Theorem 5.16 (Menger's Theorem)

Let u and v be non-adjacent vertices in a graph G. The *minimum number* of vertices in a u—v separating set equals the maximum number of internally disjoint u—v paths in G.

Proof:

We proceed by induction on the size of the graph. Certainly, the result is true vacuously for all empty graphs. Assume that the result is true for all graphs of size *less* than m, where m is a positive integer, and let G be a graph of size m. Let u and v be two non-adjacent vertices of G. Suppose that there are k vertices in a minimum u—v spanning set. Certainly, G can contain no more than k internally disjoint u—v paths. We show, in fact, that G contains k internally disjoint u—v paths. Since the result is true for $k = 0$ and $k = 1$, we may assume that $k \geq 2$. We consider three cases.

Case 1. *There exists a minimum u—v separating set U in G containing a vertex x that is adjacent to both u and v.* Then the size of the subgraph $G - x$ is less than m and $U - \{x\}$ is a minimum u—v separating set in $G - x$ consisting of $k - 1$ vertices. By the induction hypothesis, there are $k - 1$ internally disjoint u—v paths in $G - x$. These paths together with the path (u, x, v) constitute k internally disjoint u—v paths in G.

Case 2. *There exists a minimum u—v separating set W in G containing a vertex in W that is not adjacent to u and a vertex in W that is not adjacent to v.* Let $W = \{w_1, w_2, \ldots, w_k\}$. Let G_u be the subgraph of G consisting of all u—w_i paths in G, where only $w_i \in W$ for each $i (1 \leq i \leq k)$ and let G'_u be the graph obtained from G_u by adding a new vertex v' and joining v' to each vertex w_i for $1 \leq i \leq k$. Let G_v and G'_v be defined similarly, where G'_v is obtained from G_v by adding the new vertex u'.

Since W contains a vertex that is not adjacent to u and a vertex that is not adjacent to v, the size of each of the graphs G'_u and G'_v is less than m. Since W is a minimum u—v separating set in G'_u, it follows by the induction hypothesis that G'_u contains k internally disjoint u—v' paths, each consisting of a u—w_i path P_i followed by the edge $w_i v'$. Similarly, there are k internally disjoint u'—v paths in G'_v, each consisting og a w_i—v path Q_i preceeded by the edge $u' w_i$. Since W is a u—v separating set in G, the two graphs G_u and G_v have only the vertices of W in common. Therefore, the k paths obtained by following P_i by Q_i for each $i (1 \leq i \leq k)$ are internally disjoint u—v paths in G.

Case 3. *For each minimum u—v separating set S in G, either every vertex of S is adjacent to u and not adjacent to v or every vertex of S is adjacent to v and not adjacent to u.* Let $P = (u, x, y, \ldots, v)$ be a u—v geodesic in G and let $e = xy$. Consider the subgraph $G - e$ in G. Certainly, every minimum u—v separating set in $G - e$ contains $k - 1$ vertices. We claim, in fact, that a minimum u—v separating set in $G - e$ contains k vertices, fro assume to the contrary, that $G - e$ contains a minimum u—v separating set $Z = \{z_1, z_2, \ldots, z_{k-1})$. Then $Z \cup \{x\}$ is a minimum u—v separating set in G. Since x is adjacent to v. On the other hand, $Z \cup \{y\}$ is a minimum u—v separating set in G and each $z_i (1 \leq i \leq k - 1)$ is adjacent to u. This implies that y is also adjacent to u, which contradicts the fact that P is a u—v geodesic. Therefore, as claimed, k is the minimum number of vertices in a u—v separating set in $G - e$. By the induction hypothesis, there are k internally disjoint u—v separating set in $G - e$. Hence there are k internally disjoint u—v paths in G as well.