

FurtherQ

Problem Set 1

Public Edition

Version 1.0

July 2022

Problems

| | |
|-------------------|----|
| <i>Problem 1</i> | 2 |
| <i>Problem 2</i> | 3 |
| <i>Problem 3</i> | 4 |
| <i>Problem 4</i> | 6 |
| <i>Problem 5</i> | 8 |
| <i>Problem 6</i> | 10 |
| <i>Problem 7</i> | 15 |
| <i>Problem 8</i> | 18 |
| <i>Problem 9</i> | 21 |
| <i>Problem 10</i> | 23 |

Editions

PB - Public Edition

SS - Secondary School Edition

Question Difficulty

Easy - Warm up

Intermediate - Standard

Challenging - Difficult

Schadenfreude - Interesting

Extensions - Out of syllabus

Paper Info

Marked Question: 1-8 & 10

Total Marks: 105

Total Time: 180 Mins (3 hrs)

Organizing Team

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1. [Suggested Time: 7 mins | Total Marks: 7 | Easy]

A function f is defined by $f : x \mapsto \sqrt{2}(\sin x + \cos x + 1)$ for $0 \leq x \leq 2\pi$

Points P and Q are the maximum and minimum points of f respectively,

Without the use of calculus, find the coordinates P and Q

Leave your answers in exact values.

[7]

$$\text{Let } y = \sqrt{2} \sin x + \sqrt{2} \cos x + \sqrt{2}$$

$$\text{Let } R \sin(x + \alpha) = \sqrt{2} \sin x + \sqrt{2} \cos x$$

[M1]

$$R \sin \alpha = \sqrt{2}$$

$$R \cos \alpha = \sqrt{2}$$

$$R = \sqrt{2+2}$$

$$= 2$$

[M1]

$$\alpha = \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

[M1]

$$\implies \sqrt{2} \sin x + \sqrt{2} \cos x = 2 \sin \left(x + \frac{\pi}{4} \right)$$

$$\therefore y = 2 \sin \left(x + \frac{\pi}{4} \right) + \sqrt{2}$$

$$\max(y) = 2 + \sqrt{2}$$

[M1]

$$\text{when } \sin \left(x + \frac{\pi}{4} \right) = 1$$

$$\therefore x + \frac{\pi}{4} = \frac{\pi}{2}$$

$$x = \frac{\pi}{4}$$

$$\implies P = \left(\frac{\pi}{4}, 2 + \sqrt{2} \right)$$

[M1]

$$\min(y) = \sqrt{2} - 2$$

[M1]

$$\text{when } \sin \left(x + \frac{\pi}{4} \right) = -1$$

$$\therefore x + \frac{\pi}{4} = \frac{3\pi}{2}$$

$$x = \frac{5\pi}{4}$$

$$\implies Q = \left(\frac{5\pi}{4}, \sqrt{2} - 2 \right)$$

[M1]

2. [Suggested Time: 5 mins | Total Marks: 5 | Easy]

Without the use of a calculator, Solve the following equation

$$6^{2x} + 36^x = 2\sqrt[3]{6^3\sqrt[3]{216}\sqrt[3]{6}}$$

Hence find the value of $\sqrt[19]{2^{6x}}$ [5]

$$6^{2x} + 6^{2x} = 2 \left(6 \left(6^3 (6)^{\frac{1}{3}} \right)^{\frac{1}{3}} \right)^{\frac{1}{3}}$$

$$2(6^{2x}) = 2 \left(6 \left(6^{\frac{10}{3}} \right)^{\frac{1}{3}} \right)^{\frac{1}{3}} \quad [M1]$$

$$= 2 \left(6^{1 + \frac{10}{9}} \right)^{\frac{1}{3}}$$

$$= 2 \left(6^{\frac{19}{9}} \right) \quad [M1]$$

$$\implies 2x = \frac{19}{27}$$

$$\therefore x = \frac{19}{54} \quad [M1]$$

$$\sqrt[19]{2^{6x}} = \sqrt[19]{2^3 \times \frac{19}{54}}$$

$$= \left(2^{\frac{19}{9}} \right)^{\frac{1}{9}}$$

$$= 2^{\frac{1}{9}}$$

$$= \sqrt[9]{2} \quad [M1]$$

3. [Suggested Time: 15 mins | Total Marks: 10 | Intermediate]

The first three terms of the expansion $(2-x) \left(2 + \frac{x^2}{4}\right)^n (4+4x+x^2)$ is $a + ax^2 - bx^3$, where $n \in \mathbb{Z}^+, n \geq 2$. Find the values of a, b, n . [10]

$$(2-x)(4+4x+x^2) = (8-x^3) \quad [\text{M1}]$$

$$\therefore (2-x) \left(2 + \frac{x^2}{4}\right)^n (4+4x+x^2) = (8-x^3) \left(2 + \frac{x^2}{4}\right)^n \quad [\text{M1}]$$

Finding the first 2 terms of $\left(2 + \frac{x^2}{4}\right)^n$,

$$\left(2 + \frac{x^2}{4}\right)^n = \sum_{r=0}^n \binom{n}{r} (2)^{n-r} \left(\frac{x^2}{4}\right)^r \quad [\text{M1}]$$

$$\begin{aligned} &= \sum_{r=0}^n \binom{n}{r} (2)^{n-3r} (x)^{2r} \\ &= 2^n + 2^{n-3}(n)x^2 + \dots \end{aligned} \quad [\text{M1}]$$

Comparing coeff. of const. term

$$8(2^n) = a \quad (1) \quad [\text{M1}]$$

Comparing coeff. of x^2

$$8n(2^{n-3}) = a \quad (2) \quad [\text{M1}]$$

Sub (1) into (2),

$$8(2^n) = 8n(2^{n-3}) \quad [\text{M1}]$$

$$2^n - \frac{n}{8}(2^n) = 0$$

$$2^n \left(1 - \frac{n}{8}\right) = 0$$

$$\therefore 2^n = 0 \quad \text{or} \quad \frac{n}{8} = 1$$

(Rej.)

$$\text{Hence, } n = 8 \quad [\text{A1}]$$

Sub $n = 8$ into (1),

$$\begin{aligned} \therefore a &= 8(2^8) \\ &= 2048 \end{aligned} \quad [\text{A1}]$$

Comparing coeff. of x^3

$$-b = -2^8$$

$$\therefore b = 256 \quad [\text{A1}]$$

Therefore, $n = 8, a = 2048, b = 256$

4. [Suggested Time: 23 mins | Total Marks: 15 | Intermediate]

Find all the angles between 0 to 2π inclusive which satisfies

$$\left(\sqrt{3}\tan(\pi\theta) + 2\right)\frac{1}{\sqrt{3}}\sin(e\theta) + \cos(e\theta) - 1 = 1$$

Leave your answers in exact values.

[15]

Ranges

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \pi\theta \leq 2\pi^2$$

[M1]

$$0 \leq e\theta \leq 2e\pi$$

$$\implies \frac{\pi}{3} \leq e\theta + \frac{\pi}{3} \leq 2e\pi + \frac{\pi}{3}$$

[M1]

Splitting the equation

$$\left(\sqrt{3}\tan(\pi\theta) + 2\right)\frac{1}{\sqrt{3}}\sin(e\theta) + \cos(e\theta) - 1 = 1$$

$$\ln\left(\sqrt{3}\tan(\pi\theta) + 2\right)\frac{1}{\sqrt{3}}\sin(e\theta) + \cos(e\theta) - 1 = \ln 1$$

[M1]

$$\left(\frac{1}{\sqrt{3}}\sin(e\theta) + \cos(e\theta) - 1\right)\ln\left(\sqrt{3}\tan(\pi\theta) + 2\right) = 0$$

$$\therefore \frac{1}{\sqrt{3}}\sin(e\theta) + \cos(e\theta) - 1 = 0 \text{ or } \sqrt{3}\tan(\pi\theta) + 2 = 1$$

[M1]

Solving the Second sub-equation

$$\sqrt{3}\tan(\pi\theta) + 2 = 1$$

$$\tan(\pi\theta) = -\frac{1}{\sqrt{3}}$$

[M1]

$$\text{ref.}\angle = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6} \text{ rad}$$

[M1]

$$\tan(\pi\theta) < 0 \implies \pi\theta \text{ is in quad 2 or 4,}$$

$$\implies \pi\theta = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}, \frac{29\pi}{6}, \frac{35\pi}{6}$$

$$\therefore \theta = \frac{5}{6}, \frac{11}{6}, \frac{17}{6}, \frac{23}{6}, \frac{29}{6}, \frac{35}{6} \text{ rad}$$

[M2]

Solving the First sub-equation

$$\frac{1}{\sqrt{3}} \sin(e\theta) + \cos(e\theta) - 1 = 0$$

$$\text{Let } R \sin(e\theta + \alpha) = \frac{1}{\sqrt{3}} \sin(e\theta) + \cos(e\theta) \quad [\text{M1}]$$

$$R \sin(\alpha) = 1$$

$$R \cos(\alpha) = \frac{1}{\sqrt{3}}$$

$$\begin{aligned} R &= \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1} \\ &= \sqrt{\frac{4}{3}} \\ &= \frac{2}{\sqrt{3}} \end{aligned} \quad [\text{M1}]$$

$$\begin{aligned} \tan(\alpha) &= \frac{1}{\frac{1}{\sqrt{3}}} \\ &= \sqrt{3} \end{aligned}$$

$$\alpha = \tan^{-1} \sqrt{3}$$

$$= \frac{\pi}{3} \text{ rad} \quad [\text{M1}]$$

$$\therefore \frac{1}{\sqrt{3}} \sin e\theta + \cos e\theta = \frac{2}{\sqrt{3}} \sin\left(e\theta + \frac{\pi}{3}\right)$$

$$\implies \frac{2}{\sqrt{3}} \sin\left(e\theta + \frac{\pi}{3}\right) = 1$$

$$\sin\left(e\theta + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \quad [\text{M1}]$$

$$\text{ref. } \angle = \sin^{-1} \frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{3} \text{ rad} \quad [\text{M1}]$$

$$\sin\left(e\theta + \frac{\pi}{3}\right) > 0 \implies \left(e\theta + \frac{\pi}{3}\right) \text{ is in quad 1 or 2}$$

$$e\theta + \frac{\pi}{3} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}, \frac{13\pi}{3}, \frac{14\pi}{3}, \frac{\pi}{3}$$

$$\implies \theta = 0, \frac{\pi}{3e}, \frac{2\pi}{e}, \frac{7\pi}{3e}, \frac{4\pi}{e}, \frac{13\pi}{3e}, \frac{\pi}{3} \text{ rad} \quad [\text{M2}]$$

Values of θ that satisfies the equation are $0, \frac{5}{6}, \frac{11}{6}, \frac{17}{6}, \frac{23}{6}, \frac{29}{6}, \frac{35}{6}, \frac{\pi}{3e}, \frac{2\pi}{e}, \frac{7\pi}{3e}, \frac{4\pi}{e}, \frac{13\pi}{3e} \text{ rad}$

5. [Suggested Time: 25 mins | Total Marks: 16 | Intermediate]

(i) Prove the following:

$$(a) \cos^4 \theta - \sin^4 \theta = \cos 2\theta \quad [2]$$

$$LHS = (\cos^2 \theta - \sin^2 \theta)(\cos^2 \theta + \sin^2 \theta) \quad [M1]$$

$$= \cos^2 \theta - \sin^2 \theta \quad [M1]$$

$$= \cos 2\theta \quad [A.G.]$$

$$= RHS \quad \blacksquare$$

$$(b) \cos^4 \theta + \sin^4 \theta = \frac{3}{4} + \frac{1}{4} \cos 4\theta \quad [4]$$

$$LHS = (\cos^2 \theta + \sin^2 \theta)^2 - 2 \sin^2 \theta \cos^2 \theta \quad [M1]$$

$$= 1 - 2(\sin \theta \cos \theta)^2 \quad [M1]$$

$$= 1 - 2 \left(\frac{\sin 2\theta}{2} \right)^2 \quad [M1]$$

$$= 1 - \frac{1}{2} \sin^2 2\theta$$

$$= 1 - \frac{1}{2} \left(\frac{1 - \cos 4\theta}{2} \right) \quad [M1]$$

$$= 1 - \frac{1}{4} - \frac{1}{4} \cos 4\theta$$

$$= \frac{3}{4} + \frac{1}{4} \cos 4\theta \quad [A.G.]$$

$$= RHS \quad \blacksquare$$

(ii) Hence or otherwise, and **leaving your answers in exact values**, evaluate

$$(a) \int_0^\pi \cos^4 \theta \, d\theta \quad [5]$$

From (i)

$$\begin{aligned} \cos^4 \theta - \cos 2\theta &= \frac{3}{4} + \frac{1}{4} \cos 4\theta - \cos^4 \theta \\ 2 \cos^4 \theta &= \frac{3}{4} + \cos 2\theta + \frac{1}{4} \cos 4\theta \end{aligned} \quad [M1]$$

$$\therefore \int_0^\pi \cos^4 \theta \, d\theta = \frac{1}{2} \int_0^\pi \left(\frac{3}{4} + \cos 2\theta + \frac{1}{4} \cos 4\theta \right) d\theta \quad [M1]$$

$$= \frac{1}{2} \left[\frac{3}{4} \theta + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{16} \right]_0^\pi \quad [M2]$$

$$\begin{aligned} &= \frac{1}{2} \left(\left(\frac{3\pi}{4} + 0 + 0 \right) - 0 \right) \\ &= \frac{3\pi}{8} \end{aligned} \quad [M1]$$

$$(b) \int_0^\pi \sin^4 \theta \, d\theta \quad [5]$$

From (i)

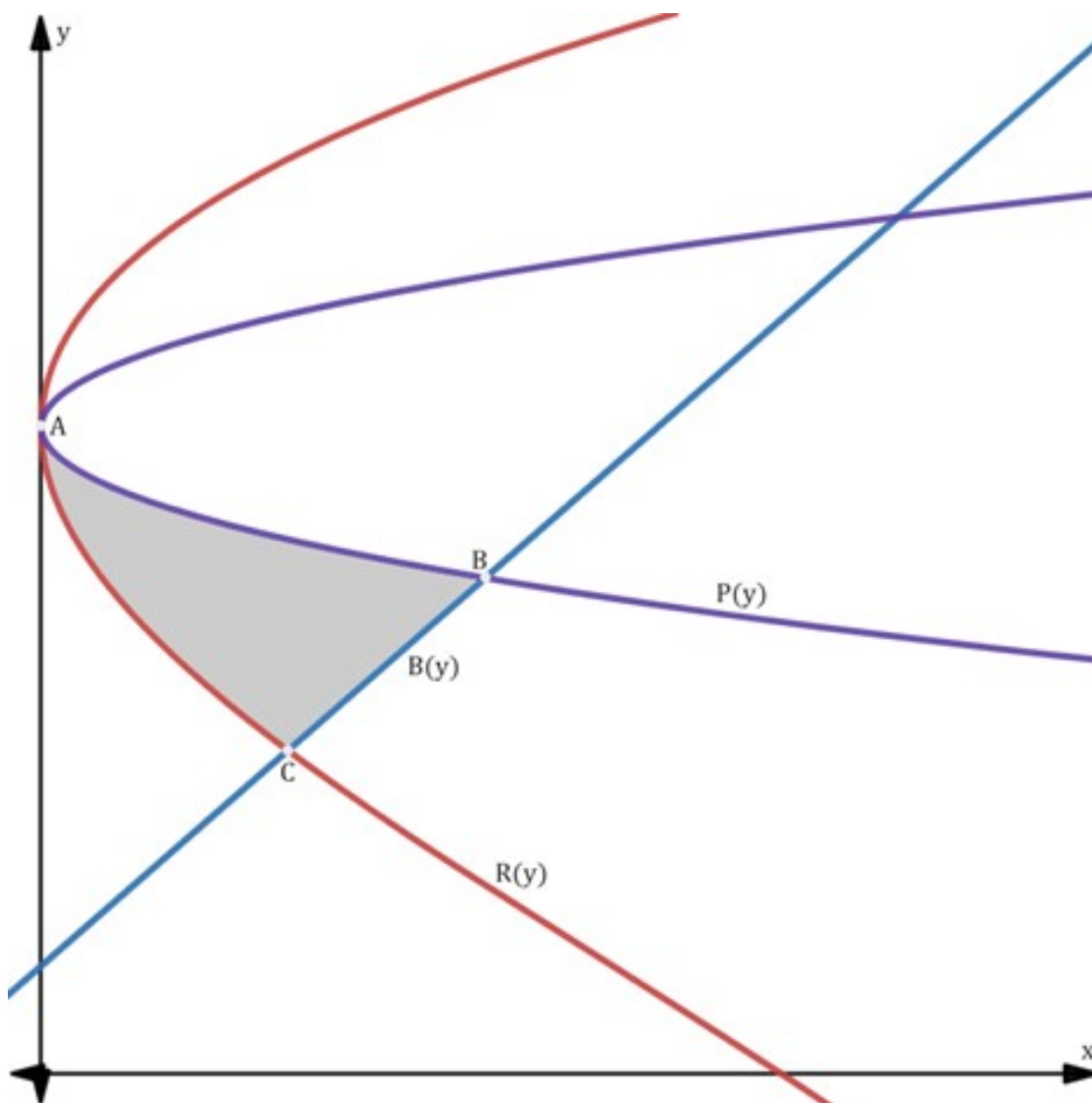
$$\begin{aligned} \sin^4 \theta + \cos 2\theta &= \frac{3}{4} + \frac{1}{4} \cos 4\theta - \sin^4 \theta \\ 2 \sin^4 \theta &= \frac{3}{4} - \cos 2\theta + \frac{1}{4} \cos 4\theta \end{aligned} \quad [M1]$$

$$\therefore \int_0^\pi \sin^4 \theta \, d\theta = \frac{1}{2} \int_0^\pi \left(\frac{3}{4} - \cos 2\theta + \frac{1}{4} \cos 4\theta \right) d\theta \quad [M1]$$

$$= \frac{1}{2} \left[\frac{3}{4} \theta - \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{16} \right]_0^\pi \quad [M2]$$

$$\begin{aligned} &= \frac{1}{2} \left(\left(\frac{3\pi}{4} + 0 + 0 \right) - 0 \right) \\ &= \frac{3\pi}{8} \end{aligned} \quad [M1]$$

6. [Suggested Time: 40 mins | Total Marks: 20 | Challenging]



Answers by Accurate drawings or graphical methods are not accepted.

The graphs are plotting y against x . Point A is a common stationary point of $R(y)$ and $P(y)$. $B(y)$ passes through the points $D(10, 2)$ and $E(1, \frac{1}{2})$.

$$P(y) = 33(y - 2)^2, \quad B(y) = R''(y)$$

Degree of polynomial $R(y)$ is 3.

Find the shaded area.

Leave your answers in exact values, in the form $a + b\sqrt{339}$, where $a, b \in \mathbb{Q}$ [20]

Degree of $R(y)$ is 3 \implies Degree of $B(y)$ is 1, therefore $B(y)$ is a linear function

Finding the Equation of $B(y)$

$$y - 2 = \frac{1}{6}(x - 10)$$

$$6y - 12 = x - 10$$

$$x = 6y - 2$$

$$\therefore B(y) = 6y - 2 \quad \text{[M1]}$$

Finding the coordinates of point A

$$P'(y) = 66(y - 2)$$

When $P(y) = 0$

$$66(y - 2) = 0$$

$$\therefore y = 2 \quad \text{[M1]}$$

Sub $y = 2$ into $P(y)$

$$\begin{aligned} \therefore x &= 33(2 - 2)^2 \\ &= 0 \end{aligned}$$

$$\implies A = (0, 2) \quad \text{[M1]}$$

Finding the equation of $R(y)$

Since $B(y) = R''(y)$

$$\begin{aligned} R'(y) &= \int B(y) \, dy \\ &= 3y^2 - 2y + c_1, \text{ where } c_1 \in \mathbb{R} \end{aligned} \quad \text{[M1]}$$

Point A is a stationary point of $R(y) \implies R'(2) = 0$

$$3(2)^2 - 2(2) + c_1 = 0$$

$$c_1 = -8 \quad \text{[M1]}$$

$$\therefore R'(y) = 3y^2 - 2y - 8$$

$$\begin{aligned}
 R(y) &= \int R'(y) \, dy \\
 &= y^3 + y^2 - 8y + c_2, \text{ where } c_2 \in \mathbb{R}
 \end{aligned}
 \tag{M1}$$

When $x = 0$, $y = 2$,

$$\begin{aligned}
 R(2) &= 0 \\
 2^3 - 22 - 8 + c_2 &= 0 \\
 \therefore c_2 &= 12 \\
 \implies R(y) &= y^3 - y^2 - 8 + 12
 \end{aligned}
 \tag{M1}$$

Given that Point C is an intercept of $R(y)$ and $B(y)$

$$\begin{aligned}
 R(y) &= B(y) \\
 y^3 - y^2 - 8y + 12 &= 6y - 2 \\
 y^3 - y^2 - 14y + 14 &= 0
 \end{aligned}
 \tag{M1}$$

$$\begin{aligned}
 \text{Let } f(x) &= y^3 - y^2 - 14y + 14 \\
 f(1) &= 0
 \end{aligned}$$

$\therefore (y - 1)$ is a factor of $f(x)$ (M1)

$$\begin{aligned}
 f(x) &= (y - 1)(y^2 - 14) \\
 &= 0
 \end{aligned}$$

$$y^2 = 14$$

$$y = \pm\sqrt{14}$$

$$\therefore y = 1 \text{ or } \pm\sqrt{14}$$

$$\text{Rej. } y = \pm\sqrt{14}$$

$$\implies y = 1 \tag{M1}$$

Sub. $y = 1$ into $B(y)$,

$$\begin{aligned}
 B(1) &= 6(1) - 2 \\
 &= 4
 \end{aligned}$$

$$\therefore C = (4, 1) \tag{M1}$$

Given that Point B is an intercept of $P(y)$ and $B(y)$

$$6y - 2 = 33(y - 2)^2 \quad [\text{M1}]$$

$$= 33y^2 - 132y + 132$$

$$33y^2 - 138y + 134 = 0$$

$$y = \frac{138 \pm \sqrt{(-138)^2 - 4(33)(134)}}{2(33)}$$

$$= \frac{138 \pm \sqrt{1356}}{2(33)}$$

$$= \frac{69 \pm \sqrt{339}}{33} \quad [\text{M1}]$$

$$\text{Reject } y = \frac{69 + \sqrt{339}}{33}, \therefore y = \frac{69 - \sqrt{339}}{33}$$

$$\text{Sub } y = \frac{69 - \sqrt{339}}{33} \text{ into } B(y)$$

$$B\left(\frac{69 - \sqrt{339}}{33}\right) = 6\left(\frac{69 - \sqrt{339}}{33}\right) - 2$$

$$= \frac{2(69 - \sqrt{339}) - 22}{11}$$

$$= \frac{116 - 2\sqrt{339}}{11}$$

$$\therefore B = \left(\frac{116 - 2\sqrt{339}}{11}, \frac{69 - \sqrt{339}}{33}\right) \quad [\text{M1}]$$

Let S be the shaded area and $\alpha = \frac{69 - \sqrt{339}}{33}$

$$S = \int_{\alpha}^2 P(y) dy + \frac{1}{2} \left(\frac{69 - \sqrt{339}}{33} - 1 \right) \left(\frac{116 - 2\sqrt{339}}{11} + 4 \right) - \int_1^2 R(y) dy$$

$$= \int_{\alpha}^2 33(y - 2)^2 dy + \frac{1}{2} \left(\frac{69 - \sqrt{339}}{33} - 1 \right) \left(\frac{116 - 2\sqrt{339}}{11} + 4 \right) - \int_1^2 y^3 - y^2 - 8y + 12 dy$$

[M1]

$$= [11(y - 2)^3]_{\alpha}^2 + \frac{1}{2} \left(\frac{36 - \sqrt{339}}{33} \right) \left(\frac{160 - 2\sqrt{339}}{11} + 4 \right) - \left[\frac{y^4}{4} - \frac{y^3}{3} - 4y^2 + 12y \right]_1^2$$

[M1]

$$= -11 \left(\frac{69 - \sqrt{339}}{33} - 2 \right)^3 + \frac{(36 - \sqrt{339})(160 - 2\sqrt{339})}{726}$$

$$- \left(\frac{2^4}{4} - \frac{2^3}{3} - 4(2)^2 + 12(2) - \left(\frac{1}{4} - \frac{1}{3} - 4 + 12 \right) \right)$$

[M1]

$$= -11 \left(\frac{3 - \sqrt{339}}{33} \right)^3 + \frac{6438 - 232\sqrt{339}}{726} - \frac{17}{12}$$

$$= \frac{-44(3078 - 366\sqrt{339}) + 99(12029 - 464\sqrt{339})}{143748}$$

$$= \frac{1055439 - 62040\sqrt{339}}{143748}$$

$$= \frac{10661}{1452} - \frac{470}{1089}\sqrt{339}$$

$$\therefore \text{the shaded area is } \left(\frac{10661}{1452} - \frac{470}{1089}\sqrt{339} \right) u^2$$

[M1]

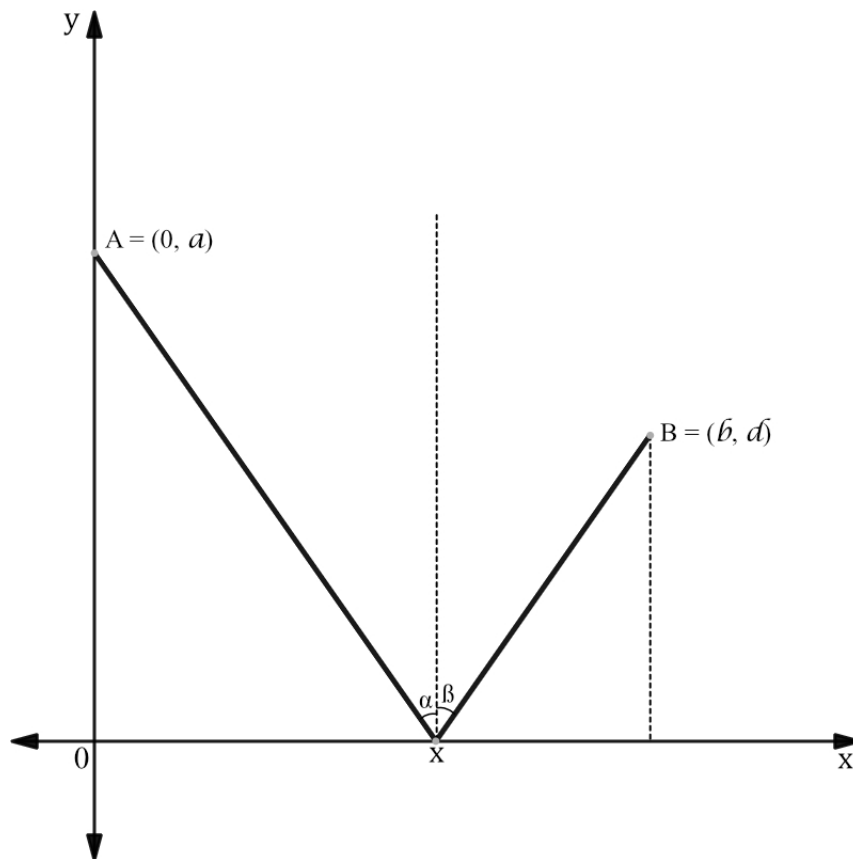
7. [Suggested Time: 20 mins | Total Marks: 10 | Challenging]

Fermat's Principle of Least Time

Fermat's Principle of Least Time states that out of all neighbouring paths available, light travels between two points along the path that requires the least time.

Consider a light ray from a source which strikes a mirror and is reflected. Let A be a point on the ray before it strikes the mirror and B be the point on the ray after reflection. v m/s is the speed of light.

A coordinate system is placed in a plane such that the x -axis runs along the mirror's surface and the point A lies on the y -axis.



Prove that the angle of incidence α is equal to the angle of reflection β .

(Proof that T is minimum is not required)

[10]

Note that $\tan \alpha = \frac{x}{a}$ and $\tan \beta = \frac{b-x}{d}$

$$\begin{aligned} T &= \frac{\text{Distance}}{\text{Speed}} \\ &= \frac{|Ax|}{v} + \frac{|Bx|}{v} \end{aligned} \quad \text{[M1]}$$

$$= \frac{1}{v} \sqrt{a^2 + x^2} + \frac{1}{v} \sqrt{d^2 + (b-x)^2} \quad \text{[M1]}$$

$$= \frac{1}{v} \left(\sqrt{a^2 + x^2} + \sqrt{d^2 + (b-x)^2} \right)$$

$$\therefore \frac{dT}{dx} = \frac{1}{v} \left(\frac{x}{\sqrt{a^2 + x^2}} - \frac{b-x}{\sqrt{d^2 + (b-x)^2}} \right) \quad \text{[M2]}$$

$$= \left(\frac{x\sqrt{d^2 + (b-x)^2} - (b-x)\sqrt{a^2 + x^2}}{v\sqrt{(a^2 + x^2)(d^2 + (b-x)^2)}} \right) \quad \text{[M1]}$$

For minimum T ,

$$\frac{dT}{dx} = 0$$

$$\frac{x\sqrt{d^2 + (b-x)^2} - (b-x)\sqrt{a^2 + x^2}}{v\sqrt{(a^2 + x^2)(d^2 + (b-x)^2)}} = 0 \quad \text{[M1]}$$

$$x\sqrt{d^2 + (b-x)^2} - (b-x)\sqrt{a^2 + x^2} = 0 \quad \text{[M1]}$$

$$x\sqrt{d^2 + (b-x)^2} = (b-x)\sqrt{a^2 + x^2}$$

$$x^2(d^2 + (b-x)^2) = (b-x)^2(a^2 + x^2) \quad \text{[M1]}$$

$$x^2d^2 + x^2(b-x)^2 - a^2(b-x)^2 - x^2(b-x)^2 = 0$$

$$x^2d^2 - a^2(b-x)^2 = 0$$

$$x^2d^2 = a^2(b-x)^2$$

$$xd = a(b-x)$$

$$\frac{x}{a} = \frac{b-x}{d}$$

$$\tan \alpha = \tan \beta \quad \text{[M1]}$$

Since both α & β are acute,

$$\implies \alpha = \beta$$

■

[M1]

8. [Suggested Time: 25 mins | Total Marks: 12 | Challenging]

A soft drink company wants to design a 500ml soda can using the least amount of material for their new Moon Shine soda. Assume that the soda can is a perfect cylinder. Using the information given below, find the cheapest material cost to produce 1000 cans. **Leave your answers in USD**

[12]

| Information List | |
|-------------------------|--|
| Price of Aluminium | 2515 USD/Metric Tonne |
| Density of Aluminium | 2.7 g/cm^3 |
| 1 Atmosphere (Pressure) | 101325 Pa |
| Room Temperature | $T_c = 28^\circ \text{C}$ |
| Temperature (Kelvin) | $T_k = T_c + 273 \text{ K}$ |
| Gas Constant (R) | $R = 8.314 \text{ m}^3 \text{ Pa/mol K}$ |

| Ideal Gas Law |
|--|
| $PV = nRT$ P – Pressure V – Volume n – Amount of Substance R – Gas Constant T – Temperature |

The Soda can is able to tolerate up to 5 atm of pressure, and has a uniform thickness of 0.01 cm. The soda releases up to $\frac{x}{10} \text{ cm}^3$ of gas for every $x \text{ ml}$ of soda under room temperature and pressure.

Let S_c & V_c be the total surface area and volume of the can respectively

$$S_c = 2\pi rh + 2\pi r^2$$

$$V_c = \pi r^2 h$$

Room temperature and pressure $\implies P = 1 \text{ atm}, T_c = 28^\circ\text{C}$

Thus, using the Ideal gas law, the number of moles of gas is

$$\begin{aligned} n &= \frac{PV}{RT} \\ &= \frac{101325 \left(\frac{50}{100^3} \right)}{8.314(273 + 28)} \\ &= 2.0245 \times 10^{-3} \end{aligned} \quad \text{[M1]}$$

Since the max pressure that the can is able to handle is 5 atm,
min volume taken up by gas is

$$\begin{aligned} V &= \frac{nRT}{P} \\ &= \frac{2.0245 \times 10^{-3}(8.314 \times (273 + 28))}{5(101325)} \\ &= 1 \times 10^{-5} \text{ m}^3 \\ &= 10 \text{ cm}^3 \end{aligned} \quad \text{[M1]}$$

Thus, $V_c = 500 + 10 = 510 \text{ cm}^3$

$$510 = \pi r^2 h$$

$$h = \frac{510}{\pi r^2} \quad \text{[M1]}$$

$$\begin{aligned} \therefore S_c &= 2\pi r \left(\frac{510}{\pi r^2} \right) + 2\pi r^2 \\ &= \frac{1020}{r} + 2\pi r^2 \\ \frac{dS_c}{dr} &= 4\pi r - \frac{1020}{r^2} \end{aligned} \quad \text{[M1]}$$

$$\text{When } \frac{dS_c}{dr} = 0$$

$$4\pi r - \frac{1020}{r^2} = 0 \quad [\text{M1}]$$

$$\frac{4\pi r^3 - 1020}{r^3} = 0$$

$$4\pi r^3 - 1020 = 0$$

$$\therefore r^3 = \frac{1020}{4\pi}$$

$$r = \sqrt[3]{\frac{255}{\pi}} \approx 4.3298 \text{ cm} \quad [\text{M1}]$$

$$\frac{d^2 S_c}{dr^2} = 4\pi + \frac{2040}{r^3}$$

$$\left. \frac{d^2 S_c}{dr^2} \right|_{r = \sqrt[3]{\frac{300}{\pi}}} = 4\pi + \frac{2040\pi}{255}$$

$$> 0 \quad [\text{M1}]$$

$$\therefore r = \sqrt[3]{\frac{255}{\pi}} \text{ gives a min. amount of aluminium used}$$

$$V_{alu} = \pi \left(0.01 + \sqrt[3]{\frac{255}{\pi}} \right)^2 \left(\frac{510}{\pi \left(\sqrt[3]{\frac{255}{\pi}} \right)^2} + 0.02 \right) - 510$$

$$\approx 3.5419 \text{ cm}^2 \quad [\text{M1}]$$

$$\text{Mass of 1 Can} = 3.5419(2.7)$$

$$= 9.56313g \quad [\text{M1}]$$

$$\text{Mass of 1k Can} = 9.56313(1000)$$

$$= 9563.13g$$

$$\approx 9.5631kg \quad [\text{M1}]$$

$$\text{Price of Aluminium in kg} = \frac{2515}{1000}$$

$$= 2.515 \text{ USD/kg} \quad [\text{M1}]$$

$$\therefore \text{Material Cost to produce 1000 Cans} = 9.5631(2.515)$$

$$\approx 24.05 \text{ USD} \quad [\text{A1}]$$

9. [Suggested Time: N.A. | Total Marks: N.A. | Schadenfreude]

Given that if x satisfies $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for some integers a_{n-1}, \cdots, a_0 , then x is irrational unless x is an integer.

Prove that $\sqrt{2} + \sqrt[3]{2}$ is irrational.

Let $x = \sqrt{2} + \sqrt[3]{2}$. Now we see that:

$$(x - \sqrt{2})^3 = 2$$

$$x^3 - 3x^2\sqrt{2} + 6x - 2\sqrt{2} = 2$$

$$x^3 + 6x - 2 = \sqrt{2}(3x^2 - 2)$$

$$(x^3 + 6x - 2)^2 = 2(3x^2 - 2)^2$$

$$x^6 - 6x^4 - 4x^3 + 36x^2 - 24x + 4 = 2(9x^4 + 12x^2 + 4)$$

$$x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4 = 0$$

Therefore, we have found a monic polynomial with integer coefficients which has $\sqrt{2} + \sqrt[3]{2}$ as its root, meaning by the equation given, $\sqrt{2} + \sqrt[3]{2}$ must either be an integer or irrational number. It is trivial that $\sqrt{2} + \sqrt[3]{2}$ is not an integer, hence it has to be an irrational number.

10. [Suggested Time: 25 mins | Total Marks: 10 | Extensions]

(i) Derive that the effective resistance, R_{eff} , of n parallel resistors is $R_{eff} = \left(\sum_{i=1}^n \frac{1}{R_i} \right)^{-1}$ [5]

Total current in a parallel circuit, $I_T = \sum_{i=1}^n I_i$ (KCL)

Voltage in a parallel circuit $V_T = V_1 = V_2 = V_3 \dots$ [M1]

By Ohm's Law, $V = IR$, it is trivial that

$$I_1 = \frac{V_1}{R_1}, I_2 = \frac{V_2}{R_2}, \dots, I_n = \frac{V_n}{R_n} \quad [M1]$$

Note that since $V_T = V_1 = V_2 = V_3 \dots$, we can rewrite the equations above as

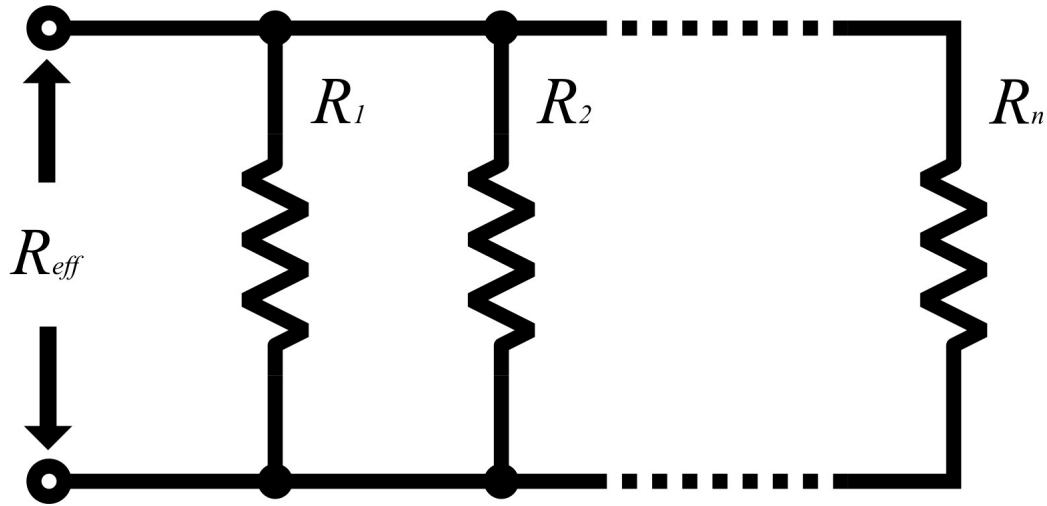
$$I_1 = \frac{V_T}{R_1}, I_2 = \frac{V_T}{R_2}, \dots, I_n = \frac{V_T}{R_n} \quad [M1]$$

Using KCL and the equations that we had derived, we can see that

$$\begin{aligned} I_T &= \sum_{i=1}^n \frac{V_T}{R_i} \\ &= V_T \sum_{i=1}^n \frac{1}{R_i} \end{aligned} \quad [M1]$$

By Ohm's law,

$$\begin{aligned} R_{eff} &= \frac{V_T}{I_T} \\ &= \frac{V_T}{V_T \sum_{i=1}^n \frac{1}{R_i}} \\ &= \frac{1}{\sum_{i=1}^n \frac{1}{R_i}} \\ &= \left(\sum_{i=1}^n \frac{1}{R_i} \right)^{-1} \quad \blacksquare \end{aligned} \quad [M1]$$



(ii)(a) Hence or otherwise, find the effective resistance of the circuit shown above.

Where $R_1 = \sqrt{1} + \sqrt{2}$, $R_2 = \sqrt{2} + \sqrt{3}$, $R_3 = \sqrt{3} + \sqrt{4} \dots$

Leave your answers in exact values, and in terms of n . [4]

We can see that the resistance of the resistors follow a series with a general term of

$$R_n = \sqrt{n} + \sqrt{n+1} \quad [\text{M1}]$$

Using the equation in part (i),

$$\begin{aligned} R_{eff} &= \left(\sum_{\alpha=1}^n \frac{1}{\sqrt{\alpha} + \sqrt{\alpha+1}} \right)^{-1} \\ &= \left(\sum_{\alpha=1}^n \frac{\sqrt{\alpha} - \sqrt{\alpha+1}}{\alpha - \alpha - 1} \right)^{-1} \\ &= \left(\sum_{\alpha=1}^n \left(\sqrt{\alpha+1} - \sqrt{\alpha} \right) \right)^{-1} \end{aligned} \quad [\text{M1}]$$

Evaluating the sum

$$\begin{aligned} \sum_{\alpha=1}^n \left(\sqrt{\alpha+1} - \sqrt{\alpha} \right) &= \left(\sqrt{2} - \sqrt{1} \right) + \\ &\quad \left(\sqrt{3} - \sqrt{2} \right) + \\ &\quad \left(\sqrt{4} - \sqrt{3} \right) + \\ &\quad \vdots \\ &\quad \left(\sqrt{n+1} - \sqrt{n} \right) \\ &= \sqrt{n+1} - 1 \end{aligned} \quad [\text{M1}]$$

Substituting the sum back into R_{eff}

$$\therefore R_{eff} = \frac{1}{\sqrt{n+1}-1} \quad \text{[M1]}$$

*The effective resistance of the circuit is $\frac{1}{\sqrt{n+1}-1}$,
where n is the number of resistors*

(ii)(b) Explain, with relevant workings, if the effective resistance of the circuit will approach a unique value as more resistors are added into the circuit. [1]

$$\text{from (ii)(a), } R_{eff} = \frac{1}{\sqrt{n+1} - 1}$$

$$\text{As } n \rightarrow \infty, \sqrt{n+1} - 1 \rightarrow \infty$$

$$\frac{1}{\sqrt{n+1} - 1} \rightarrow 0$$

$$\therefore R_{eff} \rightarrow 0$$

[M1]

Yes, R_{eff} approaches the value 0 as more resistors are added into the circuit