Chapter 5. Distributions of Functions of Random Variables

5.1. Functions of One Random Variable

1. The Chain Rule is used to take the derivative of a composite function. It has two versions:

2. Chain Rule (version 1):

$$(f(g(x)))' = f'(g(x))g'(x).$$

3. Chain Rule (version 2): If $y = y(u)$, $u = u(x)$, then $y = y(x)$, and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

4. *Fundamental Theorem of Calculus.* If

$$F(x) = \int_a^x f(t)dt,$$

then

$$F'(x) = f(x).$$

5. *Formula.* If

$$F(x) = \int_b^{\alpha(x)} f(t)dt,$$

then

$$F'(x) = f(\alpha(x))\alpha'(x).$$

(a) *Example.* If

$$F(x) = \int_1^x \cos t dt,$$

then

$$F'(x) = \cos x.$$

(b) *Example.* If

$$F(x) = \int_1^{\sin x} \cos t dt,$$

then, by the formula on the last page,

$$F'(x) = \cos(\sin x) \cdot (\sin x)' = \cos(\sin x) \cdot \cos(x).$$

6. *Formula.* If

$$F(x) = \int_{\beta(x)}^{\alpha(x)} f(t)dt,$$

then

$$F'(x) = f(\alpha(x))\alpha'(x) - f(\beta(x))\beta'(x).$$

7. *Example.* If $W \sim U(-\pi/2, \pi/2)$ and $X = \tan W$, then X has pdf

$$g(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty.$$

This distribution of X is called the Cauchy distribution.

— *Proof.* The pdf of W is

$$f(w) = \frac{1}{\pi}, \qquad -\frac{\pi}{2} < w < \frac{\pi}{2}.$$

Since $X = \tan W$, we have $W = \arctan X$. Denote by $G(x)$ the cdf of X. Then, for every real number x, we have

$$\begin{aligned} g(x) &= \frac{d}{dx} G(x) \\ &= \frac{d}{dx} P(X \leq x) \\ &= \frac{d}{dx} P(\tan W \leq x) \\ &= \frac{d}{dx} P(W \leq \arctan x) \end{aligned}$$

If follows that

$$\begin{aligned} g(x) &= \frac{d}{dx}\int_{-\pi/2}^{\arctan x} f(w)dw \\ &= \frac{d}{dx}\int_{-\pi/2}^{\arctan x} \pi^{-1}dw \\ &= \pi^{-1}\frac{d}{dx}\int_{-\pi/2}^{\arctan x} dw \\ &= \pi^{-1}\frac{d}{dx}\left(\arctan x + \frac{\pi}{2}\right) \\ &= \pi^{-1}\frac{d}{dx}(\arctan x) \\ &= \pi^{-1}\frac{1}{1+x^2}. \end{aligned}$$

QED.

8. *Formula.* If

(a) X is a continuous random variable with support (a, b);

(b) $f(x)$ is the pdf of X;

(c) $u : (a, b) \to (c, d)$ is a bijection;

(d) u is either strictly increasing or strictly decreasing;

(e) $v : (c, d) \to (a, b)$ is the inverse function of u; and

(f) $Y = u(X)$;

then, Y is a continuous random variable; the support of Y is (c, d); and the pdf of Y is

$$g(y) = f(v(y))|v'(y)|, \quad c < y < d.$$

9. *Example.* Suppose that X is a continuous random variable and $X \sim U(0,1)$. Find the pdf of $Y = 3X$.

* **Solution:** In this example, it is clear that X has support $(0,1) \subset \mathbb{R}$, and X has pdf

$$f(x) = 1, \quad 0 < x < 1.$$

We now make a summary of some other information from the problem:

$$u(x) = 3x,$$

$$\text{the inverse function of } u(x) \text{ is } v(y) = \frac{y}{3};$$

$$u : (0,1) \to (0,3) \text{ is a bijection;}$$

$$v : (0,3) \to (0,1) \text{ is also a bijection;}$$

$$\text{both } u(x) \text{ and } v(y) \text{ are increasing functions;}$$

$$X \text{ has support } (0,1);$$

$$Y \text{ has support } (0,3).$$

Now we put all this information into the formula $g(y) = f(v(y))|v'(y)|$, we get that the pdf of Y is

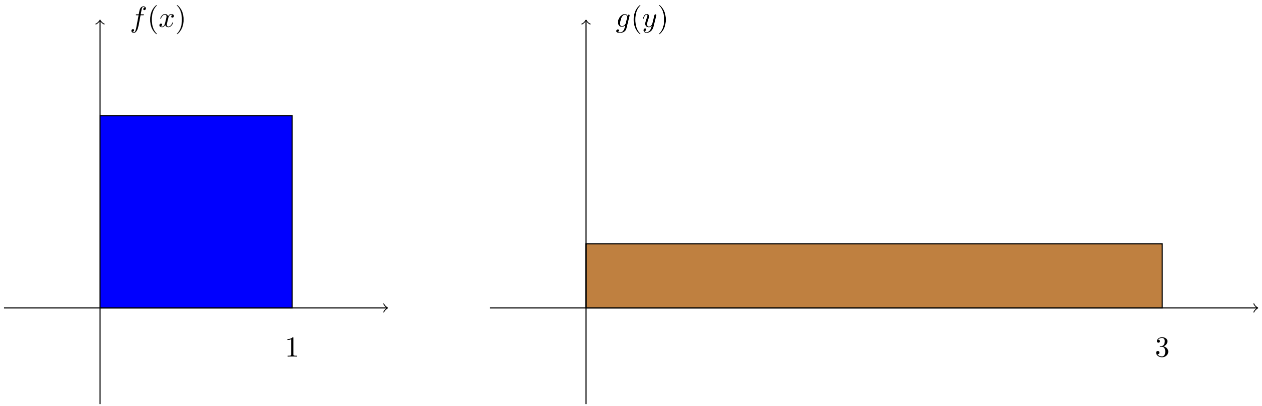
$$g(y) = f(v(y))|v'(y)| = f(y/3)|(y/3)'|, \quad 0 < y < 3, \tag{1}$$

Since $f \equiv 1$ and $(y/3)' = 1/3$, the expression fro $g(y)$ reduces to

$$g(y) = \frac{1}{3}, \quad 0 < y < 3.$$

This means that $Y = 3X \sim U(0,3)$.

The pdfs of X are Y are plotted here.



Note that, since $Y = u(X) = 3X$, the base (or support, or range) of Y is three times the base of X, and, accordingly, the density of Y is only $1/3$ of the density of X, to ensure that the total mass of the distribution is still one.

10. *Example.* Suppose that X is a continuous random variable and $X \sim U(0,1)$. Find the pdf of $Y = 2X + 3$. This example is left to the class.

11. *Example.* If X has pdf

$$f(x) = 2 - 2x, \quad 0 < x < 1,$$

and $Y = X^2$, find the pdf of Y.

* Here is the solution: Since $u(x) = x^2$, its inverse function is

$$v(y) = \sqrt{y}.$$

It is clear that X has support $A = (0,1)$, and Y has support $B = (0,1)$.

$u : A \to B$ is a bijection, and $v : B \to A$ is a also a bijection. Both u and v are increasing functions.

It is clear that $v'(y) = \frac{1}{2\sqrt{y}}$. Hence, the pdf of Y is

$$g(y) = f(v(y))|v'(y)| = (2 - 2v(y))\frac{1}{2\sqrt{y}} = \frac{1 - v(y)}{\sqrt{y}}$$

which simplifies to

$$g(y) = \frac{1}{\sqrt{y}} - 1, \quad 0 < y < 1.$$

Here, again, following our convention, only the nontrivial part of the distribution of Y is given here.

12. *Theorem.* Let X be a continuous random variable. Denote the pdf of X by $f(x)$, $-\infty < x < \infty$. If $Y = X^2$, then the pdf of Y is

$$g(y) = \frac{1}{2\sqrt{y}} \left(f(\sqrt{y}) + f(-\sqrt{y}) \right), \quad 0 < y < \infty.$$

Here, following our convention, only the nontrivial part of the distribution of Y is given here.

* Note that, this time, $f(x)$ is defined over the entire real line. Now the trouble is, the square function over the real line is not a bijection. So, this time, we have to go back and use the basic principles to derive the formula.

* **Proof.** We denote by $F(x)$ the cdf of X, and denote by $G(y)$ the cdf of Y. Since $Y = X^2$, Y cannot take negative values. Hence,

$$g(y) = 0, \quad y < 0,$$

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So, let us forget about the trivial part of the distribution for a while, and focus on the nontrival part of the distribution of Y. If $y > 0$, then

$$G(y) = P(Y \leq y) = P(X^2 \leq y).$$

It follows that

$$G(y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx.$$

Hence, for $y > 0$,

$$g(y) = G'(y) = \frac{d}{dy}\int_{-\sqrt{y}}^{\sqrt{y}} f(x)dx$$

that is,

$$\begin{aligned} g(y) &= f(\sqrt{y})(\sqrt{y})' - f(-\sqrt{y})(-\sqrt{y})' \\ &= f(\sqrt{y})(\sqrt{y})' + f(-\sqrt{y})(\sqrt{y})' \\ &= (f(\sqrt{y}) + f(-\sqrt{y}))(\sqrt{y})' \\ &= \frac{1}{2\sqrt{y}}(f(\sqrt{y}) + f(-\sqrt{y})). \end{aligned}$$

13. In the special case when $f(y)$ is an even function, we have the following theorem:

14. *Theorem.* Let X be a continuous random variable. Denote the pdf of X by $f(x)$. Suppose that the function $f(x)$ is an even function, that is,

$$f(x) = f(-x), \quad x \in \mathbb{R}.$$

If $Y = X^2$, then the pdf of Y is

$$g(y) = \frac{1}{\sqrt{y}} f(\sqrt{y}), \quad 0 < y < \infty.$$

15. *Theorem.* If $X \sim N(0, 1)$, then $X^2 \sim \chi_1^2$.

* **Proof.** Recall that the pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{x^2}{2}\right], \qquad -\infty < x < \infty.$$

It is clear that $f(x)$ is an even function. By Theorem, the pdf of $Y = X^2$ is

$$\begin{aligned} g(y) &= \frac{1}{\sqrt{y}} f(\sqrt{y}) \\ &= \frac{1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y/2} \\ &= \frac{y^{-1/2} e^{-y/2}}{\sqrt{\pi} 2^{-1/2}}, \qquad y > 0, \end{aligned}$$

which is precisely the χ_1^2 distribution.