Theorem:

- $\sim(\forall x \in D, Q(x)) \equiv$ $\exists x \in D$ s.t. $\sim Q(x)$
- $\sim(\exists x \in D, Q(x)) \equiv$ $\forall x \in D, \sim Q(x)$

Example. Negate the following. Start by writing them formally.

1. Every math class meets at 9am.

Rewrite: $\forall x \in$ {math classes}, x meets at 9am.

Negate: $\exists x \in$ {math classes} s.t. x does not meet at 9am.

Informally: Some math classes do not meet at 9am.

2. Some students take Discrete Math.

Rewrite: $\exists x \in$ {students} s.t. x takes DM.

Negate: $\forall x \in$ {students}, x does not take DM.

Informally: No students take DM.
All students do not take DM.

Notes:

1. When we say two *quantified* statements are logically equivalent, we mean they have the same truth value for **every possible substitution** of **predicate, predicate variables, and choice of domain.**
2. Say it with me now:
   - The negation of a universal statement is an
     existential statement
   - The negation of an existential statement is a
     universal statement

More examples: Negate the following.

1. Every integer has a multiplicative inverse.

   Rewrite: $\forall\, n \in \mathbb{Z}$, n has a multiplicative inverse.

   Negate: $\exists\, n \in \mathbb{Z}$ s.t. n does not have a multiplicative inverse.

2. There is a prime number greater than 1 million.

   Rewrite: $\exists\, n \in \{\text{primes}\}$ s.t. $n > 1{,}000{,}000$

   Negate: $\forall\, n \in \{\text{primes}\}$, $n \leq 1{,}000{,}000$.

Negating Universal Conditional Statements

How can we rewrite $\sim(\forall x \in D, P(x) \rightarrow Q(x))$?

universal $\Rightarrow$ existential, negate predicate

$\exists\, x \in D$ s.t. $\sim(P(x) \rightarrow Q(x))$

$\Downarrow$

becomes $\wedge$

$P(x) \wedge \sim Q(x)$

Answer: $\sim(\forall x \in D, P(x) \rightarrow Q(x)) \equiv \exists\, x \in D$ s.t. $P(x) \wedge \sim Q(x)$

Example. Negate the following.

1. For every integer n, if n is even, then $n/2$ is an integer.

Rewrite: $\forall\ n \in \mathbb{Z},\ n$ even $\Rightarrow \frac{n}{2} \in \mathbb{Z}$

Negate: $\exists\ n \in \mathbb{Z}$ s.t. n even $\underline{\underline{\text{and}}}$ $\frac{n}{2} \notin \mathbb{Z}$.

2. For every real number x, if x is not rational, then $2x$ is not rational.

Rewrite: $\forall\ x \in \mathbb{R},\ x \notin \mathbb{Q} \Rightarrow 2x \notin \mathbb{Q}$

Negate: $\exists\ x \in \mathbb{R}$ s.t. $x \notin \mathbb{Q}$ $\underline{\underline{\text{and}}}$ $2x \in \mathbb{Q}$.

Vacuous truth of universal conditional statements

Example. Negate: "All positive integers less than -5 are divisible by 3."

Rewrite: $\forall x \in \mathbb{Z}^+$, if $x < -5$ then x is divisible by 3.

Negate: $\exists x \in \mathbb{Z}^+$ s.t. $x < -5$ and x is divisible by 3.

Notice: Exactly one of these statements must be true, since a statement and its negation have opposite truth values.

The negation can't be true - there are no positive integers less than -5 - so the original must be true.