H2 Further Mathematics
Summary Notes

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Contents

0 Assumed Knowledge

0.1 Algebra

0.1.1 Completing the Square

$$\boxed{ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+c-a\left(\frac{b}{2a}\right)^2}$$

Min/Max Point $\rightarrow \left(-\frac{b}{2a}, c-a\left(\frac{b}{2a}\right)^2\right)$

0.1.2 Nature of Roots

$Discriminant < 0 \implies$ No **Real** Roots

$Discriminant = 0 \implies$ 2 **Equal** Roots

$Discriminant > 0 \implies$ 2 **Distinct** Roots

0.1.3 Vieta's Formulas (Degree of 2)

Suppose that $\alpha+\beta$ are the two roots of $ax^2+bx+c=0,\ a\neq 0$

Sum of roots $=\alpha+\beta=-\frac{b}{a}$

Product of roots $=\alpha\beta=\frac{c}{a}$

0.1.4 General Vieta's formulas

0.1.5 Polynomials

Expansions:

$(a\pm b)^2=a^2\pm 2ab+b^2$

$(a\pm b)^3=a^3\pm 3a^2b+3ab^2\pm b^3$

$a^2-b^2=(a+b)(a-b)$

$a^3\pm b^3=(a\pm b)\left(a^2\mp ab+b^2\right)$

Binomial Formula:

$$\boxed{(a+b)^n=\sum_{r=0}^{n}\binom{n}{r}a^{n-r}(b)^r}$$

0.1.6 Partial Fractions

$$\frac{f(x)}{(ax+b)(cx+d)}=\frac{A}{ax+b}+\frac{B}{cx+d},\ A,B\in\mathbb{R}$$

$$\frac{f(x)}{(ax+b)(cx+d)^2}=\frac{A}{ax+b}+\frac{B}{cx+d}+\frac{C}{(cx+d)^2},\ A,B,C\in\mathbb{R}$$

$$\frac{f(x)}{(ax+b)(cx^2+d)}=\frac{A}{ax+b}+\frac{Bx+C}{cx^2+d},\ A,B,C\in\mathbb{R}$$

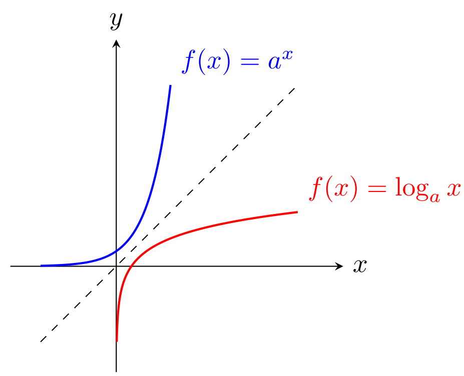
0.1.7 Factor & Remainder Theorem

Given a polynomial $f(x)$,

If $f\left(\frac{b}{a}\right)=0,\ a,b\in\mathbb{R}, a\neq 0,\ (ax-b)$ is **factor** of $f(x)$

If $f\left(\frac{b}{a}\right)=c,\ a,b\in\mathbb{R}, a\neq 0,$ the **remainder** of $f(x)$ divided by $(ax-b)$ is c

0.1.8 Logarithmic & Exponential

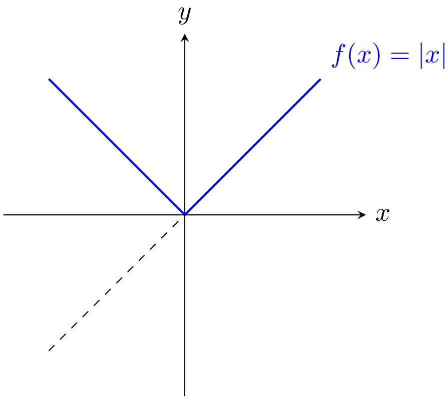


In General : $y=a^x \iff x=log_a(y),\ a>0,\ a\neq 1$

Exponential	Logarithmic
$\forall x\in\mathbb{R},\ f(x)>0$ y-intercept : $y=1$ $y=0$ is an asymptote $f(x)$ is increasing for $a>1$ $f(x)$ is decreasing for $0<a<1$	$D_f=\mathbb{R}^+$ x-intercept : $x=1$ $x=0$ is an asymptote $f(x)$ is increasing for $a>1$ $f(x)$ is decreasing for $0<a<1$

Rules of Indices	Laws of Logarithm
If $a,b,m\in\mathbb{R}^+$ $a^m * a^n=a^{m+n}$ $a^m\div a^n=a^{m-n}$ $(a^m)^n=a^{mn}$ $a^m * b^m=(a+b)^m$ $a^m\div b^m=\left(\frac{a}{b}\right)^m$	If $a,m,n\in\mathbb{R}^+,\ a>0,\ a\neq 1$ $\log_a mn=\log_a m+\log_a n$ $\log_a\frac{m}{n}=\log_a m-\log_a n$ $\log_a m^n=n\log_a m$ $\log_n m=\frac{\log_a m}{\log_a n}$

0.1.9 Modulus

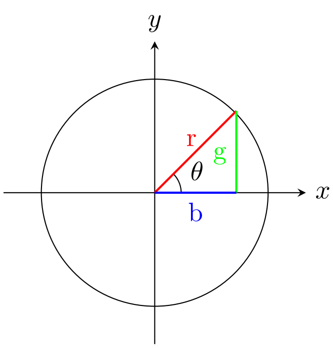


Generally, the modulus function is defined as

$$|f(x)| = \begin{cases} f(x) & \text{if } x \geq 0 \\ -f(x) & \text{if } x < 0 \end{cases}$$

0.2 Trigonometry

0.2.1 Trigo Ratios for a General Angle

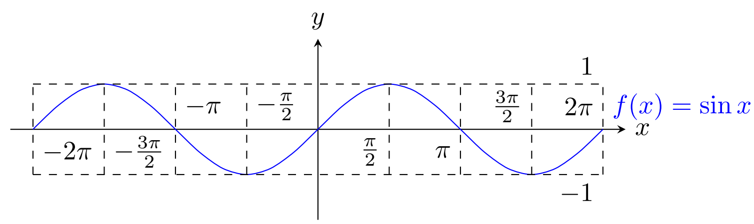


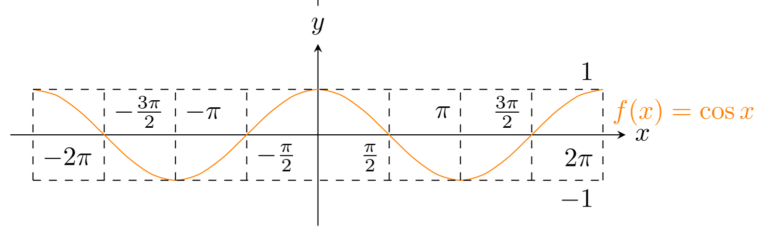
$$\sin\theta = \frac{g}{r} \qquad \csc\theta = \frac{r}{g}$$
$$\cos\theta = \frac{b}{r} \qquad \sec\theta = \frac{r}{b}$$
$$\tan\theta = \frac{g}{b} \qquad \cot\theta = \frac{b}{g}$$

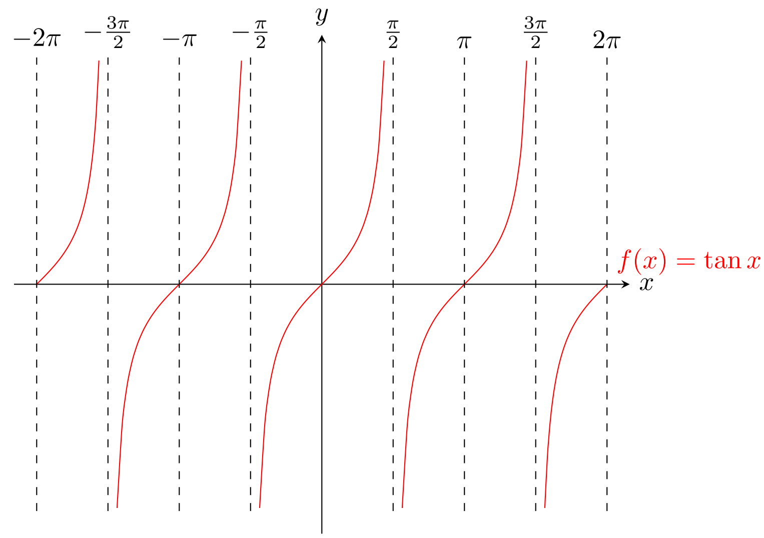
0.2.2 Principal Values

$$-\frac{\pi}{2} \leq \sin^{-1} x \ \leq \frac{\pi}{2} \text{ where } x \in [-1, 1]$$
$$0 \leq \sin^{-1} x \ \leq \pi \text{ where } x \in [-1, 1]$$
$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \text{ where } x \in \mathbb{R}$$

0.2.3 Graphs of Trigo Functions







0.2.4 Sine & Cosine Rules

Given any triangle with sides of length a, b, c and opposite angles A, B, C

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$
$$a^2 = b^2 + c^2 - 2(b)(c)\cos A$$
$$\cos A = \frac{b^2 + c^2 - a^2}{2(b)(c)}$$

0.2.5 R-Formula

A Sum or difference of Sines and cosines can be represented with a single trig function if they have the same angles

$$\boxed{a\sin\theta \pm b\cos\theta = \sin(\theta \pm \alpha)}$$

$$\boxed{a\cos\theta \mp b\cos\theta = \cos(\theta \pm \alpha)}$$

$R = \sqrt{a^2 + b^2}$ & $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

0.2.6 Basic Identities

$$\text{Area of Triangle} = \frac{1}{2}(a)(b)\sin\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

0.2.7 Sum of Angles

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$$

$$\cos(A \pm B) = \cos A\cos(B) \mp \sin(A)\sin(B)$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan(A)\tan(B)}$$

$$\sin 2A = 2\sin(A)\cos(A)$$

$$\cos 2A = \cos^2(A) \mp \sin^2(A)$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2(A)}$$

0.2.8 Fator & Reverse Factor Theorem

$$\sin A + \sin B = 2\sin\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$$

$$\sin A - \sin B = 2\cos\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$$

$$\cos A + \cos B = 2\cos\left[\frac{1}{2}(A+B)\right]\cos\left[\frac{1}{2}(A-B)\right]$$

$$\cos A - \cos B = 2\sin\left[\frac{1}{2}(A+B)\right]\sin\left[\frac{1}{2}(A-B)\right]$$

$$\boxed{\begin{aligned}\sin(A+B)\cos(A-B) &= \frac{1}{2}(\sin 2A + \sin 2B)\\ \cos(A+B)\sin(A-B) &= \frac{1}{2}(\sin 2A - \sin 2B)\\ \cos(A+B)\cos(A-B) &= \frac{1}{2}(\cos 2A + \cos 2B)\\ \sin(A+B)\sin(A-B) &= -\frac{1}{2}(\cos 2A - \cos 2B)\end{aligned}}$$

*Not in MF26

1 Graphing Techniques

1.1 Graph Features

1.1.1 Basic characteristics

Axial Intercepts

$x - intercept : y = 0$

$y - intercept : x = 0$

Stationary Points

Stationary points are points on a curve where

$$\left.\frac{dy}{dx}\right|_{x=k} = 0,\ k \in \mathbb{R}$$

The Nature of the stationary point can be determined by using the second or first derivative tests

Asymptotes

A Line or curve that a function approaches arbitrarily close to

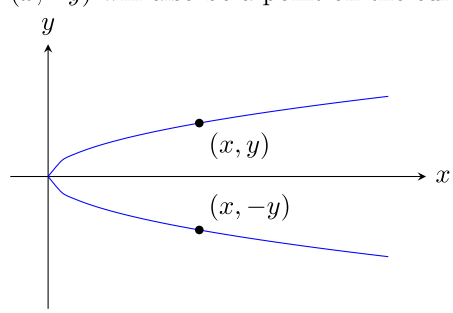
$$\begin{aligned}\text{Horizontal}: &\ \text{when } x \to \infty,\ y \to a, \text{where } a \in \mathbb{R}\ \therefore y = a\\ \text{Vertical}: &\ \text{when } y \to \infty,\ x \to b, \text{where } b \in \mathbb{R}\ \therefore x = b\\ \text{Oblique}: &\ \text{when } x \to \infty,\ y \to cx + d, \text{where } c, d \in \mathbb{R}\\ &\ \therefore y = cx + d\end{aligned}$$

A Horizontal Asymptote can be cut through, but vertical asymptotes will never be passed through

1.1.2 Symmetry

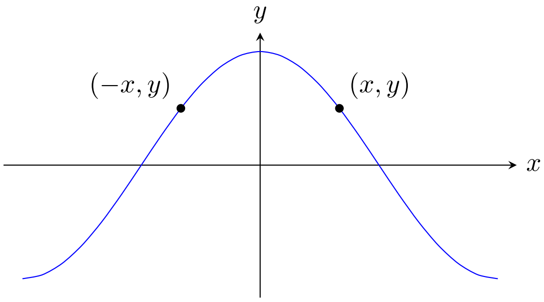
Symmetric about the x-axis

If (x, y) is a point on the curve,
$(x, -y)$ will also be a point on the curve



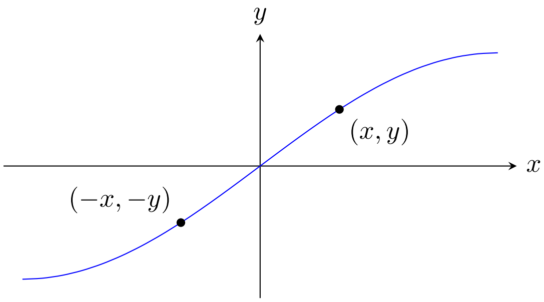
Symmetric about the y-axis (Even Functions)

Mathematically, $f(x) = f(-x)$



Symmetric about the origin (Odd Functions)

Mathematically, $f(x) = -f(x)$



1.2 Types of Graphs

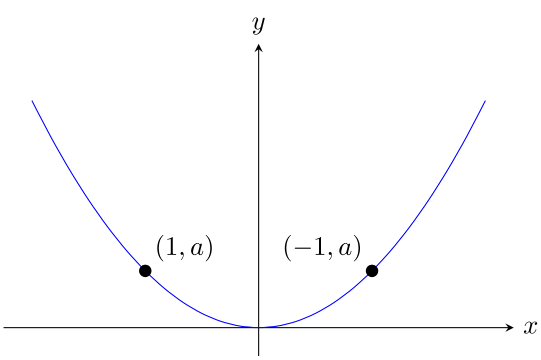
1.2.1 Power Functions

$$\boxed{\text{General Power Function : } f(x) = ax^n,\ a \in \mathbb{R}^+}$$

When n is an even positive integer

Function Type : Even Function

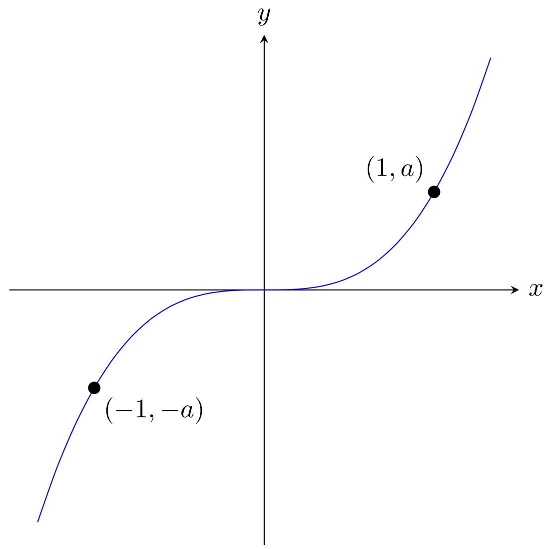
Common Points : $(-1, a)$, $(0, 0)$, $(1, a)$



When n is an odd positive integer, n $\geq$ 3

Function Type : Odd Function

Common Points : $(-1, -a)$, $(0, 0)$, $(1, a)$



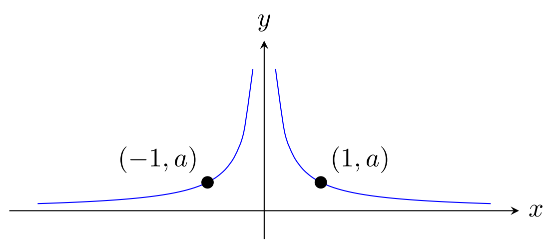
When n is an even negative integer

Function Type : Even Function

Common Points : $(-1, a)$, $(0, 0)$, $(1, a)$

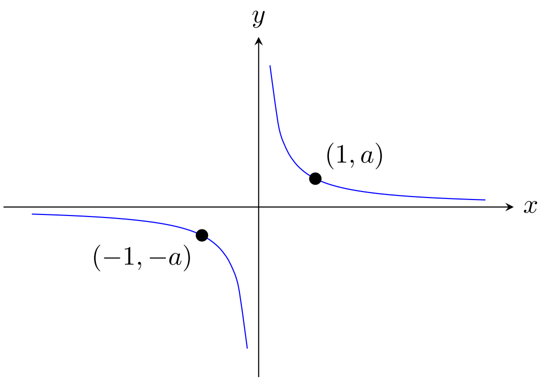
Horizontal Asymptote : $y = 0$

Vertical Vertical : $x = 0$



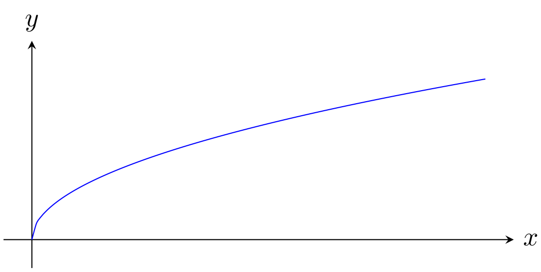
When n is an odd negative integer

$$\begin{aligned}\text{Function Type}: &\ \text{Odd Function}\\ \text{Common Points}: &\ (-1,-a),\ (0,0),\ (1,a)\\ \text{Horizontal Asymptote}: &\ y=0\\ \text{Vertical Vertical}: &\ x=0\end{aligned}$$



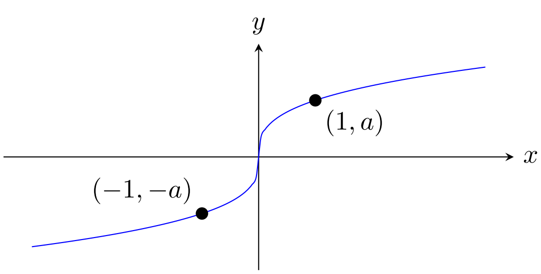
When $n \in \mathbb{Q}$ in the form $\frac{1}{k}$
Such that k is an even positive integer

Common Points : $(0,0)$, $(1,a)$



When $n \in \mathbb{Q}$ in the form $\frac{1}{k}$
Such that k is an odd positive integer

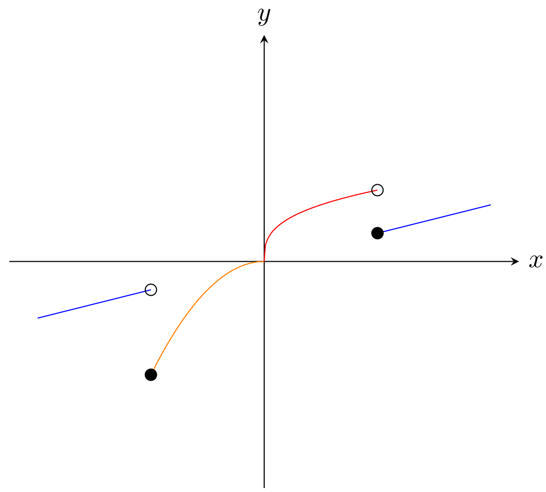
$$\begin{aligned}\text{Function Type}: &\ \text{Odd Function}\\ \text{Common Points}: &\ (-1,-a),\ (0,0),\ (1,a)\end{aligned}$$



1.2.2 Piecewise Functions

Different intervals each has a different definition

$$f(x)=\begin{cases}H(x), & \text{if } a\le x<b\\ G(x), & \text{if } b\le x<c\\ Q(x), & \text{Otherwise}\end{cases}$$

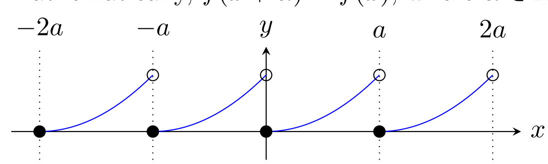


When the function is **discontinuous** between intervals,
A **Solid** dot is used if the point is on the interval, an **empty** is used if it isn't

When the function is **Continuous** between intervals, dots are not needed

1.2.3 Periodic Functions

A function that repeats itself after an interval, Mathematically, $f(x+\alpha)=f(x)$, where $\alpha \in \mathbb{R}$



In this example, the function $f(x)$ had an interval of $0 \le x < a$

1.2.4 Rational Functions

A rational function is in the form $y = \dfrac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials

Types of Rational Functions

Degree of $p(x)$ < Degree of $q(x)$

$y = 0$ is the Horizontal Asymptote

Solutions to $q(x) = 0$ will yield the Vertical Asymptote(s)

Degree of $p(x) \geq$ Degree of $q(x)$

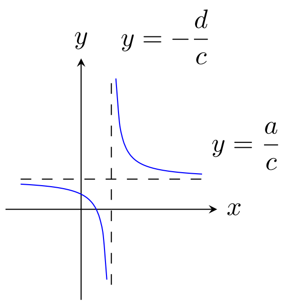
$y = \dfrac{p(x)}{q(x)} \equiv y = h(x) + \dfrac{r(x)}{q(x)}$, where $deg(r(x)) < deg(q(x))$

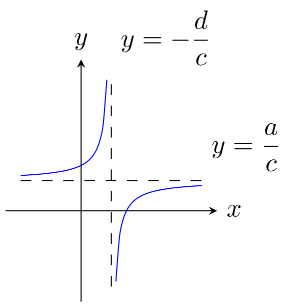
$y = h(x)$ is the Oblique Asymptote

Solutions to $q(x) = 0$ will yield the Vertical Asymptote(s)

Rectangular Hyperbola

Rectangular Hyperbola : $y = \dfrac{ax+b}{cx+d} \implies y = p + \dfrac{r}{cx+d}$,
where $c \neq 0, x \neq -\frac{d}{c}$. The possible curve shapes are





The Function is first to be rationalized by long division, then
the Asymptotes and graph features can be found with the information above

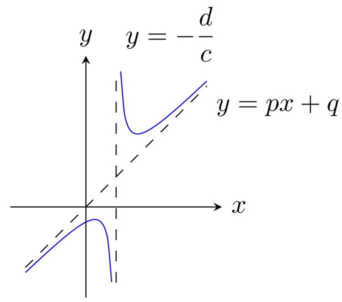
Quadratic Over Linear Functions

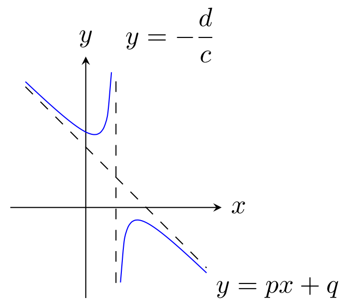
$y = \dfrac{ax^2+bx+c}{dx+f} \implies y = px + q + \dfrac{r}{dx+f}$

One Oblique Asymptote : $y = px + q$

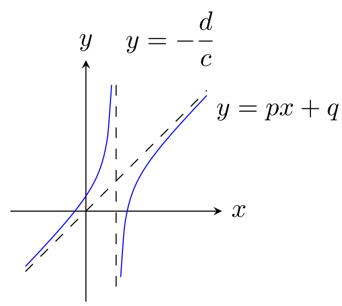
One Vertical Asymptote : $x = -\dfrac{f}{d}$

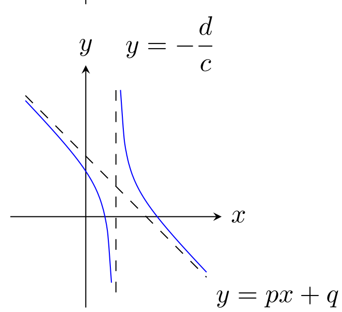
Case 1 : Two Turning Points





Case 2 : No Turning Points





Linear Over Quadratic Functions

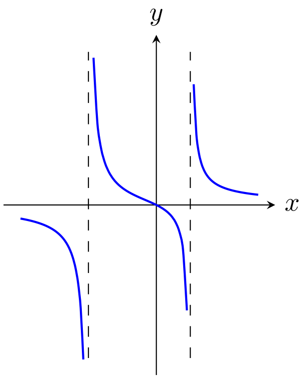
$$y = \frac{dx+f}{ax^2+bx+c} \implies y = 0 + dx + fax^2 + bx + c$$

One Horizontal Asymptote : $y = 0$

Vertical Asymptote(s) : Solve for denominator $= 0$

| Solutions = Asymptote(s)

Example : When there are 2 Vertical Asymptotes

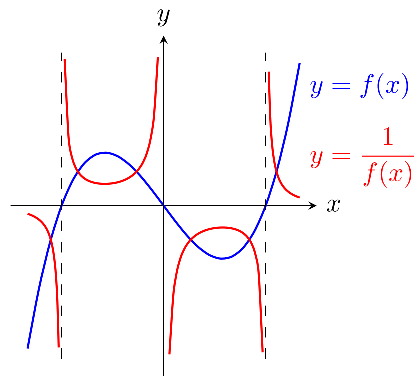


1.2.5 Reciprocal Functions

$y = f(x)$	$y = \dfrac{1}{f(x)}$
x-intercept : $x = a$	Vertical Asymptote : $x = a$
*Vertical Asymptote : $x = a$	*x-intercept : $x = a$
(a, b)	** $\left(a, \dfrac{1}{b}\right)$
Local Maxima : (a, b)	Local Minima : ** $\left(a, \dfrac{1}{b}\right)$
Local Minima : (a, b)	Local Maxima : ** $\left(a, \dfrac{1}{b}\right)$
Horizontal Asymptote : $y = b$	Horizontal Asymptote : $y = \dfrac{1}{b}$
Oblique Asymptote : $y = ax + b$	Horizontal Asymptote : $y = 0$
$f(x)$ is increasing	$\dfrac{1}{f(x)}$ is decreasing
$f(x)$ is decreasing	$\dfrac{1}{f(x)}$ is increasing

* True for **most** cases

** $b \neq 0$



1.3 Linear Transformations

1.3.1 Translation

$y = f(x - a)$

- Translation of the graph $f(x)$ by a units in the direction of the x-axis
- Add a to all of the x-coordinates

$y = f(x - a)$

- Translation of the graph $f(x)$ by $-a$ units in the direction of the x-axis
- Subtract a to all of the x-coordinates

$y = f(x) + a$ or $y - a = f(x)$

- Translation of the graph $f(x)$ by a units in the direction of the y-axis
- Add a to all of the y-coordinates

$y = f(x) - a$ or $y + a = f(x)$

- Translation of the graph $f(x)$ by $-a$ units in the direction of the y-axis
- Subtract a to all of the y-coordinates

1.3.2 Scaling

$y = f\left(\frac{x}{a}\right)$

- Scaling the graph $f(x)$ by a scale factor of a parallel to the x-axis
- Multiply a to all of the x-coordinates

$y = af(x)$ or $\frac{y}{a} = f(x)$

- Scaling the graph $f(x)$ by a scale factor of a parallel to the y-axis
- Multiply a to all of the y-coordinates

1.3.3 Reflection

$y = f(-x)$

- Reflecting the graph $f(x)$ about y-axis
- Multiply -1 to all of the x-coordinates

$y = -f(x)$

- Reflecting the graph $f(x)$ about x-axis
- Multiply -1 to all of the y-coordinates

1.3.4 Modulus Transformations

$y = |f(x)|$

- Reflect the graph $f(x)$ about y-axis
  for the regions **below the x-axis**
- Multiply -1 to the y-coordinates of the **reflected points**

$y = f(|x|)$

- Remove the graph that is in the **negative x-axis**
  Reflect the remaining graph $f(x)$ about y-axis
- Multiply -1 to the x-coordinates of the **reflected points**

$|y| = f(x)$

- Remove the graph that is in the **negative y-axis**
  Reflect the remaining graph $f(x)$ about x-axis
- Multiply -1 to the y-coordinates of the **reflected points**

1.3.5 Sequence of Tranformations

Suggested Sequence to minimise errors:

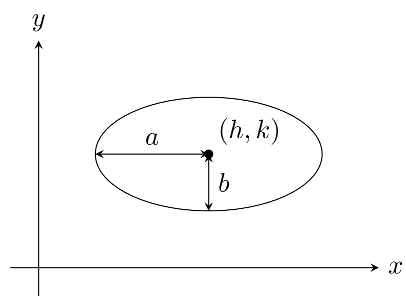
1. T_x **Horizontal** Translation
2. S_x **Horizontal** Scaling
3. S_y **Vertical** Scaling
4. T_y **Vertical** Translation

$\boxed{\text{Sequence : } T_x S_x S_y T_y}$

1.4 Conic Sections

1.4.1 Ellipses

$$\boxed{\text{Standard Form : } \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1}$$

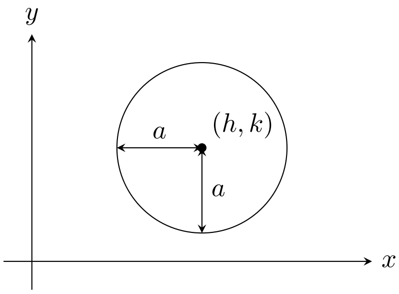


Properties

- Center : (h, k)
- Semi-Major-Axis : a
- Semi-Minor-Axis : b
- Symmetrical about $x = h$ & $y = k$

1.4.2 Circles

$$\boxed{\text{Standard Form : } (x-h)^2 + (y-k)^2 = 1}$$

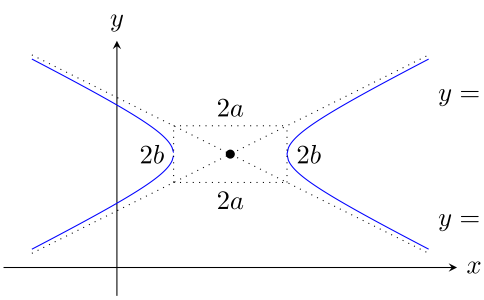


Properties

- Center : (h, k)
- Radius : a
- Symmetrical about $x = h$ & $y = k$

1.4.3 Hyperbolas

$$\boxed{\text{Standard Form : } \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1}$$



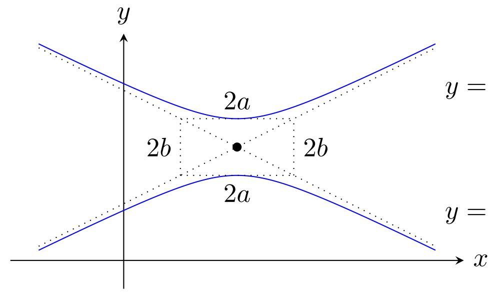
$$y = \frac{b}{a}(x-h) + k$$

$$y = -\frac{b}{a}(x-h) + k$$

Properties

- Center : (h, k)
- Symmetrical about $x = h$ & $y = k$
- Vertices are 'b' units from the center (h, k) Vertically
- Oblique Asymptotes are $y = \pm\frac{b}{a}(x-h) + k$

$$\boxed{\text{Standard Form : } \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1}$$



$$y = \frac{b}{a}(x-h) + k$$

$$y = -\frac{b}{a}(x-h) + k$$

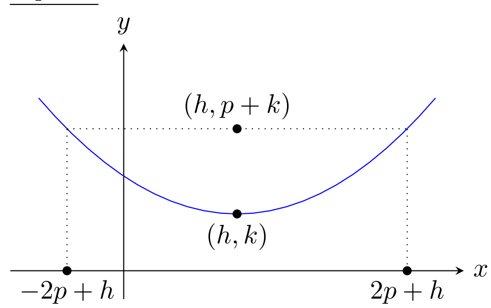
Properties

- Vertexes : $(h, k \pm b)$
- Symmetrical about $x = h$ & $y = k$
- Vertices are 'a' units from the center (h, k) Horizontally
- Oblique Asymptotes are $y = \pm\frac{b}{a}(x-h) + k$

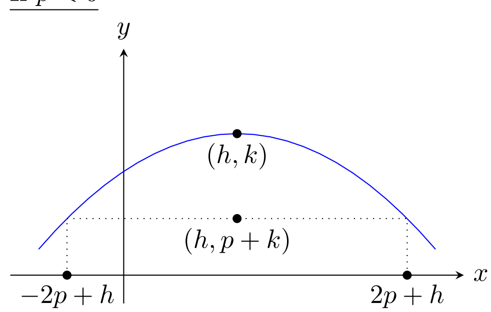
1.4.4 Parabolas

$$\boxed{\text{Standard Form : } (x-h)^2 = 4p(y-k)}$$

If $p > 0$



If $p < 0$

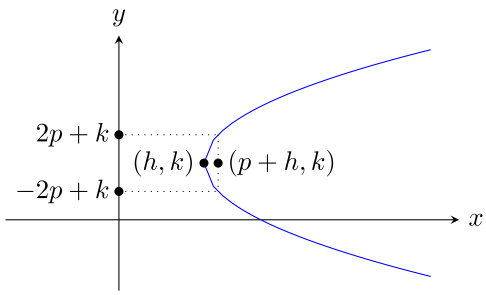


Properties

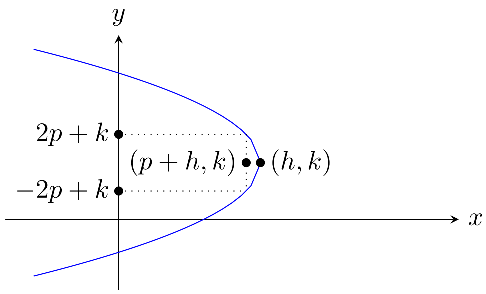
- Vertex : (h, k)
- Symmetrical about $x = h$
- If $p > 0$, $y \geq k \ \forall x(x \in \mathbb{R})$
  If $p < 0$, $y \leq k \ \forall x(x \in \mathbb{R})$

$$\boxed{\text{Standard Form : } (y-h)^2 = 4p(x-k)}$$

If $p > 0$



If $p < 0$



Properties

- Vertex : (h, k)
- Symmetrical about $y = k$
- If $p > 0$, $x \geq h \ \forall y(y \in \mathbb{R})$
  If $p < 0$, $x \leq h \ \forall y(y \in \mathbb{R})$

1.5 Parametric Equations

1.5.1 Sketching Parametric Curves

1.5.2 Finding Asymptotes of Parametric Curves

1.5.3 Conversion of Parametric to Cartesian Equations

2 Equations & Inequalities

Hello, here is some text without a meaning...

3 Functions

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4 Sequences & Series

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5 Vectors

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6 Recurrence Relations

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