Theorem: Existence and Uniqueness of Binary Integer Representations

Given any positive integer n, n has a unique representation in the form

$$n = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0$$

where r is a non-negative integer, $c_r = 1$, and $c_j = 1$ or 0 for all $j = 0, 1, 2 \ldots, r-1$.

Proof

We give separate proofs by strong mathematical induction to show first the existence and second the uniqueness of the binary representation.

Existence (proof by strong mathematical induction): Let the property $P(n)$ be the equation

$$n = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0 \qquad \longleftarrow P(n)$$

where r is a non-negative integer, $c_r = 1$, and $c_j = 1$ or 0 for all $j = 0, 1, 2, \ldots, r-1$.

Show that $P(1)$ is true:

Let $r = 0$ and $c_0 = 1$. Then $1 = c_r \cdot 2^r$, and so $n = 1$ can be written in the required form.

Show that for all integers $k \geq 1$, if $P(i)$ is true for all integers i from 1 through k, then $P(k+1)$ is also true:

Let k be an integer with $k \geq 1$. Suppose that for all integers i from 1 through k,

$$i = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0 \qquad \longleftarrow \text{inductive hypothesis}$$

where r is a non-negative integer, $c_r = 1$, and $c_j = 1$ or 0 for all $j = 0, 1, 2, \ldots, r-1$. We must show that $k+1$ can be written as a sum of powers of 2 in the required form.

Case 1 $(k+1)$ **is even**: In this case $\frac{k+1}{2}$ is an integer, and by inductive hypothesis, since $1 \geq \frac{(k+1)}{2} \leq k$, then,

$$\frac{(k+1)}{2} = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0$$

where r is a non-negative integer, $c_r = 1$, and $c_j = 1$ or 0 for all $j = 0, 1, 2, \ldots, r-1$. Multiplying both sides of the equation by 2 gives

$$k+1 = c_r \cdot 2^{r+1} + c_{r-1} \cdot 2^r + \cdots + c_2 \cdot 2^3 + c_1 \cdot 2^2 + c_0 \cdot 2$$

which is a sum of powers of 2 of the required form.

Case 2 $(k+1)$ **is odd**: in the case $\frac{k}{2}$ is an integer, and by inductive hypothesis, since $1 \leq \frac{k}{2} \leq k$, then

$$\frac{k}{2} = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0$$

where r is a non-negative integer, $c_r = 1$, and $c_j = 1$ or 0 for all $j = 0, 1, 2, \ldots, r-1$. Multiplying both sides of the equation by 2 and adding 1 gives

$$k+1 = c_r \cdot 2^{r+1} + c_{r-1} \cdot 2^r + \cdots + c_2 \cdot 2^3 + c_1 \cdot 2^2 + c_0 \cdot 2 + 1$$

which is also a sum of powers of 2 of the required form.

The preceeding arguments show that regardless of whether $k+1$ is even or ordd, $k+1$ has a presentation of the required form. [Or, in other words, $P(k+1)$ is true.]

Uniqueness: To proveuniqueness, suppose that there is an integer n with two different representations as a sum of non-negative integer powers of 2. Equating the two representations and canceling all identical terms gives

$$2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_1 \cdot 2 + c_0 = 2^s + d_{s-1} \cdot 2^{s-1} + \cdots + d_1 \cdot 2 + d_0 \qquad \longleftarrow 5.4.1$$

where r and s are non-negative integers, and each c_i and each d_i equal 0 or 1. Without loss of generality, we may assume that $r < s$. But by the formula for the sum of a geometric sequence and because $r < s$

$$2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_1 \cdot 2 + c_0 \leq 2^r + 2^{r-1} + \cdots + 2 + 1 = 2^{r+1} - 1 < 2^s$$

Thus

$$2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_1 \cdot 2 + c_0 < 2^s + d_{s-1} \cdot 2^{s-1} + \cdots + d_1 \cdot 2 + d_0$$

which contradicts equation (5.4.1). Hence the supposition is false, so any integer n has only one representation as a sum of non-negative integer powers of 2.