Example 9: For the initial value problem

$$\frac{dy}{dx} = 3y^{\frac{2}{3}}, \quad y(2) = 0$$

does the Existence and Uniqueness of Solution Theorem imply the existence of a unique solution?

Solution

Here $f(x, y) = 3y^{\frac{2}{3}}$ and $\frac{\partial f}{\partial y} = 2y^{-\frac{1}{3}}$. Unfortunately $\frac{\partial f}{\partial y}$ is not continuous or even defined when $y = 0$. Consequently, there is no rectangle containing $(2, 0)$ in which both f and $\frac{\partial f}{\partial y}$ are continuous. Because the hypotheses of the Existence and Uniquenes of Solution Theorem do not hold, we cannot use the Existence and Uniquenes of Solution Theorem to determine whether the initial value problem does or does not have a unique solution. It turns out that this initial value problem has more than one solution.