Theorem: Convergence of a Geometric Series

A geometric series with ratio r diverges when $|r| \geq 1$. If $|r| < 1$, then the series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \ |r| < 1$$

Proof:

The series diverges when $r \pm 1$. If $r \neq \pm 1$, then

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

. Multiplying by r yields

$$rS_n = ar + ar^2 + ar^3 + \ldots ar^n$$

Subtracting the second equation from the first produces $S_n - rS_n = a - ar^n$. Therefore, $S_n(1-r) = a(1-r^n)$, and the n^{th} partial sum is

$$S_n = \frac{a}{1-r}(1-r^n)$$

When $|r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$, and you obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left\{\frac{a}{1-r}(1-r^n\right\} = \frac{a}{1-r}\left\{\lim_{n\to\infty}(1-r^n)\right\} = \frac{a}{1-r}$$

which means that the series *converges* and its sum is $\frac{a}{(1-r)}$.