H2 Further Mathematics
Summary Notes

NousernameSG

May 21, 2023

Contents

* Refers to Topics that are not in the syllabus

0 Assumed Knowledge

0.1 Algebra

0.1.1 Completing the Square

$$\boxed{ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+c-a\left(\frac{b}{2a}\right)^2}$$

Min/Max Point $\rightarrow \left(-\frac{b}{2a}, c-a\left(\frac{b}{2a}\right)^2\right)$

0.1.2 Nature of Roots

Discriminant $< 0 \implies$ No **Real** Roots

Discriminant $= 0 \implies$ 2 **Equal** Roots

Discriminant $> 0 \implies$ 2 **Distinct** Roots

0.1.3 Vieta's Formulas (Degree of 2)

Suppose that $\alpha + \beta$ are the two roots of $ax^2+bx+c=0$, $a \neq 0$

Sum of roots $= \alpha+\beta = -\frac{b}{a}$

Product of roots $= \alpha\beta = \frac{c}{a}$

0.1.4 General Vieta's formulas

0.1.5 Polymonials

Expansions:

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

$a^2 - b^2 = (a+b)(a-b)$

$a^3 \pm b^3 = (a \pm b)\left(a^2 \mp ab + b^2\right)$

Binomial Formula:

$$\boxed{(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}(b)^r}$$

0.1.6 Partial Fractions

$$\frac{f(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b}+\frac{B}{cx+d}\,,\ A,B\in\mathbb{R}$$

$$\frac{f(x)}{(ax+b)(cx+d)^2} = \frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2}\,,\ A,B,C\in\mathbb{R}$$

$$\frac{f(x)}{(ax+b)(cx^2+d)} = \frac{A}{ax+b}+\frac{Bx+C}{cx^2+d}\,,\ A,B,C\in\mathbb{R}$$

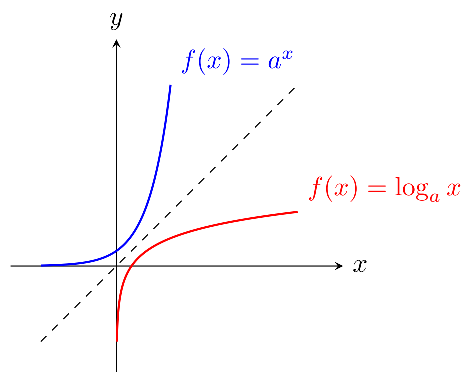
0.1.7 Factor & Remainder Theorem

Given a polynomial $f(x)$,

If $f\left(\frac{b}{a}\right)=0$, $a,b\in\mathbb{R}, a\neq 0$, $(ax-b)$ is **factor** of $f(x)$

If $f\left(\frac{b}{a}\right)=c$, $a,b\in\mathbb{R}, a\neq 0$, the **remainder** of $f(x)$ divided by $(ax-b)$ is c

0.1.8 Logarithmic & Exponential

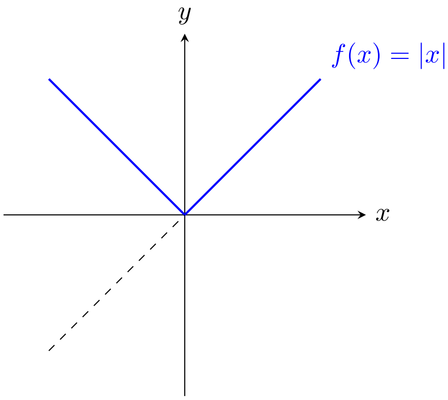


In General : $y=a^x \iff x=log_a(y)$, $a>0$, $a\neq 1$

Exponential	Logarithmic
$\forall x\in\mathbb{R},\ f(x)>0$	$D_f=\mathbb{R}^+$
y-intercept : $y=1$	x-intercept : $x=1$
$y=0$ is an asymptote	$x=0$ is an asymptote
$f(x)$ is increasing for $a>1$	$f(x)$ is increasing for $a>1$
$f(x)$ is decreasing for $0<a<1$	$f(x)$ is decreasing for $0<a<1$

Rules of Indices	Laws of Logarithm
If $a,b,m\in\mathbb{R}^+$	If $a,m,n\in\mathbb{R}^+$, $a>0$, $a\neq 1$
$a^m * a^n = a^{m+n}$	$\log_a mn = \log_a m + \log_a n$
$a^m \div a^n = a^{m-n}$	$\log_a \frac{m}{n} = \log_a m - \log_a n$
$(a^m)^n = a^{mn}$	$\log_a m^n = n\log_a m$
$a^m * b^m = (a+b)^m$	$\log_n m = \frac{\log_a m}{\log_a n}$
$a^m \div b^m = \left(\frac{a}{b}\right)^m$	

0.1.9 Modulus

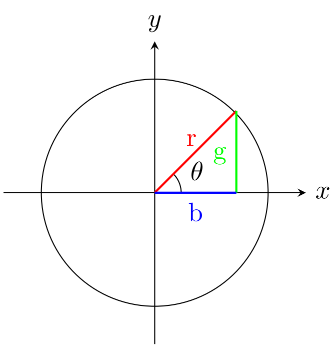


Generally, the modulus function is defined as

$$|f(x)| = \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(x) & \text{if } x < 0 \end{cases}$$

0.2 Trigonometry

0.2.1 Trigo Ratios for a General Angle

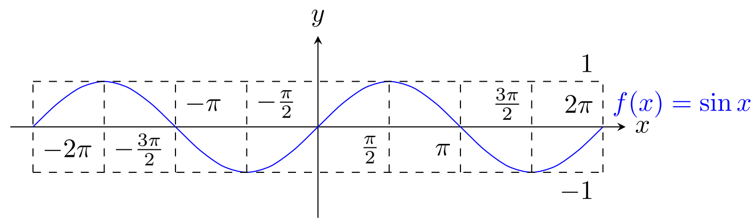


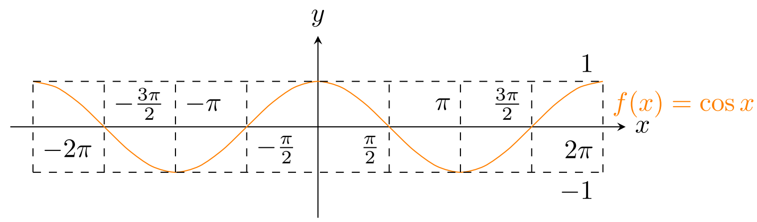
$$\sin\theta = \frac{g}{r} \qquad \csc\theta = \frac{r}{g}$$
$$\cos\theta = \frac{b}{r} \qquad \sec\theta = \frac{r}{b}$$
$$\tan\theta = \frac{g}{b} \qquad \cot\theta = \frac{b}{g}$$

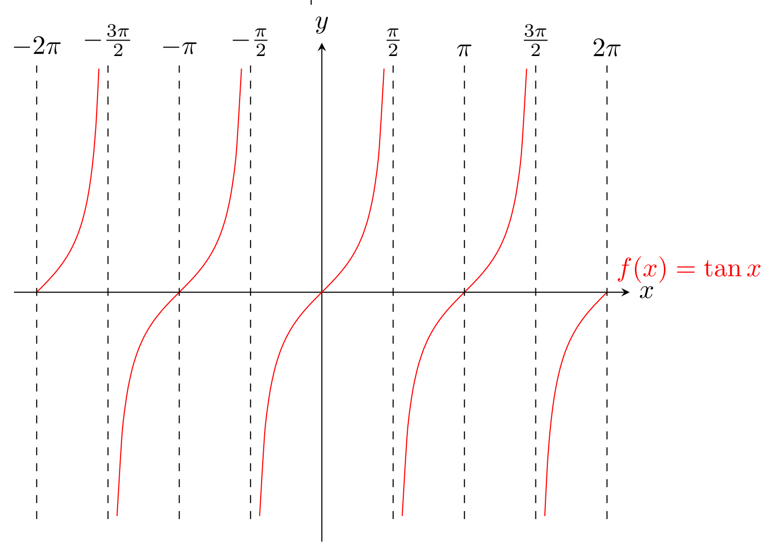
0.2.2 Principal Values

$$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2} \text{ where } x \in [-1, 1]$$
$$0 \leq \sin^{-1} x \leq \pi \text{ where } x \in [-1, 1]$$
$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \text{ where } x \in \mathbb{R}$$

0.2.3 Graphs of Trigo Functions







0.2.4 Sine & Cosine Rules

Given any triangle with sides of length a, b, c and opposite angles A, B, C

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2(b)(c)\cos A$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2(b)(c)}$$

0.2.5 R-Formula

A Sum or difference of Sines and cosines can be represented with a single trig function if they have the same angles

$$\boxed{a\sin\theta \pm b\cos\theta = \sin(\theta \pm \alpha)}$$

$$\boxed{a\cos\theta \mp b\cos\theta = \cos(\theta \pm \alpha)}$$

$R = \sqrt{a^2 + b^2}$ & $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

0.2.6 Basic Identities

$$\text{Area of Triangle} = \frac{1}{2}(a)(b)\sin\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

0.2.7 Sum of Angles

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\cos(A \pm B) = \cos A\cos(B) \mp \sin(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan(A)\tan(B)}$$

$$\sin 2A = 2\sin(A)\cos(A)$$

$$\cos 2A = \cos^2(A) \mp \sin^2(A)$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2(A)}$$

0.2.8 Fator & Reverse Factor Theorem

$$\sin A + \sin B = 2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$$

$$\sin A - \sin B = 2\cos\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$$

$$\cos A + \cos B = 2\cos\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$$

$$\cos A - \cos B = 2\sin\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$$

$$\boxed{\begin{aligned} \sin P\cos Q &= \frac{1}{2}(\sin(P+Q) + \sin(P-Q)) \\ \cos P\sin Q &= \frac{1}{2}(\sin(P+Q) - \sin(P-Q)) \\ \cos P\cos Q &= \frac{1}{2}(\cos(P+Q) + \cos(P-Q)) \\ \sin P\sin Q &= -\frac{1}{2}(\cos(P+Q) - \cos(P-Q)) \end{aligned}}$$

*Not in MF26

1 Graphing Techniques

1.1 Graph Features

1.1.1 Basic characteristics

Axial Intercepts

$x - intercept : y = 0$

$y - intercept : x = 0$

Stationary Points

Stationary points are points on a curve where

$$\left.\frac{dy}{dx}\right|_{x=k} = 0,\ k \in \mathbb{R}$$

The Nature of the stationary point can be determined by using the second or first derivative tests

Asymptotes

A Line or curve that a function approaches arbitrarily close to

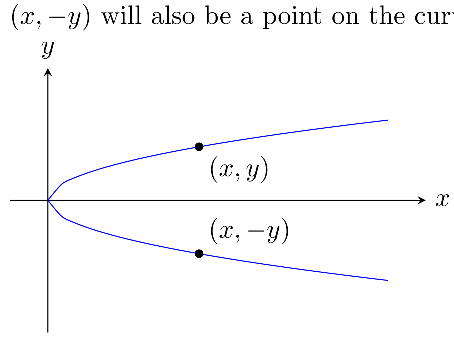
$$\begin{aligned} \text{Horizontal} &: \text{ when } x \to \infty,\ y \to a, \text{where } a \in \mathbb{R} \ \therefore y = a \\ \text{Vertical} &: \text{ when } y \to \infty,\ x \to b, \text{where } b \in \mathbb{R} \ \therefore x = b \\ \text{Oblique} &: \text{ when } x \to \infty,\ y \to cx + d, \text{where } c, d \in \mathbb{R} \\ &\qquad \therefore y = cx + d \end{aligned}$$

A Horizontal Asymptote can be cut through, but vertical asymptotes will never be passed through

1.1.2 Symmetry

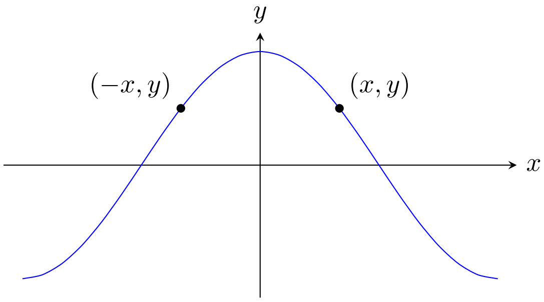
Symmetric about the x-axis

If (x, y) is a point on the curve,
$(x, -y)$ will also be a point on the curve



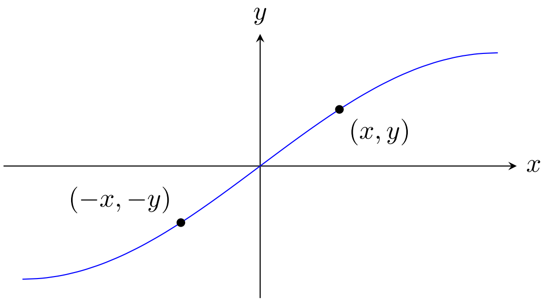
Symmetric about the y-axis (Even Functions)

Mathematically, $f(x) = f(-x)$



Symmetric about the origin (Odd Functions)

Mathematically, $f(x) = -f(x)$



1.2 Types of Graphs

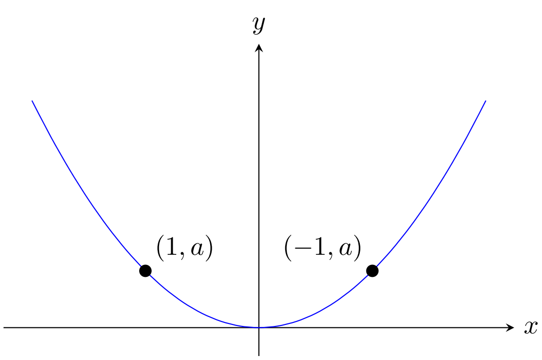
1.2.1 Power Functions

$$\boxed{\text{General Power Function}: f(x) = ax^n,\ a \in \mathbb{R}^+}$$

When n is an even positive integer

Function Type : Even Function

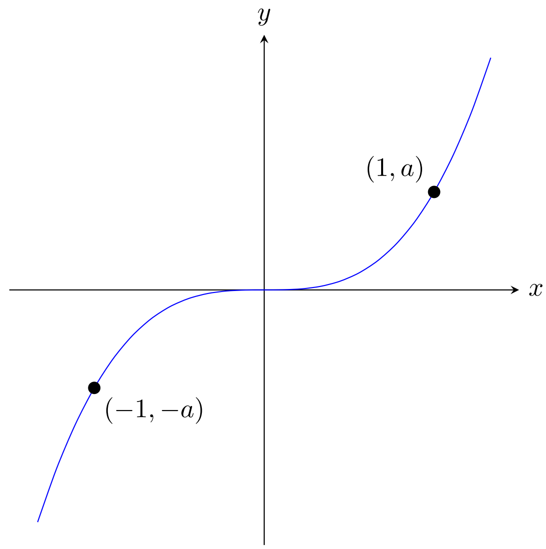
Common Points : $(-1, a)$, $(0, 0)$, $(1, a)$



When n is an odd positive integer, n $\geq$ 3

Function Type : Odd Function

Common Points : $(-1, -a)$, $(0, 0)$, $(1, a)$



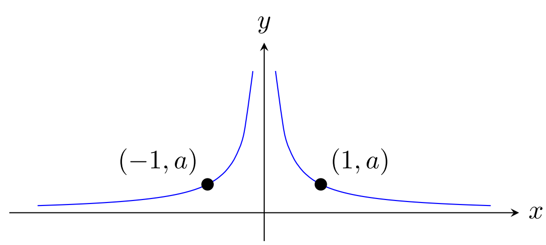
When n is an even negative integer

Function Type : Even Function

Common Points : $(-1, a)$, $(0, 0)$, $(1, a)$

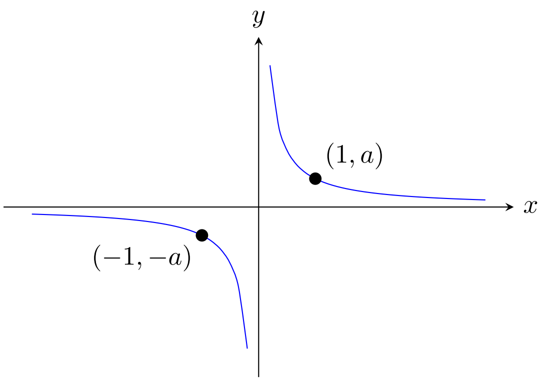
Horizontal Asymptote : $y = 0$

Vertical Vertical : $x = 0$



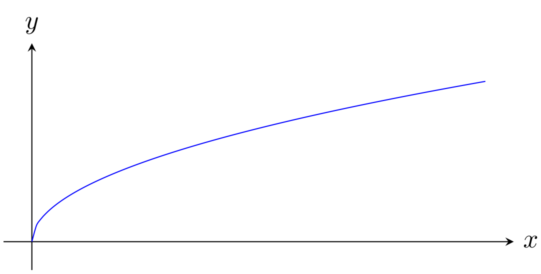
When n is an odd negative integer

$$\begin{aligned}\text{Function Type}: &\ \text{Odd Function}\\ \text{Common Points}: &\ (-1,-a),\ (0,0),\ (1,a)\\ \text{Horizontal Asymptote}: &\ y=0\\ \text{Vertical Vertical}: &\ x=0\end{aligned}$$



When $n \in \mathbb{Q}$ in the form $\frac{1}{k}$
Such that k is an even positive integer

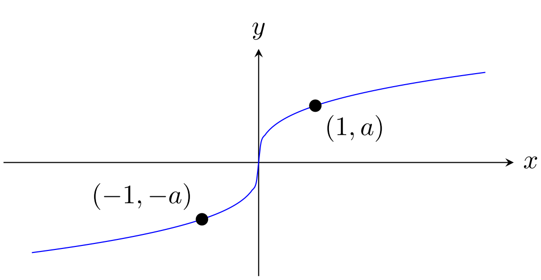
Common Points : $(0, 0)$, $(1, a)$



When $n \in \mathbb{Q}$ in the form $\frac{1}{k}$
Such that k is an odd positive integer

Function Type : Odd Function

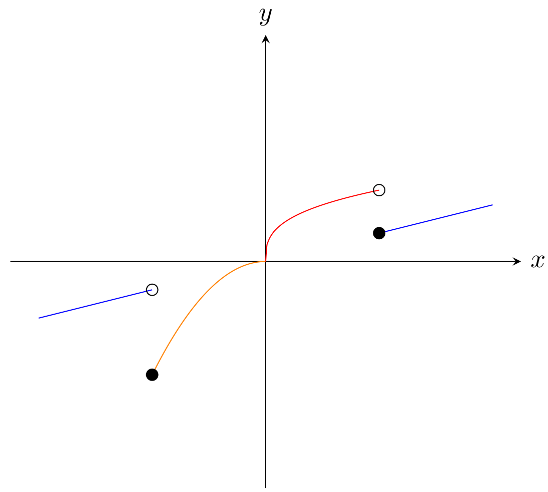
Common Points : $(-1, -a)$, $(0, 0)$, $(1, a)$



1.2.2 Piecewise Functions

Different intervals each has a different definition

$$f(x) = \begin{cases} H(x), & \text{if } a \le x < b \\ G(x), & \text{if } b \le x < c \\ Q(x), & \text{Otherwise} \end{cases}$$

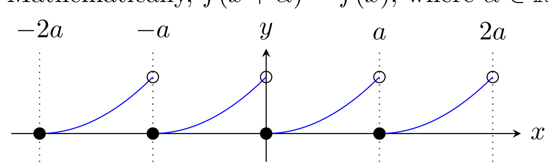


When the function is **discontinuous** between intervals,
A **Solid** dot is used if the point is on the
interval, an **empty** is used if it isn't

When the function is **Continuous** between intervals,
dots are not needed

1.2.3 Periodic Functions

A function that repeats itself after an interval,
Mathematically, $f(x + \alpha) = f(x)$, where $\alpha \in \mathbb{R}$



In this example, the function $f(x)$ had an interval of
$0 \le x < a$

1.2.4 Rational Functions

A rational function is in the form $y = \dfrac{p(x)}{q(x)}$,
where $p(x), q(x)$ are polynomials

Types of Rational Functions

Degree of $p(x)$ $<$ Degree of $q(x)$

$y = 0$ is the Horizontal Asymptote

Solutions to $q(x) = 0$ will yield the Vertical Asymptote(s)

Degree of $p(x)$ $\geq$ Degree of $q(x)$

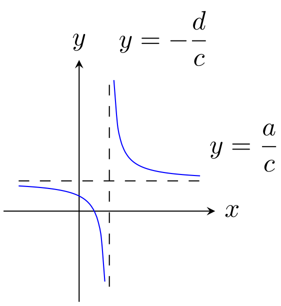
$y = \dfrac{p(x)}{q(x)} \equiv y = h(x) + \dfrac{r(x)}{q(x)}$, where $deg(r(x)) < deg(q(x))$

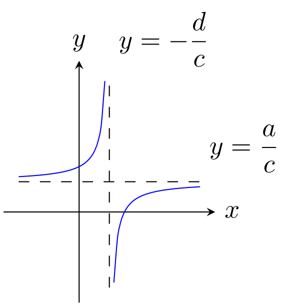
$y = h(x)$ is the Oblique Asymptote

Solutions to $q(x) = 0$ will yield the Vertical Asymptote(s)

Rectangular Hyperbola

Rectangular Hyperbola : $y = \dfrac{ax+b}{cx+d} \implies y = p + \dfrac{r}{cx+d}$,
where $c \neq 0, x \neq -\frac{d}{c}$. The possible curve shapes are





The Function is first to be rationalized by long division, then the Asymptotes and graph features can be found with the information above

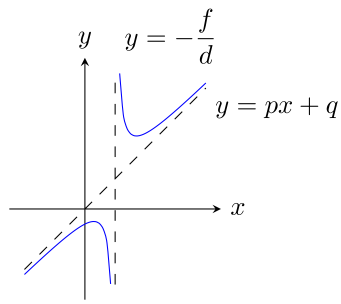
Quadratic Over Linear Functions

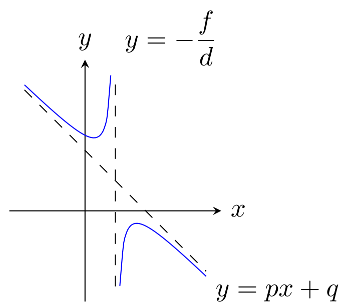
$y = \dfrac{ax^2+bx+c}{dx+f} \implies y = px + q + \dfrac{r}{dx+f}$

One Oblique Asymptote : $y = px + q$

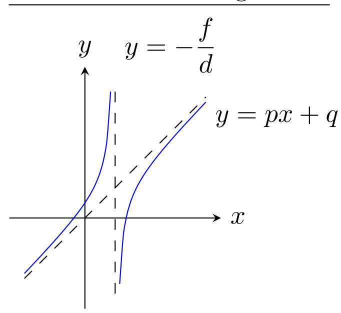
One Vertical Asymptote : $x = -\dfrac{f}{d}$

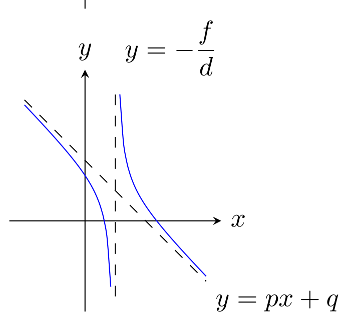
Case 1 : Two Turning Points





Case 2 : No Turning Points





Linear Over Quadratic Functions

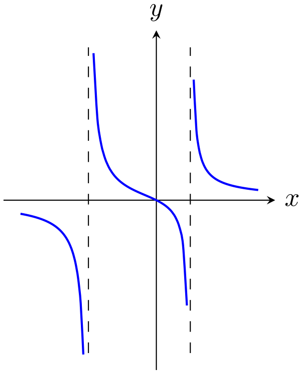
$y = \dfrac{dx+f}{ax^2+bx+c} \implies y = 0 + dx + fax^2 + bx + c$

One Horizontal Asymptote : $y = 0$

Vertical Asymptote(s) : Solve for denominator $= 0$

| Solutions $=$ Asymptote(s)

Example : When there are 2 Vertical Asymptotes

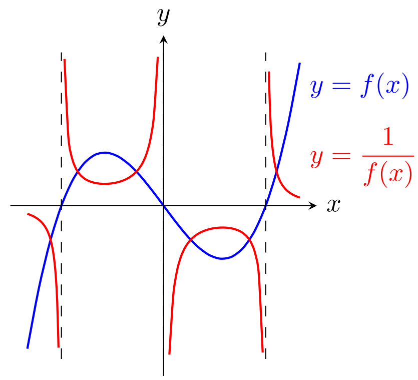


1.2.5 Reciprocal Functions

$y = f(x)$	$y = \dfrac{1}{f(x)}$
x-intercept : $x = a$	Vertical Asymptote : $x = a$
*Vertical Asymptote : $x = a$	*x-intercept : $x = a$
(a, b)	** $\left(a, \dfrac{1}{b}\right)$
Local Maxima : (a, b)	Local Minima : ** $\left(a, \dfrac{1}{b}\right)$
Local Minima : (a, b)	Local Maxima : ** $\left(a, \dfrac{1}{b}\right)$
Horizontal Asymptote : $y = b$	Horizontal Asymptote : $y = \dfrac{1}{b}$
Oblique Asymptote : $y = ax + b$	Horizontal Asymptote : $y = 0$
$f(x)$ is increasing	$\dfrac{1}{f(x)}$ is decreasing
$f(x)$ is decreasing	$\dfrac{1}{f(x)}$ is increasing

* True for **most** cases

** $b \neq 0$



1.3 Linear Transformations

1.3.1 Translation

$y = f(x - a)$

- Translation of the graph $f(x)$ by a units in the direction of the x-axis
- Add a to all of the x-coordinates

$y = f(x - a)$

- Translation of the graph $f(x)$ by $-a$ units in the direction of the x-axis
- Subtract a to all of the x-coordinates

$y = f(x) + a$ or $y - a = f(x)$

- Translation of the graph $f(x)$ by a units in the direction of the y-axis
- Add a to all of the y-coordinates

$y = f(x) - a$ or $y + a = f(x)$

- Translation of the graph $f(x)$ by $-a$ units in the direction of the y-axis
- Subtract a to all of the y-coordinates

1.3.2 Scaling

$y = f\left(\frac{x}{a}\right)$

- Scaling the graph $f(x)$ by a scale factor of a parallel to the x-axis
- Multiply a to all of the x-coordinates

$y = af(x)$ or $\frac{y}{a} = f(x)$

- Scaling the graph $f(x)$ by a scale factor of a parallel to the y-axis
- Multiply a to all of the y-coordinates

1.3.3 Reflection

$y = f(-x)$

- Reflecting the graph $f(x)$ about y-axis
- Multiply -1 to all of the x-coordinates

$y = -f(x)$

- Reflecting the graph $f(x)$ about x-axis
- Multiply -1 to all of the y-coordinates

1.3.4 Modulus Transformations

$y = |f(x)|$

- Reflect the graph $f(x)$ about y-axis
for the regions **below the x-axis**
- Multiply -1 to the y-coordinates of the **reflected points**

$y = f(|x|)$

- Remove the graph that is in the **negative x-axis**
Reflect the remaining graph $f(x)$ about y-axis
- Multiply -1 to the x-coordinates of the **reflected points**

$|y| = f(x)$

- Remove the graph that is in the **negative y-axis**
Reflect the remaining graph $f(x)$ about x-axis
- Multiply -1 to the y-coordinates of the **reflected points**

1.3.5 Sequence of Tranformations

Suggested Sequence to minimise errors:

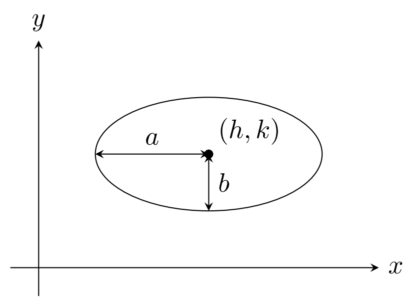
1. T_x **Horizontal** Translation
2. S_x **Horizontal** Scaling
3. S_y **Vertical** Scaling
4. T_y **Vertical** Translation

$\boxed{\text{Sequence : } T_x S_x S_y T_y}$

1.4 Conic Sections

1.4.1 Ellipses

$$\boxed{\text{Standard Form : } \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

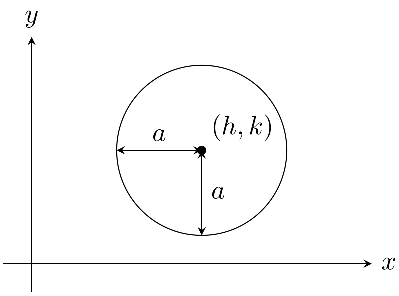


Properties

- Center : (h, k)
- Semi-Major-Axis : a
- Semi-Minor-Axis : b
- Symmetrical about $x = h$ & $y = k$

1.4.2 Circles

Standard Form : $(x-h)^2 + (y-k)^2 = 1$

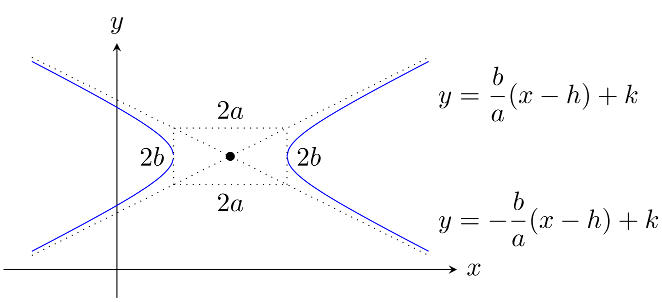


Properties

- Center : (h,k)
- Radius : a
- Symmetrical about $x = h$ & $y = k$

1.4.3 Hyperbolas

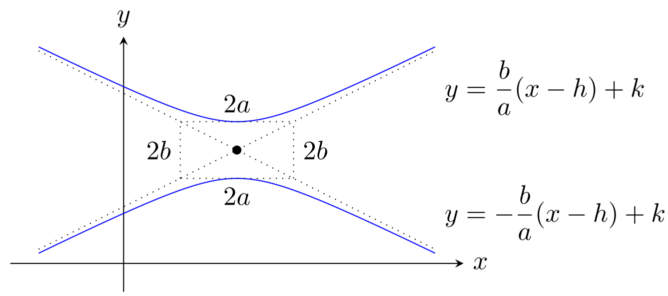
$$\text{Standard Form : } \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



Properties

- Center : (h,k)
- Symmetrical about $x = h$ & $y = k$
- Vertices are 'b' units from the center (h,k) Vertically
- Oblique Asymptotes are $y = \pm\frac{b}{a}(x-h) + k$

$$\text{Standard Form : } \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$



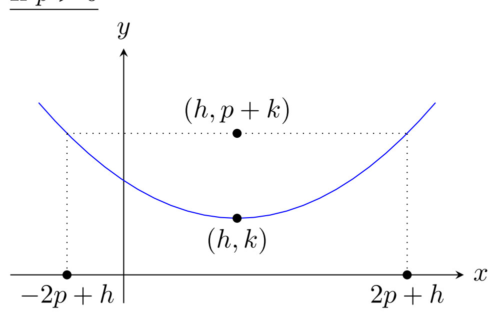
Properties

- Vertexes : $(h, k \pm b)$
- Symmetrical about $x = h$ & $y = k$
- Vertices are 'a' units from the center (h,k) Horizontally
- Oblique Asymptotes are $y = \pm\frac{b}{a}(x-h) + k$

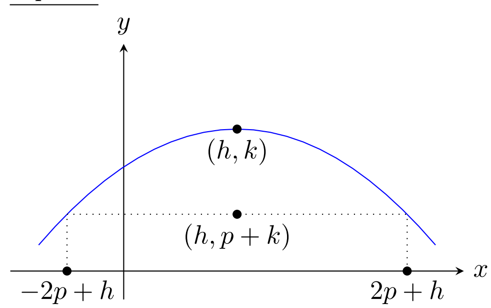
1.4.4 Parabolas

Standard Form : $(x-h)^2 = 4p(y-k)$

If $p > 0$



If $p < 0$

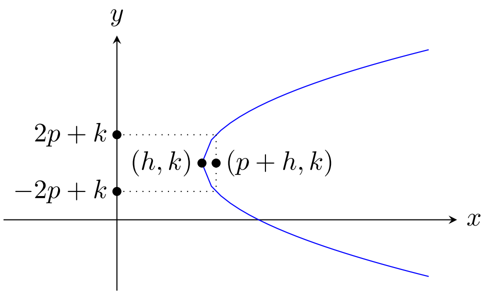


Properties

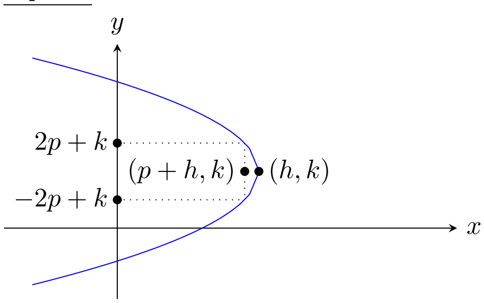
- Vertex : (h, k)
- Symmetrical about $x = h$
- If $p > 0$, $y \geq k \ \forall x (x \in \mathbb{R})$
  If $p < 0$, $y \leq k \ \forall x (x \in \mathbb{R})$

$$\boxed{\text{Standard Form : } (y-h)^2 = 4p(x-k)}$$

If $p > 0$



If $p < 0$



Properties

- Vertex : (h, k)
- Symmetrical about $y = k$
- If $p > 0$, $x \geq h \ \forall y (y \in \mathbb{R})$
  If $p < 0$, $x \leq h \ \forall y (y \in \mathbb{R})$

1.5 Parametric Equations

Expressing the variables of a function using another variable.

Cartesian Equation : $f(x, y)$

Parametric Equation : $x = f(t)$ & $y = f(t)$

1.5.1 Sketching Parametric Curves (Using GC)

Set an appropriate range for t under **Parametric** Mode and key in the $x(t)$ & $y(t)$ functions to graph it out
but draw a table of t and the respective x, y

Table example:

t	-1	0	1
x	3	1	3
y	-4	1	2

1.5.2 Finding Asymptotes of Parametric Curves

Example : $x = \dfrac{1}{t}$, $y = t + 1$ for $t > 0$

As $y \to \infty$, $t \to \infty \implies x \to 0$
$\therefore x = 0$ is a Horizontal Asymptote
As $x \to \infty$, $t \to 0 \implies y \to 1$
$\therefore y = 1$ is a Vertical Asymptote

t acts as an intermediary

1.5.3 Conversion of Parametric to Cartesian Equations

Manipulate the terms containing the additional variable (such as t) until it can be cancelled out to give an equation
with only the original variables. Trig Identities can be used too.

Example : $x = a\cos t$, $y = a\sin t$

$$x^2 = a^2\left(1 - \sin^2 t\right)$$

$$x^2 = a^2 - a^2\left(\frac{y^2}{a^2}\right)$$

$\therefore y^2 + x^2 = a^2$ [Cartesian Equation]

2 Equations & Inequalities

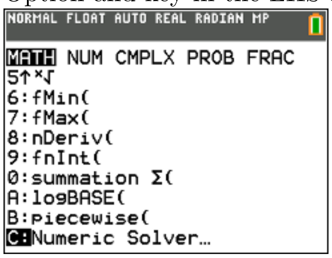
2.1 Solving Equations using GC

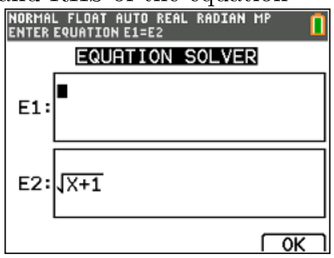
2.1.1 Graphical Method

Graph out the left and right side of the equation as 2 seperate graphs and solve for it's intercept

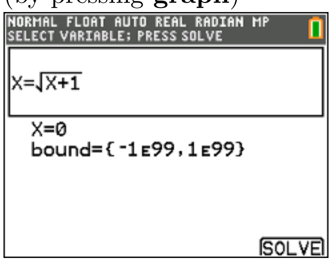
2.1.2 Equation Solver (Not Recommended)

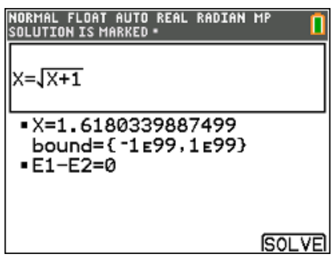
Press the **math** button and select the Numeric Solver Option and key in the LHS and RHS of the equation





Do note that you would have to key a guess (in this case it's $x = 0$) before you are able to solve for the equation (by pressing **graph**)





2.2 System of Linear Equations

An equation is linear when the variables have a power of 1

$$\text{General Form : } \sum_{i=1}^{n}(a_i x_i) = b, \text{ where } a_i, b \in \mathbb{R}$$

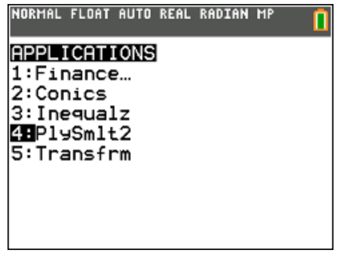
Linear Equations	Non-Linear Equations
$x - 2y + 3z = 180,$ $2x_1 + x_2 - 10x_{=}350$	$xy = 1,$ $x^2 + 3y = 1$

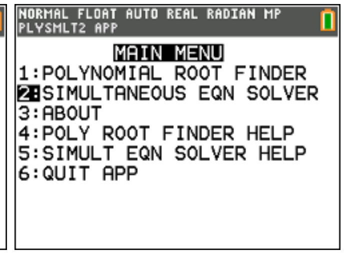
A System of Linear Equations have 3 possible outcomes

- Exactly one Solution (Unique)
- Infinitely many Solutions
- No Solutions

No Solution $\implies$ Inconsistent
At least 1 Solution $\implies$ Consistent

2.2.1 Solving with GC





We can use the simultatneous equation solver under the 'pltSmlt2' App to solve our system
When a system has infinitely many solutions
The solutions are represented with a 'free' variable
Example:

$x = z - 2$

$y = 3z + 2$

$z = z$

In this case, z is the 'free' variable that determines the values of x & y, thus it has infinitely many solutions.
The unique solution is determined by the context given

Example : If $z = 3$, then $x = 1$, $y = 11$

2.3 Inequalities

2.3.1 Basic Concepts

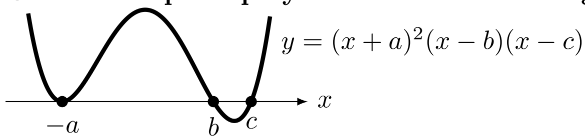
Let $a, b, c, d \in \mathbb{R}$

Properties
If $a > b$ & $b > c$, then $a > c$
If $a > b$ & $c > 0$, then $ac > bc$ & $\left(\frac{a}{c} > \frac{b}{c}\right)$
If $a > b$ & $c < 0$, then $ac < bc$ & $\left(\frac{a}{c} < \frac{b}{c}\right)$
If $a > b$ & $c > d$, then $a + c > b + d$ But $a - c > b - d$ may not be true
If $a > b > 0$ & $n > 0$, then $a^n > b^n$ & $\left(\frac{1}{a^n} < \frac{1}{d^n}\right)$
If $a > b$, then $f(a) > f(b)$ if f is monotonic increasing then $f(a) < f(b)$ if f is monotonic decreasing

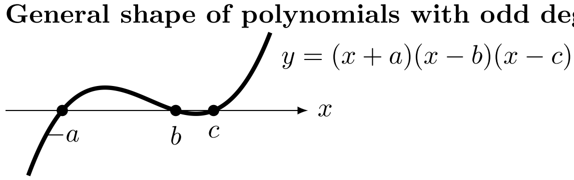
2.3.2 Polynomial Inequalities

For such inequalities in the form $f(x) \geq 0$ or other inequality signs We can solve it by looking at the general shape of the function or the test point method, after determining the roots of the equation

General shape of polynomials with even degrees



General shape of polynomials with odd degrees

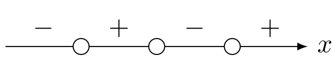


Steps to Solve the inequality

1. Make the term with the highest power is positive
2. Factorize the equation
3. Draw a number line with the roots of the equation
4. By either graph or test point, determine the sign of each section
   A section is the "gap" between roots and the ends
5. Write out the final inequality

Graph method is to mentally graph out the shape of the graph to determine the sign of each section, while test point is to test a number within that section (e.g using 1 for $0 < x < 2$) to determine the sign

Example : $(x-1)(x-2)(x-3) > 0$



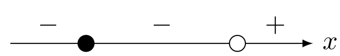
$\therefore 1 < x < 2$ or $x > 3$
$\{x \in \mathbb{R} : 1 < x < 2 \text{ or } x > 3\}$

2.3.3 Inequalities involving Rational Functions

Such inequalities can be solved with a similar method outlined in 2.3.2

1. Express the inequality in the form such that one side is 0 $\left(\text{e.g. } \dfrac{g(x)}{h(x)} > 0\right)$
2. Make the term with the highest power is positive
3. Factorize the polynomials
4. By either graph or test point, determine the sign of each section
   A section is the "gap" between roots and the ends
5. **Check that the x value(s) where the denominator = 0 isn't included**

Example : $\dfrac{(x-1)^2}{(x-3)} \geq 0$



$\therefore x > 3$ or $x = 1$
$\{x \in \mathbb{R} : x > 3 \text{ or } x = 1\}$

2.3.4 Inequalities involving Modulus Functions

Let $x, y \in \mathbb{R}$ and $a > 0$

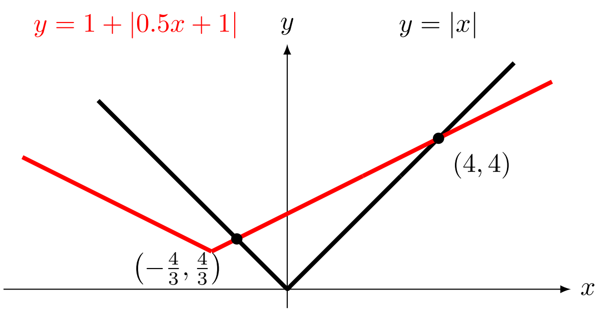
Properties
$\lvert x\rvert > a \iff x < -a$ or $x > a$ $\lvert x\rvert < a \iff -a < x < a$
$\lvert x\rvert < \lvert y\rvert \iff x^2 < y^2$ $\lvert x\rvert > \lvert y\rvert \iff x^2 > y^2$
$\lvert x^2\rvert = \lvert x\rvert^2 = x^2$ $\lvert xy\rvert = \lvert x\rvert\lvert y\rvert$ $\sqrt{x^2} = \lvert x\rvert$
$\lvert x-y\rvert = \lvert y-x\rvert$
$\left\lvert \dfrac{x}{y} \right\rvert = \dfrac{\lvert x\rvert}{\lvert y\rvert}$

Modulus Inequalities can be solved using these properties by maniplulating the inequality or by a graphical method (2.3.5)

2.3.5 Using a Graphical Method

Sketching the LHS and RHS of an inequality to determine the range of values

Example : $|x| - |0.5x + 1| > 1$
We can sketch the graphs $y = |x|$ & $y = 1 + |0.5x + 1|$



We would have to solve for the points of intercepts algebratically and by referring to our graph, we can determine the solution set to the inequality, which in this case is $x < -\dfrac{4}{3}$ or $x > 4$

2.3.6 Using Substitution

This method is usually used when solving for an inequality with a similar form to one that was solved in a previous part of a question

Example : Solve the inequality $\dfrac{4-x}{x-2} > 3$,
hence solve $\dfrac{6x-2}{1-2x} > 2$

Manipulate $\dfrac{6x-2}{1-2x} > 2$ to a similar form of $\dfrac{4-x}{x-2} > 3$

$$\frac{6x-2}{1-2x} > 2$$
$$\frac{6x-2}{1-2x} + 1 > 3$$
$$\frac{6x-2+1-2x}{1-2x} > 3$$
$$\frac{4x-1}{1-2x} > 3$$
$$\frac{\frac{1}{x}}{\frac{1}{x}}\left(\frac{4x-1}{1-2x}\right) > 3, \ x \neq 0$$
$$\frac{4-\frac{1}{x}}{\frac{1}{x}-2} > 3$$

Thus we had arrived at a similar form, and we can use the substitution $u = \frac{1}{x}$ to solve the inequality.

Note that now other than $x \neq \dfrac{1}{2}$, $x \neq 0$ too

3 Functions

3.1 Set Notation

Symbol	Meaning	Example
$\mathbb{N}$	Natural Numbers	$\{1, 2, 3, \cdots\}$
$\mathbb{Z}_0^+$	Non -ve Integers	$\{0, 1, 2, \cdots\}$
$\mathbb{Z}$	Integers	$\{\cdots, -1, 0, 1, \cdots\}$
$\mathbb{Q}$	Rationals	$\{\frac{p}{q} : p, q \in \mathbb{Z}\}$
$\mathbb{R}$	Real Numbers	$\{\cdots, -\pi, \frac{1}{2}, \sqrt{2}, 2 \cdots\}$
$\mathbb{C}$	Complex Numbers	$\{a + bi : a, b \in \mathbb{R}\}$

$$\boxed{\mathbb{N} \subset \mathbb{Z}_0^+ \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}$$

3.1.1 Interval Notation

Notation	Definition
$[a, b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$
(a, b)	$\{x \in \mathbb{R} : a < x < b\}$
$(a, b]$	$\{x \in \mathbb{R} : a < x \leq b\}$
$(-\infty, b]$	$\{x \in \mathbb{R} : x \leq b\}$
$[a, b) \cup (c, \infty)$	$\{x \in \mathbb{R} : a \leq x < b \text{ or } x > c\}$
$[a, c) \cap [b, \infty)$	$\{x \in \mathbb{R} : b \leq x < c\}$
$\mathbb{R}\backslash\{0\}$	$\{x \in \mathbb{R} : x \neq 0\}$

Notes
Interval and Set Notations are not exact replacements of each other even though they are similar

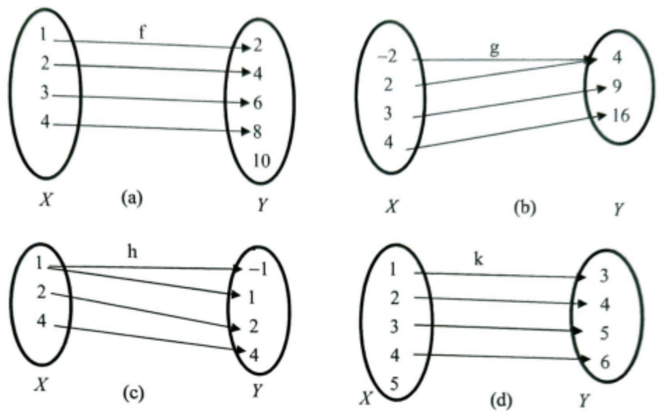
3.2 Definitions

3.2.1 Notations

Representation	Meaning
f	function
$f : A \to B$	Set Mapping
$f : x \mapsto x^2$	Rule of function
$f(x)$	Rule of function
D_f	Domain of f
R_f	Range of f

3.2.2 Relations

The Associatation between 2 Sets is a Relation



3.2.3 Definition of a Function

A **Function** is a relation from Set X to Y where every input $x \in X$ is mapped to a **unique output** $y \in Y$

With this definition, we can see from 3.2.2 that only (a) & (b) are functions. (c) is not a function as input $(x = -1)$ is mapped to $y = -1$ & $y = 1$, (d) in not a function as input $(x = 5)$ does not have a related output in Y

Function Example:

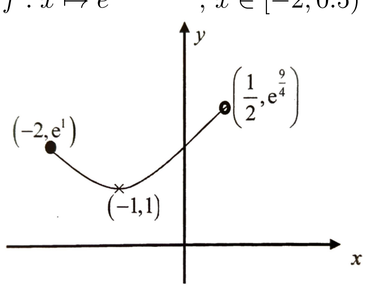
$f : x \mapsto x, x \in (-2, 2)$

Rule Domain

3.2.4 Finding Range with GC

Key the function into the GC, while taking note of the starting and ending points of the domain. If the function has turning points, it has to be taken into account as it coul affect the Range.

Example:
$f : x \mapsto e^{x^2+2x+1}$, $x \in [-2, 0.5)$



$\therefore R_f = \left[1, e^{\frac{9}{4}}\right)$

3.3 Injective Functions

3.3.1 Definition of Injection

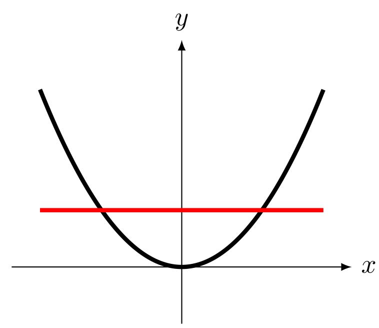
Injectivity means that every input has a **unique** output formally, $f : X \mapsto Y$ is injective iff

$$\boxed{f(a) = f(b) \implies a = b \text{ OR } a \neq b \implies f(a) \neq f(b)}$$

where $a, b, \in \mathbb{D}_f$

3.3.2 Disproving Injectivity

We only need to show that a single value violates the definition, which can be done by the horizontal line test



In this case, as the horizontal line intercepts the function at more than 1 point, the function isn't injective

3.3.3 Proving Injevtivity

To prove injectivity, we would have to use the definition
Example, $f(x) = \dfrac{1}{x-2}$, $x \in \mathbb{R}$

$$\begin{aligned} & f(a) = f(b) \\ \implies & \frac{1}{a-2} = \frac{1}{b-2} \\ \implies & a - 2 = b - 2 \\ \implies & a = b \end{aligned}$$

Thus, the function f is injective

3.4 *Surjective Functions

3.4.1 Definition of Surjection

Surjectivity means that the range of the function is an equal set to the codomain.
Formally, $f : X \mapsto Y$ is surjective iff $R_f \equiv Y$

3.5 *Bijective Functions

3.5.1 Definition of Bijection

A Bijective function is one that satisfies both injectivity and surjectivity. A function has to be bijective for the inverse to exist, however, in the context of A-Levels, injectivity suffices

3.6 Inverse Functions

An inverse function, when applied to it's respective non-inverse, gives the input of the original function, or basically undoing the function

3.6.1 Invese Function Notation

The inverse of a function f can be written as f^{-1}

3.6.2 Conditions for an inverse to exist

A function has to be bijective (or injective only in the context of A-Levels) for the inverse to exist

3.6.3 Properties of Inverse Functions

1. $f^{-1}(x)$ is a reflection of $f(x)$ about the line $y = x$
2. $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$
3. $\left(f^{-1}\right)^{-1} = f$

3.6.4 Self-Inversing functions

A self inversing function f is one where $ff = x \ \forall x \in D_f$
Example of Self-Inversing Function : $f(x) = \dfrac{-mx + a}{bx + m}$, where $a, b, m \in \mathbb{R}$
However, there is a value of a to omit, which is where $-mx + a = k(bx + m)$, $k \in \mathbb{R}/\{0\}$: or $a = -\dfrac{m^2}{b}$

3.7 Composite Functions

3.7.1 Composite Function Notation

The composition of two functions f & g can be written as $f \circ g$ or fg where $fg(x) = f[g(x)]$

3.7.2 Conditions for a composition to exist

For composition $f \circ g$ to exist, $R_g = D_f$

3.7.3 Properties of Composite Functions

1. Composition is not Commutative $f \circ g \neq g \circ f$

3.7.4 Finding Domain & Range of a Composition

3.8 Extensions

3.8.1 Properties of Functions

3.8.2 Inverse Trigonometic Functions

3.8.3 Monotonic Functions

3.8.4 *Floor and Ceiling Functions

3.8.5 *Continuity & Discontinuity

4 Sequences & Series

5 Vectors

Hello, here is some text without a meaning...

6 Recurrence Relations

Hello, here is some text without a meaning...

7 Mathematical Induction

Hello, here is some text without a meaning...

8 Differentiation

Hello, here is some text without a meaning...

9 Linear Algebra

Hello, here is some text without a meaning...

10 Integration

10.1

10.1.1