Example 4: Show that

$$\Big(x + 3x^3 \sin(y)\Big)dx + \Big(x^4 \cos(y)\Big) = 0$$

is *not* exact but that multiplying this equation by the factor x^{-1} yields an exact equation. Use this fact to solve $\Big(x + 3x^3 \sin(y)\Big)dx + \Big(x^4 \cos(y)\Big) = 0$

Solution

In $\Big(x + 3x^3 \sin(y)\Big)dx + \Big(x^4 \cos(y)\Big) = 0$, $M = x + 3x^3 \sin(y)$ and $N = x^4 \cos(y)$. Because

$$\frac{\partial M}{\partial y} = 3x^3 \cos(y) \neq 4x^3 \cos(y) = \frac{\partial N}{\partial x}$$

$\Big(x+3x^3 \sin(y)\Big)dx+\Big(x^4 \cos(y)\Big) = 0$ is not exact. When we multiply $\Big(x+3x^3 \sin(y)\Big)dx+\Big(x^4 \cos(y)\Big) = 0$ by the factor x^{-1}, we obtain

$$\Big(1 + 3x^2 \sin(y)\Big)dx + \Big(x^3 \cos(y)\Big)dy = 0$$

For this new equation, $M = 1+3x^2 \sin(y)$ and $N = x^3 \cos(y)$. If we test for exactness, we now find that

$$\frac{\partial M}{\partial y} = 3x^2 \cos(y) = \frac{\partial N}{\partial x}$$

and hence $\Big(1 + 3x^2 \sin(y)\Big)dx + \Big(x^3 \cos(y)\Big)dy = 0$ is exact. Upon solving $\Big(1 + 3x^2 \sin(y)\Big)dx + \Big(x^3 \cos(y)\Big)dy = 0$, we find that the solution is given implicitly by $x + x^3 \sin(y) = C$. Since equations $\Big(x + 3x^3 \sin(y)\Big)dx + \Big(x^4 \cos(y)\Big) = 0$ is given implicitly by $x + x^3 \sin(y) = C$.