FurtherQ
Problem Set 1

Public Edition
Version 2.0

October 2022

Problems

Editions

PB - Public Edition
SS - Secondary School Edition

Question Difficulty

Easy - Warm up
Intermediate - Standard
Challenging - Difficult
Schadenfreude - Interesting
Extensions - Out of syllabus

Organizing Team

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Paper Info

Total Time: 180 Mins (3 hrs)
Graded Questions: 1-8 & 10
Total Marks: 105

Attempted	/9
Grade	
Marks	/105
Percentage	%

1. [**Suggested Time: 7 mins | Total Marks: 7 | Easy**]
A function f is defined by $f : x \mapsto \sqrt{2}\,(\sin x + \cos x + 1)$ for $0 \le x \le 2\pi$
Points P and Q are the maximum and minimum points of f respectively,
Without the use of calculus, find the coordinates P and Q
Leave your answers in exact values. [7]

$$Let\ y = \sqrt{2}\sin x + \sqrt{2}\cos x + \sqrt{2}$$

$$Let\ R\sin(x+\alpha) = \sqrt{2}\sin x + \sqrt{2}\cos x$$ [M1]

$$R\sin\alpha = \sqrt{2}$$
$$R\cos\alpha = \sqrt{2}$$

$$R = \sqrt{2+2}$$
$$= 2$$ [M1]

$$\alpha = \tan^{-1} 1$$
$$= \frac{\pi}{4}$$ [M1]

$$\implies \sqrt{2}\sin x + \sqrt{2}\cos x = 2\sin\left(x + \frac{\pi}{4}\right)$$
$$\therefore y = 2\sin\left(x + \frac{\pi}{4}\right) + \sqrt{2}$$

$$max(y) = 2 + \sqrt{2}$$ [M1]

When $\sin\left(x + \frac{\pi}{4}\right) = 1$

$$\therefore x + \frac{\pi}{4} = \frac{\pi}{2}$$
$$x = \frac{\pi}{4}$$

$$\implies P = \left(\frac{\pi}{4}, 2 + \sqrt{2}\right)$$ [A1]

$$min(y) = \sqrt{2} - 2$$ [M1]

When $\sin\left(x + \frac{\pi}{4}\right) = -1$

$$\therefore x + \frac{\pi}{4} = \frac{3\pi}{2}$$
$$x = \frac{5\pi}{4}$$

$$\implies Q = \left(\frac{5\pi}{4}, \sqrt{2} - 2\right)$$ [A1]

2. [**Suggested Time: 5 mins | Total Marks: 5 | Easy**]
Without the use of a calculator, Solve the following equation
$6^{2x} + 36^x = 2\sqrt[3]{6\sqrt[3]{216\sqrt[3]{6}}}$
Hence find the value of $\sqrt[19]{2^{6x}}$ [5]

$$6^{2x} + 6^{2x} = 2\left(6\left(6^3(6)^{\frac{1}{3}}\right)^{\frac{1}{3}}\right)^{\frac{1}{3}}$$

$$2\left(6^{2x}\right) = 2\left(6\left(6^{\frac{10}{3}}\right)^{\frac{1}{3}}\right)^{\frac{1}{3}}$$ [M1]

$$= 2\left(6^{1+\frac{10}{9}}\right)^{\frac{1}{3}}$$

$$= 2\left(6^{\frac{19}{27}}\right)$$ [M1]

$$\implies 2x = \frac{19}{27}$$

$$\therefore x = \frac{19}{54}$$ [M1]

$$\sqrt[19]{2^{6x}} = \sqrt[19]{2^{3\times\frac{19}{54}}}$$ [M1]

$$= \left(2^{\frac{19}{9}}\right)^{\frac{1}{9}}$$

$$= 2^{\frac{1}{9}}$$

$$= \sqrt[9]{2}$$ [A1]

3. [**Suggested Time: 15 mins | Total Marks: 10 | Intermediate**]
The first three terms of the expansion $(2-x)\left(2+\frac{x^2}{4}\right)^n\left(4+2x+x^2\right)$ is $a+ax^2-bx^3$, where $n \in \mathbb{Z}^+, n \geq 2$. Find the values of a, b, n. [10]

$(2-x)\left(4+2x+x^2\right) = \left(8-x^3\right)$ [M1]

$\therefore (2-x)\left(2+\frac{x^2}{4}\right)^n\left(4+2x+x^2\right) = \left(8-x^3\right)\left(2+\frac{x^2}{4}\right)^n$ [M1]

Finding the first 2 terms of $\left(2+\frac{x^2}{4}\right)^n$,

$$\left(2+\frac{x^2}{4}\right)^n = \sum_{r=0}^{n}\binom{n}{r}(2)^{n-r}\left(\frac{x^2}{4}\right)^r \quad \text{[M1]}$$

$$= \sum_{r=0}^{n}\binom{n}{r}(2)^{n-3r}(x)^{2r}$$

$$= 2^n + 2^{n-3}(n)x^2 + \cdots \quad \text{[M1]}$$

Comparing coeff. of const. term

$8\left(2^n\right) = a \qquad (1)$

[M1]

Comparing coeff. of x^2

$8n\left(2^{n-3}\right) = a \qquad (2)$

[M1]

Sub (1) into (2),

$8\left(2^n\right) = 8n\left(2^{n-3}\right)$ [M1]

$2^n - \frac{n}{8}\left(2^n\right) = 0$

$2^n\left(1-\frac{n}{8}\right) = 0$

$\therefore 2^n = 0 \quad or \quad \frac{n}{8} = 1$

(*Rej.*)

Hence, $n = 8$ [A1]

Sub $n = 8$ into (1),

$\therefore a = 8\left(2^8\right)$

$= 2048$ [A1]

Comparing coeff. of x^3

$-b = -2^8$

$\therefore b = 256$ [A1]

Therefore, $n = 8, a = 2048, b = 256$

4. [**Suggested Time: 23 mins | Total Marks: 15 | Intermediate**]
Find all the angles between 0 to 2π inclusive which satisfies
$$\left(\sqrt{3}\tan(\pi\theta)+2\right)^{\frac{1}{\sqrt{3}}\sin(e\theta)+\cos(e\theta)-1}=1$$
Leave your answers in exact values. [15]

Ranges

$0 \leq \theta \leq 2\pi$

$0 \leq \pi\theta \leq 2\pi^2$ [M1]

$0 \leq e\theta \leq 2e\pi$

$\implies \frac{\pi}{3} \leq e\theta + \frac{\pi}{3} \leq 2e\pi + \frac{\pi}{3}$ [M1]

Splitting the equation

$\left(\sqrt{3}\tan(\pi\theta)+2\right)^{\frac{1}{\sqrt{3}}\sin(e\theta)+\cos(e\theta)-1}=1$

$\ln\left(\sqrt{3}\tan(\pi\theta)+2\right)^{\frac{1}{\sqrt{3}}\sin(e\theta)+\cos(e\theta)-1}=\ln 1$ [M1]

$\left(\frac{1}{\sqrt{3}}\sin(e\theta)+\cos(e\theta)-1\right)\ln\left(\sqrt{3}\tan(\pi\theta)+2\right)=0$

$\therefore \frac{1}{\sqrt{3}}\sin(e\theta)+\cos(e\theta)-1=0 \; or \; \sqrt{3}\tan(\pi\theta)+2=1$ [M1]

Solving the Second sub-equation

$\sqrt{3}\tan(\pi\theta)+2=1$

$$\tan(\pi\theta) = -\frac{1}{\sqrt{3}}$$ [M1]

$$ref.\angle = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}\,rad$$ [M1]

$\tan(\pi\theta) < 0 \implies \pi\theta$ is in quad 2 or 4,

$\implies \pi\theta = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}, \frac{29\pi}{6}, \frac{35\pi}{6}$

$\therefore \theta = \frac{5}{6}, \frac{11}{6}, \frac{17}{6}, \frac{23}{6}, \frac{29}{6}, \frac{35}{6}\,rad$ [A2]

Solving the First sub-equation

$$\frac{1}{\sqrt{3}}\sin(e\theta)+\cos(e\theta)-1=0$$

$$Let\ R\sin(e\theta+\alpha)=\frac{1}{\sqrt{3}}\sin(e\theta)+\cos(e\theta)$$ [M1]

$$R\sin(\alpha)=1$$

$$R\cos(\alpha)=\frac{1}{\sqrt{3}}$$

$$\begin{aligned}R&=\sqrt{\left(\tfrac{1}{\sqrt{3}}\right)^2+1}\\&=\sqrt{\frac{4}{3}}\\&=\frac{2}{\sqrt{3}}\end{aligned}$$ [M1]

$$\begin{aligned}\tan(\alpha)&=\frac{1}{\frac{1}{\sqrt{3}}}\\&=\sqrt{3}\\\alpha&=\tan^{-1}\sqrt{3}\\&=\frac{\pi}{3}\ rad\end{aligned}$$ [M1]

$$\therefore\ \frac{1}{\sqrt{3}}\sin e\theta+\cos e\theta=\frac{2}{\sqrt{3}}\sin\left(e\theta+\frac{\pi}{3}\right)$$

$$\implies \frac{2}{\sqrt{3}}\sin\left(e\theta+\frac{\pi}{3}\right)=1$$

$$\sin\left(e\theta+\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$ [M1]

$$ref.\angle=\sin^{-1}\frac{\sqrt{3}}{2}$$

$$=\frac{\pi}{3}\ rad$$ [M1]

$$\sin(e\theta+\frac{\pi}{3})>0\implies\left(e\theta+\frac{\pi}{3}\right)\text{ is in quad 1 or 2}$$

$$e\theta+\frac{\pi}{3}=\frac{\pi}{3},\frac{2\pi}{3},\frac{7\pi}{3},\frac{8\pi}{3},\frac{13\pi}{3},\frac{14\pi}{3},\frac{\pi}{3}$$

$$\implies \theta=0,\frac{\pi}{3e},\frac{2\pi}{e},\frac{7\pi}{3e},\frac{4\pi}{e},\frac{13\pi}{3e},\frac{\pi}{3}\ rad$$ [A2]

Values of θ that satisfies the equation are $0,\frac{5}{6},\frac{11}{6},\frac{17}{6},\frac{23}{6},\frac{29}{6},\frac{35}{6},\frac{\pi}{3e},\frac{2\pi}{e},\frac{7\pi}{3e},\frac{4\pi}{e},\frac{13\pi}{3e}$ rad

5. [**Suggested Time: 25 mins | Total Marks: 16 | Intermediate**]

(i) Prove the following:

(a) $\cos^4\theta - \sin^4\theta = \cos 2\theta$ [2]

$$
\begin{aligned}
LHS &= (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) \\
&= \cos^2\theta - \sin^2\theta && \text{[M1]} \\
&= \cos 2\theta && \text{[A1]} \\
&= RHS && \blacksquare
\end{aligned}
$$

(b) $\cos^4\theta + \sin^4\theta = \dfrac{3}{4} + \dfrac{1}{4}\cos 4\theta$ [4]

$$
\begin{aligned}
LHS &= (\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta \\
&= 1 - 2(\sin\theta\cos\theta)^2 && \text{[M1]} \\
&= 1 - 2\left(\frac{\sin 2\theta}{2}\right)^2 && \text{[M1]} \\
&= 1 - \frac{1}{2}\sin^2 2\theta \\
&= 1 - \frac{1}{2}\left(\frac{1-\cos 4\theta}{2}\right) && \text{[M1]} \\
&= 1 - \frac{1}{4} - \frac{1}{4}\cos 4\theta \\
&= \frac{3}{4} + \frac{1}{4}\cos 4\theta && \text{[A1]} \\
&= RHS && \blacksquare
\end{aligned}
$$

(ii) Hence or otherwise, and **leaving your answers in exact values**, evaluate

(a) $\displaystyle\int_0^{\pi} \cos^4\theta \, d\theta$ [5]

From **(i)**

$$\cos^4\theta - \cos 2\theta = \frac{3}{4} + \frac{1}{4}\cos 4\theta - \cos^4\theta$$

$$2\cos^4\theta = \frac{3}{4} + \cos 2\theta + \frac{1}{4}\cos 4\theta \quad \text{[M1]}$$

$$\therefore \int_0^{\pi} \cos^4\theta \, d\theta = \frac{1}{2}\int_0^{\pi} \left(\frac{3}{4} + \cos 2\theta + \frac{1}{4}\cos 4\theta\right) d\theta \quad \text{[M1]}$$

$$= \frac{1}{2}\left[\frac{3}{4}\theta + \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{16}\right]_0^{\pi} \quad \text{[M2]}$$

$$= \frac{1}{2}\left(\left(\frac{3\pi}{4} + 0 + 0\right) - 0\right)$$

$$= \frac{3\pi}{8}u^2 \quad \text{[A1]}$$

(b) $\displaystyle\int_0^{\pi} \sin^4\theta \, d\theta$ [5]

From **(i)**

$$\sin^4\theta + \cos 2\theta = \frac{3}{4} + \frac{1}{4}\cos 4\theta - \sin^4\theta$$

$$2\sin^4\theta = \frac{3}{4} - \cos 2\theta + \frac{1}{4}\cos 4\theta \quad \text{[M1]}$$

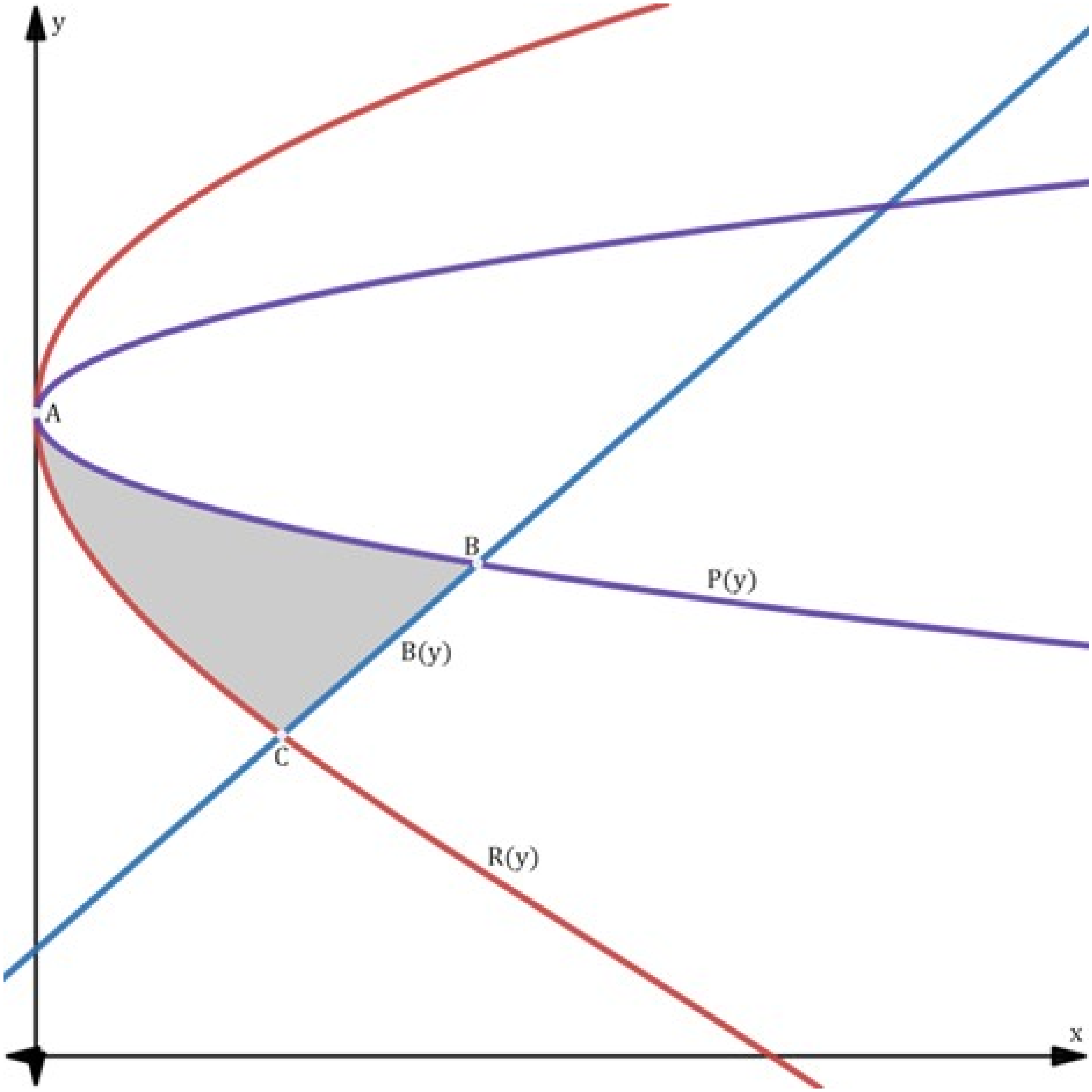
$$\therefore \int_0^{\pi} \sin^4\theta \, d\theta = \frac{1}{2}\int_0^{\pi} \left(\frac{3}{4} - \cos 2\theta + \frac{1}{4}\cos 4\theta\right) d\theta \quad \text{[M1]}$$

$$= \frac{1}{2}\left[\frac{3}{4}\theta - \frac{\sin 2\theta}{2} + \frac{\sin 4\theta}{16}\right]_0^{\pi} \quad \text{[M2]}$$

$$= \frac{1}{2}\left(\left(\frac{3\pi}{4} + 0 + 0\right) - 0\right)$$

$$= \frac{3\pi}{8}u^2 \quad \text{[A1]}$$

6. [**Suggested Time: 40 mins | Total Marks: 20 | Challenging**]



Answers by Accurate drawings or graphical methods are not accepted.
The graphs are plotting y against x. Point A is a common stationary point of R(y) and P(y). B(y) passes through the points $D(10, 2)$ and $E(1, \frac{1}{2})$.
$P(y) = 28(y-2)^2$, $B(y) = R''(y)$
Degree of polynomial R(y) is 3.

Find the shaded area. **Leave your answers in exact values.** [20]

Degree of $R(y)$ is 3 $\implies$ Degree of $B(y)$ is 1, therefore $B(y)$ is a linear function

Finding the Equation of $B(y)$

$$\text{Gradient} = \frac{\frac{1}{2} - 2}{1 - 10} = \frac{1}{6}$$ [M1]

$$y - 2 = \frac{1}{6}(x - 10)$$
$$6y - 12 = x - 10$$
$$x = 6y - 2$$
$$\therefore B(y) = 6y - 2$$ [M1]

Finding the coordinates of point A

$P'(y) = 56(y - 2)$

When $P(y) = 0$

$56(y - 2) = 0$

$\therefore y = 2$ [M1]

Sub $y = 2$ into $P(y)$

$$\therefore x = 28(2 - 2)^2 = 0$$

$\implies A = (0, 2)$ [M1]

Finding the equation of $R(y)$

Since B(y) = R"(y)

$$R'(y) = \int B(y)\, dy = 3y^2 - 2y + c_1, where\ c_1 \in \mathbb{R}$$ [M1]

Point A is a stationary point of $R(y) \implies R'(2) = 0$

$3(2)^2 - 2(2) + c_1 = 0$

$c_1 = -8$ [M1]

$\therefore R'(y) = 3y^2 - 2y - 8$

$$
\begin{aligned}
R(y) &= \int R'(y)\ dy \\
&= y^3 + y^2 - 8y + c_2, \textit{where } c_2 \in \mathbb{R} && \text{[M1]}
\end{aligned}
$$

When $x = 0,\ y = 2,$

$$
\begin{aligned}
R(2) &= 0 \\
2^3 - 22 - 8 + c_2 &= 0 \\
\therefore c_2 &= 12 && \text{[M1]} \\
\implies R(y) &= y^3 - y^2 - 8y + 12
\end{aligned}
$$

Given that Point C is an intercept of $R(y)$ and $B(y)$

$$
\begin{aligned}
R(y) &= B(y) \\
y^3 - y^2 - 8y + 12 &= 6y - 2 \\
y^3 - y^2 - 14y + 14 &= 0 && \text{[M1]}
\end{aligned}
$$

$$
\begin{aligned}
\textit{Let } f(x) &= y^3 - y^2 - 14y + 14 \\
f(1) &= 0
\end{aligned}
$$

$\therefore (y-1)$ is a factor of $f(x)$ [M1]

$$
\begin{aligned}
\implies f(x) &= (y-1)(y^2-14) \\
&= 0
\end{aligned}
$$

This implies that,

$$
\begin{aligned}
y^2 &= 14 \\
y &= \pm\sqrt{14} \\
\therefore y &= 1 \ or \pm \sqrt{14} \\
\textit{Rej. } y &= \pm\sqrt{14} \\
\implies y &= 1 && \text{[M1]}
\end{aligned}
$$

Sub $y = 1$ into $B(y)$,

$$
\begin{aligned}
B(1) &= 6(1) - 2 \\
&= 4 \\
\therefore C &= (4, 1) && \text{[M1]}
\end{aligned}
$$

Given that Point B is an intercept of P(y) and B(y)

$$6y - 2 = 28(y-2)^2 \qquad \text{[M1]}$$
$$= 28y^2 - 112y + 112$$

$$2\left(14y^2 - 59y + 57\right) = 0$$
$$2(7x-19)(2x-3) = 0$$
$$\therefore y = \frac{19}{7} \; or \; \frac{3}{2}$$

Reject $y = \frac{19}{7}$, as it implies that point B is above A

Thus, $y = \frac{3}{2}$ [M1]

Sub $y = \frac{3}{2}$ into B(y)

$$B\left(\frac{3}{2}\right) = 6\left(\frac{3}{2}\right) - 2$$
$$= 7$$

$$\therefore B = \left(7, \frac{3}{2}\right) \qquad \text{[M1]}$$

Let S be the shaded area

$$S = \int_{\frac{3}{2}}^{2} P(y)\,dy + \frac{1}{2}(4+7)\left(\frac{1}{2}\right) - \int_{1}^{2} R(y)\,dy$$
$$= \int_{\frac{3}{2}}^{2} 28(y-2)^2\,dy + \frac{11}{4} - \int_{1}^{2} y^3 - y^2 - 8y + 12\,dy \qquad \text{[M1]}$$
$$= \left[\frac{28}{3}(y-2)^3\right]_{\frac{3}{2}}^{2} + \frac{11}{4} - \left[\frac{y^4}{4} - \frac{y^3}{3} - 4y^2 + 12y\right]_{1}^{2} \qquad \text{[M1]}$$
$$= \frac{28}{3}(0) - \frac{28}{3}\left(\frac{3}{2} - 2\right)^3 + \frac{11}{4} - \left(\frac{2^4}{4} - \frac{2^3}{3} - 4(2)^2 + 12(2) - \left(\frac{1}{4} - \frac{1}{3} - 4 + 12\right)\right) \qquad \text{[M1]}$$
$$= \frac{7}{6} + \frac{11}{4} - \frac{17}{12}$$
$$= \frac{5}{2}$$

$\therefore$ The shaded area is $\left(\frac{5}{2}\right)$ $units^2$ [A1]

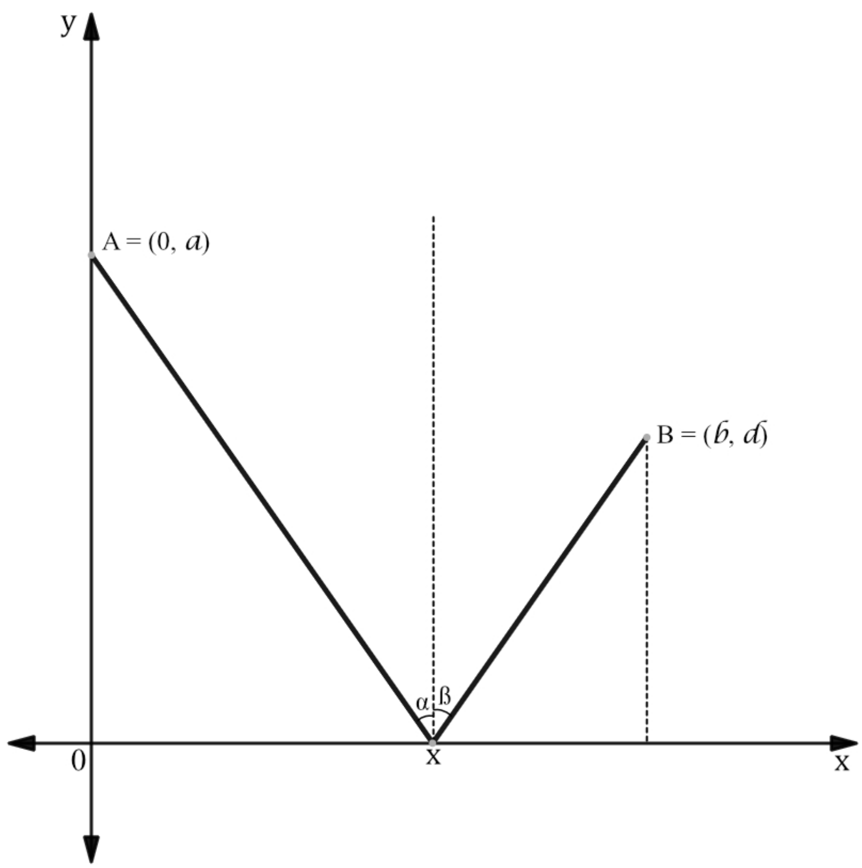
7. [**Suggested Time: 20 mins | Total Marks: 10 | Challenging**]

Fermat's Principle of Least Time

Fermat's Principle of Least Time states that out of all neighbouring paths available, light travels between two points along the path that requires the least time.

Consider a light ray from a source which strikes a mirror and is reflected. Let A be a point on the ray before it strikes the mirror and B be the point on the ray after reflection. v m/s is the speed of light.

A coordinate system is placed in a plane such that the x-axis runs along the mirror's surface and the point A lies on the y-axis.



Prove that the angle of incidence α is equal to the angle of reflection β.
(Proof that T is minimum is not required) [10]

Note that $\tan\alpha = \dfrac{x}{a}$ and $\tan\beta = \dfrac{b-x}{d}$

$$\begin{aligned} T &= \frac{Distance}{Speed} \\ &= \frac{|Ax|}{v} + \frac{|Bx|}{v} && \text{[M1]} \\ &= \frac{1}{v}\sqrt{a^2+x^2} + \frac{1}{v}\sqrt{d^2+(b-x)^2} && \text{[M1]} \\ &= \frac{1}{v}\left(\sqrt{a^2+x^2} + \sqrt{d^2+(b-x)^2}\right) \end{aligned}$$

$$\begin{aligned} \therefore \frac{dT}{dx} &= \frac{1}{v}\left(\frac{x}{\sqrt{a^2+x^2}} - \frac{b-x}{\sqrt{d^2+(b-x)^2}}\right) && \text{[M2]} \\ &= \left(\frac{x\sqrt{d^2+(b-x)^2} - (b-x)\sqrt{a^2+x^2}}{v\sqrt{(a^2+x^2)\left(d^2+(b-x)^2\right)}}\right) && \text{[M1]} \end{aligned}$$

For minimum T,

$$\begin{aligned} \frac{dT}{dx} &= 0 \\ \frac{x\sqrt{d^2+(b-x)^2} - (b-x)\sqrt{a^2+x^2}}{v\sqrt{(a^2+x^2)\left(d^2+(b-x)^2\right)}} &= 0 && \text{[M1]} \\ x\sqrt{d^2+(b-x)^2} - (b-x)\sqrt{a^2+x^2} &= 0 && \text{[M1]} \\ x\sqrt{d^2+(b-x)^2} &= (b-x)\sqrt{a^2+x^2} \\ x^2\left(d^2+(b-x)^2\right) &= (b-x)^2\left(a^2+x^2\right) && \text{[M1]} \\ x^2d^2 + x^2(b-x)^2 - a^2(b-x)^2 - x^2(b-x)^2 &= 0 \\ x^2d^2 - a^2(b-x)^2 &= 0 \\ (xd - a(b-x))\,(xd + (b-x)) &= 0 \end{aligned}$$

$$\begin{aligned} xd - a(b-x) &= 0 \\ xd &= a(b-x) \\ \frac{x}{a} &= \frac{b-x}{d} \\ \therefore \tan\alpha &= \tan\beta \end{aligned}$$

$$\begin{aligned} xd + a(b-x) &= 0 \\ xd &= -a(b-x) \\ \frac{x}{a} &= -\frac{b-x}{d} \\ \therefore \tan\alpha &= -\tan\beta \end{aligned}$$

Since both α & β are acute, it implies that both $\tan\alpha$ & $\tan\beta$ are positive,

we would have to reject $\tan\alpha = -\tan\beta$ [M1]

Hence, $\alpha = \beta$ ■

[A1]

8. [**Suggested Time: 25 mins | Total Marks: 12 | Challenging**]
A soft drink company wants to design a 500ml soda can using the least amount of material for their new Moon Shine soda. Assume that the soda can is a perfect cylinder. Using the information given below, find the cheapest material cost to produce 1000 cans. **Leave your answers in USD** [12]

Information List	
Price of Aluminium	2515 $USD/Metric\,Tonne$
Density of Aluminium	$2.7g/cm^3$
1 Atmosphere (Pressure)	101325 Pa
Room Temperature	$T_c = 28°c$
Temperature (Kelvin)	$T_k = T_c + 273$ K
Gas Constant (R)	$R = 8.314\,m^3\,Pa/mol\,K$

Ideal Gas Law
$\boldsymbol{PV = nRT}$
$P - Pressure$
$V - Volume$
$n - Amount\ of\ Substance$
$R - Gas\ Constant$
$T - Temperature$

The Soda can is able to tolerate up to 5 atm of pressure, and has a uniform thickness of 0.01 cm. The soda releases up to $\frac{x}{10}\ cm^3$ of gas for every $x\ ml$ of soda under room temperature and pressure.

Let S_c & V_c be the total surface area and volume of the can respectively

$S_c = 2\pi rh + 2\pi r^2$

$V_c = \pi r^2 h$

Room temperature and pressure $\implies P = 1\ atm,\ T_c = 28°c$

Thus, using the Ideal gas law, the number of moles of gas is

$$n = \frac{PV}{RT}$$
$$= \frac{101325\left(\frac{50}{100^3}\right)}{8.314(273+28)}$$
$$= 2.0245 \times 10^{-3}$$

[M1]

Since the max pressure that the can is able to handle is 5 atm,
min volume taken up by gas is,

$$V = \frac{nRT}{P}$$
$$= \frac{2.0245 \times 10^{-3}(8.314 \times (273+28))}{5(101325)}$$
$$= 1 \times 10^{-5}\ m^3$$
$$= 10\ cm^3$$

[M1]

$$Thus,\ V_c = 500 + 10 = 510cm^3$$
$$510 = \pi r^2 h$$
$$h = \frac{510}{\pi r^2}$$

[M1]

$$\therefore S_c = 2\pi r\left(\frac{510}{\pi r^2}\right) + 2\pi r^2$$
$$= \frac{1020}{r} + 2\pi r^2$$
$$\frac{dS_c}{dr} = 4\pi r - \frac{1020}{r^2}$$

[M1]

$$\begin{aligned}
\text{When } \frac{dS_c}{dr} &= 0 \\
4\pi r - \frac{1020}{r^2} &= 0 && \text{[M1]} \\
\frac{4\pi r^3 - 1020}{r^3} &= 0 \\
4\pi r^3 - 1020 &= 0 \\
\therefore r^3 &= \frac{1020}{4\pi} \\
r &= \sqrt[3]{\frac{255}{\pi}} \approx 4.3298\ cm && \text{[M1]} \\
\frac{d^2S_c}{dr^2} &= 4\pi + \frac{2040}{r^3} \\
\left.\frac{d^2S_c}{dr^2}\right|_{r\,=\,\sqrt[3]{\frac{300}{\pi}}} &= 4\pi + \frac{2040\pi}{255} \\
&> 0 && \text{[M1]}
\end{aligned}$$

$\therefore r = \sqrt[3]{\frac{255}{\pi}}$ gives a min. amount of aluminium used

$$\begin{aligned}
V_{alu} &= \pi\left(0.01 + \sqrt[3]{\frac{255}{\pi}}\right)^2\left(\frac{510}{\pi\left(\sqrt[3]{\frac{255}{\pi}}\right)^2} + 0.02\right) - 510 \\
&\approx 3.5419\ cm^2 && \text{[M1]}
\end{aligned}$$

$$\begin{aligned}
\text{Mass of 1 Can} &= 3.5419(2.7) \\
&= 9.56313g && \text{[M1]} \\
\text{Mass of 1k Cans} &= 9.56313(1000) \\
&= 9563.13g \\
&\approx 9.5631kg && \text{[M1]}
\end{aligned}$$

$$\begin{aligned}
\text{Price of Aluminium in kg} &= \frac{2515}{1000} \\
&= 2.515\ USD/kg && \text{[M1]}
\end{aligned}$$

$$\begin{aligned}
\therefore \text{Material Cost to produce 1000 Cans} &= 9.5631(2.515) \\
&\approx 24.05\ USD && \text{[A1]}
\end{aligned}$$

9. [**Suggested Time: N.A. | Total Marks: N.A. | Schadenfreude**]
Given that if x satisfies $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for some integers $a_{n-1}, \cdots, a_0$, then x is irrational unless x is an integer.
Prove that $\sqrt{2} + \sqrt[3]{2}$ is irrational.

Let $x = \sqrt{2} + \sqrt[3]{2}$. Now we see that,

$$\begin{aligned}
\left(x - \sqrt{2}\right)^3 &= 2 \\
x^3\sqrt{2}x^2 + 6x + 2\sqrt{2} &= 2 \\
x^3 + 6x - 2 &= \sqrt{2}\left(3x^2 - 2\right) \\
\left(x^3 + 6x - 2\right)^2 &= 2\left(3x^2 - 2\right)^2 \\
x^6 - 6x^4 - 4x^3 + 36x^2 - 24x + 4 &= 2(9x^4 + 12x^2 + 4) \\
x^6 - 6x^4 - 4x^3 + 12x^2 - 24x - 4 &= 0
\end{aligned}$$

Therefore, we have found a monic polynomial with integer coefficients which has $\sqrt{2} + \sqrt[3]{2}$ as its root, meaning by the equation given, $\sqrt{2} + \sqrt[3]{2}$ must either be an integer or irrational number. We can trivially see that $\sqrt{2} + \sqrt[3]{2}$ is not an integer, hence it has to be an irrational number.

10. [**Suggested Time: 25 mins | Total Marks: 10 | Extensions**]

(i) Deduce that the effective resistance, R_{eff}, of n parallel resistors is $R_{\text{eff}} = \left(\sum_{i=1}^{n} \frac{1}{R_i}\right)^{-1}$ [5]

Total current in a parallel circuit $I_T = \sum_{i=1}^{n} I_i \quad (KCL)$

Voltage in a parallel circuit *circuit* $V_T = V_1 = V_2 = V_3 \cdots$ [M1]

By Ohm's Law, V = IR, we can see that

$I_1 = \frac{V_1}{R_1}, I_2 = \frac{V_2}{R_2}, \cdots, I_n = \frac{V_n}{R_n}$ [M1]

Note that since $V_T = V_1 = V_2 = V_3 \cdots$, we can rewrite the equations above as

$I_1 = \frac{V_T}{R_1}, I_2 = \frac{V_T}{R_2}, \cdots, I_n = \frac{V_T}{R_n}$ [M1]

Using KCL and the equations that we had derived, we can see that

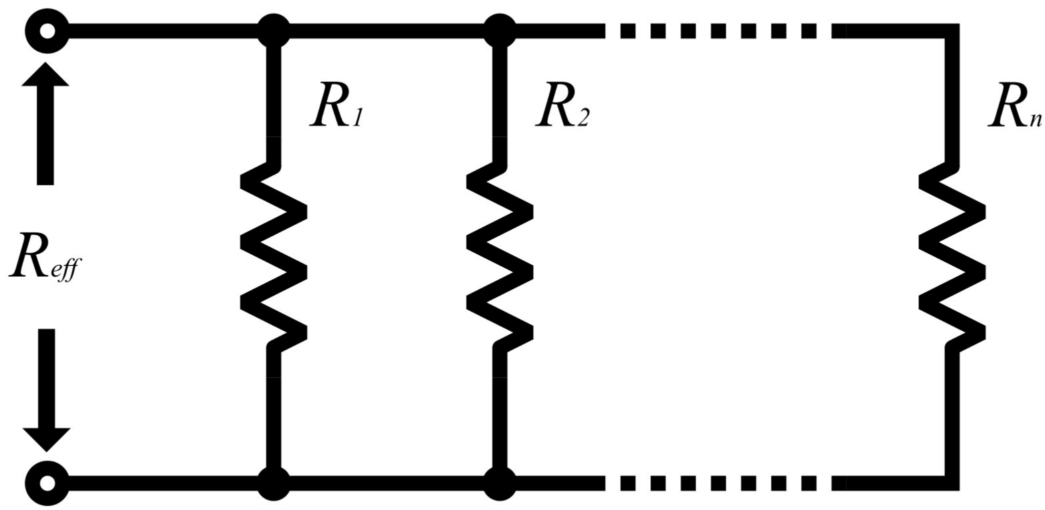
$$
\begin{aligned}
I_T &= \sum_{i=1}^{n} \frac{V_T}{R_i} \\
&= V_T \sum_{i=1}^{n} \frac{1}{R_i}
\end{aligned}
$$

[M1]

By Ohm's law,

$$
\begin{aligned}
R_{\text{eff}} &= \frac{V_T}{I_T} \\
&= \frac{V_T}{V_T \sum_{i=1}^{n} \frac{1}{R_i}} \\
&= \frac{1}{\sum_{i=1}^{n} \frac{1}{R_i}} \\
&= \left(\sum_{i=1}^{n} \frac{1}{R_i}\right)^{-1} \qquad \blacksquare
\end{aligned}
$$

[A1]



(ii)(a) Hence or otherwise, find the effective resistance of the circuit shown above. Where $R_1 = \sqrt{1} + \sqrt{2}, R_2 = \sqrt{2} + \sqrt{3}, R_3 = \sqrt{3} + \sqrt{4} \cdots$
Leave your answers in exact values, and in terms of n. [4]

We can see that the resistance of the resistors follow a series with a general term of
$R_n = \sqrt{n} + \sqrt{n+1}$ [M1]

Using the equation in part **(i)**,

$$\begin{aligned} R_{\text{eff}} &= \left(\sum_{\alpha=1}^{n} \frac{1}{\sqrt{\alpha} + \sqrt{\alpha+1}} \right)^{-1} \\ &= \left(\sum_{\alpha=1}^{n} \frac{\sqrt{\alpha} - \sqrt{\alpha+1}}{\alpha - \alpha - 1} \right)^{-1} \\ &= \left(\sum_{\alpha=1}^{n} \left(\sqrt{\alpha+1} - \sqrt{\alpha} \right) \right)^{-1} \end{aligned}$$ [M1]

Evaluating the sum

$$\begin{aligned} \sum_{\alpha=1}^{n} \left(\sqrt{\alpha+1} - \sqrt{\alpha} \right) = &\left(\sqrt{2} - \sqrt{1} \right) + \\ &\left(\sqrt{3} - \sqrt{2} \right) + \\ &\left(\sqrt{4} - \sqrt{3} \right) + \\ &\vdots \\ &\left(\sqrt{n+1} - \sqrt{n} \right) \\ =&\sqrt{n+1} - 1 \end{aligned}$$ [M1]

Substituting the sum back into R_{eff}

$\therefore R_{\text{eff}} = \dfrac{1}{\sqrt{n+1}-1}$ [A1]

The effective resistance of the circuit is $\dfrac{1}{\sqrt{n+1}-1}$,
where n is the number of resistors

(ii)(b) Explain, with relevant workings, if the effective resistance of the circuit will approach a unique value as more resistors are added into the circuit. [1]

From (ii)(a), $R_{\text{eff}} = \dfrac{1}{\sqrt{n+1}-1}$

As $n \to \infty$, $\sqrt{n+1} - 1 \to \infty$

$$\frac{1}{\sqrt{n+1}-1} \to 0$$

$$\therefore R_{\text{eff}} \to 0$$ [A1]

Yes, R_{eff} approaches the value 0 as more resistors are added into the circuit