Chapter 4. Bivariate Distributions

4.1. Distributions of Two Discrete Random Variables

1. If $X_1, X_2, \cdots, X_n$ are random variables, then the ordered n-tuple

$$(X_1, X_2, \cdots, X_n)$$

is usually called a random vector. Each X_i has a distribution. However, to understand the relations between the random variables, we need study their joint distribution.

2. Let use begin with an example:

3. *Example.* Suppose that $T \sim$ Bernoulli $(\frac{1}{2})$. Let T_1, T_2, T_3 be a random sample of size three from the population X. Then, T_1, T_2, T_3 are mutually independent, and each $T_i \sim$ Bernoulli $(\frac{1}{2})$.

   The sample space is

$$\Omega = \{000, 001, 010, 100, 110, 101, 011, 111\}.$$

   Here, for example, the entry 001 is the short notation for the event

$$(T_1 = 0, T_2 = 0, T_3 = 1).$$

   Let

$$X = X_1 + X_2, \quad Y = X_2 + X_3.$$

Then, X and Y are random variables as well. X has possible values $0, 1, 2$, and X has pmf

x	0	1	2
$f_1(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

For example,

$$f_1(0) = P(X = 0) = P(\{000, 001\}) = \frac{2}{8} = \frac{1}{4}.$$

In fact, it is clear that $X \sim$ Binomial $(2, \frac{1}{2})$. Similarly, Y has possible values $0, 1, 2$, and Y has pmf

y	0	1	2
$f_2(y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Again, it is clear that $Y \sim$ Binomial $(2, \frac{1}{2})$.

To understand the relation between X and Y, we need study their joint distribution. For each $i = 0, 1, 2$ and each $j = 0, 1, 2$, we define

$$p_{ij} = f(i, j) = P(X = i, Y = j).$$

For example,

$$f(0, 1) = P(X = 0, Y = 1) = P(\{001\}) = \tfrac{1}{8}.$$

Calculation shows that

$$\begin{aligned} &f(0,0) = \tfrac{1}{8}, \quad f(0,1) = \tfrac{1}{8}, \quad f(0,2) = 0, \\ &f(1,0) = \tfrac{1}{8}, \quad f(1,1) = \tfrac{2}{8}, \quad f(1,2) = \tfrac{1}{8}, \\ &f(2,0) = 0, \quad f(2,1) = \tfrac{1}{8}, \quad f(2,2) = \tfrac{1}{8}. \end{aligned}$$

We summary all this information into a table:

$X \backslash f \backslash Y$	0	1	2
0	$\frac{1}{8}$	$\frac{1}{8}$	0
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$

This function f is call the joint pmf of X and Y.

Note that X has range

$$S_1 = \{0, 1, 2\}\,;$$

Y has range

$$S_2 = \{0, 1, 2\}\,.$$

It follows that the random vector (X, Y) has range

$$S = S_1 \times S_2 = \{ \begin{array}{ccc} (0,0) & (0,1) & (0,2) \\ (1,0) & (1,1) & (1,2) \\ (2,0) & (2,1) & (2,2) \end{array} \}.$$

The joint pmf f can be considered as a function defined on $S = S_1 \times S_2$. Note that

$$\sum_{i \in S_i} \sum_{j \in S_2} f(i, j) = 1.$$

We can use f to calculate some probabilities — for example,

$$\begin{aligned} & P(X = Y) \\ = \; & P(X = Y = 0) + P(X = Y = 1) + P(X = Y = 2) \\ = \; & f(0,0) + f(1,1) + f(2,2) \\ = \; & \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{1}{2}. \end{aligned}$$

4. *Definition.* Suppose that X and Y are discrete random variables. Denote by S_1 the range of X, and let S_2 be the range of Y. Let $S = S_1 \times S_2$.

   For each $(x, y) \in S$, let

$$f(x, y) = P(X = x, Y = y).$$

   Then f is called the joint probability mass function of X and Y.

5. Some properties of this joint pmf f are listed below:

6. *Theorem.*

   (a) $0 \leq f(x,y) \leq 1$.

   (b) $\sum_{(x,y)\in S} f(x,y) = 1$.

   (c) If $A \subset S$, then

$$P((X,Y) \in A) = \sum_{(x,y)\in A} f(x,y).$$

7. *Definition.* X and Y are discrete random variables, S_1 is the range of X, S_2 is the range of Y. Let $S = S_1 \times S_2$. Let f be the joint probability mass function of X and Y.

   (a) For each $x \in S_1$, let

   $$f_1(x) = P(X = x) = \sum_{y \in S_2} f(x, y).$$

   Then $f_1 : S_1 \to \mathbb{R}$ is called the marginal pmf of X.

   (b) For each $y \in S_2$, let

   $$f_2(y) = P(Y = y) = \sum_{x \in S_1} f(x, y).$$

   Then $f_2 : S_2 \to \mathbb{R}$ is called the marginal pmf of Y.

(definition continues on the page . . .)

And, if, for each $(x, y) \in S$,

$$f(x, y) = f_1(x) f_2(y),$$

then we say X and Y are independent.

8. We continue to study Example 2 of this section: The joint pmf of X and Y is given in the table We summary all this information into a table:

$X\backslash f\backslash Y$	0	1	2
0	$\frac{1}{8}$	$\frac{1}{8}$	0
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$

If we add the marginal distributions of X and Y to this table, we get:

$X\backslash f\backslash Y$	0	1	2	f_1
0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{4}$
1	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{2}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
f_2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

Since

$$f(0,2) = 0, \quad f_1(0)f_2(2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16},$$

we have

$$f(0,2) \neq f_1(0)f_2(2).$$

In other words,

$$P(X = 0, Y = 2) \neq P(X = 0)P(Y = 2).$$

Therefore, X and Y are not independent.