Theorem: Alternating Series Remainder

If a convergent alternating series satisfies the condition $a_{n+1} \leq a_n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than (or equal to) the first neglected term. That is,

$$|S - S_N| = |R_N| \leq a_{N+1}$$

Proof:

The series obtained by deleting the first N terms of the given series satisfies the conditions of the Alternating Series Test and has a sum of R_N.

$$R_N = S - S_N = \sum_{n=1}^{\infty}(-1)^{n+1}a_n - \sum_{n=1}^{N}(-1)^{n+1}a_n$$

$$= (-1)^N a_{n+1} + (-1)^{N+1}a_{N+2} + (-1)^{N+2}a_{N+3} + \ldots$$

$$= (-1)^N (a_{N+1} - a_{N+2} + a_{N+3} - \ldots)$$

$$|R_N| = a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + a_{N+5} - \ldots$$

$$= a_{N+1} - (a_{N+2} - a_{N+3}) - (a_{N+4} - a_{N+5}) - \ldots$$

Consequently, $|S - S_N| = |R_N| \leq a_{N+1}$.