Theorem: Sum of a Geometric Sequence

For any real real number r excdept 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$

Proof (by mathematical induction)

Suppose r is a particular but arbitrarily chosen real number that is not equal to 1, and let the property $P(n)$ be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1} \qquad \longleftarrow P(n)$$

We must show that $P(n)$ is true for all integers $n \geq 0$. We do this by mathematical induction on n.

Show that $P(0)$ is true:

To establish $P(0)$, we must show that

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1} \qquad \longleftarrow P(0)$$

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

also because $r^1 = r$ and $r \neq 1$. Hence $P(0)$ is true

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k+1)$ is also true:

[Suppose that $P(k)$is true for a particular but arbitrarily chosen integer $k \geq 0$. That is:]

Let k be any integer with $k \geq 0$, and suppose that

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1}-1}{r-1} \qquad \longleftarrow P(k) \text{ inductive hypothesis}$$

[We must show that $P(k+1)$ is true. That is:] We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1}-1}{r-1}$$

or equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2}-1}{r-1}$$

[We will show that the left-hand side of $P(k+1)$ equals the right-hand side.] The left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=0}^{k+1} r^i &= \sum_{i=0}^{k} r^i + r^{k+1} && \text{by writing the } (k+1)\text{st term separtely from the first } k \text{ terms} \\
&= \frac{r^{k+1}-1}{r-1} + r^{k+1} && \text{by substitution from the inductive hypothesis} \\
&= \frac{r^{k+1}-1}{r-1} + \frac{r^{k+1}(r-1)}{r-1} \\
&= \frac{(r^{k+1}-1)r^{k+1}(r-1)}{r-1} \\
&= \frac{(r^{k+1}-1)r^{k+2}-(r^{k+1})}{r-1} \\
&= \frac{r^{k+2}-1}{r-1}
\end{aligned}$$

Which is the right-hand side of $P(k+1)$