Section 4.2 Homogeneous Linear Equations: The General Solution

2^{nd} order linear ode: $a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x)$, $a_0(x) \neq 0$

3^{rd} order linear ode: $a_0(x)y''' + a_1(x)y'' + a_2(x)y' + a_3(x)y = F(x)$, $a_0(x) \neq 0$

If $F(x)$ is identically zero then the equation is homogeneous. Otherwise the equation is non-homogeneous.

Definition Two functions $f_1(x)$ and $f_2(x)$ are said to be linearly dependent (L.D.) if one is a scalar multiple of the other.

Example $f_1(x) = x$ and $f_2(x) = 2x$ are L.D. because $f_1(x) = \frac{1}{2} f_2(x)$ or $f_2(x) = 2 f_1(x)$.

If $f_1(x)$ is not a scalar multiple of $f_2(x)$, $f_1(x)$ and $f_2(x)$ are said to be linearly independent (L.I.).

Definition We define the Wronskian of the functions f_1 and f_2, denoted $W[f_1, f_2]$, by

$$W[f_1, f_2] = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}$$

Notes

1. $f_1(x)$ and $f_2(x)$ are L.D. $\Leftrightarrow$ $W[f_1, f_2] = 0$ for all x.

2 If $W[f_1, f_2] \neq 0$ for all x then f_1 and f_2 are L.I..

Example 1 $f_1(x) = x$, $f_2(x) = 2x$

$$W[f_1, f_2] = \begin{vmatrix} x & 2x \\ 1 & 2 \end{vmatrix} = (x)(2) - (2x)(1) = 0 \quad \text{for all } x.$$

$\Rightarrow$ $f_1(x)$ + $f_2(x)$ are L.D.

Example 2 $f_1(x) = 1+x$, $f_2(x) = x^2$

$$W[f_1, f_2] = \begin{vmatrix} 1+x & x^2 \\ 1 & 2x \end{vmatrix} = (1+x)(2x) - (1)(x^2)$$

$$= 2x + 2x^2 - x^2 = 2x + x^2$$

$$= x(2+x) \neq 0 \quad \text{for all } x$$

$\Rightarrow$ f_1 + f_2 are L.I.

Three functions are L.D. if any one of the three is a linear combination of the remaining two.

That is, $f_1(x) = c_2 f_2(x) + c_3 f_3(x)$

where c_2 and c_3 are constants.

Notes

1. $f_1(x)$, $f_2(x)$ + $f_3(x)$ are L.D. $\Leftrightarrow$

$$W[f_1, f_2, f_3] = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = 0 \quad \text{for all } x.$$

2. If $W[f_1, f_2, f_3] \neq 0$ for all x then f_1, f_2 and f_3 are L.I.

$\Rightarrow$ Any one of the three functions cannot be expressed as a linear combination the remaining two.

Example $f_1(x) = 1$, $f_2(x) = e^x$, $f_3(x) = e^{-x}$

$$W[f_1, f_2, f_3] = \begin{vmatrix} 1 & e^x & e^{-x} \\ 0 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix}$$

$$= 1. \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix}$$

$$= 1[(e^0) - (-e^0)]$$

$$= 2 \neq 0$$

$\therefore$ f_1, f_2 and f_3 are L.I.

Definitions

① Two L.I. solutions of $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ are called a <u>fundamental set of solutions.</u>

② If $y_1(x)$ and $y_2(x)$ are L.I. solutions of the homogeneous 2nd order linear ode.

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$$

then the <u>general solution</u> is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are arbitrary constants.

<u>Note</u> Similar remarks apply to n'th order eqns.

<u>Example</u> Show that $1, e^x, e^{-x}$ are a fundamental set of solutions of $y''' - y' = 0$.
Find the general solution + a particular solution satisfying the conditions $y(0) = 2, y'(0) = 0, y''(0) = 2$.

<u>Solution</u> Let $y_1(x) = 1$, $y_2(x) = e^x$ + $y_3(x) = e^{-x}$.

Recall, $W[y_1, y_2, y_3] = 2 \neq 0$

$\Rightarrow y_1, y_2 + y_3$ are L.I.

$y_1(x) = 1$, $y_1''' = 0$, $y_1' = 0$; $y_1''' - y_1' = 0 \Rightarrow y_1$ is a soln.

$y_2(x) = e^x$, $y_2''' = e^x$, $y_2' = e^x$; $y_2''' - y_2' = 0 \Rightarrow y_2$ is a soln.

$y_3(x) = e^{-x}$, $y_3''' = -e^{-x}$, $y_3' = -e^{-x}$; $y_3''' - y_3' = 0 \Rightarrow y_3$ is a soln.

$\therefore$ $y_1(x)$, $y_2(x)$ & $y_3(x)$ is a fundamental solution set.

General solution : $y = c_1 \cdot 1 + c_2 \cdot e^x + c_3 \cdot e^{-x}$

Particular solution : $y(0) = 2$, $y'(0) = 0$, $y''(0) = 2$

$$\left.\begin{array}{llll} y = c_1 + c_2 e^x + c_3 e^{-x}, & x=0,\ y=2 & \Rightarrow & c_1 + c_2 + c_3 = 2 \\ y' = c_2 e^x - c_3 e^{-x}, & x=0,\ y'=0 & \Rightarrow & c_2 - c_3 = 0 \\ y'' = c_2 e^x + c_3 e^{-x}, & x=0,\ y''=2 & \Rightarrow & c_2 + c_3 = 2 \end{array}\right\}$$

$2c_2 = 2$

$c_2 = 1$, $c_3 = 1$, $c_1 = 0$

$\therefore$ $y = 0 \cdot 1 + 1 \cdot e^x + 1 \cdot e^{-x} = e^x + e^{-x}$.

H.W. Pages 165-166, #'s 27-33, 35