Theorem: Properties of Inverse Matrices

If A is an invertible matric, k is a positive integer, and c is a nonzero scalar, then A^{-1}, A^k, cA, and A^T are invertible and the statements below are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}$
4. $(A^T)^{-1} = (A^{-1})^T$

Proof

The key to the proofs of Properties 1,3, and 4 is the fact that the inverse of a matrix is unique. That is, if $BC = CB = I$, then C is the inverse of B.

Property 1 states that the inverse of A^{-1} is A itself. To prove this, observe that $A^{-1}A = AA^{-1} = I$, which means that A is the inverse of A^{-1}. Thus $A = (A^{-1})^{-1}$.

Similarily, Property 3 states that $\frac{1}{c}A^{-1}$ is the inverse of cA, $c \neq 0$. To prove this, use the properties of scalar multiplication.

$$(cA)\left(\frac{1}{c}A^{-1}\right) = \left(c\frac{1}{c}AA^{-1}\right) = (1)I = I$$
$$\left(\frac{1}{c}A^{-1}\right)(cA) = \left(c\frac{1}{c}AA^{-1}\right)A^{-1}A = (1)I = I$$

So $\frac{1}{c}A^{-1}$ is the inverse of (cA), which implies that $(cA)^{-1} = \frac{1}{c}A^{-1}$.