Theorem 8.4

A collection $\{S_1, S_2, \ldots S_n\}$ of non-empty finite sets has a system of distinct representatives *if and only if* for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements.

Symbolically: A collection $\{S_1, S_2, \ldots S_n\}$ of non-empty finite sets has a system of distinct representatives $\Leftrightarrow$ for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements.

Proof:

Suppose that $\{S_1, S_2, \ldots S_n\}$ has a system of distinct representatives. Then, necessarily, for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements. So only the converse needs to be verified.

Let $\{S_1, S_2, \ldots S_n\}$ be a collection of n sets such that for each integer k with $1 \leq k \leq n$, the union of any k of these sets contains at least k elements. We construct a bipartite sets $U = \{S_1, S_2, \ldots S_n\}$ and $W = S_1 \cup S_2 \cup \ldots \cup S_n$, where a vertex $S_i (1 \leq i \leq n)$ in U is adjacent to a vertex w in W if $w \in S_i$. Let X be any subset of U, where $|X| = k$ with $1 \leq k \leq n$. Since the union of any k sets contains at least k elements, $|N(X) \geq |X|$. Therefore, G satisfies Hall's condition. By Theorem 8.3, G contains a matching of cardinality n, which pairs off the sets $S_1, S_2, \ldots S_n$ with n distinct elements in $S_1 \cup S_2 \cup \ldots \cup S_n$, producing a system of distinct representatives for these sets.