Theorem: Convergence of Taylor Series

If $\lim_{n\to\infty} R_n = 0$ for all x in the interval$I$, then the Taylor Series for f converges and equals $f(x)$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Proof:

For a Taylor series, the n^{th} partial sum coincides with the n^{th} Taylor polynomial. That is, $S_n(x) = P_n(x)$. Because

$$P_n(x) - R_n(x)$$

if follows that

$$\begin{aligned} \lim_{n\to\infty} S_n(x) &= \lim_{n\to\infty} P_n(x) \\ &= \lim_{n\to\infty} [f(x) - R_n(x)] \\ &= f(x) - \lim_{n\to\infty} R_n(x) \end{aligned}$$

So, for a given x, the Taylor series (the sequence of partial sums) converges to $f(x)$ if and only if $R_n(x) \to 0$ as $n \to \infty$.