INTRODUCTION TO THE STANDARD MODEL (PHYS7645) LECTURE NOTES

LECTURE NOTES BASED ON A COURSE GIVEN BY LAWRENCE GIBBONS. THE COURSE BEGINS WITH A BRIEF OVERVIEW OF THE STANDARD MODEL AND MOVES ON TO DERIVE ITS PROPERTIES FROM FIRST PRINCIPLES.

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Chapter 1

Preface

This set of notes are based on lectures given by Lawrence Gibbons in the *Introduction to the Standard Model* course at Cornell University during Spring 2013. The course uses both Langacker's *The Standard Model and Beyond* and Peskin and Schroeder's *Introduction to Quantum Field Theory* as a reference text. I wrote these notes during lectures and as such may contain some small typographical errors and sloppy diagrams. I've attempted to proofread and fix as many of these problems as I can. If you have any questions or would like a copy of the LaTeX file feel free to let me know at ajd268@cornell.edu.

Chapter 2

Introduction

2.1 Lagrangians and Conserved Quantities

In these notes we work in natural units, $\hbar = c = 1$ (remember that $\hbar c \approx 197 MeV fm$). We will use four vectors $p^\mu = (E, \mathbf{p}), x^\mu = (t, \mathbf{x})$ with the West-coast metric:

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = g_{\mu\nu} \tag{2.1}$$

and

$$p_\mu = (E, -\mathbf{p}) \tag{2.2}$$

which gives $p^2 \equiv p^\mu p_\mu = E^2 - \mathbf{p}^2 = m^2$. We have

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \nabla) \quad ; \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = (\frac{\partial}{\partial t}, -\nabla) \tag{2.3}$$

so we have a correspondence of $p^\mu \leftrightarrow i\partial^\mu$, $E \leftrightarrow i\frac{\partial}{\partial t}$, and $\mathbf{p} \leftrightarrow -i\nabla$. This gives for non-relativistic particles,

$$\frac{p^2}{2m} + V = E \Rightarrow \qquad \text{Schrodinger equation} \tag{2.4}$$

For relativistic fields

$$E^2 = p^2 + m^2 \Rightarrow \qquad \text{KG equation} \tag{2.5}$$

or in differential operator notation,

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi = 0 \tag{2.6}$$

This is known as the Klien Gordan (KG) equation and arises from the Lagrangian density (we will show this shortly),

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2} m^2 \phi^2 \tag{2.7}$$

The total Lagrangian is

$$L = \int \mathcal{L}\, d^3x \tag{2.8}$$

and the action is

$$S_{KG} = \int L\, dt = \int d^4x \mathcal{L} \tag{2.9}$$

The Euler equation is given by

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \right) = 0 \tag{2.10}$$

Applying this the Klien Gordan Lagrangin implies that,

$$\frac{\partial \mathcal{L}_{KG}}{\partial \phi} = -m^2 \phi \qquad , \qquad \frac{\partial \mathcal{L}_{KG}}{\partial^\mu \phi} = \partial_\mu \phi \tag{2.11}$$

Thus the Klien Gordan Lagrangian gives the Klien-Gordan equation:

$$\partial^\mu \partial_\mu \phi + m^2 \phi = 0 \tag{2.12}$$

Suppose have two equal mass, real scalar fields (in other words fields that come from the Klien Gordan equation) which we denote, ϕ_1 and ϕ_2.

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi_1 \partial_\mu \phi_1 - \frac{1}{2} m^2 \phi_1^2 + \frac{1}{2}\partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2} m^2 \phi_2 \tag{2.13}$$

Suppose now we define a complex scalar field,$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ and $\phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$. This gives

$$\mathcal{L} = \partial^\mu \phi \partial_\mu \phi^* - m^2 \phi \phi^* \tag{2.14}$$

This is the same Lagrangian written under somewhat more convenient fields.

Consider $\mathcal{L}$ under $\phi \to \phi' = e^{i\alpha}\phi$ and $\phi'^* = e^{-i\alpha}\phi^*$. $\mathcal{L}$ is unchanged under this transformation. Note that if $\alpha \to \alpha(\mathbf{x}, t)$, the Lagrangian would not have remained invariant. This is a known as a global symmetry. Noether's theorem says that there is a conserved quantity which in this case can be interpreted as a charge. A complex scalar field has a charge that is conserved. This is called a $U(1)$ symmetry.

Since there is a conserved quantity in this choice of coordiantes there must have also been a conserved quantity in terms of ϕ_1 and ϕ_2. To see what this was we rewrite our original Lagrangian only in slightly more convenient notation,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \boldsymbol{\phi}^T \partial^\mu \boldsymbol{\phi} - \frac{m^2}{2}\boldsymbol{\phi}^T \boldsymbol{\phi} \tag{2.15}$$

where $\boldsymbol{\phi} \equiv (\phi_1\, \phi_2)^T$. This Lagranian is invariant under rotations between ϕ_1, ϕ_2. In other words under the transformation,

$$\boldsymbol{\phi} \to \mathcal{O}\boldsymbol{\phi} \tag{2.16}$$

where $\mathcal{O}^T = \mathcal{O}$. The conserved charge is,

$$\boldsymbol{\phi}^2 = \phi_1^2 + \phi_2^2 \tag{2.17}$$

2.2 Feynman Rules

We now briefly recall some of the Feynman rules.

2.2.1 ϕ^4 Theory

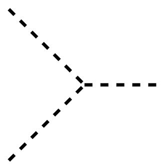
The Lagrangian for ϕ^4 theory is

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{2.18}$$

Each propagator in momentum space is,

$$\frac{i}{p^2 - m^2 + i\epsilon} \tag{2.19}$$

Each external lines contributes a trivial factor of 1 and the ϕ^4 vertex is,



$$\rightarrow -i\lambda$$

2.2.2 QED

The gauge invariant quantity in electrodynamics is the electromagnetic(EM) field tensor,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.20}$$

or in terms of the four-potential,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{2.21}$$

The kinetic Lagrangian term is given by

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{2.22}$$

Applying Euler Lagrange equations gives Maxwell's equations.

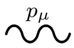
The electromagnetic gauge transformation gives,

$$A^\mu \rightarrow A^\mu + \partial^\mu X \tag{2.23}$$

where X is a field. The EM field tensor (and hence the electric and magnetic field) elements are unchanged under a gauge transformation. The physics doesn't change, but your calculations do. Typically we will use the Lorenz gauge:

$$\partial^\mu A_\mu = 0 \tag{2.24}$$

Using that gauge we end up with the photon propagator,



$$= \frac{-ig^{\mu\nu}}{p^2 + i\epsilon} \tag{2.25}$$

For an external photon,

$$\rightarrow \epsilon_\mu(p) \tag{2.26}$$

and for an outgoing external photon we have

$$\rightarrow \epsilon^*_\mu(p) \tag{2.27}$$

The polarization vector has the properties,

$$\epsilon^\mu p_\mu = 0, \quad \epsilon^\mu \epsilon^*_\mu = -1 \tag{2.28}$$

There are two common bases for the polarization vectors. We assume $\mathbf{p}$ to be along $\hat{z}$.

1.
$$\begin{cases} (0,1,0,0) \\ (0,0,1,0) \\ \frac{1}{m}(p,0,0,E) \end{cases} \tag{2.29}$$

2.
$$\begin{cases} \frac{1}{\sqrt{2}}(0,1,i,0,0) \\ -\frac{1}{\sqrt{2}}(0,1,-i,0) \\ \frac{1}{m}(p,0,0,E) \end{cases} \tag{2.30}$$

This is the helicity basis.

Chapter 3

Dirac Equation

Dirac knew of the KG equation and he realized that the negative energy solutions came from the fact that the Hamiltonian was quadratic in energy. So he set out of linearize the problem. He wanted to find an Hamiltonian that is linear in momentum.

$$H\psi = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi \tag{3.1}$$

He still wanted the solution to be consistent with relativity and have $E^2 = p^2 + m^2$. Thus he wanted

$$H^2\psi = (|\mathbf{p}|^2 + m^2)\psi \tag{3.2}$$

Now

$$H^2\psi = \left(\underbrace{\alpha_i^2}_{1} p_i^2 + \overbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}^{1} p_i p_j + \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{0} p_i m + \overbrace{\beta^2}^{1} m^2 \right)\psi \tag{3.3}$$

(where we found the relations above by comparison with above). Thus we have

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = 1 \tag{3.4}$$

and

$$\{\alpha_i, \alpha_j\} = \{\alpha_i, \beta\} = 0 \tag{3.5}$$

Since $\alpha_i^2 = 1$ we know that $\alpha_i = \alpha_i^{-1}$. We also know

$$\begin{aligned} \alpha_i\alpha_j &= -\alpha_j\alpha_i & (3.6)\\ \alpha_j &= -\alpha_i\alpha_j\alpha_i & (3.7)\\ \mathrm{Tr}(\alpha_j) &= -\mathrm{Tr}(\alpha_i\alpha_j\alpha_i) & (3.8)\\ \mathrm{Tr}(\alpha_j) &= \mathrm{Tr}(\alpha_i^2\alpha_j) & (3.9)\\ \mathrm{Tr}(\alpha_j) &= -\mathrm{Tr}(\alpha_j) & (3.10) \end{aligned}$$

Hence this implies that $\mathrm{Tr}(\alpha_j) = 0$.

To better understand our constrains on the α matrices we look for their eigenvalues. We denote their eigenvectors by $\mathbf{v}$ and their eigenvalues by λ_i. We have,

$$\alpha_i \mathbf{v} = \lambda_i \mathbf{v} \tag{3.11}$$

$$\mathbf{v} = \lambda_i^2 \mathbf{v} \tag{3.12}$$

Thus $\lambda_i = \pm 1$. The trace of a matrix is just the sum its eignevalues. Since the matrices are traceless,

$$\mathrm{Tr}\,(\alpha_i) = (1 + 1 + ... + 1) + (-1 - 1... - 1) = 0 \tag{3.13}$$

This can only happen if we have an even dimension such that we have an equal number of $+1$'s and -1's.

Since H must be Hermitian this implies that α, β are also Hermitian. There are no 2×2 solutions satisfying all the constraints. There are however an infinite number of 4×4 solutions. We mention two such representations

1. Paul-Dirac representation

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad , \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.14}$$

where recall that the Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.15}$$

2. Weyl (Chiral) representation:

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{3.16}$$

One can get from any of these solutions to any other solutions by a unitary matrix U:

$$\alpha_i' = U \alpha_i U^\dagger \quad , \quad \beta' = U \beta U^\dagger \tag{3.17}$$

Now we come back to the Dirac equation

$$E\psi = H\psi \tag{3.18}$$

$$i\frac{\partial}{\partial t}\psi = -i\vec{\alpha} \cdot \nabla\psi + \beta m\psi \tag{3.19}$$

$$i\beta\frac{\partial}{\partial t}\psi = -i\beta\vec{\alpha} \cdot \nabla\psi + m\psi \tag{3.20}$$

$$0 = (-i\beta\frac{\partial}{\partial t} + i\beta\vec{\alpha} \cdot \nabla - m)\psi \tag{3.21}$$

We define $\gamma^\mu = (\beta, \beta\vec{\alpha})$. Then we can write the Dirac equation in manifestly convariant form,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{3.22}$$

or

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{3.23}$$

We now review some properties of γ^μ.

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{3.24}$$

in particular: $(\gamma^0)^2 = (\gamma^i)^2 = -1$. Now

$$(\gamma^0)^\dagger = \beta^\dagger = \beta \tag{3.25}$$

and

$$(\gamma^i)^\dagger = (\beta\alpha_i)^\dagger = \alpha_i^\dagger\beta^\dagger = \alpha_i\beta = -\beta\alpha_i = -\gamma^i \tag{3.26}$$

so the γ_i are antihermitian. Furthermore

$$(\gamma^i)^\dagger = \alpha_i\beta = \beta^2\alpha_i\beta = \gamma^0\gamma^i\gamma^0 \tag{3.27}$$

Therefore

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{3.28}$$

Consider the adjoint of the Dirac equation.

$$[(i\partial_\mu\gamma^\mu - m)\psi]^\dagger = 0 \tag{3.29}$$

$$-i\partial_\mu\psi^\dagger(\gamma^\mu)^\dagger - m\psi^\dagger = 0 \tag{3.30}$$

$$i\partial_\mu\left(\psi^\mu\right)^\dagger\gamma^0\underbrace{\gamma^0(\gamma^\mu)^\dagger\gamma^0}_{\gamma^\mu} + m\psi^\dagger\gamma^0 = 0 \tag{3.31}$$

Define $\bar{\psi} = \psi^\dagger\gamma^0$. Then we have

$$i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0 \tag{3.32}$$

Note that this is similar to the Dirac equation only now we have a positive mass term. Our two equations are

$$\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{3.33}$$

$$\bar{\psi}(i\gamma^\mu\overleftarrow{\partial_\mu} + m)\psi = 0 \tag{3.34}$$

Summing these equations we have

$$i\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \tag{3.35}$$

Thus $\bar{\psi}\gamma^\mu\psi$ is a conserved current.

Our goal now is to determine the Lorentz behavior of different quantities. We define

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{3.36}$$

and

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right] \tag{3.37}$$

We now want to know how to boost our spinor. Recall that a four-vector gets boosted as,

$$(x')^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{3.38}$$

where Λ is a 4×4 boost/rotation matrix. We define

$$\psi'(x') = S\psi(x) \tag{3.39}$$

We must have

$$i(\gamma_{\mu}\partial'^{\mu} - m)\psi'(x') = 0 \tag{3.40}$$

$$i(\gamma_{\mu}\Lambda^{\mu}{}_{\nu}\partial^{\nu} - m)S\psi(x) = 0 \tag{3.41}$$

Multiplying by S^{-1} on the right,

$$\left(S^{-1}(i\gamma_{\mu})S\Lambda^{\mu}{}_{\nu}\partial^{\nu} - m\right)\psi = 0 \tag{3.42}$$

We'll have a manifestly invariant Dirac equation if $\gamma_{\nu} = S^{-1}\gamma_{\mu}S\Lambda^{\mu}{}_{\nu}$ or

$$S^{-1}\gamma^{\rho}S = \Lambda^{\rho}{}_{\nu}\gamma^{\nu} \tag{3.43}$$

In other words the transformation can be alternatively done on the γ^{μ} matrices which can be transformed as a four-vector!

We will find S for an infinitesimal transformation.

$$\Lambda^{\nu}{}_{\mu} = g^{\nu}{}_{\mu} + \lambda\epsilon^{\nu}{}_{\mu} + \mathcal{O}(\lambda^2) \tag{3.44}$$

where $\omega_{\mu\nu}$ is an antisymmetric tensor (as usual this condition is to preserve the scalar product).

Look for

$$S = 1 + \lambda\omega_{\mu\nu}S^{\mu\nu} \tag{3.45}$$

where $S^{\mu\nu}$ are some arbitrary 4×4 matrix. Using our defining equation for S we have,

$$(1 - \lambda\omega_{\mu\nu}S^{\mu\nu})\gamma^{\rho}(1 + \lambda\omega_{\mu\nu}S^{\mu\nu}) = (g^{\rho}{}_{\nu} + \lambda\omega^{\rho}{}_{\nu})\gamma^{\nu} \tag{3.46}$$

$$\gamma^{\rho} + \lambda\omega_{\mu\nu}\left[\gamma^{\rho}, S^{\mu\nu}\right] = \gamma^{\rho} + \lambda\omega^{\rho}{}_{\nu}\gamma^{\nu} \tag{3.47}$$

$$\omega_{\mu\nu}\left[\gamma^{\rho}, S^{\mu\nu}\right] = \omega^{\rho}{}_{\nu}\gamma^{\nu} \tag{3.48}$$

$$\omega_{\mu\nu}\left[\gamma^{\rho}, S^{\mu\nu}\right] = \omega_{\mu\nu}g^{\mu\rho}\gamma^{\nu} \tag{3.49}$$

$$\omega_{\mu\nu}\left[\gamma^{\rho}, S^{\mu\nu}\right] = \omega_{\mu\nu}\frac{1}{2}\left(\gamma^{\nu}g^{\mu\rho} - \gamma^{\mu}g^{\nu\rho}\right) \tag{3.50}$$

$$\tag{3.51}$$

Try to look for solutions for each $\mu\nu$. If we can find such solutions then the equations will still work with the contractions over $\omega_{\mu\nu}$. We are looking for term by term solutions obeying

$$[\gamma^\rho, S^{\mu\nu}] = \frac{1}{2}\left(\gamma^\nu g^{\rho\mu} - \gamma^\mu g^{\rho\nu}\right) \tag{3.52}$$

The right hand side is antisymmetric about swapping μ and ν. This can only hold if $S^{\mu\nu}$ is also antisymmetric about this transformation. The simplest anticommuting object in this space is, $\sigma^{\mu\nu}$. We write,

$$S^{\mu\nu} = \alpha\sigma^{\mu\nu} = \frac{i\alpha}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) \tag{3.53}$$

where α is for the time being an unknown constant. This gives

$$\frac{1}{2}\left(\gamma^\nu g^{\rho\mu} - \gamma^\mu g^{\rho\nu}\right) = \frac{i\alpha}{2}\left(\gamma^\rho\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right) - \left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)\gamma^\rho\right) \tag{3.54}$$

We have,

$$\gamma^\rho\gamma^\mu\gamma^\nu = 2g^{\rho\mu}\gamma^\nu - \gamma^\mu\gamma^\rho\gamma^\nu \tag{3.55}$$
$$= 2g^{\rho\mu}\gamma^\nu - 2g^{\nu\rho}\gamma^\mu + \gamma^\mu\gamma^\nu\rho \tag{3.56}$$

Thus,

$$\frac{1}{2}\left(\gamma^\nu g^{\rho\mu} - \gamma^\mu g^{\rho\nu}\right) = 2i\alpha\left(\gamma^\nu g^{\rho\mu} - \gamma^\mu g^{\nu\rho}\right) \tag{3.57}$$

This holds true if

$$\alpha = -\frac{i}{4} \tag{3.58}$$

Thus

$$S = 1 - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu} = 1 + \frac{\lambda}{8}\omega_{\mu\nu}\left[\gamma^\mu, \gamma^\nu\right] \tag{3.59}$$

3.1 Transformation Properties of Bilinears

Bilinears are terms that take the form $\bar{\psi}\sigma\psi$. We now look for their transformation properties.

We begin by studying the conjugate spinor,

$$\bar{\psi}' = \psi^\dagger S^\dagger\gamma^0 \tag{3.60}$$
$$= \psi^\dagger\gamma^0\gamma^0 S^\dagger\gamma^0 \tag{3.61}$$
$$= \bar{\psi}(\gamma^0 S^\dagger\gamma^0) \tag{3.62}$$

Studying the term in brackets,

$$\gamma^0 S^\dagger \gamma^0 = 1 + \frac{\lambda}{8}\omega_{\mu\nu}\left(\gamma^0\gamma^{\nu\dagger}\gamma^0\gamma^0\gamma^{\mu\dagger}\gamma^0 - \gamma^0\gamma^{\mu\dagger}\gamma^0\gamma^0\gamma^{\nu\dagger}\gamma^0\right) \tag{3.63}$$

$$= 1 + \frac{\lambda}{8}\left(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma^\nu\right) \tag{3.64}$$

$$= 1 - \frac{\lambda}{8}\omega_{\mu\nu}\left[\gamma^\mu, \gamma^\nu\right] \tag{3.65}$$

$$= S^{-1} \tag{3.66}$$

Hence we have

$$\bar{\psi}' = \bar{\psi} S^{-1} \tag{3.67}$$

We are now in position to study bilinear transformation properties. The simplest bilinear transforms trivially,

$$\bar{\psi}'\psi' = \bar{\psi} S^{-1} S \psi \tag{3.68}$$

$$= \bar{\psi}\psi \tag{3.69}$$

Now consider

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi} S^{-1}\gamma^\mu S \psi \tag{3.70}$$

but recall that $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$. Thus

$$\left(\bar{\psi}'\gamma^\mu\psi'\right) = \Lambda^\mu{}_\nu \left(\bar{\psi}\gamma^\nu\psi\right) \tag{3.71}$$

Hence $\bar{\psi}\gamma^\mu\psi$ is a four-vector. Further note that contracting this expression with another four-vector gives a scalar.

Thus we see that our QED Lagrangian,

$$\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \tag{3.72}$$

is manifestly Lorentz covariant since both $\bar{\psi}\gamma^\mu\partial_\mu\psi$ and $m\bar{\psi}\psi$ transform trivially under Lorentz boosts.

3.2 Spin

Consider a rotation $\delta\theta_z$ about the z axis. The Lorentz matrix for this transformation is

$$S(\delta\theta_z) = 1 - \frac{i}{2}\delta\theta_z\sigma^{12} \tag{3.73}$$

Recall from quantum mechanics that a rotation operator should have the form $e^{-iS_z\delta\theta_z} \approx 1 - iS_z\delta\theta_z$. Comparing these two equations we see that,

$$S_z = \frac{1}{2}\sigma^{12} = \frac{i}{4}(\gamma^1\gamma_2 - \gamma^2\gamma^1) = \frac{i}{2}\gamma^1\gamma^2 \tag{3.74}$$

now

$$S_z^2 = -\frac{1}{4}\gamma^1\gamma^2\gamma^1\gamma^2 \tag{3.75}$$
$$= \frac{1}{4} \tag{3.76}$$

using $\gamma_i^2 = -1$. Similarly one can show that $S_x^2 = S_y^2 = 1/4$[1]. This implies that

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{3}{4} \tag{3.77}$$

If we act on some state we have,

$$S^2 \left|s, m\right\rangle = s\,(s+1) \left|s, m\right\rangle = \frac{3}{4} \left|s, m\right\rangle \tag{3.78}$$

and hence $s = \frac{1}{2}$. Thus the Dirac equation holds only for spin 1/2 particles.

3.3 Free Particle Solutions

We will look for free particle solutions $\psi(x) = u(p)e^{-ip\cdot x}$, $\partial_\mu\psi = -ip_\mu u(p)e^{-ip\cdot x}$. Inserting this form into the Dirac equation gives,

$$(\gamma^\mu p_\mu - m)u(p) = 0 \tag{3.79}$$

We will look for what is called the Weyl or Chiral representation solutions. We use

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{3.80}$$

The Dirac equation gives

$$\begin{pmatrix} -m & E - \boldsymbol{\sigma}\cdot\mathbf{p} \\ E + \boldsymbol{\sigma}\cdot\mathbf{p} & -m \end{pmatrix} \begin{pmatrix} u_A(p) \\ u_B(p) \end{pmatrix} \tag{3.81}$$

This gives two equations

$$-mu_A + (E - \boldsymbol{\sigma}\cdot\mathbf{p})u_B = 0 \quad , \quad (E + \boldsymbol{\sigma}\cdot\mathbf{p})u_A - mu_B = 0 \tag{3.82}$$

These equations are coupled and difficult to solve in general. However, they are easy to solve in two limits. In the limit that $m \to 0$ the equations decouple yielding compact solutions. Though can be done, we alternatively study the equations in the opposing limit of $\mathbf{p} \to 0$ (the rest frame). From there we can find the general solution by boosting our spinors.

[1] You may feel the urge to extrapolate these resuls and say that the spinors are eigenstates of S_x and S_y; This is NOT true! However, spinors are eigenstates of the squares of these operators.

In this limit two linearly dependent equations,

$$m(-u_A+u_B)=0 \quad , \quad m(u_A-u_B)=0 \tag{3.83}$$

There is a symmetry between u_A and u_B in this frame that didn't exist before. We can write the general solution as,

$$u(p)=\sqrt{m}\begin{pmatrix}\chi\\ \chi\end{pmatrix} \tag{3.84}$$

where χ is a 2-spinor with $\chi^\dagger\chi=1$.

Consider a boost along z. A four-vector transforms as,

$$\begin{pmatrix}E\\ p_z\end{pmatrix}=\begin{pmatrix}m\cosh y_L\\ m\sinh y_L\end{pmatrix}=\exp\left(y_L\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)\begin{pmatrix}m\\0\end{pmatrix} \tag{3.85}$$

while a spinor in the new frame is,[Q 1: check for any more factors]

$$u(p)=\exp\left[\frac{y_L}{4}\left(\gamma^0\gamma^i-\gamma^i\gamma^0\right)\right]\sqrt{m}\begin{pmatrix}\chi\\ \chi\end{pmatrix} \tag{3.86}$$

$$=\exp\left[-\frac{y_L}{2}\begin{pmatrix}\sigma_3&0\\0&-\sigma_3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\chi\\ \chi\end{pmatrix} \tag{3.87}$$

This gives after some algebra:

$$u(p)=\begin{pmatrix}\left(\sqrt{E+p_z}\left(\frac{1-\sigma_3}{2}\right)+\sqrt{E-p_z}\left(\frac{1+\sigma_3}{2}\right)\right)\chi\\ \left(\sqrt{E+p_z}\left(\frac{1+\sigma_3}{2}\right)+\sqrt{E-p_z}\left(\frac{1-\sigma_3}{2}\right)\right)\chi\end{pmatrix} \tag{3.88}$$

We now consider a particle "spin up" along the z direction (which is right handed relative to z), $\chi=\chi^1\equiv\begin{pmatrix}1\\0\end{pmatrix}$ and "spin down" which is left handed and $\chi=\chi^2\equiv\begin{pmatrix}0\\1\end{pmatrix}$. Now $\frac{1-\sigma_3}{2}=\begin{pmatrix}0&0\\0&2\end{pmatrix}$. Thus spin up gives

$$u^1(p)=\begin{pmatrix}\sqrt{E-p_z}\chi^1\\ \sqrt{E+p_z}\chi^1\end{pmatrix} \tag{3.89}$$

and spin down gives

$$u^2(p)=\begin{pmatrix}\sqrt{E+p_z}\chi^2\\ \sqrt{E-p_z}\chi^2\end{pmatrix} \tag{3.90}$$

In the energy/massless limit we get

$$u^1(p)\to\sqrt{2E}\begin{pmatrix}0\\ \chi^1\end{pmatrix} \quad , \quad u^2(p)\to\sqrt{2E}\begin{pmatrix}\chi^2\\ 0\end{pmatrix} \tag{3.91}$$

This gives us the helicity/chirality of your particle. The helicity operator is given by

$$h=\frac{1}{2}\frac{\mathbf{\Sigma}\cdot\mathbf{p}}{|\mathbf{p}|} \tag{3.92}$$

with $\Sigma = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$.

$$hu^1(p) = \frac{1}{2}u^1(p) \quad , \quad hu^2(p) = -\frac{1}{2}u^2(p) \tag{3.93}$$

The helicity of a massive particle ($m \neq 0$) is a relative concept. You can always boost to a frame where $p_z \to -p_z$. This flips u^1 and u^2. However for a massless particle the helicity is locked in since you can't boost faster then that particle. Such fields are known as chiral fields.

3.4 Negative Energy Solutions

In the Dirac equation there exists negative energy solutions,

$$\psi = v(p)e^{ip\cdot x} \tag{3.94}$$

with $E < 0$. Then the Dirac equation gives

$$(\gamma^\mu p_\mu + m)\, v(p) = 0 \tag{3.95}$$

and for a boost along the z direction,

$$v(p) = \begin{pmatrix} \left[\sqrt{E+p_z}\left(\frac{1-\sigma_3}{2}\right) + \sqrt{E-p_z}\left(\frac{1+\sigma_3}{2}\right)\right]\eta \\ -\left[\sqrt{E-p_z}\left(\frac{1-\sigma_3}{2}\right) + \sqrt{E+p_z}\left(\frac{1+\sigma_3}{2}\right)\right]\eta \end{pmatrix} \tag{3.96}$$

where η is a 2 component spinor, independent of χ.

Below we introduce some "handy-dandy" relationships:

$$\bar{u}(p) = u^\dagger(p)\gamma^0 \tag{3.97}$$

$$\bar{v}(p) = v^\dagger(p)\gamma^0 \tag{3.98}$$

$$\bar{u}^r(p)u^s(p) = -\bar{v}^r(p)v^s(p) = 2m\delta^{rs} \tag{3.99}$$

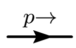
$$\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0 \tag{3.100}$$

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m \tag{3.101}$$

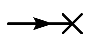
$$\sum_s v^s\bar{v}^s = \not{p} - m \tag{3.102}$$

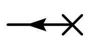
where we use Feynman's slash notation, $\not{p} \equiv p^\mu\gamma_\mu$.

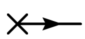
The Feynman diagram contributions are below

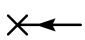


$\rightarrow \quad \dfrac{i\left(\not{p}+m\right)}{p^2 - m^2 + i\epsilon}$ (3.103)

 $\rightarrow \quad u^s(p)$ (3.104)

 $\rightarrow \quad \bar{v}^s(p)$ (3.105)

 $\rightarrow \quad \bar{u}^s(p)$ (3.106)

 $\rightarrow \quad v^s(p)$ (3.107)

Chapter 4

Gauge Invariance

4.1 Golden Rule

1. Decay of particle with mass M to n other particles, all with masses m_i. Let $\mathcal{M}$ be the amplitude for the process:

$$d\Gamma = |\mathcal{M}|^2 \frac{S(2\pi)^4}{2M} \delta^4(p - \sum_j p_{j,final}) \times \left(\frac{d^3 p_{1,f}}{(2\pi)^3 2E_{1,f}} \cdots \frac{d^3 p_{n,f}}{(2\pi)^3 2E_{n,f}} \right) \tag{4.1}$$

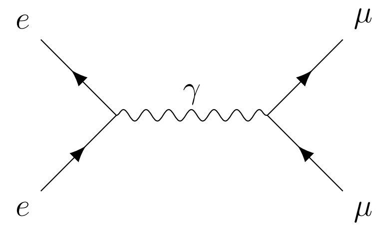
   where S is a statistical factor introduced to avoid double counting when we have identical particles in the final state (known as a symmetry factor). If you have ℓ copies of a given type of particle, then you pick up a factor of $\frac{1}{\ell!}$. e.g. if you have two muons and two electrons in the final state then you have a factor of $\frac{1}{2!2!}$ (while electrons and positrons are NOT considered identical)

2. Scattering:

$$1_i + 2_i \to 1_f + 2_f + ... + n_f \tag{4.2}$$

$$d\sigma = \frac{|\mathcal{M}|^2 \times S(2\pi)^4}{\sqrt{(p_1^\mu p_{2\mu} - m_1 m_2)^2}} \delta^4(p_{1,i} + p_{2,i} - \sum_j p_{j,f}) \times \left(\frac{d^3 p_{1,f}}{(2\pi)^3 2E_{1,f}} \right) \cdots \left(\frac{d^3 p_{n,f}}{(2\pi)^3 2E_{n,f}} \right) \tag{4.3}$$

Thus far we have only discussed non-interacting theories. We will be interested in process such as,



Currently our discussion of QED has only included,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{4.4}$$

We don't want to put interactions in by hand since there are many interactions we could have. We use gauge invariance.

4.2 Schrodinger Equation

The Schrodinger equation says

$$-\frac{1}{2m}\nabla^2\psi(\mathbf{x},t) = i\frac{\partial}{\partial t}\psi(\mathbf{x},t) \tag{4.5}$$

The energy is unchanged if $\psi(\mathbf{x},t) \to e^{i\alpha}\psi(\mathbf{x},t)$. Physical observables, $\psi^\dagger\psi \to \psi'^\dagger\psi'$ are also unchanged under this transformation. As currently stated, this no longer holds if we consider local gauge transformations, $\alpha \to \alpha(\mathbf{x},t)$. Local gauge invariance says

$$\psi(\mathbf{x},t) \to \psi'(\mathbf{x},t) = e^{i\alpha(\mathbf{x},t)}\psi(\mathbf{x},t) \tag{4.6}$$

Hence

$$\nabla\psi(x) \to e^{i\alpha(\mathbf{x},t)}\nabla\psi(\mathbf{x},t) + i(\nabla\alpha)e^{i\alpha(\mathbf{x},t)}\psi(\mathbf{x},t) \tag{4.7}$$

Suppose we conside gauge invariance to a fundamental principle of nature. We try to find a way to fix the equation above to make it invariant under such transformations.

To "fix" this we map our derivatives such that

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + ieA^0 \tag{4.8}$$

$$\nabla \to \nabla - ie\mathbf{A} \tag{4.9}$$

where A^μ is an unknown field called a "gauge field". Consider the following:

$$\left(\frac{\partial}{\partial t} + ieA^0\right)\psi(\mathbf{x},t) \to \left(\frac{\partial}{\partial t} + ie\left(A^0 + \delta A^0\right)\right)e^{i\alpha(\mathbf{x},t)}\psi(\mathbf{x},t) \tag{4.10}$$

$$= e^{i\alpha(\mathbf{x},t)}\left(\frac{\partial}{\partial t} + i\frac{\partial\alpha}{\partial t} + ieA^0 + ie\delta A^0\right)\psi(\mathbf{x},t) \tag{4.11}$$

To ensure gauge invariance we demand

$$e\delta A^0 = -\frac{\partial\alpha}{\partial t} \tag{4.12}$$

A very similar calculation can be done for the gradient. This gives

$$\delta\mathbf{A} = +\frac{1}{e}\nabla\alpha \tag{4.13}$$

In four-vector notation we require

$$\delta A^\mu = -\frac{1}{e}\partial^\mu \alpha \tag{4.14}$$

or

$$A^\mu \to A^\mu - \frac{1}{e}\partial^\mu \alpha \tag{4.15}$$

This is the standard electromagnetic field gauge transformation! So demanding gauge invariance is in some sence equivalent to adding electromagnetic interactions. Our general prescription is then to use what we call a "gauge covarient derivative". We replace

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu \tag{4.16}$$

In the case of the Schrodinger equation we write

$$-\frac{1}{2m}\left(\nabla - ie\mathbf{A}\right)^2 \psi = i\left(\frac{\partial}{\partial t} - ieA^0\right)\psi \tag{4.17}$$

$$\left(-\frac{1}{2m}\left(\nabla - ie\mathbf{A}\right)^2 + eA^0\right)\psi = i\frac{\partial}{\partial t}\psi \tag{4.18}$$

This equation is now gauge invariant.

4.3 Dirac Equation

Our Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{4.19}$$

which is not gauge invariant. To introduce interactions we invoke the prescription that we just developed, $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$. By construction we have (remember that α is a function of space-time)

$$D_\mu\psi \to D'_\mu\psi' = e^{i\alpha}D_\mu\psi \tag{4.20}$$

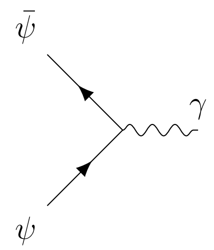
and

$$\bar{\psi} \to e^{-i\alpha}\bar{\psi} \tag{4.21}$$

we have,

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\underbrace{\bar{\psi}\gamma^\mu\psi}_{j^\mu}A_\mu \tag{4.22}$$

Its interesting the note that the conserved current we discovered earlier is the quantity that couples to the vector potential. This suggests that the conserved charge corresponding to this current is indeed the electric charge. We now have coupling terms corresponding to,



Under a gauge transformation we have,

$$\mathcal{L} \to \bar{\psi} e^{-i\alpha} e^{i\alpha} \left(i\partial^{\mu} D_{\mu} - m \right) \psi = \mathcal{L} \tag{4.23}$$

We are however still missing a kinetic term for A^{μ}. If we want to interpret A_{μ} as a photon field we need to have a term corresponding to the energy of a photon to move from place to place. This kinetic term should be quadratic in derivatives of A^{μ} and should not contain any ψ dependence. It should respect the local phase invariance which means the terms you have to use when forming it, $F^{\mu\nu}$ must have a certain form. We also require Lorentz invariance. This gives us two choices

$$\propto F^{\mu\nu} F_{\mu\nu} \tag{4.24}$$

and

$$\propto \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \tag{4.25}$$

The second term violates parity and time reversal which we know from experiment that they should hold for a photon. Thus we are left with

$$\mathcal{L}_{QED} = \bar{\psi} \left(i\gamma^{\mu} D_{\mu} - m \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{4.26}$$

Applying the Euler Lagrange equations with respect to A^{μ} gives Maxwell's equations with the source terms.

Note that there is one more term that is tempting to add,

$$-\frac{m^2}{2} A^{\mu} A_{\mu} \tag{4.27}$$

However, this does not respect local gauge invariance. Taking this is a principle of Nature it implies that the particle corresponding to the A^{μ} field must be massless!

Chapter 5

Group Theory

The principle of gauge invariance appears to hold in nature. We have seen above that using it as a guiding principle we are able to derive interactions between particles and particle properties that we know from electromagnetism. The transformation which we "gauged" or made local was a $U(1)$ transformation, i.e. simply adding a phase. This is the simplest possible local transformation and lead to the simplest Lagrangian that we see in our everyday life. Nature has more gauge invariances and knowing how to deal with them requires a strong understanding of these transformations. In physics transformations tend to form what are known as groups. We now study some Mathematics before studying more complicated theories.

5.1 Definition

A group is a set of symmetry operations on physical systems. It is a set of elements $\{a, b, c, ...\}$ such as the symmetry operations of a triangle. We will have a product operation on those elements which satisfies

1. Closure: If a and b are in the group then $a \cdot b$ is in the group.
2. Associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. Identity: There exists an element I in the group such that $a \cdot I = I \cdot a = a$
4. Inverse: For any element a in the group there exists an inverse element, a^{-1} such that $a^{-1}a = aa^{-1} = I$

Examples of groups and nongroups.

- Addition of real numbers. We show this explicitly as an example. Suppose α and β are real numbers.
  1. $\alpha + \beta \in \mathbb{R}$
  2. $\alpha + \beta = \beta + \alpha$

3. $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$
4. $\alpha \cdot \frac{1}{\alpha} = 1$

- If we have real numbers and multiplication, you don't have a group since zero doesn't have an inverse.

- $e^{i\theta}$ and multiplication, then you do form a group.

Note that there is no commutation rule. You can have non commuting groups.

5.2 Group Examples

Consider the Parity transformation. It forms a group once you include the identity. We can make what's known as a group multiplication table:

	1	P
1	1	P
P	P	1

so we have $P^{-1} = P$.

Next consider the Cyclic group. In a cyclic group each element squared is equal to a new distinct element until the final one which returns to the identity. For example:

$$Z_3 = \{e, a, a^2 = b\} \tag{5.1}$$

with $a^3 = e$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

5.3 Representations

A representation of a group ($\mathcal{G}$) is a map of linear operators (usually matrices) D acting on a vector space V such that for $g_1, g_2 \in \mathcal{G}$,

$$D(g_1)\,|g_2\rangle = |g_1 g_2\rangle \tag{5.2}$$

this is true if and only if

$$D(g_1)D(g_2) = D(g_1 g_2) \tag{5.3}$$

The dimension of the representation is $\dim(V)$. 0

5.4 Types of Representations

5.4.1 Trivial Representation

The trivial representation is for every element $g \in \mathcal{G}$,

$$D(g) = \mathbb{1} \tag{5.4}$$

where $\mathbb{1}$ denotes the identity.

Another useful representation is a reducible representation. This is the case if there is a subspace of V with elements v such that for every group element g we have,

$$D(g)D(v) \in V \tag{5.5}$$

A Completely Reducible representation is one that we can write $D(g)$ is a block diagonal form:

$$\begin{pmatrix} D_1(g) & ... & ... & 0 \\ \vdots & D_2(g) & ... & \vdots \\ \vdots & ... & ... & \vdots \\ 0 & ... & ... & D_N(g) \end{pmatrix} \tag{5.6}$$

An irreducible representation is a not-reducible representation.

5.4.2 Regular representation

A covenient representation is one known as the regular representation. In this representation the group acts on itself. In this representation we can write group elements as both vectors in the space and operators. To see how this works consider the group Z_3. It is not difficult to guess one particular 3 dimensional representation of this group,

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 = \end{pmatrix} \quad , \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad , \quad D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{5.7}$$

Now suppose we define the basis on which this representation acts by,

$$|e\rangle \equiv |e_1\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad |a\rangle \equiv |e_2\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad |b\rangle \equiv |e_3\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.8}$$

We this we have,

$$\langle e_j|D(e)|e_1\rangle = \begin{cases} 1 & j = 1 \\ 0 & j \neq 1 \end{cases} \tag{5.9}$$

$$\langle e_j|D(a)|e_1\rangle = \begin{cases} 1 & j = 2 \\ 0 & j \neq 2 \end{cases} \tag{5.10}$$

$$\langle e_j|D(b)|e_1\rangle = \begin{cases} 1 & j = 3 \\ 0 & j \neq 3 \end{cases} \tag{5.11}$$

and similarly for the group elements acting on the other basis vectors. Thus acting on a "state" of a group element performs the acting of the state. We can always write,

$$D(g_1)\,|g_2\rangle = D(g_1 \cdot g_2)\,|e\rangle \tag{5.12}$$

The states can be created by acting on the identity with group elements.

This representation will be a convenient tool when discussed non-abelian algebras.

5.5 Important Groups

One important group is $U(N)$. This is the group of unitary matrices that act on complex $N-$ vectors, $\chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}$ such that $\chi^\dagger \chi$ is unchanged.

Another important group is $SU(N)$. This is called the special unitary group. It is a subset of $U(N)$ satisfying $\det U = 1$. This still satisfies a group due to the dispersion property of the determinant:

$$U_1, U_2 \in SU(N) \Rightarrow \det(U_1 U_2) = \det(U_1)\det(U_2) = 1 \tag{5.13}$$

5.6 Group Combinations

We can also have a direct product of groups. Suppose each element of a group $\mathcal{G}$ factors into two commuting sets of operators from groups $\mathcal{G}_1, \mathcal{G}_2$ then $\mathcal{G}$ is a direct product of $\mathcal{G}_1, \mathcal{G}_2$:

$$\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2 \tag{5.14}$$

As a trivial example consider the group $U(N)$ with elements u. This group consists of all unitary matrices (which by definition must have determinant equal to a phase). Thus we can write,

$$u = e^{i\chi_s} u_s = u_s e^{i\chi_s} \tag{5.15}$$

where u_s is an element of $U(N)$ with the phase pulled out of it such that $\det u_s = 1$. Then we have $u_s \in SU(N)$. So we can write

$$U(N) = SU(N) \otimes U(1) \tag{5.16}$$

since the group of phases, $e^{i\chi_s}$, commutes with all elements in $SU(N)$.

5.7 Lie Groups

The elements of a Lie group $\mathcal{G}$ depends smoothly on a set of continuous parameters,

$$g(\boldsymbol{\alpha}) = g(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathcal{G} \tag{5.17}$$

If $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are "close to each other" in parameter space then their elements, $g(\boldsymbol{\alpha})g(\boldsymbol{\beta})$ are "close to each other" in group space[1]. We say there is a smooth mapping between the parameters and the group elements themselves.

For clarity we parametrize our group elements with an infinitesimal variable, λ such that

$$g(\lambda\boldsymbol{\alpha})\Big|_{\lambda=0} = e \quad \xrightarrow{\text{or equivantly}} \quad D(\lambda\boldsymbol{\alpha})\Big|_{\lambda=0} = \mathbb{1} \tag{5.18}$$

Near $\lambda = 0$ we can write:

$$D(\lambda\boldsymbol{\alpha}) = \mathbb{1} + i\lambda\alpha_a T_a \tag{5.19}$$

where the factor of i is a conventional convenience. By Taylor's theorem you can think of T_a as

$$T_a = -\frac{\partial D(\boldsymbol{\alpha})}{\partial \alpha_a}\Big|_{\lambda=0} \tag{5.20}$$

The T_a's are the group "generators". We can write (applying the group element an infinite number of times and taking the spacing between group elements to zero)

$$D(\boldsymbol{\alpha}) = \lim_{k\to\infty}\left(\mathbb{1} + \frac{i}{k}\alpha_a T_a\right)^k = e^{i\alpha_a T_a} \tag{5.21}$$

If we have a fixed $\boldsymbol{\alpha}$, it defines a particular "direction" in group space and we have commuting group elements,

$$g(\lambda_1\boldsymbol{\alpha})g(\lambda_2\boldsymbol{\alpha}) = g((\lambda_1 + \lambda_2)\boldsymbol{\alpha}) \tag{5.22}$$

However if you have $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$ then

$$e^{i\alpha_a T_a} e^{i\beta_b T_b} \neq e^{i(\alpha_a + \beta_a)T_a} \tag{5.23}$$

since in general T_a and T_b don't commute. However if $e^{i\boldsymbol{\alpha}_a T_a}$ and $e^{i(\boldsymbol{\beta}_b T_b)}$ are both in the group then we know that we can write

$$e^{i\alpha_a T_a} e^{i\beta_b T_b} = e^{i\delta_c T_c} \tag{5.24}$$

for some δ_c. Its possible to show (though we omit the proof here) that one can expand the above expression to 2^{nd} order and proove that,

$$[T_a, T_b] = i f_{abc} T_c \tag{5.25}$$

If f_{abc} are known then we can determine δ_c. Clearly it's also true that $f_{abc} = f_{-bac}$. The miracle is that this is sufficient to specify all the group multiplication properties. The rules $[T_a, T_b] = i f_{abc} T_c$ are known as the "Lie Algebra". The f_{abc} are known as the "structure constants".

As an example let us now go back to $U(1) = e^{i\theta} = \lim_{k\to\infty}\left(1 + \frac{i}{k}\theta\right)^k$. There is a single generator $T = 1$ and $\alpha = \{\theta\}$. If we make θ arbitrarily small we get close to not changing the phase at all. Note that to go to a local transformation we would let our parameters $\alpha_a \to \alpha_a(x)$. We want properties of T_a for $U(N)$ and $SU(N)$:

[1] The meaning of "close to each other" here is highly informal but this will do for normally be sufficient for a physicist.

1. By definition for $U(\boldsymbol{\alpha}) \in U(N)$ then
$$U(\boldsymbol{\alpha}) = e^{i\alpha_a T_a} = 1 + i\alpha_a T_a + \mathcal{O}(\alpha_a^2) \tag{5.26}$$
and
$$U^{-1}(\boldsymbol{\alpha}) = U^\dagger = (1 - i\alpha_a T_a^\dagger) + \mathcal{O}(\alpha_a^2) \tag{5.27}$$
hence
$$1 = UU^{-1} = (1 + i\alpha_a T_a)(1 - i\alpha_a T_a^\dagger) = 1 + i\alpha_a(T_a - T_a^\dagger) + \mathcal{O}(\alpha^2) \tag{5.28}$$
This can only hold if the generators are Hermitian[2],
$$T_a^\dagger = T_a \tag{5.29}$$

2. Now suppose $U(N) \in SU(N)$. In that case
$$\det U(\boldsymbol{\alpha}) = \det e^{i\alpha_a T_a} = e^{i\mathrm{Tr}(\alpha_a T_a)} = e^{i\alpha_a \mathrm{Tr}(T_a)} \tag{5.30}$$
For this to be equal to 1 we must have
$$\mathrm{Tr}(T_a) = 0 \tag{5.31}$$
Hence the $SU(N)$ generators are traceless and Hermitian.

The Jacobi identity says that
$$[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0 \tag{5.32}$$
Inserting our results above we have,
$$[T_a, [T_b, T_c]] = i f_{bcd}\,[T_a, T_d] = -f_{bcd} f_{ade} T_e \tag{5.33}$$
For the Jacobi identity to hold for every generator requires the strucutre constants to obey,
$$f_{bde} f_{ade} + f_{abd} f_{cde} + f_{cad} f_{bde} = 0 \tag{5.34}$$

5.8 $SU(2)$

We now consider our first non-abelian group. A complete set of 2×2 matrices is spanned by the Pauli Matrices:
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.35}$$

[2]This was the reason for the conventional i in the definition of $e^{i\alpha_a T_a}$. Otherwise we would have antihermitian generators

$U(2)$ has 4 generators.

For $SU(2)$ we require traceless generators (as mentioned above). Hence the identity doesn't qualify. To normalize the modulus of our structure constants to 1 we divide each matrix by 2. Then we have our $SU(2)$ generators,

$$SU(2): \quad T_a = \left\{ \frac{\sigma_1}{2}, \frac{\sigma_2}{2}, \frac{\sigma_3}{3} \right\} \tag{5.36}$$

Its easy to calculate the commutators. As an example consider,

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{5.37}$$

and

$$\sigma_2\sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{5.38}$$

which give

$$\left[\frac{\sigma_1}{2}, \frac{\sigma_2}{2} \right] = 2\frac{1}{4} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\frac{\sigma_3}{2} \tag{5.39}$$

hence we have $f_{123} = 1$. In general one can show that,

$$[T_i, T_j] = i\epsilon_{ijk}T_k \tag{5.40}$$

The $SU(2)$ structure constants are $f_{ijk} = \epsilon_{ijk}$

5.9 $SU(3)$

The $SU(3)$ dimension 3 representation is given by the Gell-Man Matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad , \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad , \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad , \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The $SU(3)$ generators are given by $T_i = \frac{\lambda_i}{2}$. The structure constants are given by

$$\begin{aligned}
&f_{123} = 1 \\
&f_{458} = f_{678} = \frac{\sqrt{3}}{2} \\
&f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2} \\
&f_{38j} = 0 \quad \forall j
\end{aligned}$$

Note that we don't have any structure constants that have both a 3 and an 8 since λ_3 and λ_8 commute.

5.10 From Group Theory to Fields

We say a field ϕ is in a representation D of $\mathcal{G}$ if and only if ϕ transforms as $\phi \to D(g)\phi$.

[Q 2: Fix up this intro.] We begin by looking at $SU(2)$. The natural way to look at $SU(2)$ is by two component spinors $\begin{pmatrix} \eta_u \\ \eta_d \end{pmatrix}$. The Pauli matrices build transformations of the spinors. We also know that the Pauli matrices give us the generators of $SU(2)$. The spinors themselves give us what's known as the "fundamental representation".

In $SU(N)$ the dimension of fundamental representation (denoted N) is also N. The anti-fundamental representation (denoted $\bar{N}$), $(U \to U^*)$, or explicitly:

$$U = 1 + i\alpha_a T_a \quad \Rightarrow \quad U^* = 1 - i\alpha_a T_a^*$$

Thus the generators of the anti-fundamental representation are $-T_a^*$. Hence the anti-fundamental and fundamental representations have the same structure constants.

A third interesting representation is the adjoint representation. One definition of this representation is in terms of the matrices T^b with elements,

$$(T^b)_{ac} = i f_{abc} \tag{5.41}$$

The dimension of the representation (the dimension of the matrices) is given by the number of generators. Another equivalent representation to the adjoint representation is a group of $N_g \times N_g$ (N_g is the number of generators) matrices that transforms as N on first index and transform as $\bar{N}$ on the second index. In the adjoint representation, each generator maps to a basis state in the vector space. [Q 3: Understand this second repersentation of the adjoint.]

We now go back to spin 1 spinors. These spinors transform as a 3 dimension $SU(2)$ representation,

$$T_1 \equiv J_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad , \quad T_2 \equiv J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$
$$T_3 \equiv J_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5.42}$$

Our J_i do not commute. Of particular interest for us is known as the Cartan Subalgebra. It is the largest set of generators that commute. In $SU(2)$ we only have only have 1 generator that simultaneously commutes with other generators in a set (in other words none of the operators commute with each other). We conventionally choose J_3 to make up the Cartan Subalgebra in this case.

The raising and lowering operators are given by

$$J^{\pm} = \frac{J_1 \pm J_2}{\sqrt{2}} \tag{5.43}$$

So

$$J_3 \left(J^{\pm} \left| m \right\rangle \right) = (m \pm 1)\, J^{\pm} \left| m \right\rangle \tag{5.44}$$

It turns out that a particular representation of $SU(2)$ is completely defined by the matrix elements that connect the various $|j, m\rangle$ states.

$$\langle j, m' | J_3 | j, m \rangle = m \delta_{m,m'} \tag{5.45}$$

$$\langle j, m' | J^{+} | j, m' \rangle = \sqrt{(j+m+1)(j-m)}\,\delta_{m',m+1} \tag{5.46}$$

$$\langle j, m' | J^{-} | j, m' \rangle = \sqrt{(j-m+1)(j+m)}\,\delta_{m',m-1} \tag{5.47}$$

In the spin j representation the matrix elements of our generators J_a are given by,

$$\left(J_a^j \right)_{k\ell} = \langle j, \underbrace{j+1-k}_{m'} | J_a | j, \overbrace{j+1-\ell}^{m} \rangle \tag{5.48}$$

where $J_a = \{J_1, J_2, J_3\}$[3].

We call the Cartan Subalgebra generators, X_i. For $SU(2)$ we have $J_3 = X_1$ For $SU(3)$ we have T_3, T_8 as the Cartan Subalgebra (as these are the ones that commute). In $SU(2)$:

$$X_1 = J_3 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \tag{5.49}$$

and in $SU(3)$:

$$X_1 = T_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad X_2 = T_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{5.50}$$

Suppose that our representation has X_i diagonalized. Then the the states of the representation can be labeled as (these are the only quantities of the state we can know at the same time so make a good label for our states)

$$|\boldsymbol{\mu}, n\rangle \tag{5.51}$$

where n is some set of quantum numbers and μ_i are defined by

$$X_i \left| \boldsymbol{\mu}, n \right\rangle = \mu_i \left| \boldsymbol{\mu}, n \right\rangle \tag{5.52}$$

[3]The addition of angular momentum can be viewed as simply the direct product of $SU(2)$ spaces involved,

$$\left| m_{1/2} \right\rangle \left| m_1 \right\rangle$$

The J_3 's of product states will add.

where $\boldsymbol{\mu} = (\mu_1, \mu_2, ..., \mu_{N_C})$ (N_C is equal to the number of generators in the Cartan Subalgebra) are the "weight vectors" of a given state and μ_i 's are the weights.

For general $SU(N)$ we have more then just one direction to raise and lower our states; We need multiple labels! For $SU(2)$, $j = 1/2$:

J_3 eigenstates	$\boldsymbol{\mu}$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\frac{1}{2}$
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$-\frac{1}{2}$

We can only raise and lower in one direction.

This is not the case in $SU(3)$:

T_3, T_8 eigenstates	$\boldsymbol{\mu}$
$\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$	$\left\{\frac{1}{2}, \frac{\sqrt{3}}{6}\right\}$
$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$	$\left\{-\frac{1}{2}, \frac{\sqrt{3}}{6}\right\}$
$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$	$\left\{0, -\frac{\sqrt{3}}{3}\right\}$

where the values of $\boldsymbol{\mu}$ are found by acting on the eigenstates with T_3 and T_8.

In the adjoint representation, we get the generalization of the raising and lowering operators. The rows and columns of the matrices (defined by $(T_b)_{ac} = -if_{abc}$) are labeled by the generator label so the dimension of the states is equal to the number of generators. This is reminiscent of the regular representation we saw earlier. As was the case there we can choose the basis of our states such that we can label them using the generators.

$|T_a\rangle = T_a |e\rangle$ is equal to the state in the adjoint representation that corresponds to generator T_a. We have

$$\alpha |T_a\rangle + \beta |T_b\rangle = |\alpha T_a + \beta T_b\rangle \tag{5.53}$$

where α, β are just numbers. We now take one of our generators and act on the state that corresponds to another generator:

$$\begin{aligned}
T_a |T_b\rangle &= |T_c\rangle \langle T_c|T_a|T_b\rangle && (5.54)\\
&= |T_c\rangle (T_a)_{bc} && (5.55)\\
&= -if_{acb} |T_c\rangle && (5.56)\\
&= if_{abc} |T_c\rangle && (5.57)\\
&= |if_{abc} T_c\rangle && (5.58)\\
&= |[T_a, T_b]\rangle && (5.59)
\end{aligned}$$

The Cartan generators, X_i, give

$$X_i |X_j\rangle = |[X_i, X_j]\rangle = 0 \tag{5.60}$$

since by definition the Cartan generators commute with each other, for any X_i in the Cartan Subalgebra the weight vector ($\boldsymbol{\mu}$) is zero.

Now consider non-Cartan generator states, $|E_{\boldsymbol{\alpha}}\rangle$,

$$X_i |E_{\boldsymbol{\alpha}}\rangle \equiv \alpha_i |E_{\boldsymbol{\alpha}}\rangle = |\alpha_i E_{\boldsymbol{\alpha}}\rangle \tag{5.61}$$

where α_i are the weights in the adjoint representation and are also known as the "roots". We can also write

$$X_i |E_{\boldsymbol{\alpha}}\rangle = |[X_i, E_{\boldsymbol{\alpha}}]\rangle \tag{5.62}$$

comparing these two results above we have,

$$[X_i, E_{\boldsymbol{\alpha}}] = \alpha_i E_{\boldsymbol{\alpha}} \tag{5.63}$$

This reminds us of $SU(2)$ where we have,

$$\left[J_3, J^{\pm}\right] = \pm J^{\pm} \tag{5.64}$$

Notice the parallel between the $E_{\boldsymbol{\alpha}}$ and the $J^{\pm}$ (in $SU(2)$, $X_1 = J^3$). Recall also that

$$J^{-} = \left(J^{+}\right)^{\dagger} \tag{5.65}$$

To find the analogous expression in $SU(N)$ note that for a unitary group the diagonal generators must be real (otherwise they couldn't be hermitian). So we can write,

$$[X_i, E_{\boldsymbol{\alpha}}]^{\dagger} = \left[E_{\boldsymbol{\alpha}}^{\dagger}, X_i\right] = -\left[X_i, E_{\boldsymbol{\alpha}}^{\dagger}\right] \tag{5.66}$$

The left hand side can also be written

$$\langle E_{\boldsymbol{\alpha}}| X_i = \alpha_i E_{\boldsymbol{\alpha}}^{\dagger} \tag{5.67}$$

hence we have

$$\left[X_i, E_{\boldsymbol{\alpha}}^{\dagger}\right] = -\alpha_i E_{\boldsymbol{\alpha}}^{\dagger} \tag{5.68}$$

which implies that,

$$E_{\boldsymbol{\alpha}}^{\dagger} = E_{-\boldsymbol{\alpha}} \tag{5.69}$$

Consider now some state given by its weight vector and some quantum numbers which we denote n.

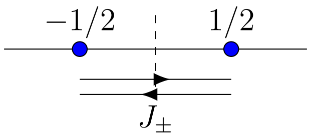
$$\begin{aligned} X_i E_{\boldsymbol{\alpha}} |\boldsymbol{\mu}, n\rangle &= [X_i, E_{\boldsymbol{\alpha}}] |\boldsymbol{\mu}, n\rangle + E_{\boldsymbol{\alpha}} X_i |\boldsymbol{\mu}, n\rangle && (5.70) \\ &= \alpha_i E_{\boldsymbol{\alpha}} |\boldsymbol{\mu}, n\rangle + \mu_i E_{\boldsymbol{\alpha}} |\boldsymbol{\mu}, n\rangle && (5.71) \\ &= (\mu_i + \alpha_i) E_{\boldsymbol{\alpha}} |\boldsymbol{\mu}, n\rangle && (5.72) \end{aligned}$$

Hence $E_{\boldsymbol{\alpha}}(E_{-\boldsymbol{\alpha}})$ are the raising (lowering) operators and $E_{\pm\boldsymbol{\alpha}} |\boldsymbol{\mu}, n\rangle$ takes state $|\boldsymbol{\mu}, n\rangle$ with a weight vector $\boldsymbol{\mu}$ to a state with weight vector $\boldsymbol{\mu} \pm \boldsymbol{\alpha}$. The root vectors $\boldsymbol{\alpha}$ give the direction that the root vectors are "raised" or "lowered".

5.11 $SU(3)$ and $SU(2)$

The weights in a given representation will represent the different states that we are in. The roots will let us move from state to state. Note that the roots are not independent of the weights. They must correspond to different states in your system since they connect them.

Consider $SU(2)$ in the fundamental representation. Our weights are just $-1/2, 1/2$ are our roots are just ± 1. Diagramically we have,



The weight diagram for the $\bar{2}$ weight diagram is identical to the 2 representation diagram. This is representative of the fact that the two representations are equivalent representations (the only differ by a Levi-Cevita tensor),

$$2 \cong \bar{2} \tag{5.73}$$

Now consider a more intricate group, $SU(3)$. We already know all the weight vectors. We listed them in a table above. To get from the weight vector $\left(1/2, \sqrt{3}/6\right)^T$ to $\left(-1/2, \sqrt{3}/6\right)$ requires a vector $(1,0)^T$. To get from these two vectors to $\left(0, -\sqrt{3}/3\right)^T$ requires vectors of $\left(1/2, \sqrt{3}/2\right)^T$ and $\left(-1/2, \sqrt{3}/2\right)^T$. These are the roots. Alternatively one can arrive at these vectors by finding the basis in which the Carton generators are diagonal. In the fundamental of $SU(3)$ this turns out to be,

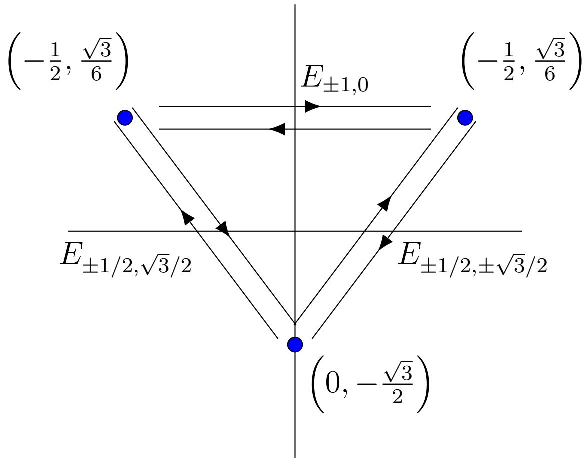
$$E_{\pm 1,0} = \frac{1}{\sqrt{2}} \left(T_1 \pm i T_2\right) \tag{5.74}$$

$$E_{\pm\frac{1}{2},\pm\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{2}} \left(T_4 \pm i T_5\right) \tag{5.75}$$

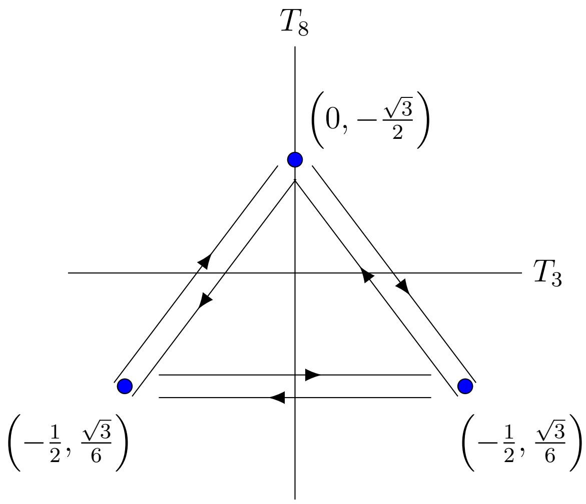
$$E_{\mp\frac{1}{2},\pm\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{2}} \left(T_6 \pm i T_7\right) \tag{5.76}$$

The first index on the $E_{\boldsymbol{\alpha}}$ labels the T_3 weight component and the second denotes the T_8 weight component.

The weight diagram shows all the weights and the transformations connecting it. It is given by



On the other hand for $SU(3)$, $\bar{3}$ is given by



This is clearly distinct from the 3 representation of $SU(3)$. The 3 states of a given weight vector in $SU(3)$ are mapped to color while the $\bar{3}$ states are mapped to anticolour.

Consider two spin-$\frac{1}{2}$ fields, in this case the J_3 's add and the J's give,

$$2 \otimes 2 = \underbrace{3}_{J=1} \oplus \overbrace{1}^{J=0} \tag{5.77}$$

For $SU(3)$ we want the T_3 and T_8 to add. To get observables we must also have the singlet state. For qq:

$$qq : 3 \otimes 3 = (6 \oplus 3) \tag{5.78}$$

We don't get a singlet state. On the other hand

$$q\bar{q} : 3 \otimes \bar{3} = (8 \oplus 1) \tag{5.79}$$

$$qqq : 3 \otimes 3 = (10 + 8 + ... + 1) \tag{5.80}$$

Thus we can get the singlet state through either a meson ($q\bar{q}$) or a baryon (qqq). One can go on and study bound states with larger numbers of quarks but these are much less common in Nature.

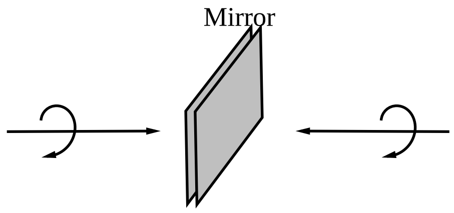
5.A Discrete Symmetries in Quantum Field Theory

This section is based on [Peskin and Schroeder(1995)] chapter 3.6. I added this section for completeness and it's results will be important later. Lorentz transformations are the group of continuous operations that keep the Minkowski interval invariant. However we can expand this definition to include Parity, $\mathcal{P}$, and Time reversal, $\mathcal{T}$:

$$\mathcal{P}\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \qquad \mathcal{T}\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

since they also keep the quantity $t^2 - \mathbf{x}^2$ invariant. We define a proper orthochronous Lorentz group by $L_+^\uparrow$ the group without Parity and Time reversal. Every quantum field theory must be invariant under these transformations. The "orthochronous improper Lorentz Group" , $L_-^\uparrow = \mathcal{P}, L_+^\uparrow$, is the Lorentz group including Parity. The "nonorthochronous proper Lorentz Group" , $L_+^\downarrow = \mathcal{T}, L_+^\uparrow$, is the Lorentz group including time reversal and finally the "nonorthochronous improper Lorentz Group" , $L_-^\downarrow = \mathcal{P}, \mathcal{T}, L_+^\uparrow$ is the Lorentz group including Time reversal and Parity. The gravitational, electromagnetic, and strong interactions turn out to be symmetric under $\mathcal{P}$ and $\mathcal{T}$. The weak interactions do not have this property but break Parity as we will see later.

The Parity operator should flip the momentum of particles but not their spin:



If we denote the annihilation operators as $a_{\mathbf{p}}^s$ then we must have $\mathcal{P}a_{\mathbf{p}}^s\mathcal{P} = \eta_a a_{-\mathbf{p}}^s$ where η_a is some phase which naively obeys $\eta_a^2 = 1$ (acting with Parity twice must give back the original value). This is actually two restrictive for some theories. In the case of Dirac fermions they always come in pairs. So it's enough to require $\eta_a^2 = \pm 1$. We will focus on Dirac fermions. In the interaction picture Dirac fermionic fields take the form,

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_{\mathbf{p}}^s u^s(p) e^{-ip\cdot x} + b_{\mathbf{p}}^{s\dagger} v^s(p) e^{ip\cdot x} \right) \tag{5.81}$$

where a^s and b^s are annihilation operators. Now consider the parity operator acting on the field:

$$\mathcal{P}\psi(x)\mathcal{P} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(\eta_a a^s_{-\mathbf{p}} u^s(p) e^{-ip\cdot x} + \eta^{b*} b^{s\dagger}_{-\mathbf{p}} v^s(p) e^{ip\cdot x} \right) \tag{5.82}$$

We now change variables using $\mathbf{p} \to -\mathbf{p}$. This gives $p \cdot x \to p \cdot (t, -\mathbf{x})$, $p \cdot \bar{\sigma} \to p \cdot \sigma$, $p \cdot \bar{\sigma} \to p \cdot \sigma$ (since $\bar{\sigma} \equiv (1, -\boldsymbol{\sigma})$). Thus we can write

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} \to \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\xi \\ \sqrt{p\cdot\sigma}\xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u(p) = \gamma^0 u(p) \tag{5.83}$$

where ξ is some spin state. Similarly we have

$$v(p) \to \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\xi \\ -\sqrt{p\cdot\sigma}\xi \end{pmatrix} = -\gamma^0 v(p) \tag{5.84}$$

so we can write

$$\mathcal{P}\psi(x)\mathcal{P} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(\eta_a \gamma^0 a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot(t,-\mathbf{x})} - \eta_b^* b^{s\dagger}_{\mathbf{p}} \gamma^0 v^s(p) e^{ip\cdot(t,-\mathbf{x})} \right) \tag{5.85}$$

If we have a Parity eigenstate then should should be equal to some constant matrix times $\psi(t, -\mathbf{x})$. This is the case if $\eta_b^* = -\eta_a$. Then we have

$$\mathcal{P}\psi(x)\mathcal{P} = \eta_a \gamma^0 \psi(t, -\mathbf{x}) \tag{5.86}$$

Notice that the positive and negative frequencies in the Dirac fermion are not independent of one another. They must transform in a related way to carefully preserve Parity.

Chapter 6

QCD

We want to write down Lagrangians. Recall that in the SM we have 6 quark flavors: u, d, c, s, t, b. Each flavor comes in 3 colors (r,g,b). We begin by considering a single quark flavor. We can write

$$\mathcal{L}_0 = \bar{q}\underbrace{\left(i\gamma^\mu\partial_\mu - m\right)}_{(i\gamma^\mu\partial_\mu - m)\mathbb{1}_{3\times 3}} q \tag{6.1}$$

with

$$q \equiv \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad , \quad \bar{q} \equiv (\bar{q}_1, \bar{q}_2, \bar{q}_3) \tag{6.2}$$

with each of q_i being one four component Dirac fermion. Explicitly we have

$$\mathcal{L}_0 = \bar{q}_1\left(i\gamma^\mu\partial_\mu - m\right) q_1 + \bar{q}_2\left(i\gamma^\mu\partial_\mu - m\right) q_2 + \bar{q}_3\left(i\gamma^\mu\partial_\mu - m\right) q_3 \tag{6.3}$$

For a global flavor $SU(3)$ transformation:

$$q \to q' = Uq \tag{6.4}$$

$$\bar{q} \to \bar{q}' = \bar{q}U^\dagger = \bar{q}U^{-1} \tag{6.5}$$

with $U \in SU(3)$. Thus

$$\mathcal{L}_0' = \bar{q}U^{-1}(i\gamma^\mu\partial_\mu - m)Uq = \bar{q}(i\gamma^\mu\partial_\mu - m)q = \mathcal{L}_0 \tag{6.6}$$

so we have a global $SU(3)$ symmetry.

Just as we did in the $U(1)$ case to produce electromagnetism we now want to consider local transformations. This involves

$$U = e^{ig\alpha_a T_a} \to e^{ig\alpha_a(x)T_a} \approx 1 + ig\alpha_a(x)T_a \tag{6.7}$$

Our local transformation is then

$$q \to Uq = (1 + ig\alpha_a(x)T_a)q \tag{6.8}$$

$$\bar{q} \to \bar{q}U^\dagger = \bar{q}(1 - ig\alpha_a(x)T_a) \tag{6.9}$$

We have

$$\partial_\mu q \to \partial_\mu q' = (1 + ig\alpha_a(x)T_a)\partial_\mu q + \underbrace{ig(\partial_\mu \alpha_a)T_a q}_{\text{extra term}} \tag{6.10}$$

We can eliminate this extra term by changing

$$\partial_\mu \to D_\mu = \partial_\mu - igT_a G^a_\mu \tag{6.11}$$

where G^a_μ is an known collection of 8 fields (in $SU(3)$). The sum over a is just a straight forward sum over the generator label (in other words $G^a_\mu = G_{a,\mu}$). We now write

$$\begin{aligned} D^\mu q \to & \left[\partial_\mu - igT_a(G^a_\mu + \delta G^a_\mu)\right](1 + ig\alpha_a T_a)q & (6.12)\\ &= (1 + ig\alpha_a T_a)\,\partial_\mu q + igT_a q\partial_\mu \alpha_a - igT_a G^a_\mu(1 + ig\alpha_b T_b)q & \\ &\quad - igT_a \delta G^a_\mu\,(1 + ig\alpha_a T_a)\,q & (6.13)\\ &= (1 + ig\alpha_a T_a)\underbrace{(\partial_\mu - igT_a G^a_\mu)}_{D_\mu} q + (1 + ig\alpha_a T_a)(iT_b G^b_\mu)q & \\ &\quad - igT_a(1 + g\alpha_b T_b)G^a_\mu q - igT_a\delta G^a_\mu q + igT_a q\partial_\mu \alpha_a & (6.14) \end{aligned}$$

where we have thrown away one of the terms of order the product of $\alpha_a G^a_\mu$ (we are assuming that both α_a and δG^a are small). Simplifying we have

$$\begin{aligned} D_\mu q \to & (1 + g\alpha_a T_a)D_\mu q - g^2\alpha_b\,[T_b, T_a]\,G^a_\mu q - igT_a\delta G^a_\mu q + igT_a q\partial_\mu\alpha_a & (6.15)\\ &= (1 + g\alpha_a T_a)D_\mu q - ig^2\alpha_b f_{bac}T_c G^a_\mu q + igT_a q\partial_\mu \alpha_a - igT_a\delta G^a_\mu q & (6.16) \end{aligned}$$

In order to have a transformation we require

$$0 = -g^2\alpha_b f_{bac}T_c G^a_\mu + gT_a\partial_\mu\alpha - gT_a\delta G^a_\mu \tag{6.17}$$

For Hermitian generators one can show that we can always choose a basis for the generators so that

$$\mathrm{tr}(T_a T_b) \propto \delta_{ab} \tag{6.18}$$

For $SU(3)$ and $SU(2)$ in the fundamental representations we have

$$\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab} \tag{6.19}$$

Mutliplying equation 6.17 by T_d and taking the trace gives,

$$\begin{aligned} 0 &= -g^2\alpha_b f_{bac}\mathrm{tr}(T_d T_c)G^a_\mu + g\mathrm{tr}(T_d T_a)\partial_\mu\alpha_a - g\mathrm{tr}(T_d T_a)\delta G^a_\mu & (6.20)\\ &= -g^2\alpha_b f_{bad}G^a_\mu + g\partial_\mu\alpha_d - g\delta G^a_\mu & (6.21) \end{aligned}$$

Relabeling our indices by $d \to a, a \to c$ and using $f_{bca} = f_{abc}$ then we have

$$\delta G^c_\mu = \partial_\mu\alpha_c - g\alpha_b f_{abc}G^a_\mu \tag{6.22}$$

In an abelian group we don't have the extra term (in such a group we don't have trace relationship).

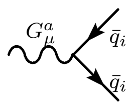
In order to have gauge invariance we require

$$G_\mu^a \to G_\mu^a + \partial_\mu \alpha_a - g\alpha_b f_{abc} G_\mu^c \tag{6.23}$$

Having the coupling strength appear in the gauge transformation means that g is independent of flavor. In other words all flavors will have the same coupling strength. Note that the gluon fields mix between one another under this transformation. This suggests (and we'll see this later too) that the gluons carry color.

In summary we have

$$\mathcal{L}_0 \to \mathcal{L} = \bar{q}\left(i\gamma^\mu \partial_\mu - m\right) q + g\bar{q} T_a G_\mu^a \gamma^a q \tag{6.24}$$

We have a vertex 

$\to$ $ig(T_a)_{ij}$. We handle color flow at the vertex.

One may ask why we know that we have $SU(3)$ and not $U(3)$. The $U(3)$ are the generators T_a of $SU(3)$ and $\mathbb{1}$. $\mathbb{1}$ commutes with everything, hence the corresponding structure constant would be $f_{9ab} = 0$. Under a $U(3)$ rotation

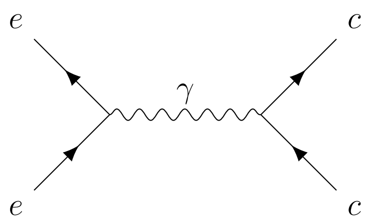
$$G_\mu^9 \to G_\mu^9 + \partial_\mu q \tag{6.25}$$

with respect to color G_μ^9 transforms like a color singlet. It carries no color charge. We would expect this 9^{th} field to be a force similar to the photon (a long range force). We do not see such force in nature.

To see this consider the following process

$$e^+ e^- \to J/\psi \to u\bar{u} \tag{6.26}$$

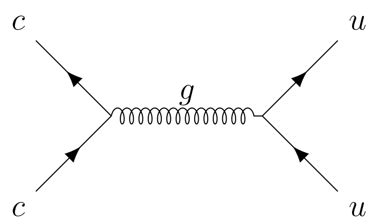
or diagrammatically we have



where

$$|c\bar{c}\rangle = \frac{1}{\sqrt{3}} \sum_i |c_i \bar{c}_i\rangle \tag{6.27}$$

is the J/ψ bound state of $c\bar{c}$. What we want to know is whether J/ψ can decay through a single gluon annihilation. In other words through for example,



The vertex terms are

$$\frac{iq}{\sqrt{3}} \sum_i \bar{c}_i \gamma^\mu c_i G^a_\mu \quad \propto \quad \alpha_a \mathrm{tr}(T_a) \tag{6.28}$$

However for $SU(3)$ the T_c 's are traceless and the vertex to lowest order is zero. On the other hand for $U(3)$ where we have the $\mathbb{1}$ term which is not traceless. Experimentally the J/ψ decays are heavily suppressed.

6.1 Gluon Kinetic Term

Our next step to building a QCD Lagrangian is to build the kinetic terms for the gluons. We will work by analogy with the photon however the different gauge transformation gives some complications. For a covarient derivative we want

$$D'_\mu U\psi = U(D_\mu \psi) \tag{6.29}$$

so that $D_\mu\psi$ transforms just like ψ. Applying this one more time we have

$$D'_\nu D'_\mu (U\psi) = D'_\nu U(D_\mu\psi) = U(D_\nu D_\mu \psi) \tag{6.30}$$

Furthermore we have from above that

$$U^{-1} D'_\mu U\psi = D_\mu \psi \tag{6.31}$$

hence

$$U^{-1} D'_\mu U = D_\mu \tag{6.32}$$

(or equivalently $D_\mu = U D'_\mu U^{-1}$). From this we know that

$$\left[D'_\mu, D'_\nu\right] = D'_\mu D'_\nu - D'_\nu D'_\mu \tag{6.33}$$

$$= U\left[D_\mu, D_\nu\right] U^{-1} \tag{6.34}$$

For $U(1)$ we have

$$\left[D_\mu, D_\nu\right]\psi = \left[\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu\right]\psi \tag{6.35}$$

$$= ie\left(\left[\partial_\mu, A_\nu\right] + \left[A_\mu, \partial_n\right]\right)\psi \tag{6.36}$$

Consider for a moment

$$[\partial_\mu, A_\nu]\,\psi = \overbrace{(\partial_\mu A_\nu)\,\psi + A_\nu \partial_\mu \psi}^{\text{product rule}} - A_\nu \partial_\mu \psi \tag{6.37}$$

$$= (\partial_\mu A_\nu)\,\psi \tag{6.38}$$

Thus we have for $U(1)$:

$$[D_\mu, D_\nu]\,\psi = ie\,(\partial_\mu A_\nu - \partial_\nu A_\mu)\,\psi \tag{6.39}$$

$$= ieF_{\mu\nu}\psi \tag{6.40}$$

The left hand side transforms as $[D_\mu, D_\nu]\,\psi \to U\,([D_\mu, D_\nu]\,\psi)$ In order for this to hold for the right hand side we require $F_{\mu\nu} \to F_{\mu\nu}$. In other words $F_{\mu\nu}$ is gauge invariant ($F_{\mu\nu}$ is the component so it can't produce factors of U). In $U(1)$ its easy to guess a form for a gauge invariant quantity since the transformation is simple. Complications arise for more complicated groups.

We now consider $SU(N)$ with the covariant derivative,

$$D_\mu = \partial_\mu - igT_a G^a_\mu \tag{6.41}$$

$$[D_\mu, D_\nu]\,\psi = \left[\partial_\mu - igT_a G^a_\mu, \partial_\nu - igT_b G^b_\nu\right] \tag{6.42}$$

$$= -ig\left\{ \overbrace{\left[\partial_\mu, T_b G^b_\nu\right]}^{\text{analogous to before}} + \left[T_a G^a_\mu, \partial_\nu\right] \right\}\psi - g^2\,[T_a, T_b]\, G^a_\mu G^b_\nu \tag{6.43}$$

$$= -ig\left\{T_b \partial_\mu G^b_\nu - T_a \partial_\nu G^a_\mu\right\}\psi - g^2 i f_{abc} T_c G^a_\mu G^b_\nu \psi \tag{6.44}$$

$$= -ig\left\{T_a \partial_\mu G^a_\nu - T_a \partial_\nu G^a_\mu + g f_{bca} T_a G^b_\mu G^c_\nu\right\}\psi \tag{6.45}$$

$$\equiv -igT_a G^a_{\mu\nu}\psi \tag{6.46}$$

where after cycling the structure constant indices we define

$$G^a_{\mu\nu} \equiv \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f_{abc} G^b_\mu G^c_\nu \tag{6.47}$$

Based on our earlier discussion we know that the combination of $igT_a G^a_{\mu\nu}$ is gauge invariant. It's important to note that we don't expect the $G^a_{\mu\nu}$ to be gauge invariant on it's own but the sum over $T_a G^a_{\mu\nu}$ should be. It turns out that the combination $G^a_{\mu\nu} G^{\mu\nu}_a$ is gauge invariant.

Our full QCD Lagrangian is given by

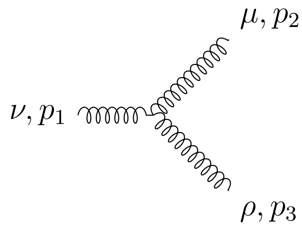
$$\mathcal{L}_{QCD} = \bar{q}\,(i\gamma^\mu \partial_\mu - m)\,q + g\bar{q}T_a G^a_\mu \gamma^\mu q - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \tag{6.48}$$

To get a feel for our kinetic term we expand it out

$$-\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a = -\frac{1}{4}(\partial_\mu G_\nu a - \partial_\nu G^a)(\partial_\mu G^\nu_a - \partial^\nu G^\mu_a) - \frac{g}{2}(\partial_\mu G^a_\nu - \partial_\nu G^a_\nu) f_{abc} G^\mu_b G^\nu_c$$
$$- \frac{g^2}{4}\left(f_{abc} f_{adc} G^b_\mu G^c_\nu G^\mu_d G^\nu_e\right) \tag{6.49}$$

In $U(1)$ gauge theories all interactions arise from adding in the covariant derivatives. However, this is not the case for non-abelian gauge theories. Here we get more interactions just between the gluon fields!

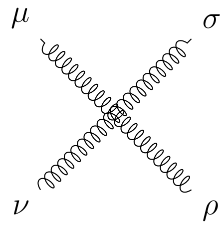
Notice the triple gauge vertex between three gluons,



this gives a vertex term of

$$-gf_{abc}\left(g_{\mu\nu}(p_1-p_2)_\rho+g_{\nu\rho}(p_2-p_2)_\mu+g_{\rho\mu}(p_3-p_1)_\nu\right) \tag{6.50}$$

The quadropole gauge boson vertex,



which gives the vertex,

$$-ig^2\left(f_{abc}f_{cde}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}\right)+f_{ace}f_{bde}\left(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\nu\rho}\right)+f_{ade}f_{cbe}\left(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\nu}g_{\rho\sigma}\right)\right) \tag{6.51}$$

You may wonder if we were a bit too quick with pulling out Feynman diagrams. We haven't taken the time to derive Feynman diagrams in non-Abelian theories and indeed there are some complications. It turns out if you carefully quantize QCD your answer doesn't quite come out right. The problem is with the gauge choice. The physical gauge ("axial") gives gluons that have only ± 1 helicity (no longitudinal polarization, i.e. helicity 0). Our interactions are still correct however our propagator is more complicated.

In our gauge, the "covariant gauge", the nonabelian turns when you quantize your theory will give you an unphysical time-like longitudinal piece. These contributions violate unitarity. However if you do the full analysis you'll find that in addition to the gluons you get what's known as "Fadeev-Popov ghosts". These are scalar fields that satisfy Fermi statistics. Fortunately these particles appear only in loops. These contributions exactly cancel the longitudinal gluons.

6.2 Running of Couplings

When doing calculations in quantum field theory the cross-sections depend on the couplings in your theory. In bare perturbation theory the couplings are typically infinite.

However, even in renormalized perturbation theory there is no clear definition of the coupling strength. In general we can renormalize our theory with renormalization conditions at any scale we choose. However, if we choose a renormalization scale far from the scale of the problem we are studying then we end up with what's known as large log corrections which spoil the perturbation theory. These large logs can be summed to all order in perturbation theory. The effect of this sum is to change the effective couplings felt by the particles depending on the scale in which the interaction takes place. This effect is small in QED however turns out to be crucial in QCD.

In QED at very small momenta we have $\alpha_{QED} \approx 1/127$ but at the Z boson scale,

$$\alpha_{QED}(\mu^2 = m_z^2) = \frac{e^2(\mu^2 = m_z^2)}{4\pi} = \frac{1}{128} \tag{6.52}$$

Doing a full quantum field theory calculation one can find the couples at different scale μ from a reference scale m_z by

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(m_z)} + b_i \ln \frac{m_z^2}{\mu^2} \tag{6.53}$$

where b_i is known as the beta function. In QED we have,

$$b_{QED} = \frac{1}{3\pi}\left[\frac{1}{9}n_{1/3} + \frac{4}{9}n_{2/3} + n_1\right] \tag{6.54}$$

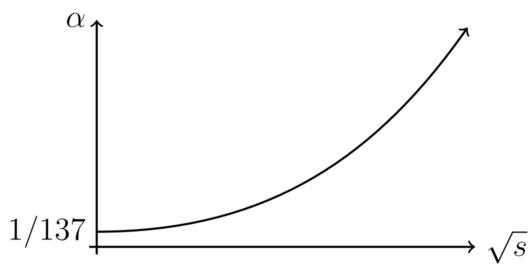
where n_q denotes the number of quarks below the energy scale of interest with charge q. For example if $m_b < \mu < m_z$ then we have

$$n_{1/3} = \overbrace{3}^{\text{flavors}} \times \underbrace{3}_{\text{color}} = 9 \tag{6.55}$$

$$n_{2/3} = 2 \times 3 = 6 \tag{6.56}$$

$$n_1 = 3 \tag{6.57}$$

At this scale we have, $b_{QED} = 0.71$. The running of the coupling is sketched below:



In QCD the situation is different. At high energies (such as around the Z scale) the coupling is reasonably perturbative,

$$\alpha_s(\mu^2 = m_z^2) = 0.12 \tag{6.58}$$

For QCD we have

$$\frac{1}{b_3} = \frac{1}{4\pi}\left(\frac{4}{3}T_r - \frac{11}{3}C_F\right) \tag{6.59}$$

where T_R depends on the gluons representation and the number of flavor. For the SU (3) octet we have, $T_R = n_F/2$. C_F depends only on the color the group. For $SU(3)$, $C_F = 3$. We have

$$b_3 = \frac{1}{4\pi}\left(\frac{2}{3}n_F - 11\right) = \frac{1}{12\pi}(2n_F - 33) \tag{6.60}$$

For $m_b < \mu < m_z$, $b \sim -0.5$. A typical scale is shown here

μ	$\alpha = \alpha_3$
m_z	0.12
m_b	0.2
m_c	0.3
m_s	1.7

An important quantitty that is other seen in the literature is the scale Λ_{QCD}. This scale can roughly be viewed as the energy scale at which perturbative QCD breaks down and the strong force becomes nonperturbative. It's typically in the range of $200-400$ MeV. This is of order a $\sim$ 1fm (the scale of a nucleus). To get a working definition of this scale is to fit α_s at some range of interest and choose Λ_{QCD} such that $\alpha_s \sim 1$. If you associate some scale

$$\Lambda_{QCD}^2 = \mu_o^2$$

where μ_0 satisfies[Q 4: what does this mean?]

$$-12\pi/(33-2n_f)\alpha(\mu_0^2) \tag{6.61}$$

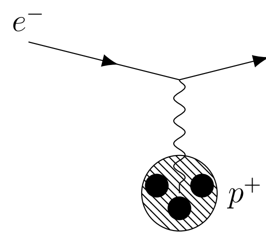
then

$$\alpha_s(\mu^2) = \frac{12\pi}{(33-2n_F)\ln\frac{\mu^2}{\Lambda_{QCD}^2}} \tag{6.62}$$

for $n_F = 5$ we have $\Lambda_{QCD} \sim 200$ MeV. If you have 3 active flavors ($n_F = 3$) then $\Lambda_{QCD} \sim 400$ MeV. Λ_{QCD} is a useful way to encapsulates all the nonperturbative behavior.

6.3 Deep Inelastic Scattering and Proton Structure

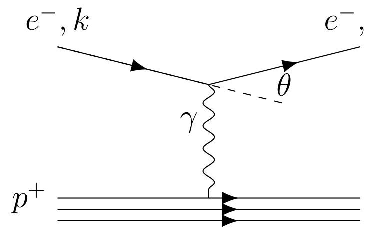
At the LHC we don't have the pleasure of simple point particle collisions. Instead we have complicated proton proton collisions. Each proton has three valence quarks (two up quarks and a down) but also contains many quarks and gluons that are popping in and out of the vacuum. These are known as "sea quark and gluons". We want to learn how to deal with such processes in a controlled way. The quarks and gluons within the proton have some distribution of motion and we want to figure out how to deal with this complication appropriately. We start by analyzing collisions with a single proton and an electron. At low energies(long wavelengths), the wavelength of the electron is too long to probe the short distance scale of the proton. Small electron wavelengths are need to resolve the proton structure (hence the need for high energy colliders). The collisions look something like,



We can have two situations

- Low energy electrons produce soft photons. The proton recoils coherently as gluons "redistribute" momenta on a time scale small compared to it's motion. These collisions do not probe the structure of the proton.
- High energy electrons can emit a hard photon which gives a lot of energy to a constituent of the proton. New particles form and a hadronic shower appears. This is called "Deep Inelastic Scattering".

We now take a closer look into these processes (see sections 5.5 in Langacker, 17.3 in Peskin). Consider the following collision between a proton and an electron



Consider the limit in which the electron mass goes to zero, i.e. $k = E(1, \hat{k}), k' = E(1, \hat{k}')$. In this case the momenta transfered is

$$q^2 = (k - k')^2 \tag{6.63}$$
$$= -E^2(1 - \cos\theta) \tag{6.64}$$

We're going to assume that $E \gg m_p$. This means in the CM (center of mass) frame each of the parton's momentum are approximately collinear with the proton (in the center of mass frame the proton will be highly boosted).

Thus we can take $|\mathbf{p}_\perp| \ll p_\parallel$ in the CM frame for the partons. With this in mind we can write $p_\mu = \xi P_\mu$, where p_μ denotes the parton momentum while P_μ denotes the proton momentum. We're interested in the "Parton Distribution Function",

$$f_f(\xi)\,, \tag{6.65}$$

which is the expected number density of finding a parton with flavor f with momentum fraction ξ in the proton. Equivalently $f_f(\xi)d\xi$ is the expected number of partons of type f with a longitudinal momentum fraction between ξ and $\xi + d\xi$.

Classicaly, $f_u(\xi)$ is a constant for all ξ and is equal to 2/3 while $f_d(\xi)$ is a constant and equal to 1/3. However, as we mentioned above this is not the whole story due to the sea quarka n gluons. f_u, f_d receive contributions from both valence and sea quarks. However, we must still have,

$$\sum_f f_f(\xi) = 1 \tag{6.66}$$

for all ξ.Suppose now you want to calculate the cross-section for a collision of an electron off of a quark with final momentum p' and initial momentum ξP. The cross section is given by

$$\sigma\left(e^{-1}(k) + p(P) \rightarrow e^-(k') + X\right) = \int_0^1 d\xi \sum_{q_f} f_{q_f}(\xi)\sigma\left(e^-(k) + q_f(\xi P) \rightarrow e^-(k') + q_f(p')\right) \tag{6.67}$$

This equation can be interpreted as follows. The cross section to see one flavor collide at a particular momentum fraction is $\sigma\left(e^-(k) + q_f(\xi P) \rightarrow e^-(k') + q_f(p')\right)$. The cross section to see a collision at any momentum fraction, ξ is this cross section multiplied by the probability of the quark carrying the momentum fraction. The total probability for a collision of a given flavor is the sum (through integration) over all possible ξ values. Lastly to get all possible flavors we have to sum over the flavor (over all 12 quark and anti-quark flavors).

We introduce the Mandlestam variables (we use hats to denote quark quantities and no hats to denote proton properties):

$$\hat{s} = (k+p)^2 = (k + \xi P)^2 \tag{6.68}$$

$$\hat{t} = q^2 = -Q^2 \tag{6.69}$$

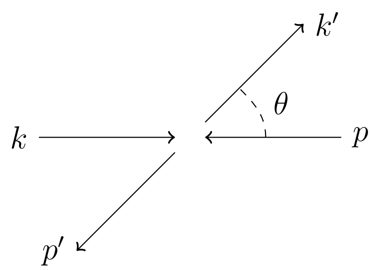
$$\hat{u} = (k - p')^2 \tag{6.70}$$

The cross section for $e^- f \rightarrow e^- f$ in the limit of $m_e^2, m_f^2 \ll E_{cm}^2$ is given by

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2 Q_f^2}{2E_{cm}^2(1-\cos\theta)^2}\left(4 + (1+\cos\theta)^2\right) \tag{6.71}$$

(see appendix 6.A for details).

We consider the collision in the CM frame:



we have

$$
\begin{aligned}
k^\mu &= \frac{E_{cm}}{2}(1,0,0,1) \\
k^{\mu\prime} &= \frac{E_{cm}}{2}(1,0,\sin\theta,\cos\theta) \\
p^\mu &= \frac{E_{cm}}{2}(1,0,0,-1) \\
p^{\mu\prime} &= \frac{E_{cm}}{2}(1,0,-\sin\theta,-\cos\theta)
\end{aligned}
$$

Now we have

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 Q_f^2}{E_{cm}^2(1-\cos\theta)^2}(4+(1+\cos\theta)^2) \tag{6.72}$$

The Mandlestam variables are

$$\hat{s} = E_{cm}^2 \tag{6.73}$$

$$\hat{t} = (k-k')^2 = -2k\cdot k' = -\frac{E_{cm}^2}{2}(1-\cos\theta) \tag{6.74}$$

$$\hat{u} = (k-p')^2 = -2k\cdot p' = -\frac{E_{cm}^2}{2}(1+\cos\theta) \tag{6.75}$$

Note that

$$\hat{s}+\hat{t}+\hat{u} = \sum_{i=1}^{4} m_i^2 \approx 0 \tag{6.76}$$

and we have

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2 Q_f^2}{\hat{s}(2\hat{t}/\hat{s})^2}\left(4+\left(\frac{2\hat{u}}{\hat{s}}\right)^2\right) \tag{6.77}$$

$$= \frac{\pi\alpha^2 Q_f^2}{\hat{s}\hat{t}^2}(\hat{s}^2+\hat{u}^2) \tag{6.78}$$

Furthermore we can rewrite the differential since,

$$\hat{t} = -\frac{\hat{s}}{2}(1-\cos\theta) \tag{6.79}$$

$$\Rightarrow \frac{d\sigma}{d\hat{t}} = \frac{d\sigma}{d\cos\theta}\frac{d\cos\theta}{d\hat{t}} = \frac{2}{\hat{s}}\frac{d\sigma}{d\cos\theta} \tag{6.80}$$

Hence we can write

$$\frac{d\sigma}{d\hat{t}} = \frac{2\pi\alpha^2 Q_f^2}{\hat{s}^2\hat{t}^2}(\hat{s}^2+(\hat{s}+\hat{t})^2) \tag{6.81}$$

$$= \frac{2\pi\alpha^2 Q_f^2}{\hat{t}^2}\left(1+\left[1+\hat{t}/\hat{s}\right]^2\right) \tag{6.82}$$

Back to the electron proton system we have

$$\hat{t} = t = q^2 \equiv -Q^2$$
$$\hat{s} = 2p \cdot k = 2\xi P \cdot k = \xi s$$

where our Mandlestam variables are now in the electron proton system (not in the quark system) and we have defined $Q^2 \equiv -t > 0$. This gives,

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 Q_f^2}{Q^4}\left(1 + \left[1 - \frac{Q^2}{\xi s}\right]^2\right) \tag{6.83}$$

This expression holds for any energy transferred Q^2, flavor, and momentum fraction ξ. Since in the proton we don't have control over the momentum fraction nor the flavor we integrate over these variables with the appropriate weighting factor, the parton distribution functions,

$$\frac{d\sigma_{tot}}{dQ^2} = \int_0^1 d\xi \sum_f f_f(\xi) Q_f^2 \frac{2\pi\alpha^2}{Q^4}\left(1 + \left[1 - \frac{Q^2}{\xi s}\right]^2\right) \tag{6.84}$$

Notice that we are now able to make any measurement we wish at a proton proton collider and as long as we know the parton distribution functions we are able to make sharp theoretical predictions!Now consider a measurement at Hera for example where they collide electrons and protons. Recall that $q \equiv k - k' = p' - p$. This is a quantity we know. We also have $k + p = k' + p' \Rightarrow p' = k - k' + p = q + p$. Hence

$$0 = p'^2 = (q + p)^2 = q^2 + 2q \cdot p = 2q \cdot (\xi P) - Q^2$$

which implies that,

$$\xi = \frac{Q^2}{2q \cdot P} \tag{6.85}$$

but $Q^2 \equiv -q^2$ is a measurable well known quantity. We also know the incoming and outgoing electrons and proton momenta. So are able to measure the momentum fraction that the parton receives in each collision. By measuring parton collisions and fitting the cross-section to equation 6.84 with $f_f(\xi)$ as a parameter we can extract the parton distribution function.

Consider the Lorentz invariant quantity,

$$y \equiv \frac{2P \cdot q}{2P \cdot k} = \frac{2P \cdot q}{s} \tag{6.86}$$

In the proton rest frame we have

$$y = \frac{q_0}{k_0} \tag{6.87}$$

Thus y is the energy transferred over initial energy of the electron, i.e. the fraction of the electron energy that is given to the proton system. Clearly we have $0 < y < 1$. We have

$$y = \frac{2\xi P \cdot q}{2\xi P \cdot k} \tag{6.88}$$

$$= \frac{2p \cdot (k - k')}{2p \cdot k} \tag{6.89}$$

$$= \frac{\hat{s} + \hat{u}}{\hat{s}} \tag{6.90}$$

$$= -\frac{\hat{t}}{\hat{s}} \tag{6.91}$$

This expression is in the parton level. We now want to switch back to the proton level. Recall that $2P \cdot q = Q^2/\xi$ (see Eq 6.85) , and hence

$$y = \frac{Q^2}{\xi} \frac{1}{2P \cdot k} = \frac{Q^2}{\xi s} \tag{6.92}$$

or

$$Q^2 = \xi y s \tag{6.93}$$

so we have

$$dQ^2 = \xi s dy \tag{6.94}$$

Going back to our cross section and using the Fundamental theorem of Calculus we can write,

$$\frac{d\sigma_{tot}}{d\xi dy} = \sum_f \xi f_f(\xi) Q_f^2 \frac{2\pi\alpha^2}{\xi^2 s} \frac{1 + (1-y)^2}{y^2} \tag{6.95}$$

This form is known as "Bjorken scaling" in which the initial parton distribution is independent of Q^2. So the $\xi f_f(\xi)$ is telling us about the structure of the proton.

Now consider some constraints in $f_f(\xi)$. The valence quarks are $2u, 1d$. Bounds states of quarks and antiquarks go in and out of the vacuum but the expected number of have a sea up quark is equal to the expected number of sea antiup quarks. Thus we must have the condition,

$$\int_0^1 \left[f_u(\xi) - f_{\bar{u}}(\xi) \right] d\xi = 2 \tag{6.96}$$

similarly if we take the expected number of d quarks and subtract the expected number of anti- d quarks from the sea we have,

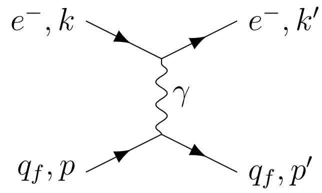
$$\int_0^1 \left[f_d(\xi) - f_{\bar{d}}(\xi) \right] d\xi = 1 \tag{6.97}$$

We have similar constraints for the other quarks. A final constraint is the constituents must sum over all momentum to the proton momentum:

$$\int_0^1 d\xi \xi \left[f_g(\xi) + f_u(\xi) + f_d(\xi) + f_c(\xi) + f_s(\xi) + ... + f_{\bar{u}}(\xi) + ... \right] = 1 \tag{6.98}$$

6.A $eq_f \to eq_f$

In this section we calculate the cross-section for an elastic collision between and electron and a quark. The only allowed channel is the t channel giving the diagram (at high energies one would need to worry about a diagram containing a Z boson as well but we omit this contribution),



The amplitude is

$$i\mathcal{M} = [\bar{u}(k')\,(-ie\gamma_\mu)\,u(k)]\left(\frac{-i}{q^2}\right)[\bar{u}(p')\,(-ieQ_f\gamma^\mu)\,u(p)] \tag{6.99}$$

$$= \frac{e^2 Q_f}{q^2}\,[\bar{u}_{k'}\gamma_\mu u_k]\,[\bar{u}_{p'}\gamma^\mu u_p] \tag{6.100}$$

squaring gives,

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\frac{e^4 Q_f^2}{(q^2)^2}\mathrm{Tr}\left[\gamma_\mu \not{k}\gamma_\nu \not{k}'\right]\mathrm{Tr}\left[\gamma^\mu \not{p}\gamma^\nu \not{p}'\right] \tag{6.101}$$

$$= \frac{8e^4 Q_f^2}{(q^2)^2}\left((k\cdot p')(k'\cdot p) + (k\cdot p)(k'\cdot p')\right) \tag{6.102}$$

The kinematics are given by,

$$k^\mu = \frac{E_{cm}}{2}(1,0,0,1)$$
$$k^{\mu\prime} = \frac{E_{cm}}{2}(1,0,\sin\theta,\cos\theta)$$
$$p^\mu = \frac{E_{cm}}{2}(1,0,0,-1)$$
$$p^{\mu\prime} = \frac{E_{cm}}{2}(1,0,-\sin\theta,-\cos\theta)$$

which gives,

$$k\cdot p = \frac{E_{cm}^2}{2} \tag{6.103}$$

$$k\cdot p' = \frac{E_{cm}^2}{4}(1+\cos\theta) \tag{6.104}$$

$$k'\cdot p = \frac{E_{cm}^2}{4}(1+\cos\theta) \tag{6.105}$$

$$k'\cdot p' = \frac{E_{cm}^2}{2} \tag{6.106}$$

Inserting in these relations we have,

$$\overline{|\mathcal{M}|^2} = \frac{E_{cm}^4 e^4 Q_f^2}{2(q^2)^2}\left(4 + (1+\cos\theta)^2\right) \tag{6.107}$$

The $2 \to 2$ cross section is given by

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{1}{16\pi^2}\frac{1}{4E_A E_B\,|v_A - v_B|}\frac{|\mathbf{p}|}{E_{cm}}}_{\text{prefactor}}\overline{|\mathcal{M}|^2} \tag{6.108}$$

For the kinematics of our problem the prefactor is just

$$= \frac{1}{64\pi^2}\frac{1}{E_{cm}^2} \tag{6.109}$$

The momentum transferred is,

$$q^2 = \frac{E_{cm}^2}{4}\left(1-\cos\theta\right)^2 \tag{6.110}$$

which gives

$$\frac{d\sigma}{d\Omega} = \frac{e^4 Q_f^2}{8\pi^2 E_{cm}^2\left(1-\cos\theta\right)^2}\left(4 + (1+\cos\theta)^2\right) \tag{6.111}$$

$$= \frac{\alpha^2 Q_f^2}{2E_{cm}^2\left(1-\cos\theta\right)^2}\left(4 + (1+\cos\theta)^2\right) \tag{6.112}$$

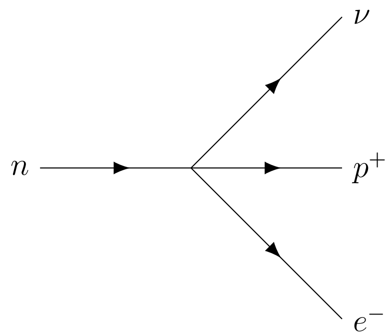
as quoted above.

Chapter 7

Weak Interactions

7.1 Building V-A Theory

We begin with a historical aside. Prior to the weak interaction, all people knew was that the neutron had a tendency to decay into a proton. Furthermore, people noticed that energy conservation appeared to be badly violated. With this in mind Pauli introduced another neutral particle (the neutrino) as a mathematical trick to preserve energy conservation. Fermi and Gamow-Teller took this particle seriously and suggested a "current-current" interaction:



They also knew about muon decay, $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$. They used what they knew about Lagrangians and went ahead to find the form the Lagrangian had to have in order to have such interactions (they assumed a vector interaction by example of QED(electrons are Dirac fermions)):

$$\mathcal{L} = -\sum_i c_i \left(\bar{\psi}_{\nu_\mu} \Gamma_i \psi_\mu\right) \left(\bar{\psi}_e \Gamma_i \psi_{\nu_e}\right) + h.c. \tag{7.1}$$

where i run over many different types of interactions and Γ_i some vertex factors that may or may not contain Lorentz indices. For fermions we have (remember that $\psi(t, -\mathbf{x}) \xrightarrow{\mathcal{P}}$

$\eta_a\gamma^0\psi$ and we take $\eta_a = 1$ for simplicity)

$$\mathcal{P}\left(\bar{\psi}_{\nu_\mu}\Gamma_i\psi_\mu\right)\left(\bar{\psi}_e\Gamma_i\psi_{\nu_e}\right)\mathcal{P} = \mathcal{P}\left(\bar{\psi}_{\nu_\mu}\mathcal{P}\mathcal{P}\Gamma_i\mathcal{P}\mathcal{P}\psi_\mu\mathcal{P}\right)\left(\mathcal{P}\bar{\psi}_e\mathcal{P}\mathcal{P}\Gamma_i\mathcal{P}\mathcal{P}\psi_{\nu_e}\right)\mathcal{P} \tag{7.2}$$

$$= \left[\bar{\psi}'_{\nu_\mu}\gamma^0\Gamma'_i\gamma^0\psi_\mu\right]\left[\bar{\psi}_e\gamma^0\Gamma'_i\gamma^0\psi_{\nu_e}\right] \tag{7.3}$$

where we defined $\psi' \equiv \psi(t, -\mathbf{x})$. If we assume Parity invariance we have the following allowed options for Γ_i.

$$\Gamma_i \in \Big\{\Gamma_3 = \mathbb{1}(\text{scalar}), \Gamma_V = \Gamma^\mu(\text{vector}), \Gamma_P = \Gamma^5(\text{pseudoscalar}),$$
$$\Gamma_A = \Gamma^5\Gamma^\mu(\text{axial-vector}), \Gamma_T = \sigma^{\mu\nu}(\text{tensor})\Big\}$$

Its not obvious that each of these terms preserve Parity. We show that this holds for a particular example of an axial-vector current $\gamma^5\gamma^\mu$:

$$\mathcal{P}\left(\bar{\psi}\gamma^5\gamma^\mu\psi\right)\left(\bar{\psi}\gamma^5\gamma^\mu\psi\right)\mathcal{P} = \left(\bar{\psi}'\gamma^0\gamma^5\gamma^\mu\gamma^0\psi'\right)\left(\bar{\psi}'\gamma^0\gamma^5\gamma_\mu\gamma^0\psi'\right) \tag{7.4}$$

$$= \left(\bar{\psi}'\gamma^5\gamma^0(2g^{\mu 0} - \gamma^0\gamma^\mu)\psi'\right)\left(\bar{\psi}'\gamma^5\gamma^0(2g^0{}_\mu - \gamma^0\gamma_\mu)\psi'\right) \tag{7.5}$$

$$= \left(\bar{\psi}'\gamma^5\gamma^\mu\psi'\right)\left(\bar{\psi}'\gamma^5\gamma^\mu\psi'\right) \tag{7.6}$$

as required.

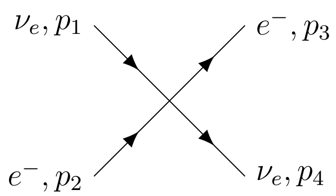
We now perform some dimensional analysis.

$$[\mathcal{L}] = 4$$
$$\mathcal{L}_{Dirac} = \bar{\psi}\partial_\mu\psi \rightarrow [\psi] = 3/2$$
$$[c_i] = 4 - 4(3/2) = -2$$

This hinted at the fact that this interaction was effective low energy theory. Effective theories tend to have dimensionful couplings. People found the spectrum for Lagrangians with all the possible vertices above and it was found that the energies were all wrong. After many years people began to rethink the assumption of Parity conservation. Once this was shed people came up with the interaction with $c_V = -\frac{G_F}{\sqrt{2}}, c_A = \frac{G_F}{\sqrt{2}}$ and a Lagrangian,

$$\mathcal{L}_{V-A} = \frac{G_F}{\sqrt{2}}\left(\bar{\psi}_{\nu_\mu}\gamma^\mu\left(1-\gamma^5\right)\psi_\mu\right)\left(\bar{\psi}_e\gamma_\mu\left(1-\gamma^5\right)\psi_{\nu_e}\right) \tag{7.7}$$

Consider the following collision: $\nu_e(p_1) + e^-(p_2) \rightarrow e^-(p_3) + \nu_e(p_4)$:



$$\mathcal{M} = \frac{G_F}{\sqrt{2}} \underbrace{\left[\bar{u}_3 \gamma^\mu \left(1-\gamma^5\right) u_1\right]}_{[\ldots]} \left[\bar{u}_4 \gamma_\mu \left(1-\gamma^5\right) u_2\right] \tag{7.8}$$

$$\begin{aligned}
[\ldots]^\dagger &= \left[u_3^\dagger \gamma^0 \gamma^\mu \left(1-\gamma^5\right) u_1\right]^\dagger && (7.9)\\
&= u_1^\dagger \left(1-\gamma^{5\dagger}\right) \gamma^{\mu\dagger} \gamma^{0\dagger} u_3 && (7.10)\\
&= u_1^\dagger \left(1-\gamma^5\right) \gamma^0 \gamma^\mu u_3 && (7.11)\\
&= \bar{u}_1 \left(1+\gamma^5\right) \gamma^\mu u_3 && (7.12)\\
&= \bar{u}_1 \gamma^\mu \left(1-\gamma^5\right) u_3 && (7.13)\\
& && (7.14)
\end{aligned}$$

and we have

$$\mathcal{M}^\dagger = \frac{G_F}{\sqrt{2}} \left[\bar{u}_1 \gamma^\nu \left(1-\gamma^5\right) u_3\right] \left[\bar{u}_2 \gamma_\nu \left(1-\gamma^5\right) u_4\right] \tag{7.15}$$

hence we have

$$\left|\mathcal{M}\right|^2 = \frac{G_F^2}{2} L^{\mu\nu} M_{\mu\nu} \tag{7.16}$$

with

$$\begin{aligned}
L^{\mu\nu} &= \left[\bar{u}_3 \gamma^\mu \left(1-\gamma^5\right) u_1\right] \left[\bar{u}_1 \gamma^\nu \left(1-\gamma^5\right) u_3\right] && (7.17)\\
M_{\mu\nu} &= \left[\bar{u}_4 \gamma_\mu \left(1-\gamma^5\right) u_2\right] \left[\bar{u}_2 \gamma_\nu \left(1-\gamma^5\right) u_4\right] && (7.18)
\end{aligned}$$

Averaging over initial spins and summing over final spins we get,

$$\overline{\left|\mathcal{M}\right|^2} = \frac{1}{2}\frac{1}{2}(16)\,\mathrm{Tr}\left[\gamma^\mu P_L \not{p}_1 \gamma^\nu P_L \left(\not{p}_3 + m_e\right)\right] \mathrm{Tr}\left[\gamma^\mu P_L \left(\not{p}_2 + m_e\right) \gamma_\nu P_L \not{p}_4\right] \tag{7.19}$$

where we define

$$P_{R/L} \equiv \frac{1}{2}\left(1 \pm \gamma^5\right) \tag{7.20}$$

We have an initial factor of 1/2 due to the electron spin and a factor of 16 to switch the $1-\gamma^5$ matrix to a projection matrix. Taking the traces gives,

$$\left|\mathcal{M}\right|^2 = 64 G_F^2 \left(p_1 \cdot p_2\right)\left(p_3 \cdot p_4\right) \tag{7.21}$$

We have,

$$\begin{aligned}
s &= \left(p_3 + p_4\right)^2 = m_e^2 + 2 p_3 \cdot p_4 && (7.22)\\
s &= \left(p_1 + p_2\right)^2 = m_e^2 + 2 p_1 \cdot p_2 && (7.23)
\end{aligned}$$

which gives,

$$\begin{aligned}
\left|\mathcal{M}\right|^2 &= 16 G_F^2 \left(s - m_e^2\right)^2 && (7.24)\\
& && (7.25)
\end{aligned}$$

In the CM frame,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\overline{|\mathcal{M}|^2} = \frac{G_F^2}{4\pi^2}\frac{\left(s-m_e^2\right)^2}{s} \sim \frac{G_F^2}{4\pi^2}s \tag{7.26}$$

If you have a point-like interaction in an s-wave (orbital angular momentum, $L=0$ state), then [Q 5: check a NRQM book to confirm this equation]

$$\frac{d\sigma}{d\Omega} = \frac{1}{s}\left|\mathcal{M}_0\right|^2 \tag{7.27}$$

where $\left|\mathcal{M}_0\right|^2$ is the probability of interaction and so unitarity implies that

$$\left|\mathcal{M}_0\right|^2 \leq 1 \tag{7.28}$$

So $\left|\mathcal{M}_0\right|^2 > 1$ is

$$s > \frac{2\pi}{G_F} \tag{7.29}$$

This theory has predicted its own demise!It doesn't make sence at these energies (which turn out to correspond $\sqrt{s} \sim 750\text{GeV}$). Something else must occur at this scale to happen to keep the amplitude from diverging. This has finally been achieved at the LHC. It all comes back to having this dimensionful coupling in our interaction. This set an energy scale at which we would expect something interesting to happen.

We now put our $V-A$ interaction into a more useful form. Recall the relations of the projection operators,

$$P_L + P_R = 1 \tag{7.30}$$
$$P_L^2 = P_L \tag{7.31}$$
$$P_R^2 = P_R \tag{7.32}$$

Furthermore we have,

$$P_L P_R = \frac{1}{4}\left(1-\gamma^5\right)\left(1+\gamma^5\right) = \frac{1}{4}\left(1-\gamma^5\right)^2 = 0 \tag{7.33}$$

Now

$$\left.\begin{matrix} \psi_L & = P_L\psi \\ \psi_R & = P_R\psi \end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix} \bar\psi_L = \bar\psi P_R \\ \bar\psi_R = \bar\psi P_L \end{matrix}\right. \tag{7.34}$$

and

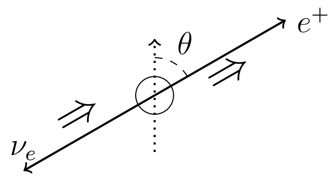
$$\psi = \psi_L + \psi_R \tag{7.35}$$

We have (note the choice of Lagrangian interaction $(V-A)$ is an experimental question)

$$\bar\psi\gamma^\mu\left(1-\gamma^5\right)\psi = 2\bar\psi\gamma^\mu P_L\psi \tag{7.36}$$
$$= 2\bar\psi P_R\gamma^\mu P_L\psi \tag{7.37}$$
$$= 2\bar\psi_L\gamma^\mu\psi_L \tag{7.38}$$

Notice that only the left handed components take part in the weak interactions. We will need to formulate our theory to specify that something different is happening between the left and right handed portions of our field. We say that the boson which mediates the charged current interaction (the W boson) maximally violates parity since it only interacts with the left handed fields.

Consider a W boson splitting into a neutrino and a positron. The neutrino must be left handed and the positron must be right handed,[Q 6: fix]



Define the $+z$ direction along W spin direction. The initial W angular momentum state is $|j,m\rangle = |1,1\rangle$. The angle momentum of the final e^+, ν_e states is fixed relative to their direction of propagation but their direction is not. Thus they can be thought of as a $|1,1\rangle$ state rotated by some angle θ,

$$R_y(\theta)\,|1,1\rangle = e^{-(\mathbf{J}\cdot\hat{y})\theta}\,|1,1\rangle \tag{7.39}$$

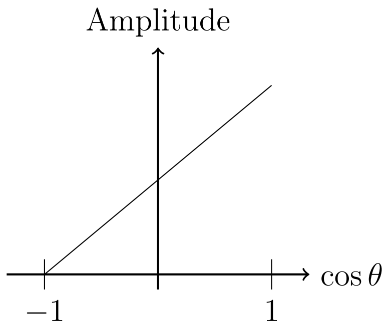
The overlap of the initial and final angular momentum states (and hence the structure of the amplitude for a collision) is,

$$\langle 1,1|R_y(\theta)|1,1\rangle = d_{m=1,m'=1}(\theta) \tag{7.40}$$

$$= \frac{1}{2}(1+\cos\theta) \tag{7.41}$$

where more generally

$$d^j_{m,m'}(\theta) \equiv \langle j,m'|R_y(\theta)|j,m\rangle \tag{7.42}$$



7.2 Moving to $SU(2)$

Thus far we have only discussed the $V-A$ structure of the weak interactions. While we know that this the theory structure at low energies we also mentioned that the theory predicts its own demize at about a TeV. So naturally we want to figure out what replaces it. The low energy behvior of the theory told us that it put left and right handed particles at different footing. Our goal now is to find some way to make this possible in our current framework.

Our Dirac Lagrangian is given by

$$\begin{aligned}\mathcal{L} &= \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi &(7.43)\\ &= \left(\bar{\psi}_L + \bar{\psi}_R\right)\left(i\partial\!\!\!/ - m\right)\left(\psi_L + \psi_R\right) &(7.44)\end{aligned}$$

Now we have

$$\bar{\psi}_L\psi_L = \bar{\psi}\left(P_R P_L\right)\psi = 0 = \bar{\psi}_R\psi_R \tag{7.45}$$

and

$$\begin{aligned}\bar{\psi}_R\gamma^\mu\partial_\mu\psi_L &= \bar{\psi}P_L\gamma^\mu\partial_\mu P_L\psi &(7.46)\\ &= \bar{\psi}\gamma^\mu\partial_\mu P_R P_L\psi &(7.47)\\ &= 0 &(7.48)\end{aligned}$$

similarly we have

$$\bar{\psi}_L\gamma^\mu\partial_\mu\psi_R = 0 \tag{7.49}$$

so we can rewrite the Dirac Lagrangian as,

$$\mathcal{L} = i\bar{\psi}_L\partial\!\!\!/\psi_L + i\bar{\psi}_R\partial\!\!\!/\psi_R - m\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right) \tag{7.50}$$

As we did in electromagnetism we want to try to have our interactions arise from a gauge symmetry. For QED the gauge transformations were just rotating the phase of the left and right handed particles equally. Which led to equal coupling for left and right handed particles with the gauge fields. Such a procedure will not work here since we want the right fields not the couple to the gauge fields at all. The only way we can do this is if we build a Lagrangian that is invariant under rotations between the left handed fields (put the left handed fields in multiplets) but leave the right-handed fields untransformed (put the right-handed fields in a singlet representation of the gauge symmetry). This forces us to avoid any coupling between left and right handed fields. This forces us to have massless particles, i.e. $m = 0$.

For now we consider only 1 generation of leptons. We will put the left handed fields into a "doublet":

$$L \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \tag{7.51}$$

The right handed fields will be in "singlet"

$$R \equiv e_R \tag{7.52}$$

In general we could consider bigger combinations of particles but we want to try a Lagrangian that's just one step more symmetric. We can then write the Dirac Lagrangian for all three particles as,

$$
\begin{aligned}
\mathcal{L}_e &= \bar{L}(i\partial\!\!\!/)L + \bar{R}(i\partial\!\!\!/)R && (7.53)\\
&= (\bar{\nu}_e)_L\, i\partial\!\!\!/\, (\nu_e)_L + (\bar{e})_L\, i\partial\!\!\!/\, (e)_L + (\bar{e})_R\, i\partial\!\!\!/\, (e)_R && (7.54)
\end{aligned}
$$

(we do not have any $m^2\psi_L\psi_R$ terms).

Lets postulate that $\mathcal{L}_e$ is invariant under $SU(2)_L \times U(1)$. Rotations in $SU(2)\times U(1)$ look like

$$
\begin{aligned}
L &\to \exp\left(ig\boldsymbol{\alpha}(x)\cdot\boldsymbol{\tau} + ig'\frac{Y_L}{2}\theta(x)\right)L && (7.55)\\
R &\to \exp\left(ig'\frac{Y_R}{2}\theta(x)\right)R && (7.56)
\end{aligned}
$$

where we defined $\tau_i \equiv \sigma_i/2$, $g \equiv SU(2)$ charge, $g' \equiv U(1)$ charge. In general the left and right handed field could have a different $U(1)$ charge but the $SU(2)$ charge must be the same for all the fields in a doublet by the definition of a charge for some transformation.

The covarient derivatives for the left handed fields are

$$
\begin{aligned}
D_L^\mu\,(SU(2)\times U(1)) &: \partial^\mu - ig\tau_i W_i^\mu - ig'\frac{Y_L}{2}B^\mu && (7.57)\\
D_R^\mu\,(U(1)) &: \partial^\mu - ig'\frac{Y_R}{2}B^\mu && (7.58)
\end{aligned}
$$

where W_i^μ are the $SU(2)$ gauge bosons and B^μ is the $U(1)$ gauge boson. Requiring a gauge symmetry we have

$$
\begin{aligned}
\mathcal{L}_e = \bar{L}\left(i\gamma^\mu\left(\partial_\mu - ig\boldsymbol{\tau}\cdot\mathbf{W}_\mu - ig'\frac{Y_L}{2}B_\mu\right)\right)L + \bar{e}_R\left(i\gamma^\mu\left(\partial_\mu - ig'\frac{Y_R}{2}B_\mu\right)\right)e_R\\
-\frac{1}{4}\mathbf{W}_{\mu\nu}\cdot\mathbf{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \qquad (7.59)
\end{aligned}
$$

where $W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon_{abc}W^b_\mu W^c_\nu$ (see 6.47). Under $SU(2)\times U(1)$ gauge transformation we have (see 6.23)

$$
\begin{aligned}
B_\mu &\to B_\mu + \partial_\mu\theta && (7.60)\\
W^a_\mu &\to W^a_\mu + \partial_\mu\alpha^a + g\,\underbrace{\epsilon_{abc}W^b_\mu\alpha^c}_{\mathbf{W}_\mu\times\boldsymbol{\alpha}} && (7.61)
\end{aligned}
$$

Recall that,

$$
\sigma_1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}
$$

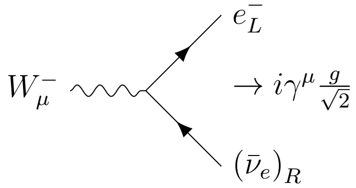
So we can write our Lagrangian explicitly as,

$$
\begin{aligned}
\mathcal{L}_e =& \bar{L}(i\gamma^\mu\partial_\mu)L + \bar{e}_R(i\gamma^\mu\partial_\mu e_R)\\
&+ \left((\bar{\nu}_e)_L\ \ \bar{e}_L\right)\frac{\gamma^\mu}{2}\begin{pmatrix} gW^3_\mu + g'Y_L B_\mu & g(W^1_\mu - iW^2_\mu)\\ g(W^1_\mu + iW^2_\mu) & -gW^3_\mu + g'Y_L B_\mu\end{pmatrix}\begin{pmatrix}(\nu_e)_L\\ e_L\end{pmatrix}\\
&+ \frac{1}{2}g'Y_R B_\mu\bar{e}_R e_R - \frac{1}{4}\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}
\end{aligned}
$$

Taking a look at our interactions we see that while W^a_μ and B_μ seemed like the natural choice for the gauge bosons earlier (they are the ones that couple directly the Pauli matrices). They are not the gauge bosons that transform the fields. Instead linear combinations of them are what act on the fields. What we observe in the laboratory are the fields that couple to each other. With this in mind we define,

$$W^{\mp} \equiv \frac{1}{\sqrt{2}}\left(W^1 \pm iW^2\right) \tag{7.62}$$

We have an interaction term given by,



It isn't clear what the other convenient linear combination of fields is as we also have a B_μ from both the left handed transformations and the right handed transformations. It turns out that the convenient combination will be

$$Z_\mu \equiv \frac{g'Y_L B_\mu + gW^3_\mu}{\sqrt{g'^2Y_L^2 + g^2}} \tag{7.63}$$

$$A_\mu \equiv \frac{gB_\mu - g'Y_L W^3_\mu}{\sqrt{g'^2Y_L^2 + g^2}} \tag{7.64}$$

It's straight forward to invert this relations and show that

$$B_\mu = \frac{g'Y_L Z_\mu + gA_\mu}{\sqrt{g'^2Y_L^2 + g^2}} \tag{7.65}$$

$$W^3_\mu = \frac{gZ_\mu - g'Y_L A_\mu}{\sqrt{g'^2Y_L^2 + g^2}} \tag{7.66}$$

We now multiply out the diagonal terms:

$$\begin{aligned}\mathcal{L}^{int}_{ee,\nu\nu} = (\bar{\nu}_e)_L\, \gamma^\mu \frac{1}{2}\left(gW^3_\mu + g'Y_L B_\mu\right)(\nu_e)_L + \bar{e}_L\gamma^\mu \frac{1}{2}\left(-gW^3_\mu + g'Y_L B_\mu\right) e_L \\ + \bar{e}_R\gamma^\mu \frac{1}{2}\left(g'Y_R B_\mu\right) e_R\end{aligned} \tag{7.67}$$

Consider the e_L term:

$$g'Y_L B_\mu - gW^3_\mu = \frac{1}{\sqrt{g'^2Y_L^2 + g^2}}\left(g'^2Y_L^2 Z_\mu + gg'Y_L A_\mu - g^2 Z_\mu + gg'Y_L A_\mu\right) \tag{7.68}$$

$$= \left(\frac{g'^2Y_L - g^2}{\sqrt{g'^2Y_L^2 + g^2}}\right) Z_\mu + \left(\frac{2gg'Y_L}{\sqrt{g'^2Y_L^2 + g^2}}\right) A_\mu \tag{7.69}$$

Repeating this process for the neutrino and right-handed particles the diagonal term takes the form,

$$\mathcal{L}^{int}_{ee,\nu\nu} = (\bar{\nu}_e)_L\,\gamma^\mu \frac{1}{2}\sqrt{g'^2Y_L^2+g^2}Z_\mu\,(\nu_e)_L + \bar{e}_L\gamma^\mu\frac{1}{2}\frac{g'^2Y_L^2-g^2}{\sqrt{g'^2Y_L^2+g^2}}Z_\mu e_L$$
$$+\bar{e}_L\gamma^\mu\frac{gg'Y_L}{\sqrt{g'^2Y_L^2+g^2}}A_\mu e_L+\bar{e}_R\gamma^\mu\frac{1}{2}\frac{g'^2Y_LY_R}{\sqrt{g'^2Y_L^2+g^2}}Z_\mu e_R+\bar{e}_R\gamma^\mu\frac{1}{2}\frac{gg'Y_R}{\sqrt{g'^2Y_L^2+g^2}}A_\mu e_R \tag{7.70}$$

We now see what our seemingly obscure linear combination of fields Z_μ, A_μ bought us. We have a field that has a similar form of coupling to left and right handed particles. We identify this field with the photon. To have the same EM couplings for e_R and e_L we require $Y_R = 2Y_L$ (compare terms 3 and 5). For the electron we define $Y_L = -1$. We summarize our results in the following table

	$T_3 = \tau_3 = \sigma_3/2$	$\frac{Y}{2}$	$T_3 + \frac{Y}{2}$
ν_e	$\frac{1}{2}$	$-\frac{1}{2}$	0
e_L	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
e_R	0	-1	-1

With our new value of Y we can write our fields in a more symmetric way,

$$Z_\mu \equiv \frac{gW^3_\mu - g'B_\mu}{\sqrt{g^2+g'^2}} \tag{7.71}$$
$$A_\mu \equiv \frac{g'W^3_\mu + gB_\mu}{\sqrt{g^2+g'^2}} \tag{7.72}$$

and we define

$$\sin\theta_w \equiv \frac{g'}{\sqrt{g^2+g'^2}} \tag{7.73}$$
$$\cos\theta_w \equiv \frac{g}{\sqrt{g^2+g'^2}} \tag{7.74}$$

where θ_w is known as the "weak" or "Weinberg" angle. We can then write

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \tag{7.75}$$
$$\Leftrightarrow \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \tag{7.76}$$

The EM interaction is given by

$$-\bar{e}_{L,R}\gamma^\mu \underbrace{\frac{gg'}{\sqrt{g^2+g'^2}}}_{e} A_\mu e_{L,R} \tag{7.77}$$

where e is the electromagnetic coupling constant and given by b

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}} = \sqrt{g^2 + g'^2} \sin\theta_w \cos\theta_w \tag{7.78}$$

Inserting in our new definitions we can write the interaction Lagrangian as

$$\begin{aligned}\mathcal{L}^{int}_{\nu\nu,ee} &= -(e) \cdot \bar{e}\gamma^\mu A_\mu e + (\bar{\nu}_e)_L \gamma^\mu \frac{(e)}{2\sin\theta_w \cos\theta_w} Z_\mu (\nu_e)_L + (e) \cdot \bar{e}_L \gamma^\mu \frac{1}{2} \frac{(2\sin^2\theta_w - 1)}{\sin\theta_w \cos\theta_w} Z_\mu e_L \\ &+ \bar{e}_R \gamma^\mu e \frac{\sin^2\theta_w}{\sin\theta_w \cos\theta_2} Z_\mu e_R \end{aligned}\tag{7.79}$$

where temporarily we write the electromagnetic coupling constant as (e) to avoid confusion with the elecron. The Z vertex factors are thus given by

$$Z_{\nu\nu} = \frac{-ie}{2\sin\theta_w \cos\theta_w} \tag{7.80}$$

$$Z_{e_L e_L} = \frac{-ie}{2\sin\theta_w \cos\theta_w} (2\sin^2\theta_w - 1) \tag{7.81}$$

$$Z_{e_R e_R} = \frac{-ie}{2\sin\theta_w \cos\theta_w} (2\sin^2\theta_w) \tag{7.82}$$

These can all be summarized by

$$Z_{ff} = \frac{ie}{\cos\theta_w \sin\theta_w} (Q_f \sin^2\theta_w - T^3_f) \tag{7.83}$$

Note that for the quarks we still have $SU(2)_L \times U(1)$. The left handed up and down quarks will form an $SU(2)$ doublet, $\begin{pmatrix} u \\ d \end{pmatrix}_L$ and the right handed fields will be $SU(2)$ singlets, u_R, d_R.

7.3 Kinetic Terms

We now consider the gauge terms of the Lagrangian.

$$\mathcal{L}^{gauge} = -\frac{1}{4}\mathbf{W}_{\mu\nu}\mathbf{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{7.84}$$

We have (recall that $B_\mu = \cos\theta_w A_\mu - \sin\theta_w Z_\mu$),

$$\begin{aligned} B_{\mu\nu}B^{\mu\nu} &= (\partial_\mu B_\nu - \partial_\nu B_\mu)(\partial^\mu B^\nu - \partial^\nu B^\mu) && (7.85)\\ &= [\cos\theta_w (\partial_\mu A_\nu - \partial_\nu A_\mu) - \sin\theta_w (\partial_\mu Z_\nu - \partial_\nu Z_\mu)] \\ &\quad \times [\cos\theta_w (\partial^\mu A^\nu - \partial^\nu A^\mu) - \sin\theta_w (\partial^\mu Z^\nu - \partial^\nu Z^\mu)] && (7.86)\\ &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\cos^2\theta_w \\ &\quad + (\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu)\sin^2\theta_w \\ &\quad + (\partial_\mu A_\nu \partial^\nu Z^\mu - \partial_\mu A_\nu \partial^\mu Z^\nu)\, 4\cos\theta_w \sin\theta_w && (7.87)\end{aligned}$$

Now consider the kinetic term for the W fields,

$$W^a_{\mu\nu}W^{\mu\nu}_a = \left(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu\right)\left(\partial^\mu W^\nu_a - \partial^\nu W^\mu_a\right) + W \text{ coupling terms} \tag{7.88}$$

where by W coupling terms we mean terms with more than 1 W fields. These will be coupling terms and we discuss these later. Simplifying for the W^3 contribution:

$$W^3_{\mu\nu}W^{\mu\nu}_3 = \left(\partial_\mu W^3_\nu - \partial_\nu W^3_\mu\right)\left(\partial^\mu W^\nu_3 - \partial^\nu W^\mu_3\right) + W \text{ coupling terms} \tag{7.89}$$

But the only difference between B_μ and W^3_μ is that W^3_μ has a positive instead of a negative sign before its $\sin\theta_w$. So we can just use the derivation above and apply $\sin\theta_w \to -\sin\theta_w$ which gives,

$$\begin{aligned} W^3_{\mu\nu}W^{\mu\nu}_3 = &(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\cos^2\theta_w \\ &+ (\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu)\sin^2\theta_w \\ &- (\partial_\mu A_\nu \partial^\nu Z^\mu - \partial_\mu A_\nu \partial^\mu Z^\nu)\, 4\cos\theta_w \sin\theta_w \end{aligned} \tag{7.90}$$

Notice the key negative sign on the term coupling the Z_μ and A_μ fields. When the two kinetic terms are added the cross-term dissapears,

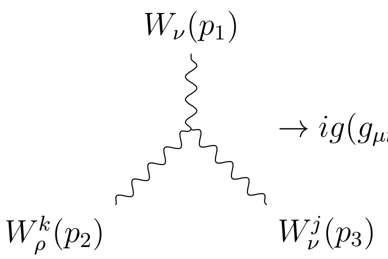
$$\begin{aligned} B_{\mu\nu}B^{\mu\nu} + W^3_{\mu\nu}W^{\mu\nu}_3 = &2(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)\cos^2\theta_w \\ &+ 2(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu)\sin^2\theta_w \end{aligned} \tag{7.91}$$

Now we see the reason for our apparently adhoc choice of field combinations earlier. A_μ and Z_μ are fields that propagate independently in space-time.

Above we ignored the $g\epsilon_{abc}W^b_\mu W^c_\nu$ term from $W^a_{\mu\nu}$ as it leads to couplings. Explicitly our kinetic term is:

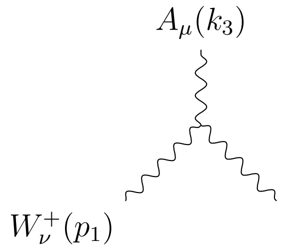
$$-\frac{1}{4}\left(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon_{abc}W^b_\mu W^c_\nu\right)\left(\partial^\mu W^\nu_a - \partial^\nu W^\mu_a + g\epsilon^{ade}W^\mu_d W^\nu_e\right) \tag{7.92}$$

Expanding this out leads to both three W_μ and four W^μ terms. [Q 7: I haven't had a chance to look over these equations and ensure there correctness to every last index. I would not use them as a reference at this point.] The trilinear vertices are,



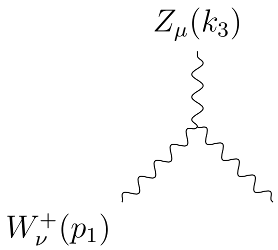
$$\to ig(g_{\mu\nu}(p_1 - p_2) + g_{\nu\rho}(p_2 - p_3) + g_{\rho\mu}(p_3 - p_1))$$

In terms of our physical fields, W^+W^-, Z, A the vertex takes the form:



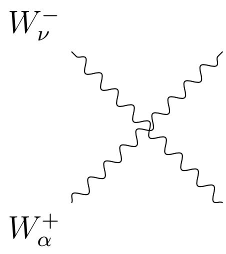
$$\to ie\left((k_1-k_2)_\mu g_{\nu\lambda}+(k_2-k_3)_\nu g_{\lambda\mu}+(k_3-k_1)_\lambda g_{\mu\nu}\right)$$

and

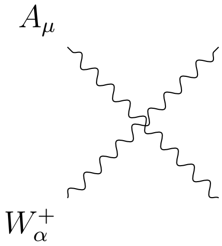


$$\to ie\cos\theta_w\left((k_1-k_2)_\mu g_{\nu\lambda}+(k_2-k_3)_\nu g_{\lambda\mu}+(k_3-k_1)_\lambda g_{\mu\nu}\right)$$

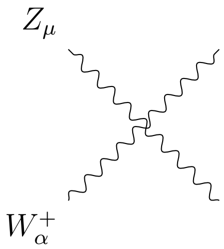
We also have the four field interactions:



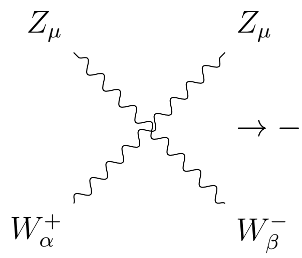
$$\to \tfrac{-ie^2}{\sin^2\theta_w}\left(2g_{\mu\nu}g_{\nu\beta}-g_{\mu\beta}g_{\alpha\nu}-g_{\mu\nu}g_{\alpha\beta}\right)$$



$$\to -ie^2\left(2g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\nu}g_{\beta\mu}\right)$$



$$\to -ie^2\cot\theta_w\left(2g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\nu}g_{\beta\mu}\right)$$



$$\rightarrow -ie^2 \cot\theta_w \left(2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right)$$

7.4 Mass Terms

Mass terms are given by a product of fields. Depending on the type of field mass terms look slightly different. We summarize the possibilities (in the Standard Model) below

J	$\mathcal{L}_{mass}$
0	$\begin{cases} -\frac{1}{2}m^2\phi^2 & \text{(real)} \\ -m^2\phi^\dagger\phi & \text{(complex)} \end{cases}$
$\frac{1}{2}$	$-m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$
1	$+\frac{1}{2}m^2A_\mu A^\mu$

The massive vector field A^μ, couples to some current j^μ

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu - j^\nu A_\nu \tag{7.93}$$

we have

$$\frac{\partial\mathcal{L}}{\partial A_\nu} = m^2A^\nu - j^\nu \tag{7.94}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \tag{7.95}$$

Plugging this into the Euler Lagrange equations we have

$$-j^\nu + m^2A^\nu + \partial_\mu \overbrace{F^{\mu\nu}}^{\partial^\mu A^\nu - \partial^\nu A^\mu} = 0 \tag{7.96}$$

$$(\Box^2 + m^2)A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu \tag{7.97}$$

Note if we have $m = 0$ then this equation would just be Maxwell's equations with a source. Lets see what will happen with $m \neq 0$. Let's take the derivative this equation

$$(\Box^2 + m^2)\partial_\nu A^\nu - \Box^2(\partial_\mu A^\mu) = \partial_\nu j^\nu \tag{7.98}$$

$$m^2\partial_\nu A^\nu = \partial_\nu j^\nu \tag{7.99}$$

So j^μ is not longer a conserved current since $m \neq 0$! There is a source or sink of charge in the theory. Here we see the reason we have gauge invariance. It is a requirement for charge conservation, which is taken to be a fundamental principle. Having massive gauge bosons breaks gauge invariance (and in turn charge conservation) and is hence forbidden in the naive formulation of gauge theories. Since we do see massive gauge bosons in Nature something must have gone wrong with this arguement. We will see how this works in the next chapter.

Since we know there do exist massive gauge bosons we quickly mention their properties. Their propagator needs to be modified,

$$-i\frac{g_{\mu\nu} - q_\mu q_\nu/m^2}{q^2 - m^2 + i\epsilon} \tag{7.100}$$

The longitudinal polarizations are allowed for $m > 0$. We pick up an extra term in the sum rule of the polarizations

$$\sum_{\lambda=1}^{3} \epsilon^\mu_\lambda \epsilon^{\star\nu}_\lambda = -g^{\mu\nu} + \frac{q^\mu q^\nu}{m^2} \tag{7.101}$$

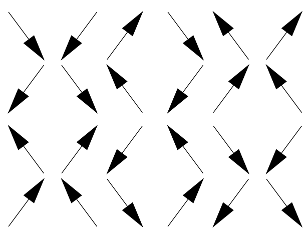
Chapter 8

Spontaneous Symmetry Breaking

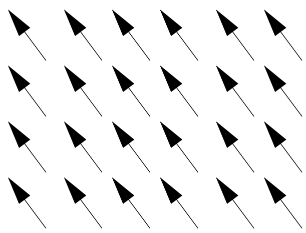
8.1 Toy Example - A $U(1)$ Global Symmetry

We need a gauge invariant Lagrangian. This leads to gauge bosons and fermions which we don't observe in the real world. To preserve gauge invariance but obtain massive particles we can implement something called Spontaneous Symmetry Breaking (SSB).

The idea behind spontaneous symmetry breaking is that the system obeys some symmetry but the ground state does not. So while overall the system is invariant under a transformation at low energies it would be invariant, dynamically breaking the symmtry. This idea occurs in a ferromagnet. At high temperature spins orient randomly:



However, at low temperatures all the spins will align in some random direction:



The rotational symmetry that we had is now broken but the particular ground state was chosen randomly.

With this analogy in mind consider a complex scalar field,

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \tag{8.1}$$

The Lagrangian is given by [1],

$$\mathcal{L}_{free} = (\partial^\mu \phi)^*(\partial_\mu \phi) \tag{8.2}$$

and a potential with a $U(1)$ global symmetry,

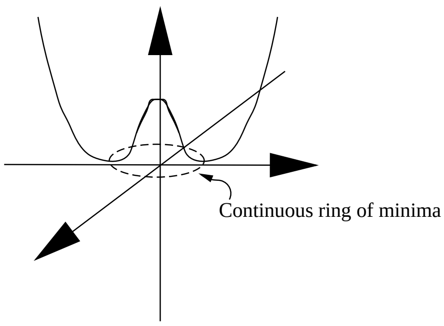
$$V(\phi) = \mu^2 \phi^* \phi + \lambda(\phi^* \phi)^2 \tag{8.3}$$

with $\lambda > 0$. Our full Lagrangian is then

$$\mathcal{L} = (\partial^\mu \phi)^*(\partial_\mu \phi) - \mu^2 \phi^* \phi - \lambda(\phi^* \phi)^2 \tag{8.4}$$

$$= \frac{1}{2}\partial^\mu \phi_1 \partial_\mu \phi_1 + \frac{1}{2}\partial^\mu \phi_2 \partial_\mu \phi_2 - \frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 \tag{8.5}$$

If $\mu^2 > 0$, then the potential has a minima at the origin; the symmetry will not be broken in the ground state. On the other hand if $\mu^2 < 0$ we have a potential of the form,



There is a small ring of minima at $\phi_1^2 + \phi_2^2 \equiv v^2$. $V(\phi)$ has a minima at

$$|\phi|^2 = \frac{\phi_1^2 + \phi_2^2}{2} = -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2} \tag{8.6}$$

where v is known as the vacuum expectation value (VEV).

Let's expand about $\phi_1 = v$, $\phi_2 = 0$. We define $\phi_1(x) = v + \eta(x)$, $\phi_2(x) = \rho(x)$. Plugging this into our Lagrangian we have

$$\mathcal{L} = \frac{1}{2}\partial^\mu \eta \partial_\mu \eta + \frac{1}{2}\partial^\mu \rho \partial_\mu \rho - \frac{\mu^2}{2}\left(v^2 + 2v\eta + \eta^2 + \rho^2\right) - \frac{\lambda}{4}\left(v^2 + 2v\eta + \eta^2 + \rho^2\right)^2 \tag{8.7}$$

$$= \frac{1}{2}\partial^\mu \eta \partial_\mu \eta + \frac{1}{2}\partial^\mu \rho \partial_\mu \rho + \frac{\lambda v^2}{2}\left(v^2 + 2v\eta + \eta^2 + \rho^2\right) - \frac{\lambda}{4}\left(v^2 + 2v\eta + \eta^2 + \rho^2\right)^2 \tag{8.8}$$

$$\begin{aligned} &= \frac{1}{2}\partial^\mu \eta \partial_\mu \eta + \frac{1}{2}\partial^\mu \rho \partial_\mu \rho + \lambda v^2 \eta^2 \left(\frac{1}{2} - 1\right) + \lambda v^2 \rho^2 \left(\frac{1}{2} - \frac{1}{2}\right) \\ &\quad - \lambda v^3 \eta (1-1) + \lambda v^4 \left(\frac{1}{2} - \frac{1}{4}\right) - \lambda v \eta \left(\eta^2 + \rho^2\right) - \frac{\lambda}{4}(\eta^2 + \rho^2)^2 \end{aligned} \tag{8.9}$$

$$= \frac{1}{2}\partial^\mu \eta \partial_\mu \eta + \frac{1}{2}\partial^\mu \rho \partial_\mu \rho - \frac{1}{2}\lambda v^2 \eta^2 + \frac{1}{4}\lambda v^4 - \lambda v \eta \left(\eta^2 + \rho^2\right) - \frac{\lambda}{4}(\eta^2 + \rho^2)^2 \tag{8.10}$$

[1] We begin by studying classical field theory and later worry about the subtleties of quantization.

Note the quadratic term for the η field:

$$-\frac{1}{2}\left(2\lambda v^2\right)\eta^2 \tag{8.11}$$

Thus we have a new field η that has a mass of $2\lambda v^2$. Further the ρ term does not have a mass!So we have one massless field which we call our "Goldstone Boson". We get a massive field for the one that goes along the real axis of ϕ and a massless field for the one along the imaginary axis. That's because the potential has no derivative along the imaginary axis. Goldstone bosons always arise with the breaking of a global symmetry. There is one Goldstone boson for each generator.

8.2 Global → local symmetry

We now move onto a more intricate example: $U(1)$ local symmetry breaking. Recall that to have a $U(1)$ symmetry we use $D_\mu = \partial_\mu - igA_\mu$ with $\phi(x) \to e^{ig\theta(x)}\phi \equiv \phi'(x)$. We have

$$\mathcal{L} = (D^\mu\phi)^*(D_\mu\phi) - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{8.12}$$

We consider the particular case of $\mu^2 < 0$. We initially have ϕ_1, ϕ_2 which form two degrees of freedom (DOF) while A_μ has two transverse polarizations so it has two DOF as well.

We rewrite

$$\phi(x) = (\phi_1(x) + i\phi_2(x)) \tag{8.13}$$

$$= r(x)e^{-i\theta(x)} \tag{8.14}$$

where $r(x)$ and $\phi(x)$ are two fields. We choose a gauge such that $\phi(x) \to e^{i\theta(x)}\phi(x) = r(x)$. This will give a gauge with $\phi(x)$ being real (as we did in the global toy example earlier). This is called the unitary gauge. We expand about the minimum of the potential denoted by v as,

$$\phi(x) = \frac{1}{\sqrt{2}}(v + h(x)), \tag{8.15}$$

where $h(x)$ is our Higg's field. Since the potential takes the same form as before so does the VEV giving,

$$v^2 = -\frac{\mu^2}{\lambda} \tag{8.16}$$

$$= \langle 0|\phi(x)|0\rangle \tag{8.17}$$

Note that you want to have a scalar vacuum since we want to want the vacuum expectation value to be Lorentz invariant (otherwise it would not be rotationally invariant and to our knowledge Nature is isotropic). Our Lagrangian becomes

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left[\,(\partial_\mu + igA_\mu)\,(v+h)\,\right]\left[\,(\partial^\mu - igA^\mu)\,(v+h)\,\right] - \frac{\mu^2}{2}\,(v+h)^2 \\
&\quad - \frac{\lambda}{4}\,(v+h)^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad (8.18) \\
&= \frac{1}{2}\partial^\mu h\partial_\mu h + \overbrace{\lambda v^2 h^2\left(-\frac{1}{2}-\frac{3}{2}\right)}^{h\text{ mass term}} + \underbrace{\frac{1}{2}g^2v^2A^\mu A_\mu}_{A_\mu\text{ mass term}} + \frac{1}{2}g^2h^2A^\mu A_\mu \\
&\quad + g^2vhA_\mu A^\mu + \lambda v^3 h(1-1) - \lambda v h^3 - \frac{\lambda}{4}h^4 \qquad (8.19)
\end{aligned}
$$

We now count degrees of freedom. The Higg's is a real scalar field and hence has 1 degree of freedom. The gauge boson is now massive. So it has 3 polarizations and hence 3 DOF. We say that the gauge boson "eats" Goldstone boson and becomes massive.

We now move on to symmetry breaking in the Standard Model (SM). The SM is an $SU(3)\times SU(2)_L\times U(1)$ theory. We focus on the $SU(2)_L\times U(1)$ sector. It can be summarized as,

doublet	$T_3=\sigma_3/2$	Y	$Q=T_3+\frac{Y}{2}$
$\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L$	$\begin{cases}\frac{1}{2}\\ -\frac{1}{2}\end{cases}$	$\begin{cases}-1\\ -1\end{cases}$	$\begin{cases}0\\ -1\end{cases}$
$\begin{pmatrix}u\\ d\end{pmatrix}_L$	$\begin{cases}\frac{1}{2}\\ -\frac{1}{2}\end{cases}$	$\begin{cases}\frac{1}{3}\\ \frac{1}{3}\end{cases}$	$\begin{cases}\frac{2}{3}\\ -\frac{1}{3}\end{cases}$
singlets			
e_R	0	-2	-1
u_R	0	$\frac{4}{3}$	$\frac{2}{3}$
d_R	0	$-\frac{2}{3}$	$-\frac{1}{3}$
Higg's sector			
$\begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix}$	$\begin{cases}\frac{1}{2}\\ -\frac{1}{2}\end{cases}$	$\begin{cases}1\\ 1\end{cases}$	$\begin{cases}1\\ 0\end{cases}$

The Higg's field take the form of

$$
\phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{pmatrix} \tag{8.20}
$$

with $\phi_1,\phi_2,\phi_3,\phi_4$ all real. The 4 DOF form 3 Goldstone bosons that the W^+, W^-, Z will eat and 1 surviving physical Higg's field which we call our Higgs.

ϕ transforms like a standard $SU(2)_L\times U(1)$ doublet,

$$
\phi(x) \to \phi'(x) = \exp\left[ig\boldsymbol{\alpha}(x)\cdot\boldsymbol{\tau} + ig'\frac{Y_H}{2}\theta(x)\right]\phi(x) \tag{8.21}
$$

Our covarient derivative takes the form

$$D^\mu = \partial^\mu - ig\boldsymbol{\tau}\cdot\mathbf{W}^\mu - ig'\frac{Y_H}{2}B^\mu \tag{8.22}$$

We assume a potential $V(\phi) = \mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$, where $\phi^\dagger\phi = \phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2$. As before we have $V(\phi)$ to be a minimum at $\phi^\dagger\phi = -\frac{\mu^2}{2\lambda} \equiv \frac{v^2}{2}$. We choose our ground state to be

$$\phi_g \equiv \langle 0|\phi|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \tag{8.23}$$

We certainly want our ground state to be electrically neutral. Recall that $Q = T_3 + \frac{Y}{2}$. Since the ground state is a $-\frac{1}{2}$ spinor we have $T_3 = -\frac{1}{2}$. Thus we require $Y_H = 1$. Note that this also forces the top component of the Higgs to have a charge of 1 as a doublet has the same quantum numbers for the top and bottom component. We expand around ϕ_g:

$$\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma_1(x) + i\sigma_2(x) \\ v + h(x) + i\eta(x) \end{pmatrix} \tag{8.24}$$

While not obvious one can show that you can pick a gauge that removes σ_1, σ_2, and η. Utilizing this we can write the field as,

$$\phi' = U\phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{8.25}$$

In this gauge the $W^\pm$ and a linear combination of W_3 and B will get a mass and the charged components and Imh completely dissappear from the theory (i.e. they get "eaten").

Our covarient derivative takes the form

$$D^\mu\phi = \begin{pmatrix} \partial^\mu - \frac{i}{2}(gW_3^\mu + g'B^\mu) & -\frac{ig}{\sqrt{2}}W^{+\mu} \\ -\frac{ig}{\sqrt{2}}W^{-\mu} & \partial^\mu + \frac{i}{2}(gW_3^\mu - g'B^\mu) \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} \tag{8.26}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{ig}{\sqrt{2}}W^{+\mu}(v+h(x)) \\ \partial^\mu h(x) + \frac{i}{2}(gW_3^\mu - g'B^\mu)(v+h(x)) \end{pmatrix} \tag{8.27}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{ig}{\sqrt{2}}W^{+\mu}(v+h(x)) \\ \partial^\mu h + \frac{i}{2}(g^2+g'^2)^{1/2}Z^\mu(v+h) \end{pmatrix} \quad \left(\begin{matrix}\text{recall the} \\ \text{definition of } Z_\mu\end{matrix}\right) \tag{8.28}$$

so we have

$$(D^\mu\phi)^* = \frac{1}{\sqrt{2}}\begin{pmatrix} i\frac{g}{\sqrt{2}}W^{-\mu}(v+h) \\ \partial^\mu h - \frac{i}{2}(g^2+g'^2)^{1/2}Z^\mu(v+h) \end{pmatrix} \tag{8.29}$$

and we have

$$(D^\mu\phi^\dagger)(D_\mu\phi) = \frac{1}{2}\left\{\frac{g^2}{2}W_\mu^+W^{-\mu}(v+h)^2 + \partial^\mu h\partial_\mu h + \frac{g^2+g'^2}{4}Z^\mu Z_\mu(v+h)^2\right\} \tag{8.30}$$

Our Lagrangian then takes the form

$$
\begin{aligned}
\mathcal{L}_\phi &= (D^\mu\phi)^\dagger(D_\mu\phi) - V(\phi) + \text{Vector boson kinetic terms} &(8.31)\\
&= \frac{g^2v^2}{4}W^-_\mu W^{+\mu} + \frac{1}{2}\partial^\mu h\partial_\mu h + \frac{g^2+g'^2}{8}v^2 Z_\mu Z^\mu + \frac{g^2 v}{2}W^-_\mu W^{+\mu}h \\
&+ \frac{g^2+g'^2}{4}vZ^\mu Z_\mu h + \frac{g^2}{4}W^-_\mu W^{+\mu}h^2 \quad + \frac{g^2+g'^2}{8}Z^\mu Z_\mu h^2 - \frac{1}{2}(2\lambda v^2)^2 - \lambda v h^3 \\
&- \frac{\lambda}{4}h^4 + \text{Vector boson kinetic terms} &(8.32)
\end{aligned}
$$

Recall that if you have a massive vector boson then it has a mass term

$$\frac{1}{2}m^2 V_\mu V^\mu \tag{8.33}$$

so we have

$$m_Z^2 = \frac{v^2}{4}(g^2+g'^2) \tag{8.34}$$

and in terms of the Weinberg angle we have

$$m_Z = \frac{v}{2}\frac{e}{\sin\theta_w\cos\theta_w} \tag{8.35}$$

and

$$m_W^2 = \frac{g^2v^2}{4} \tag{8.36}$$

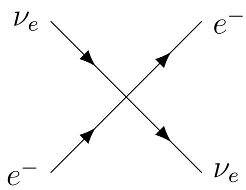
$$\Rightarrow m_W = \frac{v}{2}\frac{e}{\sin\theta_w} \tag{8.37}$$

Now note that

$$\frac{m_W}{m_Z} = \cos\theta_w \tag{8.38}$$

so we get a really simple prediction for the Weinberg angle.

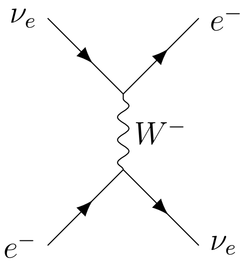
Now recall our low energy effective theory for the weak interaction,



which gave in the $V-A$ picture

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\left[\bar{u}_3\gamma^\mu(1-\gamma^5)u_1\right]\left[\bar{u}_4\gamma^\mu(1-\gamma^5)u_2\right] \tag{8.39}$$

Now consider the same diagram only with the full theory



So the full theory gives

$$\mathcal{M} = -\frac{ig^2}{8}\left(\bar{u}_3\gamma^\mu(1-\gamma^4)u_1\right)\left[\frac{-g_{\mu\nu}+\frac{q_\mu q_\nu}{m_W^2}}{q^2-m_W^2}\right]\left[\bar{u}_4\gamma^\nu(1-\gamma^5)u_2\right] \tag{8.40}$$

Now consider the low energy case of $E_{CM}^2 \ll m_W^2$. This implies that $q^2 \ll m_W^2$ and $q^\mu q^\nu \ll m_W^2$ and so we can relate the Fermi constant to the variables of the full theory,

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2} = \frac{g^2}{8g^2v^2/4}$$

which gives

$$v^2 = \frac{1}{\sqrt{2}G_F} \tag{8.41}$$

and so we have a prediction for the value of the VEV. The parameters of the Lagrangian: g, g' , v, (λ or m). Can be found through measurable parameters:

$$e, G_F, m_Z, m_W, \theta_w, ... \tag{8.42}$$

Measurements at the LHC and elsewhere have given

$$\begin{aligned}
m_W &= 80.399 \pm 0.023\text{GeV} \\
m_Z &= 91.1876 \pm 0.0021\text{GeV} \\
G_F &= (1.16637 \pm 0.00001) \times 10^{-5}\text{GeV}^2 \\
e &= \left(\frac{4\pi}{137.035999084}\right)^{1/2} \\
\sin^2\theta_w &= 022292(28)
\end{aligned}$$

which we can use to estimate our VEV:

$$v = 246 \text{ GeV}$$

Note that

$$\cos\theta_w = \left(\frac{m_W}{m_Z}\right)_{meas} = 0.8817 \pm 0.0003$$

which agrees well with the expected results:

$$\cos\theta_w = 0.8815 \pm 0.0002$$

This is our first prediction of the full theory and it is very close to the true value at tree level. The deviation is due to loop corrections. Furhermore, we also have the important SM prediction,

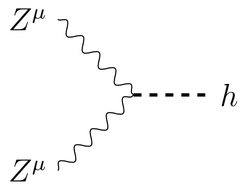
$$\rho \equiv \frac{m_Z^2 \cos^2\theta_w}{m_W} = 1 \tag{8.43}$$

This turns out to be a sensitive probe for beyond Standard Model physics.

We now look at our Lagrangian and identify our interaction terms

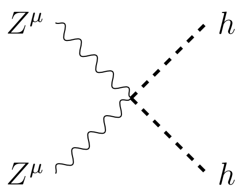
$$\frac{v^2}{8}(g^2 g'^2) Z_\mu Z^\mu \rightarrow \text{mass term} \tag{8.44}$$

The Higg's-Z interaction terms are given by [Q 8: fix these terms to be the true vertex factors not the terms in the Lagrangian]



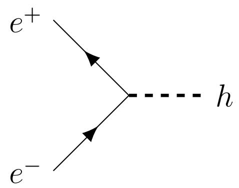
$$\rightarrow 2vh(g^2+g'^2)Z_\mu Z^\mu = \frac{m_Z e}{\sin 2\theta_w} Z^\mu Z_\mu h$$

(which gives a vertex of $\frac{2iem_Z}{\sin 2\theta_w} g_{\mu\nu}$) and

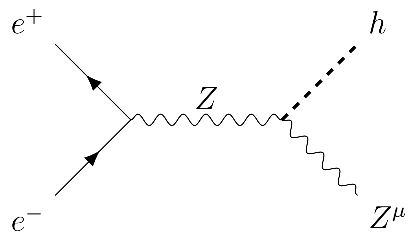


$$\rightarrow h^2(g^2+g'^2)Z_\mu Z^\mu = \frac{e^2}{2\sin^2 2\theta_w} Z^\mu Z_\mu hh$$

The Higg's production at a potential future linear collider is given by



However the most prominent channel would be what's known as the "Higgstraulung" channel:



At the LHC the "gluon fusion" channel dominates in gluon production.

8.3 Fermion Masses

So far we have made mentioned anything about fermion masses. In fact, having a theory that couples to left and right particles separately forbids such terms to be in the high energy Lagrangian!We want terms that couple ψ_R, ψ_L in gauge invariant way. This can be accomplished if the weak interaction is not a true symmetry but one that is spontaneously broken.

This can be accomplished by starting with the Higgs charged under $SU(2)$ and then breaking this symmetry. We focus the dicussion on leptons however the quark masses are generated in a completely analogous way.

When model building we obey the democratic principle; every term that is allowed by the symmetry is written. To find out whether a term is allowed we need to add the charges. One such term is given by:

$$\underbrace{(\bar{L}\phi)}_{SU(2)\text{ doublets}} e_R \tag{8.45}$$

where $L = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$, $\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$. This term is clearly invariant under $SU(3)$ as none of these fields have color. Furthermore, it is invariant under $SU(2)$ since:

$$\bar{L} \xrightarrow{SU(2)} \bar{L} \exp\left(\frac{i}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}\right) \tag{8.46}$$

$$\phi \xrightarrow{SU(2)} \exp\left(-\frac{i}{2}\boldsymbol{\theta} \cdot \boldsymbol{\sigma}\right) \phi \tag{8.47}$$

Lastly, the $U(1)_Y$ charges add as necessary:

$$Y_R + Y_\phi + Y_{\bar{L}} = -2 + 1 + 1 = 0$$

(if we forget these values they can easily be recalled using $Q = T_3 + \frac{Y}{2}$. Thus this term is invariant under all the gauge symmenries and we must write it down. The adjoint is given by

$$\left((\bar{L}\phi)e_R\right)^\dagger = e_R^\dagger \phi^\dagger \gamma^{0\dagger} L \tag{8.48}$$

$$= \bar{e}_R(\phi^\dagger L) \tag{8.49}$$

So in order to get fermion masses we add a term:

$$\mathcal{L}_{mass} = -g_e \left[(\bar{L}\phi)e_R + \bar{e}_R(\phi^\dagger L)\right] \tag{8.50}$$

Recall from our earlier discussion that a dimensionful coupling leads to problems. We check our dimensions.

$$\begin{aligned}
[\mathcal{L}] &= 4 \\
[L] = [e_R] &= \frac{3}{2} \\
[\phi] &= 1 \\
\Rightarrow [g_e] &= 0
\end{aligned}$$

as required. g_e is known as the "Yukawa coupling".

We expand around $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$ in the Unitary gauge, $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$. We get

$$\begin{aligned}
\mathcal{L}_{\phi f} &= -\frac{g_e}{\sqrt{2}}\left[\begin{pmatrix} \bar{\nu}_e & \bar{e}_L \end{pmatrix}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} e_R + \bar{e}_R \begin{pmatrix} 0 & v + h(x) \end{pmatrix}\begin{pmatrix} \nu_e \\ e_L \end{pmatrix}\right] && (8.51) \\
&= -\frac{g_e}{\sqrt{2}}\left[v(\bar{e}_L e_R + \bar{e}_R e_L) + h(\bar{e}_L e_R + \bar{e}_R e_L)\right] && (8.52)
\end{aligned}$$

which implies that

$$m_e = \frac{g_e v}{\sqrt{2}} \tag{8.53}$$

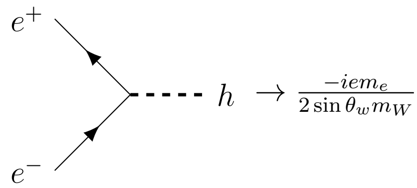
and hence $g_e = 3 \times 10^{-6}$. One of the theoretical difficulties of this model is that this coupling is so small. It is not "natural". Putting the left and right particles into a Dirac 4-component spinor,

$$\mathcal{L}_{\phi f} = -m_e \bar{e} e - \frac{m_e}{v}\bar{e} e h \tag{8.54}$$

Recall that

$$\frac{1}{v} = \frac{g}{2m_W} = \frac{e}{2m_W \sin\theta_w} \tag{8.55}$$

Thus we get the diagram for Higgs production at an e^+e^- collider,



It turns out that there is more then one way to form an $SU(2)$ invariant. Consider the term $\tilde{\phi} \equiv i\sigma_2 \phi^*$ under such transformations.

$$\begin{aligned}
i\sigma_2\phi^* \rightarrow i\sigma_2 &\left(\exp\left(-\frac{i}{2}\boldsymbol{\sigma}\cdot\theta\right)\right)^* \phi^* && (8.56) \\
= i\sigma_2 &\left(\cos\frac{\theta}{2} + i\frac{\boldsymbol{\theta}\cdot\boldsymbol{\sigma}^*}{\theta}\sin\frac{\theta}{2}\right)\phi^* && (8.57)
\end{aligned}$$

but

$$\sigma_2 \boldsymbol{\theta} \cdot \boldsymbol{\sigma}^* = \sigma_2 \left(\theta_1 \sigma^1 - \theta_2 \sigma^2 + \theta_3 \sigma^3 \right) \tag{8.58}$$
$$= - \left(\boldsymbol{\theta} \cdot \boldsymbol{\sigma} \right) \sigma_2 \tag{8.59}$$

Thus we have,

$$i\sigma_2 \phi^* \to \exp\left(-\frac{i}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\theta} \right) i\sigma_2 \phi^* \tag{8.60}$$

Thus this can be put together with a barred term to give a $SU(2)$ invariant. However, this term has a different $U(1)_Y$ charge then ϕ (we say that it is in the conjugate representation of $U(1)_Y$), thus the term $\bar{L} i\sigma_2 \phi^* e_R$ is not allowed due to hypercharge. There is in fact no other term we can write down in the lepton sector of the SM that will give fermions masses. Thus only the charged leptons can gain a mass, and not the neutrinos.

Nevertheless, if their exist uncharged right handed neutrinos then they must have hypercharge, $Y = 2\left(Q - T_3\right) = 0$. This allows the term:

$$\bar{L} \tilde{\phi} \nu_R \tag{8.61}$$

Performing a gauge transformation on the conjugate Higgs field:

$$\tilde{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix} \tag{8.62}$$

The new term gives mass to the neutrinos:

$$\mathcal{L}_{eh} = -g_e \left[\bar{L} \phi e_R + \bar{e}_R \phi^\dagger L \right] - g_\nu \left[\bar{L} \tilde{\phi} \nu_R + \bar{\nu}_R \tilde{\phi}^\dagger L \right] \tag{8.63}$$
$$= \frac{g_e}{\sqrt{2}} (v + h) \left[\bar{e}_L e_R + \bar{e}_R e_L \right] - \frac{g_\nu}{\sqrt{2}} (v + h) \left[\bar{\nu}_L \nu_R + \bar{\nu}_R \nu_L \right] \tag{8.64}$$

Note that we have assumed that the mass eigenstates are the same as the weak interaction eigenstates. However we know that this is not the case and we have mixing. We now move on to study this phenomena in detail.

Chapter 9

CKM Matrix

We now move to the quark sector. In the quark sector the interaction basis is different then the mass basis. We can choose the mass basis to be diagonal and that way particles will interact in linear combinations or we can choose the interactions to be diagonal but then the particles will be in a linear combination of masses. Typically what we call a "particle" is one with a well defined mass. We will need to know how to relate between these two bases. One basis is the weak (interaction) basis which we denote with a prime:

$$Q_L^j \equiv \begin{pmatrix} u_L^{j\prime} \\ d_L^{j\prime} \end{pmatrix}; \qquad \underbrace{u_R^{j\prime}, d_R^{j,\prime}}_{\substack{\text{Right-handed} \\ \text{singlets}}}$$

where $j = 1, 2, 3$ labels the generation. The mass basis is unprimed. The unbroken weak interaction is given by,

$$\mathcal{L}_{QH} = -\sum_{i,j=1}^{3} g_{ij} \left(\bar{Q}_L^{i\prime} \phi d_R^{j\prime} + \bar{d}_R^{j\prime} \phi^\dagger Q_L^{i\prime} \right) - \sum_{i,j=1}^{3} h_{ij} \left(\bar{Q}_L^{i\prime} \tilde{\phi} u_R^{j\prime} + \bar{u}_R^{j\prime} \tilde{\phi}^\dagger Q_L^{i\prime} \right) \tag{9.1}$$

with

$$\tilde{\phi} \equiv i\sigma_2 \phi^* \tag{9.2}$$

Introducing a VEV and choosing a gauge such that,

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}, \tilde{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} v + h(x) \\ 0 \end{pmatrix} \tag{9.3}$$

This gives

$$\mathcal{L}_{QH} = -\frac{v + h(x)}{\sqrt{2}} \left\{ \sum_{i,j=1}^{3} g_{ij} \left(\bar{d}_L^{i\prime} d_R^{j\prime} + \bar{d}_R^{j\prime} d_L^{i\prime} \right) + \sum_{i,j=1}^{3} h_{i,j} \left(\bar{u}_L^{i\prime} u_R^{j\prime} + \bar{u}_R^{j\prime} u_L^{i\prime} \right) \right\} \tag{9.4}$$

We introduce the undiagonalized mass matrices (3×3 for 3 generations):

$$M_d^{ij} = \frac{v}{\sqrt{2}} g_{ij} \tag{9.5}$$

$$M_u^{ij} = \frac{v}{\sqrt{2}} h_{ij} \tag{9.6}$$

which allows us to write the mass terms as

$$\mathcal{L}_{QH,\text{mass}} = -\left(\bar{d}'_L m_d d'_R + \bar{d}'_R m_d d'_L\right) - \left(\bar{u}'_L m_u u'_R + \bar{u}'_R m_u u'_L\right) \tag{9.7}$$

Thus we worked in the interaction basis since this was the sensible basis when discussing the interaction with the Higgs. However, now we want to go back to mass eigenstates. The physical/mass basis is related to the weak basis by some unitary transformation.[1] We have,

$$u^j_{L,R} = \left(V^u_{L,R}\right)_{j,k} u^{k\prime}_{L,R} \tag{9.8}$$

$$d^j_{L,R} = \left(V^d_{L,R}\right)_{j,K} d^{k\prime}_{L,R} \tag{9.9}$$

In these four equations we have four $n \times n$ unitary matrices, $\left\{V^u_L, V^u_R, V^d_L, V^d_R\right\}$, where n (= 3 in the SM) is the number of generations. In the physical basis we have explicitly,

$$\mathcal{L}_{QH} = -\left(1 + \frac{h}{v}\right)\left(m_d \bar{d}d + m_u \bar{u}\bar{u} + m_e \bar{c}c + m_s \bar{s}s + m_t \bar{t}t + m_b \bar{b}b\right) \tag{9.10}$$

where we have combined our left and right handed spinors into Dirac spinors,

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{9.11}$$

For brevity we denote

$$d_L = \begin{pmatrix} d_L \\ s_L \\ b_L \end{pmatrix} \tag{9.12}$$

$$u_L = \begin{pmatrix} u_L \\ c_L \\ t_L \end{pmatrix} \tag{9.13}$$

and similarly for right handed fields. We can write

$$\mathcal{L}_{QH} = -\left(\bar{d}'_L M'_d d'_R + \bar{u}'_L M_u u'_R + h.c.\right) \tag{9.14}$$

$$= -\left(\bar{d}_L M_d d_R + \bar{u}_L M_u u_R + h.c.\right) \tag{9.15}$$

[1]The question of in what basis the “particles” exist is quite subtle as we will see later in the discussion of meson oscillations.

with

$$M_d = \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix} \qquad M_u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}$$

Consider the top equation (equation in the weak basis):

$$\mathcal{L}_{QH} = \bar{d}'_L M'_d d'_R + \bar{u}'_L M_u u'_R + h.c. \tag{9.16}$$

$$= \underbrace{\bar{d}'_L V_L^{d\dagger}}_{\bar{d}_L} V_L^d M'_d V_R^{d\dagger} \underbrace{V_R^d d'_R}_{d_R} + \underbrace{\bar{u}'_L V_L^{u\dagger}}_{\bar{u}_L} V_L^u M_u V_R^{u\dagger} \underbrace{V^u u'_R}_{u_R} + h.c. \tag{9.17}$$

comparing with equation 9.15 we see that

$$M_d = V_L^d M'_d V_R^{d\dagger} \tag{9.18}$$

$$M_u = V_L^u M'_u V_R^{u\dagger} \tag{9.19}$$

The quark interaction terms are given by

$$\mathcal{L}_Q^{int} = \sum_{j=1}^{n} i\bar{Q}_L^{j\prime}\gamma^\mu \left[\partial_\mu - ig\tau_i W_\mu^i - \frac{ig'}{2} Y_L B_\mu\right] Q_L^{j\prime\prime} + \sum_{j=1}^{n} i\bar{u}_R^{j\prime}\gamma^\mu \left[\partial_\mu - i\frac{g'}{2} Y_{R,u} B_\mu\right] u_R^{j\prime}$$
$$+ \sum_{j=1}^{n} i\bar{d}_R^{j\prime}\gamma^\mu \left[\partial_\mu - i\frac{g'}{2} Y_{R,d} B_\mu\right] d_R^{j\prime} \tag{9.20}$$

There are two possible cases for the left handed terms:

1. The terms with $\mathbb{1}, \tau^3$; These are the diagonal terms and correspond to γ, Z
2. The terms with τ^1, τ^2; These are non-diagonal and give $W^\pm$

In the first case we have (we denote stuff$_\mu$ as the terms in the square brackets) for the left handed terms:

$$\sum \bar{u}_L^{j\prime}\gamma^\mu \left[\text{stuff}_\mu\right] u_L^{j\prime} \pm \bar{d}_L^{j\prime}\gamma^\mu \left[\text{stuff}_\mu\right] d_L^{j\prime} \tag{9.21}$$

Switching to the vector form for the quarks we have

$$\bar{u}'_L\gamma^\mu \left[\text{stuff}_\mu\right] u'_L \pm \bar{d}'_L\gamma^\mu \left[\text{stuff}_\mu\right] d'_L \tag{9.22}$$

We can rewrite this in terms of the unprimed basis:

$$\bar{u}_L V_L^\mu \gamma^\mu \left[\text{stuff}_\mu\right] V_L^{u\dagger} \pm \bar{d}_L V_L^d \gamma^\mu \left[\text{stuff}_\mu\right] V_L^{d\dagger} d_L \tag{9.23}$$

Now the V's act on flavor space (which is completely unrelated to stuff$_\mu$). So we can bring them across and we have

$$u_L\gamma^\mu \left[\text{stuff}_\mu\right] u_L \pm \bar{d}_L\gamma^\mu \left[\text{stuff}_\mu\right] d_L \tag{9.24}$$

Thus neutral currents remain diagonal in flavor space. The same math also follows for the right handed fields.

Now consider the second case of charged current interaction $W^{\pm} = \frac{1}{\sqrt{2}}(W^1 \mp W^2)$. Our left handed terms are:

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \begin{pmatrix} \bar{u}_L^{1\prime} & \bar{u}_L^{2\prime} & \bar{u}_L^{3\prime} \end{pmatrix} \gamma^{\mu} \begin{pmatrix} d_L^{1\prime} \\ d_L^{2\prime} \\ d_L^{3\prime} \end{pmatrix} W_{\mu}^{+} + \text{h.c.} \tag{9.25}$$

Switching to the physical basis gives:

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \bar{u}_L V_L \gamma^{\mu} V_L^{d\dagger} d_L W_{\mu}^{+} + \text{h.c.} \tag{9.26}$$

We define

$$V_{CKM} = V_L^{\mu} V_L^{d\dagger} \tag{9.27}$$

This matrix is called the "Caribbo Kobayashi Maskawa" matrix and gives

$$\mathcal{L}_{CC} = \frac{g}{\sqrt{2}} \bar{u}_L \gamma^{\mu} V_{CKM} d_L W_{\mu}^{+} \tag{9.28}$$

Note that this matrix is unitary:

$$V_{CKM}^{\dagger} V_{CKM} = V_L^{d} V_L^{\mu\dagger} V_L^{\mu} V_L^{d\dagger} \tag{9.29}$$

$$= \mathbb{1} \tag{9.30}$$

We now count the independent parameters in the CKM matrix. It is an $n \times n$ complex matrix so it has $2n^2$ initial parameters. Then we have the unitarity constraint which gives n^2 constraints.[2] Now the quark phases also constrain the matrix. We can redefine the phase of quark fields with no effect on matrix elements. The mass terms are trivially invariant under this change. This is trivial under the neutral current terms. However this is not the case for charged currents. q_L and q_R must have the same phase rotation. Our terms are of the form

$$\bar{u}_{L,i} \gamma^{\mu} V_{i,j} d_j \tag{9.31}$$

we have

$$u_{L,i} \to e^{i\phi_i} u_{L,i}$$
$$d_{L,j} \to e^{i\phi_j} d_{L,j}$$

Hence we have

$$V_{CKM} \to \begin{pmatrix} e^{i\phi_u} & 0 & 0 \\ 0 & e^{i\phi_c} & 0 \\ 0 & 0 & e^{i\phi_t} \end{pmatrix} V_{CKM} \begin{pmatrix} e^{-i\phi_d} & 0 & 0 \\ 0 & e^{-i\phi_s} & 0 \\ 0 & 0 & e^{-i\phi_b} \end{pmatrix} \tag{9.32}$$

[2] This may seem obvious since $V^{\dagger} = V^{-1}$ is n^2 equations. However, some of these equations may be redundant. The proof is more intricate but left out here.

Hence

$$V_{i,j} \to e^{-i(\phi_j-\phi_i)}V_{i,j} \tag{9.33}$$

This removes $2n-1$ degrees of freedom. Thus in total we have

$$n^2-2n+1=(n-1)^2 \tag{9.34}$$

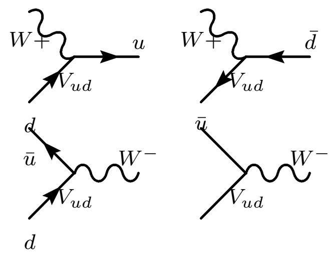
degrees of freedom.

n	free param $\to (n-1)^2$	Euler Angles	Phases
2	1	1	1
3	4	3	1
4	9	6	3

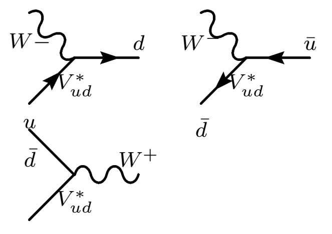
The charged current interactions are

$$\mathcal{L}_{CC}=\frac{g}{\sqrt{2}}\left\{\sum_{i,j}\bar{u}^i_L\gamma^\mu V_{i,j}d^j_L W^+_\mu+\sum_{i,j}\bar{d}^j_L\gamma^\mu V^*_{i,j}u_i W^-_\mu\right\} \tag{9.35}$$

We have the following interactions:



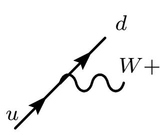
and



Consider for now 2 generations: 1 free parameter.

$$\mathcal{L}_{CC}=W^{\mu\dagger}\begin{pmatrix}\bar{u}_L\\ \bar{c}_L\end{pmatrix}\gamma^\mu\begin{pmatrix}\cos\theta_c & \sin\theta_c\\ -\sin\theta_c & \cos\theta_c\end{pmatrix}\begin{pmatrix}u_L\\ c_L\end{pmatrix}+\text{h.c.} \tag{9.36}$$

where we introduce the Cabibbo angle which is found experimentally to be $\cos\theta_c\approx 0.97$ and $\sin\theta_L\approx 0.22$. We have the interaction



$$\to \frac{ig}{\sqrt{2}}V^*_{ud}\gamma^\mu\frac{1-\gamma^5}{2} \tag{9.37}$$

One exact form is given by:

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & s_2 \\ 0 & -s_2 & c_2 \end{pmatrix} \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & s_3 \\ 0 & -s_3 & -s_3 \end{pmatrix} \tag{9.38}$$

with $c_i \equiv \cos\theta_i$ and $s_i \equiv \sin\theta_i$. The Wolfenstein parametrization simplifies this form. it is given as follows. Define $\lambda = \sin\theta_c \approx 0.22$. Then

$$V_{CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4) \tag{9.39}$$

A, ρ, η are all $\mathcal{O}(1)$ so the hierarchy is completely contained in λ.

9.1 "GIM Mechanism"

We now go into a historical aside into the Glashow, Iliopoulos, and Maiani mechanism for suppression of flavor-changing neutral currents. In 1974 they knew of quark structure u, d, s. Cabibbo said: suppose we have weak quark and lepton doublets. Since at the time they only know of 3 quarks he could only write

$$Q = \begin{pmatrix} u \\ d\cos\theta_c + s\sin\theta_c \end{pmatrix} \equiv \begin{pmatrix} u \\ d_c \end{pmatrix} \tag{9.40}$$

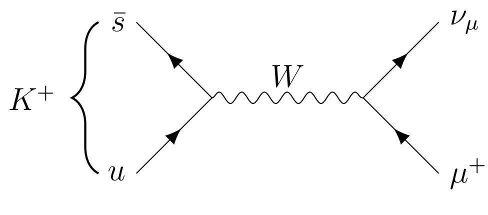
and

$$L = \begin{pmatrix} \nu_\mu \\ \mu_L \end{pmatrix} \tag{9.41}$$

Using the spin analogy he wrote the interaction

$$J^+ = g\bar{Q}\tau^\dagger Q = g\begin{pmatrix} \bar{u} & \bar{d}_c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ d_c \end{pmatrix} \tag{9.42}$$

Using this interaction he was able to explain the K^+ branching fraction:



However they saw that the K^0 decay was heavily suppressed. They had

$$\begin{aligned} H^0 &= g\bar{Q}\tau_3 Q & (9.43)\\ &\sim g(\bar{u}u - \bar{d}_c dc) & (9.44)\\ &\sim g(\bar{u}u - \bar{d}d\cos^2\theta_c - \bar{s}s\sin^2\theta_c + (\bar{d}s + \bar{s}d)\sin\theta_c\cos\theta_c) & (9.45) \end{aligned}$$

This is a flavor changed neutral current!GIM proposed a new up-type quark which they called the c quark

$$Q_2 = \begin{pmatrix} c \\ s_c \end{pmatrix} = \begin{pmatrix} c \\ -d\sin\theta_c + s\cos\theta_c \end{pmatrix} \tag{9.46}$$

with

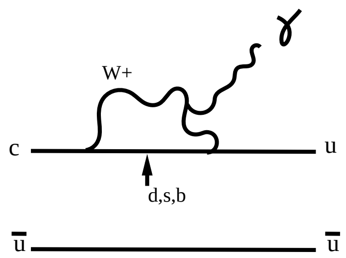
$$J^0 = \left[\bar{Q}\tau_3 Q + \bar{Q}_2\tau_2 Q_2\right] \tag{9.47}$$
$$= g\left[\bar{u}u + \bar{c}c - \bar{d}_c d_c - \bar{s}_c s_c\right] \tag{9.48}$$
$$= g\left[\bar{u}u + \bar{c}c - \bar{d}d - \bar{s}s\right] \tag{9.49}$$

They predicted the c quark mass in order to explain the observed

$$\Gamma\left(K^0 \to \mu^+\mu^-\right) \tag{9.50}$$

They predicted a charm quark mass of $\sim 1-3$GeV. The quark was successfully discovered later that same year.

Consider the following "penguin" process:



The amplitude is proportional to

$$\mathcal{M} \propto \sum_{q=d,s,b} f(m_q) V_{cq}^* V_{uq} \tag{9.51}$$

If m_d, m_s, m_b are equal (m) then

$$A \propto f(m) \sum V_{cq}^* V_{uq} \tag{9.52}$$

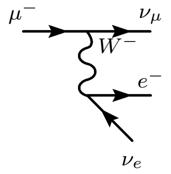
Since $V^\dagger V = \mathbb{1}$ we have $\sum V_{cq}^* V_{uq} = 0$. To see this consider the following:

$$V^\dagger V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} V_{ud}^* & V_{cd}^* & V_{td}^* \\ V_{us}^* & V_{cs}^* & V_{ts}^* \\ V_{ub}^* & V_{cb}^* & V_{tb}^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9.53}$$

This is just the GIM mechanism at work. It is essentially just a statement that CKM matrices are unitary. In practice since the masses are similar this element is just suppressed.

9.2 What Do We Know About the CKM?

For the CKM we know the magnitudes through semileptonic decays and $B^0 - \bar{B}^0$ meson mixing. Recall the muon decay lifetime from



We can make a crude estimate of the rate as follows. We know that

$$\Gamma \sim |\mathcal{M}|^2 \sim G_F^2 \tag{9.54}$$

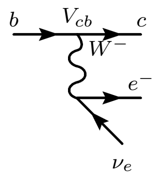
and $[\Gamma] = 1, [G_F^2] = -4$. The only other dimensionful number in the problem in the mass of the muon (since the other masses are small). By dimensional analysis we then have

$$\Gamma \sim m_\mu^5 G_F^2 \tag{9.55}$$

The true answer turns out to be

$$\Gamma = \frac{1}{192\pi^3} G_F^2 m_\mu^5 \tag{9.56}$$

Similarly the lifetime of a b quark is given by



$$\Gamma\left(\qquad \right) = \frac{m_b^5 G_F^2}{192\pi^3} |V_{cb}|^2 \left(1 + \mathcal{O}\left(\frac{1}{m_B^2}\right) + \mathcal{O}(\alpha_s) \right) \tag{9.57}$$

Thus using this diagram we can estimate the V_{cb} element. Note that there are other possible diagrams to search in, however they are either too hadronic (they form a big mess of particles and the final states are hard to determine) are suppressed due to angular momentum arguments.

There are two types of measurements

1. Inclusive measurement: Don't pick any particular hadronic final state,

$$\mathcal{M} = \langle X_c | V^\mu - A_\mu | B \rangle \frac{G_F}{\sqrt{2}} V_{cb} \langle e^- | \gamma_\mu (1 - \gamma^5) | \nu_e \rangle \tag{9.58}$$

   where the $|X_c\rangle$ can be any kind of final state that you can imagine either a single meson, or a few mesons, etc made using a charm quark (the final states will hadronize) and $|B\rangle$ is a B meson.

2. Exclusive measurement - select a particular final state. For example a D^* meson (which shortly decays into a $D + \pi$). The amplitude is

$$\mathcal{M} = \langle D^* | V^\mu - A^\mu | B \rangle \frac{G_F}{\sqrt{2}} V_{cb} \langle e^- | \gamma_\mu (1 - \gamma^5) | \nu_e \rangle \tag{9.59}$$

We have measured all the CKM elements:

- $|V_{ud}|$: We measure using neutron decay: $n \to p + e + \nu_e$. These are called "superallowed" and occur through nuclear β decay. We have $|V_{ud}| \sim 0.97377 \pm 0.00027$.

- $|V_{us}|$: We measure using $K \to \pi + e + \nu_e$ which gave $|V_{us}| = 0.2257 \pm 0.0021$ (done both exclusively and inclusively)

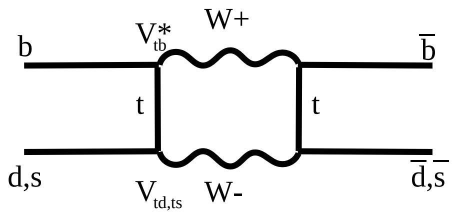
- $|V_{ub}|$: This is difficult to measure and is done using $B \to \pi + \ell + \nu$ and $B \to X_\ell \ell \nu$. The inclusive measurements and the exclusive measures different on the 2σ level: $(3.38 \pm 0.36) \times 10^{-3}$ vs. $(4.27 \pm 0.38) \times 10^{-3}$.

- $|V_{cd}|$: Measured through $D \to \pi + \ell + \nu$ and through charm production in neutrino nucleon interactions. The results are 0.230 ± 0.011.

- $|V_{cs}|$ which is given by $D \to K + \ell + \nu$. This is found to be 0.97345 ± 0.00015.

- $|V_{cb}| \sim (4.06 \pm 0.13) \times 10^{-2}$.

- We are not able to measure the other CKM matrix elements directly. Instead we can look at $B^0_{(s)} - \bar{B}^0_{(s)}$ mixing. These diagrams look like



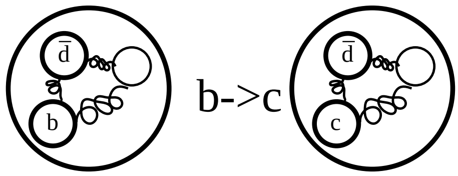
  From this we can get the elements

  - $|V_{td}| = (8.4 \pm 0.06) \times 10^{-3}$
  - $|V_{ts}| = (38.7 \pm 2.1) \times 10^{-3}$

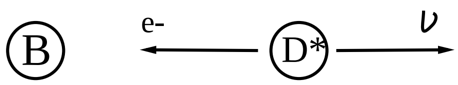
9.3 Heavy Quark Effective theory

The idea is consider the Hydrogen vs Deuterium atoms. Spectrally they look identical. The light degrees of freedom are insensitive to the heavy degrees of freedom. The extra nucleon is a source of EM potential. The fine splitting $\sim m_e \alpha^4$ and does not depend on this extra nucleon. The hyperfine splitting which is $\sim m_e \alpha^4 \frac{m_e}{m_N}$ is sensitive to this extra nucleon.

Similarly you can think of having a very heavy quark in a meson. Imagine for now that both the bottom and charm quarks are very heavy.



In this limit the b and c quarks act like static color charges. If you consider starting off with a B meson and a D^* meson that decays at rest. For zero recoil in decay it doesn't look like anything has happened and we know how to normalize.



$B \to D^* + \ell + \nu$ at zero recoil we have the form factor

$$1 + \mathcal{O}\left(\frac{1}{m_b^2}\right) + \mathcal{O}(\alpha_s) \tag{9.60}$$

9.4 Unitary Triangle

Unitarity requires

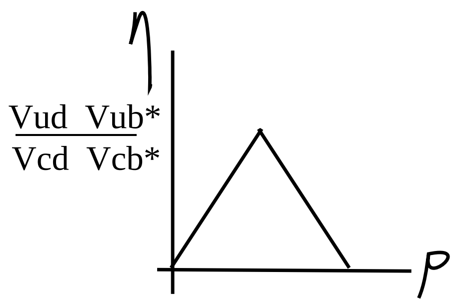
1. $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$. In practice we find 0.9999(6).
2. Consider the off diagonal 1st and 3rd rows.

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0 \tag{9.61}$$

$$\frac{V_{ud}V_{us}^*}{V_{cd}V_{vb}^*} + 1 + \frac{V_{td}V_{ts}^*}{V_{cd}V_{cb}^*} = 0 \tag{9.62}$$

$$\text{Wolfenstein parametrization:} \quad \frac{(1-\lambda^2/2)A\lambda^3(\rho + i\eta)}{-A\lambda^3} + 1 + \frac{A\lambda^3(1-\rho-i\eta)}{-A\lambda^3} = 0 \tag{9.63}$$

$$\rho + i\eta + (1 - \rho - \eta) = 1 \tag{9.64}$$



Chapter 10

Symmetries

10.1 Discrete Symmetries

The three symmetries we are interested in are time reversal ($\mathcal{T}$), parity inversion ($\mathcal{P}$), and charge conjugation ($\mathcal{C}$). CPT theorem tells us that if you have Lorentz invariance, quantum mechanics, and interactions carried by fields implies that CPT is an exact symmetry. Some of the consequences of this are as follows

- Particles and antiparticles must have the same masses
- Particles and antiparticles must have the same lifetimes

10.1.1 Time Reversal

Time reversal gives $t \to -t$ or equivalently

$$\mathcal{T}\psi(t)\overbrace{\mathcal{T}^{-1}}^{\mathcal{T}} = \psi(-t) \tag{10.1}$$

This will inverse all linear and angular momenta:

$$\begin{aligned} \mathbf{p} &\to -\mathbf{p} \\ \mathcal{L} &\to -\mathcal{L} \\ \mathbf{s} &\to -\mathbf{s} \end{aligned}$$

These kind of experiments were done at *CPLear*. We will not deal much with time reversal in these lectures.

10.1.2 Parity

Parity performs the following $\mathbf{x} \to -\mathbf{x}$ or equivalently

$$\mathcal{P}\psi(\mathbf{x})\mathcal{P} = \psi(-\mathbf{x}) \tag{10.2}$$

Parities effects on different objects is summarized below

Object type	Effect	Examples
(polar) vector	$\mathbf{v} \to -\mathbf{v}$	$(\mathbf{p}, \mathbf{E}, \mathbf{A}, \mathbf{J}, \nabla, ...)$
axial/pseudo vectors	$\mathbf{a} \to \mathbf{a}$	$\mathbf{a} = \mathbf{v}_1 \times \mathbf{v}_2$ ($\mathbf{v}_1, \mathbf{v}_2 \to$ polar vec), e.g. $\mathbf{L}, \mathbf{B}$
Scalars	$\phi \to \phi$	$\mathbf{v}_1 \cdot \mathbf{v}_2$
Pseudoscalar	$\phi \to -\phi$	$\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$

Parity acting twice must give back the original state: (denote λ as the eigenvalues of $\mathcal{P}$.

$$\mathcal{P}^2\psi = \lambda^2\psi = \psi \tag{10.3}$$

Hence the eigenvalues of $\mathcal{P}$ are ± 1. In terms of an exponential we have $e^{in_p\pi}$ with $n_p = 0, 1$. Further we have

$$\mathcal{P}Y_L^m = (-1)^L Y_L^m \tag{10.4}$$

For composite systems (multiparticles states) the parities are multiplicative (n_p's add). Two particles with intrinsic parities, n_{p_1} and n_{p_2} in some Y_L^m with $L = \ell$

$$\mathcal{P} = (-1)^{n_{p_1}+n_{p_2}+\ell} \tag{10.5}$$

For fermions f and $\bar{f}$ have opposite intrinsic parity.

$$\mathcal{P}_f \cdot \mathcal{P}_{\bar{f}} = -1 \tag{10.6}$$

Bosons and anti bosons have the same parity. For vector bosons we have $\mathcal{P} = -1$. The convention we use is to assign protons to $\mathcal{P} = +1$. This gives fermions such as quarks, e^-, μ^- ,... to $\mathcal{P} = +1$. The anti-fermions such as anti-quarks, e^+ , $\mu^+, ...$ have $\mathcal{P} = -1$. The ground state mesons have $\mathcal{P} = -1$ since they are $f_i, f\bar{f}_j$ pairs and in an s wave ($\ell = 0$). Thus they are pseudoscalars.

Ground state baryons are given by qqq in an s wave gives $\mathcal{P} = +1$ and anti-baryons in an s-wave the intrinsic parity is $\mathcal{P} = -1$.

10.1.3 Charge Conjugation

Charge conjugation changes particles to antiparticles and vice versa

$$\mathcal{C}\,|p\rangle = |\bar{p}\rangle \tag{10.7}$$

All internal quantum numbers (additive) "reverse sign". Flip the sign of T_3, Y, Q, strangeness, color, ...

Note: The only eigenstates, are particles who are their own antiparticles. For example photons are C odd. With charge conjugation we need to be a bit careful about how we define things. If we take the Dirac field:

$$\mathcal{C}\psi \overbrace{\mathcal{C}^-}^{\mathcal{C}} = -i\gamma^2\psi^* \tag{10.8}$$

For an $SU(2)$ spinor

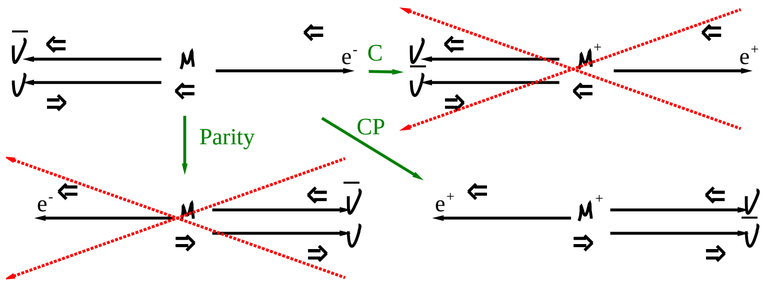
$$\mathcal{C}L\mathcal{C}^{-1} = -i\sigma^2 L_c \tag{10.9}$$

For example for the electron doublet

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\nu}_e \\ e^+ \end{pmatrix} = \begin{pmatrix} -e^+ \\ \bar{\nu}_e \end{pmatrix} \tag{10.10}$$

We swapped T^3, $Y \rightarrow -Y$ and in turn $Q = T^3 + \frac{Y}{2}$ also got flipped.

Experimentally we know that the strong and electroweak interactions don't conserve $\mathcal{C}$ and $\mathcal{P}$ separately:



CP is conserved in weak interactions. When massless fermions are involved or when fermions with degenerate masses are involved.

10.2 Isospin

Isospin is an approximate $SU(2)$ symmetry. QCD is flavor blind. $m_u \approx 3$MeV and $m_d \approx 6$MeV and $m_u - m_u \ll m_p, m_\pi$. The strong force dominates over electromagnetism since $\alpha_s \gg \alpha$. The proton is given by uud. Isospin gives

$$uud \rightarrow udd \tag{10.11}$$

or a proton changing to a neutron. The masses of proton and neutron are 938MeV and 940MeV respectively. The pions show a similar effect:

Particle	Mass (MeV)	Content
$\pi^\pm$	139	ud
π^0	135	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$
$\rho^\pm$	775.4	
ρ^0	775.5	
$K^\pm$	494	$s\bar{u}$
$\bar{K}$	498	$s\bar{d}$

The nucleon $N = \begin{pmatrix} p \\ n \end{pmatrix}$ forms an isospin doublet. The multiplet convention is that the most positive particle is in the uppermost component. This is the same as the discussion of spin angular momentum (they are both just an $SU(2)$ doublet. We have I^2, I_3 which commute with $I^2 = I_1^2 + I_2^2 + I_3^2$ and $I_i = \frac{\sigma_i}{2}$. We can view the proton and neutron as isospin states

$$|p\rangle = |\frac{1}{2}, \frac{1}{2}\rangle \tag{10.12}$$

$$|n\rangle = |\frac{1}{2}, \frac{1}{2}\rangle \tag{10.13}$$

A two particle state is given by

$$\text{triplet:} \begin{cases} |p,p\rangle = |1,1\rangle \\ \frac{1}{\sqrt{2}}\left\{|pn\rangle + |np\rangle\right\} = |1,0\rangle \\ |n,n\rangle = |1,-1\rangle \end{cases} \tag{10.14}$$

$$\text{singlet:} \left\{ \frac{1}{\sqrt{2}}\left\{|p,n\rangle - |n,p\rangle\right\} = |0,0\rangle \right. \tag{10.15}$$

For the antiparticles we need to be more careful. $\mathcal{C} = -i\sigma_2$

$$\bar{N} = \begin{pmatrix} -\bar{n} \\ \bar{p} \end{pmatrix} \tag{10.16}$$

Let's combine particle and antiparticles states. They also form isospin eigenstates.

$$\text{triplet:} \begin{cases} -|p,\bar{n}\rangle = |1,1\rangle \\ \frac{1}{\sqrt{2}}\left\{|p\bar{p}\rangle - |n\bar{n}\rangle\right\} = |1,0\rangle \\ |n,\bar{p}\rangle = |1,-1\rangle \end{cases} \tag{10.17}$$

$$\text{singlet:} \left\{ \frac{1}{\sqrt{2}}\left\{|p,\bar{p}\rangle + |n,\bar{n}\rangle\right\} = |0,0\rangle \right. \tag{10.18}$$

This descriptions works equally well for quarks. We can have $\begin{pmatrix} u \\ d \end{pmatrix}$

$$|u\rangle = |\frac{1}{2}, \frac{1}{2}\rangle \tag{10.19}$$

$$|d\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle \tag{10.20}$$

and our antiparticles get isospin $\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$

$$-|\bar{d}\rangle = |\frac{1}{2}, \frac{1}{2}\rangle \tag{10.21}$$

$$|\bar{u}\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle \tag{10.22}$$

Our pions and η meson then form isospin states

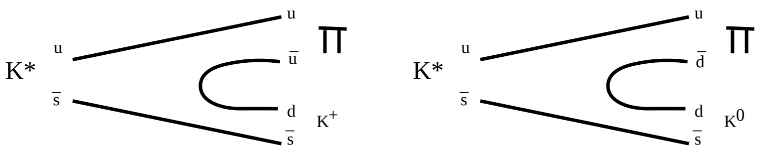
$$\text{triplet:}\begin{cases} -\left|\pi^+\right\rangle = \left|1,1\right\rangle = -\left|u\bar{d}\right\rangle \\ \left|\pi^0\right\rangle = \left|1,0\right\rangle = \frac{1}{\sqrt{2}}\left\{\left|u\bar{u}\right\rangle - \left|d\bar{d}\right\rangle\right\} \\ \left|\pi^-\right\rangle = \left|1,-1\right\rangle = \left|d\bar{u}\right\rangle \end{cases} \tag{10.23}$$

$$\text{singlet:}\left\{\left|\eta\right\rangle = \left|0,0\right\rangle = \frac{1}{\sqrt{2}}\left\{\left|u,\bar{u}\right\rangle + \left|d,\bar{d}\right\rangle\right\}\right. \tag{10.24}$$

10.3 Rates of Related Processes

10.3.1 $K^{*+} \to \pi^0 K^+$ v.s. $K^{*+} \to \pi^+ K^0$

We will now apply what we learned of is isospin to related processes. Consider the following two processes



The isospin of the different states are summarized below

$$\left|K^{*+}\right\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle \tag{10.25}$$

$$\left|K^+\pi^0\right\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|1,0\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \tag{10.26}$$

$$\left|K^0\pi^+\right\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|1,1\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle \tag{10.27}$$

The amplitude of the decay is given by

$$A\left(K^{*+} \to K^+\pi^0\right) = \left(\sqrt{\frac{2}{3}}\left\langle\frac{3}{2},\frac{1}{2}\right| - \sqrt{\frac{1}{3}}\left\langle\frac{1}{2},\frac{1}{2}\right|\right) H_s \left|\frac{1}{2},\frac{1}{2}\right\rangle \tag{10.28}$$

If we assume isospin symmetry (i.e. we assume $[H, I^2] = [H, I_3] = 0$) then we can use the Wigner Eckart theorem (proved in HW) which says that

$$\left\langle I', I_3' \middle| H \middle| I, I_3\right\rangle = \delta_{I,I'}\delta_{I_3,I_3'}A_I \tag{10.29}$$

where A_I is dependent only on I and not I_3. We then define

$$A_{1/2} = \left\langle\frac{1}{2},\frac{1}{2}\middle|H\middle|\frac{1}{2},\frac{1}{2}\right\rangle \tag{10.30}$$

so we have

$$A\left(K^{*+} \to K^+\pi^0\right) = -\frac{1}{\sqrt{3}}A_{1/2} \tag{10.31}$$

$$A\left(K^{*+} \to K^0\pi^+\right) = \sqrt{\frac{2}{3}}A_{1/2} \tag{10.32}$$

which gives

$$\frac{\Gamma\left(K^{*+} \to K^+\pi^0\right)}{\Gamma\left(K^{*+} \to K^0\pi^+\right)} = \frac{\left|A\left(K^+\pi^0\right)\right|^2}{\left|A\left(K^0\pi\right)\right|^2} = \frac{1}{2} \tag{10.33}$$

which is what we observe experimentally.

One can go about this a different way by looking at the quarks. We here ignore the small mass differences of the quarks. The π^0 meson is given by $|\pi^0\rangle = \frac{1}{\sqrt{2}}\left(-|d\bar{d}\rangle + |u\bar{u}\rangle\right)$. We created an $|u\bar{u}\rangle$ state in the $K^{*+} \to \pi^0 K^+$ reaction so the amplitude that we would indeed get a π^0 was $\frac{1}{\sqrt{2}}$. This suppression does not occur in the π^+K^0 decay. Thus we again have (in an independent method)

$$\frac{\Gamma\left(K^{*+} \to K^+\pi^0\right)}{\Gamma\left(K^{*+} \to K^0\pi^+\right)} = \frac{\left|A\left(K^+\pi^0\right)\right|^2}{\left|A\left(K^0\pi\right)\right|^2} = \frac{1}{2} \tag{10.34}$$

10.3.2 $\rho^0 \to \pi^+\pi^-$ v.s. $\rho^0 \to \pi^0\pi^0$

Consider now a different decay, the decay of a ρ^0 meson. It is a $|1,0\rangle$. We have

$$|\pi^+\pi^-\rangle = |1,1\rangle \otimes |1,-1\rangle = \sqrt{\frac{1}{6}}\,|2,0\rangle + \frac{1}{\sqrt{2}}\,|1,0\rangle + \frac{1}{\sqrt{3}}\,|0,0\rangle \tag{10.35}$$

$$|\pi^+\pi^-\rangle = |1,0\rangle \otimes |1,0\rangle = \sqrt{\frac{2}{3}}\,|2,0\rangle + 0\cdot|1,0\rangle - \frac{1}{\sqrt{3}}\,|0,0\rangle \tag{10.36}$$

We have

$$\langle\pi^+\pi^-|H|\rho^0\rangle = \frac{1}{\sqrt{2}}A_1 \tag{10.37}$$

$$\langle\pi^0\pi^0|H|\rho^0\rangle = 0\ ! \tag{10.38}$$

Thus we have complete suppression in this channel.

We can again find this amplitude in an independent method. ρ has $J = 1$. This implies that $L = 1$ due to conservation of angular momentum. π^0 's are spin zero, which implies that they are bosons. Hence they are symmetric under exchange. We must have $Y_L^m(\pi-\theta,\phi+\pi) = Y_L^m(\theta,\phi)$. They requires L to be even. Hence this is disallowed by conservation of angular momentum.

Chapter 11

Meson Spectroscopy

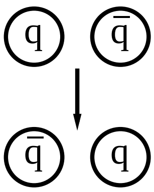
Mesons are bound states of a quark q_i, and some other quark $\bar{q}_j$. Of course there will also be other quarks that pop in and out of the vacuum all the time. Consider the D^+ meson which is made of $c\bar{d}$.

We consider

- The total spin is given by $J = L + S$
- Parity says that $\mathcal{P}Y_L^m(\theta,\phi) = Y_L^m(\pi-\theta,\phi+\pi) = (-1)^L Y_L(\theta,\phi)$. So the total parity of the $q\bar{q}$ system is

$$\mathcal{P}_{q\bar{q}} = \overbrace{(-1)}^{f\bar{f}}(-1)^L = (-1)^{L+1} \tag{11.1}$$

- Under charge conjugation (when q_i and $\bar{q}_j$ are the same flavor). Under C we have



The wave function changes sign under exchange. The fermions anticommute $\rightarrow -1$. The spatial function gives $(-1)^L$. The spin $\begin{cases} \text{singlet} & \downarrow\uparrow - \uparrow\downarrow \\ \text{triplet} & \downarrow\uparrow + \uparrow\downarrow \end{cases}$. Under $\mathcal{C}$ we have $(-1)^{L+S}$.

We summarize our understanding of the up and down quark mesons below

		J^{PC}	$I=1(u\bar{d},\bar{u},\ \frac{1}{\sqrt{2}}(u\bar{u}-\bar{d}d))$	$I=\frac{1}{2}(u\bar{s},d\bar{s},\ s\bar{u},\bar{d}s)$	$I=0(\frac{1}{\sqrt{2}}\big(u\bar{u}+d\bar{d}\big),\ s\bar{s})$	Typical Mass (MeV)
$L=0$	$S=0$	0^{-+}	$\pi^{\pm},\pi^0$	$K^{\pm},K^0,\bar{K}^0$	η,η'	~ 500
	$S=1$	1^{-+}	$\rho^{\pm},\rho^0$	K^*	ω,ϕ	~ 800
$L=1$	$S=0$	1^{+-}	b_1	$K_1(1270)^?$	$h_1(1170)$	~ 1200
	$S=1$	2^{++}	a_2	K_2^*	f_2,f_2'	~ 1400
		1^{++}	a_1	$K_1(1400)^?$	$f_1(1420),f_1(1285)$	~ 1300
		0^{++}	a_0	$K^*(1430)$	?	~ 1200

where the question marks indicate current experimental issues in finding out what states we have. Note that there are certain combinations that we just don't see in the SM. For example we don't see 0^{--}. More precisely, these states don't appear in what's known as the "static quark model".

We now come back to $\rho^0 \to \pi^+\pi^-$ for a moment. We have $\mathcal{C}(\rho^0) = -1$ and $\mathcal{C}(\pi^+\pi^-) = -\pi^-\pi^+$ ($L=1$). Moreover parity says that $\mathcal{P}(\rho^0) = -1$ and $\mathcal{P}(\pi^+\pi^-) = (-1)^L(-1)^2 = -1$. Thus $\mathcal{C}$ and $\mathcal{P}$ are both conserved which is what you need for a strong interaction. Experimentally we find

	$\pi\pi$	$\pi\pi\pi$
ρ	✓	✗
ω	✗	✓

11.1 G Parity

To understand the ρ and ω decays we need to introduce what's known as G Parity. We need two ingredients.

- In isospin states we define

$$\pi = \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix} \tag{11.2}$$

  In this basis the charge conjugation operator takes the form

$$\mathcal{C} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{11.3}$$

  where the negative signs arise from the need to have $\begin{pmatrix} -\bar{d} \\ \bar{u} \end{pmatrix}$ as the anti-particle isospin states.

- Now suppose we want to rotate by π (radians) about the I_2 axis: $R_2(\pi)$.

$$\begin{aligned} \langle \pi^0 | R_2(\pi) | \pi^0 \rangle &= \langle 1,0 | R_2(\pi) | 1,0 \rangle \\ &= d^1_{0,0}(\pi) \\ &= \cos\pi \\ &= -1 \end{aligned}$$

where one can find the $d^1_{0,0}$ coefficient in a table. We can also write

$$\begin{aligned}\langle\pi^0|R_2(\pi)|\pi^+\rangle &= \langle 1,-1|R_2(\pi)|1,1\rangle \\ &= d^1_{-1,1}(\pi) \\ &= \frac{1-\cos\pi}{2} \\ &= 1\end{aligned}$$

Note that these brakets tell us the overlap between the initial state rotated by π in the $y-axis$. This description should be described in a graduate Quantum Mechanics book. You can calculate the $d^j_{m,m'}$ factors by just finding the overlaps between spherical harmonics.

Taking these brakets together we can write our rotation operator also in matrix form:

$$R_2(\pi) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{11.4}$$

We now define a new operator G Parity operator which is the combined operation

$$G = \mathcal{C}R_2(\pi) \xrightarrow{\substack{\text{pion}\\\text{case}}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{11.5}$$

This is a general operator which we take to apply also for the nonpion case. We have

$$G\,|\pi^\pm\rangle = -\,|\pi^\pm\rangle \tag{11.6}$$
$$G\,|\pi^0\rangle = -\,|\pi^0\rangle \tag{11.7}$$

We then have

Particle	I^G
π	1^-
ρ	1^+
ω	0^-

$\mathcal{C}$ and I are both symmetries of the strong interaction. This impiles that G Parity should be conserved in a decay (if $R_2(\pi)$ and $\mathcal{C}$ are both symmetries, so must G be).

Process	ρ_0	ω
$\to\pi^+\pi^-$	$G: +1 \to (-1)^2$✓	$G: -1 \to 1$✗
$\to\pi^+\pi^-\pi^0$	$G: +1 \to (-1)^3$✗	$G: -1 \to (-1)^3$✓

Note that in practice $\Gamma_\rho \sim 150\text{MeV}$ and $\Gamma_\omega \sim 9\text{MeV}$. The reason the ρ rate is so much larger is because it's decaying into lighter particles. Thus there is more phase space to decay into.

11.2 Applications of Discrete Symmetries

Another effect of G partiy for example is by looking J/ψ. This is a $c\bar{c}$ state hence it has isospin zero. It has a G partiy of $-$. So it can only decay to an odd number of pions.

Let's consider the η meson which is the isospin singlet partner the pion. It has $I^G = 0^+$. The branching ratio is given by

$$\mathrm{Br}(\eta \to \pi^+\pi^-\pi^0) = 23\% \tag{11.8}$$

$$\mathrm{Br}(\eta \to \pi^0\pi^0\pi^0) = 32\% \tag{11.9}$$

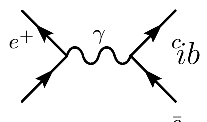
and we don't see any $\eta \to \pi^+\pi^-$. Thus this does not proceed through the strong interaction. It does in fact decay through the strong interaction. Looking at the width we see $\Gamma_\eta \sim 130\text{keV} \ll \Gamma_\omega$. This is a typical width for electromagnetic decays. The reason we don't see a $\pi^+\pi^-$ decay is as follows. The η has $J^{PC} = 0^{-+}$. The pions need to end up in a state of no angular momentum. The $\pi^+\pi^-$ need to end up in a state with $L = 0$. The Parity of the final state is given by

$$\mathcal{P}(\pi^+\pi^-) = (-1)^2(-1)^0 = 1 \tag{11.10}$$

but the Parity of the η itself is odd. Both the electromagnetic and strong interactions conserve Parity and hence it does the next "best thing" which turns out to be 3 pion decays.

Let's now take a look at the $a_1 \to \rho\pi$ decay. This decay has $J^P = 1^+ \to 1^-0^-$. So the allowed orbital angular momenta are $L = 0, 1, 2$ by angular momentum conservation. Parity gives $1 = (-1)^2(-1)^L$ which implies that $L = 0, 2$. These are both allowed and they both occur.

J/ψ is produced through:



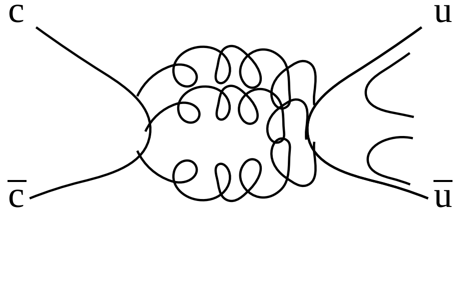
(11.11)

Since this is mediated by a photon we already know that $J^{PC} = 1^{--}$. Starting with this we now try to find the S and L for J/ψ. We know that $J = 1$. The total S possibilities are $0, 1$ (we have a two particle spin state). The Parity is -1. Hence

$$-1 = (-1)^{L+1} \tag{11.12}$$

Hence $L = 0$ which implies that $S = 1$.

The J/ψ width is $87\text{keV} \ll \Gamma_\rho, \Gamma_\omega$. If we go through all the selection rules ($I^G = 0^-$). We would expect ω - like decays. Non of the quantum numbers we have looked at forbid a strong interaction. The charm quarks have to annhilate:



To decay via the strong force at least 3 gluons are needed to be emitted. The charge conjugation quantum number needs to be conserved. A pair of gluons is C even while the J/ψ state is C odd. The charm masses are on the order of a GeV . At about 1 GeV we have $\alpha_s \sim 0.2$. Hence the vertices are at least $(0.2)^3 \sim 0.008 \sim \alpha_{EM}$ hence this decay has a rate consistent with an EM decay. This kind of suppression is known as OZI suppression and was discovered in 1963.

11.3 Neutral Kaons and CP violation

We have

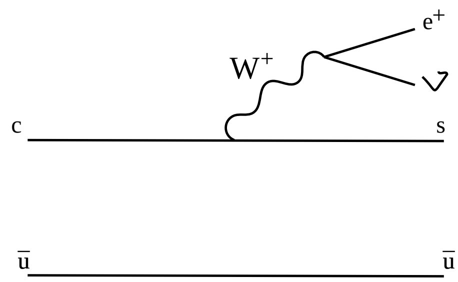
$$|K^0\rangle = |d\bar{s}\rangle \tag{11.13}$$
$$|\bar{K^0}\rangle = |s\bar{d}\rangle \tag{11.14}$$

The operation of CP on these states is

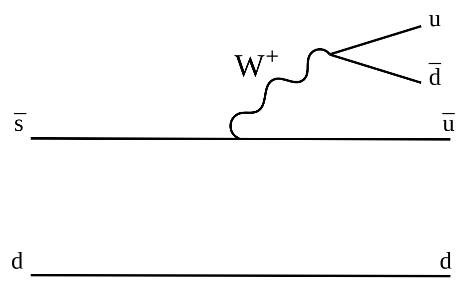
$$\begin{aligned} \mathcal{CP}\,|K^0\rangle &= \eta_K\,|\bar{K^0}\rangle \\ \mathcal{CP}\,|\bar{K^0}\rangle &= \eta_K^*\,|K^0\rangle \end{aligned} \tag{11.15}$$

where η_K is given by some phase convention, $\eta_K = e^{i\phi_K}$. We will take $\eta_K = 1$. Note that the states, $|K^0\rangle\,, |\bar{K^0}\rangle$, are not CP eigensates. This leads to the bizarre phenomenon of meson mixing.

One useful convention when deciding which state is the particle and which state is the anti-particle is as follows. We label the "particle" as the state that decays semileptonically to a positron. For example we have:



which we denote as the B^0 meson. We call the "antiparticle" the state that decays semileptoniaclly to an electron. We have Kaon decay as follows



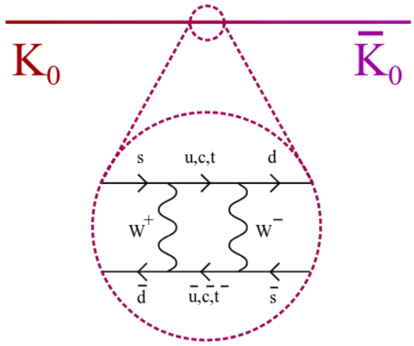
so $K^0 \to \pi^+\pi^-$ and $\bar{K}^0 \to \pi^-\pi^+$. But we can also have the reverse process such that

$$K^0 \to \begin{matrix}\pi^+\\ \pi^-\end{matrix} \to K^0 \tag{11.16}$$

as well as

$$K^0 \to \begin{matrix}\pi^+\\ \pi^-\end{matrix} \to \bar{K}^0 \tag{11.17}$$

These are called “long-range interactions”. We can also have “short-range interactions” which are mediated by W bosons. For example: [Wikipedia(2013)]



Lets consider the combination $|K_1\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right)$ and $|K_2\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\tilde{K}^0\rangle\right)$. With our phase convention we have (see equation 11.15)

$$\mathcal{CP}\,|K_1\rangle = \mathcal{CP}\frac{1}{\sqrt{2}}\left(|\bar{K}^0\rangle + |K^0\rangle\right) = |K_1\rangle \quad \text{CP even} \tag{11.18}$$

$$\mathcal{CP}\,|K_2\rangle = \mathcal{CP}\frac{1}{\sqrt{2}}\left(|\bar{K}^0\rangle - |K^0\rangle\right) = -\,|K_2\rangle \quad \text{CP odd} \tag{11.19}$$

These are now CP eigenstates. If the weak interaction conserves CP then it must mean that this linear combinations of the mass eigenstates are what decay and have a well defined lifetime, not the mass eigenstates themselves!

Consider the two pion final state. We can have either $\pi^+\pi^-$ and $\pi^0\pi^0$. Recall that pions are pseudoscalars with spin zero. Thus we have (the charge conjugation factor is given by $(-1)^{L+S}$).

$$\mathcal{CP}\left(\pi^0\pi^0\right) = (-1)^2 = 1 \tag{11.20}$$

$$\mathcal{CP}\left(\pi^-\pi^+\right) = \underbrace{(-1)^2(-1)^L}_{\mathcal{P}}\underbrace{(-1)^L}_{\mathcal{C}} = 1 \tag{11.21}$$

hence in both cases the two pion states are CP even. For the three pion final states (The $\pi^0\pi^0\pi^0$ state must be symmetric and hence $L = 0$):

$$\mathcal{CP}(\pi^0\pi^0\pi^0) = (-1)^3 = -1 \tag{11.22}$$

$$\mathcal{CP}\left(\pi^+\pi^-\pi^0\right) = \underbrace{\mathcal{CP}\left(\pi^0\right)}_{-1}\underbrace{\mathcal{CP}\left(\pi^+\pi^-\right)}_{+1}(\text{ ang. mom. factors}) \tag{11.23}$$

We have $\overbrace{\pi^0 \underbrace{\pi^+\pi^-}_{L}}^{L'}$ where L, L' sum to 0. Consider the Q of the decay. This is defined as the mass of the initial particle minus the mass of the products:

$$Q_{K\to 3\pi} = m_K - 3m_\pi \sim 70\text{MeV} \tag{11.24}$$

We have very little kinetic energy per pion. If we have $L, L' \neq 0$ then there is a large barrier for creation of the pions at the origin. This is because in general if you have angular momentum that is non zero. The wavefunction goes to zero at the origin. Recall for example the Hydrogen atom wavefunctions. Thus

$$\mathcal{CP}\left(\pi^+\pi^-\pi^0\right) = -1 \tag{11.25}$$

dominates. With this in mind we see $K_1 \to 2\pi$ and $K_2 \to 3\pi$ (they can't decay to 4π since they don't have enough mass). Further since the kinetic energy left in the 3 body decay is so small, the phase space heavily suppresses the K_2 decay. We observe a shortlived state K_S that decays to 2π that has $c\tau_S \sim 2.7$cm and a long lived state K_L with a lifetime of $c\tau_L \sim 15.5$m.

We now discuss the experiment in more detail. We produce $s\bar{s}$ pairs. These will hadronize and we will produce many different states however the states of definite strangeness that we will be produced are K^0 and $\bar{K}^0$. But the mass eigenstates are the ones that will evolve in time and do not equal the production eigenstates. The mass eigenstates are the eigenstates of the full Hamiltonian, not just of the strong interaction. These are the $|K_L\rangle$ and $|K_S\rangle$ states. Note that CPT requires that we have $m_{K^0} = m_{\bar{K}^0}$. It does not require $m_{K_S} = m_{K_L}$. Consider the time evolution of a state with definite mass and lifetime. We have

$$|A, t\rangle = e^{-\left(\frac{\Gamma}{2}+im\right)t} \underbrace{|A, t=0\rangle}_{\equiv |A\rangle} \tag{11.26}$$

The probability that we will observe $|A\rangle$ at a later time, t, is then

$$\begin{aligned} P(t) &= |\langle A|A,t\rangle|^2 \\ &= |\langle A|A\rangle|^2 e^{-\Gamma t} \end{aligned} \tag{11.27}$$

Suppose we produce a pure $|K^0\rangle$ state. In this case we have

$$\begin{aligned} |\psi(0)\rangle &= |K^0\rangle \\ &= \frac{1}{\sqrt{2}}\left(|K_S\rangle + |K_L\rangle\right) \\ |\psi(t)\rangle &= \frac{1}{\sqrt{2}}\left(|K_S\rangle e^{-(\Gamma_S/2+im_s)t} + |K_L\rangle e^{-(\Gamma_L/2+im_L)t}\right) \end{aligned} \tag{11.28}$$

and hence

$$\langle K^0|\psi(t)\rangle = \frac{1}{2}\left\{e^{-(\Gamma_S/2+im_S)t} + e^{-(\Gamma_L/2+im_L)t}\right\} \tag{11.29}$$

$$\langle \bar{K}^0|\psi(t)\rangle = \frac{1}{2}\left\{e^{-(\Gamma_S/2+im_S)t} - e^{-(\Gamma_L/2+im_L)t}\right\} \tag{11.30}$$

As expected at $t = 0$ we have only K^0 and none of it's antiparticle.

In the lab we don't see amplitudes, just intensities:

$$\begin{aligned}
I(K^0) &\equiv \left|\langle K^0|\psi(t)\rangle\right|^2 \\
&= \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + e^{-\frac{1}{2}(\Gamma_S+\Gamma_L)t}(e^{-i(m_S-m_L)t} + e^{i(m_S-m_L)t})\right] && (11.31) \\
&= \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} + 2e^{-\frac{1}{2}(\Gamma_S+\Gamma_L)t}\cos\Delta m t\right] && (11.32) \\
& && (11.33)
\end{aligned}$$

Similarly we have

$$I(\bar{K}^0) = \frac{1}{4}\left[e^{-\Gamma_S t} + e^{-\Gamma_L t} - 2e^{-\frac{1}{2}(\Gamma_S+\Gamma_L)t}\cos\Delta m t\right] \tag{11.34}$$

Now $\Delta m = (3.451 \pm 0.009) \times 10^{-12}$MeV and $\frac{\Delta m}{m} \sim 10^{-14}$.

The change in mass is given by

$$\Delta m = m_K B_K f_K^2 m_c^2 \cos^2\theta_c \sin^2\theta_c \tag{11.35}$$

where B_k is known as a bag factor which is there to account for the fact that the collision is not quite happening at the origin and the f_K is due to the long distance view of things. This is found using the box diagrams of the kaon decays we discussed earlier. It turns out the charm quark gives the biggest contribution to the loop diagram. Plugging in the values gives, $\Delta m \tau_S = 0.474$.

At CERN there was an experiment known as CPLEAR [Angelopoulos et al.(2003)] that ran

$$p\bar{p} \to K^-\pi^+K^0$$

The $K^-\pi^+$ tags the production flavor of K^0 or $\bar{K}^0$.

In 1964 Christenson, Brownin, Fitch, and Turley tested beams of K_L. They found many $K_L \to \pi^+\pi^-\pi^0$ (CP odd $\to$ CP odd) decays however they also saw a small but nonzero decay of $K_L \to \pi^+\pi^-$ which violates CP. This was the first demonstration of CP violation! The dominate mechanism for mixing is $\bar{K}^0 \leftrightarrow K^0$. Hence the $|K_L\rangle \neq |K_2\rangle$ which is a CP odd state but there's a small admixture of $|K_1\rangle$ state as well (we omit a small normalization factor):

$$|K_L\rangle \approx |K_2\rangle + \epsilon\,|K_1\rangle \tag{11.36}$$
$$|K_S\rangle \approx |K_1\rangle + \epsilon\,|K_2\rangle \tag{11.37}$$

where $|\epsilon| = 2.3 \times 10^{-3}$. Working through the amplitudes you find that the charge asymmetry

$$\begin{aligned}
\delta_L &\equiv \frac{N(\ell^+) - N(\ell^-)}{N(\ell^+) + N(\ell^-)} && (11.38) \\
&\approx 2\mathrm{Re}(\epsilon) && (11.39)
\end{aligned}$$

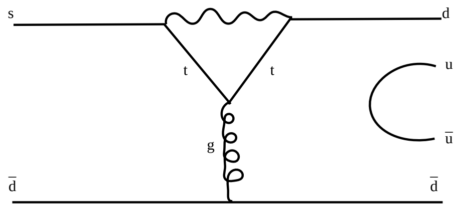
You can measure $\delta_L \approx 3.3 \times 10^{-3}$.

There is a second mechanism for direct CP violation:

$$\langle \pi\pi | H_{weak} | K_2 \rangle \neq 0$$

In K^0 system, this is parametrized by ϵ'. At this point we have $\epsilon'/\epsilon = (16.8 \pm 1.4) \times 10^{-4}$.

It was in this context that penguin diagrams came up since they dominate the decay



Another interesting phenomenon is regeneration. Suppose you pass a K_L beam through a target. The cross sections for the K and $\bar{K}$ components will interact with different cross sections since

$$\sigma(sN \to \text{stuff}) \neq \sigma(\bar{s}N \to \overline{\text{stuff}}) \tag{11.40}$$

Hence after you pass through a material the $|K_L\rangle$ initial state evolves to

$$(\text{atten. factor}) \left(|K_L\rangle + \rho\, |K_S\rangle\right) \tag{11.41}$$

So by adjusting the length of the beam as well as the type of stuff we send the Kaon beams through we can form more $|K_s\rangle$. We call this stuff the regenerator.

So far we have discussed Kaon mixing. We have observed mixing in other mesons as well. Thus far we have seen mixing in

$$D^0 - \bar{D}^0$$
$$B^0 - \bar{B}^0$$
$$B_s^0 - \bar{B}_s^0$$

In $K^0 - \bar{K}^0$ system the CP eigenstates dominate the final states. In the other meson systems, where the mesons are much heavier, there exist final states with both even and odd CP which are almost degenerate. These states are related to each other by isospin resulting in roughly equal decay widths for the CP-even and CP-odd states. For example the B meson can decay though $B \to D\pi$ and $B \to D^*\pi$, which has almost the same width. This implies that $\Gamma_S \simeq \Gamma_L$. Furthermore, since the lifetimes are small, these are tough measurements. We don't get the watch the oscillations over many meters. We have $\Delta m \gg \Gamma \gg \Delta\Gamma$.

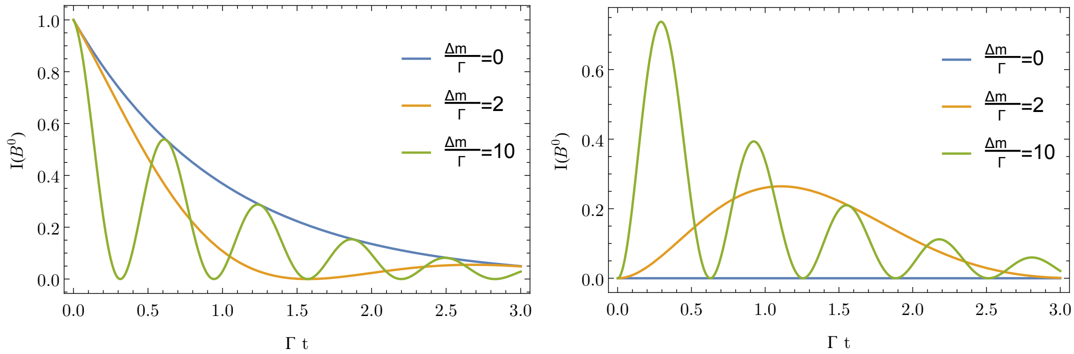
We now consider the $B^0 - \bar{B}^0$ time evolution. For this system we have $\Gamma \sim \Delta m$. Using our result earlier we have (in this system we use "light" and "heavy" instead of "short" and "long")

$$I\left(B^0\right) = \frac{1}{4}\left(e^{-\Gamma_L t} + e^{-\Gamma_H} t + 2e^{-\frac{1}{2}(\Gamma_L + \Gamma_H)t} \cos\left(\Delta t\right)\right) \tag{11.42}$$

If $\Gamma_L \sim \Gamma_H = \Gamma$ then

$$I\left(B^0\right) = \frac{1}{2}e^{-\Gamma t}\left(1+\cos\Delta m t\right) \tag{11.43}$$
$$I\left(\bar{\beta}^0\right) = \frac{1}{2}e^{-\Gamma t}\left(1-\cos\Delta m t\right) \tag{11.44}$$

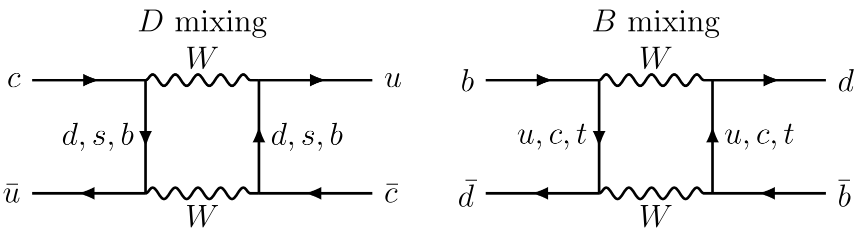
Pictorally,



We can use this asymmetry to measure Δm:

$$\frac{I(B^0)-I(\bar{B}^0)}{I(B^0)+I(\bar{B}^0)} = \cos\Delta m t \tag{11.45}$$

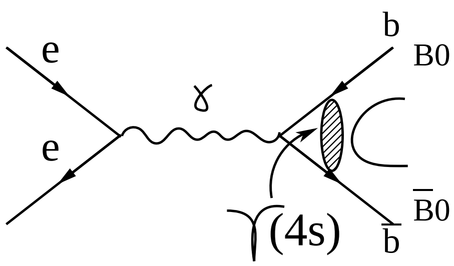
Mixing in the $D-\bar{D}^0$ is a challenge and is relatively suppressed. To understand this we compare the box diagrams for D mixing and B mixing:



Since $m_s, m_b \ll m_t$ the D loop ends up being relatively suppressed.

11.4 B factories

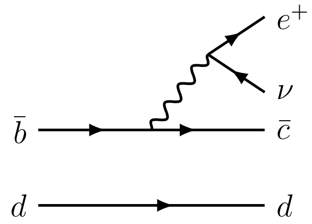
The production of B mesons take the form of



The B mesons are produced through a $\Upsilon(4s)$ meson. This meson has $J^{PC} = 1^{--}$. This is because you get a $\sim 30\%$ boost in B production

The mixing measurements are done as follows.

1. Tagging B^0 or $\bar{B}^0$ production:



   By seeing whether we had a positron or electron emitted in the reaction we can know if we had a B^0 or $\bar{B}^0$.

2. We then observe a mixing. This can be done in two ways.

   (a) The mixing is signaled by e^+e^+, $e^+\mu^+$, or $\mu^+\mu^+$. We measure χ, the time integrated mixing probability. This was done at CLEO.

   (b) A different method measures the oscillations directly. This gives Δm directly as well. The $\Upsilon(4s)$ in it's rest frame. The time constant for the decay is $\tau_t = t_{\text{decay}}$. We look for a flavor tag. The $\Upsilon(4s)$ has $J^{PC} = 1^{--}$. This implies that $B^0\bar{B}^0$ are created in a coherent final state (CP even, $L = 1$). The $L = 1$ says that we are antisymmetric under particle exchange so we have a system that will evolve like

$$\frac{1}{\sqrt{2}}\left(|B^0\rangle\,|\bar{B}^0\rangle - |\bar{B}^0\rangle\,|B^0\rangle\right)$$

   If we set $t = t_{\text{decay}} = 0$, then we know the other B meson flavor. If the first B decayed as a B^0 the second evolves until it decays as

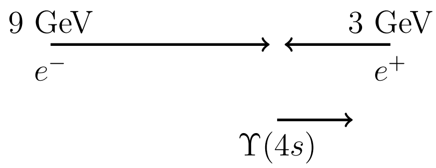
$$|\psi(t)\rangle = e^{-\Gamma t}\left(|\bar{B}^0\rangle \cos\frac{\Delta m}{2} + |B^0\rangle \sin\frac{\Delta m t}{2}\right) \tag{11.46}$$

   Working through all the mathematics of all this it turns out that this ends up being the correct description for $t < 0$

These methods are challenging. This is because $\tau \sim 1.5$ps. In symmetric B factories the $\Upsilon(4s)$ is produced at rest. This gives $p_B = 250$MeV whereas $m_B \sim 5300$MeV. This gives a distance of flight of $\gamma\tau\beta c \sim 20\mu$m, which is really hard to measure.

For this reason BaBar and Belle are antisymmetric B factories, producing boosted (and hence longer-lived B states),[1]

[1]The center of mass energy here is $\sqrt{4E_1E_2} \simeq 10.5$ GeV $\sim m_{\Upsilon(4s)}$



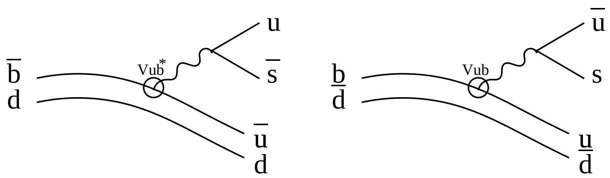
This results in average decay lengths of $L_{\text{flight}} \sim 280\mu$m.

The B_s mixing is even more challenging. For D0 and CDF B_s mixing occurs throgh $p\bar{p}$ collisions. The production state tagging is harder. Here people use a technique called "dominant fragmentation". The most energetic K^+, K^- in the jet containing the B_s has a "partner" s quark to the B_s 's.

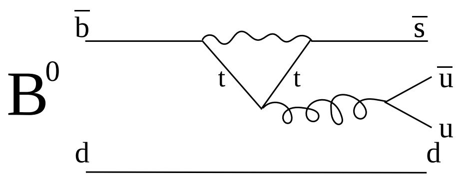
You can also try and flavor tag the other b quark jet. These tricks give a resolution of about 87fs.

11.5 CP violation in the $B^0\bar{B}^0$ system

Consider the decays of the $B^0 \to K^+\pi^-$ vs the decay of a $\bar{B}^0 \to K^-\pi^+$:



We have $V_{ub} \sim A\lambda^3 (\rho - i\epsilon)$. The B_0 decay has an amplitude of $A = A_T V_{ub}^* V_{us}$ vs the $\bar{B}$ decay which has an amplitude $\bar{A} = A_T V_{ub} V_{us}^*$ So while $V_{ub} \neq V_{ub}^*$ with only trree level we still have $|A|^2 = \left|\bar{A}\right|^2$. Thus at tree level we still can't differentiate these amplitudes. We need to use penguin diagrams:



The amplitudes are given by

$$A = A_p e^{i\delta_s} V_{tb}^* V_{ts} \tag{11.47}$$
$$\bar{A} = A_p e^{i\delta_s} V_{tb} V_{ts}^* \tag{11.48}$$

where δ_s is the strong phase difference between the tree and penguin amplitudes. This is induced by the differentiable state strong interactions. This is invarient under CP. If we define $V_{ub} = |V_{ub}|\, e^{i\gamma}$ then we have

$$A_{\mathbf{t}} = A_T\,|V_{ub}|\,|V_{us}|\,e^{-i\gamma} + A_p\,|V_{tb}|\,|V_{ts}|\,e^{i\delta_s} \tag{11.49}$$
$$\bar{A}_{bt} = A_T\,|V_{ub}|\,|V_{us}|\,e^{i\gamma} + A_p\,|V_{tb}|\,|V_{ts}|\,e^{i\delta_s} \tag{11.50}$$

So we have

$$\Gamma\left(B^0 \to K^-\pi^+\right) \propto \left(A_T\left|V_{ub}\right|\left|V_{us}\right|\right)^2 + \left(A_p\left|V_{ts}\right|\left|V_{ts}\right|\right)^2 + \left(2A_TA_p\left|V_{ub}\right|\left|V_{us}\right|\left|V_{tb}\right|\left|V_{ts}\right|\right)\cos\left(\gamma+\delta_s\right) \tag{11.51}$$

$$\Gamma\left(\bar{B}^0 \to K^+\pi^-\right) \propto (...)^2 + (...)^2 + (...)\cos\left(\gamma-\delta_s\right) \tag{11.52}$$

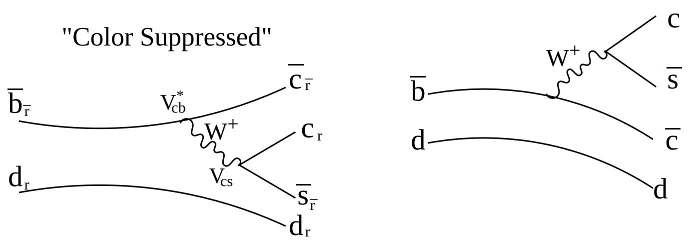
We only have CP violation if $\gamma \neq 0$ and $\delta_s \neq 0$. To measure $A_T, A_p \sim$ roughly comparable. Babar/Belle have measured this and found

$$\frac{N_{K^-\pi^+} - N_{K^+\pi^-}}{N_{K^-\pi^+} + N_{K^+\pi^-}} = -0.098 \pm 0.013 \tag{11.53}$$

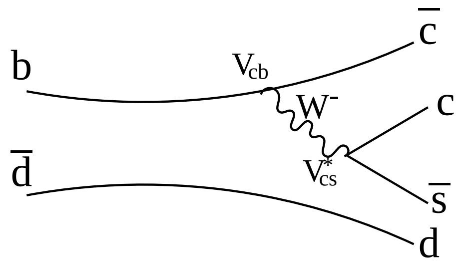
We have

$$B^0 \to \underbrace{J/\psi K_s}_{\text{CP eigenstate}} \tag{11.54}$$

which gives the two decay routes however one is color suppressed:



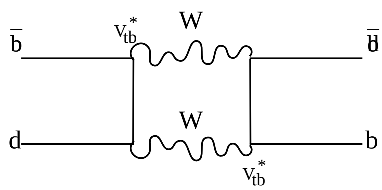
The first decay is color suppressed since the $c\,\bar{s}$ pair that is created must have a particle color configuration. For $\bar{B}^0$ decay we have:



The processes interfere at tree level with mixed and tree level processes:

$$B^0 \xrightarrow{\nearrow \bar{B}^0 \searrow} \psi K_s \tag{11.55}$$

We have



For the B^0 initial state

$$\begin{aligned}\langle B^0|\psi(t)\rangle &= \frac{1}{2}e^{-\Gamma t/2}\left(e^{-im_L t}+e^{-im_H t}\right)\\ &= e^{-\Gamma t/2}e^{-imt}\cos\frac{\Delta m t}{2}\end{aligned} \tag{11.56}$$

and

$$\begin{aligned}\langle \bar{B}^0|\psi(t)\rangle &= \frac{1}{2}e^{-\Gamma t/2}\left(e^{-m_L t}-e^{-im_H t}\right)e^{-2i\beta}\\ &= e^{-\Gamma t/2}e^{-imt}i\sin\frac{\Delta m t}{2}e^{-2i\beta}\end{aligned} \tag{11.57}$$

We should have general includes this phase earlier as well however we were sloppy since we didn't care about the phase of these amplitudes.

Now that direct and mixed amplitudes for $B^0 \to J/\psi K_s$ are given by

$$A_d = Ae^{-\Gamma t/2}e^{-imt}\cos\frac{\Delta m t}{2} \tag{11.58}$$

$$A_m = Ae^{-2i\beta}e^{\Gamma t/2}e^{-imt}i\sin\frac{\Delta m t}{2} \tag{11.59}$$

Hence we have

$$A_{tot}\left(B^0 \to J/\psi K_s\right) = Ae^{-\Gamma t/2}e^{-imt}\left(\cos\frac{\Delta m t}{2} + ie^{-2i\beta}\sin\frac{\Delta m t}{2}\right) \tag{11.60}$$

so we have

$$\begin{aligned}\text{Rate}\left(B^0 \to J/\psi K_s\right) &\propto |A_{tot}|^2\\ &= |A|^2 e^{-\Gamma t}\left(1+2\cos\frac{\Delta m t}{2}\sin\frac{\Delta m t}{2}\sin 2\beta\right)\\ &= |A|^2 e^{-\Gamma t}\left(1+\sin\Delta m t\sin 2\beta\right)\end{aligned} \tag{11.61}$$

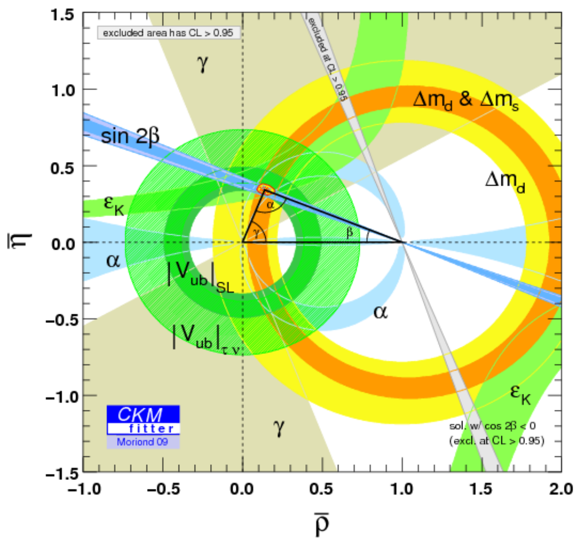
Similarly it's easy to show that

$$\text{Rate}\left(\bar{B}^0 \to J/\psi K_s\right) = |A|^2 e^{-\Gamma t}\left(1-\sin\Delta m t\sin 2\beta\right) \tag{11.62}$$

In practice we calculate

$$\begin{aligned}A_{CP}(t) &\equiv \frac{\text{Rate}\left(B^0 \to J/\psi K_s\right) - \text{Rate}\left(\bar{B}^0 \to J/\psi K_s\right)}{\text{Rate}\left(B^0 \to J/\psi K_s\right) + \text{Rate}\left(\bar{B}^0 \to J/\psi K_s\right)}\\ &= \sin\Delta m t\sin 2\beta\end{aligned} \tag{11.63}$$

All the effects of flavor physics are summarized below [Perez et al.(2010)]



11.6 Form Factors

We now consider the form factors for our weak interaction decays.

We begin by considering $\pi^+ \to \mu^+\nu$. The hadronic current is given by

$$\langle 0|V_\mu - A_\mu|\pi^+\rangle \equiv f_\pi p_\mu \tag{11.64}$$

Our amplitude is given by

$$\mathcal{M} \propto \left[\bar{u}\gamma^\mu \left(1-\gamma^5\right)_\nu\right]\left[f_\pi p_\mu\right] \tag{11.65}$$

Note that

- The amplitude must be a scalar
- The leptonic current is a 4 vector
- The hadronic current must be itself a 4 vector

Note that the pion 4-momentum is the only hadronic 4-vector that we can build the hadronic current 4-vector from. Consider the parity of the following amplitude

$$\overbrace{\langle \underbrace{0}_{+} | \underbrace{V_\mu}_{-} | \underbrace{\pi}_{-} \rangle}^{+\text{ Parity}} \tag{11.66}$$

and

$$\overbrace{\langle \underbrace{0}_{+} | \underbrace{A_\mu}_{+} | \underbrace{\pi}_{-} \rangle}^{-\text{ Parity}} \tag{11.67}$$

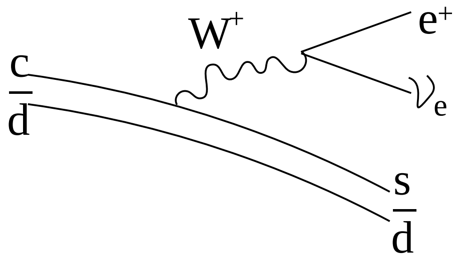
The f_π factor is just a number and p_μ is a normal vector hence $f_\pi p_\mu$ has odd parity. This means that we must have

$$\langle 0|V_\mu|\pi\rangle = 0 \tag{11.68}$$

Now consider the following reaction:

$$\text{Pseudoscalar} \to \text{Pseudoscalar} + \ell + \nu$$

In other words a semileptonic decay. As a specific example we consier $D^+ \to K^0 e^+ \nu_e$:



The amplitude is given by

$$A = \langle K^0|V^\mu - A^\mu|D^+\rangle \frac{G_F}{\sqrt{2}} V_{cs} L_\mu \tag{11.69}$$

where $L_\mu = \bar{e}\gamma^\mu(1-\gamma^5)\nu$ is the current.

Let D^+ have a momentum p^μ and K^0 have momentum k^μ. Thus

$$\langle K^0(k)|V^\mu - A^\mu|D^+(p)\rangle = p^\mu A(p\cdot k) + k^\mu B(p\cdot k)$$

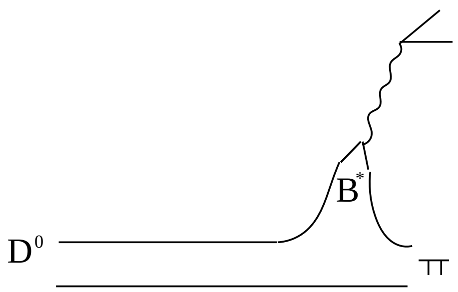
but $q^2 = (p-k)^2 = p^2 + k^2 - 2p^\mu k_\mu$. Thus

$$\langle K^0(k)|V^\mu - A^\mu|D^+(p)\rangle = \underbrace{(p^\mu + k^\mu) f_+(q^2) + (q^\mu - k^\mu) f_-(q^2)}_{\text{Parity odd}} \tag{11.70}$$

Now we need the vector contribution in order to have the overall correction parity. In other words that

$$\langle K^0|A^\mu|D^+\rangle = 0$$

This is called "vector-dominance". You can model this as a vector particle being created as an intermediate final state



We now consider a new reaction given by

$$B^0 \to D^{*-}\ell\nu$$

The hadronic current is given by

$$H^\mu = \langle D^{*-}(k)|V^\mu - A^\mu|B^0(p)\rangle \tag{11.71}$$

There are 4 vectors involved:

B momentum - p
D^* momentum - k
D^* polarization - ϵ_μ

So we can wrtie

$$H_\mu = \epsilon^*_\mu A_1(q^2) + \epsilon^*_\nu p^\nu \left(k_\mu A_2(q^2) + p_\mu A_3(q^2)\right) + \epsilon^\nu p^\alpha k^\beta \epsilon_{\mu\nu\alpha\beta} V(q^2)$$

where we don't have $\epsilon_\mu k^\mu$ term since it's equal to zero (due to the way polarization vectors work). A_1, A_2, A_3 are all axial current form factors and V is a vector current form factor.

Chapter 12

Detection

In the PDG there are many particles that we know their lifetimes, masses, etc. In order to directly detect particles in a collider they need to live long enough so they leave the collision center (radius of ~ 3cm). This requires lifetimes of order

$$\tau \sim \frac{3\text{cm}}{3 \times 10^{10}\text{cm/s}} \sim 100\text{ps}$$

With this in mind that particles we actually can detect are $e^-, \gamma, \mu^-, K^+, \pi^+, p, K_L,$ and n (as well as their antiparticles). All the other particles we detect through their decays to these particles.

For example in a B^0 "detection" we have

$$\begin{aligned} B^0 &\rightarrow D^{*^-} e^+ \nu \\ &\rightarrow D^0 \pi^- e^+ \nu \\ &\rightarrow K^- \pi^+ \pi^- e^+ \nu \end{aligned}$$

which we can finally detect.

Next we discuss the detection of the known particles listed above. For electrons we insert them in large magnetic fields. These electrons produce Breamstralung radiation. The energy of this radiation is proportional to the inverse square of the mass of the particle: $E \propto \frac{1}{m^2}$. That's why this technique only works well for electrons. The "radiation length" (x_0) is the length it takes the electron to emit $\frac{1}{e} \approx \frac{1}{3}$ of it's energy. By measuring this radiation we detect electrons.

Photons are detected through pair conversion. After the electrons have gone through x_0 they will emit a photon which will again pair produce. This product continues and is known as EM showers.

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