Theorem: The Form of a Convergent Power Series

If f is represented by a power series $f(x) = \sum a_n(x-c)^n$ for all x in an open interval I containing c. then

$$a_n = \frac{f^{(n)}(c)}{n!}$$

and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

Proof:

Consider a power series$\sum a_n(x-c)^n$ that has a radius of convergence R. Then, by Theorem 9.21, you know that the n^{th} derivative of f exists for $|x-c| < R$, and by successive differentiation you obtain the following.

$$f^{(0)}(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + a_4(x-c)^4 + \ldots$$

$$f^{(1)}(x) = a_1 + 2a_2(x-c) + 3a_(x-c)^2 + 4a_4(x-c)^3 + \ldots$$

$$f^{(2)}(x) = 2a_2 + 3!a_3(x-c) + 4 \bullet 3a_4(x-c)^2 + \ldots$$

$$f^{(3)}(x) = 3!a_3 + 4!a_4(x-c) + \ldots$$

$$\vdots$$

$$f^{(n)}(x) = n!a_n + (n+1)!a_{n+1}(x-c) + \ldots$$

Evaluating each of these derivatives at $x = c$ yields

$$f^{(0)}(c) = 0!a_0$$

$$f^{(1)}(c) = 1!a_1$$

$$f^{(2)}(c) = 2!a_2$$

$$f^{(3)}(c) = 3!a_3$$

and, in general, $f^{(n)}(c) = n!a_n$, By solving for a_n, you find the coefficients of the power series representation of $f(x)$ are

$$a_n = \frac{f^{(n)}(c)}{n!}$$