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Seminar

Physics of karst forms: rillenkarren

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Abstract

The seminar is centered on a karst form called rillenkarren¹. These are small parallel solutional channels which form on inclined rock surfaces under influence of precipitation. The purpose of my work was to find an explanation of their formation. It is not known how they form and I have not explained them either.

If rillenkarren were better understood, they might reveal something about the history of rocks on which they are found.

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1 Introduction

The form which I discuss here are rillenkarren. A dictionary [4] explains

Rillenkarren: Karren form consisting of relatively smoothly profiled shallow channels, normally about 20-30 mm apart, and separated by sharp angular ridges. It is the characteristic karren formed under subaerial conditions due to dissolution by rainwater or snow melt on bare rock lacking any soil or plant cover.

Many similar definitions and descriptions can be found in the literature (see also figure 1). But the mechanism of rillenkarren formation is usually superficial, like in the cited paragraph, omitted, or said to be unknown [1]. There are only a few proposed detailed mechanisms. All of them in fact turn out to be incomplete, so I decided to find a consistent model of rillenkarren formation.

¹also called solution flutes [1], French lapiés [2], Slovenian dežni žlebiči [3]



Figure 1: An example of rillenkarrn [5].

All my understanding of rillenkarrn on which the seminar is based originates from three weeks of work at Karst Research Institute at Scientific Research Centre of the Slovenian Academy of Science and Arts.

2 What is known about rillenkarrn

Rillenkarrn have been known for a long time. They are first mentioned in works on local geography of some alpine karsts [2]. They have been found, described and measured in many different places on various rock species. Various types of “karren” were classified according to shape and environment in which they grow [6]. Chemical aspects of their formation were studied in detail [6]. Rillenkarrn growth has even been simulated under laboratory conditions [7]. But nobody came near deriving the shape of rillenkarrn from the basic assumptions on their formation.

SKICA 3 — ZLEBIČI-VELOPOLJE

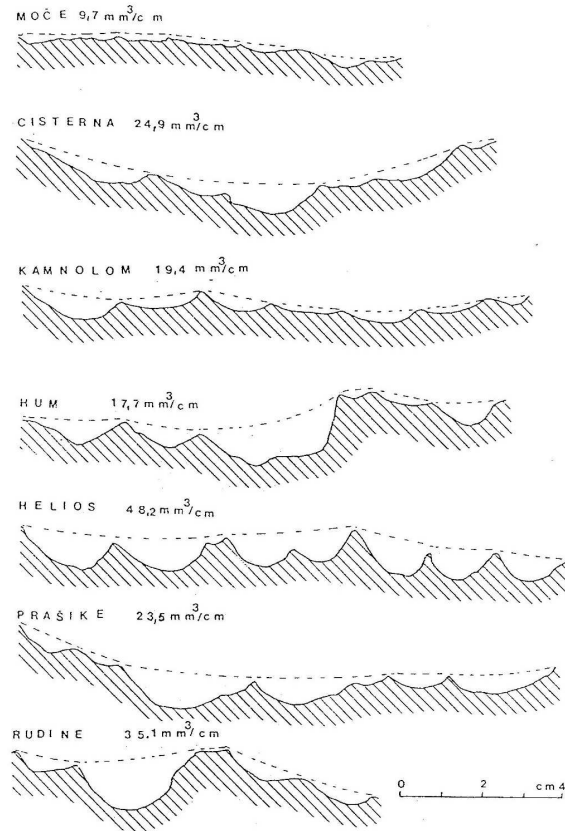


Figure 2: Some measured rillenkarren profiles [3].

2.1 Descriptions

According to [2] rillenkarren are a type of free karren. That means, they form on bare rock and not under soil cover. They are typical for sloped surfaces and have sharp crests. They are 1-2 cm deep and up to 50 cm long. They cover the surface from the upper edge downslope and terminate after a characteristic length into smooth surface called *ausgleichsfläche*.

Rillenkarren are 2-3 cm wide [6].

Rillenkarren were found in different climates, e.g. alpine, tropical, and Mediterranean [8]. They form on limestone, gypsum and salt [8].

Because they are so regular, they are easily measured. It is often done by geoscientists whose primary interest are not rillenkarren. I copied a table to give an impression of available data (see table 2.1). It is not only rillenkarren lengths, widths and depths that have been measured but sometimes their cross-sectional profiles have also been recorded (see figure 2).

Sometimes there are enough data to try to find correlation between ril-

Location	Lithology	Width cm	SD	n	Depth cm	SD	n	W/D	Length cm	SD	n	Reference
Svalbard island (Norway)	Marble	1.37	0.5	60	0.19	0.1	29	6.63	12.7	7.3	8	Unpublished data
Surprise Valley, (Canada)	Rockies Limestone	1.4	0.3	113	0.35	0.2	113	4.54	18.36	6.7	53	Unpublished data
Surprise Valley, (Canada)*	Rockies Limestone	1.132	0.2	38								Glew and Ford, 1980
Istria (Croatia)	Limestone	1.7							16			Ford and Lundberg, 1987
Istria (Croatia)	Limestone	1.54	0.4	72					16.3	4.7	36	Unpublished data
Lluc, (Spain)	Mallorca island Limestone	1.72	0.3	150	0.46	0.1	150	3.93	24.2	14	150	Mottershead, 1996a
Lluc, (Spain)	Mallorca island Limestone	1.59	0.3	30	0.44	0.1	30	3.82				Mottershead, 1996b
Lluc, (Spain)	Mallorca island Limestone	1.82	0.6	20	0.42	0.2	108	4.17	19	6.1	20	Crowther, 1998
Ses Parades, Mallorca (Spain)	island Limestone	1.76	0.6	100	0.29	0.1	100		19.5	6.9	100	Ginés, 1996
Vall den Marc, Mallorca (Spain)**	island Limestone	1.84	0.5	200	0.49	0.3	200	5.23	22.7	8.3	200	Ginés, 1996 and 1999
Tramuntana range, Mallorca (Spain) #	Limestone	1.64	0.7	2590	0.5	0.2	1200		25.3		480	Ginés, 1996
Mortero de Astrana, Cantabria (Spain)	Limestone	1.69	0.5	100	0.27	0.2	100		17.9	8.4	100	Ginés, 1996
La Safor, Valencia (Spain)	Limestone	1.8	0.5	49	0.4	0.3	49		12.5	6.1	49	Gil, 1989

Table 1: A small part of available rillenkarren morphometric data for carbonate rocks [8].

lenkarren shapes and environmental factors (slope angle [8], altitude [8], average temperature [8], precipitation per year [8] etc.).

It is said [7] that cross-sectional profile of rillenkarren is parabolic.

2.2 Experiments

It is possible to simulate rillenkarren growth experimentally under laboratory conditions and it has been done [7]. Plaster of Paris has been used instead of rocks because of numerous advantages (it is cheap, easy to handle and gives rapid results). At the same time, rillenkarren are found on gypsum in the nature and plaster of Paris is artificial form of this rock.

Plaster blocks, measuring $60 \times 36 \times 8$ cm, were put into a rainfall simulator. Numerous experiments were performed with different slopes and rainfall intensities, lasting a few hundred hours each. Rillenkarren formed and stabilised during that time. Their dimensions were measured. An ample amount of data was obtained and the analysis showed interesting correlation between slope and rill length from crest to ausgleichsfläche (see figure 3).

2.3 Theoretical models

A good model should explain the basic features of rillenkarren: the periodicity of the structure, the cross profile of a rill, the formation at a crest and extinguishing downstream [8]. Rillenkarren formation appears to be complex and it seems that several processes contribute to it.

A part of the story is chemistry. Rainwater shapes the rock mostly by dissolution [9]. In general, dissolution rate is slower where the rock solution

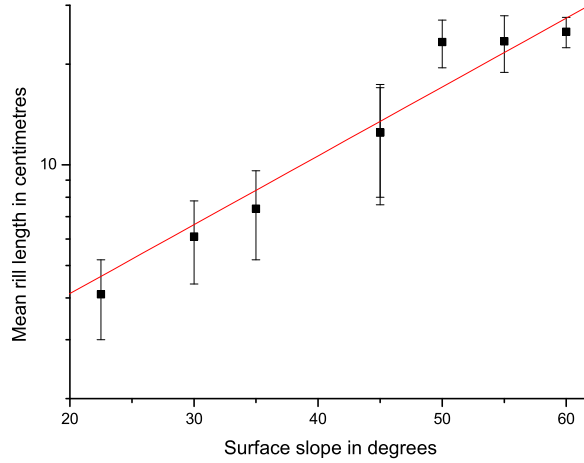


Figure 3: The relationship between surface slope and rillenkarren length obtained in the experiments with plaster [7]. The vertical axis is in logarithmic scale.

is more concentrated. The kinetics can be quite complicated [10]. However, rillenkarren form on different rocks with very different dissolution kinetics. So details of chemistry are probably not too important.

Some authors explain the onset of *ausgleichsfläche* with chemistry [6]. It is said that water gets more and more saturated when it flows downward and so stops making rills. But it has been pointed out that the concentration of solute does not really change along the slope [7].

Rillenkarren surfaces are commonly inhabited by lichens and the interior of the rock by endolithic cyanobacteria [8]. They certainly are important for corrosion and erosion. But they have no obvious effect on rillenkarren formation. They are also not present on salt rillenkarren and in laboratory experiments. So it seems that they are not crucial for rillenkarren development.

Therefore it seems possible that most of rillenkarren properties are determined by the physical processes that contribute to their formation. Physics could come into play through mechanical erosion, which may influence rillenkarren growth. But dissolution is probably more important, because the material leaves the rock mostly in dissolved form. On the other hand, the *chemical* process of dissolving the rock is governed by the concentration of the solute in water at rock-water boundary, which in turn depends on *physics* of fluid flow. It indicates that physics may reveal something about rillenkarren formation.

3 Rillenkarren from physicist's point of view

3.1 My opinion

I started explaining rillenkarren with a rough idea of how they are formed. Only then I could decide which processes are probably important and which are not and then work on the important ones.

The starting condition for rillenkarren formation is a fresh karren-free rock exposed to the elements. It takes many rainshowers for rillenkarren to develop. We know that the beginnings and ends of rainshowers are not too important because in experiments rillenkarren formed on blocks that were showered continuously.

The blocks in experiments were flat at first so the rock does not have to have a certain complicated shape at the beginning.

In the middle of a rainshower the whole stone is wet. Water is covering it with a thin flowing film. All the water that comes to the rock as rain is transported away in film flow. If the rainfall is of constant intensity the picture stays the same from minute to minute. I believe these are the conditions under which rillenkarren form (experiments are with me again).

The only thing causing differences in the dissolution rate from point to point is the concentration of dissolved mineral that constitutes the rock at the rock-water boundary. The raindrops are chemically all the same, at the moment of impact they do not contain any mineral. So the concentration of dissolved mineral in a volume element of water in the film depends solely on its history. Mass of the mineral is conserved. If we presume that neighbouring elements had similar histories and so have similar composition, diffusion is unimportant and can be neglected. Then every molecule that is dissolved in the element travels with that same element along the surface to the border of the rock. Along the way the rain is falling and the rock is being dissolved so we have continuous inflow into the element – of mineral from the rock and of fresh water from the upside. The concentration of mineral is determined by the proportion between both processes through the history of the element.

From hydrodynamical point of view the concentration of dissolved mineral does not change properties of water much. So we can neglect any effect it may have and first calculate the water flow as if all the water was the same. From the water flow we can find concentrations and dissolution rates which tell us the local speed of rock retreat. From here on we can find out if rillenkarren are forming or not. So the hydrodynamics is probably the most important part of understanding rillenkarren formation.

3.2 What does physics reveal

3.2.1 Hydrodynamics

Fluid flow is governed by Navier-Stokes equation, which is quite nasty to solve. It is especially true if we have unspecified boundary, as is the water – air boundary in our case. So I did not try to solve the equation in all its complexity when random drops are falling on a strange-formed surface.

Navier-Stokes equation for incompressible fluids is [11]

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = \mathbf{f}_{pr} - \nabla p + \eta \nabla^2 \mathbf{v}, \quad (1)$$

where ρ stands for density, \mathbf{v} for fluid velocity, t for time, \mathbf{f}_{pr} for density of external forces, p for pressure, and η for viscosity. It can of course be much simplified for our use. Some of assumptions I will make are used commonly and are together called lubrication approximation [12] and the formulas have already been derived. But I simplified things still some more.

The first approximation is ignoring the terms $\rho (\mathbf{v} \cdot \nabla) \mathbf{v}$ and $\rho \frac{\partial \mathbf{v}}{\partial t}$, thus making the equation linear in \mathbf{v} . It is justified if the inertial effects are small, which is certainly true in our case.

Because water is flowing in a thin film we can say that the situation is much more rapidly changing in perpendicular direction than in the plane of the rock surface. I introduce local Cartesian coordinate system with origin at the rock surface and z axis perpendicular to it (see figure 4). So the ∇^2 can be approximated with $\partial^2/\partial z^2$. Because water is nearly incompressible (a new approximation), we can also say it is flowing in parallel with the rock surface. These are the assumptions behind lubrication approximation. The equation 1 becomes

$$\nabla p = \eta \frac{\partial^2 \mathbf{v}}{\partial z^2} + \mathbf{f}_{pr}, \quad (2)$$

where $\mathbf{v} = (u, v, 0)$. That gives an equation for z direction:

$$\frac{\partial p}{\partial z} = (\mathbf{f}_{pr})_z. \quad (3)$$

Gravity is the only \mathbf{f}_{pr} in our system. So $(\mathbf{f}_{pr})_z = -\rho g \cos \alpha$, where α is the slope angle of the rock surface. If we label the water film thickness (perpendicular to the surface) with $h(x, y)$, we get $p(x, y, z) = p(x, y, h) + (h - z)\rho g \cos \alpha$.

I also ignored surface tension in order not to make things unnecessarily complicated. That gave me a simple boundary condition: the pressure at the water surface is constant. So we put the constant to zero (only derivatives of pressure influence water flow so constant terms do not matter), and we get $p(x, y, z) = (h - z)\rho g \cos \alpha$.

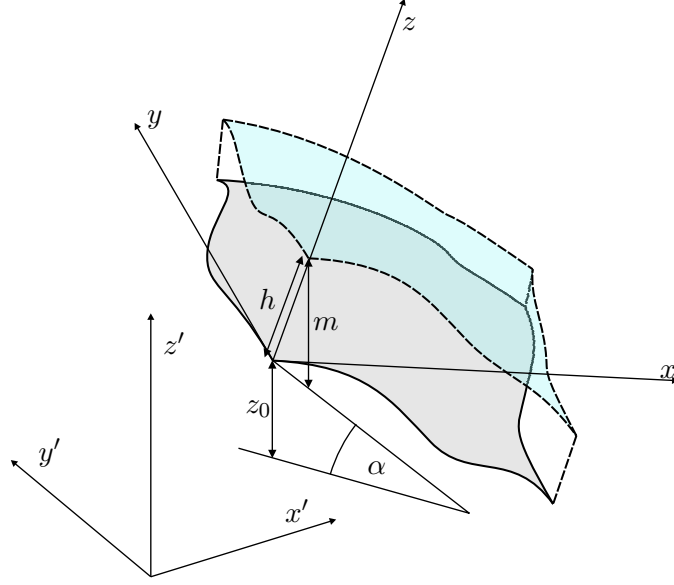


Figure 4: The coordinate systems.

Because the water flow is said to be parallel to the rock surface, the next assumption is that h only changes slowly with x and y . So I can afford the substitution

$$\nabla p = \frac{\partial p}{\partial z} \hat{e}_z \quad (4)$$

in eq. 2. I am only calculating water flow in x-y plane, so this term is now unimportant. Equation 3 becomes simply

$$\begin{aligned} \eta \frac{\partial^2 u}{\partial z^2} &= -(\mathbf{f}_{pr})_x = \rho g \frac{\partial z_0}{\partial x} \\ \eta \frac{\partial^2 v}{\partial z^2} &= -(\mathbf{f}_{pr})_y = \rho g \frac{\partial z_0}{\partial y} \end{aligned} \quad (5)$$

if the local rock elevation above some reference level is denoted by $z_0(x, y)$. By definition it is $(\frac{\partial z_0}{\partial x})^2 + (\frac{\partial z_0}{\partial y})^2 = \sin^2 \alpha$.

Everything is the same in x and y directions so it is enough to look only at the x equation: $\frac{\partial^2 u}{\partial z^2} = \frac{\rho g}{\eta} \frac{\partial z_0}{\partial x}$. By integration over z we get

$$\frac{\partial u}{\partial z} = \int \frac{\partial^2 u}{\partial z^2} dz = \int \frac{\rho g}{\eta} \frac{\partial z_0}{\partial x} dz = \frac{\rho g}{\eta} \frac{\partial z_0}{\partial x} z + A, \quad (6)$$

where I denoted integration constant by A . Integrating again, we get expression for velocity:

$$u = \int \frac{\partial u}{\partial z} dz = \int \left(\frac{\rho g}{\eta} \frac{\partial z_0}{\partial x} z + A \right) dz = \frac{\rho g}{\eta} \frac{\partial z_0}{\partial x} \frac{z^2}{2} + Az + B. \quad (7)$$

We get constants from boundary conditions. There is no slip at the rock surface, $u|_{z=0} = 0$, and no shear stress at the water surface, $\frac{\partial u}{\partial z}|_{z=h} = 0$. From here it follows $A = \frac{\rho g}{\eta} \frac{\partial z_0}{\partial x} h$ and $B = 0$.

Integrating once more, we get flow density (that is, the volume flow divided by the width over which it flows) \mathbf{j} :

$$j_x = \int_0^h u(z) dz = \int_0^h \frac{\rho g}{\eta} \frac{\partial z_0}{\partial x} \left(\frac{z^2}{2} + hz \right) dz = -\frac{\rho g}{\eta} \frac{\partial z_0}{\partial x} \frac{h^3}{3}. \quad (8)$$

We see that at a given point flow density is proportional to water film thickness to the third power. The proportionality constant depends only on water properties and rock slope. The divergence of flow density $\nabla \cdot \mathbf{j}$ equals the difference between the rain inflow and rate of change of h . In equation:

$$\nabla \cdot \mathbf{j} = v'_r - \frac{\partial h}{\partial t}, \quad (9)$$

where t stands for time and v'_r for rain intensity in z direction.

I transform the equations into a global horizontal Cartesian coordinate system. I denote horizontal coordinate axes with x' and y' and the vertical with z' (see figure 4). Instead of water film thickness h I use water depth m , which is at any point directly proportional to h : $m = h / \cos \alpha$. That way equation 9 becomes

$$\frac{1}{\cos \alpha} \nabla \cdot \mathbf{j} = v_r - \frac{\partial m}{\partial t}. \quad (10)$$

and from 8 we get

$$\frac{1}{\cos \alpha} \nabla \cdot \mathbf{j} = Am^3 + Bm^2 \left(\nabla m \cdot \frac{\nabla z_0}{|\nabla z_0|} \right). \quad (11)$$

That gives

$$\frac{1}{\cos \alpha} \nabla \cdot \mathbf{j} = Am^3 + Bm^2 \left(\nabla m \cdot \frac{\nabla z_0}{|\nabla z_0|} \right) = v_r - \frac{\partial m}{\partial t}, \quad (12)$$

where v_r stands for rain intensity in vertical direction. Capital letters A and B denote functions of only x' , y' , and $z_0(x', y')$. They do not depend on time or m .

Equation 12 is a nonlinear partial differential equation. It could be discretised into a system of algebraic equations and somehow numerically propagated through time. In principle, the rain intensity should be the sum of all the raindrops. But it is not known where and when a raindrop is going to fall so I substituted the rain intensity with its average over space and time, with a constant. Then I tried to find a stationary solution of the

equation.² I was left with the equation

$$Am^3 + Bm^2 \left(\nabla m \cdot \frac{\nabla z_0}{|\nabla z_0|} \right) - \overline{v_r} = -\frac{\partial m}{\partial t} = 0. \quad (13)$$

For linear PDEs there are standard methods of solving but for that nonlinear one I invented one by myself. I will not describe it here because it is too ineffective. I hoped the iteration would converge towards a stationary solution but in fact it did not.

The cause turned out to be quite obvious. In reality water flow at any point is controlled by the slope of water surface, and not of the rock as in equations 5. If it were not so, the water would not form pools with horizontal surface. It would flow downslopes into depressions and would not stop flowing and at the centre of depression would grow a hill of water which would not stop growing, so the equations do not have a stationary solution in general and my model is inadequate.

So I had to correct the formulas. In equation 8 I used $\frac{\partial(z_0+m)}{\partial x}$ instead of $\frac{\partial(z_0)}{\partial x}$. It is the same as using correct x and y terms and not omitting them in equation 4. Now I got equation of the form

$$Am^3 + Bm^2 \left(\nabla m \cdot \frac{\nabla z_0}{|\nabla z_0|} \right) + Cm^3 \nabla^2 m + Dm^2 (\nabla m)^2 - \overline{v_r} = 0. \quad (14)$$

Capital letters stand for functions that do not depend on m . Because the equation is more complicated than equation 13, every step now took more time. But the solution usually converged towards a good one (see figure 5).

There is a free parameter in the iterative process which states how big are the steps of the iteration. If it is set too high, they are too big and the calculated solution diverges away from the correct one. And if they are too small, it converges too slowly. The process was in fact too slow so I used multigrid technique [14]. At first I use a very coarse grid, composed of a small part of all the gridpoints. On it the solution converges fast. When I come near the correct solution, I introduce intermediate gridpoints and interpolate water surface at them. I use interpolated water surface as the starting approximation in calculating a good enough solution for the new grid. It goes relatively fast because of a good starting approximation. I then repeat the procedure until the grid is complete, full-resolution.

The whole computational cycle was still time-consuming and exhausting to use. But it worked, anyway, and gave me the description of film flow in terms of depths and flow densities. Because the approach promised to be useful I also invented some numerical rock shapes that could be compared to experiments (see figure 6).

²In reality water flowing over an incline can form propagating waves and flow in a pulsating manner. I do not know if this is important for rillenkarren or if it happens on rillenkarren at all. It is interesting that the waves are in nature damped by surface tension [13] which I ignored.

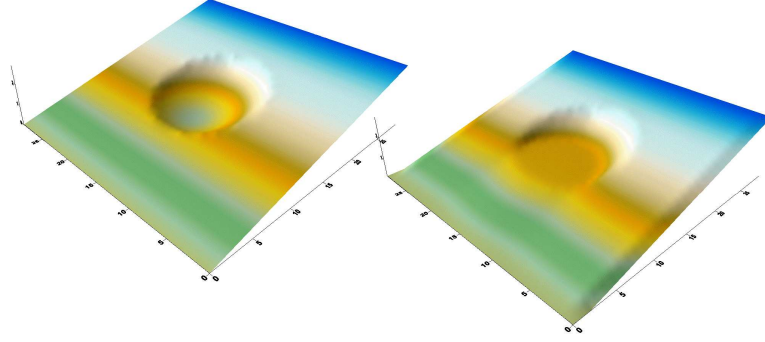


Figure 5: The left graph shows the rock surface and the right one calculated water surface. Note that at the border of the pool the angle between both surfaces is not small so the assumptions of lubrication approximation are not fulfilled. But both the real and the calculated water surfaces are nearly horizontal so I'm not worried about it.

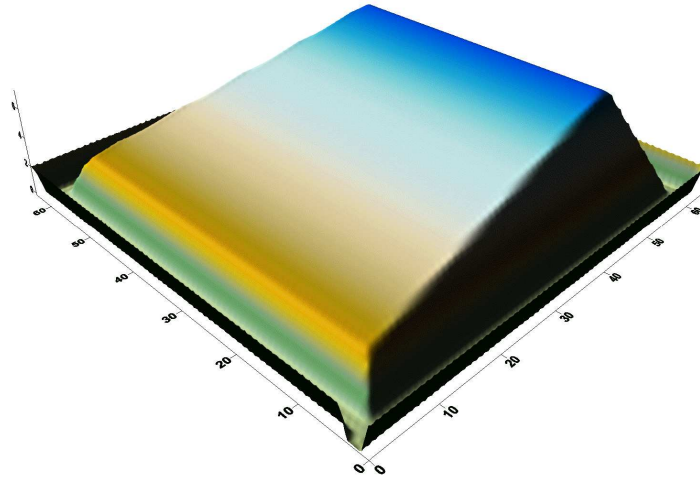


Figure 6: Elementary rock shape used with the numerical model. The channel at the border takes care of water outflow and reduces the probability of surprises.

3.2.2 Chemistry

When the film flow problem was solved, the last thing missing in the simulation of rillenkarrren development was chemistry of dissolution. I started with linear law of dissolution:

$$\frac{\partial n}{S \partial t} = \beta(c_{max} - c) \quad (15)$$

where β stands for a constant. S is surface, c_{max} is equilibrium concentration, n is chemical amount of rock. Actual concentration c is defined as $c = \frac{n}{V}$. So because of rock dissolution:

$$\frac{\partial z_0}{\partial t} = \frac{\gamma}{\cos \alpha}(c_{max} - c). \quad (16)$$

The $1/\cos \alpha$ comes from the slope of the surface because rock is dissolved perpendicular to surface, not vertically. Letter γ stands for a constant.

Concentration of solute in water was calculated using the law of conservation of mass for the rock. The amount of rock which is dissolved in a time Δt has to be flushed away during the same time (it is all in steady state). It is also true locally, if we only consider water above a small patch of rock and dissolution at that patch. The “flux” of rock through the patch into the solution equals the difference of fluxes of dissolved rock out of the place above the patch and into that place.

I assumed that water is well mixed in z direction because the film is so thin. It somewhat contradicts the assumptions of lubrication approximation that the component of water velocity perpendicular to surface is zero, but it might work anyway. For simplicity I have also kept the same grid as for the film flow calculation. Law of conservation of mass for the rock was implemented through concentrations: for every grid point I assumed some stationary concentration that was calculated as the ratio between total influx of water and that of the rock. Water flows into the patch from neighbouring patches and from the rain, and rock from neighbouring patches (already dissolved) and from the solid rock below the patch. When all this was implemented it enabled me to calculate concentration field and dissolution rate. For calculating I used iteration again.

The method introduced some anisotropy into the model. Where the slope was parallel with coordinate axes and rows of gridpoints, all the rock from the upper gridpoint flowed into one directly below it and into the next and so on. But if the surface sloped diagonally, the dissolved rock was artificially dispersed from one gridpoint into two below it and so on. But the effect is small for fine enough grids.

Then everything was put together. First the initial rock shape was fed into the model. Next water flow was calculated. It made possible for the chemistry to be calculated. Next, rock retreat was applied to the rock shape,

which gave the shape into which the rock would evolve after some time. Then the procedure was repeated for the new shape. Because of all the complexity, instabilities and slowness of the model it was only possible to make a few runs. But it was enough to prove that my model is not yet good enough – the surfaces were smoothed out instead of made wavy. In fact, that came as no real surprise.

4 Conclusion

From the fact that rillenkarren do exist it can be shown that there is some physical process that is not included in my model. The failure of the model is not simply a consequence of numerical errors.

Consider a flat sloped surface. If it is exposed to uniform rain, the concentration of dissolved rock in water film is going to be the same all over the rock. To wit, at every square centimetre of the rock there is the same inflow of rain and the same rate of dissolution if concentration is everywhere the same.

If the slope is increased, the rain inflow to a given rock surface decreases. That is why the concentration of solute in the film will also increase. If we look at rillenkarren: the sides of a rill are steeper than its bottom is, so the solution at the sides is more concentrated. The water from the sides flows to the bottom and so diminishes aggressiveness of water there. In contrast, the ridges only get fresh rainwater and should therefore retreat faster than trenches and the rillenkarren should be smoothed out.

I suspect that prediction of my model is wrong because it does not include phenomena that occur in z direction, perpendicular to the surface. It may be that accounting for layering and diffusion in that direction would change everything. Another possibility is that raindrops are important because they mix the film and so film thickness has some effect. It has already been proposed in raindrop impact and boundary layer model [8].

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