

# **CSCI 561**

## **Foundation for Artificial Intelligence**

### **Reasoning over Time**

### **Temporal Models**

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University of Southern California

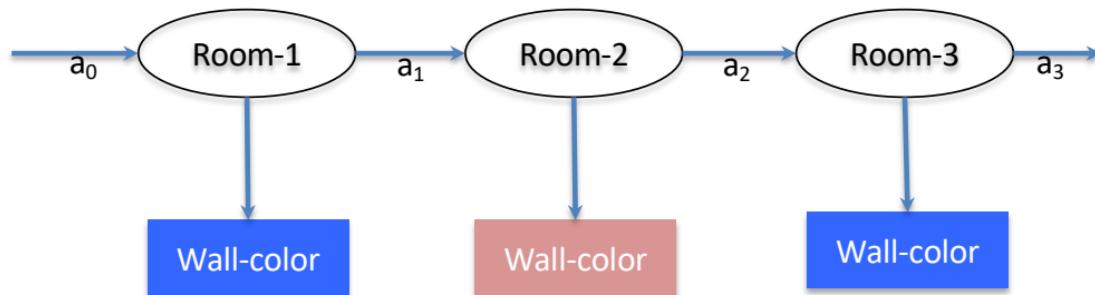
# Outline: Temporal Models

- Models with Actions and Sensors (ALFE 4-5) (aka POMDP)
  - Slides 1-20 are essential for you to understand the concepts
  - The rest will follow naturally if you do
- Markov Chains
  - No observation, no explicit actions, transit randomly
- Hidden Markov Model
  - No explicit actions, state transit randomly
- Dynamic Bayesian Networks
  - No explicit actions, States are Bayesian Networks
- Continuous State Model
  - No explicit actions, States are continuous
- POMDP
  - Discrete states with probabilistic actions, sensors, & transitions

1. ④

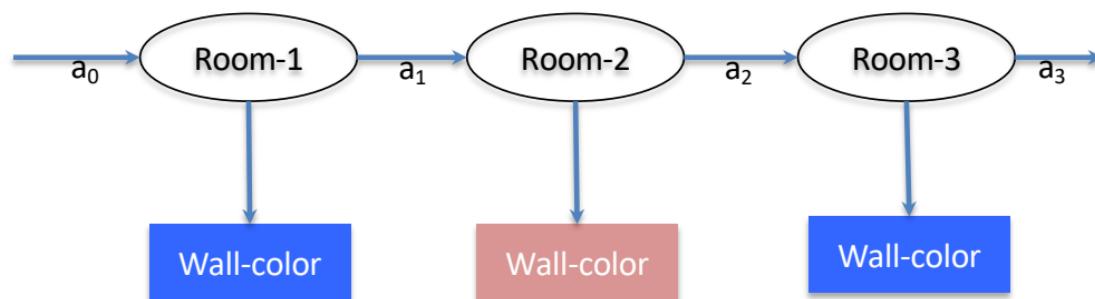
# Example of Reasoning over Time

- Speech Recognition
  - “Listening is not always equal to hearing” 😊
- Traveling through rooms with colored walls
  - “Seeing does not always tell where you are” 😊



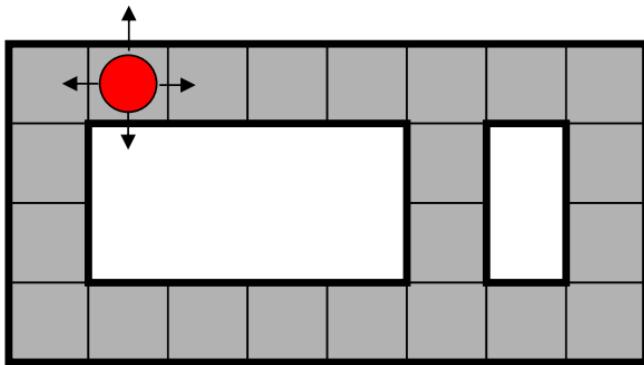
# ↳ Problem of Reasoning over Time

- Given:
  - A model of the world (e.g., a map of rooms with colored walls)
  - An experience of observations and actions from time 1 to t
- Compute (among others):
  - Which states (e.g., rooms) the robot was/is/will-be in?



# Example: Robot Localization

*Example from  
Michael Pfeiffer*



State X: Room

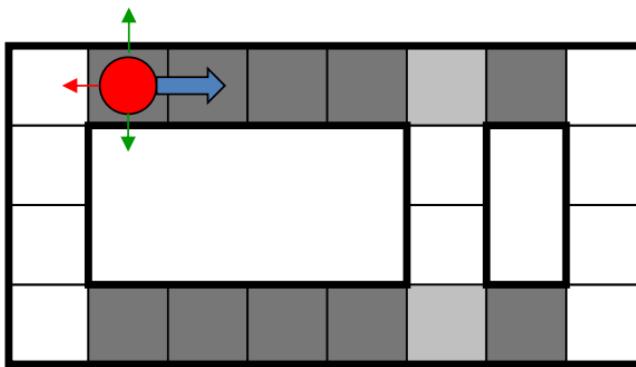


t=0 (before sensing, all states are equally likely)

Sensor model: never more than 1 mistake

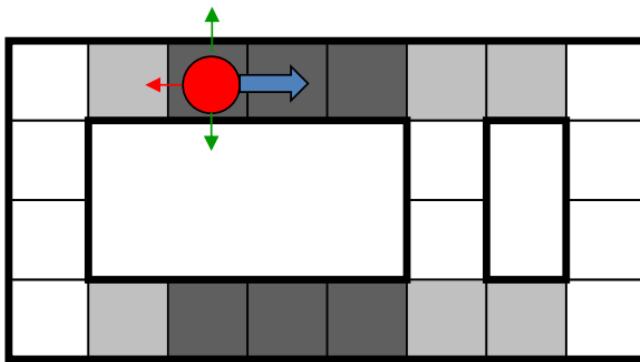
Motion model: may not execute action with small prob.

# Example: Robot Localization



$t=1$  (sensing: no room up or down)

# Example: Robot Localization



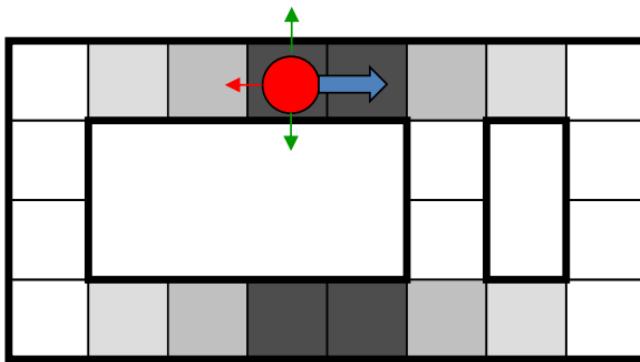
Prob

0

1

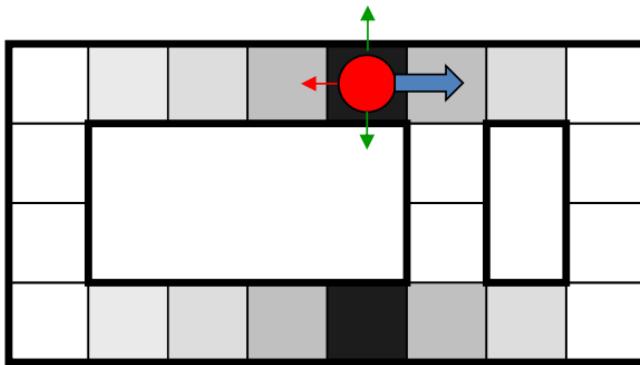
$t=2$

# Example: Robot Localization



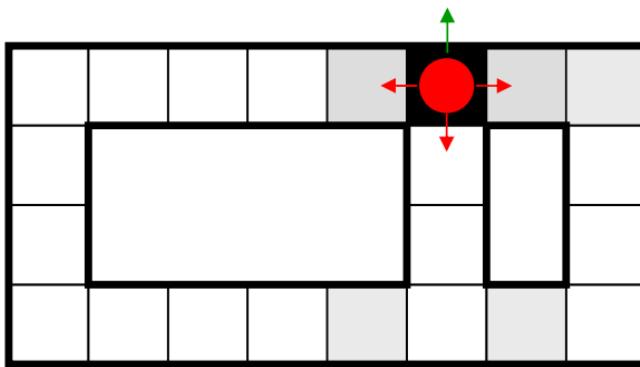
$t=3$

# Example: Robot Localization



$t=4$

# Example: Robot Localization



Prob

0

1

$t=5$

2

# Little Prince Environment ( $S, A, \Phi, \theta, \pi$ )

- Transition Probabilities  $\Phi$  (ask “Action Model”)

F	S0	S1	S2	S3
S0	0.1	0.1	0.1	0.7
S1	0.1	0.1	0.7	0.1
S2	0.7	0.1	0.1	0.1
S3	0.1	0.7	0.1	0.1

B	S0	S1	S2	S3
S0	0.1	0.1	0.7	0.1
S1	0.1	0.1	0.1	0.7
S2	0.1	0.7	0.1	0.1
S3	0.7	0.1	0.1	0.1

T	S0	S1	S2	S3
S0	0.7	0.1	0.1	0.1
S1	0.1	0.7	0.1	0.1
S2	0.1	0.1	0.1	0.7
S3	0.1	0.1	0.7	0.1

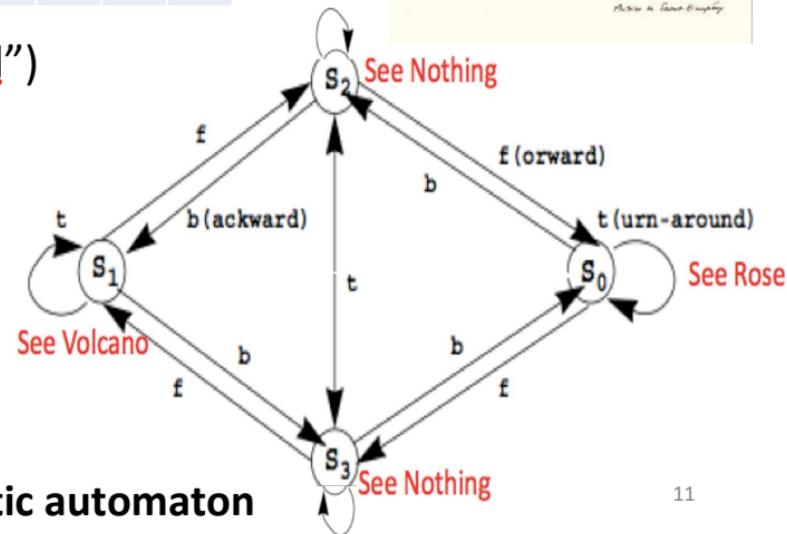


- Appearance Probabilities  $\theta$  (aka “Sensor Model”)

$\theta$	Rose	Volcano	Nothing
S0	0.8	0.1	0.1
S1	0.1	0.8	0.1
S2	0.1	0.1	0.8
S3	0.1	0.1	0.8

- Initial State Probabilities  $\pi$  (aka “Belief States”)

$\pi$	S0	S1	S2	S3
	.25	.25	.25	.25



# Little Prince in Action

amongst observations

experience

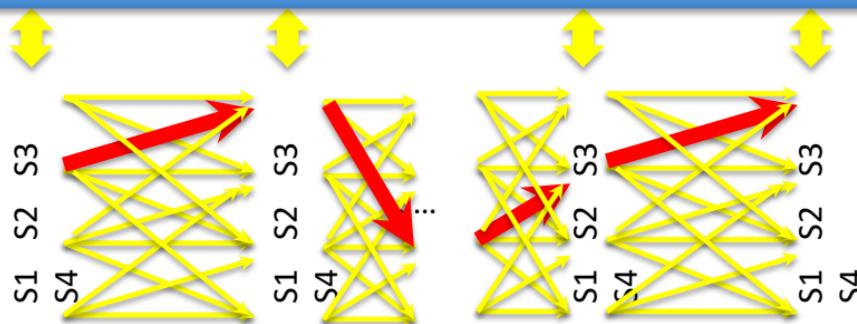
$E_{1:T}$

{rose}, fward, {none}, ..., turn, {rose}, back, {volcano}

time →

state  
s

Hidden!



- Given: "experience"  $E_{1:T}$  (time 1 through T)
- You can Infer:
  - $P(X_t | E_{1:t})$ , where (which state) am I at now? (estimation, filtering, localization)
  - $P(X_{t+k} | E_{1:t})$ , where I will be at time  $t+k$ ? (prediction)
  - $P(X_k | E_{1:t})$ , where I was at time  $k < t$ ? (smooth)
  - $P(X_{1:t} | E_{1:t})$  the probability of every state sequence that I went through? (explanation)
  - $P(x_{1:t} | E_{1:t})$  the most likely sequence of states I went through? (Viterbi algorithm)

# Some General Problems for Temporal Models



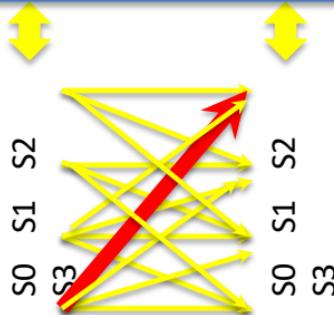
- $P(X_{1:t} | E_{1:t})$ 
  - compare all state sequences I might go through ("explanation")
- $P(X_t | E_{1:t})$ 
  - which state I am in now ("state estimation" "localization")
- $P(X_{t+k} | E_{1:t})$ 
  - which state I will be in at time t+k ("prediction")
- $P(X_k | E_{1:t})$ 
  - which state I was in at time k ( $k < t$ ) ("smoothing")
- $P(x_{1:t} | E_{1:t})$ 
  - what is the best state sequence that I went through ("viterbi")
- $P(M_t | E_{1:t})$ 
  - How correct is my model at time t ("model learning")

3

## P(X | E) Example

- Given: “experience”  $E_{1:2} = \{\text{rose}, \text{forward}, \text{nothing}\}$
- Infer: the most likely “state sequence”  $X_{1:2}$  // by comparing all

$E = \{\text{rose}\}, \text{forward}, \{\text{nothing}\}$



$E_{1:2} = \{\text{rose}, \text{forward}, \text{nothing}\}$

$O_{1:2} = \{\text{rose}, \text{nothing}\}$  // observations

$A_1 = \{\text{forward}\}$  // action sequence

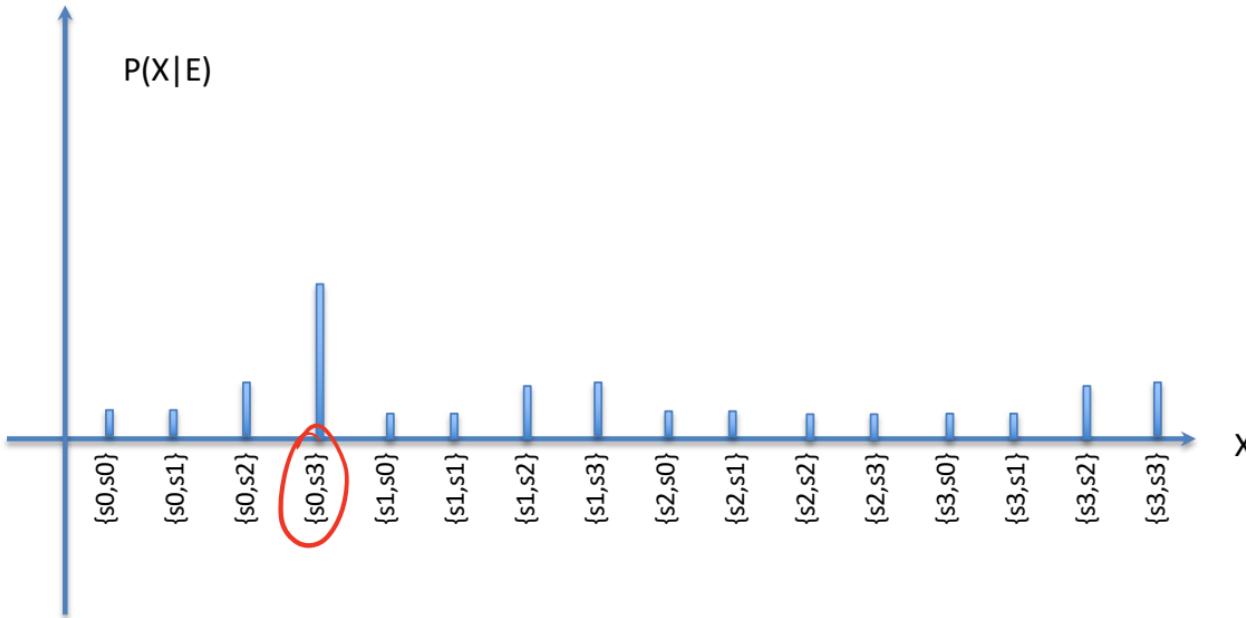
$$E_{1:2} = O_{1:2} A_1$$

Compute  $P(X | E)$ :

16 possible  $X$ , which one is most likely?

$\{\text{s0,s0}\}, \{\text{s0,s1}\}, \{\text{s0,s2}\}, \underline{\{\text{s0,s3}\}},$   
 $\{\text{s1,s0}\}, \{\text{s1,s1}\}, \{\text{s1,s2}\}, \{\text{s1,s3}\},$   
 $\{\text{s2,s0}\}, \{\text{s2,s1}\}, \{\text{s2,s2}\}, \{\text{s2,s3}\},$   
 $\{\text{s3,s0}\}, \{\text{s3,s1}\}, \{\text{s3,s2}\}, \{\text{s3,s3}\}$

# $P(X|E)$ Example in a Graph



4. ~~if~~ all possible sequence.

## Compute $P(X_{1:T} | E_{1:T})$

- $P(X_{1:T} | E)$  where  $E=OA$
- $X_{1:T}$  is all possible sequences of states
  - Let  $x=\{i_1, i_2, \dots, i_T\}$  be a sequence of states
- $P(x | E) = p(xE)/P(E) = P(xOA)/P(OA)$ 

$$= p(x|A)p(O|x A)/p(O|A)$$
 ~~$P(O|XA) P(x|A) P(A)$~~   
 ~~$P(O|A) P(x|A)$~~
- $p(x|A) = \pi_{i_1}(1)P_{i_1 i_2}[b_1]P_{i_2 i_3}[b_2] \cdots P_{i_{T-1} i_T}[b_{T-1}]$
- $p(O|x A) = \theta_{i_1}(z_1)\theta_{i_2}(z_2) \cdots \theta_{i_{T-1}}(z_{T-1})\theta_{i_T}(z_T)$

$$O = \{z_1, z_2, \dots, z_T\}, A = \{b_1, b_2, \dots, b_{T-1}\}, x = \{i_1, i_2, \dots, i_T\}$$

# In Our Example

- 16 possible  $x$
- $p(x|A)$

$\{s_0, s_0\}, \{s_0, s_1\}, \{s_0, s_2\}, \{s_0, s_3\}, \{s_1, s_0\}, \{s_1, s_1\}, \{s_1, s_2\}, \{s_1, s_3\}, \{s_2, s_0\}, \{s_2, s_1\}, \{s_2, s_2\}, \{s_2, s_3\}, \{s_3, s_0\}, \{s_3, s_1\}, \{s_3, s_2\}, \{s_3, s_3\}$

- $p(\{s_0, s_0\} | \{f\}) = \pi(s_0) p_{s_0, s_0}(f) = .25 * .1 = .025$
- $p(\{s_0, s_1\} | \{f\}) = \pi(s_0) p_{s_0, s_1}(f) = .25 * .1 = .025$
- $p(\{s_0, s_2\} | \{f\}) = \pi(s_0) p_{s_0, s_2}(f) = .25 * .1 = .025$
- $p(\{s_0, s_3\} | \{f\}) = \pi(s_0) p_{s_0, s_3}(f) = .25 * .7 = .175$
- $p(\{s_1, s_0\} | \{f\}) = \pi(s_1) p_{s_1, s_0}(f) = .25 * .1 = .025$
- $p(\{s_1, s_1\} | \{f\}) = \pi(s_1) p_{s_1, s_1}(f) = .25 * .1 = .025$
- $p(\{s_1, s_2\} | \{f\}) = \pi(s_1) p_{s_1, s_2}(f) = .25 * .1 = .025$
- $p(\{s_1, s_3\} | \{f\}) = \pi(s_1) p_{s_1, s_3}(f) = .25 * .7 = .025$
- $p(\{s_2, s_0\} | \{f\}) = \pi(s_2) p_{s_2, s_0}(f) = .25 * .1 = .025$
- $p(\{s_2, s_1\} | \{f\}) = \pi(s_2) p_{s_2, s_1}(f) = .25 * .1 = .025$
- $p(\{s_2, s_2\} | \{f\}) = \pi(s_2) p_{s_2, s_2}(f) = .25 * .1 = .025$
- $p(\{s_2, s_3\} | \{f\}) = \pi(s_2) p_{s_2, s_3}(f) = .25 * .7 = .025$
- $p(\{s_3, s_0\} | \{f\}) = \pi(s_3) p_{s_3, s_0}(f) = .25 * .1 = .025$
- $p(\{s_3, s_1\} | \{f\}) = \pi(s_3) p_{s_3, s_1}(f) = .25 * .1 = .025$
- $p(\{s_3, s_2\} | \{f\}) = \pi(s_3) p_{s_3, s_2}(f) = .25 * .1 = .025$
- $p(\{s_3, s_3\} | \{f\}) = \pi(s_3) p_{s_3, s_3}(f) = .25 * .7 = .025$

- $p(O|x_A)$

- $\Theta s_0(r) \Theta s_0(n) = .8 * .1 = .08$
- $\Theta s_0(r) \Theta s_1(n) = .8 * .1 = .08$
- $\Theta s_0(r) \Theta s_2(n) = .8 * .8 = .64$
- $\Theta s_0(r) \Theta s_3(n) = .8 * .8 = .64$
- $\Theta s_1(r) \Theta s_0(n) = .1 * .1 = .01$
- $\Theta s_1(r) \Theta s_1(n) = .1 * .1 = .01$
- $\Theta s_1(r) \Theta s_2(n) = .1 * .8 = .08$
- $\Theta s_1(r) \Theta s_3(n) = .1 * .8 = .08$
- $\Theta s_2(r) \Theta s_0(n) = .1 * .1 = .01$
- $\Theta s_2(r) \Theta s_1(n) = .1 * .1 = .01$
- $\Theta s_2(r) \Theta s_2(n) = .1 * .8 = .08$
- $\Theta s_2(r) \Theta s_3(n) = .1 * .8 = .08$
- $\Theta s_3(r) \Theta s_0(n) = .1 * .1 = .01$
- $\Theta s_3(r) \Theta s_1(n) = .1 * .1 = .08$
- $\Theta s_3(r) \Theta s_2(n) = .1 * .8 = .08$
- $\Theta s_3(r) \Theta s_3(n) = .1 * .8 = .08$

$E = \{\text{rose, forward, nothing}\}$   
 $O = \{\text{rose, nothing}\}$ ,  
 $A = \{\text{forward}\}$

Best Explanation is:  $\{s_0, s_3\}$

# Which Explanation is the Best?

- Among all possible sequences of states, the best “explanation” is the sequence of states that gives the maximal value for

$$\pi_{i_1}(1)\theta_{i_1}(z_1)P_{i_1 i_2}[b_1] \cdots \theta_{i_{T-1}}(z_{T-1})P_{i_{T-1} i_T}[b_{T-1}]\theta_{i_T}(z_T)$$

Experience:

Observations:  $o_1, o_2, \dots, o_{T-1}, o_T$

Actions:  $b_1, b_2, \dots, b_{T-1}$

Sensor models:  $\theta_i(k) = p(z_k|s_i)$ ,  $z_k \in Z$ , and  $s_i \in S$ ,

Action models:  $P_{ij}[b] = P(s_j | s_i, b)$

Explanation: State sequence:  $i_1, i_2, i_3, \dots, i_T$

# Compute Hidden State Sequence

$$O = \{z_1, z_2, \dots, z_T\}, A = \{b_1, b_2, \dots, b_T\}, I = \{i_1, i_2, \dots, i_T\}$$

- *Experience consists of both O and A*
- O is observation sequence and A is action sequence in experience
- M is the model, C is the background information

$$p(I|AMC) = \pi_{i_1}(1)P_{i_1i_2}[b_1]P_{i_2i_3}[b_2]\cdots P_{i_{T-1}i_T}[b_{T-1}]$$

$$p(O|IAMC) = \theta_{i_1}(z_1)\theta_{i_2}(z_2)\cdots\theta_{i_{T-1}}(z_{T-1})\theta_{i_T}(z_T)$$

On slide #16, we used  $x$  for  $I$ , so these two terms were written as  $p(x|A)$  and  $p(O|xA)$  there.

# Compute Hidden State Sequence

$$p(O|AMC) = \sum_I p(O, I|AMC) = \sum_I p(I|AMC)p(O|IAMC)$$

Therefore, we have

$$p(O|AMC) = \sum_{I=i_1 i_2 \dots i_T} \pi_{i_1}(1)\theta_{i_1}(z_1)P_{i_1 i_2}[b_1] \dots \theta_{i_{T-1}}(z_{T-1})P_{i_{T-1} i_T}[b_{T-1}]\theta_{i_T}(z_T)$$

Among all possible sequences in  $I$ , there is one with the maximal probability.

However, this straightforward way of computing  $p(O|AMC)$  is very expensive. It requires on the order of  $2TN^T$  calculations, since at every time  $t = 1, 2, \dots, T$ , there are  $N$  possible states to go through and for each summand about  $2T$  calculations are required. Clearly a more efficient procedure is required. One such procedure is the combination of the forward and backward procedures.

5.

## P(O|AMC) by Forward Procedure

Main idea: do not consider all possible state sequences, but every step in the experience, and compute P(O|AMC) incrementally on the time t:

$t = 1, 2, \dots, T$  as follows:

# of  $\alpha$  in every step = # of state

$$\alpha_1(i) = p(z_1, i_1 = s_i | AMC)$$

$$\alpha_2(i) = p(z_1, z_2, i_2 = s_i | AMC)$$

⋮

$$\alpha_t(i) = p(z_1, z_2, \dots, z_t, i_t = s_i | AMC)$$

⋮

$$\alpha_{T-1}(i) = p(z_1, z_2, \dots, z_{T-1}, i_{T-1} = s_i | AMC)$$

$$\alpha_T(i) = p(z_1, z_2, \dots, z_{T-1}, z_T, i_T = s_i | AMC)$$

The term  $\alpha_1(i)$  represents the probability of being in state  $s_i$  and seeing  $z_1$  at time  $t = 1$ . The term  $\alpha_t(i)$  represents the probability of seeing  $\{z_1, \dots, z_t\}$  and ending in state  $s_i$  at time  $t$ .

# Forward Procedure

The computation of  $\alpha_t(i)$  is straightforward. When  $t = 1$ , the probability of  $\alpha_1(i)$  depends only on  $\theta$  and  $\pi$ :

$$\alpha_1(i) = \pi_i \theta_i(z_1) \quad (5.9)$$

Furthermore, the value of  $\alpha_{t+1}(j)$  can be computed recursively based on the values of  $\alpha_t(i)$ . In other words, in order to be in state  $s_j$  at time  $t + 1$ , the system must have been in any previous state  $s_i$  at time  $t$  (with probability  $\alpha_t(i)$ ) and then made a transition to state  $s_j$  with probability  $P_{ij}[b_t]$ . Thus the probability of reaching state  $s_j$  at time  $t + 1$  is the sum, for all  $i \in S$ , of the product of three probabilities:  $\alpha_t(i)$ , the probability of being at state  $s_i$  at time  $t$ ;  $P_{ij}[b_t]$ , the probability of taking the transition from  $s_i$  to  $s_j$  under action  $b_t$ , and,  $\theta_j(z_{t+1})$ , the probability of seeing  $z_{t+1}$  at state  $s_j$ . That is,

**Forward Procedure**  $\alpha_{t+1}(j) = \sum_{i \in S} \alpha_t(i) P_{ij}[b_t] \theta_j(z_{t+1})$  *observation* *transition*  $(5.10)$

Finally, when  $\alpha_T(i)$  is known for all  $i$ , we can calculate the value of  $p(O|AMC)$  by summing up  $\alpha_T(i)$  on all model states. That is,

**Complexity is  $O(TN^2)$**   $p(O|AMC) = \sum_{s \in S} \alpha_T(i)$   $(5.11)$

# Forward Procedure Example

Time:  $t_1$ -----  $t_2$ -----  $t_3$ -----  
 $E = \{ \text{rose, forward, none, turn, volcano, ....} \}$

$\alpha_1(i)$ :

$$\begin{aligned}\alpha_1(s_0) &= \pi(s_0)\Theta s_0(r) = .25 * .8 = .20 \\ \alpha_1(s_1) &= \pi(s_1)\Theta s_1(r) = .25 * .1 = .025 \\ \alpha_1(s_2) &= \pi(s_2)\Theta s_2(r) = .25 * .1 = .025 \\ \alpha_1(s_3) &= \pi(s_3)\Theta s_3(r) = .25 * .1 = .025\end{aligned}$$

$\alpha_2(i)$ :

$$\begin{aligned}\alpha_2(s_0) &= \alpha_1(s_0) p_{s_0,s_0}(f) \Theta s_0(n) \\ &\quad + \alpha_1(s_1) p_{s_1,s_0}(f) \Theta s_0(n) \\ &\quad + \alpha_1(s_2) p_{s_2,s_0}(f) \Theta s_0(n) \\ &\quad + \alpha_1(s_3) p_{s_3,s_0}(f) \Theta s_0(n)\end{aligned}$$

$$\begin{aligned}\alpha_2(s_1) &= \alpha_1(s_0) p_{s_0,s_1}(f) \Theta s_1(n) \\ &\quad + \alpha_1(s_1) p_{s_1,s_1}(f) \Theta s_1(n) \\ &\quad + \alpha_1(s_2) p_{s_2,s_1}(f) \Theta s_1(n) \\ &\quad + \alpha_1(s_3) p_{s_3,s_1}(f) \Theta s_1(n)\end{aligned}$$

$$\begin{aligned}\alpha_2(s_2) &= \alpha_1(s_0) p_{s_0,s_2}(f) \Theta s_2(n) \\ &\quad + \alpha_1(s_1) p_{s_1,s_2}(f) \Theta s_2(n) \\ &\quad + \alpha_1(s_2) p_{s_2,s_2}(f) \Theta s_2(n) \\ &\quad + \alpha_1(s_3) p_{s_3,s_2}(f) \Theta s_2(n)\end{aligned}$$

$$\begin{aligned}\alpha_2(s_3) &= \alpha_1(s_0) p_{s_0,s_3}(f) \Theta s_3(n) \\ &\quad + \alpha_1(s_1) p_{s_1,s_3}(f) \Theta s_3(n) \\ &\quad + \alpha_1(s_2) p_{s_2,s_3}(f) \Theta s_3(n) \\ &\quad + \alpha_1(s_3) p_{s_3,s_3}(f) \Theta s_3(n)\end{aligned}$$

$\alpha_3(i)$ :

$$\begin{aligned}\alpha_3(s_0) &= \alpha_2(s_0) p_{s_0,s_0}(t) \Theta s_0(v) \\ &\quad + \alpha_2(s_1) p_{s_1,s_0}(t) \Theta s_0(v) \\ &\quad + \alpha_2(s_2) p_{s_2,s_0}(t) \Theta s_0(v) \\ &\quad + \alpha_2(s_3) p_{s_3,s_0}(t) \Theta s_0(v)\end{aligned}$$

$$\begin{aligned}\alpha_3(s_1) &= \alpha_2(s_0) p_{s_0,s_1}(t) \Theta s_1(v) \\ &\quad + \alpha_2(s_1) p_{s_1,s_1}(t) \Theta s_1(v) \\ &\quad + \alpha_2(s_2) p_{s_2,s_1}(t) \Theta s_1(v) \\ &\quad + \alpha_2(s_3) p_{s_3,s_1}(t) \Theta s_1(v)\end{aligned}$$

$$\begin{aligned}\alpha_3(s_2) &= \alpha_2(s_0) p_{s_0,s_2}(t) \Theta s_2(v) \\ &\quad + \alpha_2(s_1) p_{s_1,s_2}(t) \Theta s_2(v) \\ &\quad + \alpha_2(s_2) p_{s_2,s_2}(t) \Theta s_2(v) \\ &\quad + \alpha_2(s_3) p_{s_3,s_2}(t) \Theta s_2(v)\end{aligned}$$

$$\begin{aligned}\alpha_3(s_3) &= \alpha_2(s_0) p_{s_0,s_3}(t) \Theta s_3(v) \\ &\quad + \alpha_2(s_1) p_{s_1,s_3}(t) \Theta s_3(v) \\ &\quad + \alpha_2(s_2) p_{s_2,s_3}(t) \Theta s_3(v) \\ &\quad + \alpha_2(s_3) p_{s_3,s_3}(t) \Theta s_3(v)\end{aligned}$$

6.

# Backward Procedure

- Similar to forward
- But starting from the end and walking backwards
- See ALFE 5.10.1

The backward procedure is symmetric to the forward procedure. Instead of incrementing  $E$  from the beginning to the end, it increments from the end to the beginning. In particular, we have

$$\begin{aligned}\beta_{T-1}(i) &= p(z_T | AMC, i_{T-1} = s_i) \\ \beta_{T-2}(i) &= p(z_{T-1}, z_T | AMC, i_{T-2} = s_i) \\ &\vdots \\ \beta_t(i) &= p(z_{t+1}, \dots, z_{T-1}, z_T | AMC, i_t = s_i) \\ &\vdots \\ \beta_2(i) &= p(z_3, \dots, z_{T-1}, z_T | AMC, i_2 = s_i) \\ \beta_1(i) &= p(z_2, z_3, \dots, z_{T-1}, z_T | AMC, i_1 = s_i)\end{aligned}$$

Each quantity  $\beta_t(i)$  specifies the probability of seeing the sequence  $z_{t+1}, \dots, z_T$  if the state at time  $t$  is  $s_i$ .

As in the forward procedure, the value of  $\beta_{t-1}(i)$  can be easily computed. When  $t = T$ , we define

$$\beta_T(i) = 1 \text{ for all states } s_i \quad (5.12)$$

Furthermore, the value of  $\beta_{t-1}(i)$  can be computed based on the values of  $\beta_t(j)$ . In other words, in order to be in state  $s_i$  at time  $t - 1$ , the system would have to be (with probability  $\beta_t(i)$ ) in some next state  $s_j$  at time  $t$  with observation  $z_t$ , having made a transition from  $s_i$  to  $s_j$  with probability  $P_{ij}[b_{t-1}]$ . Thus, the probability of being at state  $i$  at time  $t - 1$  is the sum of the product of the following three probabilities:  $P_{ij}[b_{t-1}]$ , the probability of taking the transition from  $s_i$  to  $s_j$  under action  $b_{t-1}$ ,  $\theta_j(z_t)$ , the probability of seeing  $z_t$  at  $s_j$ , and,  $\beta_t(j)$ , the probability of seeing the rest of observations after time  $t$  given state  $j$  at time  $t$ . Thus, our equation is

$$\beta_{t-1}(i) = \sum_{j \in S} P_{ij}[b_{t-1}] \theta_j(z_t) \beta_t(j) \quad (5.13)$$

Finally, when all  $\beta_1(i)$  are known,  $p(O | AMC)$  can be computed by summing up  $\beta_1(i)$  on all model states. That is,

$$p(O | AMC) = \sum_{s_i \in S} \pi_i(1) \theta_i(z_1) \beta_1(i) \quad (5.14)$$

Thus, equations 5.12, 5.13, and 5.14 constitute the backward procedure.

# Inference: State Estimation



- $P(X_t | E_{1:t})$ 
  - Which state I am most likely in now?
  - It is the state  $s_i$  such that  $\alpha_T(i)$  is maximal
  - Viterbi algorithm could be used for this.

# Inference: State Prediction



- $P(X_{t+k} | E_{1:t})$ 
  - which state I will be in at time  $t+k$ ?
  - Depending on your action during  $(t, t+k]$
  - If you have a non-action, use that information and continue computing the future  $\alpha$  values.
  - Example/exercise: compute when  $k=1, k=2, \dots$

# Inference: State Smoothing



- $P(X_k | E_{1:t})$ 
  - which state I was in at time k (smoothing)
  - It is the state  $s_i$  such that  
 $\alpha_k(i) P_{ij}(b_{k+1}) \theta_j(z_{k+1}) \beta_{k+1}(j)$  is maximal
  - This is to say, given time  $1, \dots, k, k+1, \dots, t$ 
    - (best forward from 1 to k)
    - & (best transition from k to k+1)
    - & (best backward to k+1 from t)

# State Explanation



- $P(X_{1:t} | E_{1:t})$ 
  - what states I have been through (explanation)
  - They are the following states:
    - $t=1$ : the state  $s_i$  such that  $\alpha_1(i)$  is the maximal
    - $t=2$ : the state  $s_i$  such that  $\alpha_2(i)$  is the maximal
    - $t=T$ : the state  $s_i$  such that  $\alpha_T(i)$  is the maximal

# Model Learning



- $P(M_t | E_{1:t})$ 
  - How correct is my model of the world (learning)
  - We will teach you this later.

# Temporal Models

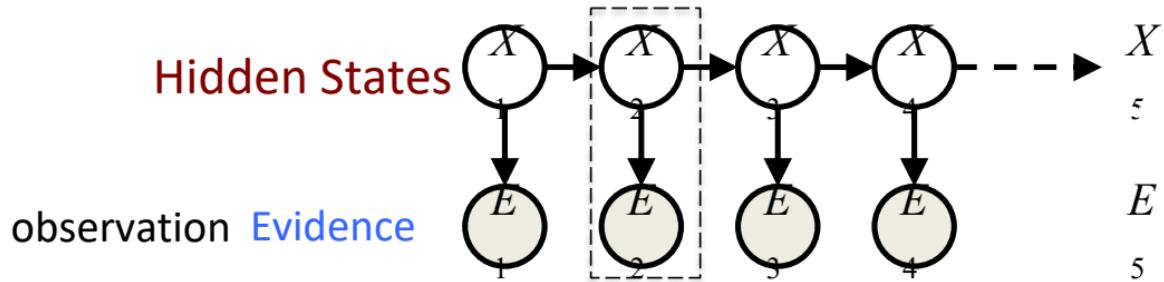
- Models with actions and sensors (ALFE 4-5) (aka POMDP)
- **Markov Chains**
  - No observation, no explicit actions, transit randomly
- **Hidden Markov Model**
  - No explicit actions, state transit randomly (e.g., predicting weather)
- Continuous State Model
  - No explicit actions, States are continuous
- Dynamic Bayesian Networks
  - No explicit actions, States are Bayesian Networks
- POMDP
  - Discrete states with probabilistic actions, sensors, & transitions

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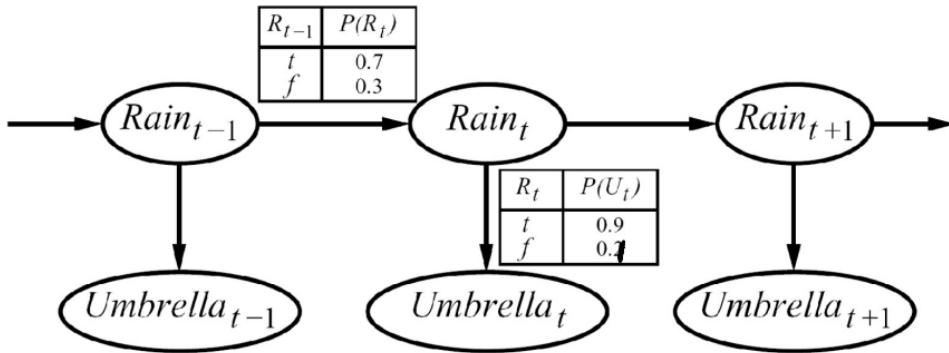
# Hidden Markov Models

- Markov chains not so useful for most agents
  - Eventually you don't know anything anymore
  - Need observations to update your beliefs
- Hidden Markov models (HMMs)
  - Underlying Markov chain over states  $S$
  - You observe outputs (effects) at each time step
  - As a Bayes' net:

Compared to the general action/sensor models,  
HMM does not have an explicit action model.



# Rain/Umbrella Example in the Book



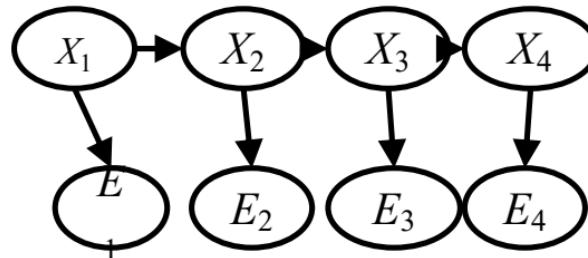
Story background:

Security guard at top-secret underground facility.  
Is it raining? No direct observation.  
But observe whether director has umbrella when coming to work.

- An HMM is defined by:
  - Initial distribution:  $P(X_1)$
  - Transitions:  $P(X|X_{-1})$
  - Emissions (sensor model):  $P(E|X)$

# Conditional Independence

- HMMs have two important independence properties:
  - Markov hidden process, future depends only on the present
  - Current observation is independent of all else, given the current state



- Quiz: does this mean that observations are independent?
  - [No, correlated by the hidden state]

# Real HMM Examples

- Speech recognition HMMs:
  - Observations are acoustic signals (continuous valued)
  - States are specific positions in specific words (so, tens of thousands)
- Machine translation HMMs:
  - Observations are words (tens of thousands)
  - States are translation options
- Robot tracking:
  - Observations are range readings (continuous/discrete)
  - States are positions on a map (continuous/discrete)

# Localization, Filtering, Monitoring

- Localization, Filtering, or monitoring, is the task of tracking the distribution  $P(X)$  (the belief state) over time
- We start with  $P(X_1)$  in an initial setting, usually uniform
- As time passes,  $1, \dots, t$ , when we have observations, we update  $P(X_t)$ :
- $P(X_t | E_{1:t})$



# Filtering Example

Day 0:

$$P(R_0) = \langle 0.5, 0.5 \rangle$$

Day 1:

$$\begin{aligned} P(R_1) &= P(r_0)P(R_1|r_0) \\ &= \alpha 0.5 \langle 0.7, 0.3 \rangle + \alpha 0.5 \langle 0.3, 0.7 \rangle \\ &= \langle 0.5, 0.5 \rangle \end{aligned}$$

observe Umbrella appears  $\exists U_1 = \text{true}$

updating with evidence for  $t=1$  gives:

$$\begin{aligned} P(R_1|u_1) &= \alpha P(u_1|R_1)P(R_1) = \alpha \langle 0.9, 0.2 \rangle \langle 0.5, 0.5 \rangle \\ &= \alpha \langle 0.45, 0.1 \rangle = \langle 0.818, 0.182 \rangle \end{aligned}$$

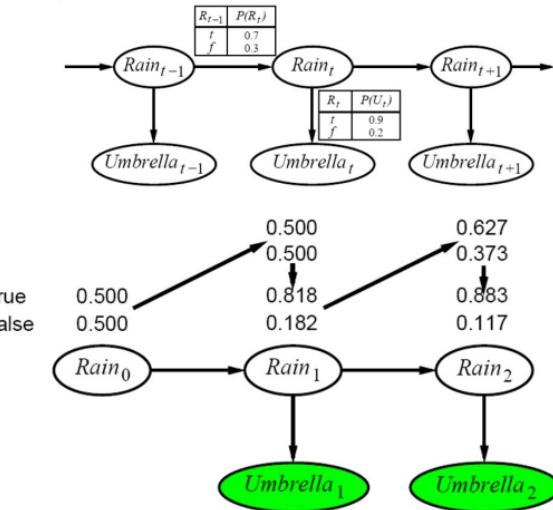
Day 2:

$$\begin{aligned} P(R_2|u_1) &= P(R_2|r_1)P(r_1|u_1) \\ &= \alpha \langle 0.818 \langle 0.7, 0.3 \rangle + 0.182 \langle 0.3, 0.7 \rangle \rangle = \langle 0.627, 0.373 \rangle \end{aligned}$$

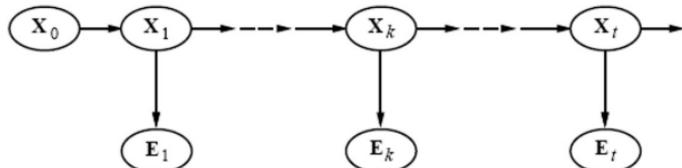
observe Umbrella appears  $\exists U_2 = \text{true}$

updating with evidence for  $t=2$  gives:

$$\begin{aligned} P(R_2|u_1, u_2) &= \alpha P(u_2|R_2)P(R_2|u_1) = \alpha \langle 0.9, 0.2 \rangle \langle 0.627, 0.373 \rangle \\ &= \alpha \langle 0.565, 0.075 \rangle = \langle 0.883, 0.117 \rangle \end{aligned}$$



# Smoothing: $P(X_k | E_{1:T})$



- Divide evidence  $e_{1:t}$  into  $e_{1:k}$ ,  $e_{k+1:t}$ :

$$\begin{aligned}
 P(X_k | e_{1:t}) &= P(X_k | e_{1:k}, e_{k+1:t}) \\
 &= \alpha P(X_k | e_{1:k}) P(e_{k+1:t} | X_k, e_{1:k}) && \text{Using Bayes rule} \\
 &= \alpha P(X_k | e_{1:k}) P(e_{k+1:t} | X_k) && \text{Using conditional independence} \\
 &= \alpha f_{1:k} b_{k+1:t}
 \end{aligned}$$

- Backward message computed by a backwards recursion:

$$\begin{aligned}
 P(e_{k+1:t} | X_k) &= \sum_{X_{k+1}} P(e_{k+1:t} | X_k, x_{k+1}) P(x_{k+1} | X_k) && \text{Conditioning on } X_{k+1} \\
 &= \sum_{X_{k+1}} P(e_{k+1:t} | x_{k+1}) P(x_{k+1} | X_k) && \text{Using conditional independence} \\
 &= \sum_{X_{k+1}} P(e_{k+1} | x_{k+1}) \boxed{P(e_{k+2:t} | x_{k+1})} P(x_{k+1} | X_k) && \\
 &&& = \text{BACKWARD}(b_{k+2:t}, e_{k+1:t})
 \end{aligned}$$

$$b_{k+1:t} = \sum_{X_{t+1}} (P(e_{k+1} | x_{k+1}) P(x_{k+1} | X_k)) b_{k+2:t}$$

# Smoothing Example

Compute estimate for rain at t=1

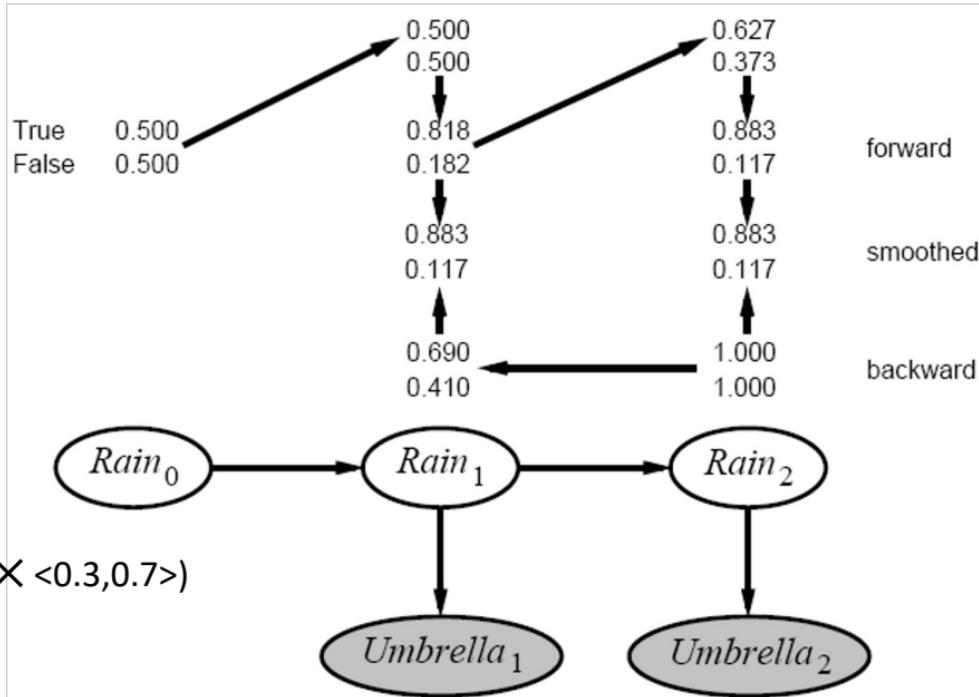
$$P(R_1|u_1, u_2) = \alpha P(R_1|u_1)P(u_2|R_1)$$

$$P(R_1|u_1) = <0.818, 0.182>$$

$$\begin{aligned} P(u_2|R_1) &= \sum_{r_2} P(u_2|r_2) P(r_2|R_1) \\ &= (0.9 \times 1 \times <0.7, 0.3>) + (0.2 \times 1 \times <0.3, 0.7>) \\ &= <0.69, 0.41> \end{aligned}$$

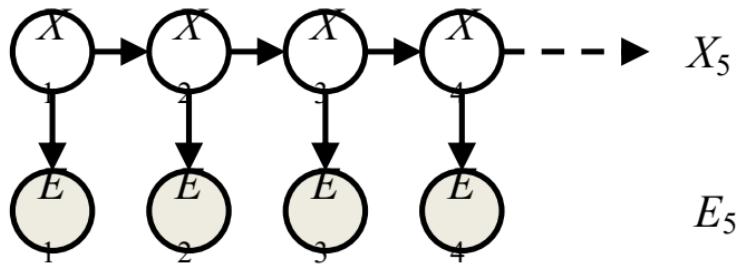
Smoothed estimate:

$$P(R_1|u_1, u_2) = \alpha <0.818, 0.182> \times <0.69, 0.41> = <0.883, 0.117>$$



- Forward-backward algorithm: cache forward messages along the way
- Time linear in  $t$  (polytree inference), space  $O(t f_j)$

# Best Explanation Queries

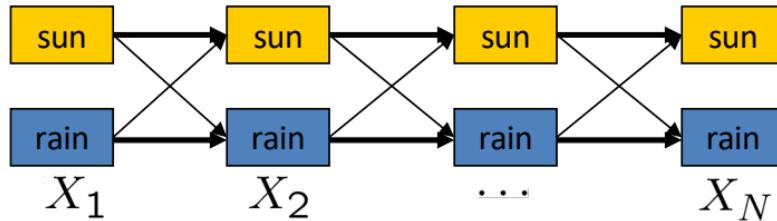


- Query: what is the most likely sequence of states?

$$\arg \max_{x_{1:t}} P(x_{1:t} | e_{1:t})$$

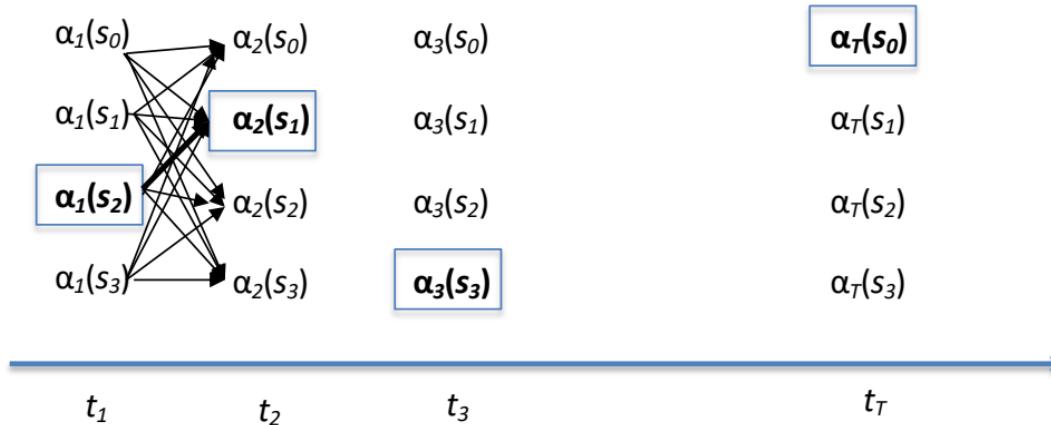
# State Path Trellis

- State trellis: graph of states and transitions over time



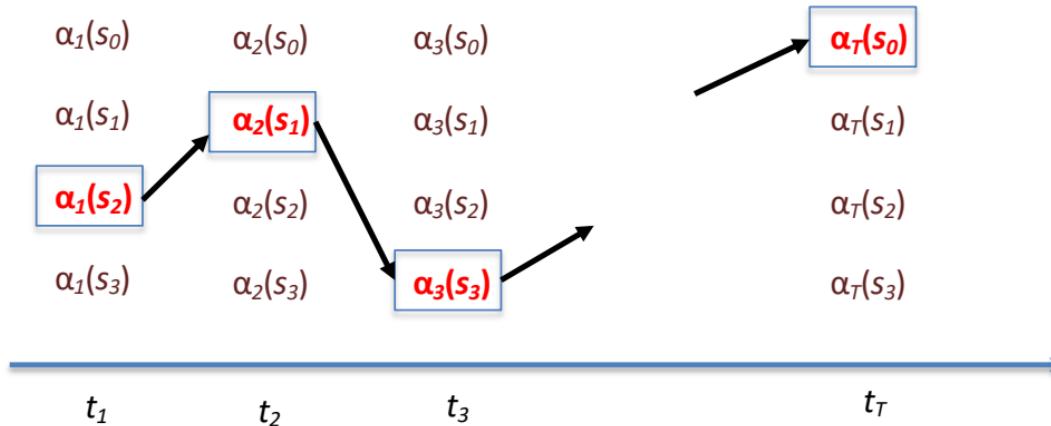
- Each arc represents some transition
- Each arc has weight  $x_{t-1} \rightarrow x_t$
- Each path is a sequence of states  $|P(x_t|x_{t-1})P(e_t|x_t)|$
- The product of weights on a path is the seq's probability
- Can think of the Forward (and now Viterbi) algorithms as computing sums of all paths (best paths) in this graph

# Forward Procedure Computes All $\alpha_t(s_i)$ on State Trellis



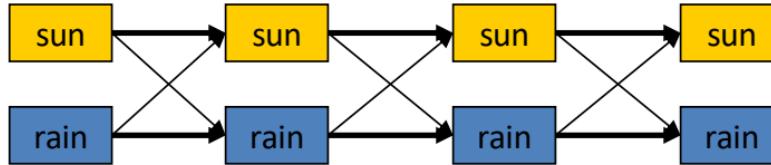
No need to compute all  $\alpha_t(s_i)$ , if we only need the best for each step  
The Viterbi algorithm does this.

# Viterbi Algorithm Computes Only the Best $\alpha_t(s_i)$ on State Trellis



No need to compute all  $\alpha_t(s_i)$ , if we only want to know the best for each step  
The Viterbi algorithm does this.

Viterbi Algorithm: choose the best state seq.  
 Similar to before  $\alpha_t(s_t)$ , but use max not sum

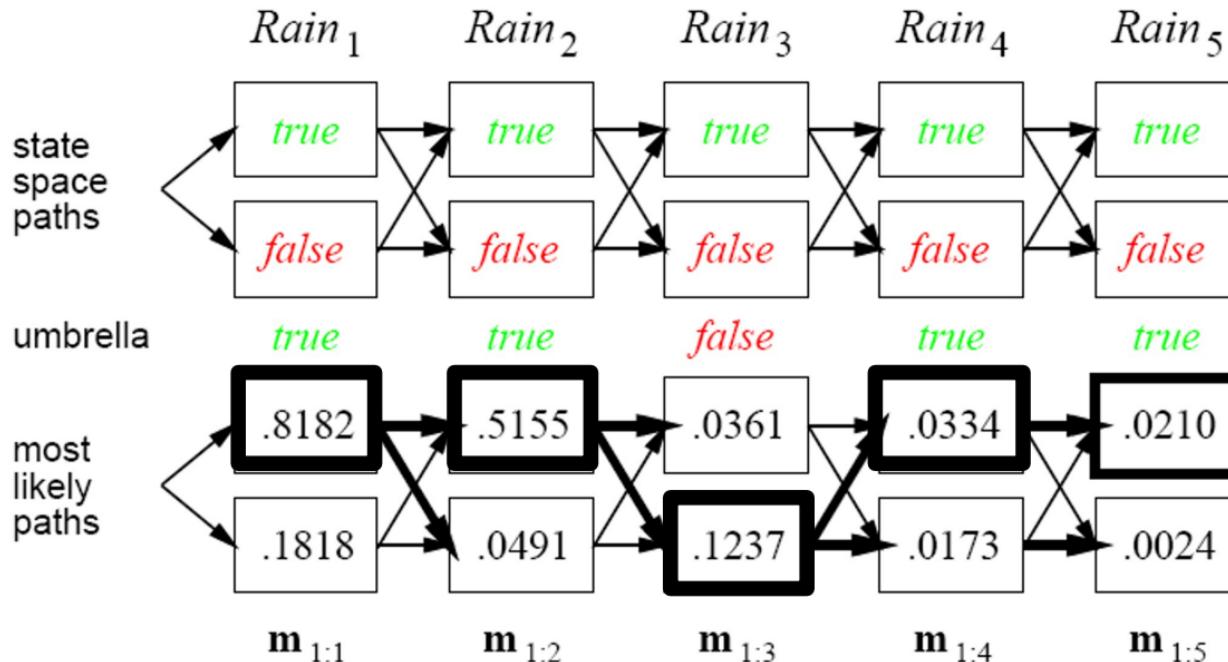
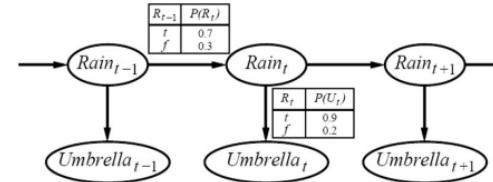


$$x_{1:T}^* = \arg \max_{x_{1:T}} P(x_{1:T} | e_{1:T}) = \arg \max_{x_{1:T}} P(x_{1:T}, e_{1:T})$$

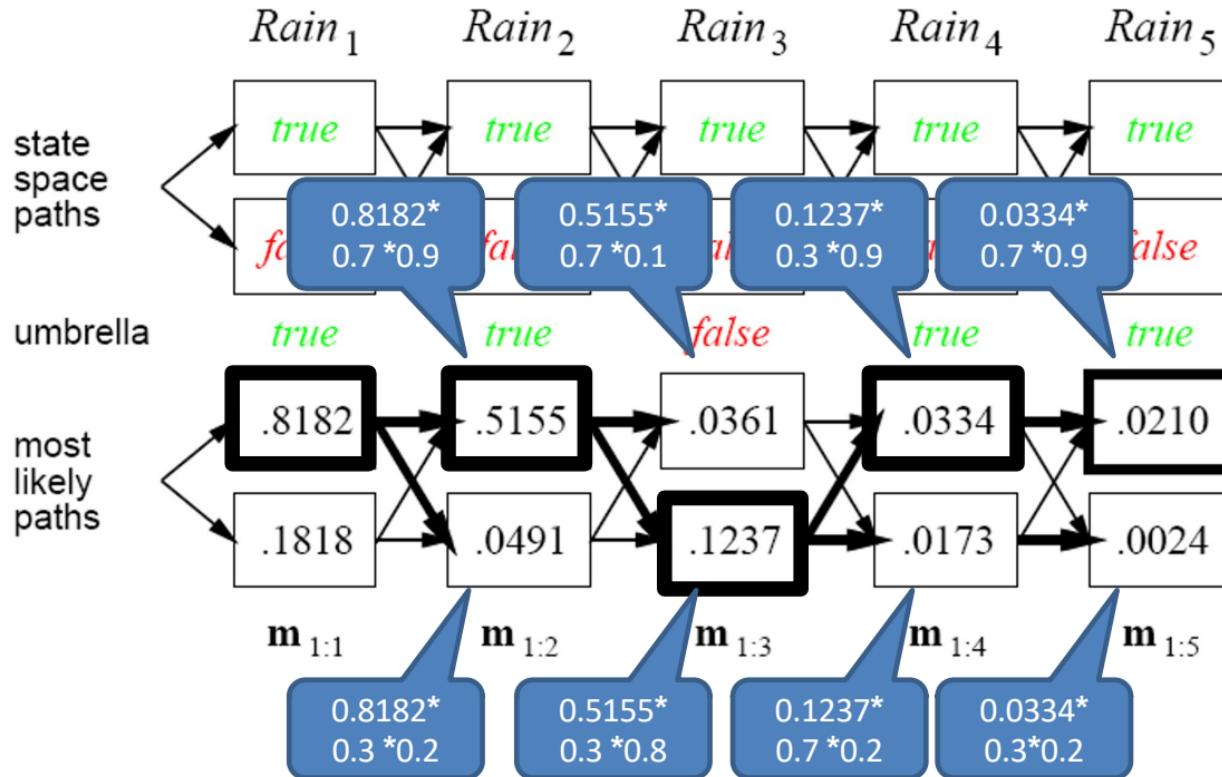

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$$\begin{aligned}
 \boxed{\alpha_t(s_t)} \quad m_t[x_t] &= \max_{x_{1:t-1}} P(x_{1:t-1}, x_t, e_{1:t}) \\
 &= \max_{x_{1:t-1}} P(x_{1:t-1}, e_{1:t-1}) P(x_t | x_{t-1}) P(e_t | x_t) \\
 &= P(e_t | x_t) \max_{x_{t-1}} P(x_t | x_{t-1}) \max_{x_{1:t-2}} P(x_{1:t-1}, e_{1:t-1}) \\
 &= P(e_t | x_t) \max_{x_{t-1}} P(x_t | x_{t-1}) \boxed{\alpha_{t-1}(s_{t-1})}_{1)}
 \end{aligned}$$

# Viterbi Example



# Viterbi Example

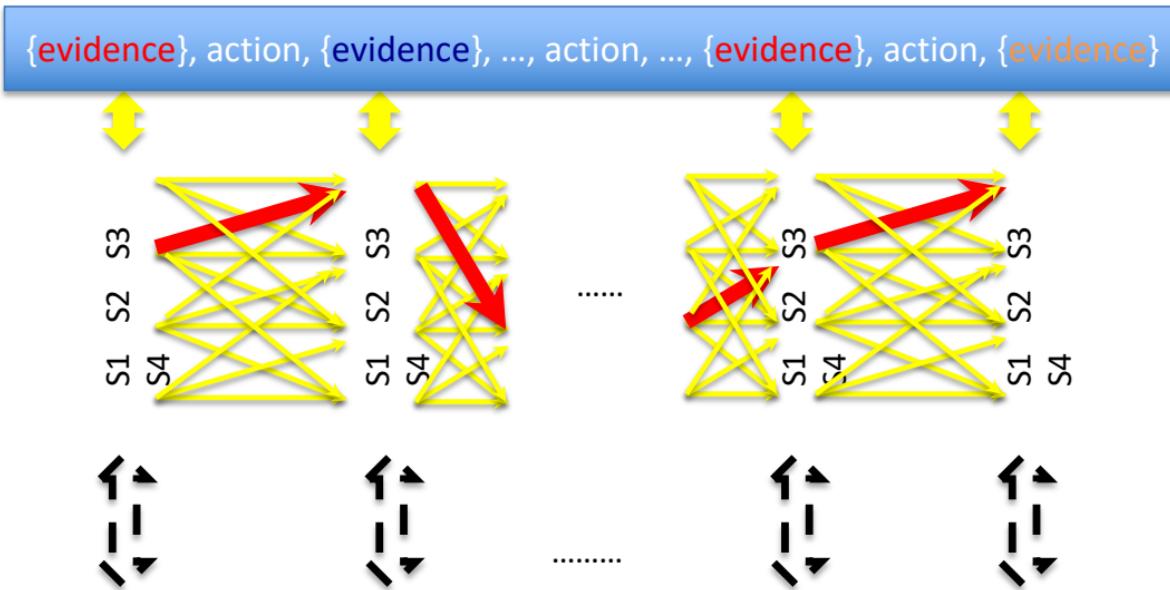


# Temporal Models

- Models with actions and sensors (ALFE 4-5) (aka POMDP)
- Markov Chains
  - No observation, no explicit actions, transit randomly
- Hidden Markov Model
  - No explicit actions, state transit randomly
- **Dynamic Bayesian Networks**
  - No explicit actions, States are Bayesian Networks
- Continuous State Model
  - No explicit actions, States are continuous
- POMDP
  - Discrete states with probabilistic actions, sensors, & transitions

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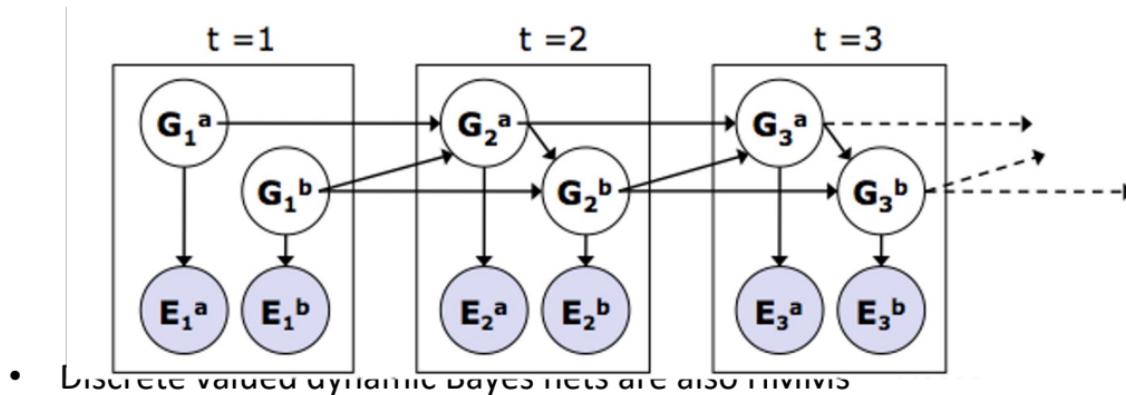
# Dynamic Bayesian Networks



Each time “slice” is a Bayesian Network with variables and CPTs

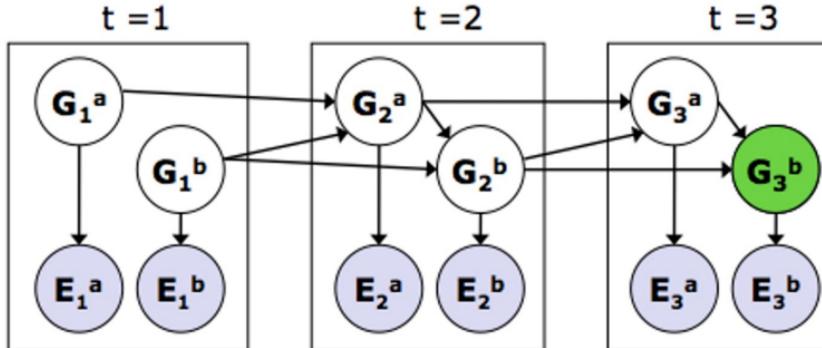
# Dynamic Bayes Nets (DBNs)

- We want to track multiple variables over time, using multiple sources of evidence
- Idea: Repeat a fixed Bayes net structure at each time
- Variables from time  $t$  can condition on those from  $t-1$



# Exact inference in DBNs

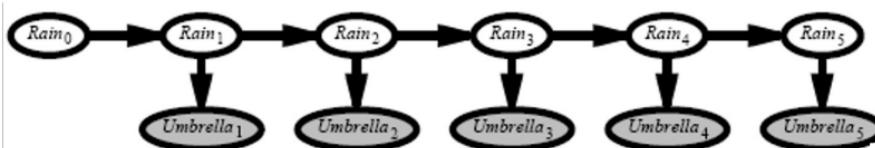
- Variable elimination applies to Dynamic Bayesian nets
- Procedure: “unroll” the network for  $T$  time steps, then eliminate variables until  $P(X_T | e_{1:T})$  is computed



- Online belief updates: Eliminate all variables from the previous time step; store factors for current time only

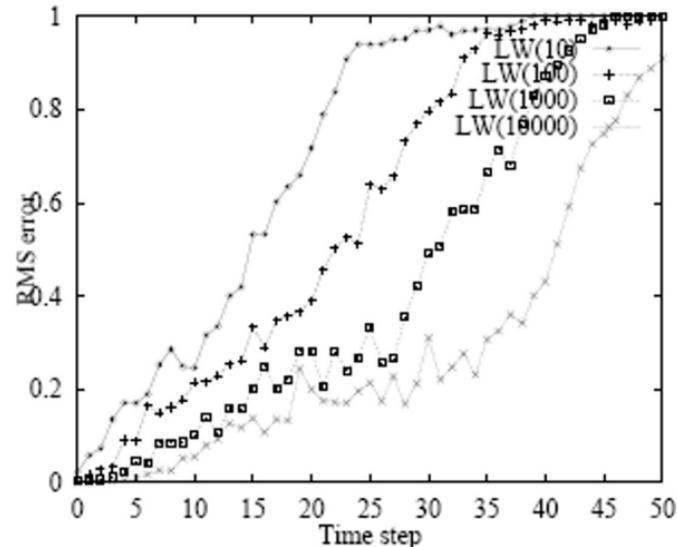
# Likelihood weighting for DBNs

Set of weighted samples approximates the belief state



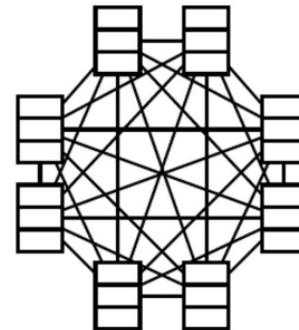
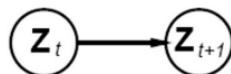
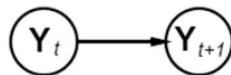
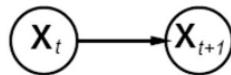
LW samples pay no attention to the evidence!

- ) fraction “agreeing” falls exponentially with t
- ) num. samples required grows exponentially with t



# DBNs vs. HMMs

- Every HMM is a single-variable DBN; every discrete DBN is an HMM

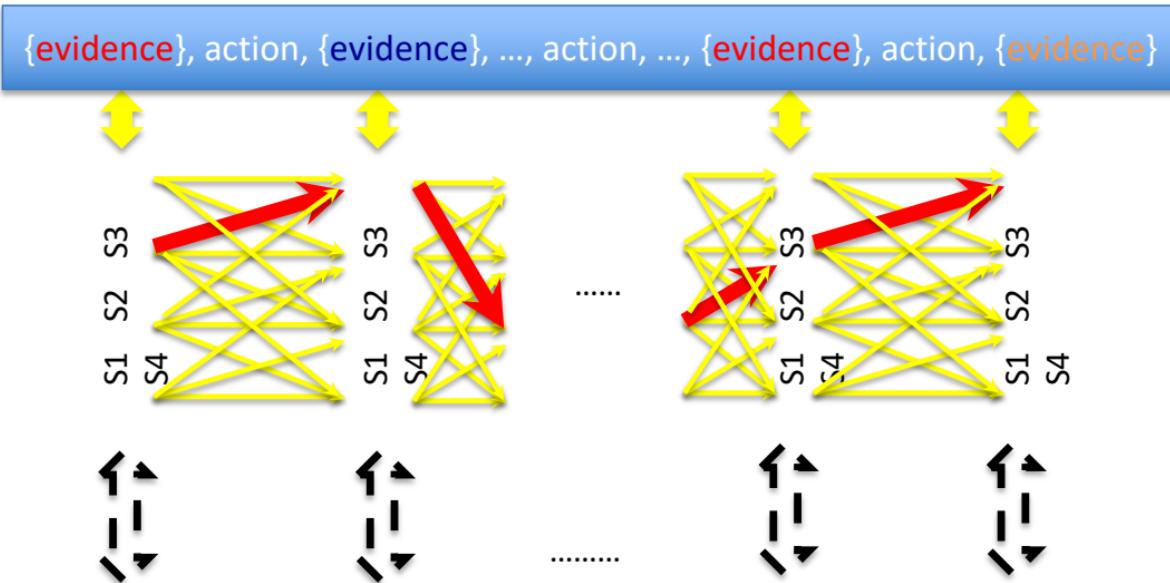


- DBN: Sparse dependencies, exponentially fewer parameters;  
e.g., 20 state variables, three parents each  
DBN has  $20 \times 2^3 = 160$  parameters, HMM has  $2^{20} \times 2^{20} = 10^{12}$

# Temporal Models

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  - No observation, no explicit actions, transit randomly
- Hidden Markov Model
  - No explicit actions, state transit randomly
- Dynamic Bayesian Networks
  - No explicit actions, States are Bayesian Networks
- **Continuous State Model**
  - No explicit actions, States are continuous
- POMDP
  - Discrete states with probabilistic actions, sensors, & transitions

# Kalman Filters



In each time slice, state variables are continuous (not discrete)

# Summary

- Temporal models use states, transitions, sensors
  - Transitions may be related to agent's actions, or spontaneous
- Markov assumptions and stationarity assumption, so we need
  - transition model  $P(X_t | X_{t-1})$  or  $P(X_t | X_{t-1}, a_{t-1})$
  - sensor model  $P(E_t | X_t)$
- Tasks: filtering, prediction, smoothing, most likely seq
  - all done recursively with constant cost per time step
- Types of models
  - HMM have a single discrete state variable
  - Dynamic Bayes nets subsume HMMs; exact update intractable
  - Other models may have internal structure driven by actions