0 Functions

0.1 Set Notation

Symbol	Meaning	Example
$\mathbb{N}$	Natural Numbers	$\{1, 2, 3, \cdots\}$
$\mathbb{Z}_0^+$	Non -ve Integers	$\{0, 1, 2, \cdots\}$
$\mathbb{Z}$	Integers	$\{\cdots, -1, 0, 1, \cdots\}$
$\mathbb{Q}$	Rationals	$\{\frac{p}{q} : p, q \in \mathbb{Z}\}$
$\mathbb{R}$	Real Numbers	$\{\cdots, -\pi, \frac{1}{2}, \sqrt{2}, 2 \cdots\}$
$\mathbb{C}$	Complex Numbers	$\{a + bi : a, b \in \mathbb{R}\}$

$\boxed{\mathbb{N} \subset \mathbb{Z}_0^+ \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}}$

0.1.1 Interval Notation

Notation	Definition
$[a, b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$
(a, b)	$\{x \in \mathbb{R} : a < x < b\}$
$(a, b]$	$\{x \in \mathbb{R} : a < x \leq b\}$
$(-\infty, b]$	$\{x \in \mathbb{R} : x \leq b\}$
$[a, b) \cup (c, \infty)$	$\{x \in \mathbb{R} : a \leq x < b \text{ or } x > c\}$
$[a, c) \cap [b, \infty)$	$\{x \in \mathbb{R} : b \leq x < c\}$
$\mathbb{R}\backslash\{0\}$	$\{x \in \mathbb{R} : x \neq 0\}$

Notes

Interval and Set Notations are not exact replacements of each other even though they are similar

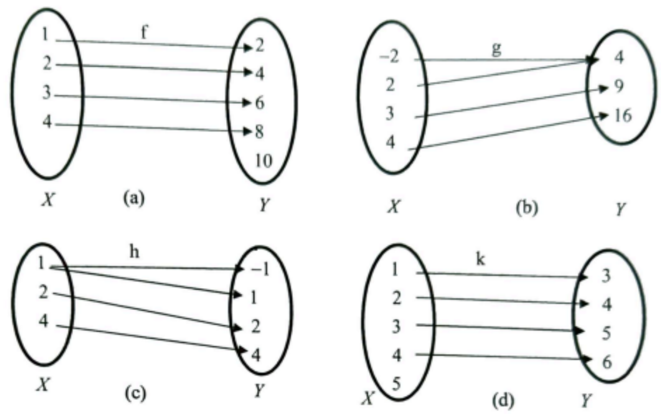
0.2 Definitions

0.2.1 Notations

Representation	Meaning
f	function
$f : A \to B$	Set Mapping
$f : x \mapsto x^2$	Rule of function
$f(x)$	Rule of function
D_f	Domain of f
R_f	Range of f

0.2.2 Relations

The Associatation between 2 Sets is a Relation



0.2.3 Definition of a Function

A **Function** is a relation from Set X to Y where every input $x \in X$ is mapped to a **unique output** $y \in Y$

With this definition, we can see from 3.2.2 that only (a) & (b) are functions. (c) is not a function as input $(x = -1)$ is mapped to $y = -1$ & $y = 1$, (d) in not a function as input $(x = 5)$ does not have a related output in Y

Function Example:

$f : x \mapsto x, x \in (-2, 2)$

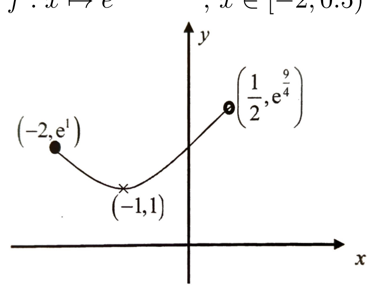
Rule Domain

0.2.4 Finding Range with GC

Key the function into the GC, while taking note of the starting and ending points of the domain. If the function has turning points, it has to be taken into account as it coul affect the Range.

Example:

$f : x \mapsto e^{x^2+2x+1}, x \in [-2, 0.5)$



$\therefore R_f = \left[1, e^{\frac{9}{4}}\right)$

0.3 Injective Functions

0.3.1 Definition of Injection

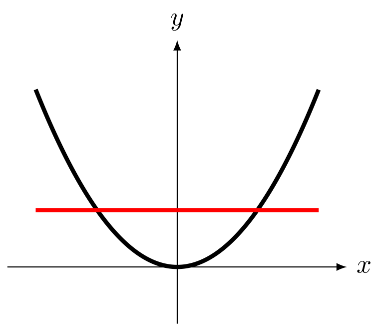
Injectivity means that every input has a **unique** output formally, $f : X \mapsto Y$ is injective iff

$$\boxed{f(a) = f(b) \implies a = b \text{ OR } a \neq b \implies f(a) \neq f(b)}$$

where $a, b, \in \mathbb{D}_f$

0.3.2 Disproving Injectivity

We only need to show that a single value violates the definition, which can be done by the horizontal line test



In this case, as the horizontal line intercepts the function at more than 1 point, the function isn't injective

0.3.3 Proving Injevtivity

To prove injectivity, we would have to use the definition
Example, $f(x) = \dfrac{1}{x-2}, x \in \mathbb{R}$

$$\begin{aligned} f(a) &= f(b) \\ \implies \frac{1}{a-2} &= \frac{1}{b-2} \\ \implies a-2 &= b-2 \\ \implies a &= b \end{aligned}$$

Thus, the function f is injective

0.4 *Surjective Functions

0.4.1 Definition of Surjection

Surjectivity means that the range of the function is an equal set to the codomain.
Formally, $f : X \mapsto Y$ is surjective iff $R_f \equiv Y$

0.5 *Bijective Functions

0.5.1 Definition of Bijection

A Bijective function is one that satisfies both injectivity and surjectivity. A function has to be bijective for the inverse to exist, however, in the context of A-Levels, injectivity suffices

0.6 Inverse Functions

An inverse function, when applied to it's respective non-inverse, gives the input of the original function, or basically undoing the function

0.6.1 Invese Function Notation

The inverse of a function f can be written as f^{-1}

0.6.2 Conditions for an inverse to exist

A function has to be bijective (or injective only in the context of A-Levels) for the inverse to exist

0.6.3 Properties of Inverse Functions

1. $f^{-1}(x)$ is a reflection of $f(x)$ about the line $y = x$
2. $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$
3. $\left(f^{-1}\right)^{-1} = f$

0.6.4 Self-Inversing functions

A self inversing function f is one where $ff = x \; \forall x \in D_f$
Example of Self-Inversing Function : $f(x) = \dfrac{-mx + a}{bx + m}$, where $a, b, m \in \mathbb{R}$
However, there is a value of a to omit, which is where $-mx + a = k(bx + m)$, $k \in \mathbb{R}/\{0\}$: or $a = -\dfrac{m^2}{b}$

0.7 Composite Functions

0.7.1 Composite Function Notation

The composition of two functions f & g can be written as $f \circ g$ or fg where $fg(x) = f[g(x)]$

0.7.2 Conditions for a composition to exist

For composition $f \circ g$ to exist, $R_g = D_f$

0.7.3 Properties of Composite Functions

1. Composition is not Commutative $f \circ g \neq g \circ f$

0.7.4 Finding Domain & Range of a Composition

0.8 Extensions

0.8.1 Properties of Functions

0.8.2 Inverse Trigonometic Functions

0.8.3 Monotonic Functions

0.8.4 *Floor and Ceiling Functions

0.8.5 *Continuity & Discontinuity