Example 7: As shown in Example 2, the function $\phi(x) = c_1 e^{-x} + c_2 e^{2x}$ is a solution to

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that the initial conditions

$$y(0) = 2 \quad \text{and} \quad \frac{dy}{dx}(0) = -3$$

are satisfied.

Solution

To determine the constants c_1 and c_2, we first compute $\frac{d\phi}{dx}$ to get $\frac{d\phi}{dx} = -c_1 e^{-x} + 2c_2 e^{2x}$. Substituting in our initial conditions gives the following systems of equations:

$$\begin{cases} \phi(0) = c_1 e^0 + c_2 e^0 = 2 \\ \dfrac{d\phi}{dx}(0) = -c_1 e^0 + 2c_2 e^0 = -3 \end{cases} \quad \text{or} \quad \begin{cases} c_1 + c_2 = 2 \\ -c_1 + 2c_2 = -3 \end{cases}$$

Adding the last two equations yields $3c_2 = -1$, so $c_2 = -\frac{1}{3}$. Since $c_1 + c_2 = 2$, we find $c_1 = \frac{7}{3}$. Hence, the solution to the initial value problem is $\phi(x) = (\frac{7}{3})e^{-x} - (\frac{1}{3})e^{2x}$.