

# (DRAFT) Implicit enumeration with dual bounds obtained from approximation algorithms

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## 1 Introduction

The field of discrete optimisation (CITE) studies problems that model discrete decision making. Mixed-Integer Programming refers to a set of modelling and solving techniques for such discrete problems (CITE) by formulating them as Mixed-Integer Programs (MIPs). For many problems, this approach is the most efficient known method to find a solution. The standard MIP formulation, for maximisation, is as follows:

$$\begin{array}{ll}\text{maximise} & cx + hy \\ \text{subject to} & Ax + Gy \leq b \\ & x \geq 0 \text{ integral} \\ & y \geq 0\end{array}$$

Where  $x$  and  $y$  are the sets of decision variables that we assign values to with the purpose of maximising the objective function  $cx + hy$ . Generally, objective functions are constrained by linear inequalities, and variables can be specified to be integer or continuous.

Optimisation problems with linear constraints and continuous variables are known as Linear Programs (LPs), which are both solvable in polynomial time and very efficiently in practice (CITE). In this report, we consider maximisation problems whose constraints are all linear. MIPs, on the other hand, are characterised by having one or more variables constrained to be integer, generally resulting in an NP-Hard problem (CITE). By removing the integer constraints on variables in a MIP, we obtain its associated *LP relaxation*.

Because the feasible region of continuous solutions naturally subsumes the set of feasible solutions for integers, we know, for optimal value for the LP relaxation  $z_{LP}$  and optimal value to the MIP  $z$ , that

$$z \leq z_{LP}$$

Since  $z_{LP}$  is at least as large as our optimal value,  $z$  is provided with an upper bound, which we refer to as the *dual bound*. The state-of-the-art method for solving MIPs uses this relationship to search for what are known as *primal solutions* that are feasible to the MIP, and employs dual bounds to implicitly enumerate through these. We find primal solutions by forming *subproblems*, constrained versions of the original problem, and solving the LP relaxation of these. If the solved LP relaxation is feasible to the original MIP, then we have found a primal solution, and so have an associated lower bound on our optimal value, which we refer to as the *primal bound*.

The process of choosing a variable to constrain is known as *branching*, and is generally informed by information obtained from the LP relaxation's solution. Branching can form a number of subproblems, which are all considered *branches* of the parent problem. If we find that a given subproblem's dual bound is lower than the MIP's primal bound, then we can safely *prune* this branch as we know further evaluation and constraining of this subproblem cannot yield a better primal solution. Otherwise, if this subproblem has a dual bound above our primal bound, we continue constraining and solving the relaxation until a new feasible solution is found, or a further constrained subproblem of this is pruned.

This constitutes the core of the Branch and Bound approach, which we formally describe below: let  $(x^i, y^i)$  be  $x$  and  $y$  values associated with the optimal solution to linear program  $LP_i$ ,  $(x^*, y^*)$  denote an optimal solution to the MIP;  $\underline{z}$  denote the lower bound on the optimal value and  $z^*$  the optimal solution of the MIP,

and let  $\mathcal{L}$  denote the list of nodes of the Branch and Bound tree yet to be solved (i.e. not pruned nor branched on).

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**Algorithm 1:** Branch and Bound algorithm for Mixed-Integer Programming

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**Data:** A Mixed-Integer Program.

**Result:** The optimal solution value, and the associated solution

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1 Set  $\mathcal{L} := N_0$ , set  $\underline{z} := -\infty$ , set  $(x^*, y^*) := \emptyset$ ;
2 while  $\mathcal{L}$  is not empty do
3   Select a node  $N_i$  from  $\mathcal{L}$ , by some node selection scheme, deleting it from  $\mathcal{L}$ ;
4   Solve  $LP_i$  to get solution value  $z_i$  and  $(x^i, y^i)$  if it exists;
5   if  $LP_i$  is infeasible, i.e.  $z_i = -\infty$  then
6     Go to step 2;
7   end
8   else
9     Let  $(x^i, y^i)$  be an optimal solution of  $LP_i$  and  $z_i$  be its objective value
10  end
11  if  $z_i \leq \underline{z}$  then
12    Go to step 2;
13  else
14    if  $(x^i, y^i)$  is feasible to the MIP then
15      Set  $\underline{z} := z_i$ ;
16      Set  $(x^*, y^*) := (x^i, y^i)$ ;
17      Go to step 2;
18    end
19  end
20  else
21    Branching: From  $LP_i$  construct 2 linear programs  $LP_{i1}, LP_{ik}$  with smaller feasible regions whose
        union does not contain  $(x^i, y^i)$ , but contains all the solutions of  $LP_i$  with  $x \in \mathbb{Z}$ , by choosing a
        variable to constrain;
22    Add the corresponding new nodes  $N_{i1}, N_{ik}$  to  $\mathcal{L}$  and go to step 2.
23  end
24  Return  $\underline{z}$ ;
25 end

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While finding optimal solutions with Branch and Bound is in worst case exponential-time, for some problems there exist Approximation Algorithms (AAs) which can provide feasible solutions in as fast as polynomial-time. To achieve this, AAs exchange a guarantee on optimality for a guarantee the *quality* of the solution, within some factor of the optimal value.

An  $\alpha$ -approximation is an AA that guarantees that the solution value will always be within a constant factor  $\alpha < 1$  of the optimal solution; i.e. for an approximate solution value  $z_A$ , it must be that  $\alpha z \leq z_A \leq z$ . As a result, for a maximisation problem, we can analytically derive an upper bound on the optimal value from the lower bound guarantee on the optimal solution by simple algebraic manipulation. In this way, we can use AAs in lieu of an LP relaxation to obtain valid dual bounds on our optimal solution. Because we are leveraging guarantees on the optimal value, the dual bounds we obtain are also guaranteed to be within a level of accuracy. This is distinct from LP relaxations, which can provide arbitrarily poor dual bounds.

When conducting a Branch and Bound with AAs, an important observation is that the general strategies which have contributed to the success of the standard Branch and Bound with LPs must be addressed. In particular, methods of branching and node selection do not have a clear translation with respect to AAs. In this project, we investigate methods of effectively using dual bounds provided by AAs for the 0,1 Knapsack. To do this, we investigate known approximation schemes and methods to construct high quality dual bounds from them, as well as devising branching strategies within its Branch and Bound.

## 2 Literature Review

We now give a brief literature review of the research and improvements to the Branch and Bound approach, as well as an overview of details and approaches with AAs that are central to the premise of this project.

### 2.1 Branch and Bound improvements and strategies

While our previous description of the Branch and Bound approach captured the core components, there still remain many aspects which are uncertain: which variables to constrain for branching, alternative methods to relax a solution, and node selection schemes. These problems are in fact non-trivial and represent active research areas within Mixed-Integer Programming. In light of this, we provide an overview of known strategies and the body of research that comprises the current understanding in the field.

#### 2.1.1 Branching Strategies

Deriving useful subproblems is central to the Branch and Bound approach, as it provides us with the partial enumeration required to establish bounds on an optimal solution value. A basic strategy is to branch on the variable with the largest fractional component, however [1] has shown this to be as bad as random selection. It is possible to make inferences based on the properties of each variable’s fractional component, as shown by [23] by the proposal of “use penalties”. This was used to inform the branching process where “up penalties” and “down penalties” were calculated for each variable based on their respective fractional components, after which the variable with the highest penalty in any direction was chosen. However, [17] showed with computational experiments that such approaches may have limited use, while approaches involving issuing “pseudocosts” as proposed by [2] may have advantages.

In an attempt to provide a categorisation for branching methods, [18] proposed two groupings: *binary/non-binary* strategies, and *wide* strategies. Both problem types and tree depth provided the ideas central to these classifications.

A detailed survey of the literature and the current open questions in research is also provided by [18], where the reader is directed for further reading.

#### 2.1.2 Node selection schemes

Talk about best bound?

#### 2.1.3 Bounding optimal solutions

As we have seen, a natural method of finding dual bound on optimal solutions can be done with LP relaxations. Solving LPs has been efficiently possible since the introduction of Dantzig’s Simplex method [6]. Despite its efficiency, though, LP relaxations can provide arbitrarily poor bounds in this context: such an approach for finding the Vertex Cover of a graph will be generally far from optimal (CITE?). However, this is only one of many known ways to obtain valid dual bounds within Mixed Integer Programming: other examples include Semidefinite Programming relaxations (see [15, 24]), and Lagrangian relaxations [8].

#### 2.1.4 Extensions, improvements, and related techniques

MIP solvers in the state-of-the-art often combine Branch and Bound strategies with various extensions to improve performance. One common extension to the method includes using cutting planes [9] to produce the Branch-and-Cut algorithm. The cutting plane approach, introduced by [5], and extended to general Integer Programs (IPs) in [9], involves solving a LP relaxation of a program and formulates a constraint, a

cutting plane, that separates this solution from the rest of the search space. A Branch and Cut approach [20, 21], then, is a Branch and Bound with the ability to choose to add a cutting plane instead of branching on variable. For a more detailed treatment on this topic, the reader is referred to [4]. Other key methods for extending the Branch and Bound are with presolving techniques [16] and using primal heuristics [3].

Another related concept is the idea of warm starting a solution in linear programming linked to integer programming by [22]. In short, warm starting is a method to exploit information gained about the problem from previous computations to inform future ones. One simple method for this is to use previous computation in finding a starting primal bound in a Branch and Bound.

## 2.2 Approximation algorithms for the Knapsack Problem

As mentioned, the case study chosen for experimentation within this project is the 0,1 Knapsack Problem (KP). Known to be NP-Hard (CITE), KP has been studied extensively both in to solving optimally and approximately (see, for example [13, 10, 12, 14, 7, 19]), and as such, it serves well as a topic for analysis. We describe the formulation for KP as follows: given  $n$  items associated weights  $w_i$  and values  $v_i$ , and Knapsack capacity  $W$ , our program is

$$\begin{aligned} & \text{minimize} && \sum_{i=1}^n v_i x_i \\ & \text{subject to} && \sum_{i=1}^n w_i x_i \leq W, \text{ and } x_i \in \{0, 1\} \end{aligned}$$

It is well known that KP lends itself well to a Dynamic Programming (DP) approach, although there are many ways to conduct this. One of the most well known examples, performing a DP by values as presented by [25], can find a solution in  $O(n^2 P)$  time for the most profitable item's value  $P$ , which we now describe: let  $A(i, p)$  be the minimal weight of the solution to KP with only the first  $i$  items available, where the total value is exactly  $p$ . If no such set exists<sup>1</sup>, then  $A(i, p) = \infty$ .

The DP recurrence can then be defined as follows:

$$A(i+1, p) = \begin{cases} \min\{A(i, p), w_{i+1} + A(i, p - v_{i+1})\}, & \text{if } v_{i+1} < p \\ A(i+1, p) = A(i, p) & \text{otherwise} \end{cases} \quad (1)$$

The optimal solution value can then be found by finding the value  $\max\{p \mid A(n, p) \leq W\}$ .

Being NP-Hard, KP does not admit a polynomial time algorithm, however we can construct a *polynomial time approximation scheme* (PTAS). A PTAS is guaranteed to give a solution value within a factor of  $(1 - \epsilon)$  of optimality, and have running time bounded by a polynomial in  $n$ , for fixed values of  $\epsilon$ . A classic example of a PTAS was presented by (cite Sahni), with time complexity  $O(n^{1/\epsilon})$ .

If we restrict the definition of a PTAS further, so that the running time is bounded by a polynomial in both  $n$  and  $\epsilon$ , we obtain a *fully polynomial time approximation scheme* (FPTAS). This means we can choose an  $\epsilon$  that gives us a high quality solution while still being polynomially bounded.

It is common for FPTAS's for KP to achieve polynomially bounded run time by adjusting the input problem to reduce its complexity, and using a DP to then solve this adjusted problem. As such, the construction of an FPTAS normally involves two distinct steps: a preprocessing step to form the simpler problem, and a step where this is solved to find an approximate value for the original problem. We now give an overview of the range of FPTAS's described that led to the state of the art.

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<sup>1</sup>that is, the first  $i$  items cannot yield a value of  $p$  and so cannot have an associated minimal weight

### 2.2.1 Ibarra and Kim's FPTAS (1975)

**TODO:** describe the reasoning behind letting  $K = \frac{\epsilon P}{n}$

What is now a standard method of constructing an FPTAS for KP was first described by Ibarra and Kim [11]. While there are several components at play in their proposal algorithm, the core of what creates the approximation involves a scaling the values of all items down to by a factor dependant on  $\epsilon$ . The distillation of their algorithm, as described by [25], is presented below:

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**Algorithm 2:** FPTAS for Knapsack

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**Data:** A KP problem instance, and error parameter  $\epsilon$ .

**Result:** The Weight and value pair of the optimal solution

- 1 Given  $\epsilon$ , let  $K = \frac{\epsilon P}{n}$ ;
  - 2 For each object  $i$ , define adjusted values  $v' = \lfloor \frac{v_i}{K} \rfloor$ ;
  - 3 Run a DP with these adjusted item values;
  - 4 Return the approximation solution set,  $S'$ , provided by the DP;
- 

By using the DP recurrence (1) described in the previous section, we can obtain an FPTAS for the Knapsack. We now present the argument for this as described by [25].

**Lemma 1.** *The set,  $S'$ , output by the algorithm satisfies:*

$$\text{value}(S') \geq (1 - \epsilon) \cdot \text{OPT}$$

*Proof.* Let  $z_{S'}$  be the value of the solution set  $S'$  obtained from the FPTAS, and  $z_O$  be the value of the optimal solution set  $O$ . Also, let  $z'_{S'}$  be the value of the solution set with adjusted/truncated values value the FPTAS's solution set  $S'$ , and  $z'_O$  be the optimal solution set with similarly adjusted value for all items.

Because of the flooring operation truncating off at most  $K$  we know that the difference between  $z_O$  and  $K \cdot z'_O$  would be at most  $nK$ . Furthermore, since  $z'_{S'}$  is the optimal value for the adjusted profits,  $z'_O$  cannot exceed it.

So,

$$z_O - nK \leq K \cdot z'_O \leq K \cdot z'_{S'} \leq z_{S'} \leq z_O$$

Since  $K = \frac{\epsilon P}{n}$ ,

$$z_O - nK = z_O - \epsilon P$$

Finally, since  $P \leq z_O$ ,

$$(1 - \epsilon) \cdot z_O \leq z_{S'} \leq z_O$$

□

**Theorem 1.** *Algorithm 2 is an FPTAS for KP*

*Proof.* Assume that it is the case. Therefore by Proof By Assumption we have proven it to be true. □

**Use Rhee's thesis to help me do this**

By refining our approach, we can improve our runtime further.

- Preprocessing: Talk about dynamic programming on expensive items

- Computing: Talk about greedy approach on cheap items
- Results in  $O(mP')$  time algorithm

The remainder of the description given by [11] extends this core approach by improving the nuance of the preprocessing step, which we now describe.

The first improvement results from acknowledging that while a given input problem will potentially have many items with very small value, the bulk of the value of an optimal solution will likely be from high value items (**is this really the actual reason?**). We establish a threshold  $t$  such that the DP will only compute on items with value greater than this. This means we can amend our scaling factor in light of the reduced the range of item values that our DP considers.

Within the set of smaller valued items, we conduct a greedy search with the remaining space, having completed the algorithm, which is guaranteed to be a  $1/2$ -approximation (**I think this is right.. but verify this**).

The chosen value for  $t$ , in their case, is  $\frac{1}{3}\epsilon\tilde{P}$  (**why? maybe it's not necessary to introduce this**).

**Lemma 2.** *This is a  $1/2$ -app lol*

*Proof.* Assume that it is the case. Therefore by Proof By Assumption we have proven it to be true. □

### 2.2.2 Lawler (1978)

What he changed about the DP.

- Solution pairs as presented by WS
- Amended rounding procedure
- Complexity

### 2.2.3 Magazine and Oguz (1981)

More changes to the last two

- Based on the DP recurrence (1)
- Builds on both Lawler and Ibarra & Kim
- Space and time complexities

### 2.2.4 Kellerer and Pferschy (1999)

Use both Rhee and their informal description to fuel the discussion of this part

### 2.2.5 Kellerer and Pferschy (2001)

Vector merging procedure

## 3 Design and Analysis

In light of the approaches described in the literature to both Branch and Bound and AAs, our objective is to now find a method of constructing high-quality dual bounds. Intuitively, the guarantees which help define given FPTAS's for KP are not proven with such applications in mind, so there are potential extensions to the analysis which we can use to find stronger dual bounds. In this section, we demonstrate analysis methods which have been undertaken to this end.

### 3.1 Analytic derivation of dual bounds a priori

To form a basis to our discussion, we reintroduce the bounds obtained by the FPTAS originally presented by Ibarra and Kim in [11]. In particular, recall that we found

$$(1 - \epsilon) \cdot z_O \leq z_{S'} \leq z_O$$

Which means that before running the approximation we know *a priori* the range for the optimal value:

$$z_O \in [z_{S'}, \frac{z'_{S'}}{1 - \epsilon}]$$

This provides us with the following first attempt at an amended Branch and Bound algorithm:

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**Algorithm 3:** Branch and Bound with Approximation Algorithms

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**Data:** A MIP instance, and an AA for the given problem.

**Result:** The Weight and value pair of the optimal solution

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1 Let  $\bar{z} :=$  approximate solution;
2 while  $\mathcal{L}$  is not empty do
3   Choose a node  $N_i$  from  $\mathcal{L}$ , and delete it from  $\mathcal{L}$ ;
4   Run the AA, deriving upper bound (UB) and lower bound (LB) in the process;
5   if  $UB \leq \bar{z}$  then
6     Go to step 3;
7   end
8   else
9     if  $LB > \bar{z}$  then
10       $\bar{z} := LB$ ;
11    end
12  end
13  if for  $N_i$ 's parent's  $UB$ ,  $UB_p$ ,  $UB > UB_p$  then
14    Set  $UB := UB_p$ 
15  end
16  else
17    Choose a variable to branch on, then generate and enqueue new child nodes;
18  end
19  Return  $\bar{z}$ ;
20 end

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### 3.2 Derivation of dual bounds a posteriori

We can obtain improvements to our dual bounds by observing information obtained following completion of the FPTAS, however, i.e. *a posteriori*. To do this, we first observe that the approximate solution set  $S'$  will have a profit of  $p(S')$ . This value will be the value of the adjusted profits  $p'(S')$  scaled up by  $K$ , plus any profit  $\omega$  lost from the flooring operation, i.e.

$$p(S') = Kp'(S') + \omega$$

**Todo the same analysis but with omega involves.**

This analysis leads to the insight that

$$OPT \leq \frac{(p(S')\omega)}{1 - \epsilon}$$

Which gives us

$$OPT \leq p(S') + Kn - \omega$$

### 3.3 Improved a posteriori dual bounds

Another approach is if we round up

### 3.4 Truncation branching strategy

## 4 Implementation and results

### 4.1 Methods for implemnetation

Benchmarks lol

### 4.2 What is a logical way to lay all this out?

## 5 Todos

(Put warm starting in future work)

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