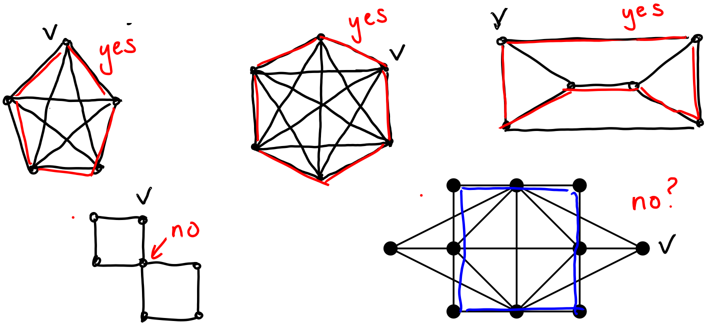
Section 6.2: Hamiltonian Graphs

could also say "closed walk"; then it's just like Eulerian circuit

Question. Can you find a cycle in the graphs below that visits every vertex exactly once?



Definition

A *Hamiltonian cycle* in a graph G is a cycle that contains every vertex. If G contains a Hamiltonian cycle, then G is *Hamiltonian.*

Hamiltonian cycles are much more interesting!

- Determining whether an arbitrary graph contains a Hamiltonian cycle is NP-complete
- We can say G is Eulerian if and only if G is connected and every vertex has even degree.
- There is no such characterization of Hamiltonian graphs - no (nontrivial) necessary and sufficient condition.
- But people really care about this problem! What should we do?
- Answer: Compile a list of necessary conditions, and compile a list of sufficient conditions.

Some easy necessary conditions

If G is Hamiltonian, then (BLANK).

- G is connected
- $\delta(G) = 2$ (if $G \neq K_1$)
- G is not a tree (unless $G = K_1$)
- G has no cut-vertex, i.e. G is 2-connected
- G has no cut-edge, i.e. G is 2-edge-connected
- Something about how degree 2 vertices interact (see blue cycle on slide 1)

none of these are sufficient.

- G contains C_n as a subgraph, where $n = |V(G)|$

This is actually a restatement of the definition, so it is necessary and sufficient

A more interesting necessary condition

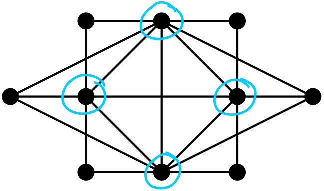
Theorem

If G is a Hamiltonian graph, then for every nonempty set $S \subseteq V(G)$, $G - S$ has at most $|S|$ components.

Before we prove it: What is the contrapositive?

If there exists a nonempty set $S \subseteq V(G)$ such that $G-S$ has more than $|S|$ components, then G is NOT Hamiltonian.

use this to argue convincingly that a graph is not Hamiltonian



Delete the blue vertices, 6 isolated vertices remain.

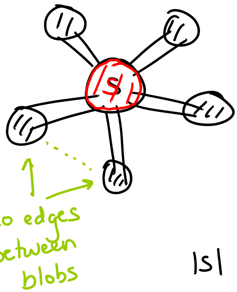
$|S| = 4$, but $G-S$ has 6 components, so no Hamiltonian cycle.

The proof

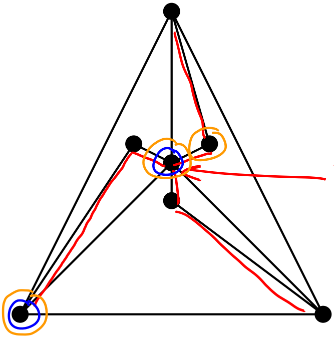
Theorem

If G is a Hamiltonian graph, then for every nonempty set $S \subseteq V(G)$, $G - S$ has at most $|S|$ components.

<u>Proof</u>: We prove the contrapositive. Suppose $G-S$ has at least $|S|+1$ components for some $S \subseteq V(G)$. A Hamiltonian cycle would visit the vertices of each component of $G-S$, and at least one vertex of S would be needed each time a component is entered. Since there are at least $|S|+1$ components and only $|S|$ vertices in S, no Hamiltonian cycle exists.



Why isn't this condition sufficient?



No Hamiltonian cycle, since degree 2 vertices would need both incident edges visited.

Degree 3, can't extend to a cycle.

However: The only vertex cuts of size 2 isolate a vertex, creating only 2 components (theorem doesn't apply)

Cuts of size 3 look like orange, they leave 3 components.

Once $|S|=4$, there are only 4 vertices in $G-S$, so there can't be more than 4 components.

SO: Not Hamiltonian, even though there IS NOT one of these sets S

Some easy sufficient conditions

If (BLANK), then G is Hamiltonian.

- $G = C_n$, or $G = K_n$, or $G = K_{n,n}$

- Conjecture: $G = G_1 \times G_2$, where G_1 and G_2 are Hamiltonian (Exploration assignment!)

- Conjecture: G is regular of even degree and 2-connected (Works for 2-regular, probably not in general)
  ? Exploration assignment!

A more interesting sufficient condition

Theorem (Ore's Theorem.)

Let G be a graph or order $n \geq 3$. If $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v, then G is Hamiltonian.

- Proof is by contradiction, using an *extremal example.*
- Among all counterexamples, choose one with the *most edges.* Call is G.
- Why? This means that G is not Hamiltonian, but if we add *any edge* to G, then G *becomes* Hamiltonian.
- So: G must at least contain a *path* containing all the vertices. (Called a *Hamiltonian path.*)