Section 7.4 Inverse Laplace Transform

Definition If $\mathcal{L}[f(t); t \to s] = F(s)$ we say that $f(t)$ is the inverse Laplace Transform of $F(s)$ and we write $f(t) = \mathcal{L}^{-1}[F(s); s \to t]$.

Examples

1. (a) $\mathcal{L}^{-1}\left[\frac{1}{s}; s \to t\right] = 1$

(b) $\mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}; s \to t\right] = t^n \quad ; n = 1, 2, 3, 4, \ldots$

(c) $\mathcal{L}^{-1}\left[\frac{1}{s-a}; s \to t\right] = e^{at}$

(d) $\mathcal{L}^{-1}\left[\frac{b}{s^2+b^2}; s \to t\right] = \sin(bt)$

(e) $\mathcal{L}^{-1}\left[\frac{s}{s^2+b^2}; s \to t\right] = \cos(bt)$

2. Use the results of Example 1 to determine the following:

(a)
$$\mathcal{L}^{-1}\left[\frac{1}{s^4}; s \to t\right] = \mathcal{L}^{-1}\left[\frac{1}{3!}\frac{3!}{s^4}; s \to t\right]$$
$$= \frac{1}{6}\mathcal{L}^{-1}\left[\frac{3!}{s^4}; s \to t\right]$$
$$= \frac{1}{6}t^3$$

(b) $\mathcal{L}^{-1}\left[\frac{4}{s} + \frac{6}{s^5} - \frac{1}{s+8} \,;\, s \to t\right]$

$= 4\,\mathcal{L}^{-1}\left[\frac{1}{s} \,;\, s\to t\right] + 6\,\frac{1}{24}\,\mathcal{L}^{-1}\left[\frac{24}{s^5} \,;\, s\to t\right]$

$- \mathcal{L}^{-1}\left[\frac{1}{s-(-8)} \,;\, s\to t\right]$

$= 4 + \frac{1}{4}\,t^4 - e^{-8t}$

(c) $\mathcal{L}^{-1}\left[\frac{(s+2)^2}{s^3} \,;\, s\to t\right]$

$= \mathcal{L}^{-1}\left[\frac{s^2+4s+4}{s^3} \,;\, s\to t\right]$

$= \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{4}{s^2} + \frac{4}{s^3} \,;\, s\to t\right]$

$= \mathcal{L}^{-1}\left[\frac{1}{s} + 4\cdot\frac{1}{s^2} + 2\,\frac{2!}{s^3} \,;\, s\to t\right]$

$= 1 + 4t + 2t^2$

(d) $\mathcal{L}^{-1}\left[\frac{1}{5s-2} \,;\, s\to t\right]$

$= \mathcal{L}^{-1}\left[\frac{1}{5s-2}\,\frac{1/5}{1/5} \,;\, s\to t\right]$

$= \mathcal{L}^{-1}\left[\frac{1}{5}\cdot\frac{1}{s-2/5} \,;\, s\to t\right]$

$= \frac{1}{5}\,e^{2t/5}$

(e) $\mathcal{L}^{-1}\left[\dfrac{10s-2}{s^2+16}\ ;\ s\to t\right]$

$= \mathcal{L}^{-1}\left[\dfrac{10s}{s^2+16} - \dfrac{2}{s^2+16}\ ;\ s\to t\right]$

$= \mathcal{L}^{-1}\left[10\,\dfrac{s}{s^2+4^2} - 2\cdot\dfrac{1}{4}\,\dfrac{4}{s^2+4^2}\ ;\ s\to t\right]$

$= 10\cos(4t) - \frac{1}{2}\sin(4t)$

Partial Fractions

(1) $\dfrac{1}{(s-1)(s+2)(s+4)} = \dfrac{A}{s-1} + \dfrac{B}{s+2} + \dfrac{C}{s+4}$

$1 = A(s+2)(s+4) + B(s-1)(s+4) + C(s-1)(s+2)$

Let $s=1$: $1 = A(3)(5) \Rightarrow A = 1/15$

Let $s=-2$: $1 = B(-3)(2) \Rightarrow B = -1/6$

Let $s=-4$: $1 = C(-5)(-2) \Rightarrow C = 1/10$

$\therefore \dfrac{1}{(s-1)(s+2)(s+4)} = \dfrac{1/15}{s-1} - \dfrac{1/6}{s+2} + \dfrac{1/10}{s+4}$

(2) $\dfrac{2s^2-9s-9}{s(s+3)(s-3)} = \dfrac{A}{s} + \dfrac{B}{s+3} + \dfrac{C}{s-3}$

$$2s^2 - 9s - 9 = A(s+3)(s-3) + Bs(s-3) + Cs(s+3)$$

Let $s=0$: $-9 = A(3)(-3) \rightarrow A = 1$

Let $s=-3$: $18+27-9 = B(-3)(-6) \rightarrow B = 2$

Let $s=3$: $18-27-9 = C(3)(6) \rightarrow C = -1$

$$\therefore \frac{2s^2 - 9s - 9}{s(s+3)(s-3)} = \frac{1}{s} + \frac{2}{s+3} - \frac{1}{s-3}$$

(3) $$\frac{2s^2 - 2s - 1}{s^2(s-1)} = \frac{As + B}{s^2} + \frac{C}{s-1}$$

$$2s^2 - 2s - 1 = (As+B)(s-1) + Cs^2$$

Let $s=0$: $-1 = B(-1) \rightarrow B = 1$

Let $s=1$: $2-2-1 = C(1) \rightarrow C = -1$

Let $s=-1$: $2+2-1 = (-A+1)(-2) - 1(-1)^2$

$$-3 = (1-A)(-2) - 1$$

$$4 = -2 + 2A$$

$$A = 3$$

$$\frac{2s^2 - 2s - 1}{s^2(s-1)} = \frac{3s+1}{s^2} - \frac{1}{s-1} = \frac{3}{s} + \frac{1}{s^2} - \frac{1}{s-1}$$

4 $$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s$$
$$1 = s^2(A+B) + sC + A$$

$$\left.\begin{array}{ll} s^2: & A+B=0 \\ s^1: & C=0 \\ s^0: & A=1 \end{array}\right\} A=1,\ B=-1,\ C=0$$

$$\therefore \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

Further Examples of Inverse Transforms

1 $$\mathcal{L}^{-1}\left[\frac{1}{s^2+s-20}\ ;\ s\to t\right]$$

$$\frac{1}{s^2+s-20} = \frac{1}{(s+5)(s-4)} = \frac{A}{s+5} + \frac{B}{s-4}$$

$$1 = A(s-4) + B(s+5)$$

Let $s=-5$: $1 = A(-9) \Rightarrow A = -1/9$

Let $s=4$: $1 = B(9) \Rightarrow B = 1/9$

$$\therefore \mathcal{L}^{-1}\left[\frac{1}{s^2+s-20}\ ;\ s\to t\right] = \mathcal{L}^{-1}\left[\frac{-1/9}{s+5} + \frac{1/9}{s-4}\ ;\ s\to t\right]$$

$$= \mathcal{L}^{-1}\left[-\frac{1}{9}\,\frac{1}{s-(-5)} + \frac{1}{9}\,\frac{1}{s-4}\,;\, s\to t\right] = -\frac{1}{9}e^{-5t} + \frac{1}{9}e^{4t}$$

② $\mathcal{L}^{-1}\left[\dfrac{s-1}{s^2(s^2+1)}\,;\, s\to t\right]$

$$\frac{s-1}{s^2(s^2+1)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$s-1 = (As+B)(s^2+1) + (Cs+D)s^2$$

$$= As^3 + Bs^2 + As + B + Cs^3 + Ds^2$$

$$= s^3(A+C) + s^2(B+D) + sA + B$$

$$\left.\begin{array}{l} s^3 : A+C=0 \\ s^2 : B+D=0 \\ s^1 : A=1 \\ s^0 : B=-1 \end{array}\right\} \quad A=1,\ B=-1,\ C=-1,\ D=1$$

$\therefore \mathcal{L}^{-1}\left[\dfrac{s-1}{s^2(s^2+1)}\,;\, s\to t\right]$

$$= \mathcal{L}^{-1}\left[\frac{s-1}{s^2} + \frac{-s+1}{s^2+1}\,;\, s\to t\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s^2} - \frac{s}{s^2+1} + \frac{1}{s^2+1}\,;\, s\to t\right]$$

$$= 1 - t - \cos t + \sin t$$

3. $$\mathcal{L}^{-1}\left[\frac{3s+5}{s^2+2s+10} ; s \to t\right]$$

$$F(s) = \frac{3(s+1)+2}{(s+1)^2+3^2} = 3\,\frac{s+1}{(s+1)^2+3^2} + \frac{2}{3}\,\frac{3}{(s+1)^2+3^2}$$

Recall $\mathcal{L}[e^{at}\cos bt ; t \to s] = \dfrac{s-a}{(s-a)^2+b^2}$

and, $\mathcal{L}[e^{at}\sin bt ; t \to s] = \dfrac{b}{(s-a)^2+b^2}$

Hence, $$f(t) = \mathcal{L}^{-1}\left[3\,\frac{s+1}{(s+1)^2+3^2} + \frac{2}{3}\,\frac{3}{(s+1)^2+3^2} ; s \to t\right]$$

$$= 3e^{-t}\cos(3t) + \frac{2}{3}e^{-t}\sin(3t)$$

4. $$\mathcal{L}^{-1}\left[\frac{s+2}{s^3-1} ; s \to t\right]$$

$$F(s) = \frac{s+2}{s^3-1} = \frac{s+2}{(s-1)(s^2+s+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}$$

$$s+2 = A(s^2+s+1) + (Bs+C)(s-1)$$

$s=1$: $3 = A(1+1+1) \Rightarrow A=1$

$s=0$: $2 = A(1) + C(-1) \Rightarrow C=-1$

$s^2: \quad 0 = A + B \rightarrow B = -1$

$$\therefore F(s) = \frac{1}{s-1} + \frac{-s-1}{s^2+s+1}$$

$$= \frac{1}{s-1} - \frac{(s+1)}{(s+\frac{1}{2})^2 + 3/4}$$

$$= \frac{1}{s-1} - \frac{s+1/2}{(s+1/2)^2 + (\sqrt{3}/2)^2} - \frac{1/2}{(s+\frac{1}{2})^2 + (\sqrt{3}/2)^2}$$

$$= \frac{1}{s-1} - \frac{(s+1/2)}{(s+1/2)^2 + (\sqrt{3}/2)^2} - \frac{1}{2}\frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2 + (\sqrt{3}/2)^2}$$

$$= \frac{1}{(s-1)} - \frac{(s+1/2)}{(s+\frac{1}{2})^2 + (\sqrt{3}/2)^2} - \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2 + (\sqrt{3}/2)^2}$$

$$\therefore f(t) = e^{t} - e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

5 $$F(s) = \frac{2+4s-2s^2}{s^3-s^2-s+1}$$

$$s^3 - s^2 - s + 1 = s^2(s-1) - (s-1) = (s-1)(s^2-1) = (s-1)^2(s+1)$$

$$F(s) = \frac{2+4s-2s^2}{(s-1)^2(s+1)} = \frac{A}{(s-1)^2} + \frac{B}{(s-1)^1} + \frac{C}{(s+1)}$$

$$\therefore 2+4s-2s^2 = A(s+1) + B(s-1)(s+1) + C(s-1)^2$$

$s=1: \quad 2+4-2 = A(2) \Rightarrow A = 2$

$s=-1$: $2-4-2 = C(-2)^2 \Rightarrow C=-1$

$s=0$: $2 = A(1) + B(-1) + C(1)$

$2 = 2 - B - 1 \Rightarrow B=-1$

$$\therefore F(s) = \frac{2}{(s-1)^2} - \frac{1}{(s-1)} - \frac{1}{(s+1)}$$

<u>Recall</u> $\mathcal{L}[e^{at}; t \to s] = \frac{1}{s-a}$

and, $\mathcal{L}[t^n e^{at}; t \to s] = \frac{n!}{(s-a)^{n+1}}$

$$\therefore f(t) = \mathcal{L}^{-1}\left[2\frac{1}{(s-1)^2} - \frac{1}{s-1} - \frac{1}{s+1}; s \to t\right]$$

$$= 2te^t - e^t - e^{-t}$$

<u>HW</u> Pages 374 – 375

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