Introduction to Parallel and Distributed Computing

Selected Parallel Algorithms
Dynamic Programming

Lecture 12, Spring 2024

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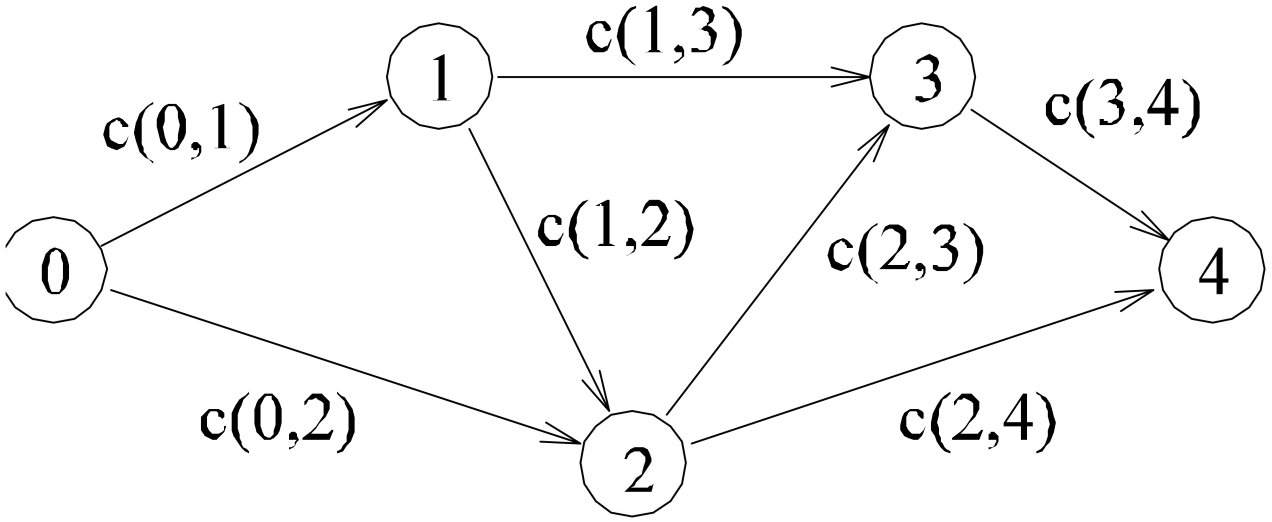
Overview of Serial Dynamic Programming

- Dynamic programming (DP) is used to solve a wide variety of discrete optimization problems such as scheduling, string-editing, packaging, and inventory management.
- Break problems into subproblems and combine their solutions into solutions to larger problems.
- In contrast to divide-and-conquer, there may be relationships across subproblems.

Dynamic Programming: Example

- An acyclic graph for which the shortest path between nodes 0 and 4 is to be computed.



$$f(4) = \min\{f(3) + c(3,4), f(2) + c(2,4)\}.$$

Dynamic Programming: Example

- Consider the problem of finding a shortest path between a pair of vertices in an acyclic graph.
- An edge connecting node i to node j has cost c(i,j).
- The graph contains n nodes numbered 0,1,…, n-1, and has an edge from node i to node j only if $i < j$. Node 0 is source and node n-1 is the destination.
- Let f(x) be the cost of the shortest path from node 0 to node x.

$$f(x) = \begin{cases} 0 & x = 0 \\ \min\limits_{0 \le j < x} \{f(j) + c(j,x)\} & 1 \le x \le n-1 \end{cases}$$

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Dynamic Programming

- The solution to a DP problem is typically expressed as a minimum (or maximum) of possible alternate solutions.
- If r represents the cost of a solution composed of subproblems x_1, x_2,…, x_l, then r can be written as

$$r = g(f(x_1), f(x_2), \ldots, f(x_l)).$$

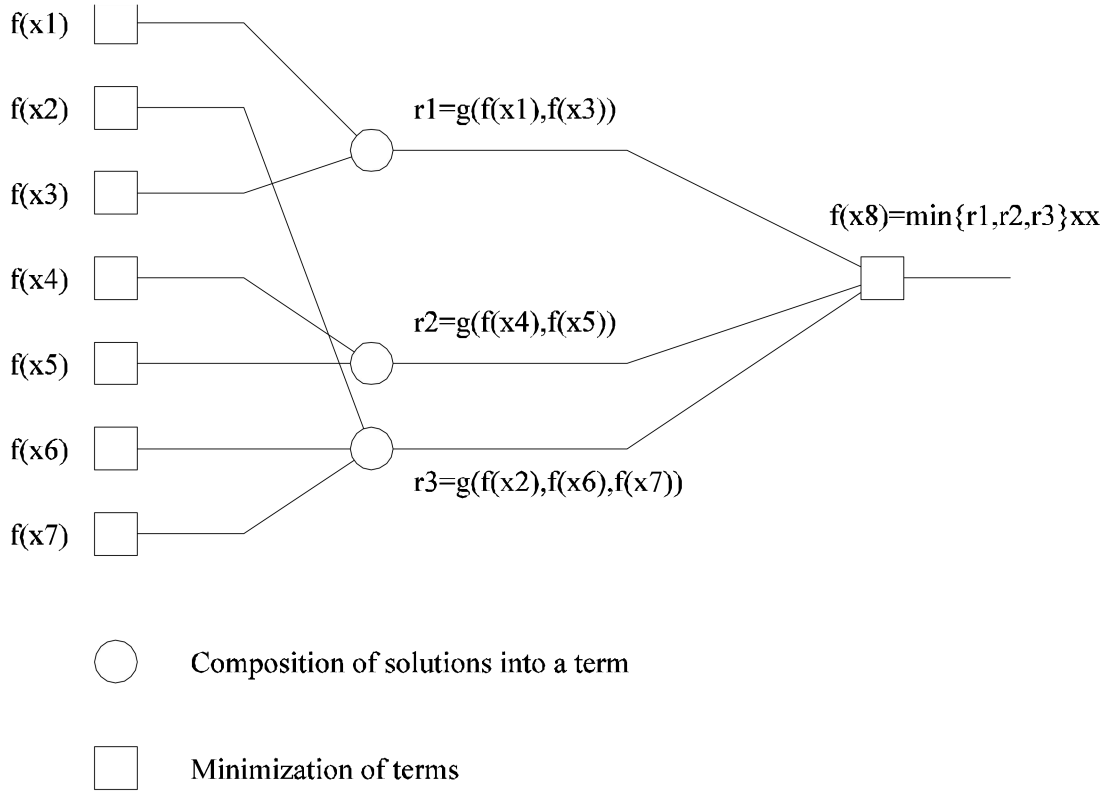
  Here, g is the composition function.

- If the optimal solution to each problem is determined by composing optimal solutions to the subproblems and selecting the minimum (or maximum), the formulation is said to be a DP formulation.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Dynamic Programming: Example

- The computation and composition of subproblem solutions to solve problem f(x8).



Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

DP: Monadic vs. Polyadic

- The recursive DP equation is also called the functional equation or optimization equation.
- In the equation for the shortest path problem the composition function is f(j) + c(j,x). This contains a single recursive term (f(j)). Such a formulation is called monadic.
- If the RHS has multiple recursive terms, the DP formulation is called polyadic.

DP: Serial vs. Non-serial

- The dependencies between subproblems can be expressed as a graph.
- If the graph can be levelized (i.e., solutions to problems at a level depend only on solutions to problems at the previous level), the formulation is called serial, else it is called non-serial.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

DP Categories: {monadic, polyadic} × {serial, non-serial}

- Based on these two criteria, we can classify DP formulations into four categories
  - serial-monadic (e.g., 0/1 knapsack)
  - non-serial-monadic (e.g., longest common subsequence)
  - serial-polyadic (e.g., Floyd's all-pair shortest paths)
  - non-serial-polyadic (e.g., optimal matrix parenthesization)
- This classification is useful since it identifies concurrency and dependencies that guide parallel formulations.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

a Concrete Example: Smith-Waterman

- **Linear gap penalty**

$$H_{i,j} = \max \begin{Bmatrix} 0 \\ H_{i-1,j} + g \\ H_{i,j-1} + g \\ H_{i-1,j-1} + W(a_i, b_j) \end{Bmatrix}$$

- **Affine gap penalty**

$$E_{i,j} = \max \begin{Bmatrix} E_{i,j-1} - G_{\text{ext}} \\ H_{i,j-1} - G_{\text{init}} \end{Bmatrix}$$

$$F_{i,j} = \max \begin{Bmatrix} F_{i-1,j} - G_{\text{ext}} \\ H_{i-1,j} - G_{\text{init}} \end{Bmatrix}$$

$$H_{i,j} = \max \begin{Bmatrix} 0 \\ E_{i,j} \\ F_{i,j} \\ H_{i-1,j-1} + W(a_i, b_j) \end{Bmatrix}$$

Richard Barnes, "A Review of the Smith-Waterman GPU Landscape," Master Report, Berkeley, 2020

Algorithmic Tricks/Improvements

- Search space reduction
- Query profile (uses texture cache)
- Data layouts
- Input-size dependent choice of algorithms
- Speculation
- Storage reduction
- Processing order
- Fine-tuning block/thread counts
- SIMD instructions
- Use of CPU and GPU
- Use of local memory
- Multi-GPU
- Calculation time prediction equation
- Pipelining
- Recompile GPU code on the fly
- Use of BWT
- Tiling
- Block pruning

https://github.com/r-barnes/sw_comparison

LTDP & Rank Convergence

- LTDP definition
  - its solution and the solutions of its subproblems are in the domain of the tropical semiring
  - the subproblems can be grouped into a sequence of stages, such that the solution to a subproblem in a stage depends on only the solutions in the previous stage
  - this dependence is linear in the tropical semiring
- Examples
  - 1) Longest Common Subsequence; 2) Viterbi
  - 3) Needleman-Wunsch; 4) Smith-Waterman
- Empirical rank convergence help parallelization

* S. Maleki, M. Musuvathi, and T. Mytkowicz, "Parallelizing dynamic programming through rank convergence," ACM SIGPLAN Notice, 2014.
* S. Maleki, M. Musuvathi, and T. Mytkowicz, "Efficient parallelization using rank convergence in dynamic programming algorithms," CACM, 2016.

Topic Overview

- Overview of Serial Dynamic Programming
- Serial Monadic DP Formulations
- Nonserial Monadic DP Formulations
- Serial Polyadic DP Formulations
- Nonserial Polyadic DP Formulations

Serial Monadic DP Formulations

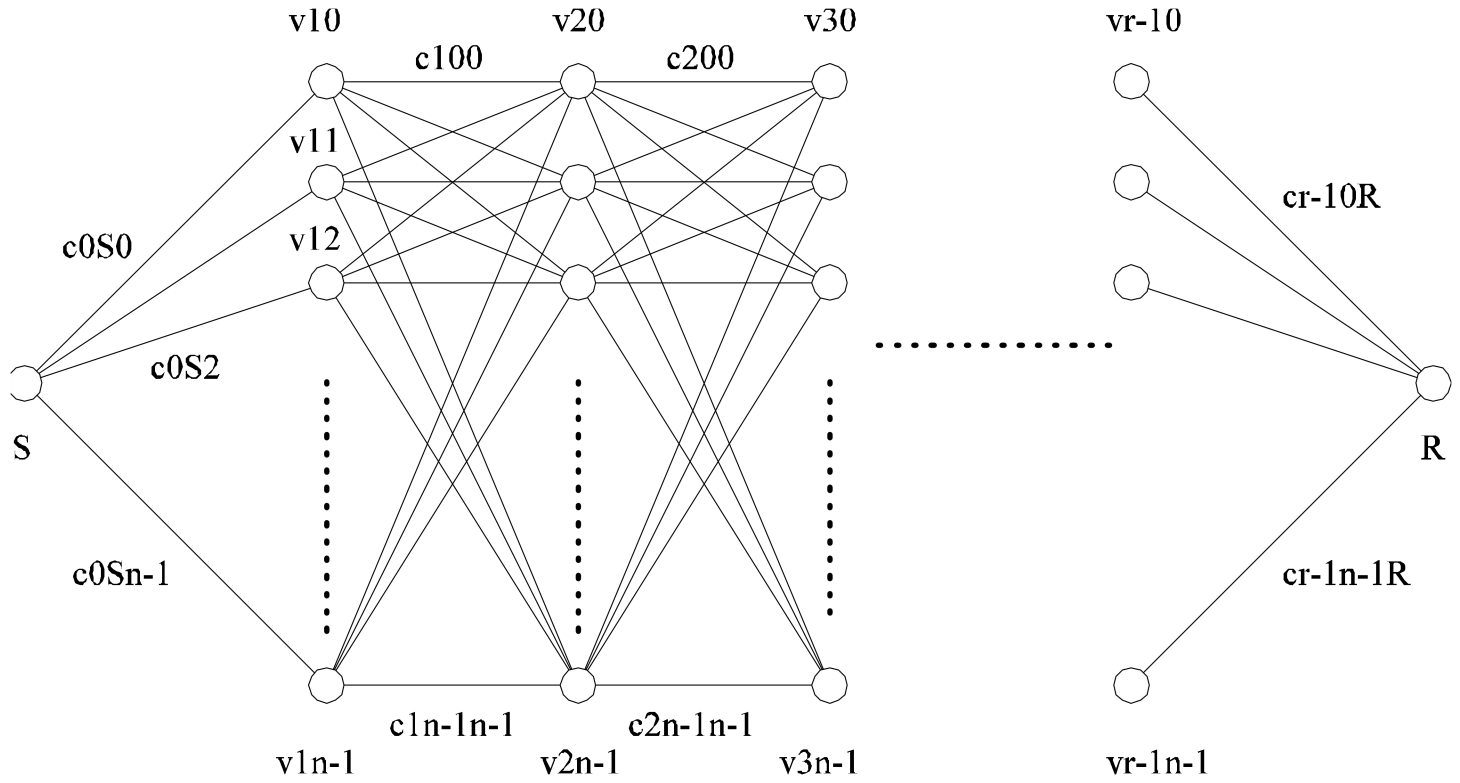
- It is difficult to derive canonical parallel formulations for the entire class of formulations.
- For this reason, we select two representative examples, the shortest-path problem for a multistage graph and the 0/1 knapsack problem.
- We derive parallel formulations for these problems and identify common principles guiding design within the class.

Shortest Path for a Multistage Graph

- Special class of shortest path problem where the graph is a weighted multistage graph of $r + 1$ levels.
- Each level is assumed to have n levels and every node at level i is connected to every node at level $i + 1$.
- Levels zero and r contain only one node, the source and destination nodes, respectively.
- The objective of this problem is to find the shortest path from S to R.

Shortest Path for a Multistage Graph

- An example of a serial monadic DP formulation for finding the shortest path in a graph whose nodes can be organized into levels.



Shortest Path for a Multistage Graph

- The i-th node at level l in the graph is labeled v_i^l and the cost of an edge connecting v_i^l to node v_j^{l+1} is labeled $c_{i,j}^l$.
- The cost of reaching the goal node R from any node v_i^l is represented by C_i^l.
- If there are n nodes at level l, the vector

  $[C_0^l, C_1^l, \ldots, C_{n-1}^l]^T$ is referred to as C^l. Note that

  $C^0 = [C_0^0]$.
- We have $C_i^l = \min \{(c_{i,j}^l + C_j^{l+1}) \mid j \text{ is a node at level } l + 1\}$

Shortest Path for a Multistage Graph

- Since all nodes v_j^{r-1} have only one edge connecting them to the goal node R at level r,
- the cost C_j^{r-1} is equal to $c_{j,R}^{r-1}$.
- We have:

$$C^{r-1} = [c_{0,R}^{r-1}, c_{1,R}^{r-1}, \ldots, c_{n-1,R}^{r-1}].$$

Notice that this problem is serial and monadic.

Shortest Path for a Multistage Graph

- The cost of reaching the goal node R from any node at level l is ($0 < l < r - 1$) is

$$C_0^l = \min\{(c_{0,0}^l + C_0^{l+1}), (c_{0,1}^l + C_1^{l+1}), \ldots, (c_{0,n-1}^l + C_{n-1}^{l+1})\},$$

$$C_1^l = \min\{(c_{1,0}^l + C_0^{l+1}), (c_{1,1}^l + C_1^{l+1}), \ldots, (c_{1,n-1}^l + C_{n-1}^{l+1})\},$$

$$\vdots$$

$$C_{n-1}^l = \min\{(c_{n-1,0}^l + C_0^{l+1}), (c_{n-1,1}^l + C_1^{l+1}), \ldots, (c_{n-1,n-1}^l + C_{n-1}^{l+1})\}.$$

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Shortest Path for a Multistage Graph

- We can express the solution to the problem as a modified sequence of matrix-vector products.
- Replacing the addition operation by minimization and the multiplication operation by addition, the preceding set of equations becomes:

$$C^l = M_{l,l+1} \times C^{l+1},$$

where C^l and C^{l+1} are $n \times 1$ vectors representing the cost of reaching the goal node from each node at levels l and $l + 1$.

Shortest Path for a Multistage Graph

- Matrix $M_{l,l+1}$ is an n x n matrix in which entry (i, j) stores the cost of the edge connecting node i at level l to node j at level l + 1.

$$M_{l,l+1} = \begin{bmatrix} c^l_{0,0} & c^l_{0,1} & \cdots & c^l_{0,n-1} \\ c^l_{1,0} & c^l_{1,1} & \cdots & c^l_{1,n-1} \\ \vdots & \vdots & & \vdots \\ c^l_{n-1,0} & c^l_{n-1,1} & \cdots & c^l_{n-1,n-1} \end{bmatrix}.$$

- The shortest path problem has been formulated as a sequence of r matrix-vector products.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Parallel SP for a Multistage Graph

- We can parallelize this algorithm using the parallel algorithms for the matrix-vector product.
- $\Theta(n)$ processing elements can compute each vector C^l in time $\Theta(n)$ and solve the entire problem in time $\Theta(rn)$.
- In many instances of this problem, the matrix M may be sparse. For such problems, it is highly desirable to use sparse matrix techniques.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

0/1 Knapsack Problem

- We are given a knapsack of capacity c and a set of n objects numbered 1, 2, …, n. Each object i has weight w_i and profit p_i.
- Let $v = [v_1, v_2, \ldots, v_n]$ be a solution vector in which $v_i = 0$ if object i is not in the knapsack, and $v_i = 1$ if it is in the knapsack.
- The goal is to find a subset of objects to put into the knapsack so that

$$\sum_{i=1}^{n} w_i v_i \leq c$$

(that is, the objects fit into the knapsack) and

$$\sum_{i=1}^{n} p_i v_i$$

is maximized (that is, the profit is maximized).

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

0/1 Knapsack Problem

- Let F[i,x] be the maximum profit for a knapsack of capacity x using only objects {1,2,…,i}. The DP formulation is:

$$F[i,x] = \begin{cases} 0 & x \geq 0, i = 0 \\ -\infty & x < 0, i = 0 \\ \max\{F[i-1,x], (F[i-1,x-w_i] + p_i)\} & 1 \leq i \leq n \end{cases}$$

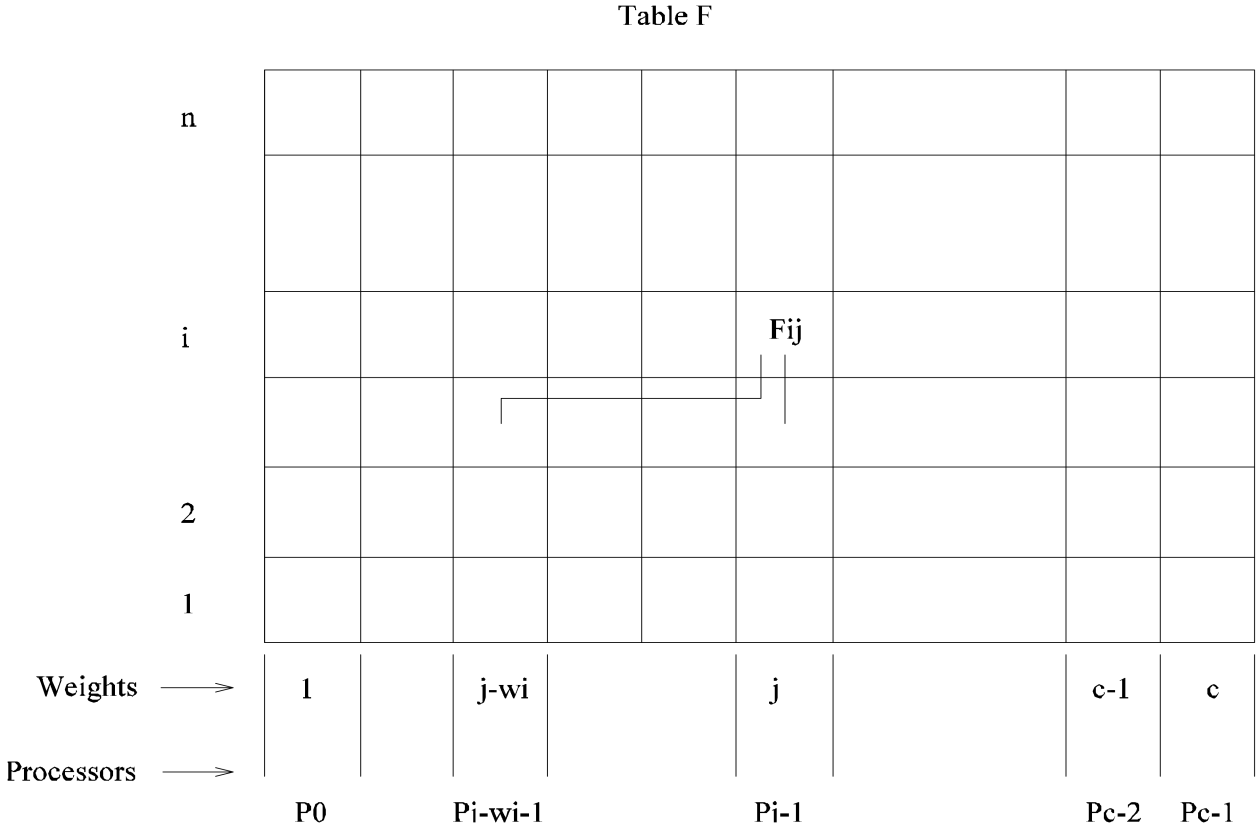
Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

0/1 Knapsack Problem

- Construct a table F of size $n \times c$ in row-major order.
- Filling an entry in a row requires two entries from the previous row: one from the same column and one from the column offset by the weight of the object corresponding to the row.
- Computing each entry takes constant time; the sequential run time of this algorithm is $\Theta(nc)$.
- The formulation is serial-monadic.

0/1 Knapsack Problem

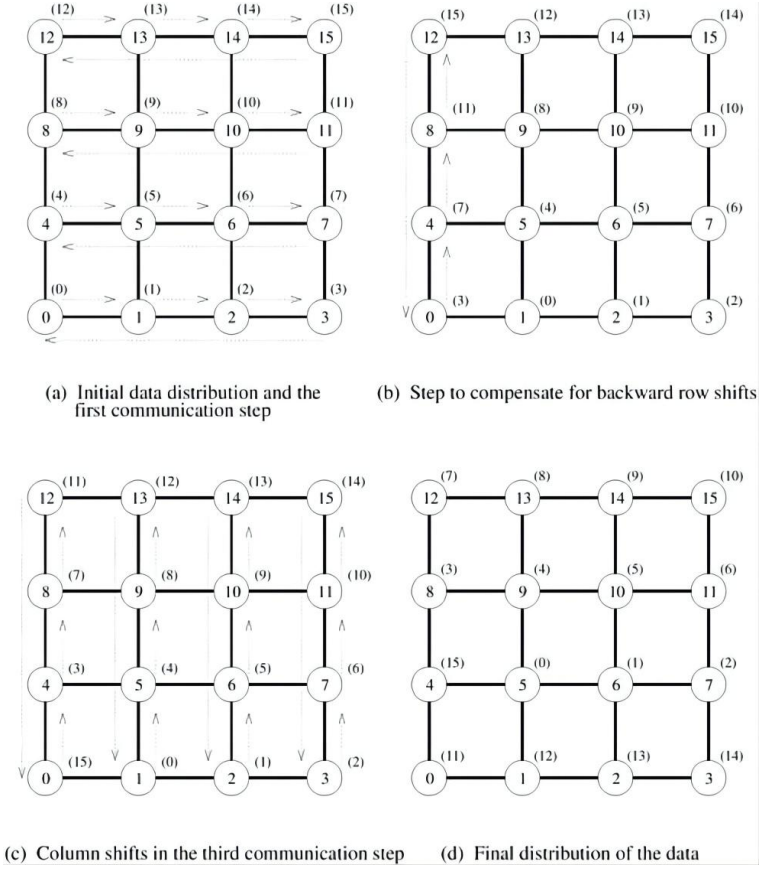
- Computing entries of table F for the 0/1 knapsack problem. The computation of entry F[i,j] requires communication with processing elements containing entries F[i-1,j] and F[i-1,j-w_i].



Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

0/1 Knapsack Problem: Circular q-Shift

- Ring: $(t_s+mt_w)\ \min(q, p-q)$
- Mesh: $(t_s+mt_w)\ (\sqrt{q}+1)$



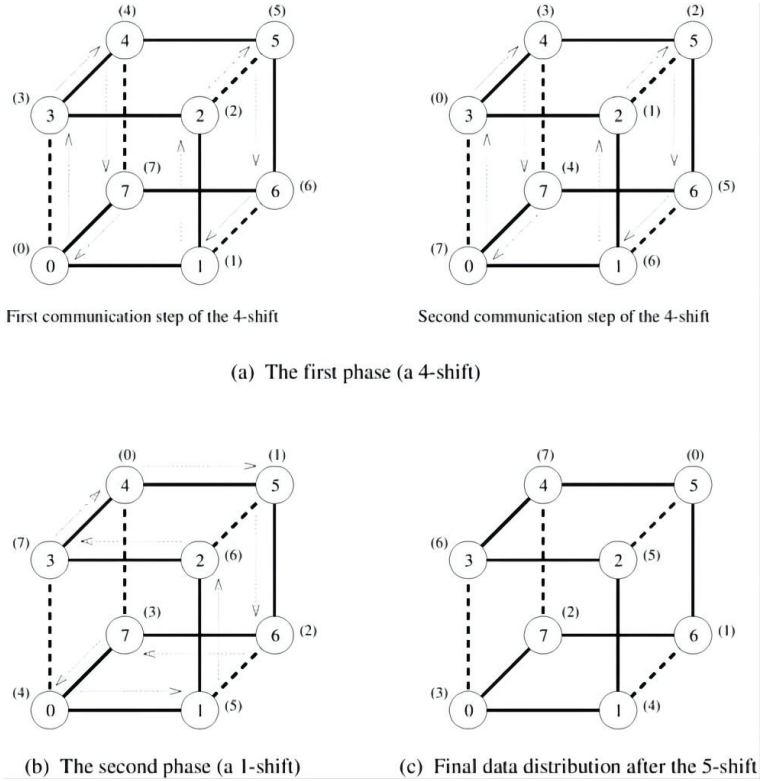
(a) Initial data distribution and the first communication step

(b) Step to compensate for backward row shifts

(c) Column shifts in the third communication step

(d) Final distribution of the data

- Hypercube: $(t_s+mt_w)\ \log p$
  - for long msg with bi-directioanl channels (E-cube routing): $t_s + m\ t_w$



First communication step of the 4-shift

Second communication step of the 4-shift

(a) The first phase (a 4-shift)

(b) The second phase (a 1-shift)

(c) Final data distribution after the 5-shift

examples of the 5-shift on a mesh and a hypercube

0/1 Knapsack Problem

- Using c processors in a PRAM, we can derive a simple parallel algorithm that runs in O(n) time by partitioning the columns across processors.
- In a distributed memory machine, in the j-th iteration, for computing F[j,r] at processing element P_{r-1}, F[j-1,r] is available locally but $F[j-1,r-w_j]$ must be fetched.
- The communication operation is a circular shift and the time is given by $(t_s + t_w) \log c$. The total time is therefore $t_c + (t_s + t_w) \log c$.
- Across all n iterations (rows), the parallel time is $O(n \log c)$. Note that this is not cost optimal.

0/1 Knapsack Problem

- Using p-processing elements, each processing element computes c/p elements of the table in each iteration.
- The corresponding shift operation takes time $(2t_s + t_w c/p)$, since the data block may be partitioned across two processors, but the total volume of data is c/p.
- The corresponding parallel time is $n(t_c c/p + 2t_s + t_w c/p)$, or $O(nc/p)$ (which is cost-optimal).
- Note that there is an upper bound on the efficiency of this formulation: $1 / (1 + t_w/t_c)$

Nonserial Monadic DP Formulations: Longest-Common-Subsequence

- Given a sequence $A = \langle a_1, a_2, \ldots, a_n \rangle$, a subsequence of A can be formed by deleting some entries from A.
- Given two sequences $A = \langle a_1, a_2, \ldots, a_n \rangle$ and $B = \langle b_1, b_2, \ldots, b_m \rangle$, find the longest sequence that is a subsequence of both A and B.
- If $A = \langle c,a,d,b,r,z \rangle$ and $B = \langle a,s,b,z \rangle$, the longest common subsequence of A and B is $\langle a,b,z \rangle$.

Longest-Common-Subsequence Problem

- Let F[i,j] denote the length of the longest common subsequence of the first i elements of A and the first j elements of B. The objective of the LCS problem is to find F[n,m].
- We can write:

$$F[i,j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ F[i-1, j-1] + 1 & \text{if } i,j > 0 \text{ and } x_i = y_j \\ \max\{F[i, j-1], F[i-1, j]\} & \text{if } i,j > 0 \text{ and } x_i \neq y_j \end{cases}$$

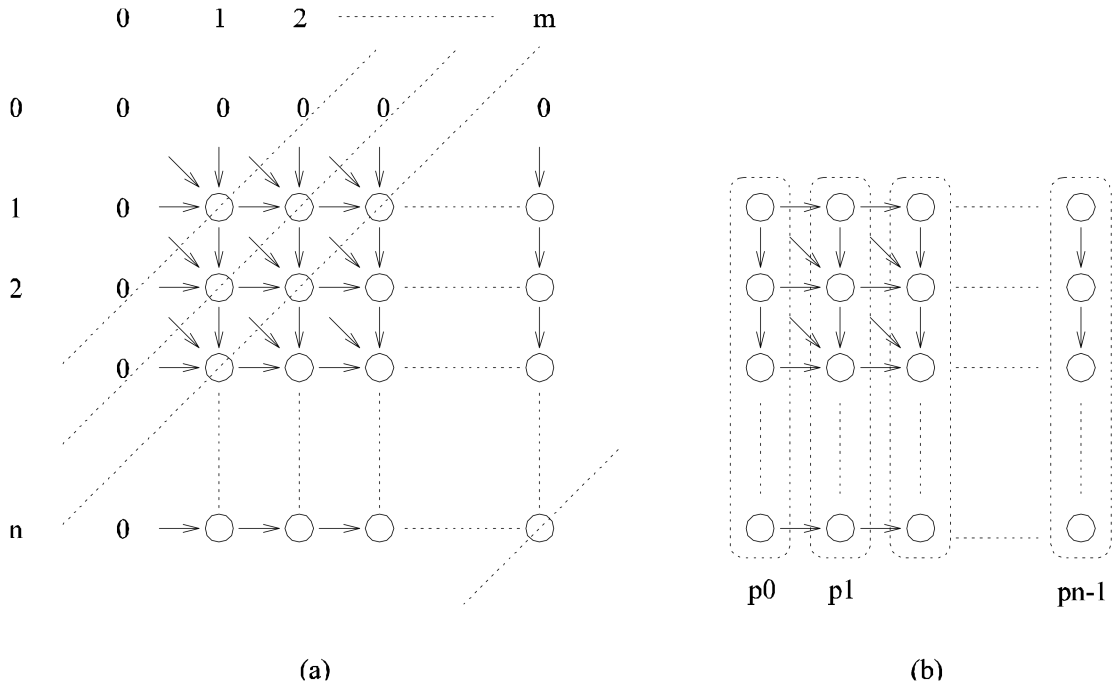
Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Longest-Common-Subsequence Problem

- The algorithm computes the two-dimensional F table in a row- or column-major fashion. The complexity is $\Theta(nm)$.
- Treating nodes along a diagonal as belonging to one level, each node depends on two subproblems at the preceding level and one subproblem two levels prior.
- This DP formulation is nonserial monadic.

Longest-Common-Subsequence Problem

- (a) Computing entries of table for the longest-common-subsequence problem. Computation proceeds along the dotted diagonal lines.
- (b) Mapping elements of the table to processing elements.



Longest-Common-Subsequence: Example

- Consider the LCS of two amino-acid sequences H E A G A W G H E E and P A W H E A E. For the interested reader, the names of the corresponding amino-acids are A: Alanine, E: Glutamic acid, G: Glycine, H: Histidine, P: Proline, and W: Tryptophan.

		H	E	A	G	A	W	G	H	E	E
	0	0	0	0	0	0	0	0	0	0	0
P	0	0	0	0	0	0	0	0	0	0	0
A	0	0	0	1	1	1	1	1	1	1	1
W	0	0	0	1	1	1	2	2	2	2	2
H	0	1	1	1	1	1	2	2	3	3	3
E	0	1	2	2	2	2	2	2	3	4	4
A	0	1	2	3	3	3	3	3	3	4	4
E	0	1	2	3	3	3	3	3	3	4	5

- The F table for computing the LCS of the sequences. The LCS is A W H E E.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Parallel Longest-Common-Subsequence

- Table entries are computed in a diagonal sweep from the top-left to the bottom-right corner.
- Using n processors in a PRAM, each entry in a diagonal can be computed in constant time.
- For two sequences of length n, there are $2n-1$ diagonals.
- The parallel run time is $\Theta(n)$ and the algorithm is cost-optimal.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Parallel Longest-Common-Subsequence

- Consider a (logical) linear array of processors. Processing element P_i is responsible for the $(i+1)$-th column of the table.
- To compute $F[i,j]$, processing element P_{j-1} may need either $F[i-1,j-1]$ or $F[i,j-1]$ from the processing element to its left. This communication takes time $t_s + t_w$.
- The computation takes constant time (t_c).
- We have: $$T_P = (2n-1)(t_s + t_w + t_c).$$
- Note that this formulation is cost-optimal, however, its *efficiency* is upper-bounded by 0.5!
- Can you think of how to fix this?

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Serial Polyadic DP Formulation: Floyd's All-Pairs Shortest Path

- Given weighted graph G(V,E), Floyd's algorithm determines the cost $d_{i,j}$ of the shortest path between each pair of nodes in V.
- Let $d_{i,j}^{k}$ be the minimum cost of a path from node i to node j, using only nodes $v_0, v_1, \ldots, v_{k-1}$.
- We have:

$$d_{i,j}^{k} = \begin{cases} c_{i,j} & k = 0 \\ \min\left\{d_{i,j}^{k-1}, \left(d_{i,k}^{k-1} + d_{k,j}^{k-1}\right)\right\} & 0 \leq k \leq n-1 \end{cases}.$$

- Each iteration requires time $\Theta(n^2)$ and the overall run time of the sequential algorithm is $\Theta(n^3)$.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Serial Polyadic DP Formulation: Floyd's All-Pairs Shortest Path

- A PRAM formulation of this algorithm uses n^2 processors in a logical 2D mesh. Processor $P_{i,j}$ computes the value of $d_{i,j}^{k}$ for $k=1,2,\ldots,n$ in constant time.
- The parallel runtime is $\Theta(n)$ and it is cost-optimal.
- The algorithm can easily be adapted to practical architectures

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Nonserial Polyadic DP Formulation: Optimal Matrix-Parenthesization Problem

- When multiplying a sequence of matrices, the order of multiplication significantly impacts operation count.
- Let $C[i,j]$ be the optimal cost of multiplying the matrices $A_i,\ldots A_j$.
- The chain of matrices can be expressed as a product of two smaller chains, $A_i,A_{i+1},\ldots,A_k$ and $A_{k+1},\ldots,A_j$.
- The chain $A_i,A_{i+1},\ldots,A_k$ results in a matrix of dimensions $r_{i-1} \times r_k$, and the chain $A_{k+1},\ldots,A_j$ results in a matrix of dimensions $r_k \times r_j$.
- The cost of multiplying these two matrices is $r_{i-1}r_kr_j$.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Optimal Matrix-Parenthesization Problem

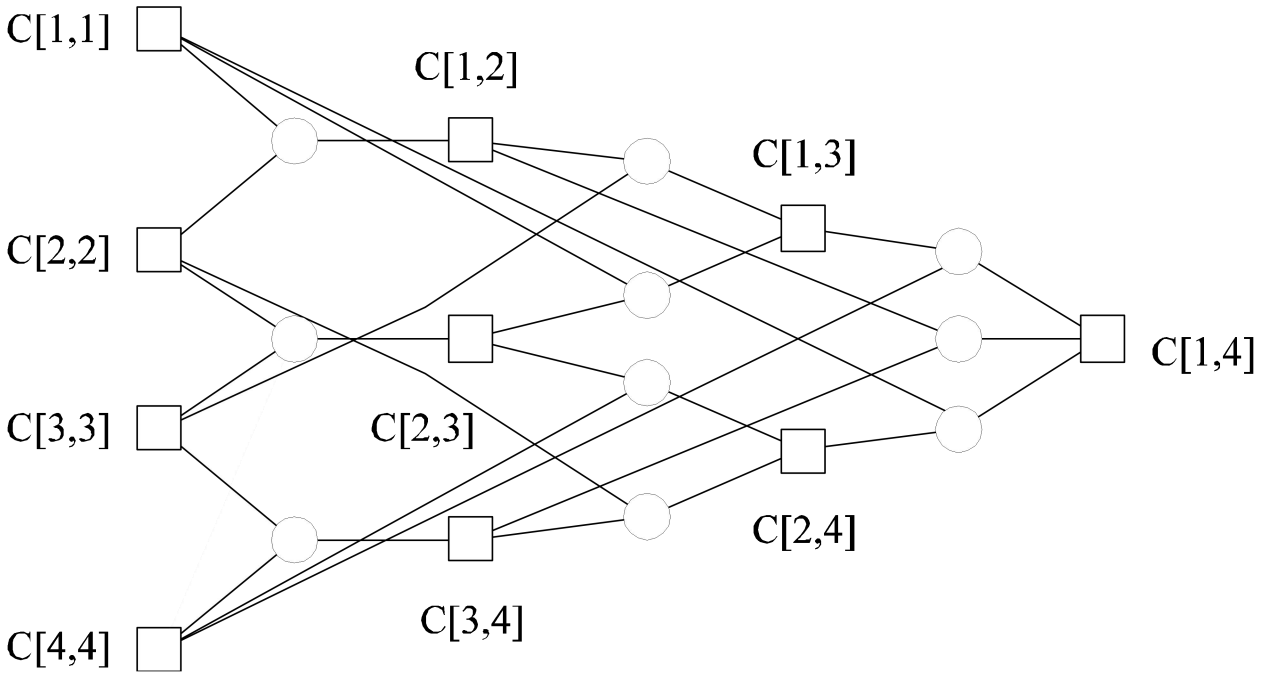
- We have:

$$C[i,j] = \begin{cases} \min\limits_{i \le k < j} \{C[i,k] + C[k+1,j] + r_{i-1} r_k r_j\} & 1 \le i < j \le n \\ 0 & j = i, 0 < i \le n \end{cases}$$

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Optimal Matrix-Parenthesization Problem

- A nonserial polyadic DP formulation for finding an optimal matrix parenthesization for a chain of four matrices. A square node represents the optimal cost of multiplying a matrix chain. A circle node represents a possible parenthesization.



Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Optimal Matrix-Parenthesization Problem

- The goal of finding C[1,n] is accomplished in a bottom-up fashion.
- Visualize this by thinking of filling in the C table diagonally. Entries in diagonal l corresponds to the cost of multiplying matrix chains of length l+1.
- The value of C[i,j] is computed as $\min\{C[i,k] + C[k+1,j] + r_{i-1}r_k r_j\}$, where k can take values from i to j-1.
- Computing C[i,j] requires that we evaluate (j-i) terms and select their minimum.
- The computation of each term takes time t_c, and the computation of C[i,j] takes time $(j-i)t_c$. Each entry in diagonal l can be computed in time lt_c.

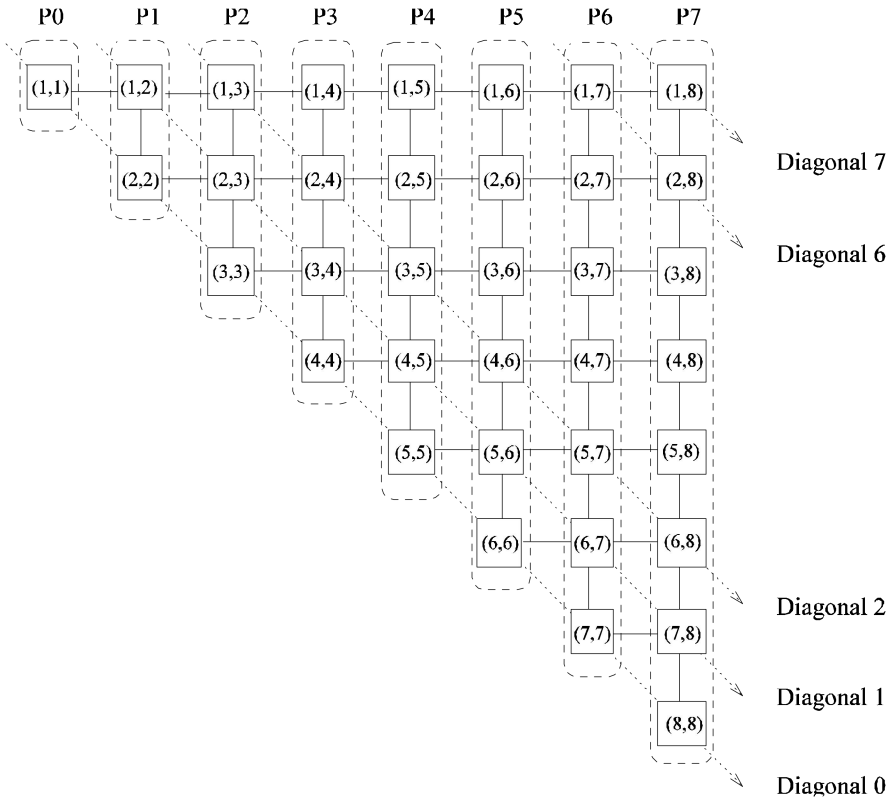
Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Optimal Matrix-Parenthesization Problem

- The algorithm computes $(n-1)$ chains of length two. This takes time $(n-1)t_c$; computing $n-2$ chains of length three takes time $(n-2)t_c$. In the final step, the algorithm computes one chain of length n in time $(n-1)t_c$.
- It follows that the serial time is $\Theta(n^3)$.

Optimal Matrix-Parenthesization Problem

- The diagonal order of computation for the optimal matrix-parenthesization problem.



Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Parallel Optimal Matrix-Parenthesization Problem

- Consider a logical ring of processors. In step l, each processor computes a single element belonging to the l-th diagonal.
- On computing the assigned value of the element in table C, each processor sends its value to all other processors using an all-to-all broadcast.
- The next value can then be computed locally.
- The total time required to compute the entries along diagonal l is $lt_c+t_s\log n+t_w(n-1)$.
- The corresponding parallel time is given by:

$$T_P = \sum_{l=1}^{n-1}(lt_c + t_s \log n + t_w(n-1)),$$

$$= \frac{(n-1)(n)}{2}t_c + t_s(n-1)\log n + t_w(n-1)^2.$$

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Parallel Optimal Matrix-Parenthesization Problem

- When using p (<n) processors, each processor stores n/p nodes.
- The time taken for all-to-all broadcast of n/p words is

$$t_s \log p + t_w n(p-1)/p \approx t_s \log p + t_w n$$

and the time to compute n/p entries of the table in the l-th diagonal is $lt_c n/p$.

$$T_P = \sum_{l=1}^{n-1} (lt_c n/p + t_s \log p + t_w n),$$

$$= \frac{n^2(n-1)}{2p} t_c + t_s(n-1)\log p + t_w n(n-1).$$

$$T_P = \Theta(n^3/p) + \Theta(n^2)$$

- This formulation can be improved to use up to n(n+1)/2 processors using pipelining.
  - due to the non-serial nature of the problem

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Discussion of Parallel Dynamic Programming Algorithms

- By representing computation as a graph, we identify three sources of parallelism: i) parallelism within nodes, ii) parallelism across nodes at a level, and iii) pipelining nodes across multiple levels. The first two are available in serial formulations and the third one in non-serial formulations.

- Data locality is critical for performance. Different DP formulations, by the very nature of the problem instance, have different degrees of locality.

Grama et al., "Introduction to Parallel Computing," Addison Wesley, 2003

Summary

- Several types of Dynamic Programming
  - Serial Monadic DP Formulations
    - example: shortest path for a multistage graph
    - example: 0/1 knapsack problem
  - Nonserial Monadic DP Formulations
    - example: longest-common-subsequence problem
  - Serial Polyadic DP Formulations
    - example: all-pairs shortest path
  - Nonserial Polyadic DP Formulations
    - example: optimal matrix-parenthesization
- Sources of parallelism
  - (serial) within a subproblem; and among intra-level subproblems
  - (nonserial) pipelining