Theorem: Taylor's Theorem

If a function f is diferrentiable through order $n+1$ in an interval I containing c, then, for each x in I, there exists z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)(z)}}{(n+1)!}(x-c)^{n+1}$$

Proof:

To find $R_n(x)$, fix x in $I(x \neq c$ and write $R_n(x) = f(x) - P_n(x)$, where $P_n(x)$ is the n^{th} Taylor Polynomial for $f(x)$. Then let g be a function of t deined by

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \ldots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-c)^{n+1}}$$

The derivative simplifies to

$$g'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-c)^{n+1}}$$

for all t between c and x.

Moreover, for a fixed x,

$$g(c) = f(x) - [P_n(x) - R_n(x)] = f(x) - f(x) = 0$$

and

$$g(x) = f(x) - 0 - \ldots - 0 = f(x) = 0$$

From Rolle's Theorem, Here is a number z between c and x such that $g'(z) = 0$. Substituting z for t in the equation for $g'(t)$ and then solving for $R_n(x)$, you obtain

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$$

Finally, because $g(c) = 0$, you have

$$0 = f(x) - f(c) - f'(c)(x-c) - \ldots - \frac{f^{(n)}}{n!}(x-c)^n - R_n(x)$$

$$f(x) = f(c) + f'(c)(x-c) + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R(x)$$