

H2 Further Mathematics

Summary Notes

NusernameSG

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0 Assumed Knowledge

0.1 Algebra

0.1.1 Completing the Square

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} \right)^2 + c - a \left(\frac{b}{2a} \right)^2$$

$$\text{Min/Max Point} \rightarrow \left(-\frac{b}{2a}, c - a \left(\frac{b}{2a} \right)^2 \right)$$

0.1.2 Nature of Roots

Discriminant $< 0 \implies$ No **Real** Roots

Discriminant $= 0 \implies$ 2 **Equal** Roots

Discriminant $> 0 \implies$ 2 **Distinct** Roots

0.1.3 Vieta's Formulas (Degree of 2)

Suppose that $\alpha + \beta$ are the two roots of $ax^2 + bx + c = 0$, $a \neq 0$

$$\text{Sum of roots} = \alpha + \beta = -\frac{b}{a}$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a}$$

0.1.4 General Vieta's formulas

0.1.5 Polynomials

Expansions:

$$(a \pm b)^2 = a^2 \pm 2ab + b^2$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

Binomial Formula:

$$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$$

0.1.6 Partial Fractions

$$\frac{f(x)}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}, A, B \in \mathbb{R}$$

$$\frac{f(x)}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}, A, B, C \in \mathbb{R}$$

$$\frac{f(x)}{(ax+b)(cx^2+d)} = \frac{A}{ax+b} + \frac{Bx+C}{cx^2+d}, A, B, C \in \mathbb{R}$$

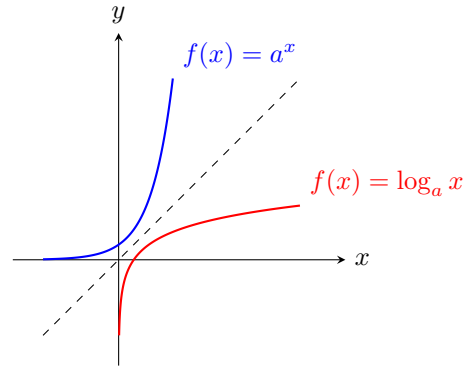
0.1.7 Factor & Remainder Theorem

Given a polynomial $f(x)$,

If $f\left(\frac{b}{a}\right) = 0$, $a, b \in \mathbb{R}, a \neq 0$, $(ax - b)$ is **factor** of $f(x)$

If $f\left(\frac{b}{a}\right) = c$, $a, b \in \mathbb{R}, a \neq 0$, the **remainder** of $f(x)$ divided by $(ax - b)$ is c

0.1.8 Logarithmic & Exponential

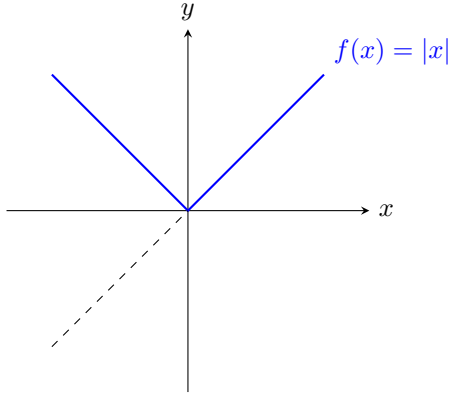


In General : $y = a^x \iff x = \log_a(y)$, $a > 0$, $a \neq 1$

Exponential	Logarithmic
$\forall x \in \mathbb{R}, f(x) > 0$	$D_f = \mathbb{R}^+$
y-intercept : $y = 1$	x-intercept : $x = 1$
$y = 0$ is an asymptote	$x = 0$ is an asymptote
$f(x)$ is increasing for $a > 1$	$f(x)$ is increasing for $a > 1$
$f(x)$ is decreasing for $0 < a < 1$	$f(x)$ is decreasing for $0 < a < 1$

Rules of Indices	Laws of Logarithm
If $a, b, m \in \mathbb{R}^+$	If $a, m, n \in \mathbb{R}^+, a > 0, a \neq 1$
$a^m * a^n = a^{m+n}$	$\log_a mn = \log_a m + \log_a n$
$a^m \div a^n = a^{m-n}$	$\log_a \frac{m}{n} = \log_a m - \log_a n$
$(a^m)^n = a^{mn}$	$\log_a m^n = n \log_a m$
$a^m * b^m = (a + b)^m$	$\log_n m = \frac{\log_a m}{\log_a n}$
$a^m \div b^m = \left(\frac{a}{b}\right)^m$	

0.1.9 Modulus

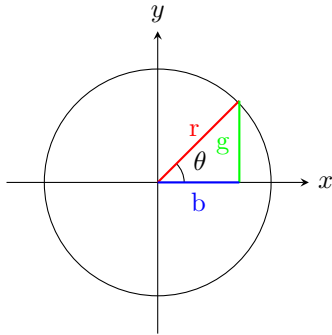


Generally, the modulus function is defined as

$$|f(x)| = \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(x) & \text{if } x < 0 \end{cases}$$

0.2 Trigonometry

0.2.1 Trigo Ratios for a General Angle



$$\sin \theta = \frac{g}{r} \quad \csc \theta = \frac{r}{g}$$

$$\cos \theta = \frac{b}{r} \quad \sec \theta = \frac{r}{b}$$

$$\tan \theta = \frac{g}{b} \quad \cot \theta = \frac{b}{g}$$

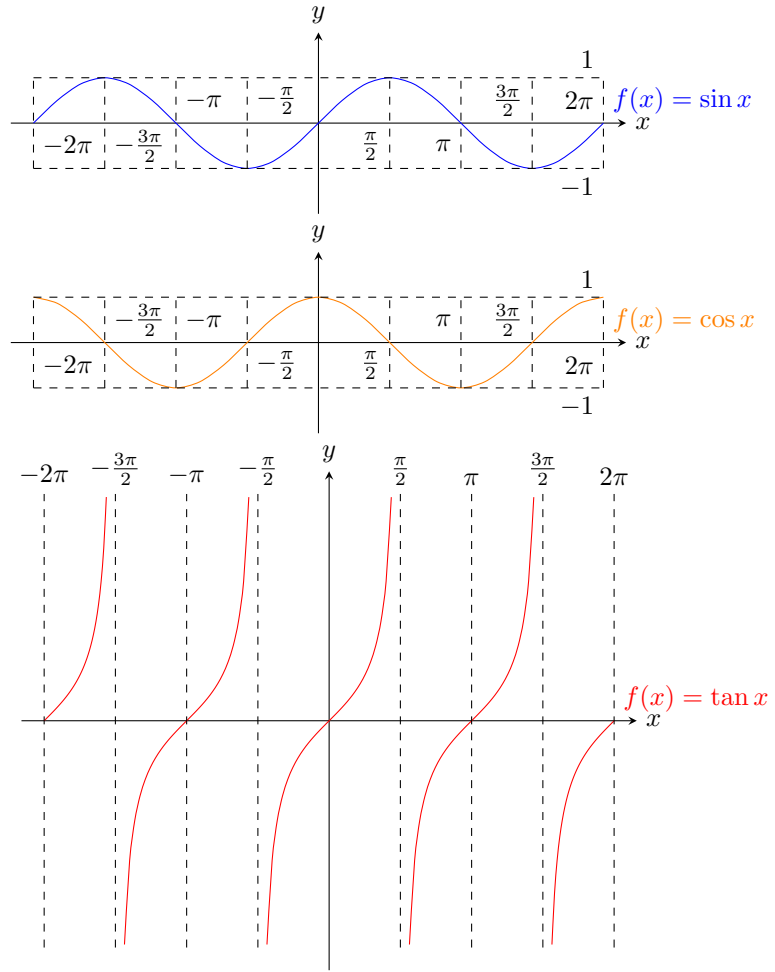
0.2.2 Principal Values

$$-\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2} \text{ where } x \in [-1, 1]$$

$$0 \leq \sin^{-1} x \leq \pi \text{ where } x \in [-1, 1]$$

$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \text{ where } x \in \mathbb{R}$$

0.2.3 Graphs of Trigo Functions



0.2.4 Sine & Cosine Rules

Given any triangle with sides of length a, b, c and opposite angles A, B, C

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2(b)(c) \cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2(b)(c)}$$

0.2.5 R-Formula

A Sum or difference of Sines and cosines can be represented with a single trig function if they have the same angles

$$a \sin \theta \pm b \cos \theta = \sin(\theta \pm \alpha)$$

$$a \cos \theta \mp b \sin \theta = \cos(\theta \pm \alpha)$$

$$R = \sqrt{a^2 + b^2} \text{ \& } \alpha = \tan^{-1} \left(\frac{b}{a} \right)$$

0.2.6 Basic Identities

$$\text{Area of Triangle} = \frac{1}{2}(a)(b) \sin \theta$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

0.2.7 Sum of Angles

$$\sin(A \pm B) = \sin(A) \cos(B) \pm \cos(A) \sin(B)$$

$$\cos(A \pm B) = \cos A \cos(B) \mp \sin(A) \sin(B)$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan(A) \tan(B)}$$

$$\sin 2A = 2 \sin(A) \cos(A)$$

$$\cos 2A = \cos^2(A) \mp \sin^2(A)$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2(A)}$$

0.2.8 Fator & Reverse Factor Theorem

$$\sin A + \sin B = 2 \sin \left[\frac{1}{2}(A + B) \right] \cos \left[\frac{1}{2}(A - B) \right]$$

$$\sin A - \sin B = 2 \cos \left[\frac{1}{2}(A + B) \right] \sin \left[\frac{1}{2}(A - B) \right]$$

$$\cos A + \cos B = 2 \cos \left[\frac{1}{2}(A + B) \right] \cos \left[\frac{1}{2}(A - B) \right]$$

$$\cos A - \cos B = 2 \sin \left[\frac{1}{2}(A + B) \right] \sin \left[\frac{1}{2}(A - B) \right]$$

$$\sin(A + B) \cos(A - B) = \frac{1}{2} (\sin 2A + \sin 2B)$$

$$\cos(A + B) \sin(A - B) = \frac{1}{2} (\sin 2A - \sin 2B)$$

$$\cos(A + B) \cos(A - B) = \frac{1}{2} (\cos 2A + \cos 2B)$$

$$\sin(A + B) \sin(A - B) = -\frac{1}{2} (\cos 2A - \cos 2B)$$

*Not in MF26

1 Graphing Techniques

1.1 Graph Features

1.1.1 Basic characteristics

Axial Intercepts

x - intercept : $y = 0$

y - intercept : $x = 0$

Stationary Points

Stationary points are points on a curve where

$$\left. \frac{dy}{dx} \right|_{x=k} = 0, k \in \mathbb{R}$$

The Nature of the stationary point can be determined by using the second or first derivative tests

Asymptotes

A Line or curve that a function approaches arbitrarily close to

Horizontal : when $x \rightarrow \infty, y \rightarrow a$, where $a \in \mathbb{R} \therefore y = a$

Vertical : when $y \rightarrow \infty, x \rightarrow b$, where $b \in \mathbb{R} \therefore x = b$

Oblique : when $x \rightarrow \infty, y \rightarrow cx + d$, where $c, d \in \mathbb{R}$
 $\therefore y = cx + d$

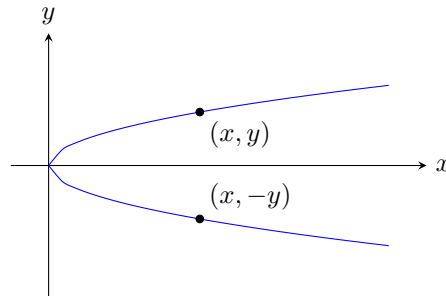
A Horizontal Asymptote can be cut through, but vertical asymptotes will never be passed through

1.1.2 Symmetry

Symmetric about the x-axis

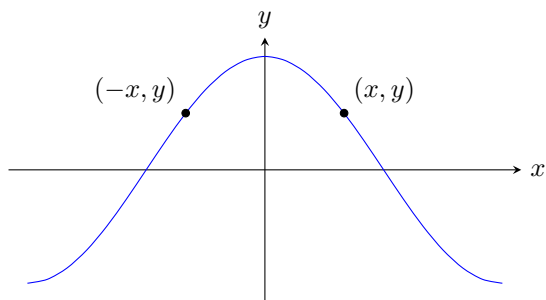
If (x, y) is a point on the curve,

$(x, -y)$ will also be a point on the curve



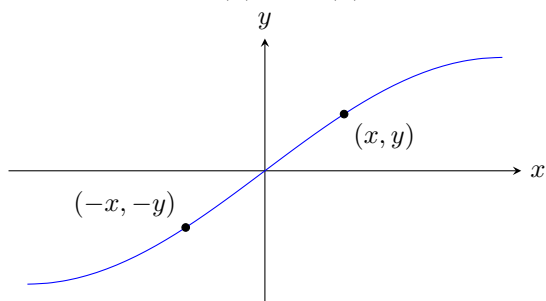
Symmetric about the y-axis (Even Functions)

Mathematically, $f(x) = f(-x)$



Symmetric about the origin (Odd Functions)

Mathematically, $f(x) = -f(-x)$



1.2 Types of Graphs

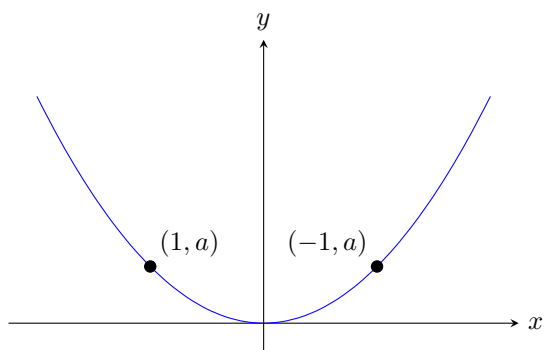
1.2.1 Power Functions

General Power Function : $f(x) = ax^n, a \in \mathbb{R}^+$

When n is an even positive integer

Function Type : Even Function

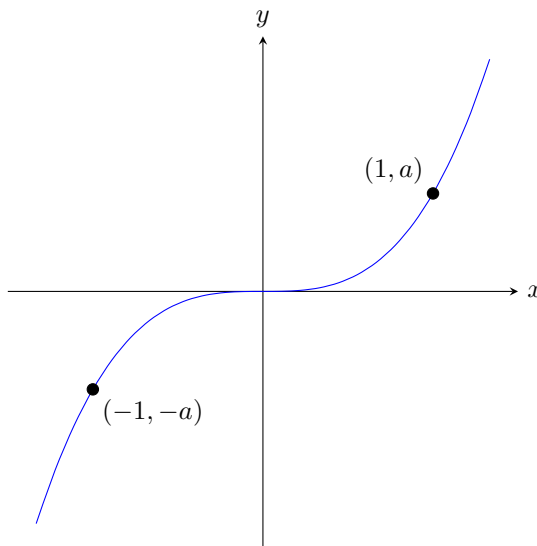
Common Points : $(-1, a), (0, 0), (1, a)$



When n is an odd positive integer, $n \geq 3$

Function Type : Odd Function

Common Points : $(-1, -a), (0, 0), (1, a)$



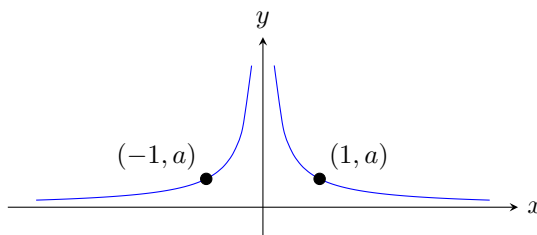
When n is an even negative integer

Function Type : Even Function

Common Points : $(-1, a), (0, 0), (1, a)$

Horizontal Asymptote : $y = 0$

Vertical Asymptote : $x = 0$



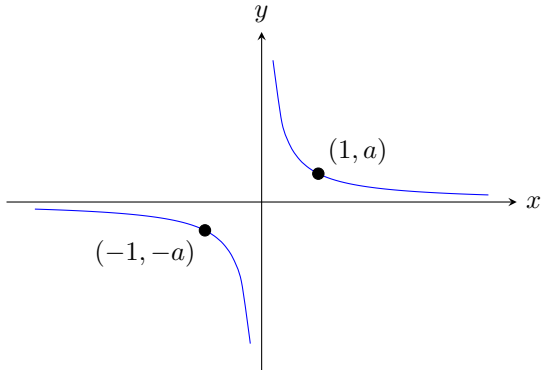
When n is an odd negative integer

Function Type : Odd Function

Common Points : $(-1, -a)$, $(0, 0)$, $(1, a)$

Horizontal Asymptote : $y = 0$

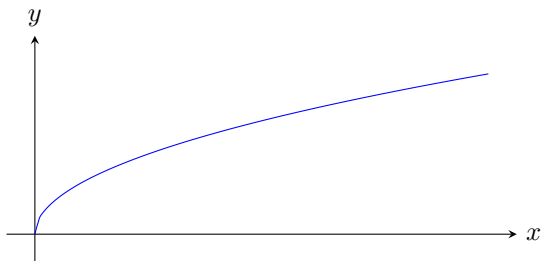
Vertical Asymptote : $x = 0$



When $n \in \mathbb{Q}$ in the form $\frac{1}{k}$

Such that k is an even positive integer

Common Points : $(0, 0)$, $(1, a)$

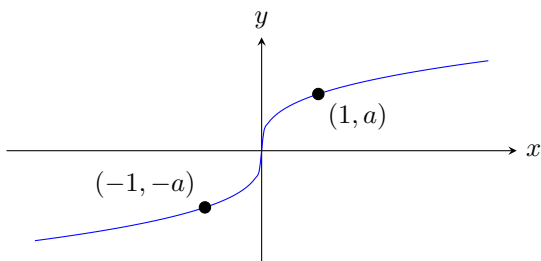


When $n \in \mathbb{Q}$ in the form $\frac{1}{k}$

Such that k is an odd positive integer

Function Type : Odd Function

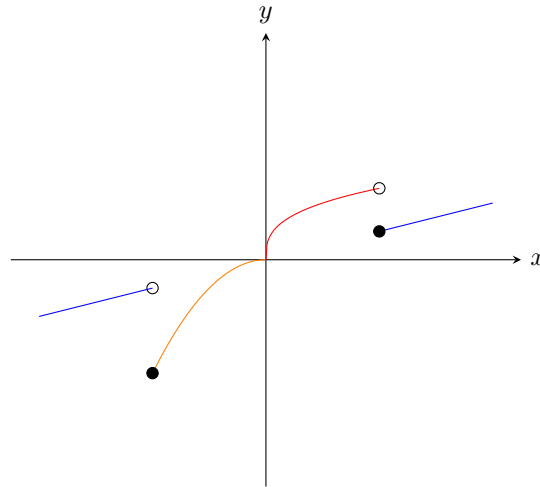
Common Points : $(-1, -a)$, $(0, 0)$, $(1, a)$



1.2.2 Piecewise Functions

Different intervals each has a different definition

$$f(x) = \begin{cases} H(x), & \text{if } a \leq x < b \\ G(x), & \text{if } b \leq x < c \\ Q(x), & \text{Otherwise} \end{cases}$$



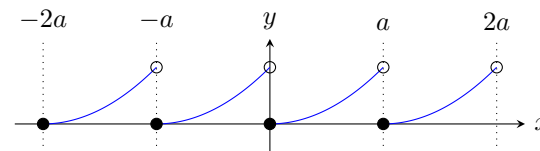
When the function is **discontinuous** between intervals, A **Solid** dot is used if the point is on the interval, an **empty** is used if it isn't

When the function is **Continuous** between intervals, dots are not needed

1.2.3 Periodic Functions

A function that repeats itself after an interval,

Mathematically, $f(x + \alpha) = f(x)$, where $\alpha \in \mathbb{R}$



In this example, the function $f(x)$ had an interval of $0 \leq x < a$

1.2.4 Rational Functions

A rational function is in the form $y = \frac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials

Types of Rational Functions

Degree of $p(x) < \text{Degree of } q(x)$

$y = 0$ is the Horizontal Asymptote

Solutions to $q(x) = 0$ will yield the Vertical Asymptote(s)

Degree of $p(x) \geq \text{Degree of } q(x)$

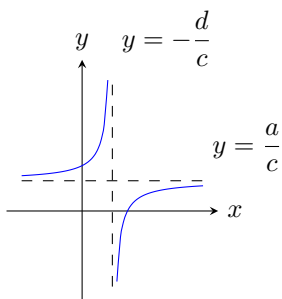
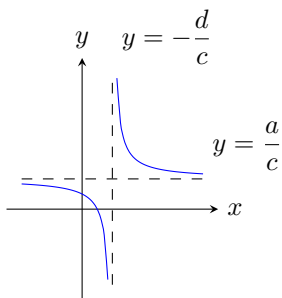
$y = \frac{p(x)}{q(x)} \equiv y = h(x) + \frac{r(x)}{q(x)}$, where $\deg(r(x)) < \deg(q(x))$

$y = h(x)$ is the Oblique Asymptote

Solutions to $q(x) = 0$ will yield the Vertical Asymptote(s)

Rectangular Hyperbola

Rectangular Hyperbola : $y = \frac{ax+b}{cx+d} \Rightarrow y = p + \frac{r}{cx+d}$,
where $c \neq 0, x \neq -\frac{d}{c}$. The possible curve shapes are



The Function is first to be rationalized by long division, then the Asymptotes and graph features can be found with the information above

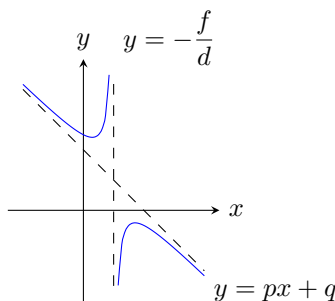
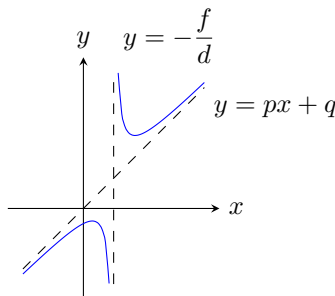
Quadratic Over Linear Functions

$$y = \frac{ax^2 + bx + c}{dx + f} \Rightarrow y = px + q + \frac{r}{dx + f}$$

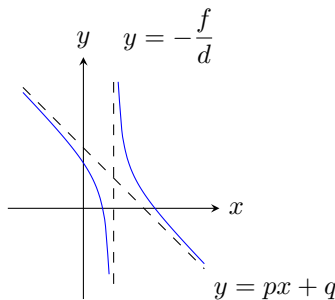
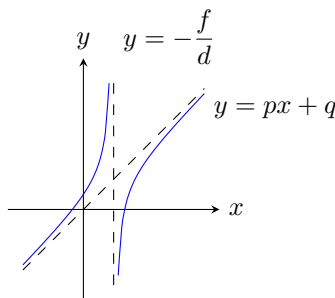
One Oblique Asymptote : $y = px + q$

One Vertical Asymptote : $x = -\frac{f}{d}$

Case 1 : Two Turning Points



Case 2 : No Turning Points



Linear Over Quadratic Functions

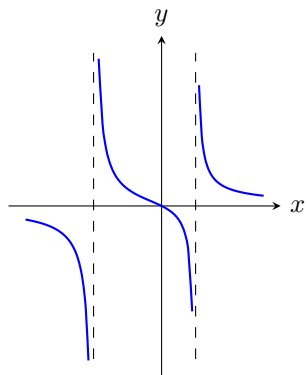
$$y = \frac{dx + f}{ax^2 + bx + c} \implies y = 0 + dx + f ax^2 + bx + c$$

One Horizontal Asymptote : $y = 0$

Vertical Asymptote(s) : Solve for denominator = 0

| Solutions = Asymptote(s)

Example : When there are 2 Vertical Asymptotes

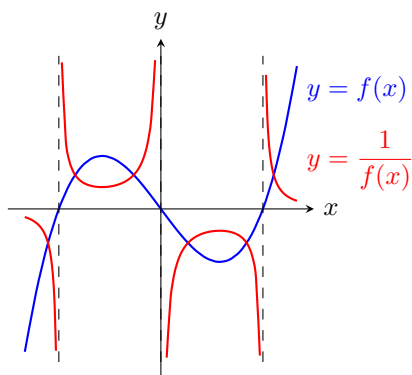


1.2.5 Reciprocal Functions

$y = f(x)$	$y = \frac{1}{f(x)}$
x -intercept : $x = a$	Vertical Asymptote : $x = a$
*Vertical Asymptote : $x = a$	* x -intercept : $x = a$
(a, b)	** $\left(a, \frac{1}{b}\right)$
Local Maxima : (a, b)	Local Minima : ** $\left(a, \frac{1}{b}\right)$
Local Minima : (a, b)	Local Maxima : ** $\left(a, \frac{1}{b}\right)$
Horizontal Asymptote : $y = b$	Horizontal Asymptote : $y = \frac{1}{b}$
Oblique Asymptote : $y = ax + b$	Horizontal Asymptote : $y = 0$
$f(x)$ is increasing	$\frac{1}{f(x)}$ is decreasing
$f(x)$ is decreasing	$\frac{1}{f(x)}$ is increasing

* True for **most** cases

** $b \neq 0$



1.3 Linear Transformations

1.3.1 Translation

$$y = f(x - a)$$

- Translation of the graph $f(x)$ by a units in the direction of the x -axis
- Add a to all of the x -coordinates

$$y = f(x - a)$$

- Translation of the graph $f(x)$ by $-a$ units in the direction of the x -axis
- Subtract a to all of the x -coordinates

$$y = f(x) + a \text{ or } y - a = f(x)$$

- Translation of the graph $f(x)$ by a units in the direction of the y -axis
- Add a to all of the y -coordinates

$$y = f(x) - a \text{ or } y + a = f(x)$$

- Translation of the graph $f(x)$ by $-a$ units in the direction of the y -axis
- Subtract a to all of the y -coordinates

1.3.2 Scaling

$$\underline{y = f\left(\frac{x}{a}\right)}$$

- Scaling the graph $f(x)$ by a scale factor of a parallel to the x -axis
- Multiply a to all of the x -coordinates

$$\underline{y = af(x) \text{ or } \frac{y}{a} = f(x)}$$

- Scaling the graph $f(x)$ by a scale factor of a parallel to the y -axis
- Multiply a to all of the y -coordinates

1.3.3 Reflection

$$\underline{y = f(-x)}$$

- Reflecting the graph $f(x)$ about y -axis
- Multiply -1 to all of the x -coordinates

$$\underline{y = -f(x)}$$

- Reflecting the graph $f(x)$ about x -axis
- Multiply -1 to all of the y -coordinates

1.3.4 Modulus Transformations

$$\underline{y = |f(x)|}$$

- Reflect the graph $f(x)$ about y -axis for the regions **below the x -axis**
- Multiply -1 to the y -coordinates of the **reflected points**

$$\underline{y = f(|x|)}$$

- Remove the graph that is in the **negative x -axis**
Reflect the remaining graph $f(x)$ about y -axis
- Multiply -1 to the x -coordinates of the **reflected points**

$$\underline{|y| = f(x)}$$

- Remove the graph that is in the **negative y -axis**
Reflect the remaining graph $f(x)$ about x -axis
- Multiply -1 to the y -coordinates of the **reflected points**

1.3.5 Sequence of Transformations

Suggested Sequence to minimise errors:

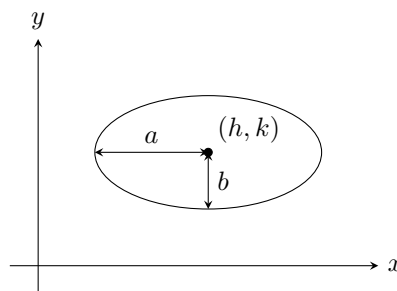
1. T_x **Horizontal** Translation
2. S_x **Horizontal** Scaling
3. S_y **Vertical** Scaling
4. T_y **Vertical** Translation

Sequence : $T_x S_x S_y T_y$

1.4 Conic Sections

1.4.1 Ellipses

Standard Form : $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

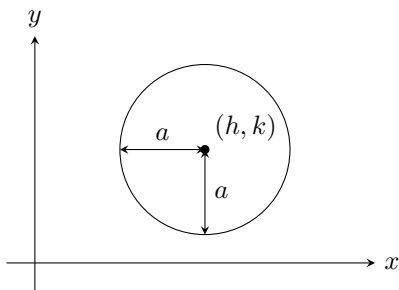


Properties

- Center : (h, k)
- Semi-Major-Axis : a
- Semi-Minor-Axis : b
- Symmetrical about $x = h$ & $y = k$

1.4.2 Circles

$$\text{Standard Form : } (x - h)^2 + (y - k)^2 = 1$$

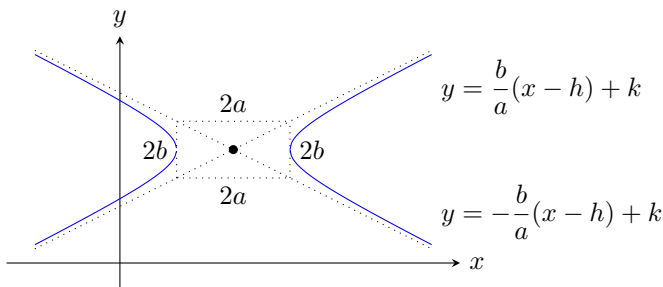


Properties

- Center : (h, k)
- Radius : a
- Symmetrical about $x = h$ & $y = k$

1.4.3 Hyperbolas

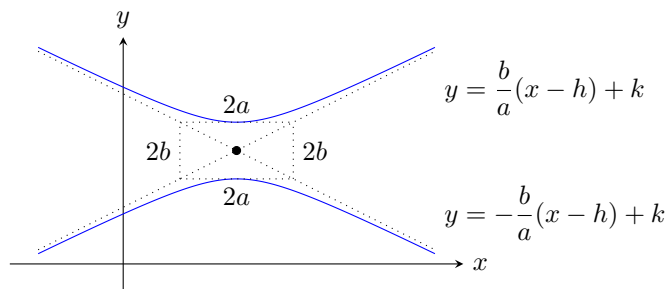
$$\text{Standard Form : } \frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$$



Properties

- Center : (h, k)
- Symmetrical about $x = h$ & $y = k$
- Vertices are ' b ' units from the center (h, k)
Vertically
- Oblique Asymptotes are $y = \pm \frac{b}{a}(x - h) + k$

$$\text{Standard Form : } \frac{(y - k)^2}{b^2} - \frac{(x - h)^2}{a^2} = 1$$



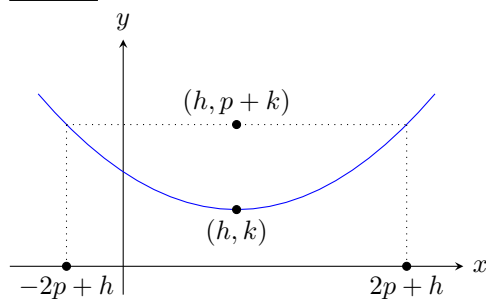
Properties

- Vertices : $(h, k \pm b)$
- Symmetrical about $x = h$ & $y = k$
- Vertices are ' a ' units from the center (h, k)
Horizontally
- Oblique Asymptotes are $y = \pm \frac{b}{a}(x - h) + k$

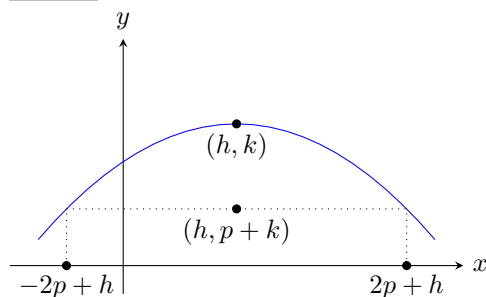
1.4.4 Parabolas

$$\text{Standard Form : } (x - h)^2 = 4p(y - k)$$

If $p > 0$



If $p < 0$

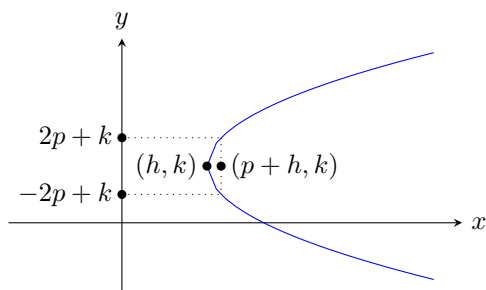


Properties

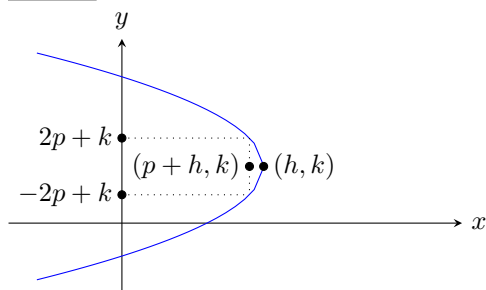
- Vertex : (h, k)
- Symmetrical about $x = h$
- If $p > 0, y \geq k \forall x(x \in \mathbb{R})$
If $p < 0, y \leq k \forall x(x \in \mathbb{R})$

Standard Form : $(y - h)^2 = 4p(x - k)$

If $p > 0$



If $p < 0$



Properties

- Vertex : (h, k)
- Symmetrical about $y = k$
- If $p > 0, x \geq h \forall y(y \in \mathbb{R})$
If $p < 0, x \leq h \forall y(y \in \mathbb{R})$

1.5 Parametric Equations

Expressing the variables of a function using another variable.

Cartesian Equation : $f(x, y)$

Parametric Equation : $x = f(t)$ & $y = f(t)$

1.5.1 Sketching Parametric Curves (Using GC)

Set an appropriate range for t under **Parametric** Mode and key in the $x(t)$ & $y(t)$ functions to graph it out but draw a table of t and the respective x, y

Table example:

t	-1	0	1
x	3	1	3
y	-4	1	2

1.5.2 Finding Asymptotes of Parametric Curves

Example : $x = \frac{1}{t}, y = t + 1$ for $t > 0$

As $y \rightarrow \infty, t \rightarrow \infty \implies x \rightarrow 0$

$\therefore x = 0$ is a Horizontal Asymptote

As $x \rightarrow \infty, t \rightarrow 0 \implies y \rightarrow 1$

$\therefore y = 1$ is a Vertical Asymptote

t acts as an intermediary

1.5.3 Conversion of Parametric to Cartesian Equations

Manipulate the terms containing the additional variable (such as t) until it can be cancelled out to give an equation

with only the original variables. Trig Identities can be used too.

Example : $x = a \cos t, y = a \sin t$

$$x^2 = a^2 (1 - \sin^2 t)$$

$$x^2 = a^2 - a^2 \left(\frac{y^2}{a^2} \right)$$

$\therefore y^2 + x^2 = a^2$ [Cartesian Equation]

2 Equations & Inequalities

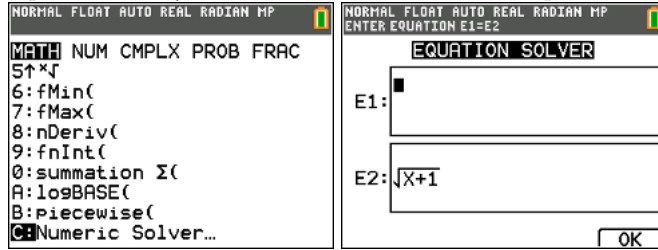
2.1 Solving Equations using GC

2.1.1 Graphical Method

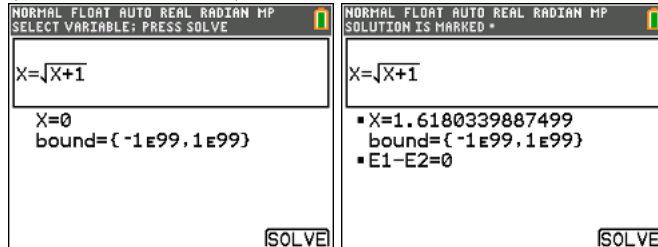
Graph out the left and right side of the equation as 2 separate graphs and solve for it's intercept

2.1.2 Equation Solver (Not Recommended)

Press the **math** button and select the Numeric Solver Option and key in the LHS and RHS of the equation



Do note that you would have to key a guess (in this case it's $x = 0$) before you are able to solve for the equation (by pressing **graph**)



2.2 System of Linear Equations

An equation is linear when the variables have a power of 1

$$\text{General Form : } \sum_{i=1}^n (a_i x_i) = b, \text{ where } a_i, b \in \mathbb{R}$$

Linear Equations	Non-Linear Equations
$x - 2y + 3z = 180,$	$xy = 1,$
$2x_1 + x_2 - 10x = 350$	$x^2 + 3y = 1$

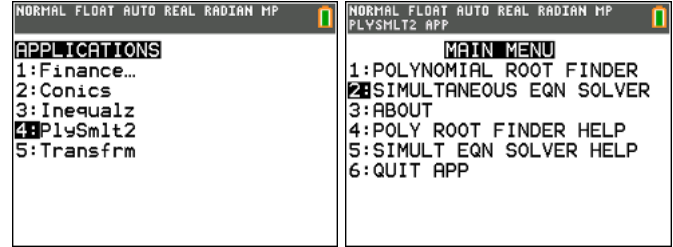
A System of Linear Equations have 3 possible outcomes

- Exactly one Solution (Unique)
- Infinitely many Solutions
- No Solutions

No Solution \implies Inconsistent

At least 1 Solution \implies Consistent

2.2.1 Solving with GC



We can use the simultaneous equation solver under the 'PlySmlt2' App to solve our system

When a system has infinitely many solutions

The solutions are represented with a 'free' variable

Example:

$$x = z - 2$$

$$y = 3z + 2$$

$$z = z$$

In this case, z is the 'free' variable that determines the values of x & y , thus it has infinitely many solutions.

The unique solution is determined by the context given

Example : If $z = 3$, then $x = 1, y = 11$

2.3 Inequalities

2.3.1 Basic Concepts

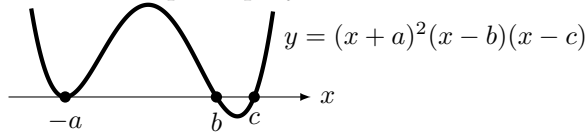
Let $a, b, c, d \in \mathbb{R}$

Properties
If $a > b$ & $b > c$, then $a > c$
If $a > b$ & $c > 0$, then $ac > bc$ & $\left(\frac{a}{c} > \frac{b}{c}\right)$
If $a > b$ & $c < 0$, then $ac < bc$ & $\left(\frac{a}{c} < \frac{b}{c}\right)$
If $a > b$ & $c > d$, then $a + c > b + d$ But $a - c > b - d$ may not be true
If $a > b > 0$ & $n > 0$, then $a^n > b^n$ & $\left(\frac{1}{a^n} < \frac{1}{b^n}\right)$
If $a > b$, then $f(a) > f(b)$ if f is monotonic increasing then $f(a) < f(b)$ if f is monotonic decreasing

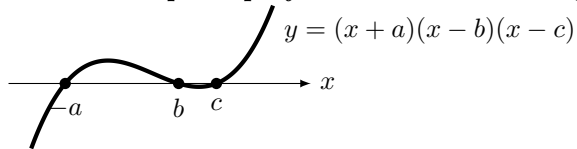
2.3.2 Polynomial Inequalities

For such inequalities in the form $f(x) \geq 0$ or other inequality signs We can solve it by looking at the general shape of the function or the test point method, after determining the roots of the equation

General shape of polynomials with even degrees



General shape of polynomials with odd degrees



Steps to Solve the inequality

1. Factorize the equation
2. Draw a number line with the roots of the equation
3. By either graph or test point, determine the sign of each section
A section is the "gap" between roots and the ends
4. Write out the final inequality

Graph method is to mentally graph out the shape of the graph to determine the sign of each section, while test point is to test a number within that section (e.g using 1 for $0 < x < 2$) to determine the sign

2.3.3 Inequalities involving Rational Functions

2.3.4 Inequalities involving Modulus Functions

2.3.5 Using a Graphical Method

3 Functions

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4 Sequences & Series

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5 Vectors

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6 Recurrence Relations

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