Section 7.3 Properties of the Laplace Transform

The First Translation Theorem

$$\mathcal{L}[e^{at} f(t) ; t \to s] = \mathcal{L}[f(t) ; t \to s-a]$$

Proof $\mathcal{L}[e^{at} f(t) ; t \to s]$

$$= \int_0^\infty e^{at} f(t) e^{-st} \, dt$$

$$= \int_0^\infty f(t) e^{-(s-a)t} \, dt$$

$$= \mathcal{L}[f(t) ; t \to s-a]$$

Examples (1) $\mathcal{L}[e^{at} t^n ; t \to s]$, $n = 1, 2, 3, \ldots$

$$= \mathcal{L}[t^n ; t \to s-a]$$

$$= \frac{n!}{(s-a)^{n+1}} , \quad s-a > 0$$

(2) $\mathcal{L}[e^{at} \sin bt ; t \to s]$

$$= \mathcal{L}[\sin bt ; t \to s-a]$$

$$= \frac{b}{(s-a)^2 + b^2} , \quad s-a > 0$$

(3) $\mathcal{L}[e^{at} \cos bt ; t \to s]$

$$= \mathcal{L}[\cos bt ; t \to s-a]$$

$$= \frac{s-a}{(s-a)^2 + b^2} , \quad s-a > 0$$

Transforms of Derivatives

Proposition 1 $\mathcal{L}[f'(t);t\to s] = s\bar{F}(s) - f(0)$

Proof $\mathcal{L}[f'(t);t\to s] = \int_0^\infty f'(t)e^{-st}\,dt$

$$= \lim_{N\to\infty}\int_0^N f'(t)e^{-st}\,dt \qquad u=e^{-st},\ dv=f'(t)\,dt$$
$$du=-se^{-st}dt,\ v=f(t)$$

$$= \lim_{N\to\infty}\left\{\left[e^{-st}f(t)\right]_{t=0}^{t=N} - \int_0^N -se^{-st}f(t)\,dt\right\}$$

$$= \lim_{N\to\infty}\left\{e^{-sN}f(N) - f(0) + s\int_0^N e^{-st}f(t)\,dt\right\}$$

$$= 0 - f(0) + s\int_0^\infty e^{-st}f(t)\,dt\,,\quad s>0$$

$$= s\bar{F}(s) - f(0)$$

Proposition 2 $\mathcal{L}[f''(t);t\to s] = s^2\bar{F}(s) - sf(0) - f'(0)$.

Proof $\mathcal{L}[f''(t);t\to s]$

$$= \mathcal{L}\left[\frac{d}{dt}f'(t);t\to s\right]$$

$$= s\,\mathcal{L}[f'(t);t\to s] - f'(0)$$

$$= s\left[s\bar{F}(s) - f(0)\right] - f'(0)$$

$$= s^2\bar{F}(s) - sf(0) - f'(0)$$

<u>Example</u> Apply the result of Proposition 1 to the function $f(t) = \sin(bt)$.

<u>Solution</u> $\mathcal{L}[f'(t);\, t \to s] = s\,F(s) - f(0)$

$$\mathcal{L}[b\cos(bt);\, t \to s] = s\,\frac{b}{s^2+b^2} - \sin(0)$$

$$\therefore\ b\,\mathcal{L}[\cos(bt);\, t \to s] = \frac{bs}{s^2+b^2}$$

$$\therefore\ \mathcal{L}[\cos(bt);\, t \to s] = \frac{s}{s^2+b^2}$$

<u>Derivatives of Transforms</u>

Recall, in Example 2 ⓔ of Section 7.2 it was established that

$$\frac{d^n}{ds^n} e^{-st} = (-1)^n t^n e^{-st}$$

$$\therefore\ (-1)^n \frac{d^n}{ds^n} e^{-st} = (-1)^{2n} t^n e^{-st} = t^n e^{-st}$$

Hence, $\mathcal{L}[t^n f(t);\, t \to s]$

$$= \int_0^\infty t^n f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t) \cdot (-1)^n \frac{d^n}{ds^n} e^{-st}\, dt$$

$$= (-1)^n \int_0^\infty \frac{d^n}{ds^n}\left(f(t)\, e^{-st}\right) dt$$

$$= (-1)^n \frac{d^n}{ds^n} \int_0^\infty f(t)\, e^{-st}\, dt$$

$$= (-1)^n \frac{d^n}{ds^n} F(s).$$

$$\therefore \mathcal{L}\left[t^n f(t)\,;\, t \to s\right] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\left[f(t)\,;\, t \to s\right]$$

<u>Example</u> $\mathcal{L}\left[t^2 \sin 3t\,;\, t \to s\right]$

$$= (-1)^2 \frac{d^2}{ds^2} \mathcal{L}\left[\sin 3t\,;\, t \to s\right]$$

$$= \frac{d}{ds}\frac{d}{ds}\frac{3}{(s^2+9)}$$

$$= \frac{d}{ds}\frac{d}{ds} 3(s^2+9)^{-1}$$

$$= \frac{d}{ds} -3(s^2+9)^{-2}\, 2s$$

$$= \frac{d}{ds}\frac{-6s}{(s^2+9)^2}$$

$$= \frac{(s^2+9)^2(-6) - (-6s)\,2(s^2+9)\,2s}{(s^2+9)^4}$$

$$= \frac{-6s^2 - 54 + 24s^2}{(s^2+9)^3}$$

$$= \frac{18(s^2-3)}{(s^2+9)^3}$$

<u>HW</u> Page 365
#'s 1–9, 13, 17, 23, 25