Example 4: Show that

$$x + y + e^{xy} = 0$$

is an implicit solution to the nonlinear equation

$$\left(1 + xe^{xy}\right)\frac{dy}{dx} + 1 + ye^{xy} = 0$$

Solution

First, we observe that we are unable to solve $x+y+e^{xy} = 0$ directly for y in terms of x alone. However, for $x + y + e^{xy} = 0$ to hold, we realize that any change in x requires a change in y,so we expect the relation $x + y + e^{xy} = 0$ to define implicitly at least one function $y(x)$. This is difficult to show directly but can be rigorously verified using the **implicit function theorem** of advanced calculus, which guarantes that such a function $y(x)$ exists that is also differentiable.

Once we know that y is a differentiable function of x, we can use te technique of implicit differentiation. Indeed, from $x + y + e^{xy} = 0$ we obtain on differentiating with respect to x and applying the product and chain rules,

$$\frac{d}{dx}(x + y + e^{xy}) = 1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right) = 0$$

or

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0$$

which is identical to the differential equation $\left(1+xe^{xy}\right)\frac{dy}{dx}+1+ye^{xy} = 0$. Thus, relation $x+y+e^{xy} = 0$ is an implicit solution on some interval guaranteed by the implicit function theorem.