Theorem 6.11

Let G be a graph of order $n \geq 3$. If for every integer j with $1 \leq j \leq \frac{n}{2}$, the number of vertices of G with degree at most j is less than j, then G is Hamiltonian.

Proof:

We show that $C(G)$ is complete. Assume, to the contray, that this is not the case. Among all pairs of non-adjacent vertices $C(G)$, let u, w be a pair for which $deg_{C(G)}u + deg_{C(G)}w$ is maximum. Necessarily, $deg_{C(G)}u + deg_{C(G)}w \leq n-1$. We may also assume that $deg_{C(G)}u \leq deg_{C(G)}w$. Let $deg_{C(G)}u = k$. Thus $k \leq \frac{n-1}{2}$ and so.

$$deg_{C(G)}w \leq n-k-1$$

Let W be the set of all vertices distinct from w that are not adjacent to w. Therefore, $u \in W$. Observe that if $v \in W$, then $deg_{C(G)}v \leq k$, for otherwise

$$deg_{C(G)}v + deg_{C(G)}w > deg_{C(G)} + deg_{C(G)}w$$

contradicting the defining property of the pair u, w. Therefore, the degree of every vertex of W is at most k. So by hypothesis, $|W| \leq k-1$. Hence

$$deg_{C(G)}w \geq (n-1)-(k-1) = n-k$$

which contradicts Theorem 6.1.