Exotic fermions in a three Higgs extension of the Standard Model – Exercises

December 9, 2017

1 Symmetry breaking and the Higgs mechanism (until February)

1.1 An abelian model

1.1.1 Pure Scalar theory

Consider the following model Lagrangian, where Φ is a complex scalar field

$$\mathcal{L} = \left[\partial^\mu \Phi\right]^* \partial_\mu \Phi - V(\Phi)$$

with potential

$$V(\Phi) = \frac{\lambda}{4} \left(\Phi^\star \Phi\right)^2 + \mu^2 \Phi^\star \Phi$$

where λ, μ^2 are both real parameters (i.e. μ^2 can also be negative... the square is just a conventional notation to indicate that the parameter has dimensions of squared mass)

1. Analyse the potential

   (a) Write a Mathematica notebook where you can provide as input fixed values $\left\{\lambda, \mu^2\right\}$ and make a 3D plot of $V(\Phi)$ as a function of the real and the imaginary part of Φ. For which ranges of values of the input parameters can you have more that one stationary point?

   (b) Verify that the Lagrangina is invariant under the following global $U(1)$ transformation:

   $$\Phi \to e^{i\alpha}\Phi$$

   where α is a real constant. To show invarince you need to prove that $\mathcal{L}(\Phi) = \mathcal{L}\left(e^{i\alpha}\Phi\right)$. How is this symmetry reflected in the shape of the 3D plots you've made?

   (c) Classify analytically all the stationary points using the real decomposition (ϕ_1, ϕ_2) where

   $$\Phi = \frac{1}{\sqrt{2}} \left(\phi_1 + i\phi_2\right)$$

   according to the ranges of $\left\{\lambda, \mu^2\right\}$. Verify your classification graphically using you Mathematica notebook.

   (d) At each stationary point, compute the Hessian matrix (or mass matrix)

   $$\partial^2_{ij} V \to \begin{pmatrix} \frac{\partial^2 V}{\partial^2 \phi_1} & \frac{\partial^2 V}{\partial \phi_1 \partial \phi_2} \\ \frac{\partial^2 V}{\partial \phi_2 \partial \phi_1} & \frac{\partial^2 V}{\partial^2 \phi_2} \end{pmatrix} \equiv \left[\mathbf{M^2}\right]_{ij}$$

Compute the eigenvalues and eigenvectors of this matrix. Knowing that the eigenvalues of the mass matrix correspond to the physical particle masses associated to the field fluctuations (along the eigen-vectors) around the vacuum state (the minimum), what can you say about the two scalar particles associated with this complex scalar Field?

(e) Consider a choice of vacuum (minimum) of the following form, with v a real constant

$$\begin{aligned}\phi_1(x) &= v+\varphi_1(x)\\ \phi_2(x) &= 0+\varphi_2(x)\end{aligned}$$

i. Insert this in the potential and organise the various terms in increasing powers of the physical fields φ_i.

ii. Assuming that at the point $\varphi_1=\varphi_2=0$ we have a minimum, obtain a relation between v and the parameters $\{\lambda,\mu^2\}$. Use this relation to simplify the various coefficients in the expansion of the potential in powers of φ_i.

iii. Can you identify the masses of the two physical fields φ_i just by looking at your simplified potential? What is the connection with the Hessian matrix you found above?

(f) By looking at each monomial in the potential, identify the possible scalar interaction vertices and the associated tree-level processes. Verify that the normalisation of the kinetic terms for each φ_i is canonical.

(g) Verify that the $U(1)$ symmetry after choosing the vacuum is now broken, i.e. if we define a new complex field fluctuating around the minimum

$$\varphi=\frac{1}{\sqrt{2}}\left(\varphi_1+i\varphi_2\right)$$

the potential does NOT obey the symmetry

$$\varphi\to e^{i\alpha}\varphi\,.$$

If you look back at the potential in terms of φ_i, can you see any residual symmetry? What are the consequence of such a symmetry for the allowed physical processes? (look at all allowed vertices).

(h) How many massless particles (or states) are present at the minimum (or vacuum)?
NOTE: This is a particular case of Goldstone's theorem that says: "Whenever a continuous symmetry group of the scalar potential is broken to a smaller group (less symmetric), then the final scalar potential will contain one massless particle (a Goldstone) for each symmetry generator that is broken".
In this case there is only one continuous parameter (the α angle) to characterise the symmetry (one symmetry generator only), so there is only one symmetry to be broken. Later we will see in the Standard Model (SM) an example with more broken symmetry generators.

2. Conside a new modified model where the $U(1)$ symmetry is explicitly broken in the potential by new terms if we include, for example,

$$V(\Phi)\to V(\Phi)+\mu_1^2\left(\Phi^2+(\Phi^*)^2\right)$$

(a) Verify that the potential is invariant under two independent $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetries

$$\begin{aligned}\phi_1 &\to -\phi_1 \qquad (\Leftrightarrow \Phi\to-\Phi^*)\\ \phi_2 &\to -\phi_2 \qquad (\Leftrightarrow \Phi\to\Phi^*)\end{aligned}$$

but that the $U(1)$ symmetry is no longer obeyed.

(b) Classify the possible types of minimae for the potential of this scalar theory and calculate the physical scalar masses at each choice of minimum. Compare the result with the $U(1)$ symmetric theory ($\mu_1^2=0$). What's different? Are there any Goldstones? Can you explain why?

1.1.2 Higgs mechanism with an abelian gauge field

Consider the following model Lagrangian

$$\mathcal{L} = [\mathcal{D}^\mu \Phi]^* \mathcal{D}_\mu \Phi - V(\Phi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

with

$$\mathcal{D}_\mu \equiv \partial_\mu + iqA_\mu \quad , \quad V(\Phi) = \frac{\lambda}{4} \left(\Phi^\star \Phi\right)^2 + \mu^2 \Phi^\star \Phi$$

1. Analyse the Lagrangian_
   (a) Verify that the electromagnetic tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is invariant under a local transformation (where α now depends on spacetime position)

   $$A_\mu \to A_\mu - \partial_\mu \alpha(x)$$

   (b) Show that the Lagrangian is still invariant under a global $U(1)$ transformation

   $$\Phi \to e^{i\beta} \Phi$$

   with β constant.

   (c) Replace $\beta \to \beta(x)$ (i.e. make the $U(1)$ local) and determine the shift (or amount by which the $U(1)$ is violated) in the Lagrangian .

   (d) Using these results:
      i. Find the transformation that must be simultaneously applied to A_μ so that the total shift in the Lagranginan is zero, so that invariance is restored.
      ii. In other words... how must we choose $\beta(x)$ so that $U(1)$ transformation of Φ, can compensate for a gauge transformation of the electromagnetic field as defined with $\alpha(x)$.
      iii. Physically what is the role of q. Geometrically (i.e. from the point of view of the symmetry transformations) what is the role of q.

   (e) Compute the Euler-Lagrange equations to obtain the classical equations of motion for A_μ and for Φ:
      i. Which equations do you get for Φ and for A_μ when you turn off the electromagnetic interaction and the scalar field self interactions $\{q \to 0, \lambda \to 0\}$?
      ii. Write both equations (for Φ and for A_μ) in a form that identifies: a) on the left hand side the free theory part, b) on the right side the source terms that depend on the interaction couplings.

2. Note that the scalar potential is precisely the same as in Sect. 1.1.1, thus the stationary points classification is exactly the same. However, in that section, we did not analyse another important theoretical requirement so that the theory is stable. Thus:
   (a) Determine what are the ranges for $\left\{\lambda, \mu^2\right\}$ such that the potential $V(\Phi)$ is bounded from below (i.e. there is a minimum of energy for the scalar field).

3. Consider the choice of vacuum through the shift:

   $$\Phi = \frac{1}{N} \left(v + h(x) + iG(x)\right)$$

   with v, h, G real and N a normalisation constant.

(a) Analyse the kinetic term $\partial_\mu \Phi \partial^\mu \Phi^*$. If you insert the expansion above what must be the choice for N such that the kinetic terms of the new field fluctuations h, G are canonically normalised (i.e. normalised as a Klein-Gordon free field)?

(b) With the N you found adapt the results of previous exercices and for the purely scalar terms of the Lagrangian:

i. Write the minimum conditions.

ii. Write the masses of the eigenstates h, G

iii. Write the scalar interaction vertices.

(c) Looking at the remaining terms coming from shifting Φ and from the term in iqA_μ:

i. Determine the new term that is generated for A_μ. How does it compare with a mass term for a Klein-Gordon field?

ii. Write the gauge transformations for the fields A_μ, h, G. Can you get rid of G by appropriately choosing $\alpha(x)$? Assuming that gauge (i.e. just setting $G = 0$ from start) obtain the Lagrangian for the physical fields h, A_μ(see also Griffiths).

iii. Separate the various parts of the Lagrangian, in particular the "free parts" (i.e. without interactions) from the interaction terms (see also Griffiths).

(d) Identify the physical masses and simplify them by using the minimum conditions. What happened to the Goldstone? (hint: what is the number of polarisations of a massless vector field (like the photon)? What about a massive vector field?).

1.1.3 Mass generation for fermions

In the Standard Model, the masses of fermions are also generated dynamically with the Higgs field. The reason why we cannot write directly fermion mass terms in the Lagrangian for the SM is due to very tight constraints on the possible type of field contractions that are allowed. These constraints, in turn, result from the various gauge symmetries that we are forced to impose to describe the various fundamental forces (electromagnetic, strong nuclear force and weak nuclear force). The fermion fields, being "charged" under such symmetries, can only appear in the Lagrangian in neutral combinations (called singlet operators). The Dirac mass terms will turn out not to be allowed and a different mechanism is called to the rescue using the Higgs boson in Yukawa terms involving one Higgs field and two fermion fields. we start by illustrating the principle with a simplified model:

1. **Weyl fermions:** Consider Dirac's theory for a fermion Ψ. The gamma matrices can be represented in the weyl representation as:

$$\gamma^\mu \to (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$$

with

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

wher σ^i are the 3 pauli matrices ($i = 1, 2, 3$), and

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

From this we define

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

with $\sigma^\mu \to (\mathbf{1}, \sigma^i)$ and $\bar{\sigma}^\mu \to (\mathbf{1}, -\sigma^i)$.

(a) Starting from the Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi$$

define

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

where ψ_L and ψ_R are 2-dimensional spinors. Show that we can write the Lagrangian as

$$\mathcal{L} = \psi_L^\dagger i\bar{\sigma}^\mu\partial_\mu\psi_L + \psi_R^\dagger i\sigma^\mu\partial_\mu\psi_R - m\left(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L\right)$$

Note that the mass term mixes the two L and R spinors.

(b) Re-write the Dirac equation using these 2-spinors and show that it is equivalent to the system

$$\begin{cases} i\sigma^\mu\partial_\mu\psi_R = m\psi_L \\ i\bar{\sigma}^\mu\partial_\mu\psi_L = m\psi_R \end{cases}$$

What happens when $m = 0$?

(c) **Expansion as Left (L) Weyl spinors:** Let us define

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \equiv \begin{pmatrix} \psi_L \\ (\xi^\star)_R \end{pmatrix} \equiv \begin{pmatrix} \psi \\ \bar{\xi} \end{pmatrix} = \begin{pmatrix} \psi_\alpha \\ \bar{\xi}^{\dot{\beta}} \end{pmatrix}$$

where we have defined a new notation with a bar to be equivalent to the complex conjugate. In this notation, the labels L and R refer, respectively, to a left handed or right handed 2-spinor and the greek indices run over $\alpha = 1, 2$ and $\dot{\beta} = 1, 2$. Here we note that if ψ is left handed (L) and $\bar{\xi}$ is right handed (R), their complex conjugates will have opposite chirality, i.e. $\bar{\psi}$ will be R and ξ will be L. It is in this sense that we are expanding Ψ in terms of the left handed spinors ψ and χ.

i. Complex conjugate the first equation above and use the identity $\left(\sigma^i\right)^\star = -\sigma^2\sigma^i\sigma^2$ to show that the system is equivalent to

$$\begin{cases} i\bar{\sigma}^\mu\partial_\mu\chi = -mi\sigma^2\bar{\psi} \\ i\bar{\sigma}^\mu\partial_\mu\psi = -mi\sigma^2\bar{\chi} \end{cases}$$

if we define the charge conjugate of $\bar{\xi}$ as

$$\chi = i\sigma^2\xi$$

(Note: Here we could also define the charge conjugate of the field ψ but we would end up with a right handed field, i.e. $\bar{\eta} = -i\sigma^2\bar{\psi}$. Because we want to represent everything with left handed fields we make the choice to use χ. The less conventional option to use Right fields is explored in an exercise below so that you can contrast and be absolutely clear.)

ii. Show that, up to a surface term, the Dirac Lagrangian is then written with L 2-spinors only: as.

$$\mathcal{L} = \psi^\dagger i\bar{\sigma}^\mu\partial_\mu\psi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi - m\left(\chi\psi + c.c.\right)$$

where we define the product of two left handed Weyl spinors as (Note also that: as numbers, the spinor components anti-commute since they describe fermions, so this product is actually symmetric)

$$\chi\psi \equiv \chi^T\left(i\sigma^2\right)\psi \equiv \chi_\alpha\epsilon^{\alpha\beta}\psi_\beta = \chi_2\psi_1 - \chi_1\psi_2 \quad .$$

By looking at the kinetic terms, we see that for L fermions we have$\bar{\sigma}^\mu$ in the kinetic part of the Lagrangian.

(d) **Expansion as R (right) Weyl spinors:** We could have decide instead to work with Righ handed Weyl spinors. To that purpose we would use the complex conjugate fields and we would repeat the first step of the previous exercise but now keeping $\bar{\xi}$ and using $\bar{\eta}$ (the charge conjugate field of ψ defined above):

i. Go back and apply the complex conjugate of the second equation of the Dirac system and use the identity $\left(\sigma^i\right)^\star = -\sigma^2\sigma^i\sigma^2$ to show that the system is also equivalent to

$$\begin{cases} i\sigma^\mu\partial_\mu\bar{\xi} = mi\sigma^2\eta \\ i\sigma^\mu\partial_\mu\bar{\eta} = mi\sigma^2\xi \end{cases}$$

ii. Show that, up to a surface term, the Dirac Lagrangian is now

$$\mathcal{L} = \bar{\eta}^\dagger i\sigma^\mu\partial_\mu\bar{\eta} + \bar{\xi}^\dagger i\sigma^\mu\partial_\mu\bar{\xi} - m\left(\bar{\eta}\bar{\xi} + c.c.\right)$$

where, again, we have

$$\bar{\eta}\bar{\xi} \equiv \bar{\eta}^T\left(-i\sigma^2\right)\bar{\xi} \equiv \bar{\eta}^{\dot{\alpha}}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}^{\dot{\beta}} = \bar{\eta}^{\dot{2}}\bar{\xi}^{\dot{1}} - \bar{\eta}^{\dot{1}}\bar{\xi}^{\dot{2}}$$

and the dotted indices convention to differentiate the Right tensors from the Left tensors (Note: The actual reason for this convention is that the Right spinors transform in a complex conjugate spinor representation of the Lorentz group.).

(e) NOTE: In all that follows we will be using Left spinors to represent all fermion fields because it is a common convention (especially in theories with supersymmetry). Nevertheless... if you compare the two formulations, you will note that they are very similar up to placing bars and dots in the appropriate places and a minus sign in the definition of charge conjugation.

Now consider the following Lagrangian with two Weyl fermions, an abelian gauge field B_μ, i.e. a $U(1)$, with an associated strength tensor $B_{\mu\nu}$ and a second abelian field C_μ with associated strength tensor $C_{\mu\nu}$ (both similar to $F_{\mu\nu}$). One can think of this model as a simplified example of the Higgs mechanism in the SM where, besides generating a mass for one gauge boson, we will also obtain a massless gauge boson (analogous to the photon). Simultaneously we will introduce the mass generation mechanism for fermions.

The Lagrangian for the system is

$$\begin{aligned} \mathcal{L} \;=\; & \left[\mathfrak{D}^\mu\Phi\right]^*\mathfrak{D}_\mu\Phi - V(\Phi) \\ & -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}C^{\mu\nu}C_{\mu\nu} \\ & +i\psi^\dagger\bar{\sigma}^\mu d_\mu\psi + i\chi^\dagger\bar{\sigma}^\mu\mathcal{D}_\mu\chi \\ & +y\Phi^\star\psi\chi + c.c. \end{aligned}$$

where now the gauge covariant derivatives are defined

$$\begin{aligned} \mathfrak{D}_\mu &\equiv \partial_\mu + igB_\mu + ig'C_\mu \\ d_\mu &\equiv \partial_\mu + igB_\mu \\ \mathcal{D}_\mu &\equiv \partial_\mu + ig'C_\mu \end{aligned}$$

On the first line of the Lagrangian we have the scalar sector, on the second line the gauge sector, on the third line two Weyl fermions and on the fourth line the Yukawa interactions that are allowed by the gauge symmetries. Note that when building a theory, we first start by defining the transformation properties (which is the same as saying that we define the symmetries) and then write ALL allowed interactions with dimension up to four. (Note that scalar fields and gauge boson fields have both dimension 1 and fermions have dimension 2/3).

1. Identify the gauge charges of each field by looking at their covariant derivatives (in particular note which fields are $B_\mu$neutral and $C_\mu$neutral). Can you say what are the charges of the complex conjugated fields?

2. Repeat the calculations you did in the previous exercise for the Higgs mechanism in the bosonic sector using the first two lines of the Lagrangian. In particular:

   (a) Focus on the simplest vacuum choice as we did before.

   (b) Determine the gauge choice that eliminates the Goldstone and adopt that gauge.

   (c) If one defines

   $$\begin{aligned} g &= \frac{e}{\sin\theta} \\ g' &= \frac{e}{\cos\theta} \end{aligned}$$

   adopting the gauge mentioned in the previous point, write the bosonic sector Lagrangina in terms of the new fields

   $$\begin{aligned} A_\mu &= \sin\theta\, B_\mu + \cos\theta\, C_\mu \\ Z_\mu &= \cos\theta\, B_\mu - \sin\theta\, C_\mu \end{aligned}$$

   (d) Using the definitions above eliminate g, g' in terms of e, θ

3. Now look at the fermionic sector:

   (a) Why is it that we cannot include in the Lagrangian a mass term of the form $m\,(\psi\chi + c.c.)$ like in the Dirac theory? Which symmetries would this violate? (Hint: Look at the charges of the fields involved and/or how this term transforms).

   (b) How does the term $\Phi^\star\psi\chi$ transform under the two gauge transformations associated with the fields B_μ and C_μ? Is it clear now how the laws of conservation of charge are implemented at the level of the Lagrangian?

   (c) Considering the vacuum choice that you have made in the analysis of the bosonic sector:

      i. Use the field expansions for the new vector fields and for the scalar field and expand the fermionic sector Lagrangian in terms of the fields $h, A_\mu, Z_\mu, \psi, \chi$.

      ii. Can you now identify mass terms for the fermions? Can you convert back the pair of fields ψ, χ in a single Dirac field (*)? What is the electric charge of that field (noting that the electromagnetic field is A_μ because it is the one remaining massless)? What are the interaction terms with the massive Z_μ field? What about with the Higgs field h?

1.2 The Higgs mechanism in the Standard Model

1.2.1 The non-abelian $SU(2)$ group and Yang-Mills theories

One defines a multiplicative group as being a set G such that the binary operation $ab = c\,:\, G\times G \to G$ obeys the following conditions:

- (associativity) for all a, b, $c \in G$

$$(ab)\,c = a\,(bc)$$

- (identity) there is an element $e \in G$ such that

$$ea = a = ae$$

- (inverse element) for each $a \in G$ there is an element $a^{-1} \in G$ such that

$$aa^{-1} = a^{-1}a = e\,.$$

These three conditions define a mutiplicative symmetry group. A special case is an abelian group, i.e. if all elements $a_1, \cdots, a_n$ commute, i.e., $[a_i, a_j] = 0$, otherwise the group is non-abelian. We will be interested in continuous Lie groups (for intuition think for example of the group of rotations in 2D that depends on a continuous parameter the angle α but in general we may have higher dimensional groups/parameters). Another set of quantities that relate to the structure of the Lie group are the structure constants. They are defined from its Lie algebra. In the case of matrix groups the elements of the Lie algebra can be defined as the set of matrices that when exponentiated give an element of the Lie group, i.e. $a = e^{-i\tau}$ (where $-i$ is conventional). Then the structure constants f_{ijk} are defined such that for elements $\tau_1, \tau_2, \ldots$ of the Lie algebra we have

$$[\tau_i, \tau_j] = i \sum_k f_{ijk} \tau_k$$

For an abelian group, the Lie algebra commutator is also zero.
The Lie algebra is basically the tangent space of the Lie group close (i.e. continuously connected) to the identity, so it can be thought of as the matrices used to expand the exponential clode to the identity matrix (i.e. $e^x \simeq 1 + x + \ldots$). We will see this with a specific example below.

1. $SU(N)$ group. This is defined as the set of N by N unitary complex matrices U with unit determinant, i.e.

$$\begin{aligned} U^\dagger U &= \mathbf{1} \\ \det U &= 1 \end{aligned}$$

The first condition defines a matrix group to be **unitary,** $U(N)$, and the added condition on the determinant makes it a **special unitary group**, $SU(N)$.

(a) Considering three elements U_1, U_2, U_3 and the definition of inverse and identity check that this set of matrices forms a group.

(b) How many real parameters are needed to span a general N by N complex matrix?

(c) By counting the number of constraints that define this group, can you find the number of independent parameters that you need to parametrise a general $SU(N)$ matrix? (Hint: you should get $N^2 - 1$. This corresponds to the number of independent generators of the Lie algebra).

2. $SU(2)$ group. Consider now the case $N = 2$.

(a) Let us look at the Lie algebra of $SU(2)$ and its structure constants by working backwards. [1] We start with the Pauli matrices, that you are told to be the Lie algebra generators of $SU(2)$, and we rebuild back the group elements by exponentiation. Given the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

you are told that the Lie algebra generators are $\tau^i = \frac{\sigma^i}{2}$

(b) Show that this Lie algebra is associated with a non-abelian Lie group and compute the structure constants.

[1] NOTE: It is also possible to obtain the structure constants and generators by using a parametrisation of these matrices and expanding near the identity but it is simpler to just check the result.

(c) Define now the exponentiation of the Lie algebra

$$U(\boldsymbol{\omega}) = e^{-i\boldsymbol{\omega}\cdot\frac{\boldsymbol{\sigma}}{2}}$$

where $\omega = (\omega_1, \omega_2, \omega_3)$ is a vector of parameters. Using the series expansion of the exponential of a matrix show that

$$U(\boldsymbol{\omega}) = \cos\left(\tfrac{\omega}{2}\right) - i\left(\hat{\omega}\cdot\boldsymbol{\sigma}\right)\sin\left(\tfrac{\omega}{2}\right)$$

where $\hat{\omega} \equiv \frac{\boldsymbol{\omega}}{\omega}$ and where you can use the identity $(\boldsymbol{\omega}\cdot\boldsymbol{\sigma})^2 = \boldsymbol{\omega}\cdot\boldsymbol{\omega} + i\boldsymbol{\sigma}\cdot(\boldsymbol{\omega}\times\boldsymbol{\sigma}) = \omega^2$.

(d) How many generators did you expect from the counting done in the exercise for $SU(N)$? Does it match?

3. Now let us see the connection with Physics. Consider the Lagrangian of a Yang-Mills theory

$$\begin{aligned}\mathcal{L}_{YM} &= i\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi - \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\,,\\ &= i\overline{\psi}_A\gamma^\mu\left[D_\mu\right]^A{}_B\,\psi^B - m\overline{\psi}_A\psi^A - \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a\,,\end{aligned}$$

here ψ^A is a multiplet of Dirac spinors (in the first line we have suppressed the indices[2]). The gauge covariant derivative is defined:

$$\begin{aligned} D_\mu &= \partial_\mu - ig\tau_a A^a_\mu\,.\\ \Leftrightarrow \left[D_\mu\right]^A{}_B &= \delta^A_B\partial_\mu - ig\left[\tau_a\right]^A{}_B\,A^a_\mu\,.\end{aligned}$$

Here $\tau_a \equiv \sigma_a/2$ with $a = 1, \ldots N^2-1$ are the Lie algebra generators of $SU(N)$ with structure constants define f_{abc}. A^a_μ is the Yang Mills non-abelian field and its field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^a{}_{bc}A^b_\mu A^c_\nu$$

(a) Compute the commutator

$$\left[D_\mu, D_\nu\right]\,.$$

and identify the field strength on the right hand side.

(b) Use the result in the previous exercise and specialise to the case of $SU(2)$ (use the Pauli matrices commutation relations)

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\varepsilon^{bca}A^b_\mu A^c_\nu.$$

(c) Why is it that in the $U(1)$ case the third term is zero?

(d) Identify the interaction vertices for this theory by expanding the Lagrangian and draw the respective Feynman diagrams (Use solid lines for fermion lines and wavy lines for gauge boson lines). What's the main difference compared to the abelian (i.e. $U(1)$) case?

(e) (Optional) Show that that $\mathcal{L}_{YM}$ for the $SU(2)$ theory is invariant under local infinitesimal gauge transformations given by (the transformation shift pareameters $\omega^a(x)$ are supposed to be infinitesimal so truncate at leading order in this parameter):

$$\begin{aligned} \psi &\to \psi + \delta\psi \text{ with } \delta\psi = i\omega^a(x)\,\tau_a\psi\\ (\text{and } \overline{\psi} &\to \overline{\psi} + \delta\overline{\psi} \text{ with } \delta\overline{\psi} = -i\omega^a(x)\,\overline{\psi}\tau_a)\\ A^a_\mu &\to A^a_\mu + \delta A^a_\mu \text{ with } \delta A^a_\mu = -\varepsilon^{bca}\omega^b(x)\,A^c_\mu - \tfrac{1}{g}\partial_\mu\omega^a(x)\\ F^a_{\mu\nu} &\to F^a_{\mu\nu} + \delta F^a_{\mu\nu} \text{ with } \delta F^a_{\mu\nu} = -\varepsilon^{bca}\omega^b(x)\,F^c_{\mu\nu}\end{aligned}$$

(f) Show that the assumption of gauge invariance (given that the field transform as above) forbid a gauge boson mass term $\mathcal{L}_m = m_A^2 A^a_\mu A^\mu_a$.

[2] Also note that we are using the Einstein summation convention for these indices that are assumed to be Euclidean, so up and down is the same, the position of the indices up or down is only used to make it clear when there are dummy indices being summed over.

1.2.2 The electroweak sector in the SM

The construction of the SM Lagrangian follows largely the principles outlined in the previous sections. The main difference compared to the Abelian model of section 1.1 that was used to illustrate the basic principles (gauge symmetries, Higgs mechanism and mass generation for fermions by choosing appropriate Yukawa terms) are the same. The main difference is that the weak forces (and also QCD) have to be described by non-abelian gauge theories. This requires choosing an appropriate set of fields, one for each particle, in appropriate representations of the gauge symmetries. To make the construction clearer it is simpler, again, to discuss first the bosonic sector.

SM Bosonic sector In the following table we have the bosonic fields that are used in the SM to describe the various bosons. We will be ignoring QCD (thus the gluons and the $SU(3)$) in the discussion because it does not interfere with the Higgs mechanism. In this table, in red, the $SU(2)$ triplet representation is denoted **3** and doublets by **2**. The bold face **1** denotes an $SU(2)$ singlet, i.e. a field that does not couple to their $SU(2)$ gauge field A^a_μ. The $U(1)_Y$ charges (this is not electromagnetism at this stage!) are denoted in green.

Gauge Bosons and Scalars in the SM				
Names		Spin 0	Spin 1	$SU(3)_C \times SU(2)_L \times U(1)_Y$
Gluons		$\times$	g	**8**, **1**, 0
A Bosons		$\times$	A^1, A^2, A^3	**1**, **3**, 0
B Boson		$\times$	B	**1**, **1**, 0
Higgs (1 generation)	ϕ	(G^+, ϕ^0)	$\times$	**1**, **2**, 1

1. The bosonic sector Lagrangian of the SM electroweak interactions is

$$\mathcal{L}_{EW} = (D_\mu\Phi)^\dagger (D^\mu\Phi) - V\left(\Phi^\dagger\Phi\right) - \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \tfrac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

where the scalar potential is

$$V\left(\Phi^\dagger\Phi\right) = \lambda\left(\Phi^\dagger\Phi\right)^2 + \mu^2\Phi^\dagger\Phi$$

and the Higgs field, Φ, is an $SU(2)$ complex scalar doublet given by (where we also note its real decomposition)

$$\Phi = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \equiv \tfrac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{pmatrix}.$$

Its gauge covariant derivative for $SU(2)_L \times U(1)_Y$ is (with $Y = 1$)

$$D_\mu = \partial_\mu - ig\tau_a A^a_\mu - ig'\tfrac{Y}{2}B_\mu$$

and the field strength tensors are

$$\begin{aligned} F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\varepsilon^a{}_{bc}A^b_\mu A^c_\nu \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\,. \end{aligned}$$

$\mathcal{L}_{EW}$ is by construction invariant under $SU(2)_L \times U(1)_Y$ transformations.

(a) Study first the scalar potential of Φ:

i. Classify all the stationary points. In which case is the symmetry preserved (symmetric phase)? And broken (broken phase)?

ii. Compute the Hessian for each stationary point and determine its eigenvalues. In which case do you get Goldstones? How many?

iii. Choose a minimum with (v real amd positive)

$$\langle\Phi\rangle = \tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} \ ,$$

identify which of the following $SU(2)$ generator combinations τ_1, τ_2, $Q = \tau_3 + \frac{1}{2}$ and $Q^{\perp} = \tau_3 - \frac{1}{2}$, leave this choice of vacuum doublet value invariant and which one don't. Knowing that in the SM the original symmetry is not broken completely by the Higgs mechanism, but instead it breaks as follows $SU(2)_L \times U(1)_Y \to U(1)_Q$ does the result make sense?
Note: A generator $\hat{\Omega}$ leaves the vacuum invariant whenever $\hat{\Omega}\langle\Phi\rangle = 0$.

2. Note now that we can always perform a gauge transformation through the following replacements

$$\begin{aligned}
\Phi(x) &\rightarrow e^{i\frac{\tau_a \xi^a(x)}{\sqrt{2}v}}\Phi(x) \\
A^a_\mu(x) &\rightarrow A^a_\mu(x) + \frac{1}{g\sqrt{2}v}\partial_\mu \xi^a(x)
\end{aligned}$$

to bring the scalar field into the simpler form

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v + h(x)\end{pmatrix}$$

This is called the *unitary gauge* which we assume in the remainder. Note that this choice corresponds to setting the 3 Goldstone field fluctuations to zero.

(a) Verify that from the two first terms in $\mathcal{L}_{EW}$ you get

$$\begin{aligned}
\mathcal{L}_1 &= \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\left(-2\mu^2\right) h^2 \\
&+ \frac{1}{2}\left(\frac{gv}{2}\right)^2 \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu}\right) \\
&+ \frac{1}{2}\frac{v^2}{2}\left(gA^3_\mu - g' B_\mu\right)\left(gA^{3\mu} - g' B^\mu\right) + \cdots
\end{aligned}$$

(b) Write the terms on the last line of $\mathcal{L}_1$ as a quadratic form in

$$X = \left(A^3_\mu, B_\mu\right) \quad .$$

Diagonalise the quadratic form and show that you get the normalised eigenvectors

$$\begin{aligned}
\mathcal{A}_\mu &= \frac{1}{\sqrt{g^2 + g'^2}}\left(g' A^3_\mu + g B_\mu\right) \\
Z_\mu &= \frac{1}{\sqrt{g^2 + g'^2}}\left(g A^3_\mu - g' B_\mu\right)
\end{aligned}$$

with eigenvalues 0 and $\frac{1}{2}v^2\left(g^2 + g'^2\right)$ respectively. Can you guess which fields/particles are these?

(c) Define also

$$\begin{aligned}
A^1_\mu &= \tfrac{1}{\sqrt{2}}\left(W^+_\mu + W^-_\mu\right) \\
A^2_\mu &= \tfrac{i}{\sqrt{2}}\left(W^+_\mu - W^-_\mu\right)
\end{aligned}$$

Determine the mass term for the complex vector $W^{\pm}_\mu$.

(d) Count the number of massive vector bosons you obtained and of massless vector bosons. Compare with the number of Goldstones that you eliminated by working in Unitary gauge and explain what happened.

(e) Similarly to the abelian model define

$$
\begin{aligned}
g &= \frac{e}{\sin\theta_W} \\
g' &= \frac{e}{\cos\theta_W}
\end{aligned}
$$

and show that

$$
\begin{aligned}
\mathcal{A}_\mu &= \sin\,\theta_W\, A^3_\mu + \cos\,\theta_W\, B_\mu \\
Z_\mu &= \cos\,\theta_W\, A^3_\mu - \sin\,\theta_W\, B_\mu
\end{aligned}
$$

Expand the cubic terms in the interaction Lagrangian containing the fields $W^\pm_\mu, Z_\mu$ and A_μ and use the expressions above to obtain a result in terms of e and θ_W. Knowing now that A_μ is the photon what can you say about the electric charges of $W^\pm_\mu, Z_\mu$.

SM fermionic sector Now we are going to look at the mass generation mechanism for fermions in the SM. In the following table we have the fermionic fields that are used in the SM to describe the various fermions. In this table, the $SU(2)$ doublets are indicated in blue by **2**. The bold face **1** denotes an $SU(2)$ singlet, i.e. a field that does not couple to the $SU(2)$ gauge field A^a_μ. The $U(1)_Y$ charges are again denoted in green. All fermions here are Weyl fermions (see discussion below in the next exercise).

Fermions in the SM		
Names	Spin 1/2	$SU(3)_C \times SU(2)_L \times U(1)_Y$
Quarks (3 generations)	$Q=(u_L, d_L)$	**3**, **2**, $1/3$
	u_R	$\bar{\mathbf{3}}$, **1**, $-4/3$
	d_R	$\bar{\mathbf{3}}$, **1**, $2/3$
Leptons (3 generations)	$L=(\nu_{eL}, e_L)$	**1**, **2**, -1
	e_R	**1**, **1**, 2

We have already seen in the more simple abelian model (Sect.1.1.3) that fermionic mass terms may or may not be allowed depending on the gauge charges of the various Weyl fermions (2-spinors) in the model. In the first exercise we revisit how this constraint arises in an example with SU(2).

1. Let us consider a mass term for a Dirac fermion where we have separated the left handed and right handed parts. Consider the Dirac fermion decomposed as Weyl 2-spinors

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

where $\psi_L$and ψ_R are two independent 2-spinors. From here on, in the SM, we will be using the standard notation where the **Left handed** Weyl 2-spinor that gives the **Left handed** part of the full Dirac spinor has a subscript **L** i.e. in this example ψ_L; and the **also Left handed** Weyl 2-spinor that gives the **RIGHT handed** part of the full Dirac spinor has a subscript **R** but with a bar i.e. in this example $\overline{\psi}_R$. This means that ALL Weyl spinors that we will use are LEFT handed even if they have an R subscript (with a bar)! Only when they are barred and have an L subscript or when they have an R subscript without a bar they become right handed (so for example $\overline{\psi}_L$ and ψ_R are right handed Weyl 2-spinors). The rule is simple:

Spinors without bars and an L subscript or with bars and R subscript are all left handed and spinors with bars and L subscript or without bars and R subscript are all right handed.

A Dirac mass term in Weyl notaton is then, as we have seen before (recall the contraction is done with $\epsilon^{\alpha\beta}$, i.e. $\psi_L\overline{\psi}_R \equiv (\psi_L)_\alpha\, \epsilon^{\alpha\beta}\left(\overline{\psi}_R\right)_\beta$ and $\overline{\psi}_L\psi_R \equiv \left(\overline{\psi}_L\right)_{\dot\alpha} \epsilon^{\dot\alpha\dot\beta}\left(\overline{\psi}_R\right)_{\dot\beta}$)

$$\mathcal{L}_m = -m\left(\psi_L\overline{\psi}_R + \overline{\psi}_L\psi_R\right)$$

In the SM the fermion fields will also be "charged" under the gauge forces determined by the gauge group $SU(2)_L \times U(1)_Y$. This means that they may be doublets of $SU(2)_L$ or be charged under the abelian $U(1)_Y$. Now, the subscript L in $SU(2)_L$ in the SM refers to the general fact that the Weyl fermions that give the left handed parts of each fermion in the SM are doublets of $SU(2)_L$, i.e. $\left(\psi_L^A\right)_\alpha$ with $A = 1, 2$, or just ψ_L^A if we suppress the spinor index. However the Weyl fermions that give the right handed part of each fermion in the SM and that are labeled with R are singlets of $SU(2)_L$. So $(\psi_R)_\alpha$(or just ψ_R) does not have any extra gauge index!

(a) Given the electroweak gauge symmetry group $SU(2)_L \times U(1)_Y$, where Weyl fermions with an L label are $SU(2)_L$ doublets and Weyl fermions with an R label are singlets of $SU(2)_L$ show that $\mathcal{L}_m$ is not invariant under infinitesimal $SU(2)_L$transformations of the two weyl fermions ψ_L, ψ_R that form a Dirac spinor which are given by

$$\begin{aligned} \psi_L &\to \psi_L + \delta\psi_L \,\text{with}\, \delta\psi_\mathrm{L} = i\omega^a\tau_a\psi_L \\ \psi_R &\to \psi_R\,. \end{aligned}$$

NOTE: these transformations mean that ψ_L is in a weak isospin $\frac{1}{2}$ representation and ψ_R has isospin 0. This is called isospin in analogy with spin angular momentum whose associated symmetry group is also $SU(2)$.

(b) Regarding the $U(1)_Y$ group, here Y denotes the weak hypercharge (this is formaly similar to the electric charge). If $\psi_L$has $Y = Y_L$ and ψ_R has $Y = Y_R$ their $U(1)_Y$ infinitesimal transformations are

$$\psi_{L,R} \to \psi_{L,R} + \delta\psi_{L,R} \,\text{where}\, \delta\psi_\mathrm{L,R} = i\tfrac{\omega}{2}Y_{L,R}\psi_{L,R}$$

Show that in general $\mathcal{L}_m$ is also not invariant under these transformations. As a curiosity find the special case where it is invariant.

(c) Given the results in this exercise, what can you say about fermion mass terms in the SM electroweak theory with $SU(2)_L \times U(1)_Y$?

2. Let us consider now a simplified version of the fermionic sector of the SM. Here we have to include the electron and its neutrino. In the SM they are organised in an $SU(2)_L$doublet of Weyl fermions and an $SU(2)_L$ singlet Weyl fermion

$$E = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \quad \text{with } Y = -1 \qquad ; \qquad e_R \text{ with } Y = 2$$

The Lagrangian is

$$\mathcal{L}_e \;=\; \overline{E}_L\left(i\bar\sigma^\mu D_\mu\right)E_L + \bar e_R\left(i\bar\sigma^\mu D_\mu\right)e_R - \left[y_e\,\Phi\cdot\overline{E}_L e_R + h.c.\right] ,$$

where the last Yukawa term and the first terms can be explicitly expanded as

$$\begin{aligned} \Phi\cdot\overline{E}_L e_R &= (\Phi)_A\left(\overline{E}_L\right)^A_{\dot\alpha}\epsilon^{\dot\alpha\dot\beta}\left(e_R\right)_{\dot\beta} \\ \overline{E}_L\left(i\bar\sigma^\mu D_\mu\right)E_L &= \left(\overline{E}_L\right)_{\dot\alpha} i\left(\bar\sigma^\mu\right)^{\dot\alpha\alpha} D_\mu\left(E_L\right)_\alpha \end{aligned}$$

where all implicit contraction of spinor indices are clear. Note that we use doted and undoted indices in right-handed and left-handed spinor respectivelly which transform differently under Lorentz transformations. Note that we cannot contract dotted and undotted inices at the cost of baddly violating Lorentz symmetry. Note also that the $\overline{\sigma}^{\mu}$matrices transform Right-handed 2-spinors into its Left-handed counterparts and vice-versa.

Expand again the Higgs field around the vacuum in the unitary gauge and:

(a) Find the mass terms generated. What is the mass of the electron and of the neutrino.

(b) Find the electric charges of the electron and of the neutrino by expanding the gauge covariant derivative and identifying the terms in $\mathcal{A}_{\mu}$.

(c) Considering the intraction terms involving these fermions and also the Z_{μ} and $W_{\mu}^{\pm}$ fields, which forces interact with the electron? And with the neutrino?

(d) Besides the fact that we are using a simplified version with only one family of leptons do you find another obvious shortcoming of the SM in terms of the properties it preedicts for the electron and the neutrino? (Hint: Inspect the conclusions you obtained in the previous lines).

(e) If you introduce two more copies of $\mathcal{L}_e$, say $\mathcal{L}_{\mu}$ and $\mathcal{L}_{\tau}$ for the muon flavour and for the tau flavour, what can you conclude about the Yukawa couplings y_{μ} and y_{τ} ?

3. Let us consider now a simplified version of the quark sector of the SM. Here we include one generation of quarks up and one generation of quarks down. In the SM they are also organised in an $SU(2)_L$doublet of Weyl fermions and $SU(2)_L$ singlets of Weyl fermions

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad \text{with } Y = \tfrac{1}{3} \qquad ; \qquad u_R \text{ with } Y = -\tfrac{4}{3}; \qquad d_R \text{ with } Y = \tfrac{2}{3}\,.$$

The Yukawa Lagrangian for the down-type quarks of the first generation is now given by

$$\mathcal{L}_{Y_{d_1}} \quad = \quad -\left[y_d\,\Phi\cdot\overline{Q}_L d_R + h.c.\right]\ .$$

Expand again the Higgs field around the vacuum in the unitary gauge and:

(a) Find the mass terms generated for the down-type quarks.

(b) Show that following Yukawa term for the up quark is not allowed by electroweak symmetry

$$-\left[y_u\,\Phi\cdot\overline{Q}_L u_R + h.c.\right]\ .$$

(c) Redefine the Higgs field showing that

$$\widetilde{\Phi} = i\sigma^2\Phi^{\dagger} = \begin{pmatrix} \phi^0 \\ -\phi^- \end{pmatrix}$$

where $\phi^- \equiv (\phi^+)^*$. What is the hypercherge of $\widetilde{\Phi}$?

(d) Show that the Yukawa term for the up quark

$$-\left[y_u\,\widetilde{\Phi}\cdot\overline{Q}_L u_R + h.c.\right]$$

is invariant under the electroweak symmetry and determine the mass of the up quark.

4. Consider now the case of two generations of quarks and the Yukawa interactions written in the charge eigenbasis

$$\mathcal{L}_{q_2} = -\sum_{i,j}^{2} \widetilde{\Phi} \cdot \overline{Q}_L^i \left(Y_u\right)_{ij} u_R^j - \sum_{i,j}^{2} \Phi \cdot \overline{Q}_L^i \left(Y_d\right)_{ij} d_R^j$$

where we have for the charge eigenstates

$$Q_L^1 \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \; Q_L^2 \equiv \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \; u_R^1 \equiv u_R\,, \; u_R^2 \equiv c_R\,, \; d_R^1 \equiv d_R\,, \; d_R^2 \equiv s_R\,.$$

(a) Physical states, which we typically denote as particle, has a well defined mass. Therefore, to obtain the physical mass eigenstates we need to re-write the Higgs interactions (or Yukawa interactions) in a diagonla form. To do so we diagonalize both Yukawa matrices using bi-unitaty transformations of the form

$$\begin{aligned} Y_u^D &= \mathcal{U}_L Y_u \mathcal{U}_R^\dagger \Leftrightarrow \left(Y_u^D\right)_{aa} = \left(\mathcal{U}_L\right)_{ai} \left(Y_u\right)^{ij} \left(\mathcal{U}_R^*\right)_{ja} \\ Y_d^D &= \mathcal{D}_L Y_d \mathcal{D}_R^\dagger \Leftrightarrow \left(Y_d^D\right)_{aa} = \left(\mathcal{D}_L\right)_{ai} \left(Y_d\right)^{ij} \left(\mathcal{D}_R^*\right)_{ja} \end{aligned}$$

with $\mathcal{U}_{L,R}$ and $\mathcal{D}_{L,R}$ unitary matrices. Show that we can write

$$\begin{aligned} \left(Y_u\right)^{kl} &= \left(\mathcal{U}_L^*\right)^{ka} \left(Y_u^D\right)_{aa} \left(\mathcal{U}_R\right)^{al} \\ \left(Y_d\right)^{kl} &= \left(\mathcal{D}_L^*\right)^{ka} \left(Y_d^D\right)_{aa} \left(\mathcal{D}_R\right)^{al} \end{aligned}$$

(b) Use the last result to diagonalize $\mathcal{L}_{q_2}$ and, after expanding the Higgs field around the vacuum, show that the physical quark mass eigenstates (written with primes) are given by

$$\begin{aligned} u_L'^a &= \sum_{i=1}^{2} u_L^i \left(\mathcal{U}_L^*\right)^{ia} \\ u_R'^a &= \sum_{j=1}^{2} \left(\mathcal{U}_R\right)^{aj} u_R^j \\ d_L'^a &= \sum_{i=1}^{2} d_L^i \left(\mathcal{D}_L^*\right)^{ia} \\ u_R'^a &= \sum_{j=1}^{2} \left(\mathcal{D}_R\right)^{aj} d_R^j \end{aligned}$$

What are the masses of the u, d, c, and s quarks? Knowing the tabulated values (see Particle Data Group tables online) what can you conclude in terms of the strength of the interaction with the Higgs field?

(c) Given the kinetic terms for quarks

$$\mathcal{L}_{\text{kin}}^{\text{quarks}} = \overline{Q}_L^i \left(i\bar{\sigma}^\mu D_\mu\right) Q_L^i + \overline{d}_R^i \left(i\bar{\sigma}^\mu D_\mu\right) d_R^i + \overline{u}_R^i \left(i\bar{\sigma}^\mu D_\mu\right) u_R^i$$

expand the first term to show that

$$\mathcal{L}_{\text{kin}}^{\text{Q}_\text{L}^\text{i}} = \overline{Q}_L^i \left(i\bar{\sigma}^\mu \partial_\mu\right) Q_L^i + e\mathcal{A}_\mu J_{\text{EM}}^\mu + g\left(W_\mu^+ J_W^{\mu +} + W_\mu^- J_W^{\mu -} + Z_\mu J_Z^{\mu 0}\right)$$

where:

i. the electromagnetic currents are given by

$$J^{\mu}_{\text{EM}} = \overline{u}^{i}_{L} q_u e \bar{\sigma}^{\mu} u^{i}_{L} + \overline{d}^{i}_{L} q_d e \bar{\sigma}^{\mu} d^{i}_{L}$$

whith $q_u = \frac{2}{3}$ and $q_d = -\frac{1}{3}$. What are those values?

ii. the neutral currents are given by

$$J^{\mu 0}_{\text{Z}} = \frac{g}{\cos\theta_W} \left[\overline{u}^{i}_{L} \bar{\sigma}^{\mu} \left(\tfrac{1}{2} - \tfrac{2}{3}\sin^2\theta_W \right) u^{i}_{L} + \overline{d}^{i}_{L} \bar{\sigma}^{\mu} \left(-\tfrac{1}{2} + \tfrac{1}{3}\sin^2\theta_W \right) d^{i}_{L} \right]$$

iii. the charged currents are given by

$$J^{\mu +}_{W} = \frac{1}{\sqrt{2}} \left(\overline{u}^{i}_{L} \bar{\sigma}^{\mu} d^{i}_{L} \right) \quad \text{and} \quad J^{\mu -}_{W} = \frac{1}{\sqrt{2}} \left(\overline{d}^{i}_{L} \bar{\sigma}^{\mu} u^{i}_{L} \right)$$

- Hint: Use the the expansion of the covariant derivative writing explicitly the Pauli matrices. After expanding in terms of the $\left(A^1_\mu,\, A^2_\mu,\, A^3_\mu,\, B_\mu\right)$ gauge fields rotate it to the physical gauge boson basis $\left(W^+_\mu,\, W^-_\mu,\, Z_\mu,\, \mathcal{A}_\mu\right)$.

(d) Defining the quark mixing matrix $V^{ab} = \left(\mathcal{U}^*_L\right)^{ai} \left(\mathcal{D}_L\right)^{ib}$, rewrite the electromagnetic, neutral and charged currents in terms of the physical quarks u'^{a}_{L} and d'^{a}_{L}. Which currents allow for flavour changing interactions? Which currents can be simultaneously diagonalized with the Higgs interactions?

(e) With the above results, draw the Feynman diagrams for each of the interactions in $\mathcal{L}^{\text{Q}^{\text{i}}_{\text{L}}}_{\text{kin}}$ writing the respective Feynman rules for all physical quarks. To do this, note first that since V^{ab} is a 2×2 mixing matrix it can be parametrized by what is denoted as Cabibbo mixing

$$V = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}$$

whith θ_c the Cabibbo angle. For example, in a vertex proportional to $\overline{u}'^{a}_{L} \bar{\sigma}^{\mu} V^{ab} d'^{b}_{L}$ expand it as $\overline{u}'^{a}_{L} \bar{\sigma}^{\mu} \left(V^{a1} d'_L + V^{a2} s'_L \right)$.

This result obtained for two generations of quarks can be generalized to the three generations and the new 3×3 mixing matrix is the well known Cabibbo-Kobayashi-Maskawa or V_{CKM} matrix. This matrix is dominated by the Cabibbo block and the $\left(V_{\text{CKM}}\right)^{33}$ component which represents the mixing between the top and the bottom quarks. The remaining entries are subdominant but still observable and phenomanologically relevant.

2 A 3HDM model with exotic fermions inspired by T-GUT scenarios (From February)