Link Analysis (Part I)

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With slide contributions from
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OUTLINE

Problem and Motivation

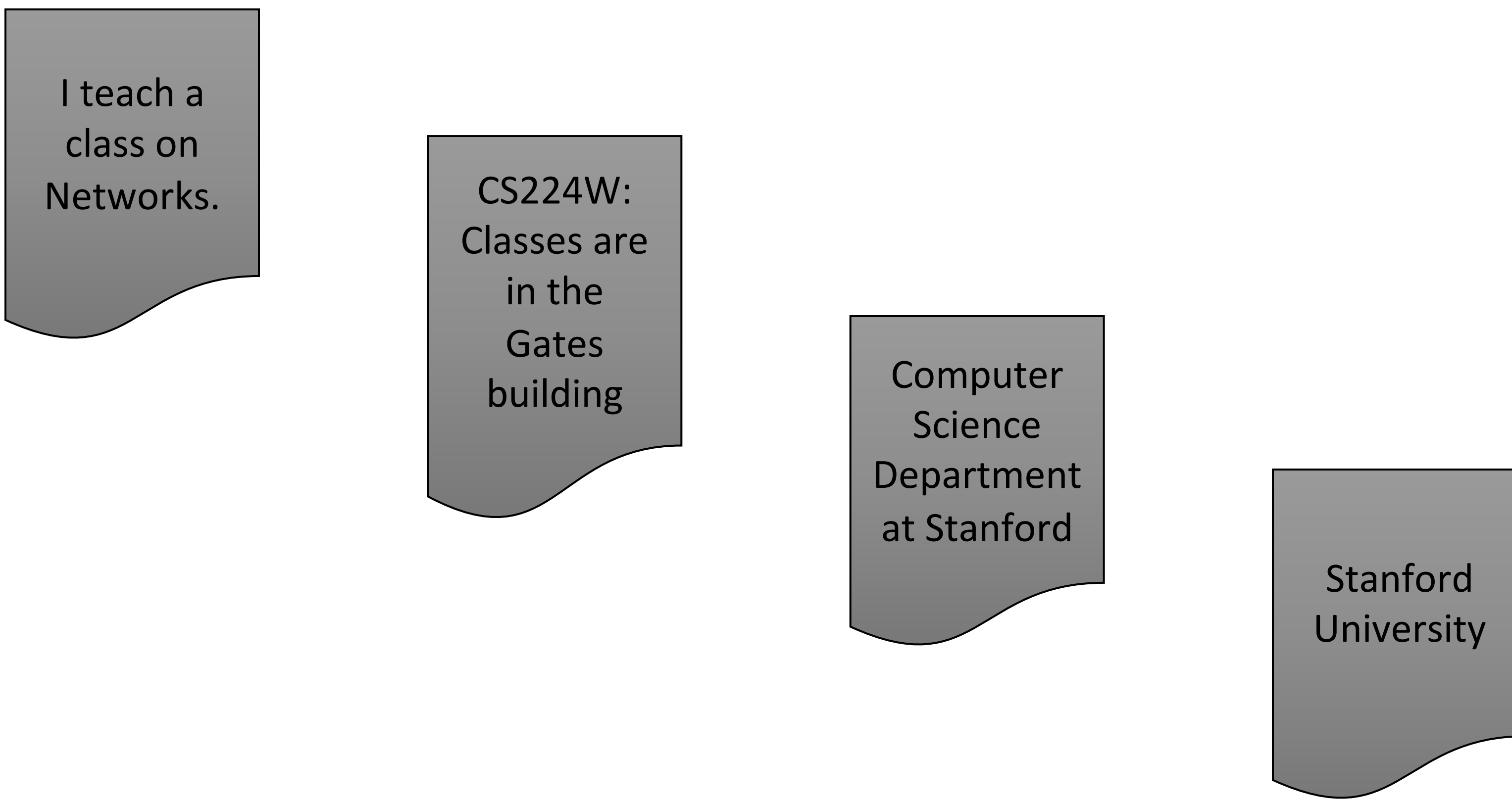
- How to rank a web page?

Three Approaches

1. Page Rank
2. Topic-Specific (personalized) Page Rank
3. Web Spam Detection

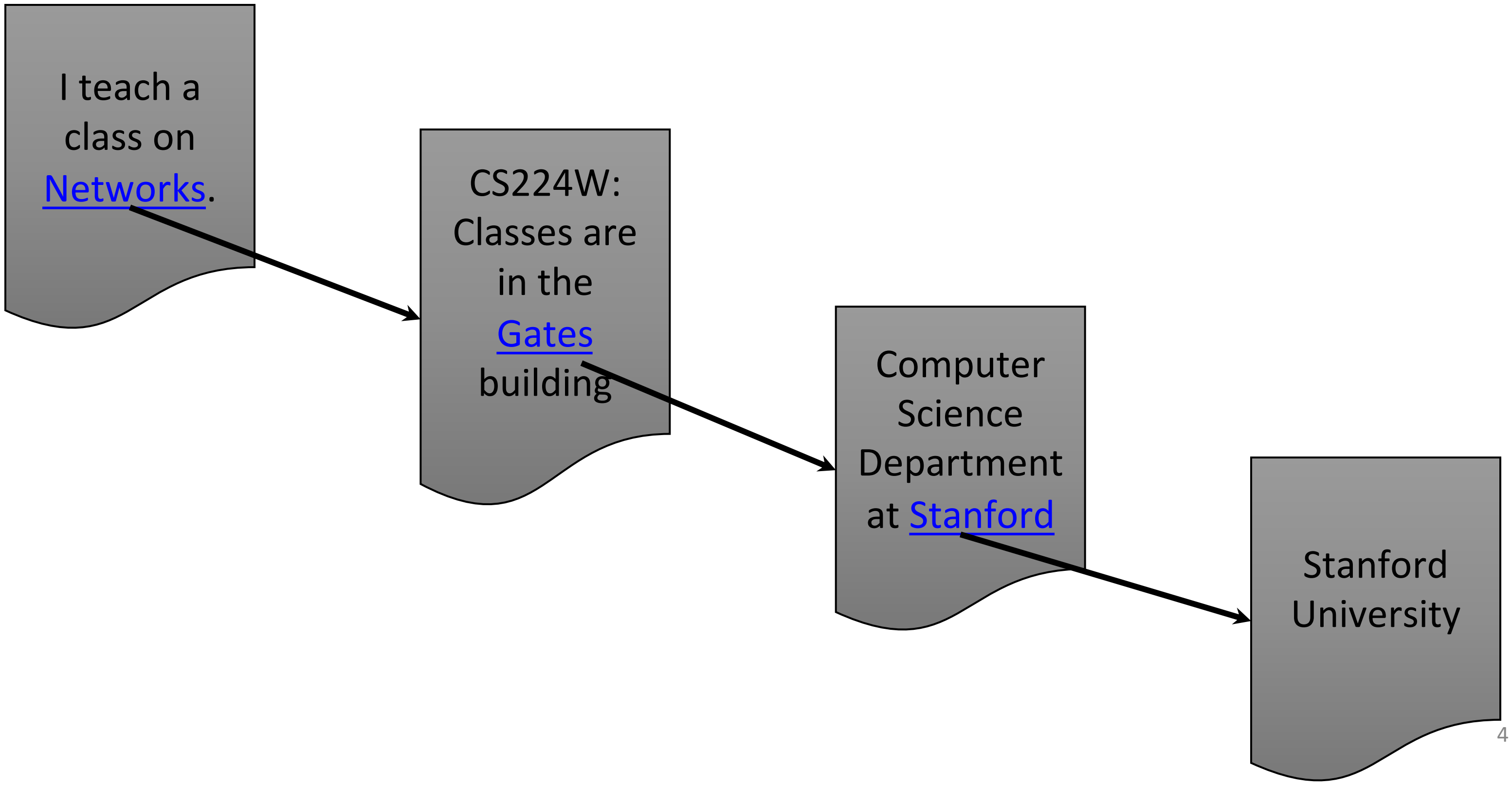
Web as a Graph

- **Web as a directed graph:**
  - **Nodes: Webpages**
  - **Edges: Hyperlinks**

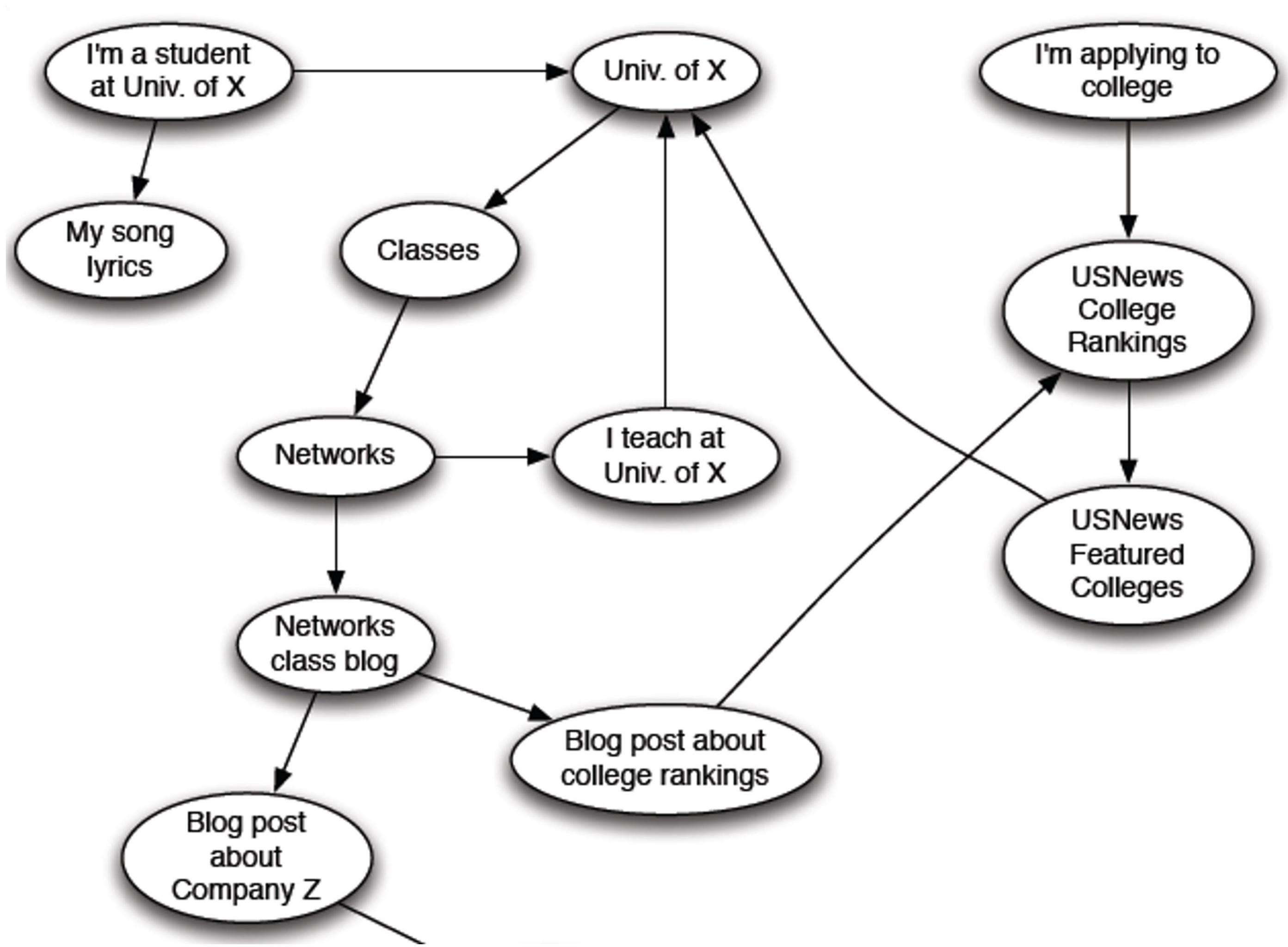


Web as a Graph

- **Web as a directed graph:**
  - **Nodes: Webpages**
  - **Edges: Hyperlinks**

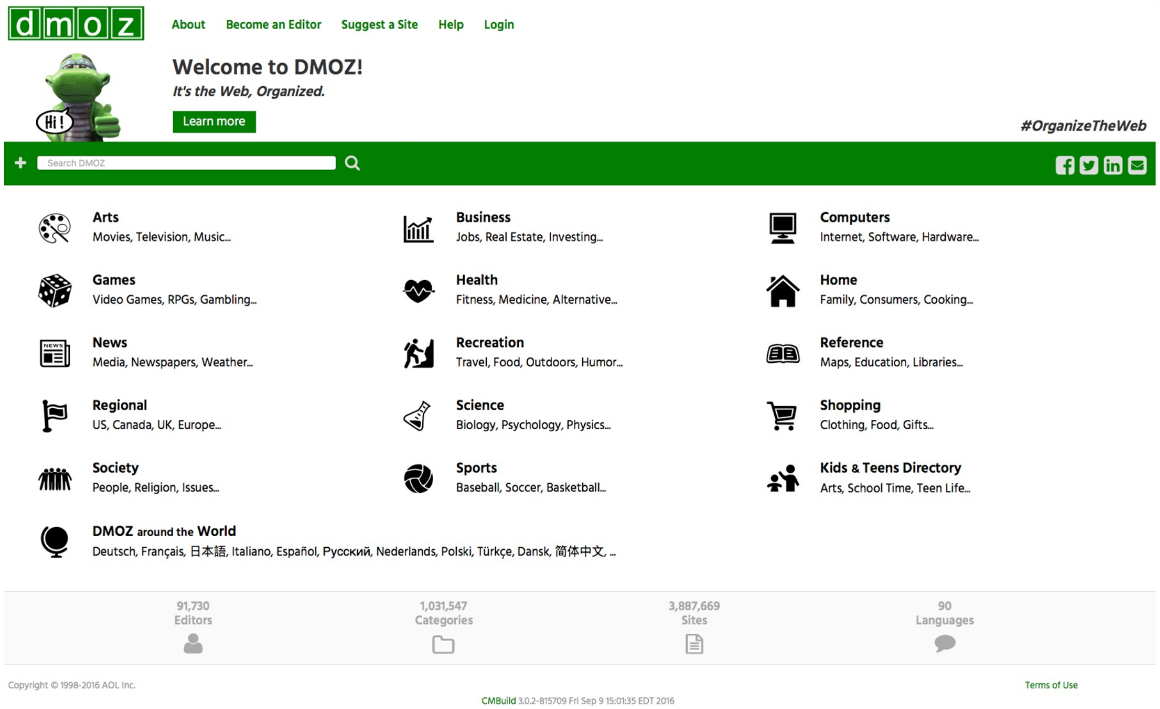


Web as a Directed Graph



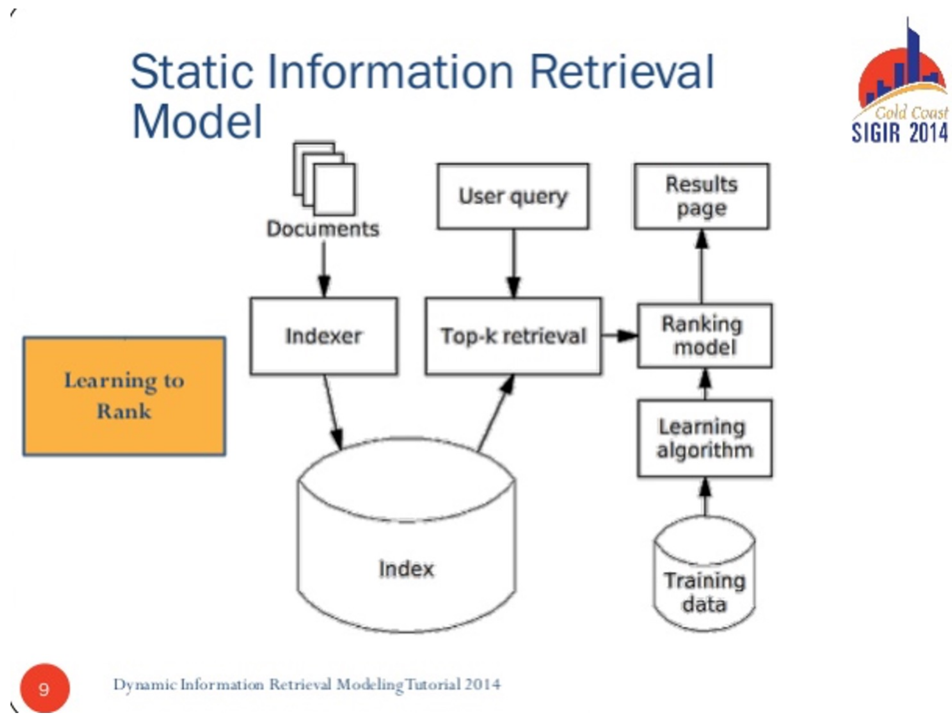
Broad Question

- **How to organize the Web?**
- **First try:** Human curated **Web directories**
  - Yahoo, DMOZ, LookSmart



Broad Question (Cont’d)

- **How to organize the Web?**
- **Second try: Web Search**
  - **Information Retrieval** investigates: Find relevant docs in a small and trusted set
    - Newspaper articles, patents, etc.



But: Web is **huge**, full of untrusted documents, random things, web spam, etc.

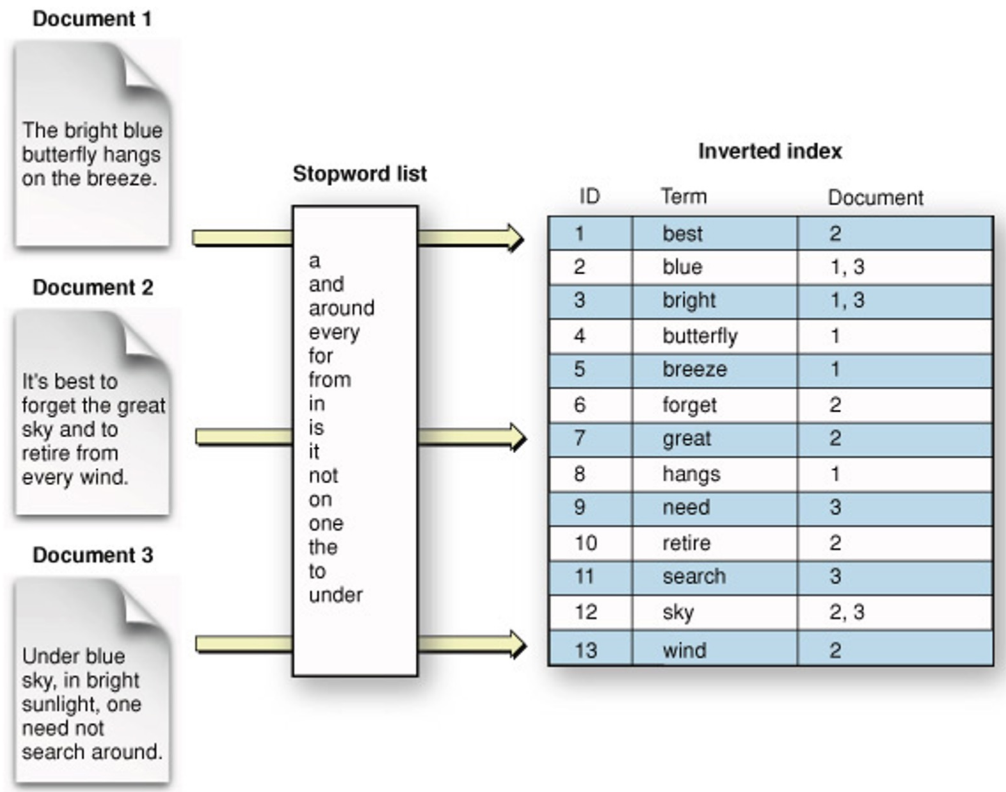
Early Web Search

- Keywords extracted from web pages
  - E.g., title, content
  - Used to build inverted index
- Queries are matched with web pages
  - Via lookup in the inverted index
  - Pages ranked by occurrences of query keywords

```
"a":      {2}
"banana": {2}
"is":     {0, 1, 2}
"it":     {0, 1, 2}
"what":   {0, 1}
```

Inverted Index

- Problem: susceptible to term spam



https://developer.apple.com/library/mac/documentation/UserExperience/Conceptual/SearchKitConcepts/searchKit_basics/searchKit_basics.html

Term Spam

- Disguise a page as something it is not about
  - E.g., adding thousands of keyword “movies”
  - Actual content may be some advertisement
  - Fool search engine to return it for query “movies”

- May even fade spam words into background

- Spam pages may be based on top-ranked pages

Web Search: Two Challenges

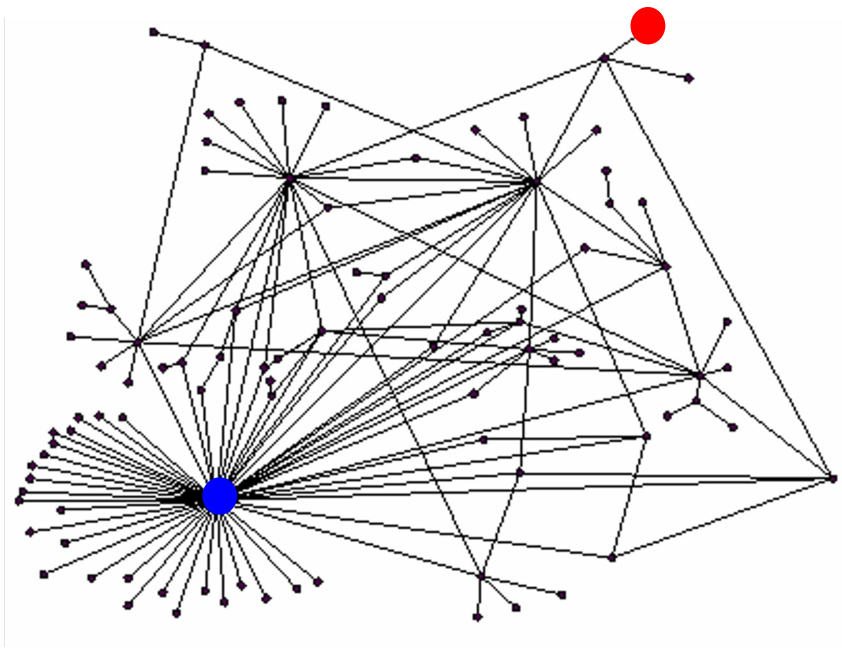
Two challenges of web search:

- **(1) Web contains many sources of information**
  Who to "trust"?
  - **Trick:** Trustworthy pages may point to each other!
- **(2) What is the "best" answer to query "newspaper"?**
  - No single right answer
  - **Trick:** Pages that actually know about newspapers might all be pointing to many newspapers

Ranking Nodes on the Graph

- **All web pages are not equally "important"**
  blog.bob.com vs. www.usc.edu

- There is large diversity in the web-graph node connectivity
- **Let's rank the pages by the link structure!**



Link Analysis Algorithms

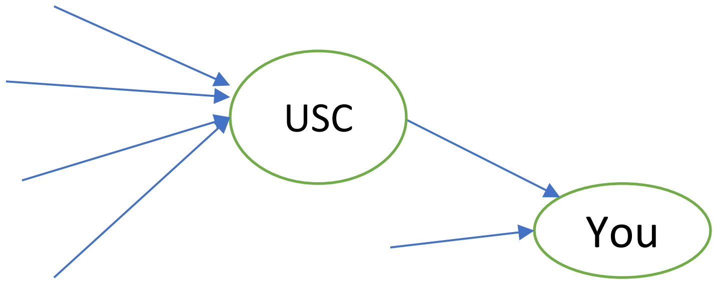
- We will cover the following **Link Analysis approaches** for computing **importance** of nodes in a graph:
  - Page Rank
  - Topic-Specific (Personalized) Page Rank
  - Web Spam Detection Algorithms

PageRank:
The “Flow” Formulation

PageRank: Combating Term Spam

- Key idea: rank pages by linkage too
  - How many pages point to a page
  - How important these pages are

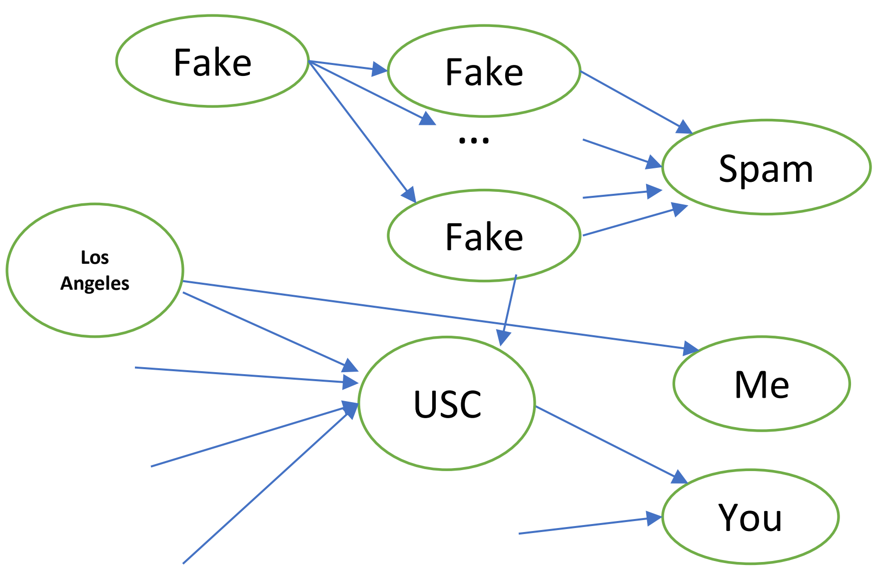
  => PageRank



- USC.edu can be important
  - because many pages point to it
- Your home page can be important
  - If it is pointed to by USC ☺️

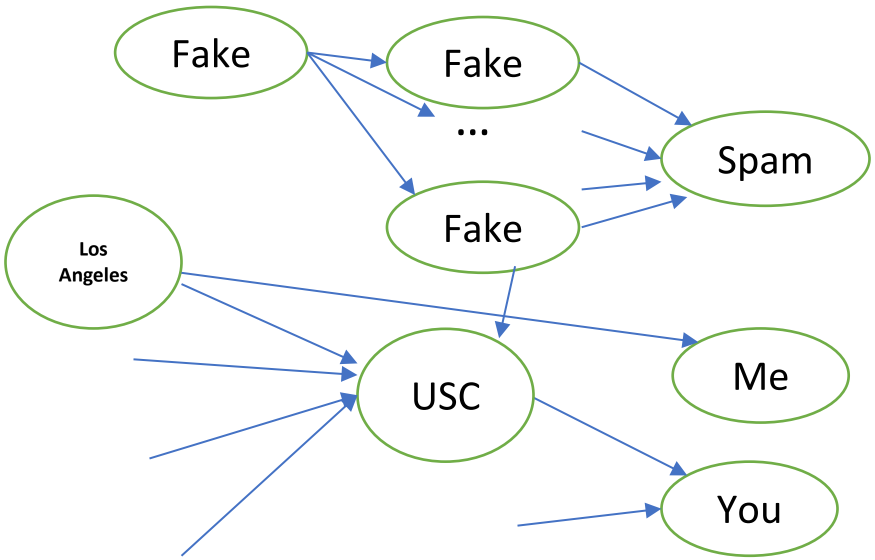
Random Surfer Model

- Random surfer of web
  - starts from any page
  - follows its outgoing links randomly
- Page is important if it attracts a large # of surfers



PageRank

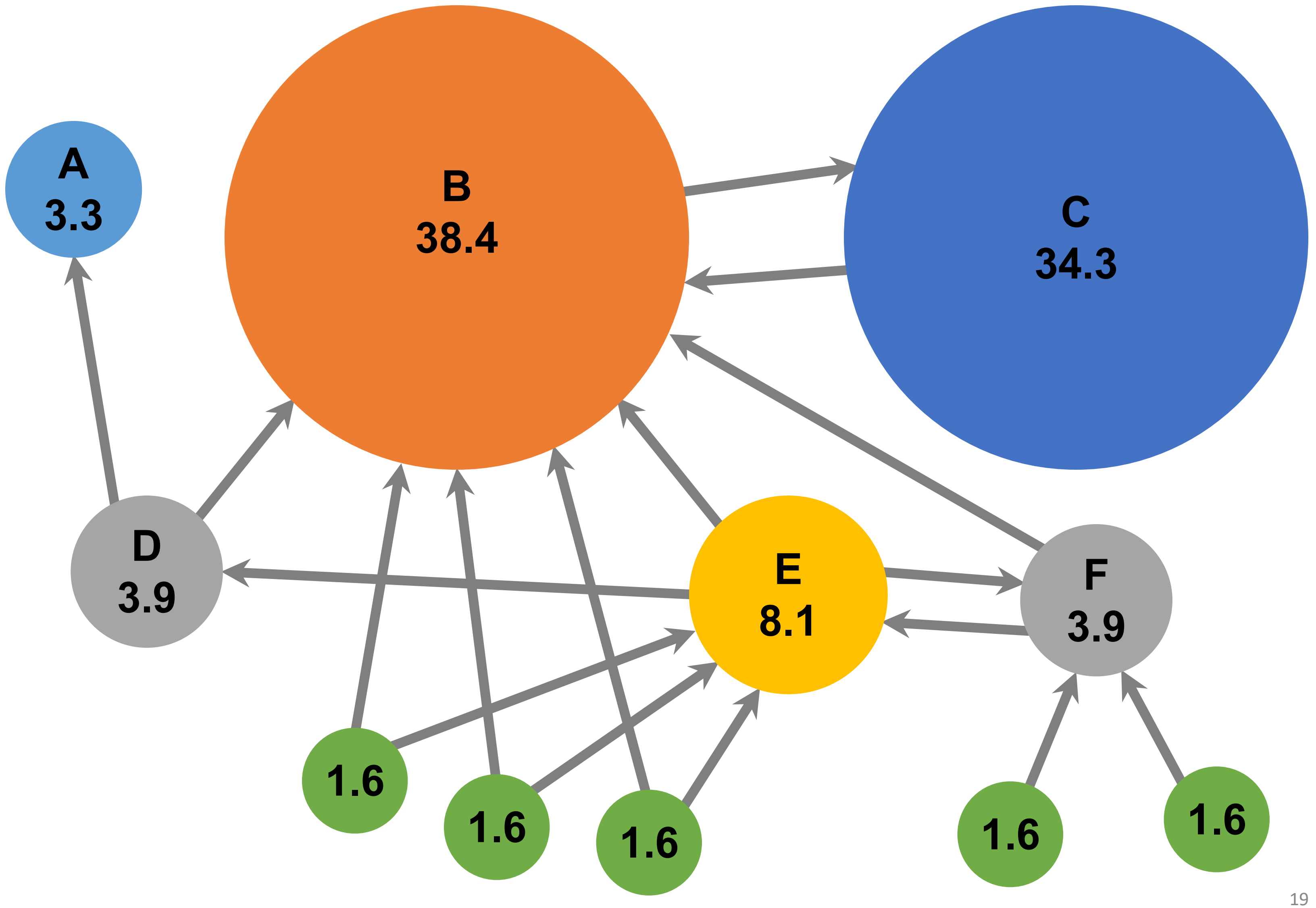
- What is the Probability that a random surfer lands on the page?



Intuition

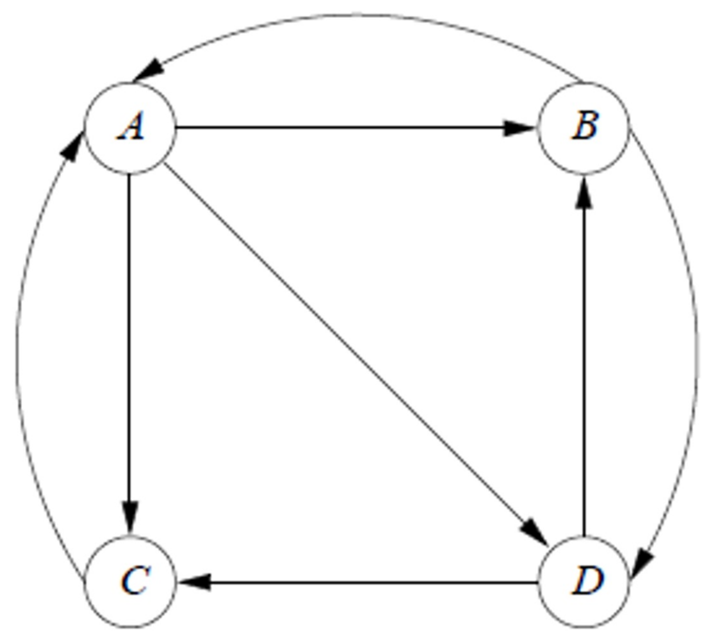
- If a page is important, then
  - many other pages may directly/indirectly link to it
  - random surfer can easily find it
- Spam pages are less connected
  - So less chance to attract random surfer
- Random surfer model more robust than manual approach
  - A collective voting scheme

Example: PageRank Scores



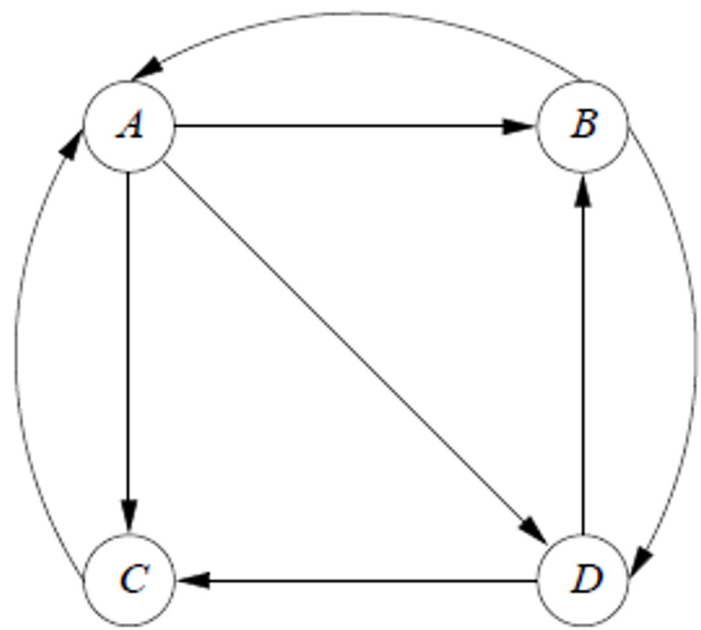
Assumption: A Strongly Connected Web Graph

- Nodes = pages
- Edges = hyperlinks between pages
- every node is reachable from every other node



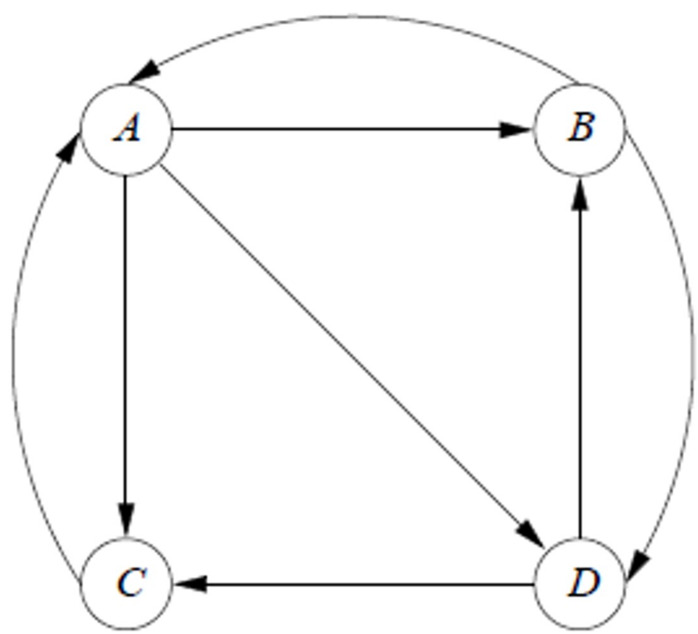
Model: Random Surfer on the Graph

- Can start at any node
  - If at A, can next go to B, C, or D, each with 1/3 prob.
  - If at B, can go to A and D, each with 1/2 prob.
  - So on…



Random Surfer Property: Memoryless

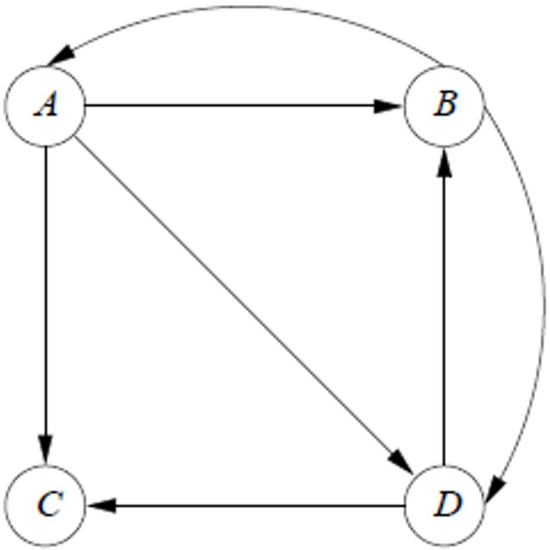
- Where to go from node X is not affected by how the surfer got to X



Extreme Case: Dead End

- Dead end: a page with no edges out
  - Absorb PageRanks
  - PageRank => 0 for any page that can reach the dead end (including the dead end itself)

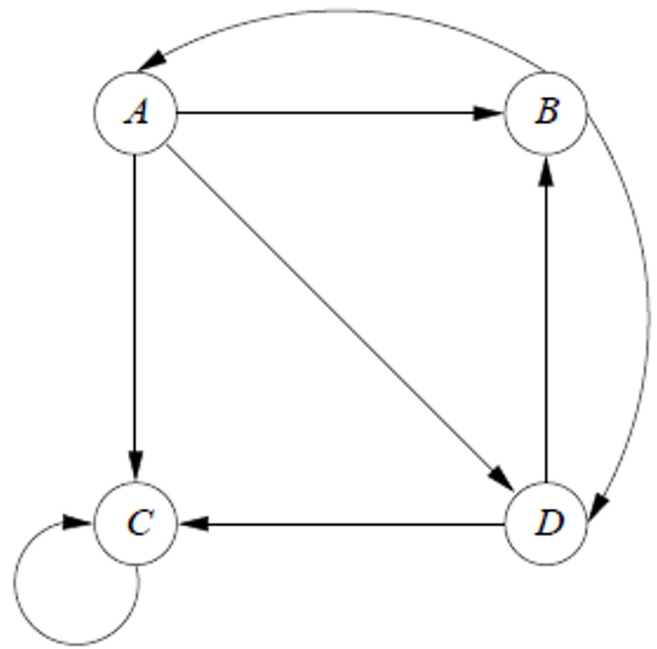
Dead end



Extreme Case: Spider Trap

- Group of pages with no edges going out of group
  - Absorb all PageRanks (rank of C =>1, others =>0)
  - Surfer can never leave, once trapped
  - Can have > 1 such trap nodes

Spider trap



PageRank: Formulation Details

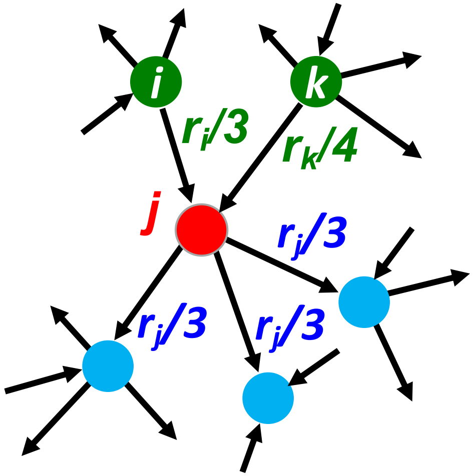
PageRank: Links as Votes

- **Idea: Links as votes**
  - **Page is more important if it has more links**
    - In-coming links? Out-going links?
- **Think of in-links as votes:**
  - www.stanford.edu has 23,400 in-links
  - www.inf553.cs.usc.edu has 1 in-link
- **Are all in-links equal?**
  - **Links from important pages count more**
  - A Recursive question!

Simple Recursive Formulation

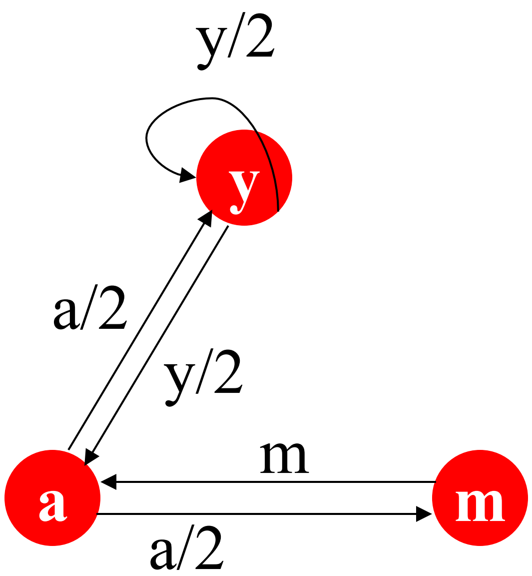
- Each link's vote is proportional to the **importance** of its source page
- If page ***j*** with importance r_j has ***n*** out-links, each link gets r_j/n votes
- Page ***j***'s own importance is the sum of the votes on its in-links

$$r_j = r_i/3 + r_k/4$$



PageRank: The "Flow" Model

- **A "vote" from an important page is worth more**
- **A page is important if it is pointed to by other important pages**
- **Define a "rank" r_j for page j**



$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d}_i}$$

d_i = out-degree of node i

"Flow" equations:

$$r_y = r_y/2 + r_a/2$$

$$r_a = r_y/2 + r_m$$

$$r_m = r_a/2$$

Solving the flow equations

"Flow" equations:

$$\mathbf{r_y = r_y/2 + r_a/2}$$
$$\mathbf{r_a = r_y/2 + r_m}$$
$$\mathbf{r_m = r_a/2}$$

- **3 equations, 3 unknowns, no constants**
  - No unique solution
  - All solutions equivalent modulo scale factor
- **Additional constraint forces uniqueness**
  - **Add some constant constraints: e.g., $r_y + r_m + r_a = 1$**
  - Solve for unique solutions: $r_y = 2/5$, $r_a = 2/5$, $r_m = 1/5$
- **Gaussian elimination method (in the later slides) works for small examples, but we need a better method for large graphs**

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

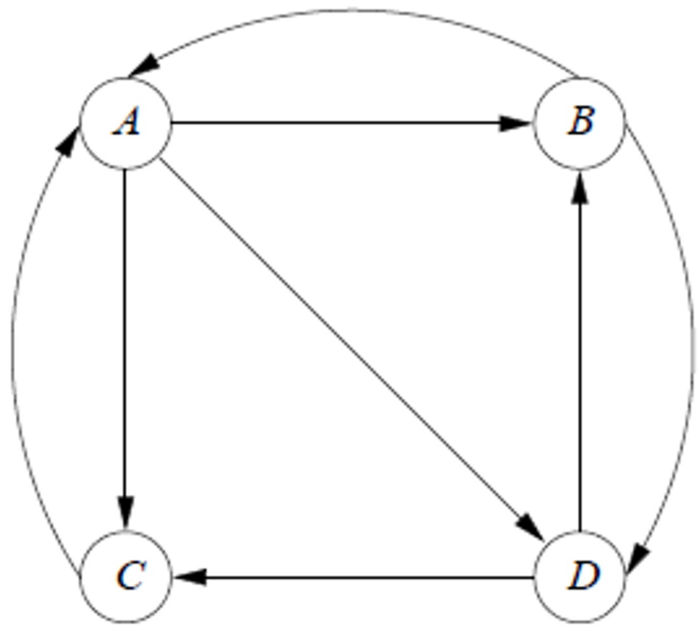
PageRank: Matrix formulation

- **Stochastic Transition (or adjacency) Matrix M**
- **Suppose page j has n outlinks**
  - If outlink j **->** i, then $\mathbf{M_{ij}=1/n}$
  - Else $\mathbf{M_{ij}=0}$
- **M** is a **column stochastic matrix**
  - Each column sums to 1

Transition Matrix

- M[i,j] = prob. of going from node j to node i
  - If j has k outgoing edges, prob. for each edge = 1/k

$$
M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}
\begin{array}{c}
\begin{array}{cccc} A & B & C & D \end{array} \\
\begin{bmatrix}
0 & 1/2 & 1 & 0 \\
1/3 & 0 & 0 & 1/2 \\
1/3 & 0 & 0 & 1/2 \\
1/3 & 1/2 & 0 & 0
\end{bmatrix}
\end{array}
$$



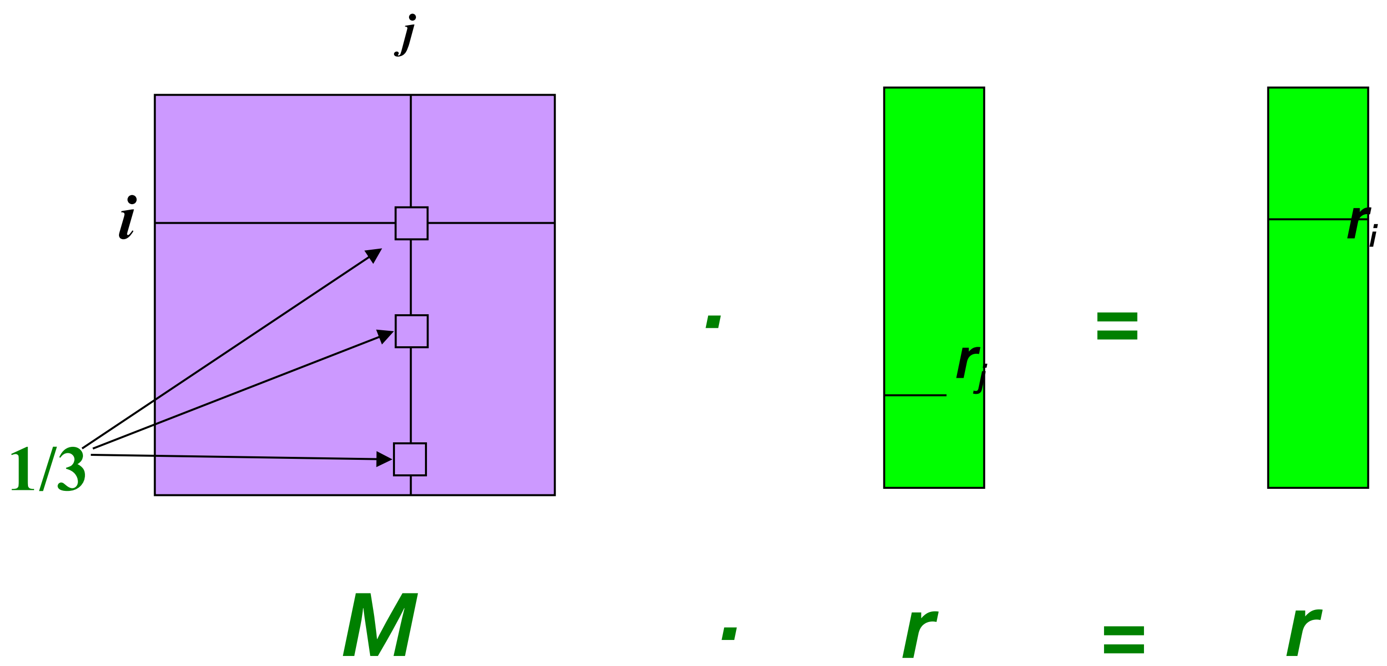
PageRank: Matrix formulation (Cont'd)

- **Stochastic Transition (or adjacency) Matrix M**
- **Suppose page j has n outlinks**
  - If outlink j **->** i, then $\mathbf{M_{ij}=1/n}$
  - Else $\mathbf{M_{ij}=0}$
- **M** is a **column stochastic matrix**
  - Each column sums to 1
- **Rank vector r** is a vector with one entry per web page
  - **r_i is the importance score of page I**
- The flow equations can be written as

$$\boxed{\mathbf{r = Mr}}$$

Example

- **Flow equation in matrix form: Mr = r**
- Suppose page *j* links to 3 pages, including *i*



Stationary Distribution

- Limiting probability distribution of random surfer
  - PageRanks are based on limiting distribution
  - the probability distribution will converge eventually
- Requirement for its existence
  - Graph is strongly connected: a node can reach any other node in the graph

  => Cannot have dead ends, spider traps

Eigenvectors and Eigenvalues

- An **eigenvector** of a square matrix $\mathbf{A}$ is a non-zero vector $\mathbf{v}$ that, when the matrix multiplies $\mathbf{v}$, yields the same as when some scalar multiplies $\mathbf{v}$, the scalar multiplier often being denoted by λ
- That is:

$$\mathbf{Av} = \lambda\mathbf{v}$$

- The number λ is called the **eigenvalue** of $\mathbf{A}$ corresponding to $\mathbf{v}$

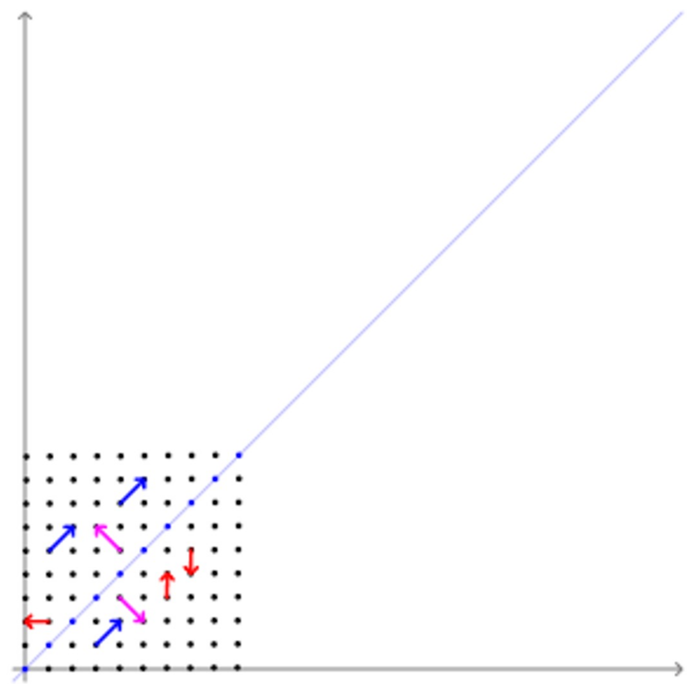
Eigenvalues and Eigenvectors Example

- The transformation matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ preserves the direction of vectors parallel to $\mathbf{v} = (1,-1)^T$ (in purple) and $\mathbf{w} = (1,1)^T$ (in blue). The vectors in red are not parallel to either eigenvector, so, their directions are changed by the transformation.

$$A\mathbf{v} = \lambda\mathbf{v}$$

http://setosa.io/ev/eigenvectors-and-eigenvalues/

https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors



Eigenvector Formulation

- **The flow equations can be written** limiting distribution

$$\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$$

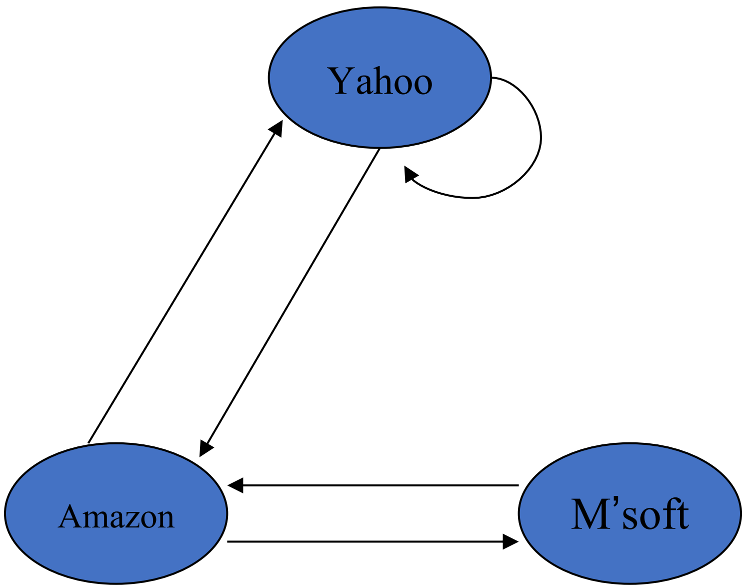
- So the **rank vector** ***r*** is an **eigenvector** of the stochastic web matrix ***M***
  - ***r*** is M's first or principal eigenvector, with corresponding eigenvalue *1*
  - Largest eigenvalue of ***M*** is **1** since ***M*** is column stochastic (with non-negative entries)
    - *We know **r** is unit length and each column of **M** sums to one*

NOTE: ***x*** is an eigenvector with the corresponding eigenvalue **λ** if:

$$\boldsymbol{Ax} = \boldsymbol{\lambda x}$$

- **We can now efficiently solve for *r***
  - **1. Power Iteration: https://en.wikipedia.org/wiki/Power_iteration**
  - **2. Use the principal eigenvector**

Example



$$\begin{array}{c|ccc|} & y & a & m \\ \hline y & 1/2 & 1/2 & 0 \\ a & 1/2 & 0 & 1 \\ m & 0 & 1/2 & 0 \\ \hline \end{array}$$

r = Mr

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix}$$

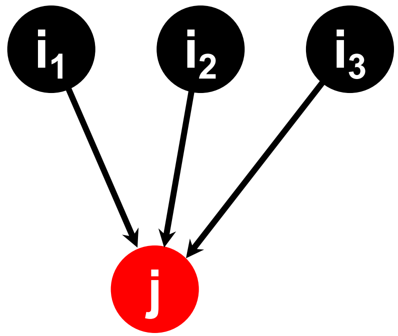
Two Approaches for Solving Mv = v

- Power Iteration
  - Repeatedly solving $Mv^k = v^{k+1}$
- Finding the Eigenvectors
  - Solving $\boldsymbol{Mv} = \boldsymbol{\lambda v}$*, or* $(\boldsymbol{M} - \boldsymbol{I})\boldsymbol{v} = 0$
  - Gaussian Elimination

Random Walk Interpretation

Imagine a random web surfer:

- At any time t, surfer is on some page ***i***
- At time t+1, the surfer follows an out-link from i uniformly at random
- Ends up on some page j linked from i
  - Process repeats indefinitely



$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d}_{\mathrm{out}}(\mathrm{i})}$$

Let:

$p(t)$ … vector whose i^{th} coordinate is the prob. that the surfer is at page i at time **t**

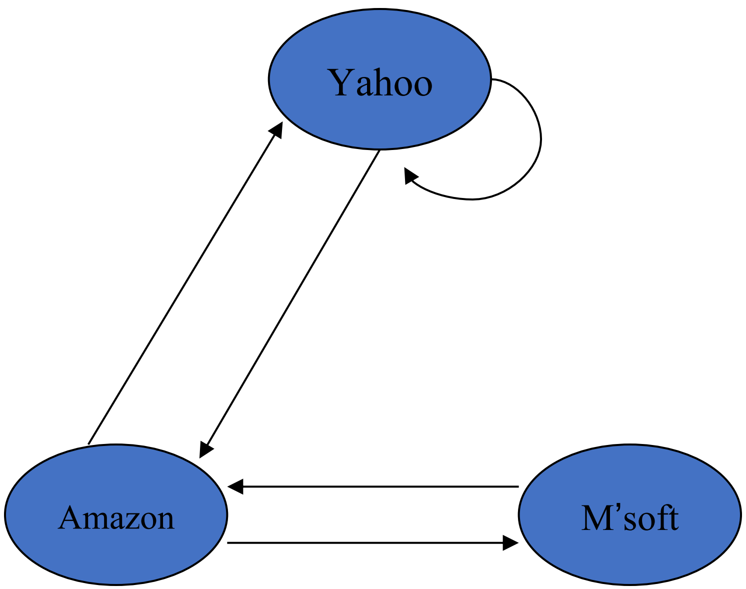
- So, $p(t)$ is a probability distribution over pages
  - -> rank vector

Power Iteration Method

- Simple iterative scheme (aka relaxation)
- Suppose there are N web pages
- Initialize: $\mathbf{r}^0 = [1/N, \ldots, 1/N]^T$
- Iterate: $\mathbf{r}^{k+1} = \mathbf{M}\mathbf{r}^k$
- Stop when $|\mathbf{r}^{k+1} - \mathbf{r}^k|_1 < \epsilon$

- $|\mathbf{x}|_1 = \sum_{1 \le i \le N} |x_i|$ is the L_1 norm
  - Can use any other vector norm e.g., Euclidean

Power Iteration Example 1



$$
\begin{array}{c|ccc|}
 & y & a & m \\
\hline
y & 1/2 & 1/2 & 0 \\
a & 1/2 & 0 & 1 \\
m & 0 & 1/2 & 0 \\
\hline
\end{array}
$$

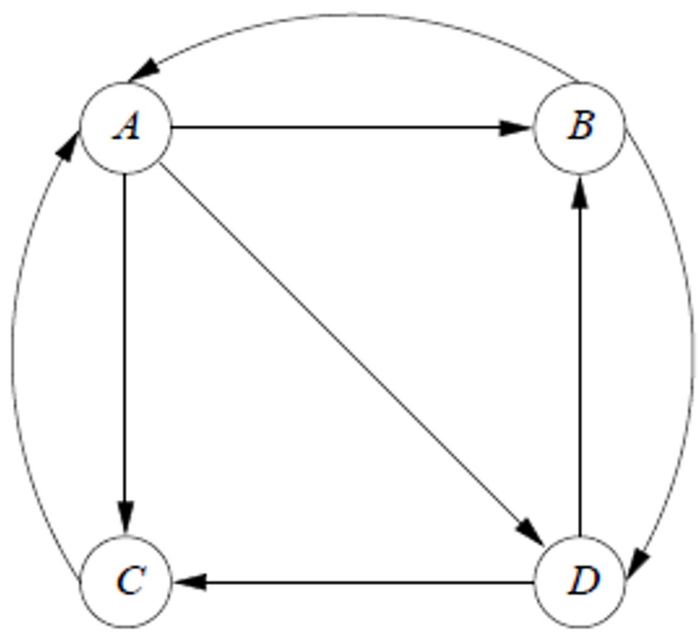
$\mathbf{r}^{k+1} = \mathbf{M}\mathbf{r}^{k}$

$$
\begin{array}{c}
r_y \\ r_a \\ r_m
\end{array}
=
\begin{array}{c}
1/3 \\ 1/3 \\ 1/3
\end{array}
\quad
\begin{array}{c}
1/3 \\ 1/2 \\ 1/6
\end{array}
\quad
\begin{array}{c}
5/12 \\ 1/3 \\ 1/4
\end{array}
\quad
\begin{array}{c}
3/8 \\ 11/24 \\ 1/6
\end{array}
\quad
\ldots
\quad
\begin{array}{c}
2/5 \\ 2/5 \\ 1/5
\end{array}
$$

Power Iteration Example 2

- M[i,j] = prob. of going from node j to node i
  - If j has k outgoing edges, prob. for each edge = 1/k

$$M = \begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$$



Prob. of Locations of Surfer

- Represented as a column vector, v
- Initially, surfer can be at any page with equal probability

$$v^0 = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

Limiting Distribution

- $v^1 = Mv^0$
- $v^2 = Mv^1 (= M^2 v^0)$
- …
- $v^i = Mv^{i-1} (= M^i v^0)$
- …
- $v = Mv$

(from some step k on, v does not change any more)

Compute Next Distribution

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \end{array} \quad \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

- $v^1 = Mv^0$
  - $v_i^1 = \sum_{j=1}^{n} M_{ij} v_j^0$
- E.g., $v_0^1 = 0 * \frac{1}{4} + \frac{1}{2} * \frac{1}{4} + 1 * \frac{1}{4} + 0 * \frac{1}{4} = \frac{3}{8}$
  - i.e., prob. at A is 3/8 (or 9/24) after step 1

$$v^0 = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \quad v^1 = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 9/24 \\ 5/24 \\ 5/24 \\ 5/24 \end{bmatrix}$$

$\boldsymbol{V^2 = ?}$ $\boldsymbol{V^3 = ?}$

Solving Mv = v by Power Iteration

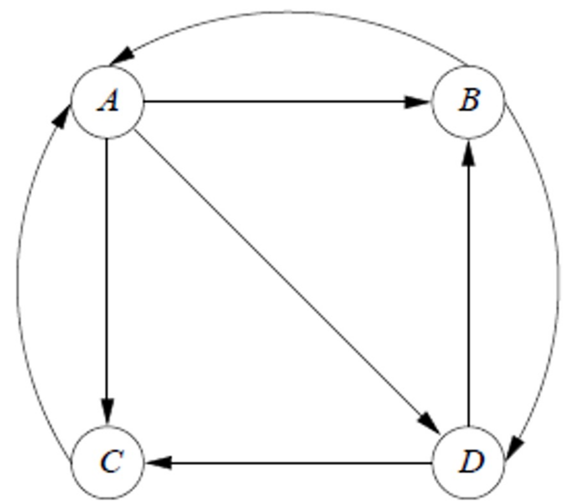
- Note: $\sum_{i=1}^{n} v_i^k$=1, after every step k
- Usually stop when little change btw iterations
- In practice, 50-75 iterations for Web

$$\begin{bmatrix}1/4\\1/4\\1/4\\1/4\end{bmatrix} \begin{bmatrix}9/24\\5/24\\5/24\\5/24\end{bmatrix}$$

NOTE: x is an eigenvector with the corresponding eigenvalue λ if:

$$Ax = \lambda x$$

$$\begin{bmatrix}11/32\\7/32\\7/32\\7/32\end{bmatrix}, \dots, \begin{bmatrix}3/9\\2/9\\2/9\\2/9\end{bmatrix}$$



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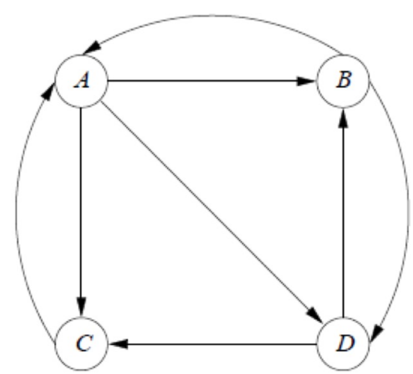
- Finding the Eigenvectors
  - Solving $\boldsymbol{Mv} = \boldsymbol{\lambda v}$, *or* $(\boldsymbol{M} - \boldsymbol{I})\boldsymbol{v} = 0$
  - Gaussian Elimination

Solving Mv = v by Finding the Principal Eigenvector (when λ=1)

- $Mv=\lambda v$
  - λ: eigenvalue, ν: eigenvector
- M is a (left) stochastic matrix
  - Each column adds up to 1
- Largest eigenvalue of a stochastic matrix is λ=1
  - Corresponding eigenvector is the <u>principal</u> eigenvector
  - Proof in the next few slides

$$M = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} \;A\; & \;B\; & \;C\; & \;D\; \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix} \end{array}$$



Characteristic Polynomial

Calculating the Eigenvalues

- $Mv = \lambda v$
- $(M - \lambda I)v = 0$
- v is not a null vector , so
  - $\det(M - \lambda I) = 0$

$$M = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

I is the identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} -\lambda & 1/2 & 1 & 0 \\ 1/3 & -\lambda & 0 & 1/2 \\ 1/3 & 0 & -\lambda & 1/2 \\ 1/3 & 1/2 & 0 & -\lambda \end{vmatrix} = 0$$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= aei + bfg + cdh - ceg - bdi - afh.$$

Compute the determinant using cofactors in this column (next slide)

Characteristic Polynomial

$$\begin{vmatrix} \frac{1}{3} & -\lambda & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & -\lambda \end{vmatrix} - \lambda \begin{vmatrix} -\lambda & \frac{1}{2} & 0 \\ \frac{1}{3} & -\lambda & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & -\lambda \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & -\lambda \end{vmatrix} -1/2 \begin{vmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{vmatrix} - \lambda(-1/2 \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} \end{vmatrix} - \lambda \begin{vmatrix} -\lambda & \frac{1}{2} \\ \frac{1}{3} & -\lambda \end{vmatrix})$$

$$= \lambda(-\frac{1}{3}\lambda - \frac{1}{6}) - \lambda[-\frac{1}{2}(-\frac{1}{2}\lambda - \frac{1}{6}) - \lambda(\lambda^2 - \frac{1}{6})]$$

$$= \lambda^4 - \frac{3}{4}\lambda^2 - \frac{\lambda}{4} = \lambda(\lambda-1)(\lambda^2 + \lambda + \frac{1}{4}) = \lambda(\lambda-1)(\lambda + \frac{1}{2})^2$$

Characteristic Polynomial

- $Mv = \lambda v$
- $(M - \lambda I)v = 0$
- $\det(M - \lambda I) = 0$

$$M = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1/2 & 1 & 0 \\ 1/3 & -\lambda & 0 & 1/2 \\ 1/3 & 0 & -\lambda & 1/2 \\ 1/3 & 1/2 & 0 & -\lambda \end{vmatrix} = 0$$

$$\lambda(\lambda\text{-}1)(\lambda + \tfrac{1}{2})^2 = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = \lambda_4 = -\frac{1}{2}$$

Solving Mv = v by Finding the Principal Eigenvector (when λ=1)

- $Mv = \lambda v$
- $(M - \lambda I)v = 0$
- $\det(M - \lambda I) = 0$

$$M = \begin{bmatrix} 0 & 1/2 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

$\lambda(\lambda\text{-}1)(\lambda + \frac{1}{2})^2 = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = \lambda_4 = -\frac{1}{2}$

- Problem becomes solving $(M - I)v = 0$
- Use Gaussian elimination

Solving Mv = v by Gaussian Elimination

- $(M - I)v = 0$

$$\begin{bmatrix} -1 & 1/2 & 1 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & -1 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ \frac{1}{3} & -1 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{1}{3} & -1 \end{bmatrix}$$

Solving Mv = v by Gaussian Elimination

- (M – I)v = 0

$$\begin{bmatrix} -1 & 1/2 & 1 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & -\frac{5}{6} & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{1}{3} & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{6}{15} & -\frac{3}{5} \\ 0 & \frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \\ 0 & \frac{2}{3} & \frac{1}{3} & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{6}{15} & -\frac{3}{5} \\ 0 & 0 & -\frac{3}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & -\frac{3}{5} \end{bmatrix}$$

Solving Mv = v by Gaussian Elimination

- (M – I)v = 0

$$\begin{bmatrix} -1 & 1/2 & 1 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{6}{15} & -\frac{3}{5} \\ 0 & 0 & -\frac{3}{5} & \frac{3}{5} \\ 0 & 0 & \frac{3}{5} & -\frac{3}{5} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -1 & 0 \\ 0 & 1 & -\frac{6}{15} & -\frac{3}{5} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Solving Mv = v by Gaussian Elimination

- (M – I)v = 0

$$\begin{bmatrix} -1 & 1/2 & 1 & 0 \\ 1/3 & -1 & 0 & 1/2 \\ 1/3 & 0 & -1 & 1/2 \\ 1/3 & 1/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \left\{ \begin{array}{l} v_1 - 3/2\, v_4 = 0 \\ v_2 - v_4 = 0 \\ v_3 - v_4 = 0 \end{array} \right.$$

$$v_1 + v_2 + v_3 + v_4 = 1$$

$$v_1 = 1/3$$

$$v_2 = v_3 = v_4 = 2/9$$

The "principal" eigenvector is [1/3, 2/9, 2/9, 2/9], when λ = 1

Comparison of Solutions

- $Mv = v$
  - Solution 1: power iteration
  - Solution 2: Gaussian elimination

- Power iteration
  - Complexity: $O(kn^2)$, where k = # of iterations

- Gaussian elimination
  - Complexity: $O(n^3)$
  - $n^2+(n-1)^2+\ldots+2^2$

Next Week

- Link Analysis (Part II)
  - Solving the flow model: $\boldsymbol{Mv} = \boldsymbol{v}$
    - Power Iteration and Gaussian Elimination
    - Dead-ends and spider traps (teleporting and taxation)
    - PageRank for Search Engines (MapReduce)
  - Topic-specific: (augment for general-popularity)
  - Trust-Rank: (fighting against link spams)
  - Hubs-and-Authorities: (multiple importances)