# Limits of Rational and Rational Form Functions

## Limit of a Rational Function

A *Rational Function* is a function formed by the quotient or ratio of two *Polynomial Functions*:

$$R(x) = \frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are each *Polynomial Functions*.

Referring to the topic document *Limit of a Function*, it is readily apparent that a *Rational Function* is formed by the arithmetic combination of “Basic Functions”. In addition, the domain of a *Rational Function* comprises all *Real* numbers except for values of $x$ where the denominator *Polynomial Function* ($Q(x)$ in the equation above) becomes zero.

From this we can conclude:

***Limit of a Rational Function***

If $P(x)$ and $Q(x)$ are *Polynomial Functions*, $R(x)$ is the *Rational Function* $P(x)/Q(x)$, and $c$ is a *Real* number such that $Q(c) \neq 0$, then:

$$\lim_{x\to c} R(x) = \lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{\lim\limits_{x\to c} P(x)}{\lim\limits_{x\to c} Q(x)} = \frac{P(c)}{Q(c)} = R(c)$$

For values of $c$ where $Q(c) = 0$, refer to the discussion below.

Note that a *Polynomial Function* is a *Rational Function* with the denominator equal to the constant 1 (which is, by definition, a polynomial); the domain of a *Polynomial Function* comprises all *Real* numbers.

When $Q(c) = 0$, direct substitution cannot be used to determine the limit of the *Rational Function*. The limit may or may not exist as follows:

1. **For $\boldsymbol{Q(c) = 0}$ and $\boldsymbol{P(c) \neq 0}$:** The limit does not exist. Graphically, there is a vertical asymptote at $x = c$ and the *Rational Function* tends to either $-\infty$ or $+\infty$ as the limit value is approached on either side.

2. **For $Q(c) = 0$ and $P(c) = 0$:** The limit is the indeterminate $0/0$. This means each polynomial has the factor $(x - c)$ occurring *at least* once. "Factoring out" every common pair of the factor $(x - c)$ from both polynomials will result in an "equivalent" *Rational Function* whose limit at $x = c$ is now resolvable by direct substitution and will be the same as the original *Rational Function* (per the *Functions That Agree at All But One Point Theorem*):

   a. If the resolved limit exists, we say there is a "hole" in the original *Rational Function* at $x = c$ (graphically the function looks "smooth and unbroken" in the neighborhood of $x = c$).

   b. If the resolved limit does not exist (a non-zero divided by zero), we simply have a vertical asymptote at $x = c$, as described in (1) above.

**Example 1**

$$L = \lim_{x \to -1} \frac{5x^3 + 4x - 5}{2x^2 - 2x + 1}$$

Directly substituting the limit value $x = -1$:

$$L = \frac{5(-1)^3 + 4(-1) - 5}{2(-1)^2 - 2(-1) + 1} = \frac{-14}{5} = -\frac{14}{5} \quad \blacksquare$$

**Example 2 (the "Student's Nightmare")**

$$L = \lim_{x \to 1} \frac{x^7 - 5x^6 + 7x^5 - x^4 + x^3 - 9x^2 + 7x - 1}{x^8 - 2x^7 + x^6 + 2x^5 - 4x^4 - x^3 + 8x^2 - 7x + 2}$$

Directly substituting the limit value $x = 1$:

$$L = \frac{1 - 5 + 7 - 1 + 1 - 9 + 7 - 1}{1 - 2 + 1 + 2 - 4 - 1 + 8 - 7 + 2} = \frac{0}{0}$$

Since the limit is indeterminate, we know that each polynomial in this *Rational Function* has the factor $(x - 1)$. Using either *Long Division* or *Synthetic Division*, we factor each polynomial in the *Rational Function* of the limit (the work behind the factoring is not shown):

$$L = \lim_{x \to 1} \frac{(x-1)(x^6 - 4x^5 + 3x^4 + 2x^3 + 3x^2 - 6x + 1)}{(x-1)(x^7 - x^6 + 2x^4 - 2x^3 - 3x^2 + 5x - 2)}$$

We can "factor out" the common pair of factors, resulting in the "equivalent" function whose limit at $x = 1$ is the same as the original *Rational Function* by the *Functions That Agree at All But One Point Theorem:*

$$L = \lim_{x \to 1} \frac{x^6 - 4x^5 + 3x^4 + 2x^3 + 3x^2 - 6x + 1}{x^7 - x^6 + 2x^4 - 2x^3 - 3x^2 + 5x - 2}$$

Again, directly substituting the limit value $x = 1$:

$$L = \frac{1 - 4 + 3 + 2 + 3 - 6 + 1}{1 - 1 + 2 - 2 - 3 + 5 - 2} = \frac{0}{0}$$

Oops, we again get an indeterminate result. This does not always happen. We simply repeat the process of factoring out the next common pair of $(x - 1)$ factors, and keep doing so if necessary, to resolve the limit:

$$L = \lim_{x \to 1} \frac{(x-1)(x^5 - 3x^4 + 2x^2 + 5x - 1)}{(x-1)(x^6 + 2x^3 - 3x + 2)}$$

$$L = \lim_{x \to 1} \frac{x^5 - 3x^4 + 2x^2 + 5x - 1}{x^6 + 2x^3 - 3x + 2}$$

Direct substitution as before:

$$L = \frac{1 - 3 + 2 + 5 - 1}{1 + 2 - 3 + 2} = \frac{4}{2} = 2$$

The limit is resolved and it exists, therefore, we conclude that:

$$L = \lim_{x \to 1} \frac{x^7 - 5x^6 + 7x^5 - x^4 + x^3 - 9x^2 + 7x - 1}{x^8 - 2x^7 + x^6 + 2x^5 - 4x^4 - x^3 + 8x^2 - 7x + 2} = 2 \quad \blacksquare$$

Graphing the original *Rational Function* indeed shows the limit as $x \to 1$ is 2 and that there is no asymptote or unusual anomaly at $x = 1$. Thus, there is a "hole" in the otherwise "smooth and unbroken" graph in the neighborhood of $x = 1$.