Section 7.5 — Solving Initial Value Problems

Recall, $\mathcal{L}[f'(t)\,;\,t\to s] = s\bar{f}(s) - f(0)$

and, $\mathcal{L}[f''(t)\,;\,t\to s] = s^2\bar{f}(s) - s f(0) - f'(0)$

Solve the following initial value problems:

1. $y' - 2y = e^{5t}$, $y(0) = 3$

Take L.T.'s : $\mathcal{L}[y'\,;\,t\to s] - 2\mathcal{L}[y\,;\,t\to s] = \mathcal{L}[e^{5t}\,;\,t\to s]$

$$s\bar{y} - y(0) - 2\bar{y} = \frac{1}{s-5}$$

$$\bar{y}(s-2) = 3 + \frac{1}{s-5} = \frac{3s-14}{s-5}$$

$$\therefore \bar{y} = \frac{3s-14}{(s-2)(s-5)} = \frac{A}{s-2} + \frac{B}{s-5}$$

$$\therefore \quad 3s-14 = A(s-5) + B(s-2)$$

Let $s = 2$: $\quad 6-14 = A(-3) \Rightarrow A = 8/3$

Let $s = 5$: $\quad 15-14 = B(3) \Rightarrow B = 1/3$

$$\therefore \bar{y} = \frac{8/3}{s-2} + \frac{1/3}{s-5}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{8}{3}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s-5}\,;\,s\to t\right] = \frac{8}{3}e^{2t} + \frac{1}{3}e^{5t}$$

2 $y'' - 2y' - 8y = 0$
$y(0) = 3\ ,\ y'(0) = 6$

Take L.T.'s : $\mathcal{L}[y'' ; t\to s] - 2\mathcal{L}[y' ; t\to s] - 8\mathcal{L}[y ; t\to s] = 0$

$$\therefore\ s^2\bar{y} - sy(0) - y'(0) - 2\left(s\bar{y} - y(0)\right) - 8\bar{y} = 0$$

$$s^2\bar{y} - 3s - 6 - 2s\bar{y} + 6 - 8\bar{y} = 0$$

$$\bar{y}\left(s^2 - 2s - 8\right) = 3s$$

$$\bar{y} = \frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$$

$$3s = A(s+2) + B(s-4)$$

Let $s = 4$: $12 = A(6) \Rightarrow A = 2$
Let $s = -2$: $-6 = B(-6) \Rightarrow B = 1$

$$\therefore\ \bar{y} = \frac{2}{s-4} + \frac{1}{s+2}$$

$$\therefore\ y(t) = \mathcal{L}^{-1}\left[2\,\frac{1}{s-4} + \frac{1}{s+2}\ ;\ s\to t\right]$$

$$= 2e^{4t} + e^{-2t}$$

3 $y'' + y = e^{-2t}\sin t$
$y(0) = 0\ ,\ y'(0) = 0$

Tak L.T.'s : $s^2\bar{y} - sy(0) - y'(0) + \bar{y} = \dfrac{1}{(s+2)^2+1}$

$$\bar{y}(s^2+1) = \frac{1}{s^2+4s+5}$$

$$\therefore \bar{y} = \frac{1}{(s^2+1)(s^2+4s+5)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4s+5}$$

$$\therefore 1 = (As+B)(s^2+4s+5) + (Cs+D)(s^2+1)$$

$$= As^3 + 4As^2 + 5As + Cs^3 + Ds^2$$
$$+ Bs^2 + 4Bs + 5B + Cs + D$$

$$= (A+C)s^3 + (4A+B+D)s^2$$
$$+ (5A+4B+C)s + (5B+D)$$

$$\therefore \left.\begin{array}{r} A+C=0 \\ 4A+B+D=0 \\ 5A+4B+C=0 \\ 5B+D=1 \end{array}\right\} \rightarrow D = 1-5B$$

$$\left.\begin{array}{r} A+C=0 \\ 4A+B+(1-5B)=0 \\ 5A+4B+C=0 \end{array}\right\} \quad \left.\begin{array}{r} A+C=0 \\ 4A-4B=-1 \\ 5A+4B+C=0 \end{array}\right\} \rightarrow C=-A$$

$$\left.\begin{array}{r} 4A-4B=-1 \\ 5A+4B-A=0 \end{array}\right\} \quad \left.\begin{array}{r} 4A-4B=-1 \\ 4A+4B=0 \end{array}\right\} \rightarrow B=-A$$

$$\left.4A+4A=-1\right\} \Rightarrow A = -1/8$$

$B = 1/8$
$C = -A = 1/8$
$D = 1 - 5B = 3/8$

$$\therefore \bar{y} = \frac{-\frac{1}{8}s + \frac{1}{8}}{s^2+1} + \frac{\frac{1}{8}s + \frac{3}{8}}{s^2+4s+5}$$

$$= -\frac{1}{8}\frac{s}{s^2+1} + \frac{1}{8}\frac{1}{s^2+1} + \frac{1}{8}\frac{(s+3)}{s^2+4s+5}$$

$$= -\frac{1}{8}\frac{s}{s^2+1} + \frac{1}{8}\frac{1}{s^2+1} + \frac{1}{8}\frac{(s+2)+1}{(s+2)^2+1}$$

$$= -\frac{1}{8}\frac{s}{s^2+1} + \frac{1}{8}\frac{1}{s^2+1} + \frac{1}{8}\frac{s+2}{(s+2)^2+1} + \frac{1}{8}\frac{1}{(s+2)^2+1}$$

$$\therefore y(t) = -\frac{1}{8}\cos t + \frac{1}{8}\sin t + \frac{1}{8}e^{-2t}\cos t + \frac{1}{8}e^{-2t}\sin t$$

$$= \frac{1}{8}(\sin t - \cos t) + \frac{1}{8}e^{-2t}(\cos t + \sin t)$$

<u>Example 4.</u> Solve the IVP $y'' - 2y' + y = 12t$
$y(0) = 4, \; y'(0) = 1$

<u>Solution</u> Take L.T.'s : $\mathcal{L}[y'' - 2y' + y; t\to s] = \mathcal{L}[12t; t\to s]$

Recall, $\mathcal{L}[t^n; t\to s] = \frac{n!}{s^{n+1}}, \; s > 0$

$$s^2\bar{y} - sy(0) - y'(0) - 2(s\bar{y} - y(0)) + \bar{y} = 12\frac{1}{s^2}$$

$$s^2\bar{y} - 4s - 1 - 2s\bar{y} + 8 + \bar{y} = 12/s^2$$

$$\bar{y}(s^2-2s+1) = \frac{12}{s^2} + 4s - 7 = \frac{12+4s^3-7s^2}{s^2}$$

$$\therefore \bar{y} = \frac{12+4s^3-7s^2}{s^2(s-1)^2} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{(s-1)^2} + \frac{D}{s-1}$$

$$4s^3-7s^2+12 = A(s-1)^2 + B(s-1)^2 s + Cs^2 + D(s-1)s^2$$
$$= A(s^2-2s+1) + B(s^3-2s^2+s)$$
$$+ Cs^2 + D(s^3-s^2)$$
$$= s^3(B+D) + s^2(A-2B+C-D)$$
$$+ s(-2A+B) + A$$

s^3 : $B+D=4$ → $24+D=4 \Rightarrow D=-20$

s^2 : $A-2B+C-D=-7$

s : $-2A+B=0$ → $B=24$

s^0 : $A=12$ → $A=12$

$12-48+C+20=-7$

$-16+C=-7$

$C=9$

$$\therefore \bar{y} = \frac{12}{s^2} + \frac{24}{s} + \frac{9}{(s-1)^2} - \frac{20}{(s-1)}$$

$$\therefore y(t) = \mathcal{L}^{-1}\left[12\frac{1}{s^2} + 24\frac{1}{s} + 9\frac{1}{(s-1)^2} - 20\frac{1}{s-1} ; s \to t\right]$$

$$= 12t + 24 + 9te^t - 20e^t$$

HW Page 382, #'s 1-21 odd

Inverse Form of the First Translation Theorem

$$\mathcal{L}[e^{at} f(t); t \to s] = \mathcal{L}[f(t); t \to s-a] = F(s-a)$$

$$\therefore \mathcal{L}^{-1}[F(s-a); s \to t] = e^{at} f(t) = e^{at} \mathcal{L}^{-1}[F(s); s \to t]$$

$$\therefore \mathcal{L}^{-1}[F(s-a); s \to t] = e^{at} \mathcal{L}^{-1}[F(s); s \to t]$$

Examples

① $$\mathcal{L}^{-1}\left[\frac{24}{(s+3)^5}; s \to t\right] = e^{-3t} \mathcal{L}^{-1}\left[\frac{24}{s^5}; s \to t\right]$$

$$= e^{-3t} t^4$$

② $$\mathcal{L}^{-1}\left[\frac{9}{(s-4)^2 + 81}; s \to t\right] = e^{4t} \mathcal{L}^{-1}\left[\frac{9}{s^2 + 9^2}; s \to t\right]$$

$$= e^{4t} \sin(9t)$$

③ $$\mathcal{L}^{-1}\left[\frac{s+2}{s^2 + 8s + 20}; s \to t\right] = \mathcal{L}^{-1}\left[\frac{(s+4) - 2}{(s+4)^2 + 4}; s \to t\right]$$

$$= e^{-4t} \mathcal{L}^{-1}\left[\frac{s-2}{s^2+4}; s \to t\right]$$

$$= e^{-4t} \mathcal{L}^{-1}\left[\frac{s}{s^2+2^2} - \frac{2}{s^2+2^2}; s \to t\right]$$

$$= e^{-4t}(\cos 2t + \sin 2t)$$