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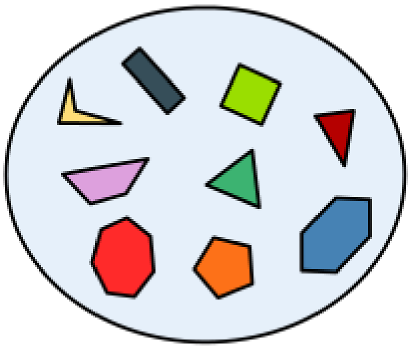
Set (mathematics)

In mathematics, a **set** is a collection of distinct objects, considered as an object in its own right. For example, the numbers 2, 4, and 6 are distinct objects when considered separately, but when they are considered collectively they form a single set of size three, written {2,4,6}. The concept of a set is one of the most fundamental in mathematics. Developed at the end of the 19th century, set theory is now a ubiquitous part of mathematics, and can be used as a foundation from which nearly all of mathematics can be derived. In mathematics education, elementary topics from set theory such as Venn diagrams are taught at a young age, while more advanced concepts are taught as part of a university degree.



A set of polygons in an Euler diagram

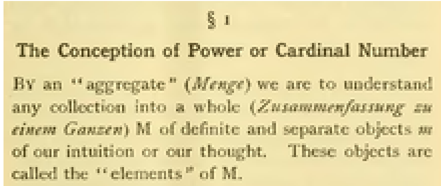
The German word *Menge*, rendered as "set" in English, was coined by Bernard Bolzano in his work *The Paradoxes of the Infinite*.

Contents

Definition

A set is a well-defined collection of distinct objects. The objects that make up a set (also known as the set's *elements* or *members*) can be anything: numbers, people, letters of the alphabet, other sets, and so on. Georg Cantor, one of the founders of set theory, gave the following definition of a set at the beginning of his *Beiträge zur Begründung der transfiniten Mengenlehre*:[1]

> A set is a gathering together into a whole of definite, distinct objects of our perception [Anschauung] or of our thought—which are called elements of the set.



§ 1

The Conception of Power or Cardinal Number

BY an "aggregate" (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) M of definite and separate objects *m* of our intuition or our thought. These objects are called the "elements" of M.

Passage with a translation of the original set definition of Georg Cantor. The German word *Menge* for *set* is translated with *aggregate* here.

Sets are conventionally denoted with capital letters. Sets *A* and *B* are equal if and only if they have precisely the same elements.[2]

For technical reasons, Cantor's definition turned out to be inadequate; today, in contexts where more rigor is required, one can use axiomatic set theory, in which the notion of a "set" is taken as a primitive notion and the properties of sets are defined by a collection of axioms. The most basic properties are that a set can have elements, and that two sets are equal (one and the same) if and only if every element of each set is an element of the other; this property is called the *extensionality of sets*.

Describing sets

There are two ways of describing, or specifying the members of, a set. One way is by intensional definition, using a rule or semantic description:

> *A* is the set whose members are the first four positive integers.
> *B* is the set of colors of the French flag.

The second way is by extension – that is, listing each member of the set. An extensional definition is denoted by enclosing the list of members in curly brackets:

> $C = \{4, 2, 1, 3\}$
> $D = \{\text{blue, white, red}\}$.

One often has the choice of specifying a set either intensionally or extensionally. In the examples above, for instance, $A = C$ and $B = D$.

In an extensional definition, a set member can be listed two or more times, for example, {11, 6, 6}. However, per extensionality, two definitions of sets which differ only in that one of the definitions lists members multiple times define the same set. Hence, the set {11, 6, 6} is identical to the set {11, 6}. Moreover, the order in which the elements of a set are listed is irrelevant (unlike for a sequence or tuple). We can illustrate these two important points with an example:

$$\{6, 11\} = \{11, 6\} = \{11, 6, 6, 11\} .$$

For sets with many elements, the enumeration of members can be abbreviated. For instance, the set of the first thousand positive integers may be specified extensionally as

$$\{1, 2, 3, ..., 1000\},$$

where the ellipsis ("...") indicates that the list continues in the obvious way.

The notation with braces may also be used in an intensional specification of a set. In this usage, the braces have the meaning "the set of all ...". So, *E* = {playing card suits} is the set whose four members are ♠, ♦, ♥, and ♣. A more general form of this is set-builder notation, through which, for instance, the set *F* of the twenty smallest integers that are four less than perfect squares can be denoted

$$F = \{n^2 - 4 : n \text{ is an integer; and } 0 \leq n \leq 19\}.$$

In this notation, the colon (":") means "such that", and the description can be interpreted as "*F* is the set of all numbers of the form $n^2 - 4$, such that *n* is a whole number in the range from 0 to 19 inclusive." Sometimes the vertical bar ("|") is used instead of the colon.

Membership

If B is a set and x is one of the objects of B, this is denoted $x \in B$, and is read as "x belongs to B", or "x is an element of B". If y is not a member of B then this is written as $y \notin B$, and is read as "y does not belong to B".

For example, with respect to the sets $A = \{1,2,3,4\}$, $B = \{\text{blue, white, red}\}$, and $F = \{n^2 - 4 : n \text{ is an integer; and } 0 \leq n \leq 19\}$ defined above,

> $4 \in A$ and $12 \in F$; but
> $9 \notin F$ and green $\notin B$.

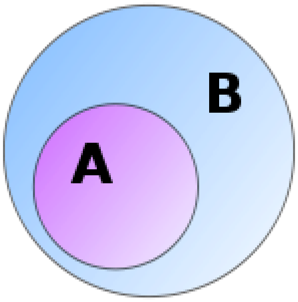
Subsets

If every member of set A is also a member of set B, then A is said to be a *subset* of B, written $A \subseteq B$ (also pronounced *A is contained in B*). Equivalently, we can write $B \supseteq A$, read as *B is a superset of A*, *B includes A*, or *B contains A*. The relationship between sets established by $\subseteq$ is called *inclusion* or *containment*.

If A is a subset of, but not equal to, B, then A is called a *proper subset* of B, written $A \subsetneq B$ (*A is a proper subset of B*) or $B \supsetneq A$ (*B is a proper superset of A*).

The expressions $A \subset B$ and $B \supset A$ are used differently by different authors; some authors use them to mean the same as $A \subseteq B$ (respectively $B \supseteq A$), whereas others use them to mean the same as $A \subsetneq B$ (respectively $B \supsetneq A$).

Examples:



A is a **subset** of *B*

- The set of all men is a proper subset of the set of all people.
- $\{1, 3\} \subseteq \{1, 2, 3, 4\}$.
- $\{1, 2, 3, 4\} \subseteq \{1, 2, 3, 4\}$.

The empty set is a subset of every set and every set is a subset of itself:

- $\varnothing \subseteq A$.
- $A \subseteq A$.

Every set is a subset of the universal set:

- $A \subseteq U$.

An obvious but useful identity which can often be used to show that two seemingly different sets are equal:

- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

A partition of a set S is a set of nonempty subsets of S such that every element x in S is in exactly one of these subsets.

Power sets

The power set of a set S is the set of all subsets of S. The power set contains S itself and the empty set because these are both subsets of S. For example, the power set of the set $\{1, 2, 3\}$ is $\{\{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \varnothing\}$. The power set of a set S is usually written as $P(S)$.

The power set of a finite set with n elements has 2^n elements. For example, the set $\{1, 2, 3\}$ contains three elements, and the power set shown above contains $2^3 = 8$ elements.

The power set of an infinite (either countable or uncountable) set is always uncountable. Moreover, the power set of a set is always strictly "bigger" than the original set in the sense that there is no way to pair every element of S with exactly one element of $P(S)$. (There is never an onto map or surjection from S onto $P(S)$.)

Every partition of a set S is a subset of the powerset of S.

Cardinality

The cardinality $|S|$ of a set S is "the number of members of S." For example, if $B = \{blue, white, red\}$, then $|B| = 3$.

There is a unique set with no members, called the *empty set* (or the *null set*), which is denoted by the symbol ∅ (other notations are used; see empty set). The cardinality of the empty set is zero. For example, the set of all three-sided squares has zero members and thus is the empty set. Though it may seem trivial, the empty set, like the number zero, is important in mathematics. Indeed, the existence of this set is one of the fundamental concepts of axiomatic set theory.

Some sets have infinite cardinality. The set **N** of natural numbers, for instance, is infinite. Some infinite cardinalities are greater than others. For instance, the set of real numbers has greater cardinality than the set of natural numbers. However, it can be shown that the cardinality of (which is to say the number of points on) a straight line is the same as the cardinality of any segment of that line, of the entire plane, and indeed of any finite-dimensional Euclidean space.

Special sets

There are some sets or kinds of sets that hold great mathematical importance and are referred to with such regularity that they have acquired special names and notational conventions to identify them. One of these is the empty set, denoted {} or ∅. A set with exactly one element, x, is a unit set, or singleton, {x}.[2]

Many of these sets are represented using blackboard bold or bold typeface. Special sets of numbers include

- **P** or $\mathbb{P}$, denoting the set of all primes: **P** = {2, 3, 5, 7, 11, 13, 17, ...}.
- **N** or $\mathbb{N}$, denoting the set of all natural numbers: **N** = {0, 1, 2, 3, ...} (sometimes defined excluding 0).
- **Z** or $\mathbb{Z}$, denoting the set of all integers (whether positive, negative or zero): **Z** = {..., −2, −1, 0, 1, 2, ...}.
- **Q** or $\mathbb{Q}$, denoting the set of all rational numbers (that is, the set of all proper and improper fractions): **Q** = $\{a/b : a, b \in$ **Z**, $b \neq 0\}$. For example, 1/4 ∈ **Q** and 11/6 ∈ **Q**. All integers are in this set since every integer a can be expressed as the fraction $a/1$ (**Z** ⊊ **Q**).
- **R** or $\mathbb{R}$, denoting the set of all real numbers. This set includes all rational numbers, together with all irrational numbers (that is, algebraic numbers that cannot be rewritten as fractions such as $\sqrt{2}$, as well as transcendental numbers such as π, *e*).
- **C** or $\mathbb{C}$, denoting the set of all complex numbers: **C** = $\{a + bi : a, b \in$ **R**$\}$. For example, $1 + 2i \in$ **C**.
- **H** or $\mathbb{H}$, denoting the set of all quaternions: **H** = $\{a + bi + cj + dk : a, b, c, d \in$ **R**$\}$. For example, $1 + i + 2j - k \in$ **H**.

Positive and negative sets are denoted by a superscript - or +. For example $\mathbb{Q}^+$ represents the set of positive rational numbers.

Each of the above sets of numbers has an infinite number of elements, and each can be considered to be a proper subset of the sets listed below it. The primes are used less frequently than the others outside of number theory and related fields.

Basic operations

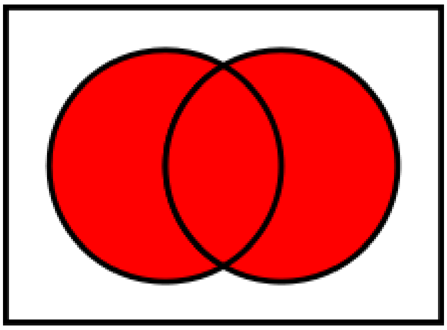
There are several fundamental operations for constructing new sets from given sets.

Unions

Two sets can be "added" together. The *union* of A and B, denoted by $A \cup B$, is the set of all things that are members of either A or B.

Examples:

- $\{1, 2\} \cup \{1, 2\} = \{1, 2\}$.
- $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$.
- $\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$



The **union** of A and B, denoted $A \cup B$

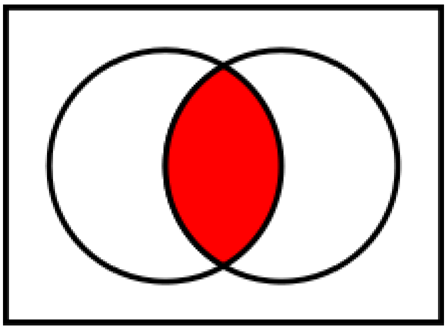
Some basic properties of unions:

- $A \cup B = B \cup A$.
- $A \cup (B \cup C) = (A \cup B) \cup C$.
- $A \subseteq (A \cup B)$.
- $A \cup A = A$.
- $A \cup U = U$.
- $A \cup \varnothing = A$.
- $A \subseteq B$ if and only if $A \cup B = B$.

Intersections

A new set can also be constructed by determining which members two sets have "in common". The *intersection* of A and B, denoted by $A \cap B$, is the set of all things that are members of both A and B. If $A \cap B = \varnothing$, then A and B are said to be *disjoint*.

Examples:



The **intersection** of A and B, denoted $A \cap B$.

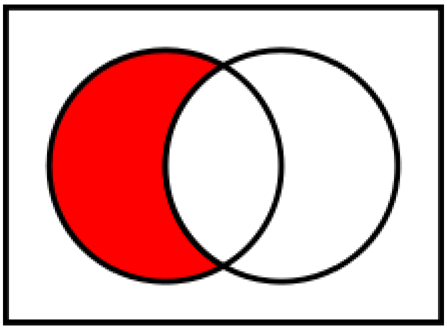
- $\{1, 2\} \cap \{1, 2\} = \{1, 2\}$.
- $\{1, 2\} \cap \{2, 3\} = \{2\}$.

Some basic properties of intersections:

- $A \cap B = B \cap A$.
- $A \cap (B \cap C) = (A \cap B) \cap C$.
- $A \cap B \subseteq A$.
- $A \cap A = A$.
- $A \cap U = A$.
- $A \cap \varnothing = \varnothing$.
- $A \subseteq B$ if and only if $A \cap B = A$.

Complements

Two sets can also be "subtracted". The *relative complement* of B in A (also called the *set-theoretic difference* of A and B), denoted by $A \setminus B$ (or $A - B$), is the set of all elements that are members of A but not members of B. Note that it is valid to "subtract" members of a set that are not in the set, such as removing the element *green* from the set $\{1, 2, 3\}$; doing so has no effect.



The **relative complement** of B in A

In certain settings all sets under discussion are considered to be subsets of a given universal set U. In such cases, $U \setminus A$ is called the *absolute complement* or simply *complement* of A, and is denoted by A'.

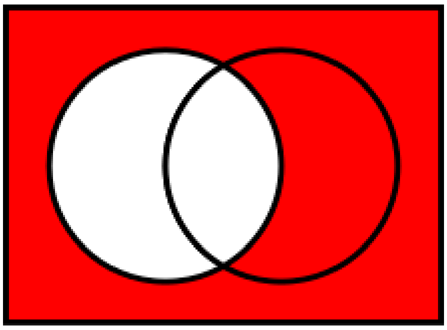
- $A' = U \setminus A$

Examples:

- $\{1, 2\} \setminus \{1, 2\} = \varnothing$.
- $\{1, 2, 3, 4\} \setminus \{1, 3\} = \{2, 4\}$.
- If U is the set of integers, E is the set of even integers, and O is the set of odd integers, then $U \setminus E = E' = O$.

Some basic properties of complements:

- $A \setminus B \neq B \setminus A$ for $A \neq B$.
- $A \cup A' = U$.
- $A \cap A' = \varnothing$.
- $(A')' = A$.
- $\varnothing \setminus A = \varnothing$.
- $A \setminus \varnothing = A$.
- $A \setminus A = \varnothing$.
- $A \setminus U = \varnothing$.
- $A \setminus A' = A$ and $A' \setminus A = A'$.
- $U' = \varnothing$ and $\varnothing' = U$.
- $A \setminus B = A \cap B'$.
- if $A \subseteq B$ then $A \setminus B = \varnothing$.

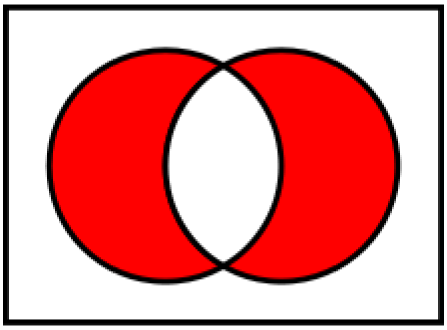


The **complement** of *A* in *U*

An extension of the complement is the symmetric difference, defined for sets *A*, *B* as

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A).$$

For example, the symmetric difference of {7,8,9,10} and {9,10,11,12} is the set {7,8,11,12}. The power set of any set becomes a Boolean ring with symmetric difference as the addition of the ring (with the empty set as neutral element) and intersection as the multiplication of the ring.



The **symmetric difference** of *A* and *B*

Cartesian product

A new set can be constructed by associating every element of one set with every element of another set. The *Cartesian product* of two sets *A* and *B*, denoted by $A \times B$ is the set of all ordered pairs (a, b) such that *a* is a member of *A* and *b* is a member of *B*.

Examples:

- {1, 2} × {red, white, green} = {(1, red), (1, white), (1, green), (2, red), (2, white), (2, green)}.
- {1, 2} × {1, 2} = {(1, 1), (1, 2), (2, 1), (2, 2)}.
- {a, b, c} × {d, e, f} = {(a, d), (a, e), (a, f), (b, d), (b, e), (b, f), (c, d), (c, e), (c, f)}.

Some basic properties of Cartesian products:

- $A \times \varnothing = \varnothing$.
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
- $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

Let *A* and *B* be finite sets; then the cardinality of the Cartesian product is the product of the cardinalities:

- $|A \times B| = |B \times A| = |A| \times |B|$.

Applications

Set theory is seen as the foundation from which virtually all of mathematics can be derived. For example, structures in abstract algebra, such as groups, fields and rings, are sets closed under one or more operations.

One of the main applications of naive set theory is constructing relations. A relation from a domain *A* to a codomain *B* is a subset of the Cartesian product $A \times B$. Given this concept, we are quick to see that the set *F* of all ordered pairs (x, x^2), where *x* is real, is quite familiar. It has a domain set **R** and a codomain set that is also **R**, because the set of all squares is subset of the set of all real numbers. If placed in functional notation, this relation becomes $f(x) = x^2$. The reason these two are equivalent is for any given value, *y* that the function is defined for, its corresponding ordered pair (y, y^2) is a member of the set *F*.

Axiomatic set theory

Although initially naive set theory, which defines a set merely as *any well-defined* collection, was well accepted, it soon ran into several obstacles. It was found that this definition spawned several paradoxes, most notably:

- Russell's paradox—It shows that the "set of all sets that *do not contain themselves*," i.e. the "set" $\{x : x \text{ is a set and } x \notin x\}$ does not exist.
- Cantor's paradox—It shows that "the set of all sets" cannot exist.

The reason is that the phrase *well-defined* is not very well-defined. It was important to free set theory of these paradoxes because nearly all of mathematics was being redefined in terms of set theory. In an attempt to avoid these paradoxes, set theory was axiomatized based on first-order logic, and thus **axiomatic set theory** was born.

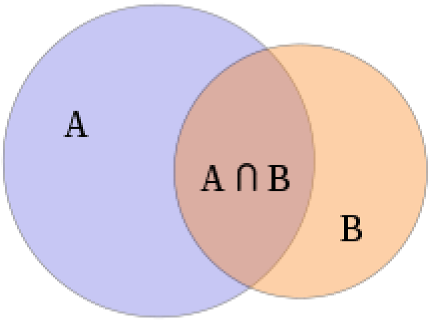
For most purposes, however, naive set theory is still useful.

Principle of inclusion and exclusion

The inclusion–exclusion principle is a counting technique that can be used to count the number of elements in a union of two sets, if the size of each set and the size of their intersection are known. It can be expressed symbolically as

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

A more general form of the principle can be used to find the cardinality of any finite union of sets:



The inclusion-exclusion principle can be used to calculate the size of the union of sets: the size of the union is the size of the two sets, minus the size of their intersection.

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = & \left(|A_1| + |A_2| + |A_3| + \ldots |A_n|\right) \\ & - \left(|A_1 \cap A_2| + |A_1 \cap A_3| + \ldots |A_{n-1} \cap A_n|\right) \\ & + \ldots \\ & + (-1)^{n-1} \left(|A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n|\right). \end{aligned}$$

De Morgan's laws

Augustus De Morgan stated two laws about sets.

If A and B are any two sets then,

- **(A ∪ B)′ = A′ ∩ B′**

The complement of A union B equals the complement of A intersected with the complement of B.

- **(A ∩ B)′ = A′ ∪ B′**

The complement of A intersected with B is equal to the complement of A union to the complement of B.

See also

- Set notation
- Mathematical object
- Alternative set theory
- Axiomatic set theory

- Category of sets
- Class (set theory)
- Dense set
- Family of sets
- Fuzzy set
- Internal set
- Mereology
- Multiset
- Naive set theory
- Principia Mathematica
- Rough set
- Russell's paradox
- Sequence (mathematics)
- Taxonomy
- Tuple

Notes

1. "Eine Menge, ist die Zusammenfassung bestimmter, wohlunterschiedener Objekte unserer Anschauung oder unseres Denkens – welche Elemente der Menge genannt werden – zu einem Ganzen". "Archived copy" (http://www.brinkmann-du.de/mathe/fos/fos01_03.htm). Archived (https://web.archive.org/web/20110610133240/http://brinkmann-du.de/mathe/fos/fos01_03.htm) from the original on 2011-06-10. Retrieved 2011-04-22.
2. Stoll, Robert. *Sets, Logic and Axiomatic Theories*. W. H. Freeman and Company. p. 5.

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