Trigonometric Substitution

For Integrands Containing Certain Radicals

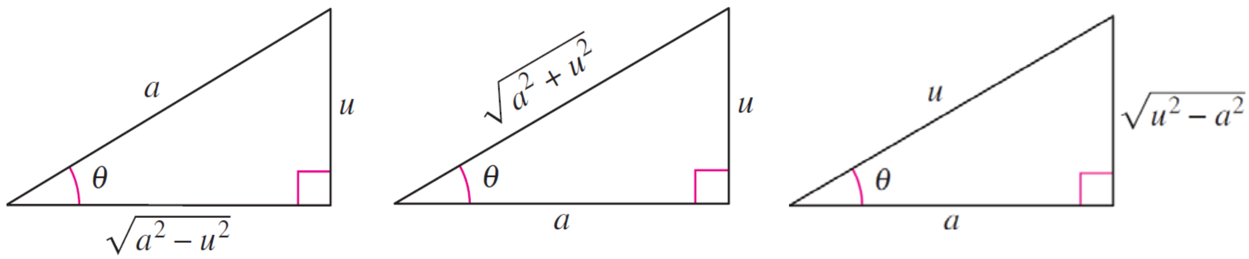
***u* is the variable of integration; *θ* is the substitution variable**
***a* is a positive constant ($a = \sqrt{a^2}$)**

Integrand Radical	u Interval where Radical defined	Forward Substitution (Transform u in Integrand into θ)	θ Interval	Back Substitution (Transform θ in antiderivative back to u. Refer to notes.)
$\sqrt{a^2-u^2}$	$-a \le u \le a$	$\sqrt{a^2-u^2} \;\to\; a\cos\theta$ $u \;\to\; a\sin\theta$ $du \;\to\; a\cos\theta\, d\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$\theta \;\to\; \sin^{-1}\frac{u}{a}$ $\sin\theta \;\to\; \frac{u}{a}$ $\cos\theta \;\to\; \frac{\sqrt{a^2-u^2}}{a}$ $\tan\theta \;\to\; \frac{u}{\sqrt{a^2-u^2}}$
$\sqrt{a^2+u^2}$	$-\infty < u < \infty$	$\sqrt{a^2+u^2} \;\to\; a\sec\theta$ $u \;\to\; a\tan\theta$ $du \;\to\; a\sec^2\theta\, d\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$\theta \;\to\; \tan^{-1}\frac{u}{a}$ $\sin\theta \;\to\; \frac{u}{\sqrt{a^2+u^2}}$ $\cos\theta \;\to\; \frac{a}{\sqrt{a^2+u^2}}$ $\tan\theta \;\to\; \frac{u}{a}$
$\sqrt{u^2-a^2}$	u positive, $u \ge a$	$\sqrt{u^2-a^2} \;\to\; a\tan\theta$ $u \;\to\; a\sec\theta$ $du \;\to\; a\sec\theta\tan\theta\, d\theta$	$0 \le \theta < \frac{\pi}{2}$	$\theta \;\to\; \cos^{-1}\frac{a}{u}$ or $\sec^{-1}\frac{u}{a}$ $\sin\theta \;\to\; \frac{\sqrt{u^2-a^2}}{u}$ $\cos\theta \;\to\; \frac{a}{u}$ $\tan\theta \;\to\; \frac{\sqrt{u^2-a^2}}{a}$
$\sqrt{u^2-a^2}$	u negative, $u \le -a$	$\sqrt{u^2-a^2} \;\to\; -a\tan\theta$ $u \;\to\; a\ \sec\theta$ $du \;\to\; a\sec\theta\tan\theta\, d\theta$	$\frac{\pi}{2} < \theta \le \pi$	$\theta \;\to\; \cos^{-1}\frac{a}{u}$ or $\sec^{-1}\frac{u}{a}$ $\sin\theta \;\to\; \frac{\sqrt{u^2-a^2}}{-u}$ $\cos\theta \;\to\; \frac{a}{u}$ $\tan\theta \;\to\; \frac{\sqrt{u^2-a^2}}{-a}$

Notes:

1. For back substitution, the three other trigonometric functions not shown in the back substitution column: $\csc\theta$, $\sec\theta$, and $\cot\theta$, are the reciprocals of $\sin\theta$, $\cos\theta$, and $\tan\theta$, respectively.
2. There are two approaches to evaluate definite integrals. The first is to find the indefinite integral (the antiderivative) in the original variable of the definite integral, and then evaluate the antiderivative between the original limits. The other approach is to transform the limits of integration through the process. However, since the original limits of integration might not be associated with trigonometric special angles, it is recommended using the indefinite integral approach.

The technique of *Trigonometric Substitution* is best understood from the following three right triangles relating $u, a,$ and θ.



Example 1

Integrate:

$$\int \frac{dx}{\sqrt{1-x^2}}$$

Substitutions (refer to the Table):

$$a = \sqrt{1} = 1$$

$$\sqrt{1-x^2} \;\rightarrow\; \cos\theta$$

$$x \;\rightarrow\; \sin\theta$$

$$dx \;\rightarrow\; \cos\theta\, d\theta$$

Transform the indefinite integral from variable x to variable θ:

$$\int \frac{dx}{\sqrt{1-x^2}} \;\rightarrow\; \int \frac{\cos\theta\, d\theta}{\cos\theta}$$

Simplify and integrate:

$$\int \frac{\cos\theta\, d\theta}{\cos\theta} = \int d\theta = \theta + C$$

Back substitute (again refer to the Table) to transform from variable θ back to variable x:

$$\theta + C \;\rightarrow\; \sin^{-1} x + C$$

Therefore,

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C \quad \blacksquare$$

Example 2

Integrate:

$$\int \frac{dx}{x^2\sqrt{4+x^2}}$$

Substitutions (refer to the Table):

$$a = \sqrt{4} = 2$$

$$\sqrt{4+x^2} \;\rightarrow\; 2\sec\theta$$

$$x \;\rightarrow\; 2\tan\theta$$

$$dx \;\rightarrow\; 2\sec^2\theta\, d\theta$$

Transform the indefinite integral from variable x to variable θ:

$$\int \frac{dx}{x^2\sqrt{4+x^2}} \;\rightarrow\; \int \frac{2\sec^2\theta\, d\theta}{(2\tan\theta)^2\cdot 2\sec\theta}$$

Simplify:

$$\int \frac{2\sec^2\theta\, d\theta}{(2\tan\theta)^2\cdot 2\sec\theta} = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\, d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta$$

Integrate the trigonometric integral above right using u-substitution (don't confuse this with *Trigonometric Substitution*!). Set:

$$u = \sin\theta$$

$$du = \cos\theta\, d\theta$$

$$\frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}\int \frac{du}{u^2} = -\frac{1}{4u} + C = -\frac{1}{4\sin\theta} + C$$

Back substitute (again refer to the Table) to transform from variable θ back to variable x:

$$-\frac{1}{4\sin\theta} + C \;\rightarrow\; -\frac{\sqrt{4+x^2}}{4x} + C$$

Therefore,

$$\int \frac{dx}{x^2\sqrt{4+x^2}} = -\frac{\sqrt{4+x^2}}{4x} + C \quad \blacksquare$$

Example 3

Evaluate:

$$\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$$

First, find the indefinite integral:

$$\int \frac{\sqrt{x^2-4}}{x}\,dx$$

Substitutions (refer to the Table; note that x is positive in the definite integral, so we use the substitutions given in the third row of the Table!):

$$a = \sqrt{4} = 2$$
$$\sqrt{x^2-4} \;\rightarrow\; 2\tan\theta$$
$$x \;\rightarrow\; 2\sec\theta$$
$$dx \;\rightarrow\; 2\sec\theta\tan\theta\,d\theta$$

Transform the indefinite integral from variable x to variable θ:

$$\int \frac{\sqrt{x^2-4}}{x}\,dx \;\rightarrow\; \int \frac{2\tan\theta}{2\sec\theta}\cdot 2\sec\theta\tan\theta\,d\theta$$

Simplify:

$$\int \frac{2\tan\theta}{2\sec\theta}\cdot 2\sec\theta\tan\theta\,d\theta = 2\int \tan^2\theta\,d\theta = 2\int (\sec^2\theta - 1)d\theta$$
$$= 2\int \sec^2\theta\,d\theta - 2\int d\theta = 2\tan\theta - 2\theta + C$$

Back substitute (again refer to the Table) to transform from variable θ back to variable x:

$$2\tan\theta - 2\theta + C \;\rightarrow\; \sqrt{x^2-4} - 2\cos^{-1}\frac{2}{x} + C$$

Therefore,

$$\int \frac{\sqrt{x^2-4}}{x}\,dx = \sqrt{x^2-4} - 2\cos^{-1}\frac{2}{x} + C \quad (\text{for } x \geq 2)$$

Evaluating the definite integral using the *Fundamental Theorem of Calculus*:

$$\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx = \left[\; \sqrt{x^2-4} - 2\cos^{-1}\frac{2}{x}\;\right]_2^4$$

$$= \left[\sqrt{12} - 2\cos^{-1}\frac{1}{2}\right] - [0 - 2\cos^{-1} 1] = \sqrt{12} - 2\cos^{-1}\frac{1}{2} + 2\cos^{-1} 1$$

$$= \sqrt{12} - 2 \cdot \frac{\pi}{3} + 2 \cdot 0 = \sqrt{12} - \frac{2\pi}{3}$$

Therefore,

$$\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx = \sqrt{12} - \frac{2\pi}{3} \quad \blacksquare$$