\boldsymbol{i} + \boldsymbol{g}(\boldsymbol{t})\boldsymbol{j}$, where $\boldsymbol{f}$ and $\boldsymbol{g}$ are differentiable functions of $\boldsymbol{t}$, then

Plane

$$r'(t) = f'(t)i + g'(t)j$$

2. If $\boldsymbol{r}(\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{t})\boldsymbol{i} + \boldsymbol{g}(\boldsymbol{t})\boldsymbol{j} + \boldsymbol{h}(\boldsymbol{t})\boldsymbol{k}$, where $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$ are differentiable functions of $\boldsymbol{t}$, then

Space

$$r'(t) = f'(t)i + g'(t)j + h'(t)k$$

Properties of the Derivative

Let $\boldsymbol{r}$ and $\boldsymbol{u}$ be differentiable vector-valued functions of $\boldsymbol{t}$, let $\boldsymbol{w}$ be a differentiable real-valued function of $\boldsymbol{t}$, and let $\boldsymbol{c}$ be a scalar.

$$\frac{d}{dt}[c\boldsymbol{r}(t)] = c\boldsymbol{r}'(t)$$

$$\frac{d}{dt}[\boldsymbol{r}(t) \pm \boldsymbol{u}(t)] = \boldsymbol{r}'(t) \pm \boldsymbol{u}'(t)$$

$$\frac{d}{dt}[w(t)\boldsymbol{r}(t)] = w(t)\boldsymbol{r}'(t) + w'(t)\boldsymbol{r}(t)$$

$$\frac{d}{dt}[\boldsymbol{r}(t) \cdot \boldsymbol{u}(t)] = \boldsymbol{r}(t) \cdot \boldsymbol{u}'(t) + \boldsymbol{r}'(t) \cdot \boldsymbol{u}(t)$$

$$\frac{d}{dt}[\boldsymbol{r}(t) \times \boldsymbol{u}(t)] = \boldsymbol{r}(t) \times \boldsymbol{u}'(t) + \boldsymbol{r}'(t) \times \boldsymbol{u}(t)$$

$$\frac{d}{dt}[\boldsymbol{r}(w(t))] = \boldsymbol{r}'(w(t))w'(t)$$

If $\boldsymbol{r}(t) \cdot \boldsymbol{r}(t) = c$, then $\boldsymbol{r}(t) \cdot \boldsymbol{r}'^{(t)} = 0$

Note Property 5 applies only to three-dimensional vector-valued functions because the cross product is not defined for two-dimensional vectors.

Definition of Integration of Vector-Valued Functions

1. If $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j}$, where f and g are continuous on $[a, b]$, then the **indefinite integral** (**antiderivative**) of $\boldsymbol{r}$ is

Plane

$$\int \boldsymbol{r}(t)\,dt = \left[\int f(t)\,dt\right]\boldsymbol{i} + \left[\int g(t)\,dt\right]\boldsymbol{j}$$

And its **definite integral** over the interval over the interval $a \le t \le b$ is

$$\int_a^b \boldsymbol{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\boldsymbol{i} + \left[\int_a^b g(t)\,dt\right]\boldsymbol{j}$$

2. If $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j} + h(t)\boldsymbol{k}$, where If f, g and h are continuous on $[a, b]$, then the **indefinite integral** (**antiderivative**) of $\boldsymbol{r}$ is

Space

$$\int \boldsymbol{r}(t)\,dt = \left[\int f(t)\,dt\right]\boldsymbol{i} + \left[\int g(t)\,dt\right]\boldsymbol{j} + \left[\int h(t)\,dt\right]\boldsymbol{k}$$

And its **definite integral** over the interval over the interval $a \le t \le b$ is

$$\int_a^b \boldsymbol{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\boldsymbol{i} + \left[\int_a^b g(t)\,dt\right]\boldsymbol{j} + \left[\int_a^b h(t)\,dt\right]\boldsymbol{k}$$

Definitions of Velocity and Acceleration

If x and y are twice-differentiable functions of t, and $\boldsymbol{r}$ is a vector-valued function given by $\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$, then the velocity vector, acceleration vector, and speed at time t are as follows

$$velocity = \boldsymbol{v}(t) = \boldsymbol{r}'(t) = x'(t)\boldsymbol{i} + y'(t)\boldsymbol{j}$$

$$acceleration = \boldsymbol{a}(t) = \boldsymbol{r}''(t) = x''(t)\boldsymbol{i} + y''(t)\boldsymbol{j}$$

$$speed = \|\boldsymbol{v}(t)\| = \|\boldsymbol{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

Position Vector

$$r(t) = -\frac{1}{2}gt^2 j + tv_0 + r_0$$

Position Vector for a Projectile

Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation θ is described by the vector function

$$r(t) = (v_0 \cos(\theta) t i) + \left[h + (v_0 \sin(\theta))t - \frac{1}{2}gt^2\right] j$$

Where g is the acceleration due to gravity.

Definition of Unit Tangent Vector

Let C be a smooth curve represented by r on an open interval I. The **unit tangent vector** $T(t)$ at t is defined as

$$T(t) = \frac{r'(t)}{\|r'(t)\|}, \quad r'(t) \neq 0$$

Definition of Principal Unit Normal Vector

Let C be a smooth curve represented by r on an open interval I. If $T(t) \neq 0$, then the **principal normal vector** at t is defined as

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

Acceleration Vector

If $r(t)$ is the position vector for a smooth curve C and $N(t)$ exists, then the acceleration vector $a(t)$ lies in the plane determined by $T(t)$ and $N(t)$

Tangential and Normal Components of Acceleration

If $r(t)$ is the position vector for a smooth curve C [for which $N(t)$ exists], then the tangential and normal components of acceleration are as follows

$$a_T = \frac{d}{dt}[\|v\|] = a \cdot T = \frac{v \cdot a}{\|v\|}$$

$$a_N = \|v\|\|T'\| = a \cdot N = \frac{\|v \times a\|}{\|v\|} = \sqrt{\|a\|^2 - a_T{}^2}$$

Note that $a_N \geq 0$. The normal component of acceleration is also called the **centripetal component of acceleration**.

Arc Length of a Space Curve

If C is a smooth curve given by $r(t) = x(t)i + y(t)j + z(t)k$ on an interval $[a, b]$, then the arc length of C on the interval is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

$$= \int_a^b \|r'(t)\|\ dt$$

Definition of Arc Length Function

Let C be a smooth curve given by $r(t)$ defined on the closed interval $[a, b]$. For $\leq t \leq b$, the **arc length function** is

$$s(t) = \int_a^t \|r'^{(u)}\|\ du$$

$$= \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\ du$$

The arc length s is called the **arc length parameter**.

Arc Length Parameter

If $\boldsymbol{C}$ is a smooth cure given by

Plane Curve

$$\boldsymbol{r}(\boldsymbol{s}) = \boldsymbol{x}(\boldsymbol{s})\boldsymbol{i} + \boldsymbol{y}(\boldsymbol{s})\boldsymbol{j}$$

Or

Space Curve

$$\boldsymbol{r}(\boldsymbol{s}) = \boldsymbol{x}(\boldsymbol{s})\boldsymbol{i} + \boldsymbol{y}(\boldsymbol{s})\boldsymbol{j} + \boldsymbol{z}(\boldsymbol{s})\boldsymbol{k}$$

Where $\boldsymbol{s}$ is the arc length parameter, then

$$\|\boldsymbol{r}'(\boldsymbol{s})\| = \boldsymbol{1}$$

Moreover, if $\boldsymbol{t}$ is *any* parameter for the vector-valued function $\boldsymbol{r}$ such that $\|\boldsymbol{r}'(\boldsymbol{s})\| = \boldsymbol{1}$, then $\boldsymbol{t}$ must be the arc length parameter.

Definition of Curvature

Let $\boldsymbol{C}$ be a smooth curve (in the plane *or* in space) given $\boldsymbol{r}(\boldsymbol{s})$, where $\boldsymbol{s}$ is the arc length parameter. The **curvature** $\boldsymbol{K}$ at $s$is

$$\boldsymbol{K} = \left\|\frac{\boldsymbol{dT}}{\boldsymbol{ds}}\right\| = \|\boldsymbol{T}'(\boldsymbol{s})\|$$

Formulas for Curvature

If $\boldsymbol{C}$ is a smooth curve given by $\boldsymbol{r}(\boldsymbol{t})$, then the curvature $\boldsymbol{K}$ of $\boldsymbol{C}$ at is

$$\boldsymbol{K} = \left\|\frac{\boldsymbol{T}'(\boldsymbol{t})}{\boldsymbol{r}'(\boldsymbol{t})}\right\| = \frac{\|\boldsymbol{r}'(\boldsymbol{t}) \times \boldsymbol{r}''(\boldsymbol{t})\|}{\|\boldsymbol{r}'(\boldsymbol{t})\|^3}$$

Curvature in Rectangular Coordinates

If $\boldsymbol{C}$ is the graph of a twice-differentiable function given by $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$, then the curvature $\boldsymbol{K}$ at the point $(\boldsymbol{x}, \boldsymbol{y})$ is

$$\boldsymbol{K} = \frac{|\boldsymbol{y}''|}{[\boldsymbol{1} + (\boldsymbol{y}')^2]^{3/2}}$$

Acceleration, Speed, and Curvature

If $\boldsymbol{r}(\boldsymbol{t})$ is the position vector for a smooth curve $\boldsymbol{C}$, then the acceleration vector is given by

$$\boldsymbol{a}(\boldsymbol{t}) = \frac{\boldsymbol{d^2s}}{\boldsymbol{dt^2}}\boldsymbol{T} + \boldsymbol{K}\left(\frac{\boldsymbol{ds}}{\boldsymbol{dt}}\right)^2 \boldsymbol{N}$$

Where $\boldsymbol{K}$ is the curvature of $\boldsymbol{C}$ and $\boldsymbol{ds/dt}$ is the speed.

Summary of Velocity, Acceleration, and Curvature

Unless noted otherwise, let $\boldsymbol{C}$ be a curvature (in the plane or in space) given by the position vector

Curve in the Plane

$$\boldsymbol{r}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t})\boldsymbol{i} + \boldsymbol{y}(\boldsymbol{t})\boldsymbol{j}$$

Or

Curve in Space

$$\boldsymbol{r}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t})\boldsymbol{i} + \boldsymbol{y}(\boldsymbol{t})\boldsymbol{j} + \boldsymbol{z}(\boldsymbol{t})\boldsymbol{k}$$

Where $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ are twice-differentiable functions of $\boldsymbol{t}$

Velocity vector, speed, and acceleration vector

Velocity Vector

$$\boldsymbol{v}(\boldsymbol{t}) = \boldsymbol{r}'(\boldsymbol{t})$$

Speed

$$\|\boldsymbol{v}(\boldsymbol{t})\| = \|\boldsymbol{r}'(\boldsymbol{t})\| = \frac{\boldsymbol{ds}}{\boldsymbol{dt}}$$

Acceleration Vector

$$\boldsymbol{a}(\boldsymbol{t}) = \boldsymbol{r}'^{\prime(t)} = \boldsymbol{a_T}\boldsymbol{T}(\boldsymbol{t}) + \boldsymbol{a_N}\boldsymbol{N}(\boldsymbol{t})$$

$$= \frac{\boldsymbol{d^2s}}{\boldsymbol{dt^2}}\boldsymbol{T}(\boldsymbol{t}) + \boldsymbol{K}\left(\frac{\boldsymbol{ds}}{\boldsymbol{dt}}\right)^2 \boldsymbol{N}(\boldsymbol{t})$$

Components of Acceleration

$$a_T = a \cdot T = \frac{v \cdot a}{\|v\|} = \frac{d^2 s}{dt^2}$$

$$a_N = a \cdot N = \frac{v \times a}{\|v\|} = \sqrt{\|a\|^2 - a_T{}^2}$$

Formulas for Curvature in the plane

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

Formulas for Curvature in the Plane or in Space

$$K = \|T'(s)\| = \|r''(s)\|$$

$$K = \left\|\frac{T'(t)}{r'(t)}\right\| = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$$K = \frac{a(t) \cdot N(t)}{\|v\|^2}$$

Cross product formulas apply only to curves in space.

Definition of a Function of Two Variables

Let D be a set of ordered pairs of real numbers. If to each ordered pair (x, y) in D there corresponds a unique real number $f(x, y)$, then f is a **function of x and y**. The set D is the **domain** of f, and the corresponding set of values for $f(x, y)$ is the **range** of f. For the function

$$z = f(x, y)$$

x and y are called the **independent variables** and z is called the **dependent variable**.

Operations on Functions with Several Variables

Sum or Difference

$$(f \pm g)(x, y) = f(x, y) \pm g(x, y)$$

Product

$$(fg)(x, y) = f(x, y)g(x, y)$$

Quotient

$$\left(\frac{f}{g}\right)(x, y) = \frac{f(x, y)}{g(x, y)}, \quad g(x, y) \neq 0$$

Composition on Functions with Several Variables

$$(g \circ h)(x, y) = g\big(h(x, y)\big)$$

Open Disk

$$\{(x, y)\}: \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

Closed Disk

$$\{(x, y)\}: \sqrt{(x - x_0)^2 + (y - y_0)^2} \leq \delta$$

Definition of he Limit of a Function of wo Variables

Let f be a function of two variables defined, except possibly (x_0, y_0) and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

If for each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

Whenever

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

Definition of Continuity of a Function of Two Variables

A function f of two variables is **continuous at a point** (x_0, y_0) in an open region R if $f(x,y)$ is defined and is equal to the limit of $f(x,y)$ as (x,y) approaches (x_0, y_0). That is

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$$

The function f is **continuous in the open region R** if it is continuous at every point in R.

Continuous Function of Two Variables

If k is a real number $f(x,y)$ and $g(x,y)$ are continuous at (x_0, y_0), then the following functions are also continuous at (x_0, y_0).

Scalar Multiple:

$$kf$$

Sum or Difference:

$$f \pm g$$

Product:

$$fg$$

Quotient:

$$\frac{f}{g}, \quad g(x_0, y_0) \neq 0$$

Continuity of a Composite Function

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function given by $(g \circ h)(x,y) = g\big(h(x,y)\big)$ is continuous at (x_0, y_0) That is,

$$\lim_{(x,y)\to(x_0,y_0)} g\big(h(x,y)\big) = g(h(x_0, y_0)\,)$$

Open Sphere

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < \delta^2$$

Definition of Continuity of a Function of Three Variables

A function f of three variables is **continuous at a point** (x_0, y_0, z_0) is defined and is equal to the limit of $f(x,y,z)$ as (x,y,z) approaches (x_0, y_0, z_0). That is

$$\lim_{(x,y,z)\to\mathbf{t}\,(x_0,y_0,z_0)} f(x,y,z) = f(x_0, y_0, z_0)$$

The function f is **continuous in the open region** R if it is continuous at every point in R.

Definition of Partial Derivative of a Function of Two Variables

If $z = f(x,y)$, then the first **partial derivatives** of f with respect to x and y are the functions f_x and f_y defined by

Partial Derivative with respect to x

$$f_x = \lim_{\Delta x\to 0} \frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}$$

Partial Derivative with respect to y

$$f_y = \lim_{\Delta y\to 0} \frac{f(x, y + \Delta y) - f(x,y)}{\Delta y}$$

Provided the limits exists.

Notation for First Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

Partial derivative with respect to x

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z \frac{\partial z}{\partial x}$$

and

Partial Derivative with respect to y

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z \frac{\partial z}{\partial y}$$

The first partials evaluated at the point (a, b) are denoted by

$$\frac{\partial z}{\partial x} |(a, b) = f_x(x, y)$$

and

$$\frac{\partial z}{\partial y} |(a, b) = f_y(x, y)$$

Differentiate twice with respect to x

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

Differentiate twice with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Differentiate first with respect to x and then with respect to y

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

Differentiate first with respect to y and then with respect to x

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$$

Equality of Mixed Partial Derivatives

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk R, then for every (x, y) in R,

$$f_{xy}(x, y) = f_{yx}(x, y)$$

Increment of z

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

Definition of Total Differential

If $z = f(x, y)$ and Δx and Δy are increments of x and y, then the **differentials** of the independent variables x and y are

$$dx = \Delta x$$

And

$$dy = \Delta y$$

And the **total differential** of the dependent variable z is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = f_x(x, y) dx + f_y(x, y) dy$$

Definition of Differentiability

A function $\boldsymbol{f}$ given by $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$ is **differentiable** at $(\boldsymbol{x_0},\boldsymbol{y_0})$ if $\Delta\boldsymbol{z}$ can be written in the form

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

Where both $\boldsymbol{\varepsilon_1}$ and $\boldsymbol{\varepsilon_2} \to \mathbf{0}$ as

$$(\Delta x,\Delta y) \to (0,0)$$

The function $\boldsymbol{f}$ is **differentiable in a region $\boldsymbol{R}$** if it is differentiable at each point in $\boldsymbol{R}$.

Sufficient Condition for Differentiability

If $\boldsymbol{f}$ is a function of $\boldsymbol{x}$ and $\boldsymbol{y}$, where $\boldsymbol{f_x}$ and $\boldsymbol{f_y}$ are continuous in an open region $\boldsymbol{R}$, then $\boldsymbol{f}$ is differentiable on $\boldsymbol{R}$.

Differentiability Implies Continuity

If a function of $\boldsymbol{x}$ and $\boldsymbol{y}$ is differentiable at $(\boldsymbol{x_0},\boldsymbol{y_0})$, the it is continuous at $(\boldsymbol{x_0},\boldsymbol{y_0})$.

Chain Rule: One Independent Variable

Let $\boldsymbol{w} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$, where $\boldsymbol{f}$ is a differentiable function of $\boldsymbol{x}$ and $\boldsymbol{y}$. If $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{t})$ and $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{t})$, where $\boldsymbol{g}$ and $\boldsymbol{h}$ are differentiable functions of $\boldsymbol{t}$, then $\boldsymbol{w}$ is a differentiable function of $\boldsymbol{t}$, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Chain Rule: Two Independent Variables

Let $\boldsymbol{w} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})$, where $\boldsymbol{f}$ is a differentiable function of $\boldsymbol{x}$ and $\boldsymbol{y}$. If $\boldsymbol{x} = \boldsymbol{g}(\boldsymbol{s},\boldsymbol{t})$ and $\boldsymbol{y} = \boldsymbol{h}(\boldsymbol{s},\boldsymbol{t})$, such that the first partials $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, and $\partial y/\partial t$ all exists, then $\partial w/\partial s$ and $\partial w/\partial t$ exist and are given by

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial x}{\partial s}\frac{\partial y}{\partial s}$$

And

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial x}{\partial y}\frac{\partial y}{\partial t}$$

Chain Rule: Implicit Differentiation

If the equation $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{0}$ defines $\boldsymbol{y}$ as a differentiable function of $\boldsymbol{x}$, then

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)},\quad F_x(x,y) \neq 0$$

If the equation $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{0}$ defines $\boldsymbol{z}$ implicitly as a differentiable function of $\boldsymbol{x}$ and $\boldsymbol{y}$, then

$$\frac{\partial z}{\partial x} = \frac{F_x(x,y,z)}{F_z(x,y,z)}$$

And

$$\frac{\partial z}{\partial y} = \frac{F_y(x,y,z)}{F_z(x,y,z)},\quad F_z(x,y,z) \neq 0$$

Definition of Directional Derivative

Let $\boldsymbol{f}$ be a function of two variables $\boldsymbol{x}$ and $\boldsymbol{y}$ and let $\boldsymbol{u} = \cos(\theta)\,\boldsymbol{i} + \sin(\theta)\,\boldsymbol{j}$ be a unit vector. Then the **directional derivative of $\boldsymbol{f}$** in the direction of $\boldsymbol{u}$, denoted by $\boldsymbol{D_u f}(\boldsymbol{x},\boldsymbol{y})$ is

$$D_u f(x,y)$$

$$= \lim_{t\to 0}\frac{f\big(x + t\cos(\theta),\, y + t\sin(\theta) - f(x,y)\big)}{t}$$

Provided the limit exists.

Directional Derivative

If $\boldsymbol{f}$ is a differentiable function of $\boldsymbol{x}$ and $\boldsymbol{y}$, then the directional derivative of $\boldsymbol{f}$ in the direction of the unit vector $\boldsymbol{u} = \cos(\boldsymbol{\theta})\,\boldsymbol{i} + \sin(\boldsymbol{\theta})$ is

$$D_u f(x, y) = f_x(x, y)\cos(\theta) + f_y(x, y)\sin(\theta)$$

Definition of Gradient of a Function of Two Variables

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. Then the **gradient of f**, denoted by $\nabla f(x, y)$, is the vector

$$\nabla f(x, y) = f_x(x, y)i + f_y(x, y)j$$

(The symbol ∇f is read as 'del$f$.") Another notation for the gradient is given by **grad** $f(x, y)$. Note that for each (x, y), the gradient $\nabla f(x, y)$ is a vector in the plane (not a vector in space).

Alternative Form of the Directional Derivative

If f is a differentiable function of x and y, then the directional derivative of f in the direction of the unit vector u is

$$D_u f(x, y) = \nabla f(x, y) \cdot u$$

Properties of the Gradient

Let f be differentiable at the point (x, y).

1. If $\nabla f(x, y) = 0$, then $D_u f(x, y) = 0$ for all u.

2. The direction of *maximum increase* of f is given by $\nabla f(x, y)$. The maximum value of $D_u f(x, y)$ is

$$||\nabla f(x, y)||$$

3. The direction of *minimum increase* of f is given by $-\nabla f(x, y)$. The maximum value of $D_u f(x, y)$ is

$$-||\nabla f(x, y)||$$

Gradient is Normal to Level Curves

If f is differentiable at (x_0, y_0) and $\nabla f(x, y) \neq 0$, then $\nabla f(x, y)$ is normal to the level curve through (x_0, y_0).

Directional Derivative and Gradient for Three Variables

Let f be a function of x, y and z with continuous first partial derivatives. The **directional derivative of f** in the direction of a unit vector

$$u = ai + bj + ck$$

Is given by

$$D_u f(x, y, z)$$

$$= af(x, y, z) + bf_x(x, y, z) + cf_y(x, y, z)$$

The **gradient of f** is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)i + f_y(x, y, z)j + f_z(x, y, z)k$$

Properties of the gradient are as follows.

1. $D_u f(x, y, z) = \nabla f(x, y, z) \cdot u$

2. If $\nabla f(x, y, z) = 0$ then $D_u f(x, y, z) = 0$ for all u

3. The direction of *maximum increase* of f is given by $\nabla f(x, y, z)$. The maximum value of $D_u f(x, y, z)$ is

$$||\nabla f(x, y, z)||$$

4. The direction of *minimum increase* of f is given by $-\nabla f(x, y, z)$. The maximum value of $D_u f(x, y, z)$ is

$$-||\nabla f(x, y, z)||$$

Definitions of Tangent Plane and Normal Line

Let F be differentiable at the point $P(x_0, y_0, z_0)$ on the surface S given by $F(x, y, z) = 0$ such that

$$\nabla F(x, y, z) \neq 0$$

1. The plane through P that is normal to $\nabla F(x_0, y_0, z_0)$ is called the **tangent plane to S** at $\mathbf{P}$.

2. The line through $\mathbf{P}$ having the direction of $\nabla F(x_0, y_0, z_0)$ is called the **normal line to S** at P

Equation of Tangent Plane

If F is differentiable at (x_0, y_0, z_0), then an equation of the tangent plane to the surface given at $F(x, y, z)$ is

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Angle of inclination of a plane

$$\cos(\theta) = \frac{|n \cdot k|}{\|n\|\|k\|} = \frac{|n \cdot k|}{\|n\|}$$

Gradient is Normal to Level Surfaces

If F is differentiable at (x_0, y_0, z_0) and

$$\nabla F(x, y, z) \neq 0$$

Then $\nabla F(x, y, z) \neq 0$ is normal to the level surface through (x_0, y_0, z_0).

Extreme Value Theorem

Let f be a continuous function of two variables x and y defined on a closed bounded region R in the **xy-plane**.

1. There is least one point in R at which f takes on minimum value.

2. There is least one point in R at which f takes on maximum value.

Definition of Relative Extrema

Let f be a function defined on a region R containing (x_0, y_0).

1. The function f has a **relative minimum** at (x_0, y_0) if

$$f(x, y) \geq f(x_0, y_0)$$

For all (x, y) in *open disk* containing (x_0, y_0).

2. The function f has a **relative maximum** at (x_0, y_0) if

$$f(x, y) \leq f(x_0, y_0)$$

For all (x, y) in *open disk* containing (x_0, y_0).

Definition of Critical Point

Let f be defined on an open region R containing (x_0, y_0). The point (x_0, y_0) is a **critical point** of f if one of the following is true.

1. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$

2. $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist

Relative Extrema Occur Only at Critical Points

If f has a relative extremum at (x_0, y_0) on an open region R, then (x_0, y_0) is a critical point of f.

Second Partials Test

Let f have continuous second partial derivatives on an open region containing a point (a, b) for which

$$f_x(a, b) = 0$$

And

$$f_y(a, b) = 0$$

To test for relative extrema of f, consider the quantity

$$d = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

1. If $d > 0$ and $f_{xx}(a, b) > 0$, then f has a **relative minimum** at (a, b)
2. If $d > 0$ and $f_{xx}(a, b) < 0$, then f has a **relative maximum** at (a, b)
3. If $d < 0$, then $(a, b, f(a, b))$ is a **saddle point**.
4. The test is inconclusive if $d = 0$

Sum of the Squared errors

$$S = \sum_{i=1}^{n} [f(x_i - y_i)]^2$$

Least Squares Regression Line

The **least squares regression line** for $\{(x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)\}$ is given by $f(x) = ax + b$, where

$$a = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} {x_i}^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

And

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i - a \sum_{i=1}^{n} x_i\right)$$

Lagrange's Theorem

Let f and g have a continuous first partial derivative such that f has an extremum at point (x_0, y_0) on the smooth constraint curve $g(x, y) = c$
If $\nabla g(x_0, y_0) \neq 0 = \lambda \nabla g(x_0, y_0)$

Method of Lagrange Multipliers

Let $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfy the hypothesis of LaGrange's Theorem, and let $\boldsymbol{f}$ have a minimum or maximum subject to the constraint $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{c}$. To find the minimum or maximum of $\boldsymbol{f}$, use these steps.

1. Simultaneously solve the equations $\nabla \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{\lambda}\nabla \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{c}$ by solving the following system of equations.

$$f_x(x,y) = \lambda \nabla g(x,y)$$

$$f_y(x,y) = \lambda \nabla g_y(x,y)$$

$$g(x,y) = c$$

2. Evaluate $\boldsymbol{f}$ at each solution point obtained in the first step. The greatest value yields the maximum of $\boldsymbol{f}$ subject to the constraints $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{c}$, and the least value yields the minimum of $\boldsymbol{f}$ subject to the constraints $\boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{c}$

Area of a Region in the Plane

1. If $\boldsymbol{R}$ is defined by $\boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{b}$ and $\boldsymbol{g_1}(\boldsymbol{x}) \le \boldsymbol{y} \le \boldsymbol{g_2}(\boldsymbol{x})$,where $\boldsymbol{g_1}(\boldsymbol{x})$ and $\boldsymbol{g_2}(\boldsymbol{x})$ are continuous on $[\boldsymbol{a},\boldsymbol{b}]$, then the area of $\boldsymbol{R}$ is

Vertically simple

$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\ dx$$

2. If $\boldsymbol{R}$ is defined by $\boldsymbol{c} \le \boldsymbol{y} \le \boldsymbol{d}$ and $\boldsymbol{h_1}(\boldsymbol{y}) \le \boldsymbol{x} \le \boldsymbol{h_2}(\boldsymbol{y})$,where $\boldsymbol{h_1}(\boldsymbol{x})$ and $\boldsymbol{h_2}(\boldsymbol{x})$are continuous on $[\boldsymbol{c},\boldsymbol{d}]$, then the area of $\boldsymbol{R}$ is

Horizontally simple

$$A = \int_c^d \int_{h_1(x)}^{h_2(x)} dx\ dy$$

Definition of Double Integral

If $\boldsymbol{f}$ is defined on a closed, bounded region $\boldsymbol{R}$ in the **xy-plane**, then the **double integral of $\boldsymbol{f}$ over $\boldsymbol{R}$** is

$$\int_R \int f(x,y)\ dA = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i, y_i)\, \Delta A_i$$

Provided the limit exists. If the limit exists, then $\boldsymbol{f}$ is **integrable** over $\boldsymbol{R}$

Volume of a Solid Region

If $\boldsymbol{f}$ is integrable over a plane region $\boldsymbol{R}$ and $(\boldsymbol{x},\boldsymbol{y})$ in $\boldsymbol{R}$, then the volume of the solid region that lies above $\boldsymbol{R}$ and below the graph of $\boldsymbol{f}$ is

$$V = \int_R \int f(x,y)\, dA$$

Properties of Double Integrals

Let f and g be continuous over a closed, bounded plane region R, and let c be a constant.

$$\int_R \int cf(x,y)\,dA = c\int_R \int f(x,y)\,dA$$

$$\int_R \int [f(x,y) \pm g(x,y)]\,dA$$

$$= \int_R \int f(x,y)\,dA \pm \int_R \int g(x,y)\,dA$$

$$\int_R \int f(x,y)\,dA \geq 0, \quad if\ f(x,y)0$$

$$\int_R \int f(x,y)\,dA \geq \int_R \int g(x,y)\,dA,$$
$$if\ f(x,y) \geq g(x,y)$$

$$\int_R \int f(x,y)\,dA$$

$$= \int_{R_1} \int f(x,y)\,dA + \int_{R_2} \int f(x,y)\,dA,$$

Where R is the union of two non-overlapping sub-regions $R_1$and R_2.

Fubini's Theorem

Let f be continuous on a plane region R.

1. If R is defined by $a \leq x \leq b$ and $g_1(x) \leq x \leq g_2$, where $g_1$and g_2 are continuous on $[a, b]$, then

$$V = \int_R \int f(x,y)\,dA$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} dy \int_R \int f(x,y)\,dy\,dx$$

2. If R is defined by $c \leq x \leq d$ and $h_1(x) \leq x \leq h_2$, where $h_1$and h_2 are continuous on $[c, d]$, then

$$V = \int_R \int f(x,y)\,dA$$

$$= \int_c^d \int_{h_1(x)}^{h_2(x)} dy \int_R \int f(x,y)\,dx\,dy$$

Definition of the Average Value of a Function Over a Region

If f is integrable over the plane region R, then the **average value** of f over R is

$$Average\ value = \frac{1}{A}\int_R \int f(x,y)\,dA$$

Where A is the area of R.

Polar sectors

$$R = \{(r,\theta): r_1 \leq r \leq r_2, \theta_1 \leq \theta \leq \theta_2\}$$

Change of Variables to Polar Form

Let $\boldsymbol{R}$ be a plane region consisting of all points $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{r}\cos(\boldsymbol{\theta}),\ \boldsymbol{r}\sin(\boldsymbol{\theta}))$ satisfying the conditions $\mathbf{0} \leq \boldsymbol{g_1}(\boldsymbol{\theta}) \leq \boldsymbol{r} \leq \boldsymbol{g_2}(\boldsymbol{\theta}), \boldsymbol{\alpha} \leq \boldsymbol{\theta} \leq \boldsymbol{\beta}$, where $\mathbf{0} \leq (\boldsymbol{\beta} - \boldsymbol{\alpha}) \leq \mathbf{2}\boldsymbol{\pi}$. If $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ are continuous on $[\boldsymbol{\alpha}, \boldsymbol{\beta}]$ and $\boldsymbol{f}$ is continuous on $\boldsymbol{R}$, then

$$\int_R \int f(x,y)\, dA = \int_\alpha^\beta f(r\cos(\theta), r\sin(\theta)\, r)\, dr\, d\theta$$

Definition of Mass of a Planar Lamina of Variable Density

If $\boldsymbol{\rho}$ is a continuous density function on the lamina corresponding to a plane region $\boldsymbol{R}$, then the mass $\boldsymbol{m}$ of the lamina is given by

Variable Density

$$m = \int_R \int \rho(x,y)\, dA$$

Moments and Center of Mass of a Variable Density Planar Lamina

let $\boldsymbol{\rho}$ be a continuous density function on the planar lamina If $\boldsymbol{R}$. The **moments of mass** with respect to the **x-** and **y-axis** are

$$M_x = \int_R \int (y)\rho(x,y)\, dA$$

And

$$M_y = \int_R \int (x)\, \rho(x,y)\, dA$$

If $\boldsymbol{m}$ is the mass of the lamina, then the **center of mass** is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

If $\boldsymbol{R}$ represents a simple plane region rather than a lamina, then the point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is called the **centroid** of the region.

Definition of Surface Area

If $\boldsymbol{f}$ and its first partial derivatives are continuous on the closed region $\boldsymbol{R}$ in the **xy-plane**, then the **area of the surface** $\boldsymbol{z} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y})$ over $\boldsymbol{R}$ is defined as

$$Surface\ area = \int_R \int dS$$

$$= \int_R \int \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

Definition of Triple Integral

If $\boldsymbol{f}$ is continuous over a bounded solid region $\boldsymbol{Q}$, then the **triple integral of $\boldsymbol{f}$ over $\boldsymbol{g}$** is defined as

$$\iiint_Q f(x,y,z)\, dV = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} (x_i, y_i, z_i)\, \Delta V_i$$

Provided the limit exists. The **volume** of the solid region $\boldsymbol{Q}$ is given by

$$Volume\ of\ Q = \iiint_Q dV$$

Evaluation by Iterated Integrals

Let $\boldsymbol{f}$ be continuous on a solid region $\boldsymbol{Q}$ defined by

$$a \leq x \leq b$$

$$h_1(x) \leq y \leq h_2(x)$$

$$g_1(x,y) \leq z \leq g_2(x,y)$$

Where $\boldsymbol{h_1}, \boldsymbol{h_2}, \boldsymbol{g_1}$ and $\boldsymbol{g_2}$ are continuous functions. Then

$$\iiiint_Q f(x,y,z)\, dV$$

$$= \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\ dz\, dy\, dx$$

Triple Integral in Cylindrical form

$$\int\int_Q\int f(x,y,z)\,dV$$

$$= \int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos(\theta),r\sin(\theta))}^{h_2(r\cos(\theta),r\sin(\theta))} f(r\cos(\theta),r\sin(\theta),z)r\,dz\,dr\,d\theta$$

Triple Integrals in Spherical Coordinates

$$\int\int_Q\int f(x,y,z)\,dV$$

$$= \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\rho\sin(\phi)\cos(\theta),\rho\sin(\phi)\sin(\theta))\rho^2\ \sin(\phi)\,d\rho\,d\theta$$

Definition of the Jacobian

If $x = g(u,v)$ and $y = h(u,v)$, then the **Jacobian** of x and y with respect to u and v, denoted by $\partial(x,y)/\partial(u,v)$, is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Change of Variables for Double Integrals

Let R be a vertically or horizontally simple region in the **xy-plane**, and let S be a vertically or horizontally simple region in the **uv-plane**. Let T from S to R be given by $T(u,v) = (x,y) = \big(g(u,v),h(u,v)\big)$, where g and h have continuous first partial derivatives. Assume that T is one-to-one except possibly on the boundary of S. If f is continuous on R, and $\partial(x,y)/\partial(u,v)$ is non-zero on S, then

$$\int_R\int f(x,y)\,dx\,dy$$

$$= \int_S\int f\big(g(u,v),h(u,v)\big)\,|g(u,v),h(u,v)|du\,dv$$

Definition of Vector Field

A **vector field over a plane region** R is a function that assigns a vector $F(x,y)$ to each point in R.

A **vector field over a solid region Q in space** is a function $F(x,y,z)$ to each point in Q**.**

Definition of Inverse Square Field

Let $r(t) = x(t)i + y(t)j + z(t)k$ be a position vector. The vector field F is an **inverse square field** if

$$F(x,y,z) = \frac{k}{\|r\|^2}u$$

Where k is a real number and

$$u = \frac{r}{\|r\|}$$

Is a unit vector in the direction of r.

Definition of Conservative Vector Field

A vector field F is called **conservative** when there exists a differentiable function f such that $F = \nabla f$. The function f is called the **potential function** for F.

Test for Conservative Vector Field in the Plane

Let M and N have continuous first partial derivatives on an open disk R. The vector field $F(x,y) = Mi + Nj$ is conservative if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Definition of Curl of a Vector Field

The curl of $\boldsymbol{F}(x,y,z) = \boldsymbol{Mi} + \boldsymbol{Nj} + \boldsymbol{Pk}$ is

Curl

$$\boldsymbol{F}(x,y,z) = \nabla \times \boldsymbol{F}(x,y,z)$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k}$$

If curl $\boldsymbol{F} = \boldsymbol{0}$, then $\boldsymbol{F}$ is said to be **irrotational**.

Test for Conservation Vector Field in Space

Suppose that $\boldsymbol{M}, \boldsymbol{N}$, and $\boldsymbol{P}$ have continuous first partial derivatives in an open sphere $\boldsymbol{Q}$ in space. The vector field

$$\boldsymbol{F}(x,y,z) = \boldsymbol{Mi} + \boldsymbol{Nj} + \boldsymbol{Pk}$$

Is conservative if and only if

Curl $\boldsymbol{F}(x,y,z) = \boldsymbol{0}$.

That is, $\boldsymbol{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z},$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Definition of Divergence of a Vector Field

The **divergence** of $\boldsymbol{F}(x,y) = \boldsymbol{Mi} + \boldsymbol{Nj}$ is

Plane

$$div\ \boldsymbol{F}(x,y,z) = \nabla \boldsymbol{F}(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial x}$$

The **divergence** of $\boldsymbol{F}(x,y) = \boldsymbol{Mi} + \boldsymbol{Nj} + \boldsymbol{Pk}$ is

Space

$$div\ \boldsymbol{F}(x,y,z) = \nabla \boldsymbol{F}(x,y,z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial x} + \frac{\partial P}{\partial x}$$

If $div\ \boldsymbol{F} = \boldsymbol{0}$, then $\boldsymbol{F}$ is said to be **divergence free**.

Divergence and Curl

If $\boldsymbol{F}(x,y,z) = \boldsymbol{Mi} + \boldsymbol{Nj} + \boldsymbol{Pk}$ is a vector field and $\boldsymbol{M}, \boldsymbol{N}$ and $\boldsymbol{P}$ have continuous second partial derivatives, then

$$div(curl\ \boldsymbol{F}) = \boldsymbol{0}$$

Definition of Line Integral

If $\boldsymbol{f}$ is defined in a region containing a smooth curve $\boldsymbol{C}$ of finite length, then the **line integral of $\boldsymbol{f}$ along $\boldsymbol{C}$** is given by

Plane

$$\int_C f(x,y,z)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i,)\,\Delta s_i$$

Or

Space

$$\int_C f(x,y)\,ds = \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\,\Delta s_i$$

Provided the limit exists

Evaluation of a Line Integral as a Definite Integral

Let f be continuous in a region containing a smooth curve C. If C is given by $r(t) = x(t)i + y(t)j$, where $a \le t \le b$, then

$$\int_C f(x,y)\,ds$$

$$= \int_a^b f\big(x(t), y(t)\big)\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

If C **is given by** $r(t) = x(t)i + y(t)j + z(t)k$, where $a \le t \le b$, then

$$\int_C f(x,y,z)\,ds$$

$$= \int_a^b f\big(x(t), y(t), y(t)\big)\sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt$$

Definition of the Line Integral of a Vector Field

Let F be a continuous vector field defined on a smooth curve C given by

$$r(t), a \le t \le b$$

The **line integral** of F on C is given by

$$\int_C F \cdot dr = \int_C F \cdot T\ ds$$

$$= \int_a^b f\big(x(t), y(t), z(t)\big) \cdot r'(t)\,dt$$

Fundamental Theorem of Line Integrals

Let C be a piecewise smooth curve lying in an open region R and given by

$$r(t) = x(t)i + y(t)j, a \le t \le b$$

If $F(x,y) = Mi + Nj$ is conservative in R, and M and N are continuous in R, then

$$\int_C F \cdot dr = \nabla f \cdot dr = f\big(x(b), y(b)\big) - f\big(x(a), y(a)\big)$$

Where f is a potential function of F. That is $F(x,y) = \nabla f(x,y)$.

Independence of Path and Conservative Vector Fieldsl

If F is continuous on an open connected region, then the line integral

$$\int_C F \cdot dr$$

Is independent of path if and only if F is conservative.

Equivalent Conditions

Let $F(x,y,z) = Mi + Nj + Pk$ have continuous first partial derivatives in an open connected region R, and let C be a piecewise smooth curve in R. The conditions listed below are equivalent.

1. F is conservative. That is, $F = \nabla f$ for some function f.

2. $\int_C F \cdot dr$ is independent of path.

$\int_C F \cdot dr = 0$ for *every closed* curve C in R

Green's Theorem

Let $\boldsymbol{R}$ be a simply connected region with a piecewise smooth boundary $\boldsymbol{C}$, oriented counterclockwise (that is, $\boldsymbol{C}$ is traversed *once* so that the region $\boldsymbol{R}$ always lies to the left). If $\boldsymbol{M}$ and $\boldsymbol{N}$ have continuous first partial derivatives in an open region containing $\boldsymbol{R}$, then

$$\int_C M\,dx + N\,dy = \int_R \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

Line Integral for Area

If $\boldsymbol{R}$ is a plane region bounded by a piecewise smooth simple closed curve $\boldsymbol{C}$, oriented counterclockwise, then the area of $\boldsymbol{R}$ is given by

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

Definition of Parametric Surface

Let $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$ be functions of $\boldsymbol{u}$ and $\boldsymbol{v}$ that are continuous on a domain $\boldsymbol{D}$ in the **uv-plane**. The set of points $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ given by

Parametric Surface

$$r(uv) = x(u,v)i + y(u,v)j + z(u,v)k$$

Is called a **parametric surface**. The equations

Parametric equations

$$x = x(u,v), y = y(u,v), and\ z = z(u,v)$$

Are the **parametric equations** for the surface.

Normal Vector to a Smooth Parametric Surface

Let $\boldsymbol{S}$ be a smooth parametric surface

$$r(uv) = x(u,v)i + y(u,v)j + z(u,v)k$$

Defined over an open region $\boldsymbol{D}$ in the **uv-plane**. Let $(\boldsymbol{u_0}, \boldsymbol{v_0})$ be a point in $\boldsymbol{D}$. A normal vector at the point

$$(x_0, y_0, z_0) = \big(x(u_0, v_0), y(u_0, v_0), z(u_0, v_0)\big)$$

Is given by

$$N = r_u(u_0, v_0) \times r_v(u_0, v_0) = \begin{vmatrix} i & j & k \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

Area of a Parametric Surface

Let $\boldsymbol{S}$ be a smooth parametric surface

$$r(uv) = x(u,v)i + y(u,v)j + z(u,v)k$$

Defined over an open region $\boldsymbol{D}$ in the **uv-plane**. If each point on the surface $\boldsymbol{S}$ corresponds to exactly one point in the domain $\boldsymbol{D}$, then the **surface area** of $\boldsymbol{S}$ is given by

$$Surface\ area = \int_S \int dS = \|r_u \times r_v\|\, dA$$

Where

$$r_u = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k$$

And

$$r_v = \frac{\partial x}{\partial u} i + \frac{\partial y}{\partial u} j + \frac{\partial z}{\partial u} k$$

Evaluating a Surface Integral

Let $\boldsymbol{S}$ be a surface given by $\boldsymbol{z} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})$ and let $\boldsymbol{R}$ be its projection onto the **xy-plane**. If $\boldsymbol{g}, \boldsymbol{g_x}$ and $\boldsymbol{g_y}$ are continuous on $\boldsymbol{R}$ and $\boldsymbol{f}$ is continuous on $\boldsymbol{S}$, then the surface integral of $\boldsymbol{f}$ over $\boldsymbol{S}$ is

$$\int_S \int f(x, y, z)\ dS$$

$$= \int_R \int f\big(x, y, g(x, y)\big) \sqrt{1 + [g_x(x, y)]^2 + [g_x(x, y)]^2}\ dA$$

Definition of Flux Integral

Let $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \boldsymbol{Mi} + \boldsymbol{Nj} + \boldsymbol{Pk}$, where $\boldsymbol{M}, \boldsymbol{N}$, and $\boldsymbol{P}$ have continuous first partial derivatives on the surface $\boldsymbol{S}$ oriented by a unit normal vector $\boldsymbol{N}$. The **flux integral of $\boldsymbol{F}$ across $\boldsymbol{S}$** is given by

$$\int_S \int F \cdot N\ dS$$

Evaluating a Flux Integral

Let $\boldsymbol{S}$ be an oriented surface given by $\boldsymbol{z} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y})$ and let $\boldsymbol{R}$ be its projection onto the **xy-plane**.

Oriented upward

$$\int_S \int F \cdot N\ dS$$

$$= \int_R \int \big[-g_x(x, y)i - g_y(x, y)j + k\big]\ dA$$

Oriented downward

$$\int_S \int F \cdot N\ dS$$

$$= \int_R \int \big[g_x(x, y)i + g_y(x, y)j - k\big]\ dA$$

For the first integral, the surface is oriented upward, and for the second the integral, the surface is oriented downward.

Summary of Line Surface Integrals

Line Integrals

$$ds = \|r'(t)\|\,dt = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [x'(t)]^2}\,dt$$

Scalar Form

$$\int_C f(x,y,z)\ ds$$

$$= \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2 + [y'(t)]^2 + [x'(t)]^2}\,dt$$

Vector Form

$$\int_C F\cdot dr = \int_C F\cdot T\,ds$$

$$= \int_a^b f(x(t),y(t),z(t))\cdot r'(t)\ dt$$

Surface Integrals $[z = g(x,y)]$

$$dS = \sqrt{1 + [g_x(x,y)]^2 + [g_x(x,y)]^2}\,dA$$

Scalar Form

$$\int_S \int f(x,y,z)\ dS$$

$$= \int_R \int f(x,y,g(x,y))\sqrt{1 + [g_x(x,y)]^2 + [g_x(x,y)]^2}\,dA$$

Vector Form (upward normal)

$$\int_S \int F\cdot N\ dS$$

$$= \int_R \int [-g_x(x,y)i - g_y(x,y)j + k]\ dA$$

Surface Integrals (parametric form)

$$dS = \|r_u(uv)\times r_v(uv)\|\,dA$$

Scalar form

$$\int_S \int f(x,y,z)\ dS =$$

$$= \int_D \int f(x(u,v),y(u,v),z(u,v))\ \|r_u(u,v) \times r_v(uv)\|\,dA$$

Vector form (upward normal)

$$\int_S \int F\cdot N\ dS = \int_D \int F\cdot r_u \times r_v\ dA$$

The Divergence Theorem

Let Q be a solid region bounded by a closed surface S oriented by a unit normal vector directed outward from Q. If F is a vector fielded whose component functions have continuous first partial derivatives in Q, then

$$\int_S \int F\cdot N\ dS = \int\int_Q \int div\,F\ dV$$

Stokes's Theorem

Let S be an oriented surface with unit normal vector N, bounded by a piecewise smooth simple closed curve C with positive orientation. If F is a vector field whose components functions have continuous first partial derivatives on an open region containing S and C, then

$$\int_C \int F\cdot dr = \int\int_C \int (curl\,F)\cdot N\ ds$$

Summary of Integration Formulas

Fundamental Theorem of Calculus

$$\int_{a}^{b} f(x)\ dx = F(b) - F(a)$$

Green's Theorem

$$\int_{C} M\,dx + N\,dy = \int_{R}\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \int_{C} F \cdot ds\,dx = \int_{C} F \cdot dr = \int_{R}\int (curl\,F) \cdot k\ da$$

Divergence Theorem

$$\int_{S}\int F \cdot N\ dS = \int\int_{Q}\int div\,F\ dV$$

Fundamental Theorem of Line Integrals

$$\int_{C} F \cdot dr = \nabla f \cdot dr = f\big(x(b), y(b)\big) - f\big(x(a), y(a)\big)$$

Stokes's Theorem

$$\int_{C}\int F \cdot dr = \int\int_{C}\int (curl\,F) \cdot N\ ds$$