Announcements

- Homework: please start early
- Please do not copy from anywhere and work on your code yourself. Our detection agents are super smart!
- Please don't post any material from this class, including your homework and related materials, to any public places, such as GitHub or others, on the Internet !
- (Note: GitHub **private** repositories are OK)

Modularity (review)

- **Modularity of partitioning S of graph G:**
  - **Q ∝ $\sum_{s \in S}$ [(# edges within group *s*) – (expected # edges within group *s*)]**
  - $Q(G,S) = \frac{1}{2m} \sum_{s \in S} \sum_{i \in s} \sum_{j \in s} \left(A_{ij} - \frac{k_i k_j}{2m} \right)$

Normalizing cost.: -1<Q<1

All possible pairs of nodes in s: both (i,j),(j,i), but not (i,i),(j,j)

The actual existence of the edge in s

The expected existence of the edge (based on G)

- Hints
  - Determine *m* and *ki*, *kj* based on *G* (before cuttings)
  - Determine *Aij* based on *s* in *S* (after cuttings)
  - If s={n_x} is a singleton, then use 2*(0 - $k_x k_x/2m$)

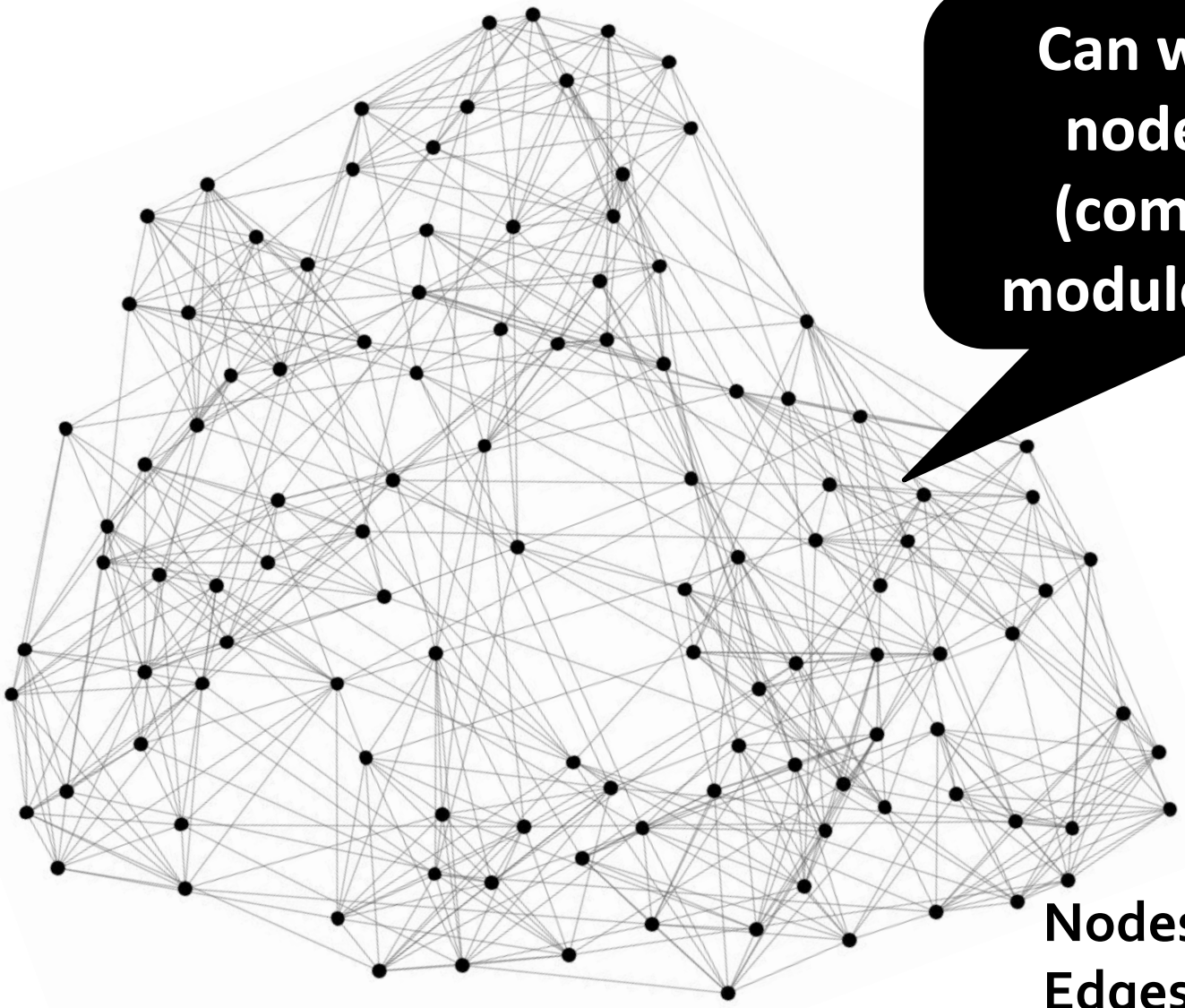
Analysis of Large Graphs: Overlapping Communities

Professor Wei-Min Shen
University of Southern California

Thanks for source slides and material to:
J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets
http://www.mmds.org

Note to other teachers and users of these slides: We would be delighted if you found this our material useful in giving your own lectures. Feel free to use these slides verbatim, or to modify them to fit your own needs. If you make use of a significant portion of these slides in your own lecture, please include this message, or a link to our web site: http://www.mmds.org

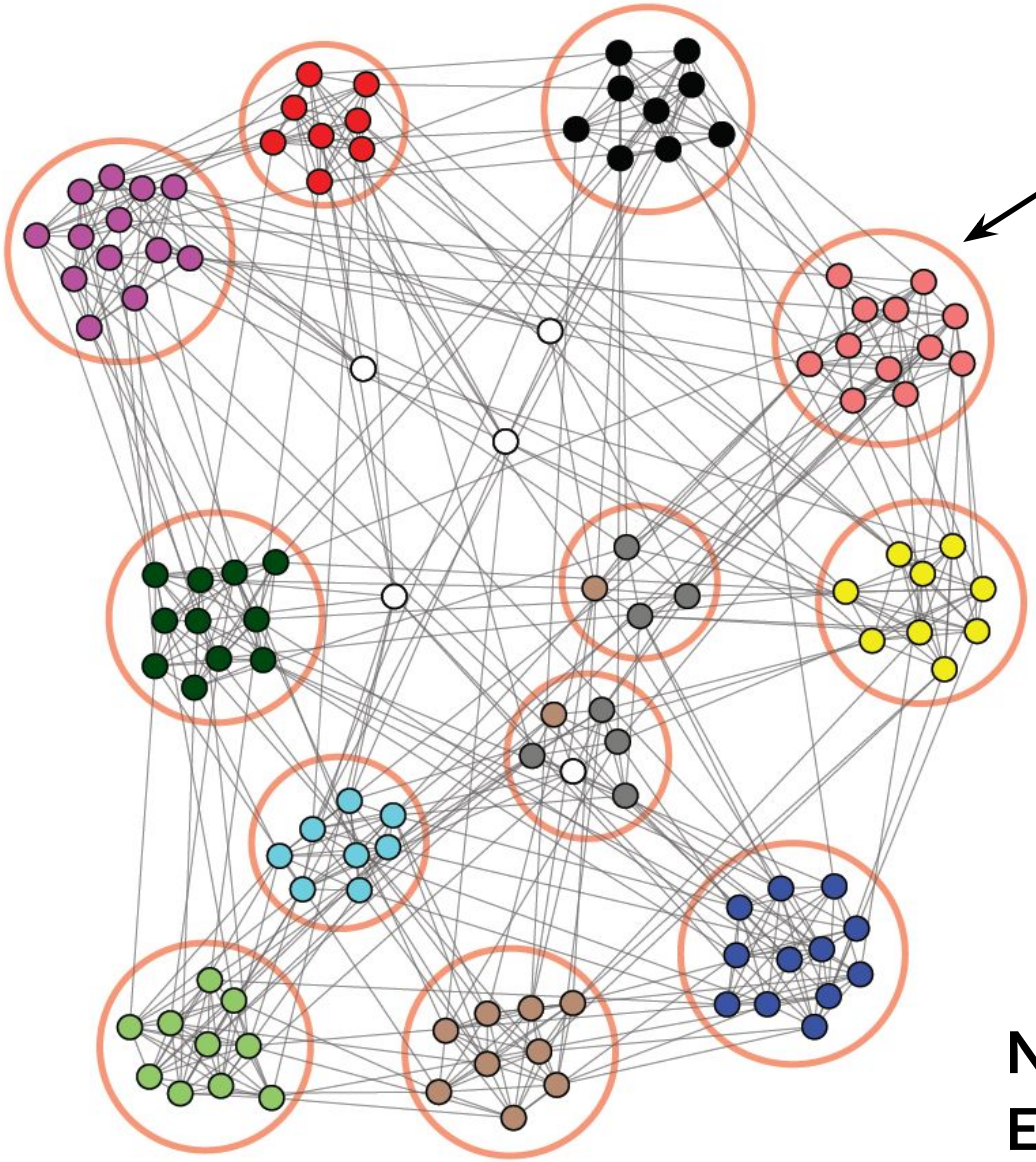
Identifying Communities



Can we identify node groups? (communities, modules, clusters)

Nodes: Football Teams
Edges: Games played

NCAA Football Network

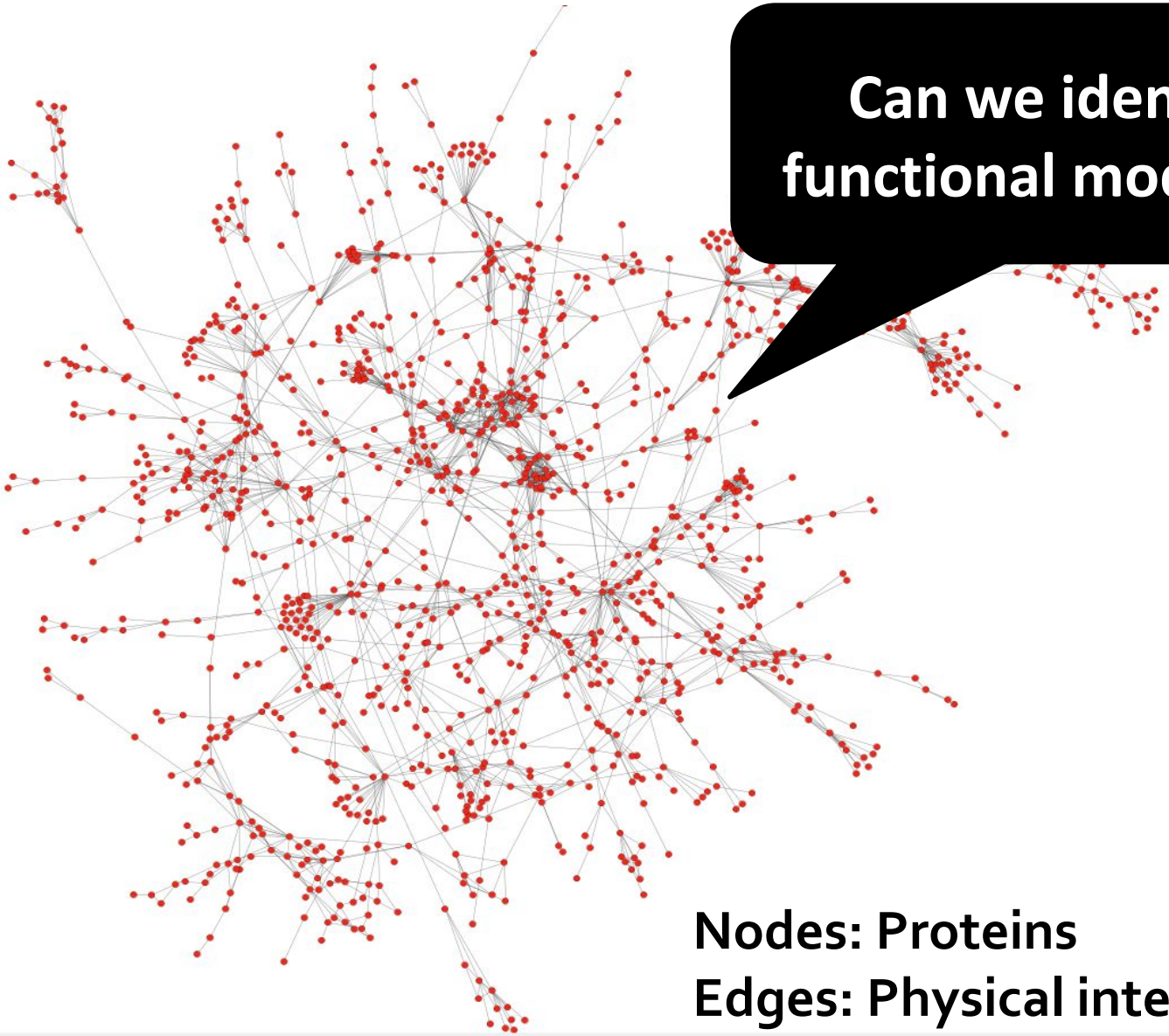


NCAA conferences

- Mid American
- Big East
- Atlantic Coast
- SEC
- Conference USA
- Big 12
- Western Athletic
- Pacific 10
- Mountain West
- Big 10
- Sun Belt
- Independents

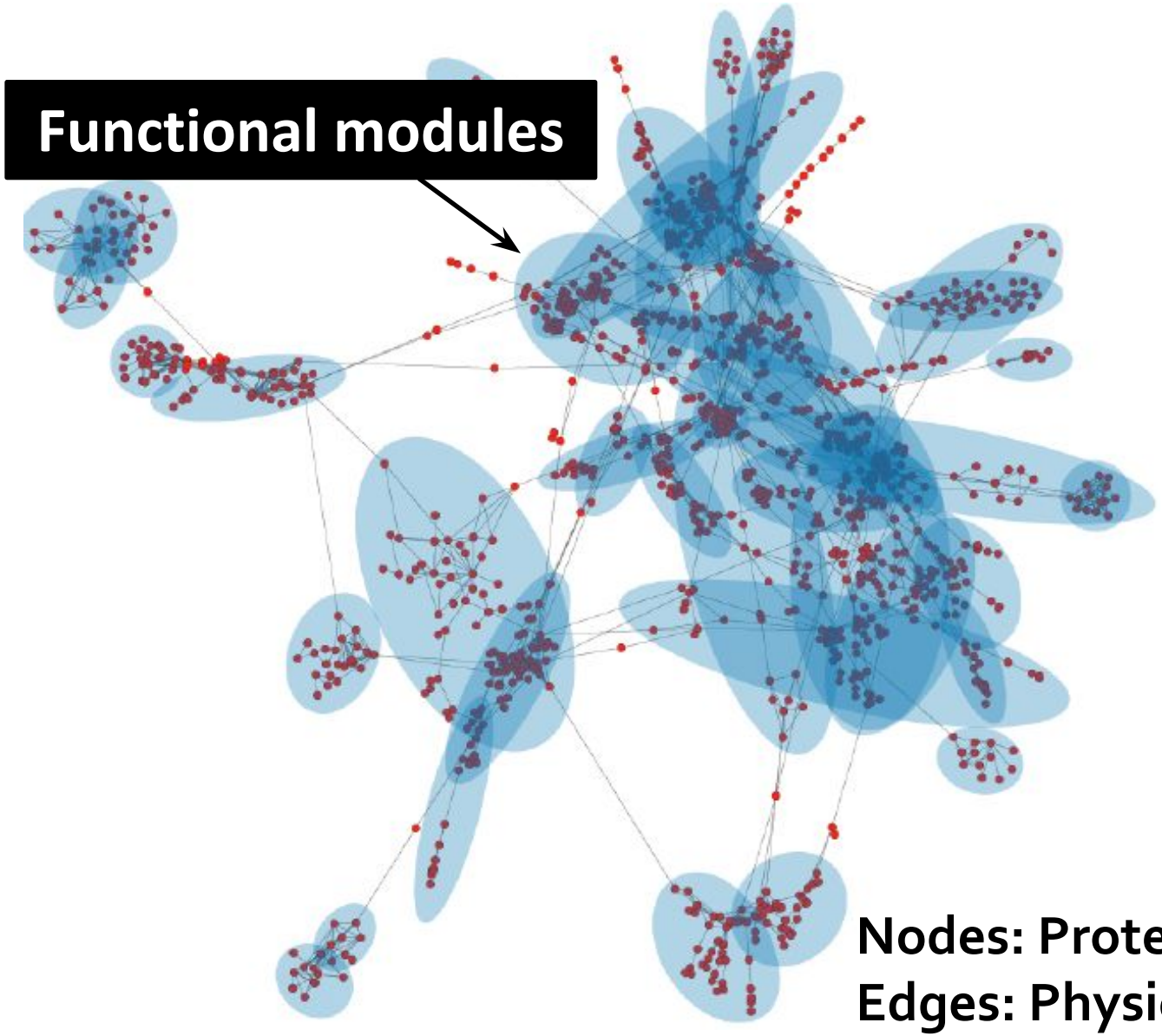
Nodes: Football Teams
Edges: Games played

Protein-Protein Interactions



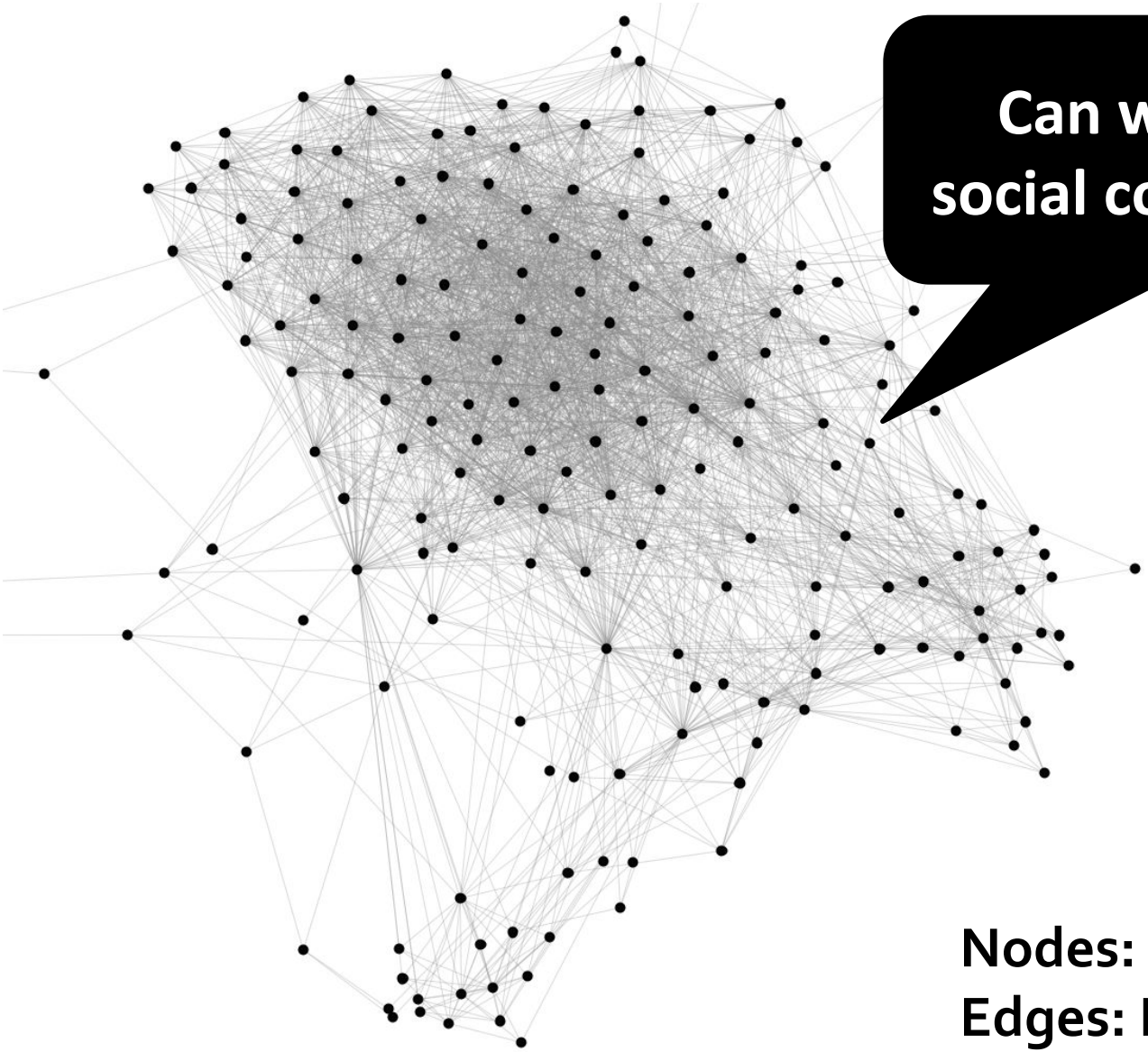
Nodes: Proteins
Edges: Physical interactions

Protein-Protein Interactions



Nodes: Proteins
Edges: Physical interactions

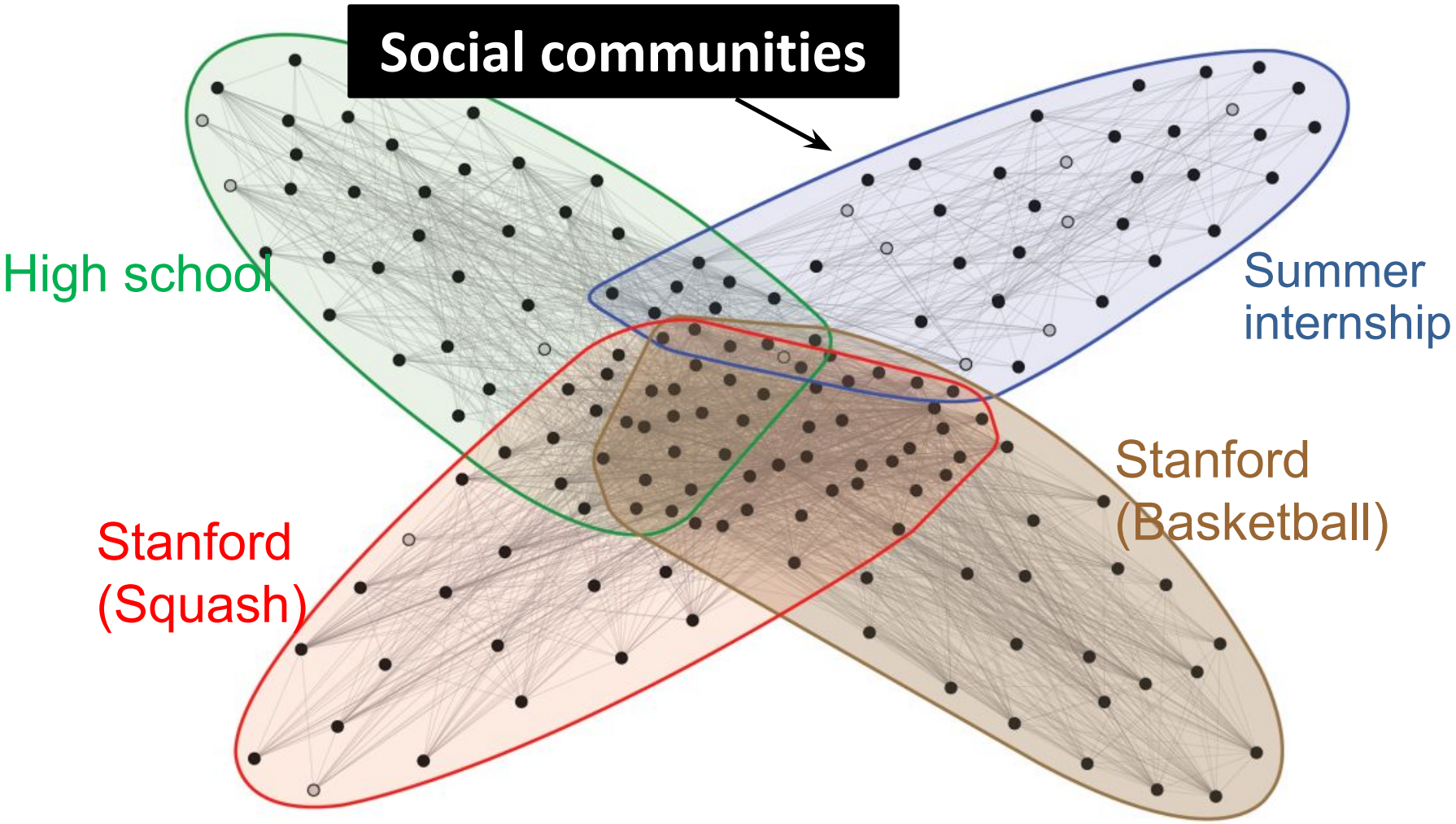
Facebook Network



Can we identify social communities?

Nodes: Facebook Users
Edges: Friendships

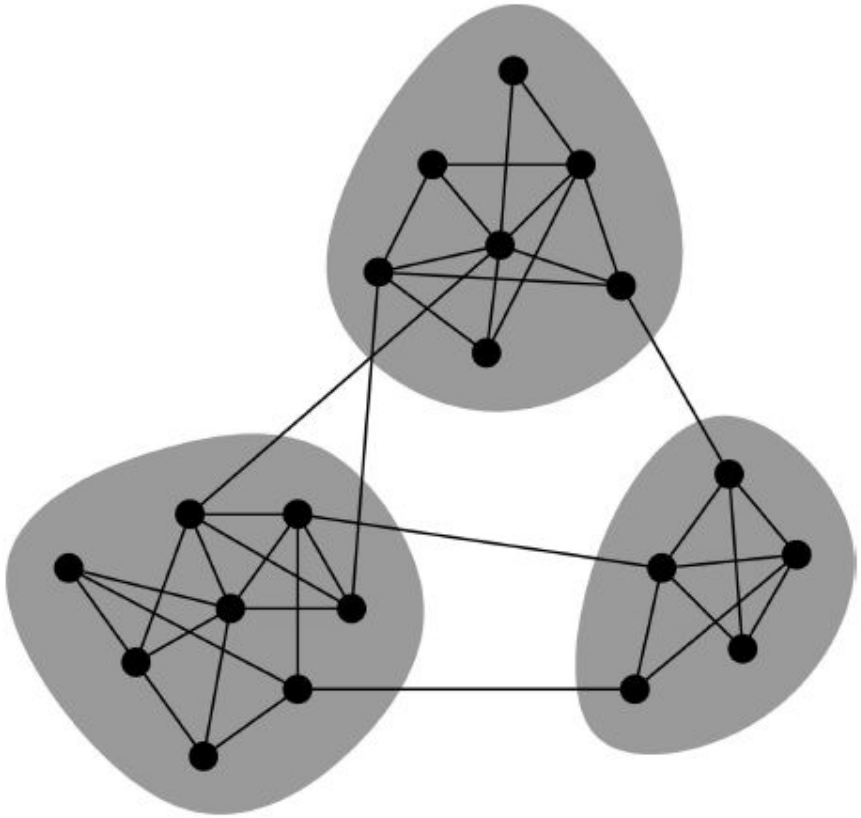
Facebook Network

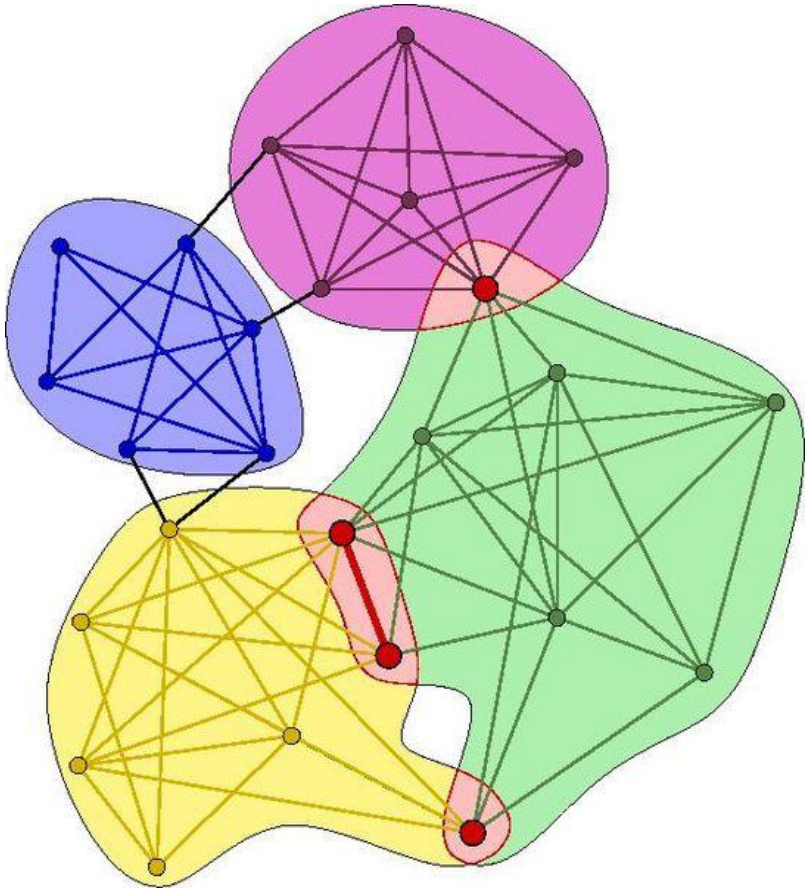


Nodes: Facebook Users
Edges: Friendships

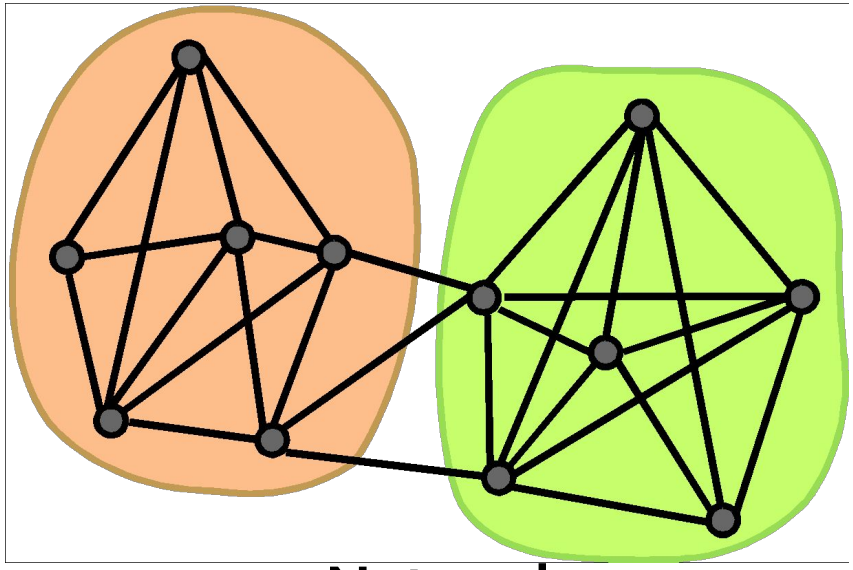
Overlapping Communities

- **Non-overlapping vs. overlapping communities**

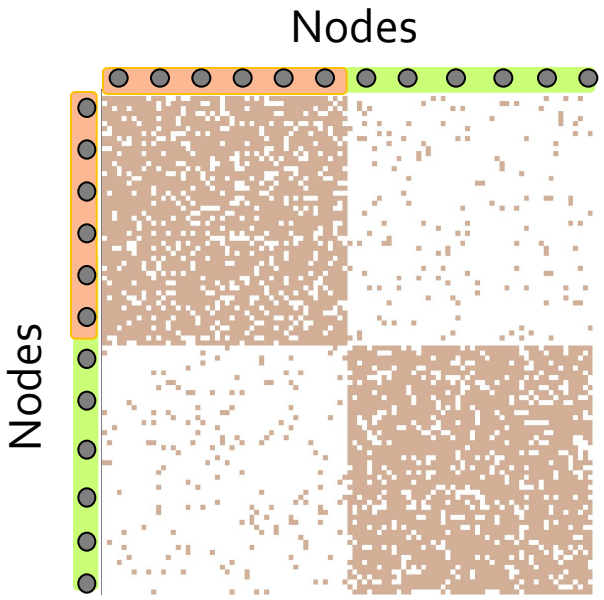




Non-overlapping Communities



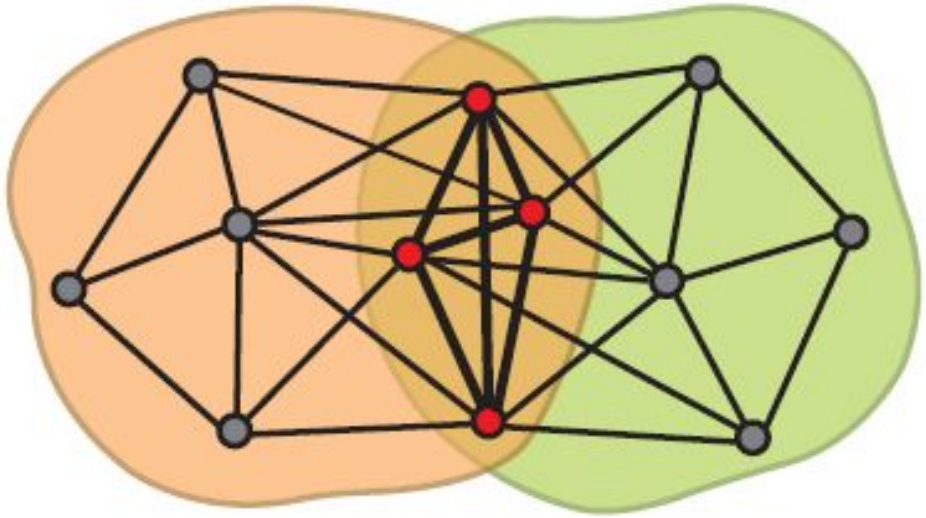
Network

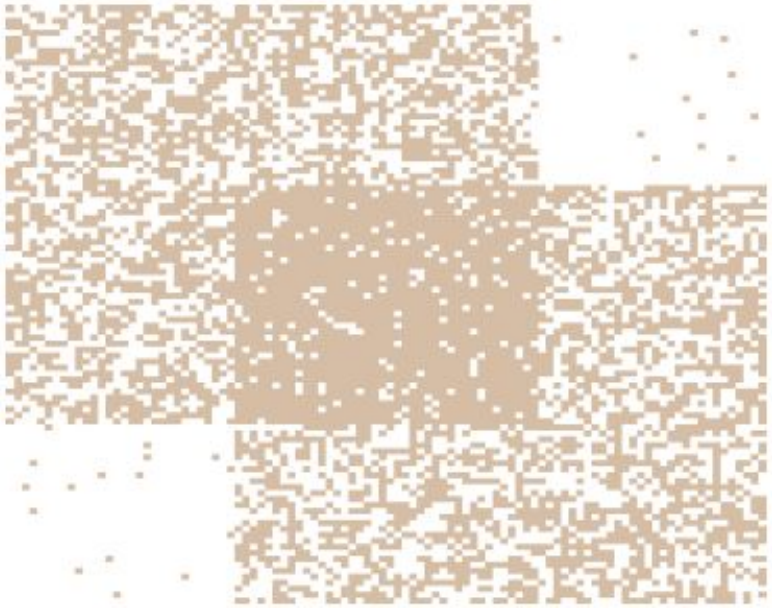


Adjacency matrix

Communities as Tiles!

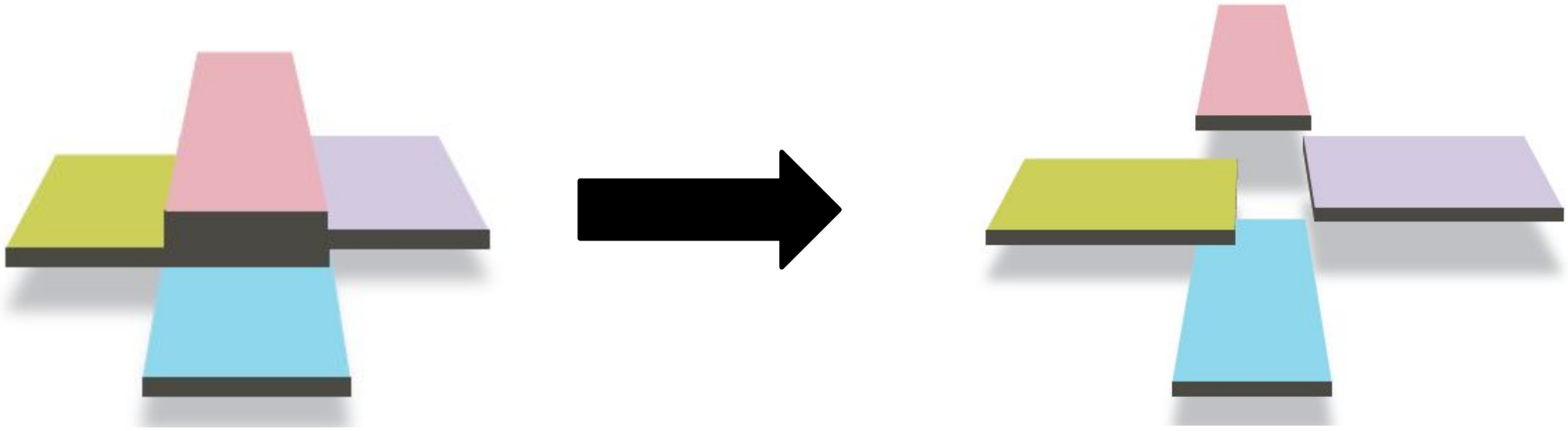
- **What is the structure of community overlaps:**
  Edge density in the overlaps is higher!





Communities as "tiles"

Recap so far…

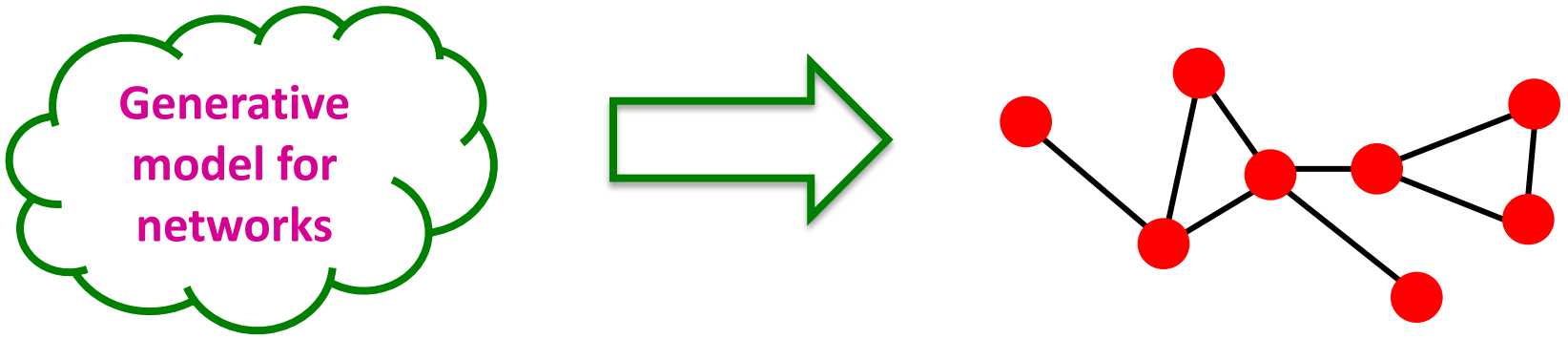


Communities in a network

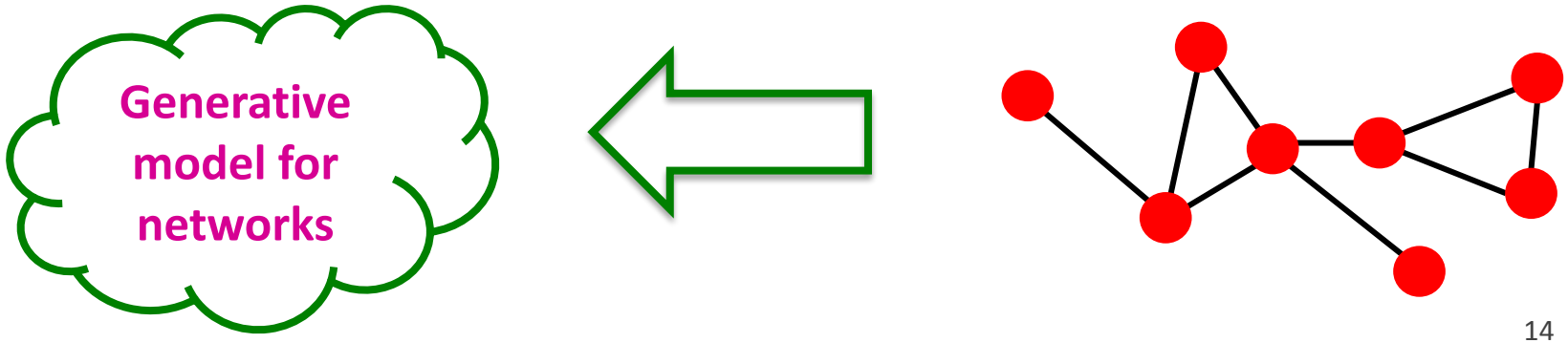
This is what we want!

Plan of attack

- **1) Given a model, we generate the network:**

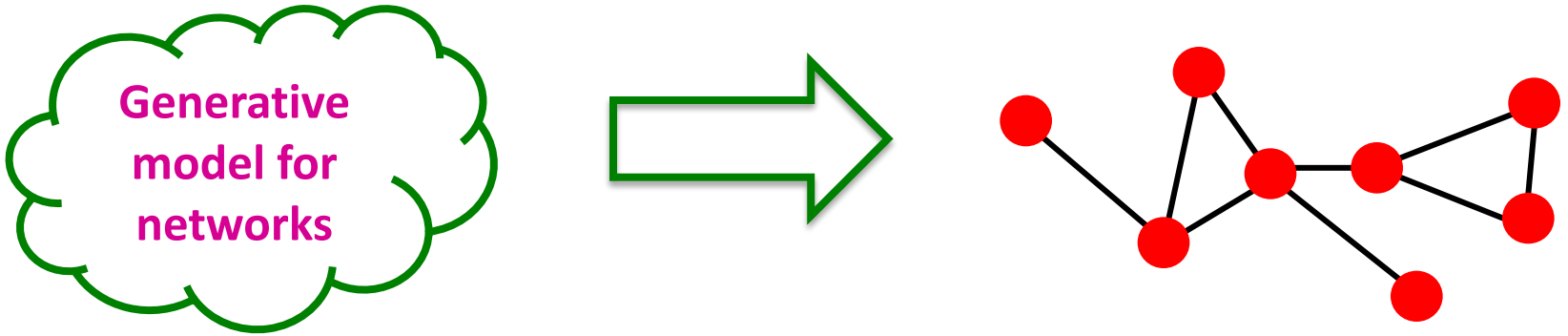


- **2) Given a network, find the "best" model**



Model of networks

- **Goal: Define a model that can generate networks**
  - The model will have a set of "parameters" that we will later want to estimate (and detect communities)



- **Q: Given a set of nodes, how do communities "generate" edges of the network?**

Maximum-Likelihood Estimation

Maximum-Likelihood Estimation

- Assume a generative process (the model M)
  - for creating instances of some artifact, for example, "friends graphs" $G_1, \ldots, G_{25}, \ldots$
- The model has parameters
  - for determining the probability of generating any particular instance of the artifact
  - the "likelihood" of the parameter values

$P(G_5 | M) = ?$

- The value of the parameters that gives the largest value of the likelihood is the "correct" model for the observed artifact

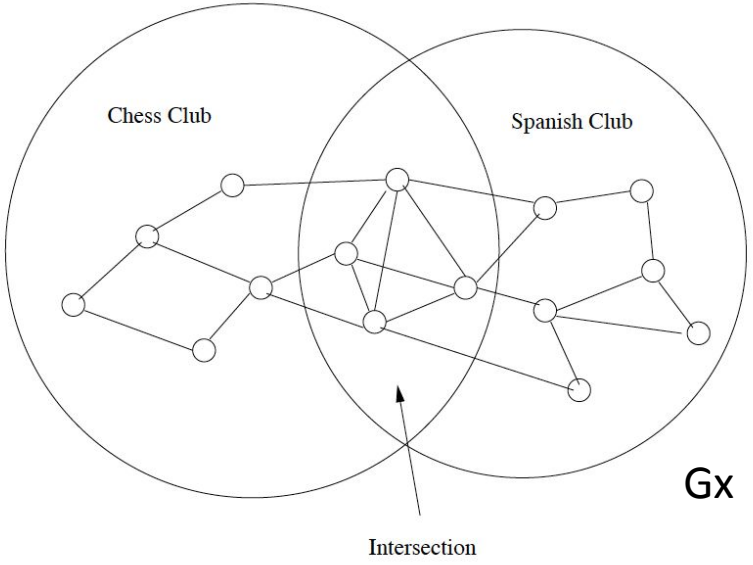
What is the simplest model for a graph?
How many parameters does it have?

Simple MLE Example (1)

- Model: Each edge is present with probability *p* with the presence or absence of each edge chosen independently
- Parameter: *p* *// this model M has only one parameter*
- Example: a graph with 15 nodes and 23 edges

- How many pairs of nodes do we have in total? $C_{15}^{2} = 105$
- If each edge is connected by a probability *p, the probability of having the graph on the right is*

$$P(G_x|M) = p^{23}(1-p)^{82}$$



Gx

Simple MLE Example (2)

- **NOW, the other way around**
- Given this graph, what is the best value for *p* ?
- Take the first derivative of the probability function $p^{23}(1-p)^{82}$ and set it to 0 for finding the maximum/minimum of *p*, as follows

$$23p^{22}(1-p)^{82} - 82p^{23}(1-p)^{81} = 0$$

$$p^{22}(1-p)^{81}\big(23(1-p) - 82p\big) = 0$$

- Maximum or minimum of $p^{23}(1-p)^{82}$ *happens when*
  - Either p=0 or 1 or 23(1-p)-82p = 0
    - The likelihood of generating the graph in the previous slide is maximized when 23(1-p)-82p = 0
- *That is:* $p = 23/105$, therefore,

$P(G_x|M) = p^{23}(1-p)^{82}$=(23/105)^23 * (1-23/105)^82 = 1.0656135e-24

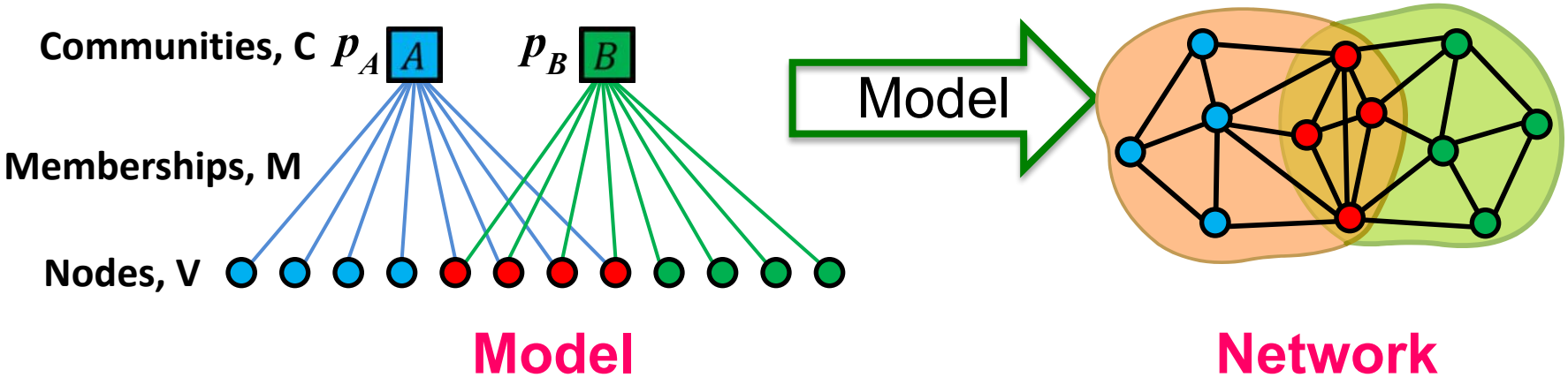
MLE Approach

- Given a graph *G*
  - Write up a probability function *f*(*p*) of *G*
  - To find the best *p* for *f*(*p*):
    - Set the derivative of *f*(*p*) to be 0: $f'(p) = 0$
    - Solve the equation $f'(p)=0$ to get *p*
- *The result p will be the "best" model for G*

Affiliation-Graph Model

Community-Affiliation Graph

Each community has a model!



- **Generative model B(V, C, M, $\{p_c\}$) for graphs:**
  - Nodes **V**, Communities **C**, Memberships **M**
  - Each community **c** has a model which is a probability p_c
  - Later we fit the model to networks to detect communities

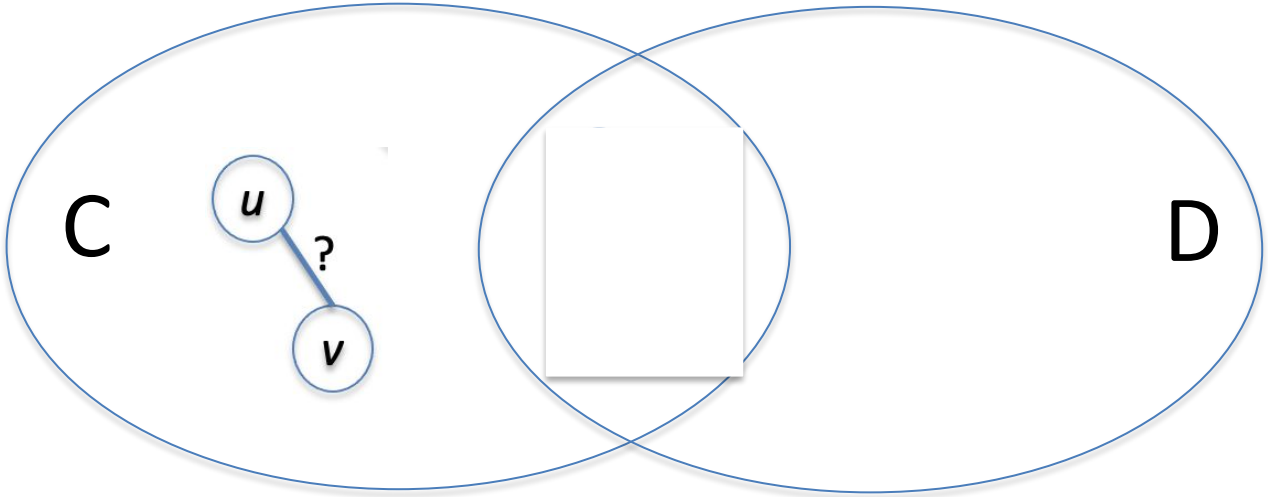
Affiliation-Graph Model (AGM)

- The Community-Affiliation Graph
  - Given
    - A number of communities, and
    - A number of individuals (nodes)
  - Two sets of parameters
    - The membership parameters
      - Each community can have any set of individuals as members (binary memberships)
    - A probability *Pc* for each community *C*
      - The probability that two members of a community C are connected because they are both members of C

Node: If a pair of nodes is in two or more communities, then there is an edge between them if any of the communities of which both are members justifies that edge according to rule 3 (see the next slide for an example)

Should *u* and *v* have an edge?

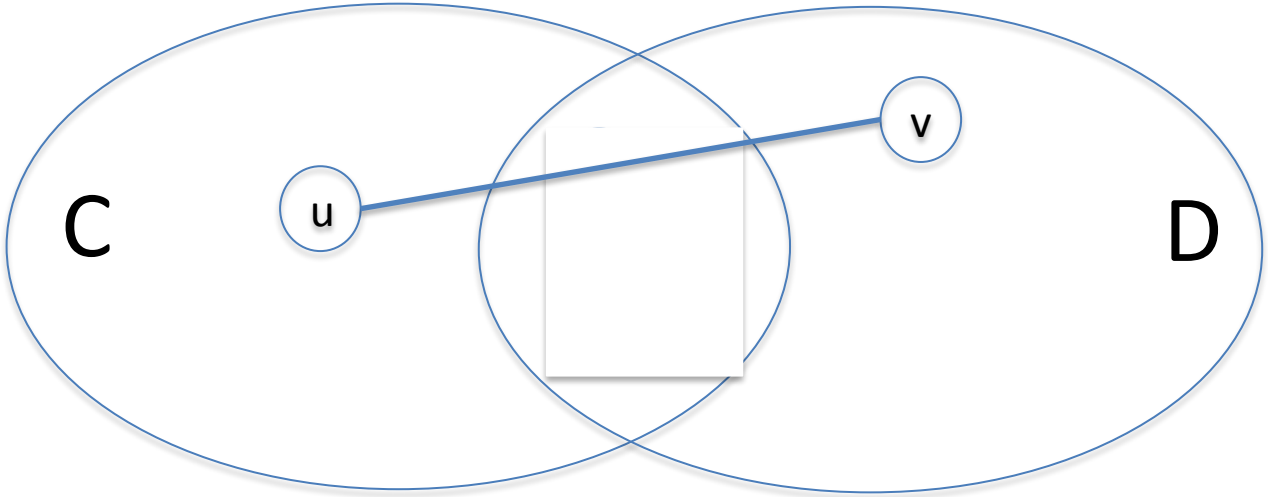
- Two communities C and D
  - Node u and v are in a single community



  - What is the probability of (*u*, *v*) exists: P_C

Should *u* and *v* have an edge?

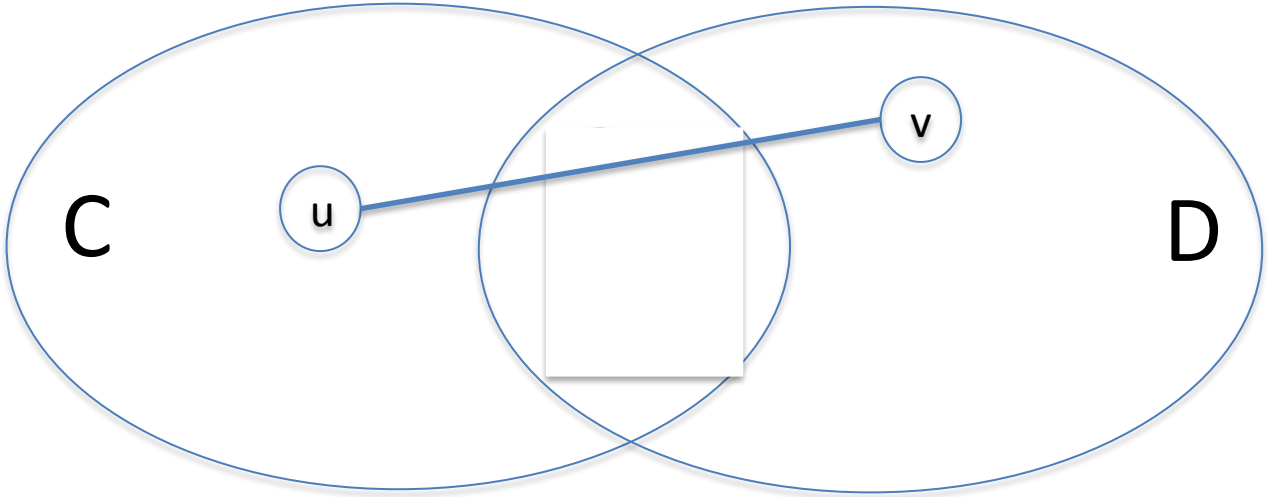
- Two communities C and D
  - Node u and v are in two different communities



  - What is the probability of (*u*, *v*) exists: ?

Should *u* and *v* have an edge?

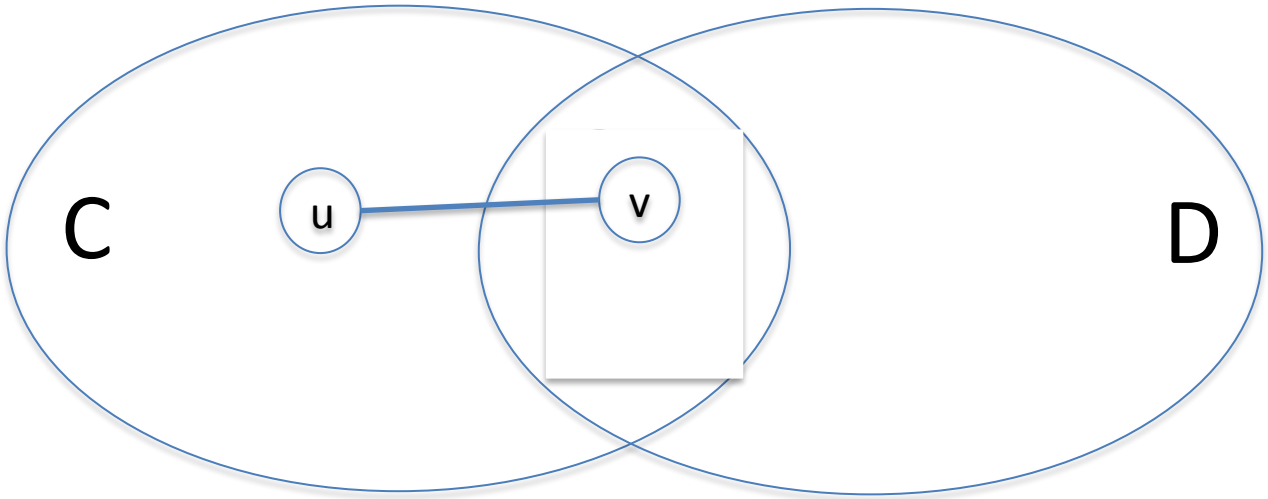
- Two communities C and D
  - Node u and v are in two different communities



  - What is the probability of (*u*, *v*) exists: ? // as small as possible

Should *u* and *v* have an edge?

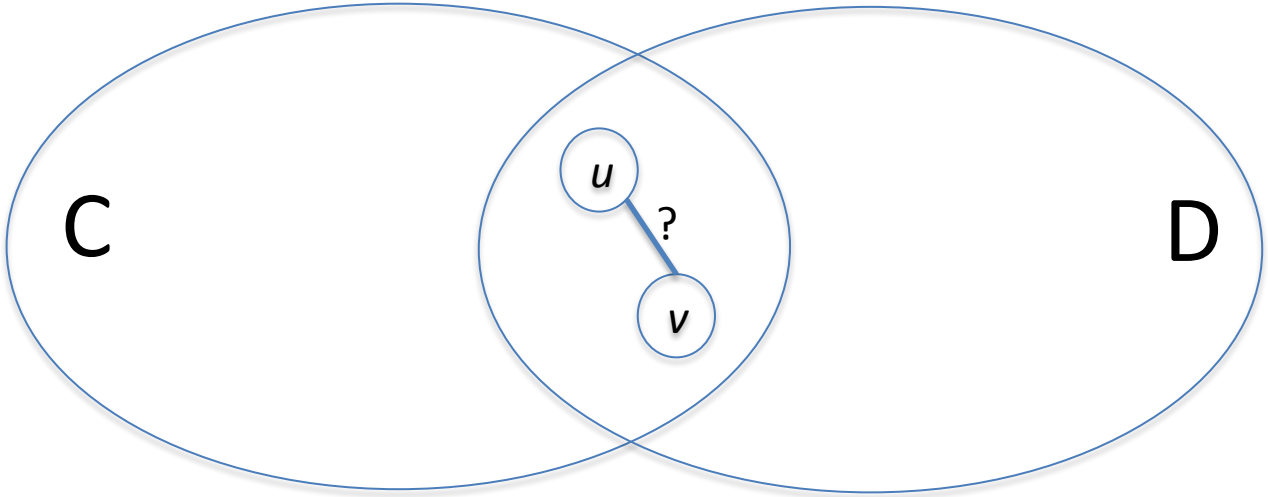
- Two communities C and D
  - Node u and v are in as follows



  - What is the probability of (*u*, *v*) exists: ?

Should *u* and *v* have an edge?

- Two communities C and D
  - Node u and v are in both communities



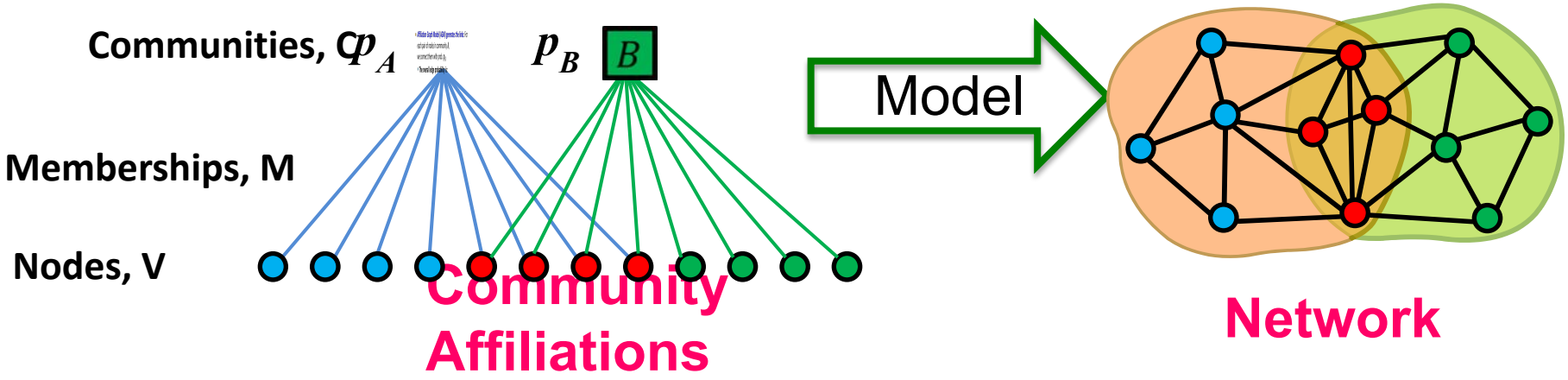
  - What is the probability of no edge between *u, v:* $(1 - P_C)(1 - P_D)$
  - What is the probability of (u, v) exists: $1 - (1 - P_C)(1 - P_D)$

AGM: Generative Process (1)

- Nodes: *u*, *v*; Edge: (*u*, *v*)
  - *u* and *v* are in both communities *C* and *D* but not in other communities
  - Given the probability that two nodes in *C* (or *D*) are connected is P_C (or P_D)
  - The probability of no edge between *u, v* is $(1 - P_C)(1 - P_D)$
  - The probability of (*u, v*) exists is $1 - (1 - P_C)(1 - P_D)$
- General case: If *u* and *v* are members of a set *M* of communities, the probability of an edge between *u* and *v* is:

$$p_{uv} = 1 - \prod_{C \text{ in } M} (1 - p_C)$$

AGM: Generative Process (2)



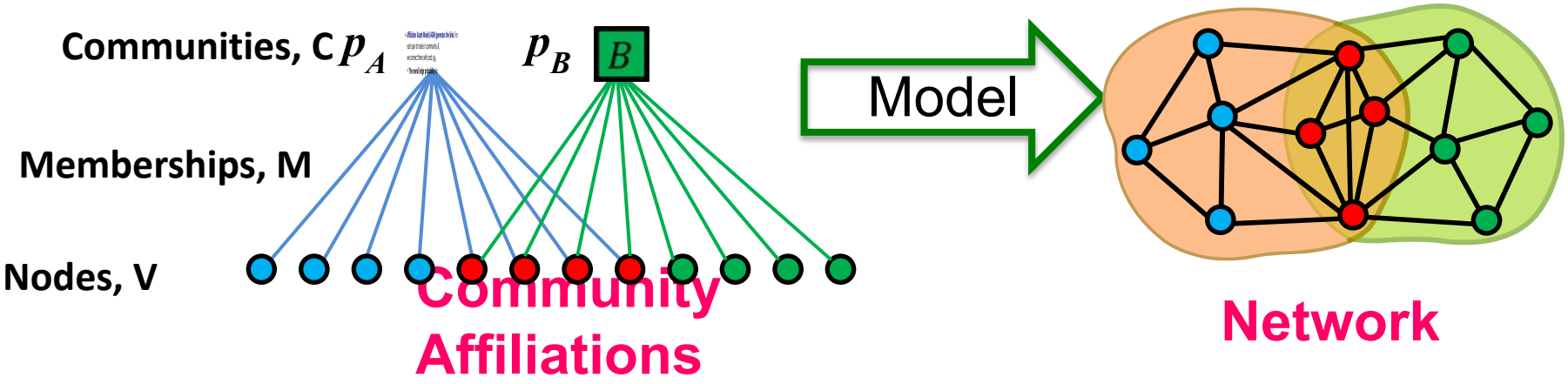
- Affiliation Graph Model (AGM) Generates the links:
  - For each pair of nodes *u* and *v* in community C, we connect them with probability p_c, then the overall edge probability is:

$$P(u,v) = 1 - \prod_{c \in M_u \cap M_v} (1 - p_c)$$

    - where M_u is the set of community u belongs to
  - If u and v share no community, then we let $P(u,v) = \varepsilon$, where ε is very tiny number // very unlikely to have such an edge

Think of this as an "OR" function: If at least 1 community says "YES" we create an edge

AGM: Generative Process (2)



- **Affiliation Graph Model (AGM) generates the links:**

For each pair of nodes in community A, we connect them with prob. p_A

- **The overall edge probability between any u and v:**

$$P(u,v) = 1 - \prod_{c \in M_u \cap M_v} (1 - p_c)$$

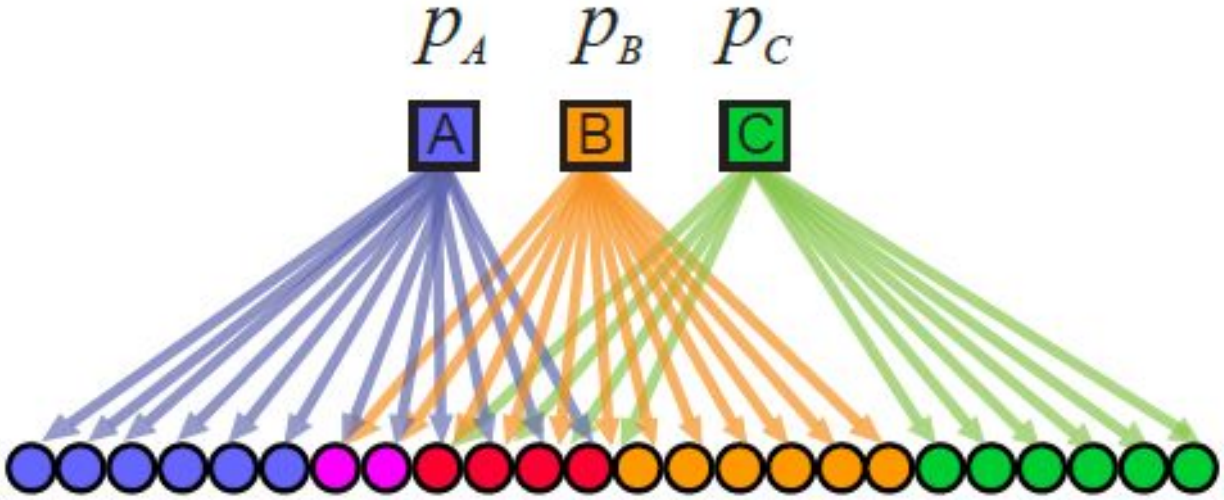
epsilon ε is a very tiny number

M_u … set of communities node u belongs to

If u, v share no communities: $P(u,v) = \varepsilon$

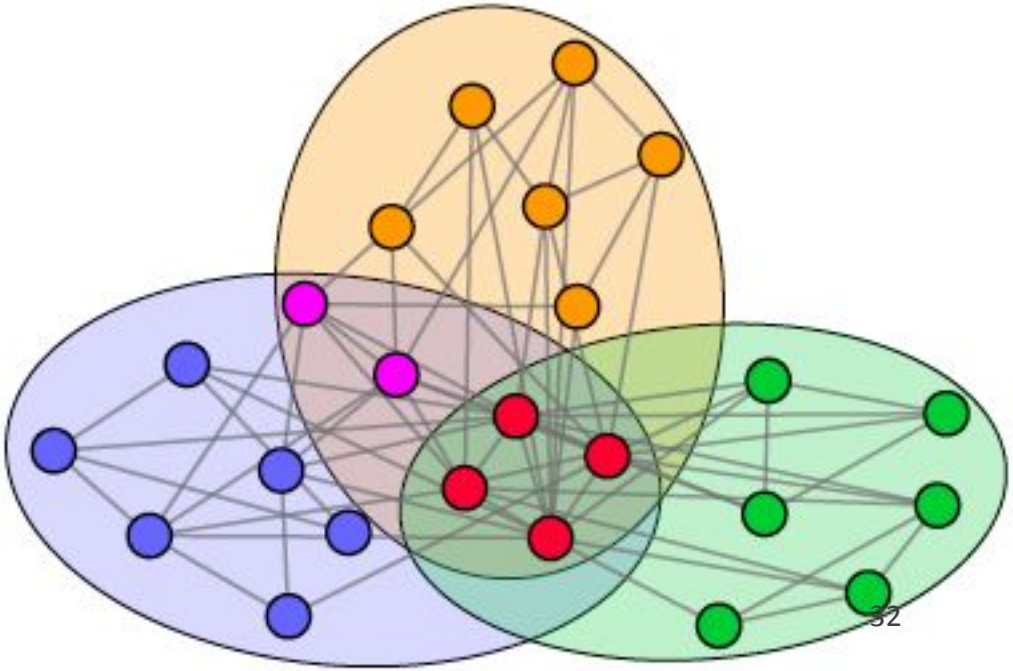
Think of this as an "OR" function: If at least 1 community says "YES" we create an edge

Recap: AGM networks



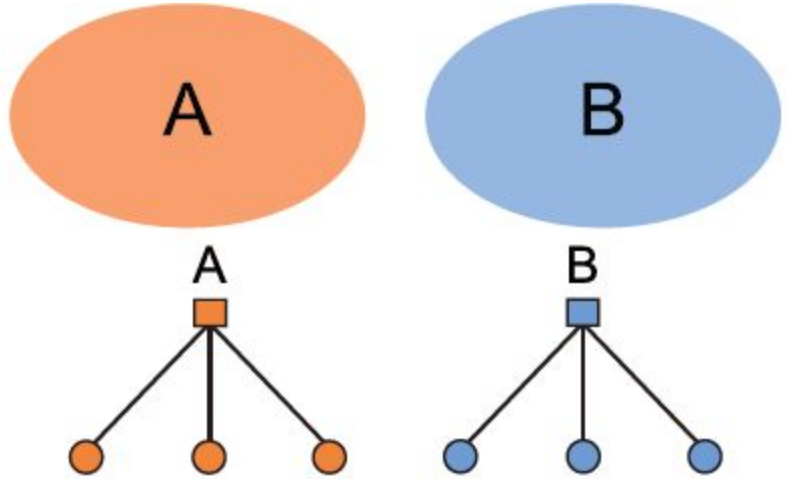
Model

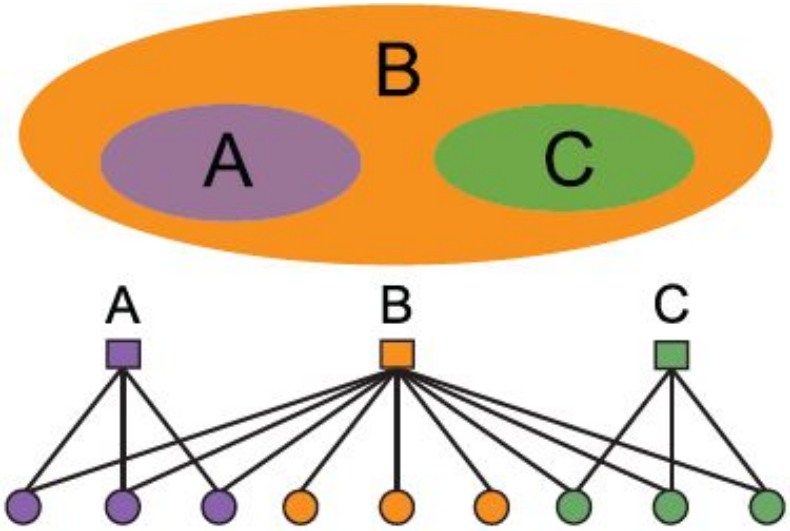
Network

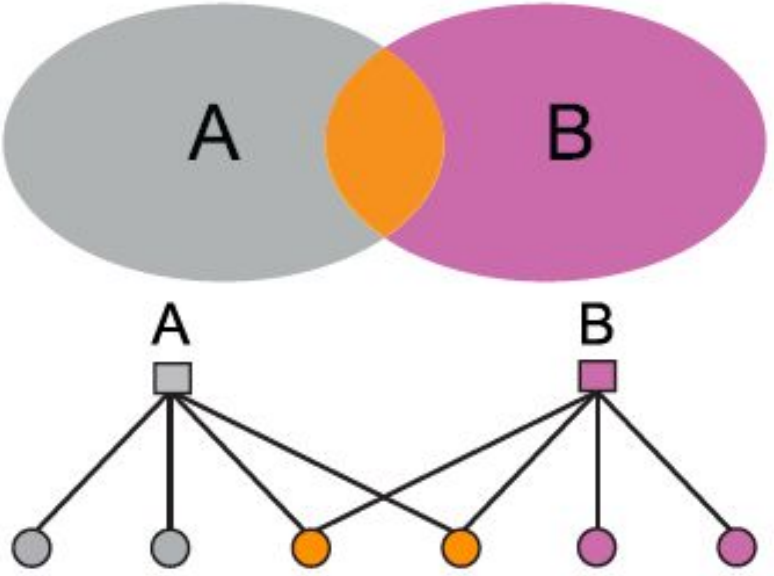


AGM: Flexibility

- **AGM can express a variety of community structures:**
  Non-overlapping, Overlapping, Nested



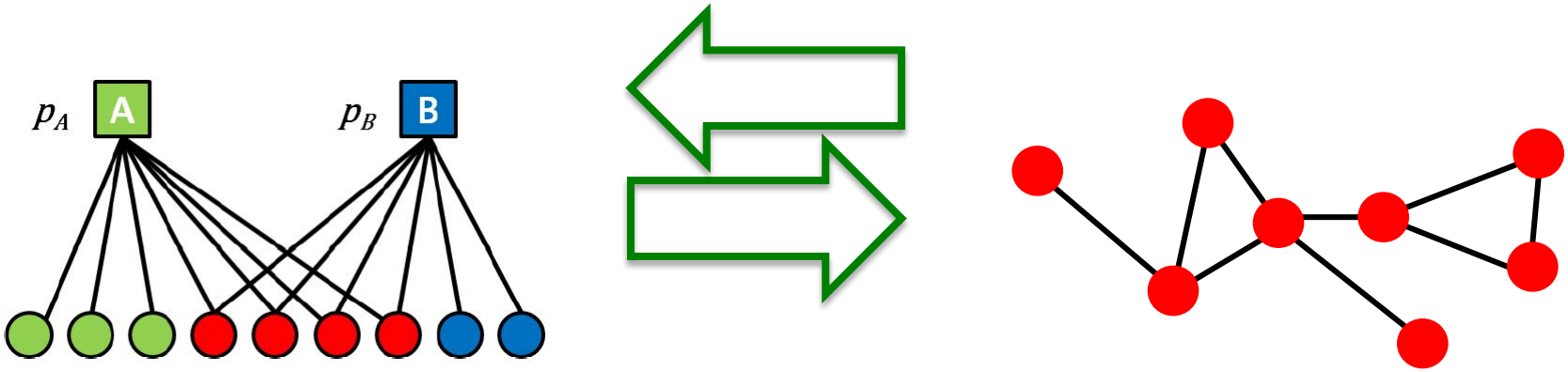




HOW DO WE DETECT COMMUNITIES WITH AGM?

Detecting Communities

- **Detecting communities with AGM:**



Given a Graph $G(V, E)$, find the Model

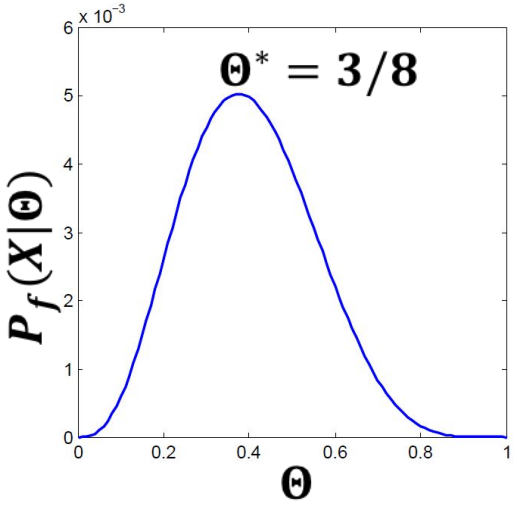
1) Affiliation graph ***M***
2) Number of communities **C**
3) Parameters $\boldsymbol{p_c}$

Maximum Likelihood Estimation

- **Maximum Likelihood Principle (MLE):**
  - **Given:** Data $\boldsymbol{X}$
  - **Assumption:** Data is generated by some model $\boldsymbol{f}(\boldsymbol{\Theta})$
    - $\boldsymbol{f}$ **… model**
    - $\boldsymbol{\Theta}$ **… model parameters**
  - Want to estimate $\boldsymbol{P_f(X|\Theta)}$**:**
    - The probability that our model $\boldsymbol{f}$ (with parameters $\boldsymbol{\Theta}$) generated the data
  - **Now let's find the most likely <u>model</u> (including the parameters) that could have generated the data:** $\arg\max_{\Theta} \boldsymbol{P_f(X|\Theta)}$

Example: MLE

- **Imagine we are given a set of coin flips**
- **Task:** Figure out the bias of a coin!
  - **Data:** Sequence of coin flips: $\boldsymbol{X = [1, 0, 0, 0, 1, 0, 0, 1]}$
  - **Model:** $\boldsymbol{f(\Theta)}$ = return 1 with prob. Θ, else return 0
  - **What is $\boldsymbol{P_f(X|\Theta)}$?** Assuming coin flips are independent
    - **So,** $\boldsymbol{P_f(X|\Theta) = P_f(1|\Theta) * P_f(0|\Theta) * P_f(0|\Theta) \ldots * P_f(1|\Theta)}$
      - What is $\boldsymbol{P_f(1|\Theta)}$? **Simple,** $\boldsymbol{P_f(1|\Theta) = \Theta}$
    - **Then,** $\boldsymbol{P_f(X|\Theta) = \Theta^3(1-\Theta)^5}$
    - **For example:**
      - $\boldsymbol{P_f(X|\Theta = 0.5) = 0.003906}$
      - $\boldsymbol{P_f\left(X\middle|\Theta = \frac{3}{8}\right) = 0.005029}$
  - **What did we learn?** Our data was most likely generated by coin with bias $\boldsymbol{\Theta = 3/8}$



AGM Example (1)

- Using AGM for modeling the graph below
- Assumptions (preset some of the parameters):
  - There are two communities C and D with P_C and P_D
  - We have determined (or are using as a temporary hypothesis) the community memberships:
    - $C = \{w, x, y\}$ and $D = \{w, y, z\}$

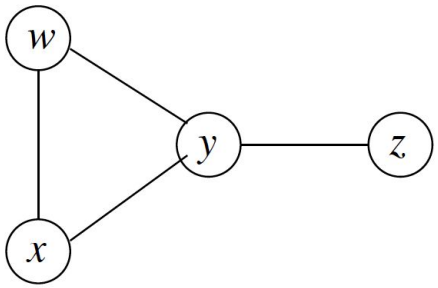


Figure 10.20: A social graph

AGM Example (2)

- Given $C = \{w, x, y\}$ and $D = \{w, y, z\}$
  - For w and x, $M_{wx} = \{C\}$
  - $P_{wx} = 1 - (1 - P_c) = P_c$
  - And
  - $P_{xy} = P_C$ and $P_{yz} = P_{wz} = P_D$
  - $P_{wy} = 1 - (1 - P_C)(1 - P_D)$
  - P_{xz} = a tiny value (very low probability)

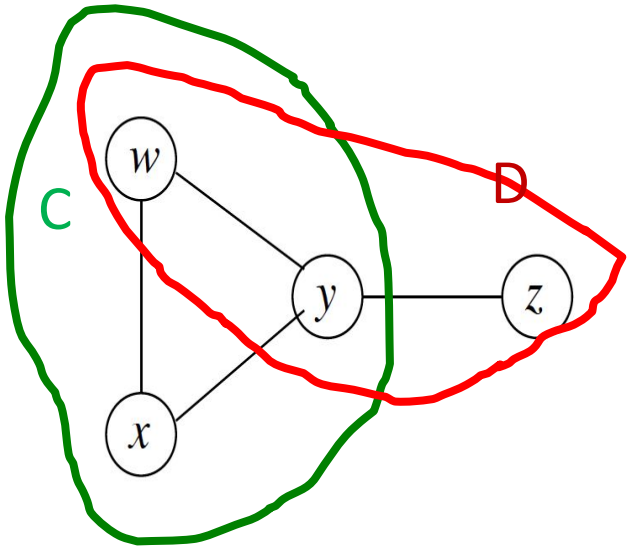


Figure 10.20: A social graph

- The likelihood of this graph given the membership assumption
- $P(G|AGM) = P_{wx} P_{wy} P_{xy} P_{yz} (1-P_{wz})(1-P_{xz})$
  $= (P_C)^2 P_D (P_C + P_D - P_C P_D)(1-P_D)(1-\varepsilon)$

AGM Example (3)

- Now, Find the values of P_C and P_D that maximize:

$$(p_C)^2 p_D(p_C + p_D - p_C p_D)(1 - p_D)(1 - \epsilon)$$

  - The last factor can be dropped because ε is negligible
- You can solve it by setting its derivative = 0
- Or
- You can solve it by the following estimated reasoning
  - *Let* P_C should be as large as possible
  - Given P_C = 1, then the above expression becomes $p_D(1\text{-}p_D)$
  - $p_D(1\text{-}p_D)$ *is maximal when* P_D=0.5
  - Thus, the maximum likelihood for the graph in Fig. 10.20 occurs when
    - Members of C are certain to have an edge between them and
    - There is a 50% chance that joint membership in D will cause an edge between the members

Maximum-Likelihood Estimation for Graphs

- **How do we do MLE for graphs?**
  - **Model generates a probabilistic adjacency matrix**
  - We then flip all the entries of the probabilistic matrix to obtain the **binary adjacency matrix A**

For every pair of nodes u, v AGM gives the prob. p_{uv} of them being linked

0	0.10	0.10	0.04
0.10	0	0.02	0.06
0.10	0.02	0	0.06
0.04	0.06	0.06	0

Flip biased coins

A

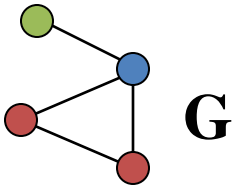
0	1	0	0
1	0	1	1
0	1	0	1
0	1	1	0

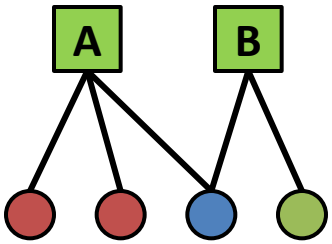
- **The likelihood of AGM generating graph G:**

$$P(G \mid \Theta) = \prod_{(u,v)\in E} P(u,v) \prod_{(u,v)\notin E} (1 - P(u,v))$$

Graphs: Likelihood $P(G|\Theta)$

- Given graph $G(V,E)$ and Θ, we calculate likelihood that Θ generated G: $P(G|\Theta)$





Θ=B(V, C, M, $\{p_c\}$)

0	0.9	0.9	0
0.9	0	0.9	0
0.9	0.9	0	0.9
0	0	0.9	0

0	1	1	0
1	0	1	0
1	1	0	1
0	0	1	0

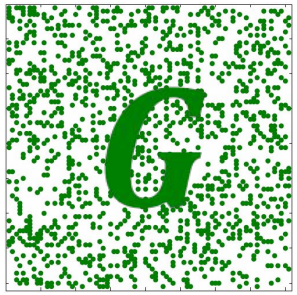
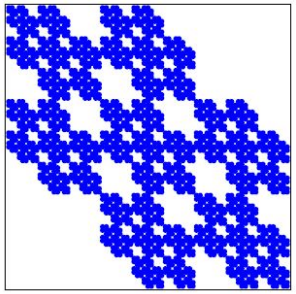
G

$P(G|\Theta)$

$$P(G \mid \Theta) = \prod_{(u,v)\in E} P(u,v) \prod_{(u,v)\notin E} (1 - P(u,v))$$

MLE for Graphs

- **Our goal:** Find $\boldsymbol{\Theta = B(V, C, M, \{p_C\})}$ **such that:**

$$\arg\max_{\Theta} P($$  $$|$$  $$)$$

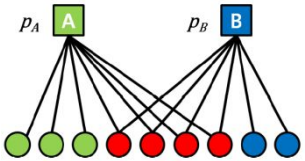
- **How do we find $\boldsymbol{B(V, C, M, \{p_C\})}$ that maximizes the likelihood?**

MLE for AGM

- **Our goal is to find $B(V, C, M, \{p_C\})$ such that:**

$$\arg\max_{B(V,C,M,\{p_C\})} \prod_{u,v \in E} P(u,v) \prod_{uv \notin E} (1 - P(u,v))$$

- **Problem:** Finding **B** means finding the bipartite affiliation network.

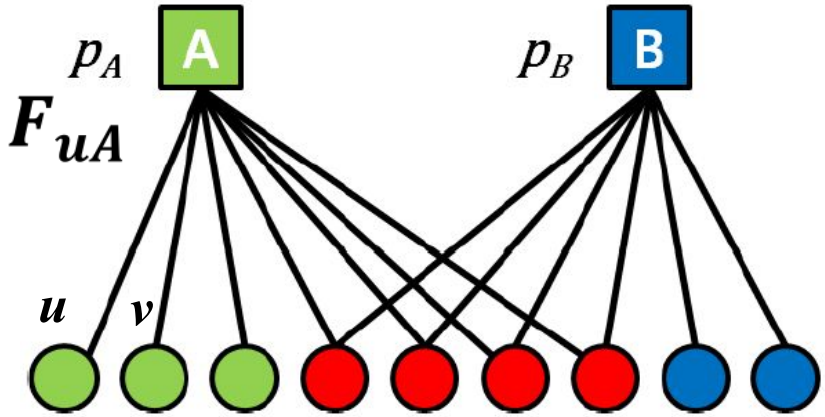


  - **There is no nice way to do this.**
  - **Fitting $B(V, C, M, \{p_C\})$ is too hard, let's change the model (so it is easier to fit)!**

From AGM to **BigCLAM** (1)

- **Relaxation: Memberships have strengths (avoid discrete membership changes)**

Membership is not binary any more



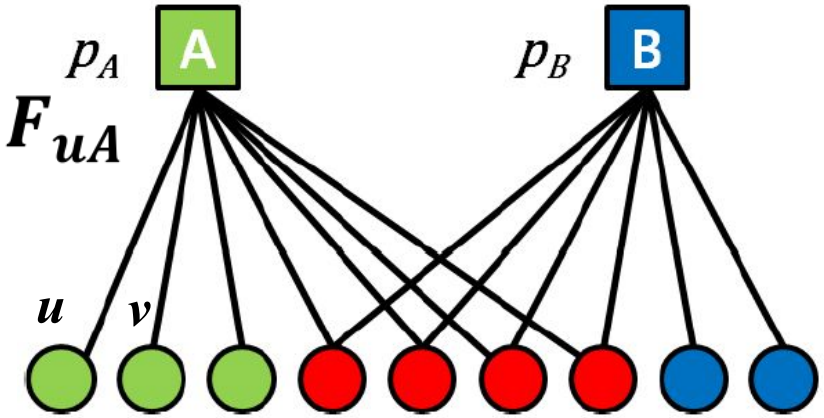
- F_{uA}: The membership strength of node u to community A (> 0) ($F_{uA} = 0$: no membership)
- **Each community A links nodes independently.**

Define: Let us define

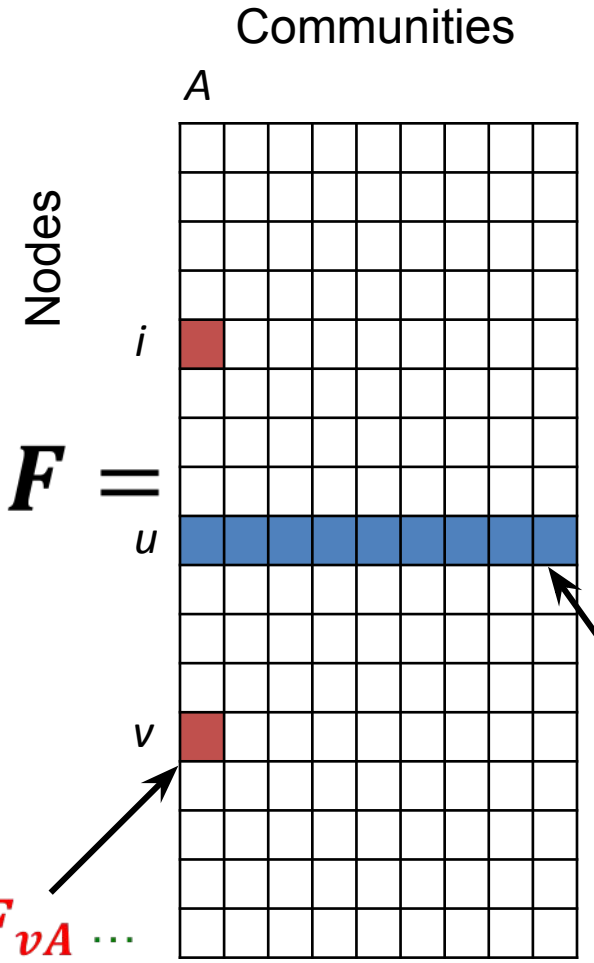
$$P_A(u, v) = 1 - \exp(-F_{uA} \cdot F_{vA})$$

From AGM to BigCLAM (2)

- $P_A(u,v) = 1 - \exp(-F_{uA} \cdot F_{vA})$
- If *u* and *v* are not in the same community
  - $P_A(u,v) = 1 - \exp(0) = 0$
- If F_{uA} *and* F_{vA} *are both large numbers:*
  - $P_A(u,v) = 1 - e^{-ALargeNumber}$ **= 1 – ASmallNumber**
- If F_{uA} *and* F_{vA} *are both small numbers:*
  - $P_A(u,v) = 1 - e^{-ASmallNumber}$ **= 1 – ALargeNumber**



Community Membership Strength Matrix *F*



- $P_A(u,v) = 1 - \exp(-F_{uA} \cdot F_{vA})$
  - **Probability of connection is proportional to the product of strengths**
    - **Notice:** If one node doesn't belong to the community ($F_{uC} = 0$) then $P(u,v) = 0$
- Prob. that **at least one** common community C links the nodes:
  - $P(u,v) = 1 - \prod_C (1 - P_C(u,v))$

From AGM to BigCLAM

- Community A links nodes u, v independently:

$$P_A(u,v) = 1 - \exp(-F_{uA} \cdot F_{vA})$$

- Then prob. at least one common C links them:

$$P(u,v) = 1 - \prod_C (1 - P_C(u,v))$$
$$= 1 - \exp(-\sum_C F_{uC} \cdot F_{vC})$$
$$= 1 - \exp(-F_u \cdot F_v^T)$$

- **Example F matrix:**

F_u: 0 1.2 0 0.2

F_v: 0.5 0 0 0.8

F_w: 0 1.8 1 0

Node community membership strengths

Then: $F_u \cdot F_v^T = 0.16$ // 0.2 x 0..8

And: $P(u,v) = 1 - exp(-0.16) = 0.14$

But: $P(u,w) = 0.88$ // 1-exp(-(1.2*1.8))

$P(v,w) = 0$

BigCLAM: How to find *F*

- **Task: Given a network $G(V, E)$, estimate F**
  - **Find F that maximizes the likelihood:**

$$arg\ max_F \prod_{(u,v)\in E} P(u,v) \prod_{(u,v)\notin E} (1 - P(u,v))$$

    - **where: $P(u,v) = 1 - \exp(-F_u \cdot F_v^T)$**
    - Many times we take the logarithm of the likelihood, and call it log-likelihood: $l(F) = \log P(G|F)$

- **Goal: Find F that maximizes $l(F)$:**

$$l(F) = \sum_{(u,v)\in E} \log(1 - \exp(-F_u F_v^T)) - \sum_{(u,v)\notin E} F_u F_v^T$$

BigCLAM: V1.0

$$l(F_u) = \sum_{v \in \mathcal{N}(u)} \log(1 - \exp(-F_u F_v^T)) - \sum_{v \notin \mathcal{N}(u)} F_u F_v^T$$

- **Compute gradient of a single row F_u of F:**

$$\nabla l(F_u) = \sum_{v \in \mathcal{N}(u)} F_v \frac{\exp(-F_u F_v^T)}{1 - \exp(-F_u F_v^T)} - \sum_{v \notin \mathcal{N}(u)} F_v$$

- **Coordinate gradient ascent:**
  - **Iterate over the rows of F:**
    - Compute gradient $\nabla l(F_u)$ of row u (while keeping others fixed)
    - Update the row F_u: $F_u \leftarrow F_u + \eta \nabla l(F_u)$
    - Project F_u back to a non-negative vector: If $F_{uC} < 0$: $F_{uC} = 0$
- **This is slow!** Computing $\nabla l(F_u)$ takes linear time!

$\mathcal{N}(u)$.. Set out outgoing neighbors

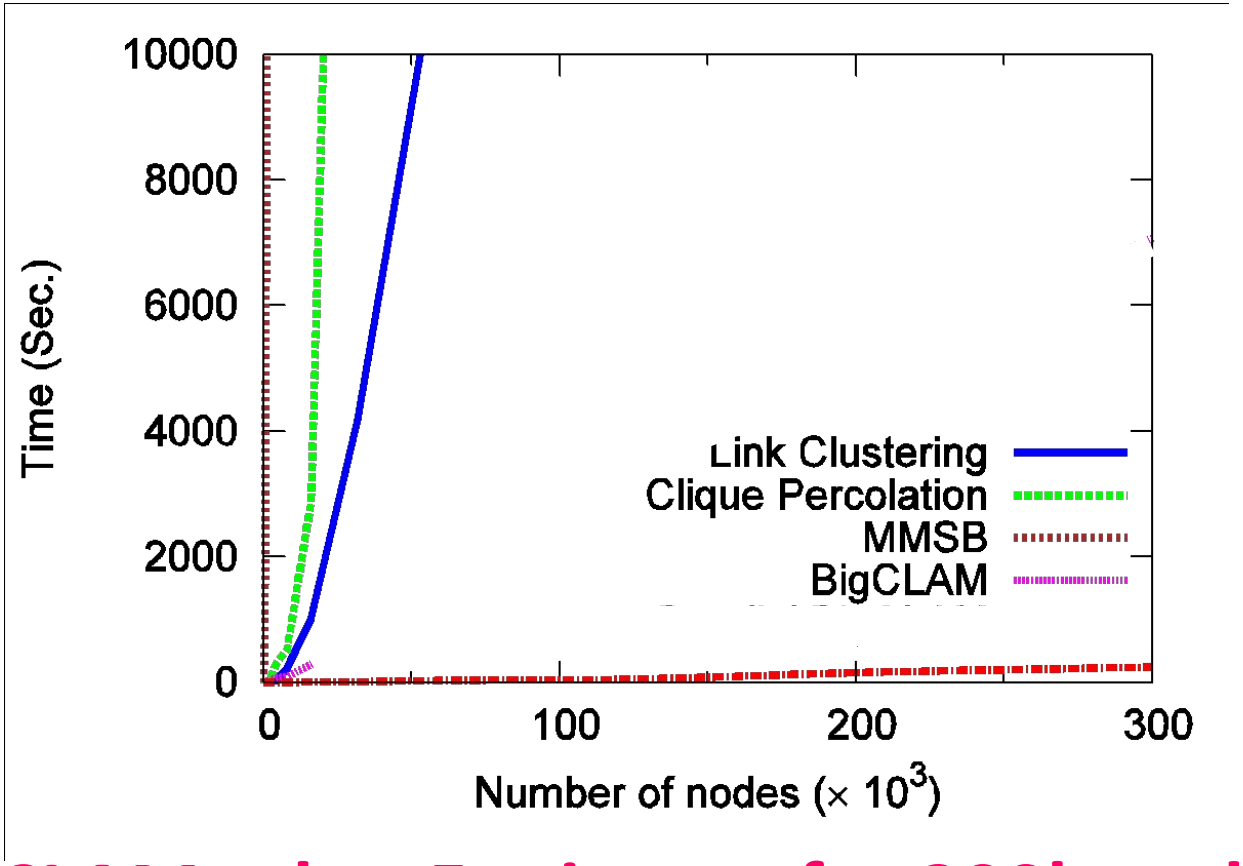
BigCLAM: V2.0

- **However, we notice:** (By examining the F matrix)

$$\sum_{v \notin \mathcal{N}(u)} F_v = (\sum_{v} F_v - F_u - \sum_{v \in \mathcal{N}(u)} F_v)$$

  - We cache $\sum_v \boldsymbol{F_v}$
  - So, computing $\sum_{v \notin \mathcal{N}(u)} \boldsymbol{F_v}$ now takes **linear time** in the degree $|\mathcal{N}(\boldsymbol{u})|$ of $\boldsymbol{u}$
    - In networks degree of a node is much smaller to the total number of nodes in the network, so this is a significant speedup!

BigClam: Scalability



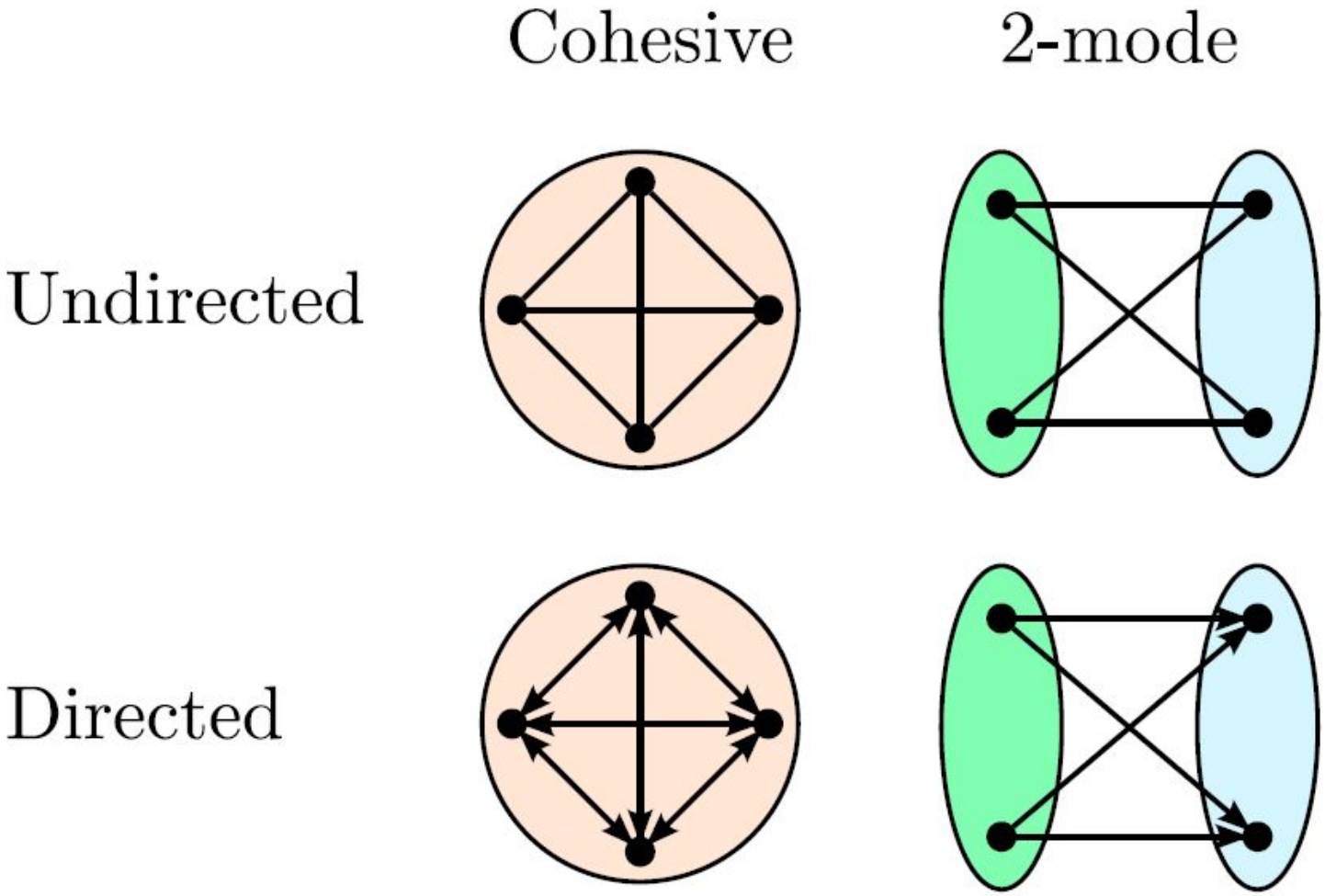
- **BigCLAM takes 5 minutes for 300k node nets**
  - **Other methods take 10 days**
- **Can process networks with 100M edges!**

Summary of BigCLAM

- Given G(V, E), compute B(C,M,{p_c}) such that P(G|B) is maximum
  - But it is too hard to find C, M, and {p_c} this way
- V1.0. Change to a model easier to find
  - Replace C and M by F // binary □ strength
  - Define $P(u,v) = 1-\exp(-F_{uA}F_{vA})$
  - Compute F such that P(G|F) is maximum
- V2.0. Speed up the computation of $\sum_{v \notin \mathcal{N}(u)} F_v$
  - By computing $\sum_{v \in \mathcal{N}(u)} F_v$

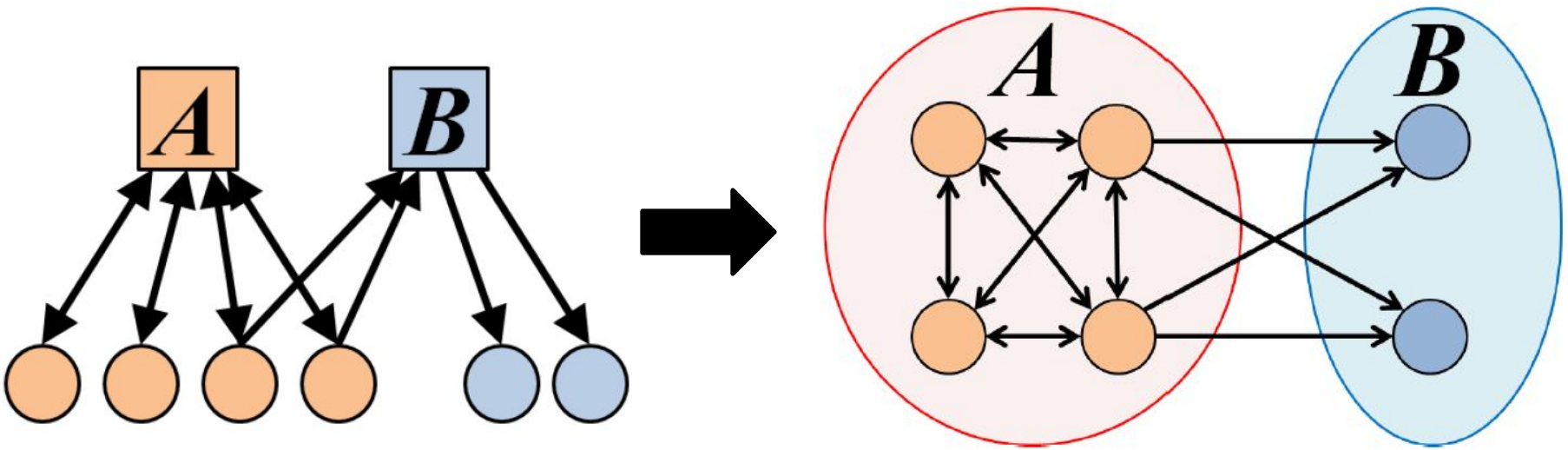
EXTENSION:
DIRECTED MEMBERSHIPS

Extension: Beyond Clusters

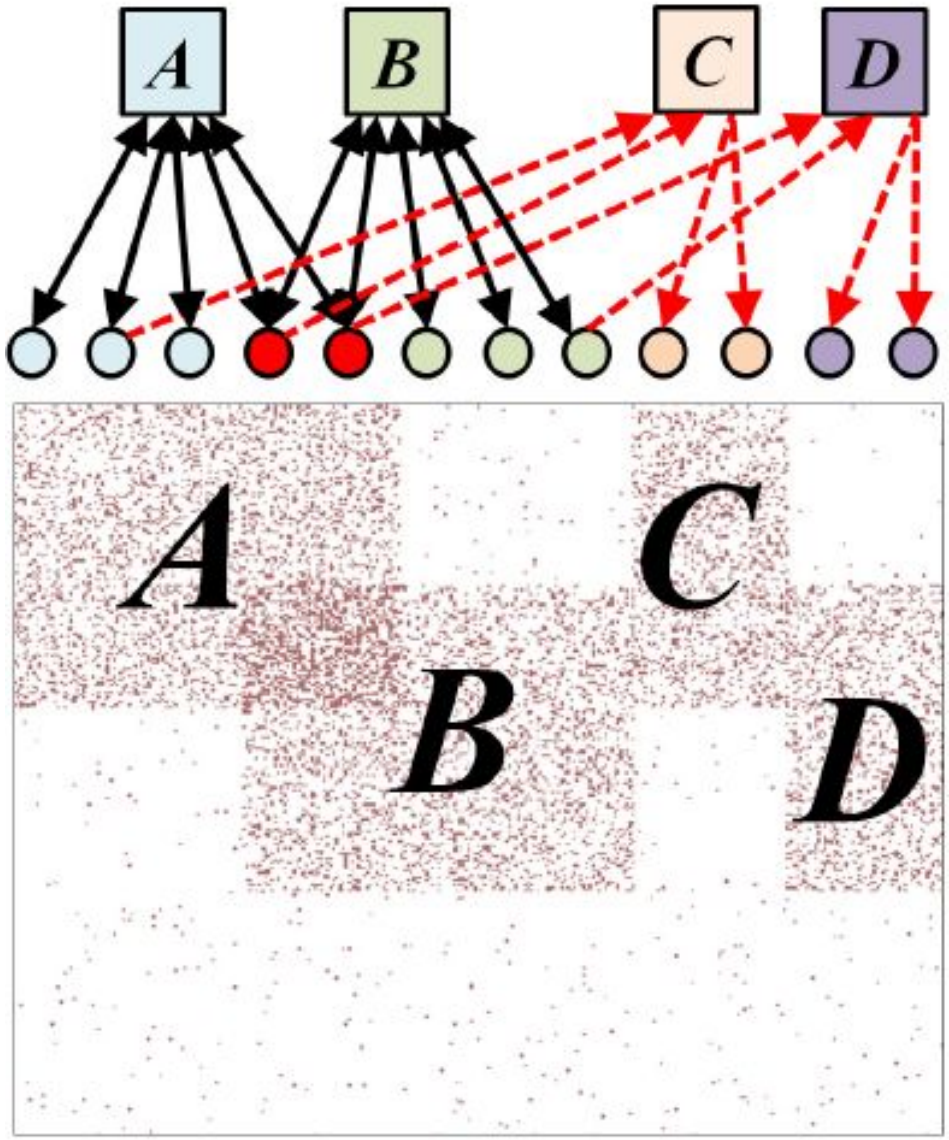


Extension: Directed AGM

- **Extension:**
  Make community membership edges directed!
  - **Outgoing membership:** Nodes "**sends**" edges
  - **Incoming membership:** Node "**receives**" edges

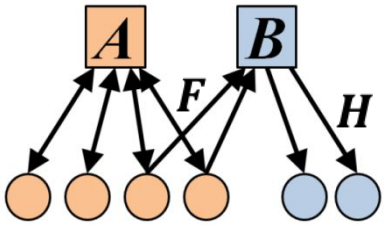


Example: Model and Network



Directed AGM

- **Everything is almost the same except now we have 2 matrices: F and H** // two membership matrixes
  - F… out-going community memberships
  - H… in-coming community memberships



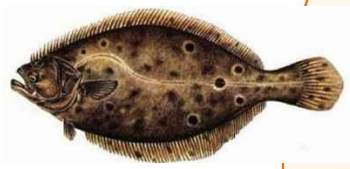
- **Edge prob.: $P(u,v) = 1 - exp(-F_u H_v^T)$**

- **Network log-likelihood:**

$$l(F,H) = \sum_{(u,v)\in E} \log(1 - \exp(-F_u H_v^T)) - \sum_{(u,v)\notin E} F_u H_v^T$$

which we optimize the same way as before

Predator-prey Communities



bluefish
striped bass
summer flounder
weakfish
white perch
spot
hogchoker
blue crab
catfish
croaker
menhaden
bay anchovy
nereis
macoma spp.
mya arenaria

More Details at…

- Overlapping Community Detection at Scale: A Nonnegative Matrix Factorization Approach by J. Yang, J. Leskovec. *ACM International Conference on Web Search and Data Mining (WSDM)*, 2013.
- Detecting Cohesive and 2-mode Communities in Directed and Undirected Networks by J. Yang, J. McAuley, J. Leskovec. *ACM International Conference on Web Search and Data Mining (WSDM)*, 2014.
- Community Detection in Networks with Node Attributes by J. Yang, J. McAuley, J. Leskovec. *IEEE International Conference On Data Mining (ICDM)*, 2013.