A Guide to NOI for Beginners (Draft)

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Github: https://github.com/NoviScl/NOI

Reference Book: http://www.ituring.com.cn/book/1044

Practice Solutions: https://github.com/yogykwan/acm-challenge-workbook/tree/master/src

Foreword

This guide is designed specifically for those who already learned the basics of programming. I assume that you already know how to code in C++ and understand the concept of time complexity, as well as some other basic data structures like linked list, stack, queue, BST. (If not, there are lots of great tutorials/MOOCs online.)

We understand that it is hard to teach oneself competitive programming. So I made this guide and recorded a series of videos explaining it. This guide is meant for you to quickly get started on solving NOI problems and get familiar with some important algorithms. There is no guarantee that you can get a medal at NOI after reading this. But I am sure you can improve your knowledge and problem solving skills if you read this guide carefully and do lots of practice for every topic. Feel free to contact me if you find any mistakes or have doubts, I am always happy to discuss with you.

1. Time Complexity

One important thing in NOI is time complexity. There is always a time limit for the problems and if your algorithm is too slow, you will get Time Limit Exceeded (TLE).

Usually the time limit is 1 second. You can substitute the maximum possible number of data witin the given range to yur algorithm's time complexity to get an exstimated number of operations needed. For example, if your algorithm is O(n^2) and the data range is $n \leq 1000$, then the max operations needed is around 10^6. The number of operations can be done in 1 second is around 10^9. So:

- $\leq 10^9$: no problem
- $> 10^9$: probably TLE (but I've seen exceptions...)

Now we will use a simple example to see how this works in practice.

E.g.1 Pick Numbers

You are given n different integers $k_1, \ldots, k_n$. You need to pick 4 numbers from them. You can pick the same number any times. If the sum of the 4 numbers you pick is m, you output YES, otherwise output NO.

$1 \leq n \leq 1000$, $1 \leq m \leq 10^8$, $1 \leq k_i \leq 10^8$

Solution 1: Brute Force

```
#include <cstdio>
using namespace std;

const int MAX_N = 1002;

int main(){
    int n, m, k[MAX_N];

    scanf("%d %d", &n, &m);
    for(int i=0; i<n; i++){
        scanf("%d", &k[i]);
    }

    bool f = false;

    // brute force all possibilities
    for(int a=0; a<n; a++){
        for(int b=0; b<n; b++){
            for(int c=0; c<n; c++){
                for(int d=0; d<n; d++){
                    if(k[a] + k[b] + k[c] + k[d] == m){
                        f = true;
                    }
                }
            }
        }
    }

    if(f) printf("YES");
    else printf("NO");
}
```

The complexity for this solution is $O(n^4)$, so the estimated number of operations is 10^{12}, which will definitely get you TLE.

One way to improve it: when you have chosen the first three numbers, you also know what the last number should be in order for the sum to be m. Hence you can use binary search to fund the desired number.

Solution 2: Binary Search the last number

```
#include <cstdio>
#include <algorithm>
using namespace std;

const int MAX_N = 1002;

int n, m, k[MAX_N];

bool binary_search(int x){
    int l=0, r=n;

    while(r-l>=1){
        int i = (l+r)/2;
        if(k[i]==x) return true;
        else if(k[i]<x) l = i+1;
        else r = i;
    }

    return false;
}

void solve(){
    //must sort before BS
    sort(k, k+n);

    bool f = false;

    for(int a=0; a<n; a++){
        for(int b=0; b<n; b++){
            for(int c=0; c<n; c++){
                if(binary_search(m-k[a]-k[b]-k[c])){
                    f = true;
                }
            }
        }
    }

    if(f) printf("YES");
    else printf("NO");
}
```

The complexity is now improved to $O(n^3 \log n)$. But this is still too slow.

One way to improve: when we have chosen the first two numbers, we also know the sum of the other two numbers in order to get a total sum of m. Hence, we can binary search among all possible sums of two given numbers to find if the desired sum is present.

Solution 3: Binary Search the sum of the last two numbers

```
#include <cstdio>
#include <algorithm>
using namespace std;

const int MAX_N = 1002;

int n, m, k[MAX_N];

// sum of two numbers
int kk[MAX_N * MAX_N];

void solve(){
    for(int c=0; c<n; c++){
        for(int d=0; d<n; d++){
            kk[c*n + d] = k[c] + k[d];
        }
    }

    sort(kk, kk+n*n);

    bool f = false;
    for(int a=0; a<n; a++){
        for(int b=0; b<n; b++){
            // use STL BS
            if(binary_search(kk, kk+n*n, (m-k[a]-k[b]))){
                f = true;
            }
        }
    }

    if(f) printf("YES");
    else printf("NO");
}
```

Complexity: sorting n^2 numbers: O($n^2 \log n$) , nested loop with binary search: O($n^2 \log n$)

So overall complexity is: O($n^2 \log n$), which is acceptable.

2. Search

2.1 Recursion with Memoization

Naive recursion is often slow because it computes the same elements many times, which is a waste of time. For example, when calcuating the fibonacci number, we compute fib(8) and fib(9) to get fib(10). However, while computing fib(9), we need to compute fib(8) again.

One way to avoid this is to store all computed values in a table for future use. This technique is called memoization.

Example: Fibonacci number

```
int memo[MAX_N + 1] = {0};

int fib(int n){
    if(n<=1) return n;
    if(memo[n]!=0) return memo[n];
    return memo[n] = fib(n-1) + fib(n-2);
}
```

2.2 Stack

Stack is already implemented in STL.

To use Stack in STL:

```
#include <stack>
#include <cstdio>
using namespace std;

int main(){
    stack<int> s;
    s.push(2);
    s.push(3);
    printf("%d\n", s.top()); //3
    s.pop();
    printf("%d\n", s.top()); //2
}
```

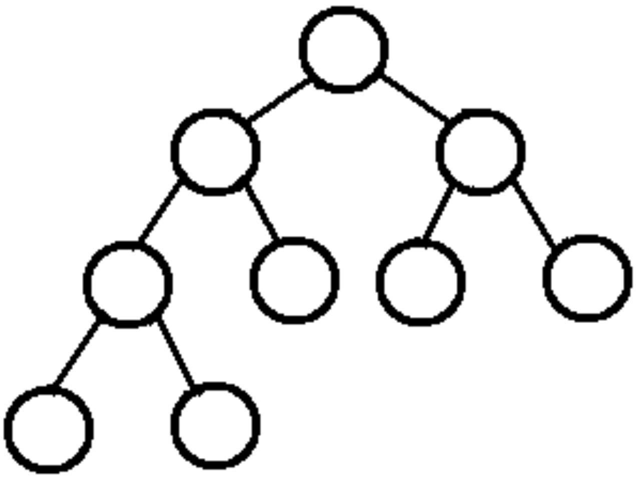
2.3 Queue

To use Queue in STL:

```cpp
#include <queue>
#include <cstdio>
using namespace std;

int main(){
    queue<int> que;
    que.push(1);
    que.push(2);
    printf("%d\n", que.front()); //1
    que.pop();
    printf("%d\n", que.front()); //2
}
```

2.4 Depth-first Search (DFS)

We use a binary tree to illustrate DFS.



DFS starts from the root and goes all the way down to the leftmost leaf node, then returns back to the previous layer, travels through the second leaf node, then returns back to the previous layer, and so on.

DFS is usually implemented by recursion.

E.g.1 Sum

You are given n integers $a_1, .., a_n$, determine if it is possible to choose some of them (each number can only be used once) so that their sum is k.

$1 \le n \le 20, -10^8 \le a_i \le 10^8, -10^8 \le k \le 10^8$

Since for each given number, we can choose to either take or not take, this is essentially searching through a binary tree.

```
const int MAX_N = 21;
int a[MAX_N];
int n, k;

bool dfs(int i, int sum){
    // leaf node, note n not (n-1)
    if(i==n) return sum == k;

    // not take a[i]
    if(dfs(i+1, sum)) return true;

    // take a[i]
    if(dfs(i+1, sum+a[i])) return true;

    return false;
}

void solve(){
    if(dfs(0, 0)) printf("YES");
    else printf("NO");
}
```

Total possible cases (number of leaf nodes) is 2^{n+1}, so complexity $O(2^n)$.

E.g.2 Lake Counting (POJ 2386)

Given a $N * M$ field, some areas some water after a rain ('W' : water, ' . ' : normal land). Connected areas with water (including diagonally adjacent) are counted as one puddle. Output the number of pubbles in the field.

$N, M \leq 100$

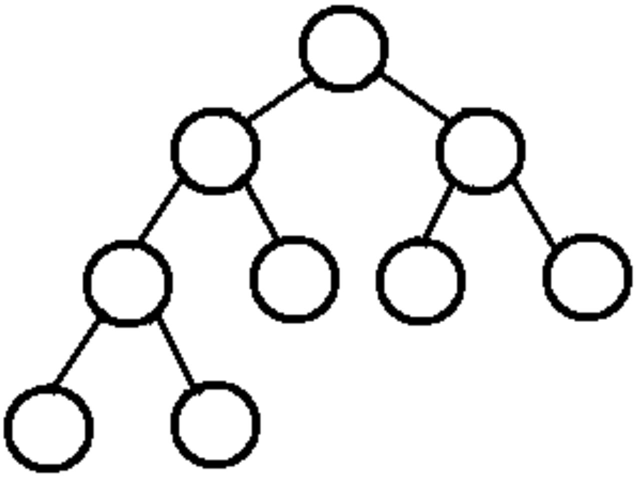
Starting from one area with water, we can use DFS to find all areas with water connected with this area. We count the number of such connected puddles while setting already counted areas to normal to avoid repetition.

```
const int MAX_N = 101;
int N, M;
char field[MAX_N][MAX_N];

void dfs(int x, int y){
    // change this area to normal
    field[x][y] = '.';

    // check all 8 adjacent areas
    for(int dx=-1; dx<=1; dx++){
        for(int dy=-1; dy<=1; dy++){
            int nx = x + dx, ny = y + dy;
            if(nx>=0 && nx<N && ny>=0 && ny<M && field[nx][ny]=='W')
                dfs(nx, ny);
        }
    }
}

void solve(){
    int res = 0;
    for(int i=0; i<N; i++){
        for(int j=0; j<M; j++){
            if(field[i][j] == 'W'){
                dfs(i, j);
                res++;
            }
        }
    }

    printf("%d\n", res);
}
```

Since every area is only searched once (after once water is set to normal and won't be searched again), time complexity is $O(N * M)$.

2.5 Breadth-first Search (BFS)

We use a binary tree to illustrate BFS.



BFS searches from the nearest nodes to the farthest nodes. In the binary tree example, starting from the root, BFS first goes to the two nodes on the next layer which are the closest to it. Then it goes to the nodes on the third layer, and so on. BFS is usually implemented by queue.

E.g.1 Maze Runner

You are given a $N * M$ maze consisting of obstacles and normal lands. ('#' : obstacle, ' . ' : land, 'S': starting point, 'G': goal). Each step you can move left, right, up or down. Find the minimum number of steps needed from starting point to the goal.

$N, M \leq 100$

```
const int MAX_N = 101;
const int INF = 9999999;

typedef pair<int, int> P;

char maze[MAX_N][MAX_N];
int N, M;
int sx, sy;  //start pt
int gx, gy;  //goal pt

int d[MAX_N][MAX_N];

int dx[4] = {1, 0, -1, 0}, dy[4] = {0, 1, 0, -1};

int bfs(){
    queue<P> que;

    for(int i=0; i<N; i++){
        for(int j=0; j<N; j++){
            d[i][j] = INF;
        }
    }

```

```
    que.push(P(sx, sy));
    d[sx][sy] = 0;

    while(que.size()){
        P cur = que.front();
        que.pop();

        if(cur.first==gx && cur.second==gy) break;

        for(int i=0; i<4; i++){
            int nx = cur.first + dx[i], ny = cur.second + dy[i];

            // available and not visited
            if(nx>=0 && nx<N && ny>=0 && ny<M && maze[nx][ny]!='#' && d[nx][ny]==INF){
                que.push(P(nx, ny));
                d[nx][ny] = d[cur.first][cur.second] + 1;
            }
        }
    }

    return d[gx][gy];
}


void solve(){
    int res = bfs();
    printf("%d\n", res);
}
```

Each point in the maze has entered the queue **at most** once. Hence complexity is $O(N * M)$.

2.6 Pruning and Backtracking

In DFS, if at a certain state we realize that this state will definitely not generate a correct answer, then we do not need to continue with this state any more, we can just seach the next possible state instead. This is called pruning.

Usually DFS is used to search for solution over a tree structure. Generally, this algorithm can be used to search over any problem space and it is called backtracking.

Example: Find all permutations of N numbers

```
int total = 0;  //#permutations
const int N = 4;  //use 4 as an example
int numbers[N], used[N], res[N];

void permutate(int ith){
    if(ith==N){
        for(int i=0; i<N; i++){
            cout<<res[i];
        }
        cout<<endl;
        total++;
        return;
    }

    // find availble numbers
    for(int i=0; i<N; i++){
        if(!used[i]){
            res[ith] = nums[i];
            used[i] = 1;
            permutate(ith+1);
            //set free for future use
            used[i]=0;
        }
    }
}

int main(){
    for(int i=0; i<N; i++){
        cin>>nums[i];
    }
    memset(used, 0, sizeof(used));
    permutate(0);
    cout<<total;
}
```

For permutation, it might be easier to directly use next_permutation function. Note that you should use do while in this case, otherwise you will miss the original array case.

Example: next_permutation

```cpp
#include <algorithm>
#include <iostream>
using namespace std;

const int N = 4;
int nums[N] = {1, 2, 3, 4}, total=0;

int main(){
    do{
        total++;
        for(int i=0; i<N; i++){
            cout<<nums[i];
        }
        cout<<endl;
    }while(next_permutation(nums, nums+N));
    cout<<total;  //24
}
```

(Practice list: refer to the recommended repo)

3. Greedy Algorithm

In greedy algorithm, we choose the current best solution at each step.

E.g.1 Job Arrangement

There are n jobs, each job starts from time s_i and ends at time t_i . If you choose a job, you must not do any other jobs during its full period (including s_i and t_i). Find the maximum number of jobs you can do.

$1 \leq N \leq 10^5, 1 \leq s_i \leq t_i \leq 10^9$

Intuition: we want to finish the current job as early as possible so that we have more time for other jobs. Every time we choose the job with the earliest ending time and not clashing with previously chosen jobs.

```
const int MAX_N = 100002;

int N, S[MAX_N], T[MAX_N];

pair<int, int> jobs[MAX_N];

void solve(){
    // sorts the first element in pair by default
    // should sort by end time
    for(int i=0; i<N; i++){
        jobs[i].first = T[i];
        jobs[i].second = S[i];
    }

    sort(jobs, jobs+N);

    // t: end time of prev chosen job
    int ans=0, t=0;
    for(int i=0; i<N; i++){
        if(t < jobs[i].second){
            ans++;
            t = jobs[i].first;
        }
    }

    printf("%d\n", ans);
}
```

Rigorous proofs of the correctness of the algorithm are possible but will not be covered here.

E.g.2 Smallest String (POJ 3617)

Given a string S with length N (all characters are uppercase), and an empty string T. Every time you can either remove the first or last character from S and append to the end of T. Construct the T with the minimum alphabetic order.

$1 \leq N \leq 2000$

Intuition: everytime choose the smaller character from the first and last character of S . If they are the same, compare the next character, do so until there is a difference (if all equal then doesn't matter).

(Consider a simple example: zabz).

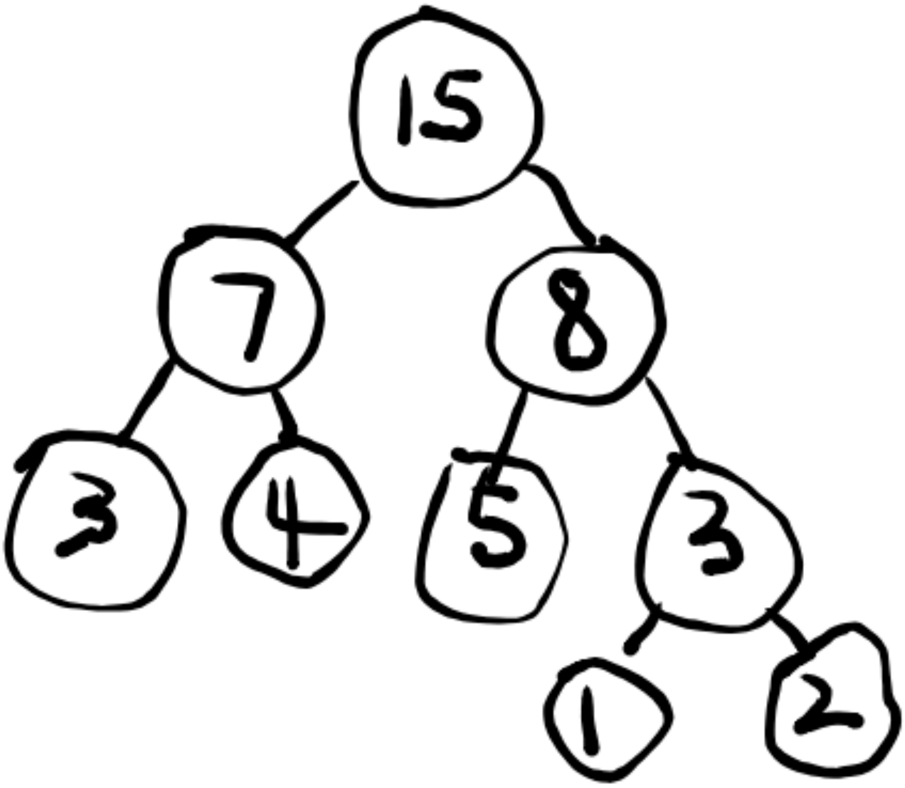
```
const int MAX_N = 2002;
int N;
char S[MAX_N + 1];

void solve(){
    int count = 0;
    int a = 0, b = N - 1;

    while(a<=b){
        bool left = false;

        for(int i=0; a+i<=b-i; i++){
            if(S[a+i] < S[b-i]){
                left = true;
                break;
            }
            else if(S[a+i] > S[b-i]){
                left = false;
                break;
            }
        }

        if(left) putchar(S[a++]);
        else putchar(S[b--]);
        count++;
        if(count%80==0) putchar('\n');
    }
}
```

E.g.3 Fence Repair (POJ 3253)

You need to cut a board into N pieces, with lengths $L_1, \ldots, L_N$. The sum of all the cut boards should be the same of the original board. The cost of cutting a board into 2 pieces equals to the length of the board. For example, if you want to cut a board with length 21 into boards with lengths 5, 8, 8, you can first cut it into 13 and 8 (cost: 21), then cut 12 into 5 and 8 (cost: 13).

Find the minimum cost of cutting the board.

$1 \leq N \leq 2 * 10^4, 0 \leq L_i \leq 5 * 10^4$



Intuition: Cutting the board is like splitting the node into two child nodes. The total cost is the sum of all non-leaf nodes, which also equals to the sum of (leaf node value)*(leaf node depth). Therefore, to get minimum total cost, we want the least value leaf nodes to have the largest depth.

Suppose we already have the cut boards $L_1, .., L_N$, then the shortest and second shorted board (suppose they are L_1, L_2) should be brothers (note it is impossible for a node to have only one child node) and from the same parent node. Then we replace them with $(L_1 + L_2)$ and continue the process until there is only one board left.

```
typedef long long ll;

int N, L[MAX_N];

void solve(){
    ll ans = 0;

    while(N > 1){
        int min1=0, min2=1;
        // min1: shortest, min2: second shortest
        if(L[min1] > L[min2]) swap(min1, min2);

        for(int i=2; i<N; i++){
            if(L[i] < L[min1]){
                min2 = min1;
                min1 = i;
            }
            else if(L[i] < L[min2]){
                min2 = i;
            }
        }

        int t = L[min1] + L[min2];
        ans += t;

        // replace min1 with t
        // swap min2 with last ele and del it
        if(min1 == N-1) swap(min1, min2);
        L[min1] = t;
        L[min2] = L[N-1];
        N--;
    }

    printf("%lld\n", ans);
}
```

Complexity O(n^2). This can be further improved with priority queue (every time pop two front, push their sum).

```
typedef long long ll;

int N, L[MAX_N];

void solve(){
    ll ans = 0;

    // small root heap
    priority_queue<int, vector<int>, greater<int> > que;

    for(int i=0; i<N; i++){
        que.push(L[i]);
    }

    while(que.size() > 1){
        int l1, l2;
        l1 = que.top();
        que.pop();
        l2 = que.top();
        que.pop();

        ans += l1+l2;
        que.push(l1+l2);
    }

    printf("%lld\n", ans);
}
```

Complexity: O($N \log N$)

4. Dynamic Programming

There are two approaches, top-down (recursion), bottom-up (iteration). I find it best to explain DP with examples. Most examples I list here are must-know for NOI.

E.g.1 0-1 Knapsack

You have n items each with weight w_i and value v_i. Your bag has max wieght capacity W. Find the max value of items that can be put in the bag.

$1 \leq n \leq 100, 1 \leq w_i, v_i \leq 100, 1 \leq W \leq 10^4$

Let dp(i, j) denote the max value the bag can have using only the first i items and with capacity j.

For each item, we either take or do not take (if capcacity). So we can just choose the max from those two options.

Top-down: recursion with memoisation

```
const int MAX_N = 102;
const int MAX_W = 10002;

int dp[MAX_N][MAX_W];

int rec(int i, int j){
    if(dp[i][j]>=0){
        return dp[i][j];
    }

    int res;
    // end case
    if(i==0){
        res = 0;
    }
    // can't take w[i]
    else if(j < w[i]){
        res = rec(i-1, j);
    }
    // j>=w[i]
    else{
        res = max(rec(i-1, j), rec(i-1, j-w[i])+v[i]);
    }

    return dp[i][j]=res;
}

void solve(){
    memset(dp, -1, sizeof(dp));
    printf("%d\n", rec(N, W));
}
```

In the bottom-up approach, we fill up the dp table in order.

(Note: sometimes I start from index 1 when inputing data.)

```
int dp[MAX_N][MAX_W];

void solve(){
    // no item or no capacity: 0
    memset(dp, 0, sizeof(dp));

    for(int i=1; i<=N; i++){
        for(int j=1; j<=W; j++){
            if(j < W[i]){
                dp[i][j] = dp[i-1][j];
            }
            else{
                dp[i][j] = max(dp[i-1][j], dp[i-1][j-w[i]]+v[i]);
            }
        }
    }
}
```

Complexity: O(nW)

We can see the that the most important part of DP is to identify the subproblem states and update equations.

In this problem, the subproblem state is the range of items available and the capacity. The update equations can then be deduced easily.

We can save some space by using a rolling array.

```
int dp[MAX_W + 1];

void solve(){
    memset(dp, 0, sizeof(dp));
    for(int i=1; i<=n; i++){
        for(int j=W; j>=w[i]; j--){
            dp[j] = max(dp[j], dp[j-w[i]]+v[i]);
        }
    }
    printf("%d\n", dp[W]);
}
```

Note that we need to go from right to left in the inner loop in order to use the values from previous i.

E.g.2 Longest Common Subsequence (LCS)

Find the length of the longest common subsequence of two string. For example, the LCS of 'abcd' and 'becd' is 'bcd'.

String length: $1 \le n, m \le 1000$

Let dp(i, j) denote the length of LCS of substrings $s_1 \dots s_i$ and $t_1 \dots t_j$.

If $s_i = t_j$: dp(i, j) = max(dp(i-1, j-1)+1, dp(i-1, j), dp(i, j-1))

Else: dp(i, j) = max(dp(i-1, j), dp(i, j-1))

```
int n, m;
char s[MAX_N], t[MAX_M];

int dp[MAX_N+1][MAX_M+1];

void solve(){
    memset(dp, 0, sizeof(dp));
    for(int i=1; i<=n; i++){
        cin>>s[i];
    }
    for(int i=1; i<=m; i++){
        cin>>t[i];
    }

    for(int i=1; i<=MAX_N; i++){
        for(int j=1; j<=MAX_M; j++){
            dp[i][j] = max(dp[i-1][j], dp[i][j-1]);
            if(s[i]==t[j]){
                dp[i][j] = max(dp[i][j], dp[i-1][j-1]+1);
            }
        }
    }

    cout<<dp[n][m];
}
```

E.g.3 Unbounded Knapsack

You have n types of items each with weight w_i and value v_i. Your bag has max wieght capacity W. Find the max value of items that can be put in the bag. Note that you can take unlimited number of copies of each type of item.

$1 \le n \le 100, 1 \le w_i, v_i \le 100, 1 \le W \le 10^4$

In this case, we need to add another inner loop to find the best number of copies to take within the capacity.

```c
int dp[MAX_N + 1][MAX_W + 1];

void solve(){
    memset(dp, 0, sizeof(dp));

    for(int i=1; i<=n; i++){
        for(int j=1; j<=W; j++){
            for(int k=0; k*w[i]<=j; k++){
                dp[i][j] = max(dp[i][j], dp[i-1][j - k*w[i]] + k*v[i]);
            }
        }
    }
    printf("%d\n", dp[n][W]);
}
```

Complexity: O(nW^2)

We can further improve this algorithm. Note that choosing k in $dp[i][j]$ is the same as choosing $(k-1)$ in $dp[i][j-w[i]]$ (take one copy of ith item). Hence we can use this to reduce repeated calculation.

$dp[i][j]$

$= \max\{dp[i-1][j-k*w[i]] + k*v[i] | k \geq 0\}$

$= \max(dp[i][j], \max\{dp[i-1][j-k*w[i]] + k*v[i] | k \geq 1\})$ (either not take or take at least one)

$= \max(dp[i][j], \max\{dp[i-1][(j-w[i]) - k*w[i]] + k*v[i] | k \geq 0\})$ (take out one from k)

$= \max(dp[i][j], dp[i][j-w[i]] + v[i])$

(This can come from observation and intuition as well.)

```
void solve(){
    memset(dp, 0, sizeof(dp));

    for(int i=1; i<=n; i++){
        for(int j=1; j<=W; j++){
            if(j<w[i]){
                dp[i][j] = dp[i-1][j];
            }
            else{
                dp[i][j] = max(dp[i-1][j], dp[i][j-w[i]]+v[i]);
            }
        }
    }
    printf("%d\n", dp[n][W]);
}
```

This can also be improved by using a rolling array.

```
int dp[MAX_W + 1];

void solve(){
    memset(dp, 0, sizeof(dp));

    for(int i=1; i<=n; i++){
        for(int j=w[i]; j<=W; j++){
            dp[j] = max(dp[j], dp[j-w[i]]+v[i]);
        }
    }

    printf("%d\n", dp[W]);
}
```

Note that in the inner loop we go from left to right to use the computed values at this i iteration.

E.g.4 0-1 Knapsack 2

You have n items each with weight w_i and value v_i. Your bag has max wieght capacity W. Find the max value of items that can be put in the bag.

$$1 \leq n \leq 100, 1 \leq w_i \leq 10^7, 1 \leq v_i \leq 100, 1 \leq W \leq 10^9$$

The difference of this problem with the first 0-1 knapsack is that the range for w_i and W become much larger and O(nW) will get TLE.

Notice that the value of v is rather small this time. Let dp(i, j) denote the minimum weight needed to get total value of j choosing only from the first i items. Similarly, for each item, we either take or do not take.

Our update equation will then be: $dp[i][j] = min(dp[i-1][j], dp[i-1][j-v[i]] + w[i])$

Note that when $i = 0, j > 0; dp[i][j] = inf$, where inf is a very large number.

The final answer is then the maximum j that makes $dp[i][j] \leq W$.

```
const int INF = 99999999;
int dp[MAX_N + 2][MAX_N * MAX_V + 1];

void solve(){
    // memset only works for 0 and 1
    fill(dp[0], dp[0]+MAX_N*MAX_V+1, INF);
    dp[0][0] = 0;

    for(int i=1; i<=n; i++){
        for(int j=1; j<=MAX_N*MAX_V; j++){
            if(j<v[i]){
                dp[i][j]=dp[i-1][j];
            }
            else{
                dp[i][j]=min(dp[i-1][j], dp[i-1][j-v[i]]+w[i]);
            }
        }
    }

    int res=0;
    for(int i=0; i<=MAX_N*MAX_V; i++){
        if(dp[n][i]<=W) res=i;
    }
    printf("%d\n", res);
}
```

Complexity: O($n \sum_i v_i$)

E.g.5 Sum

Given n different intergers a_i, each can be taken at most m_i times. Determine if it's possible to choose among them so that their sum is K.

$1 \leq n \leq 100, 1 \leq a_i, m_i \leq 10^5, 1 \leq K \leq 10^5$

Let dp(i, j) denote the number of ways to choose only from the first i numbers to get sum j .

We have: $dp[i][j] = \sum dp[i-1][j-k*a_i], 0 \leq k \leq m_i, k*a_i \leq j$

```
int n;
int K;
int a[MAX_N];
int m[MAX_N];

bool dp[MAX_N+1][MAX_K+1];

void solve(){
    memset(dp, 0, sizeof(dp));
    for(int i=0; i<=n; i++){
        dp[i][0] = 1;
    }

    for(int i=1; i<=n; i++){
        for(int j=1; j<=K; j++){
            for(int k=0; k<=m[i] && k*a[i]<=j; k++){
                dp[i][j] += dp[i-1][j-k*a[i]];
            }
        }
    }

    if(dp[n][K]) cout<<"YES";
    else cout<<"NO";
}
```

Complexity: O($K\sum_i m_i$)

By redesigning the problem state and formulation, we can actually improve the complexity.

Let dp(i, j) denote the max number of the ith number left when choosing from the first i numbers to get sum j.

We then have the new update equation:

$dp[i][j] =$

m_i, if $dp[i-1][j] \geq 0$

-1, if $j < a_i$ or $dp[i][j-a_i] \leq 0$ (can't take at least one a_i)

$dp[i][j-a_i]-1$, other cases ($dp[i][j-a_i] \geq 1$)

```
int dp[NAX_K + 1];

void solve(){
    memset(dp, -1, sizeof(dp));
    dp[0] = 0;
    for(int i=1; i<=n; i++){
        for(int j=0; j<=K; j++){
            if(dp[j]>=0){
                dp[j] = m[i];
            }
            else if(j<a[i] || dp[j-a[i]]<=0){
                dp[j]=-1;
            }
            else{
                dp[j] = dp[j-a[i]]-1;
            }
        }
    }

    if(dp[K]>=0) cout<<"YES";
    else cout<<"NO";
}
```

Now the complexity is reduced to O(nK).

E.g.6 Longest Increasing Subsequence (LIS)

Given a sequence with n numbers: $a_0, \ldots, a_{n-1}$. Find the length of the LIS of the sequence. (LIS: a subsequence where $a_i < a_j$ for any $i < j$.)

Let dp[i] denote: the length of the longest LIS ending with a_i

We have: $dp[i] = \max(1, dp[j] + 1 | j < i, a_j < a_i)$

```
int n;
int a[MAX_N];
int dp[MAX_N];

void solve(){
    int res=0;
    for(int i=0; i<n; i++){
        dp[i]=1;
        for(int j=0; j<i; j++){
            if(a[j]<a[i])
                dp[i] = max(dp[i], dp[j]+1);
        }
        res = max(res, dp[i]);
    }

    cout<<res;
}
```

Complexity: O(n^2)

Another way to think of the problem is: if the length of the subsequence is fixed, we want the last number of the sequence to be small so that more larger numbers can be appended.

Let dp[i] denote the minimum end number of a LIS with length i , INF if impossible.

```
const int INF = 99999999;
int n;
int a[MAX_N];
int dp[MAX_N];

void solve(){
    fill(dp, dp+n, INF);

    int res=0;
    for(int i=1; i<=n; i++){
        for(int j=0; j<n; j++){
            if(i==1 || dp[i-1]<a[j]){
                dp[i] = min(dp[i], a[j]);
            }
        }
        if(dp[i]<INF){
            res = max(res, i);
        }
    }

    cout<<res;
}
```

The complexity is still O(n^2)

Observation: in this case, the DP array will be **stricty increasing**, each a_j will only be updated at most once. We just need to decide where a_j should be in the DP array, which can is the lower_bound of the array.

```
int dp[MAX_N];

void solve(){
    fill(dp, dp+n, INF);
    for(int i=0; i<n; i++){
        *lower_bound(dp, dp+n, a[i]) = a[i];
    }
    cout<<lower_bound(dp, dp+n, INF)-dp;
}
```

Complexity: O(nlogn)

E.g.7 Split numbers

Split n identical items into less than or equal to m groups. Find the number of ways to split mod M.

$1 \leq m \leq n \leq 1000, 2 \leq M \leq 10000$

Such problem is called the m-splitting number of n.

Let $dp[i][j]$ denote the i splitting number of j.

A naive thought would be to take out k from j first and split the rest $(j-k)$ into $(i-1)$ groups.

$dp[i][j] = \sum_{k=0}^{j} dp[i-1][j-k]$

However, this is wrong because it counted repeatedly. For example, it will count $1+1+2$ and $1+2+1$ as two different ways.

Consider the m splitting number of n, a_i $(\sum_{i=1}^{m} a_i = n)$. If for every i, $a_i > 0$, then $\{a_i - 1\}$ denotes the m splitting of $(n-m)$ (subtracting 1 from each of the m group). If there is $a_i = 0$, then it denotes the $(m-1)$ (at least one group is gone) splitting of n.

So we have: $dp[i][j] = dp[i][j-i] + dp[i-1][j]$

```
int n, m;
int dp[MAX_M + 1][MAX_N + 1];

void solve(){
    dp[0][0] = 1;
    for(int i=1; i<=m; i++){
        for(int j=0; j<=n; j++){
            if(j >= i){
                dp[i][j] = (dp[i-1][j] + dp[i][j-i])%M;
            }
            else{
                // must have a_i = 0
                dp[i][j] = dp[i-1][j];
            }
        }
    }
}
```

Complexity: O(nm)

E.g.8 Take numbers

There are n types of items, the ith type has a_i copies. Items of the same type are counted as the same. How many ways are there to take m items from them? Output the result mod M.

$1 \leq n \leq 1000, 1 \leq m \leq 1000, 1 \leq a_i \leq 1000, 2 \leq M \leq 10000$

Let $dp[i][j]$ denoate the number of ways to take j items from the first i types only.

To take j items from the first i types, we can first take $(j-k)$ items from the first ($i-1$) types and take k items of the ith type:

$dp[i][j] = \sum_{k=0}^{\min(j,a[i])} dp[i-1][j-k]$

The complexity of this is O(nm^2)

A common trick in such summation is to use previously calculated values.

We observe that:

$dp[i][j] = dp[i-1][j] + dp[i-1][j-1] + \ldots + dp[i-1][j-a_i]$

$dp[i][j-1] = dp[i-1][j-1] + dp[i-1][j-2] + \ldots + dp[i-1][j-a_i] + dp[i-1][j-1-a_i]$

Thus we have:

$dp[i][j] = dp[i][j-1] + dp[i-1][j] - dp[i-1][j-1-a_i]$

```
int n, m;
int a[MAX_N+1];

int dp[MAX_N+1][MAX_M+1];

void solve(){
    memset(dp, 0, sizeof(dp));

    //always have one way to take nothing
    for(int i=0; i<=n; i++){
        dp[i][0] = 1;
    }

    for(int i=1; i<=n; i++){
        for(int j=1; j<=m; j++){
            if(j-1-a[i]>=0){
                //add M to avoid negative
                dp[i][j] = (dp[i][j-1]+dp[i-1][j]-dp[i-1][j-1-a[i]]+M)%M;
            }
            else{
                dp[i][j] = (dp[i][j-1]+dp[i-1][j])%M;
            }
        }
    }
    cout<<dp[n][m];
}
```

Complexity: O(nm)

5. Data Structure

5.1 Heap (Priority Queue)

With heap, you can insert and get the smallest element within O($\log n$) time.

Heap is a complete binary tree where the parent nodes' values are always smaller than or equal to the child nodes' value. (The other way round for big root heap.)

Example:

```cpp
#include <queue>
#include <vector>
#include <iostream>
using namespace std;

struct cmp{
    bool operator()(int a, int b){
        return a > b;
    }
};

int main(){
    priority_queue<int> pque;

    pque.push(3);
    pque.push(5);
    pque.push(1);

    while(!pque.empty()){
        cout<<pque.top()<<endl; // 5 3 1
        pque.pop();
    }

    priority_queue<int, vector<int>, greater<int>> que;

    que.push(3);
    que.push(5);
    que.push(1);

    while(!que.empty()){
        cout<<que.top()<<endl; // 1 3 5
        que.pop();
    }

    priority_queue<int, vector<int>, cmp> Q;

    Q.push(3);
    Q.push(5);
    Q.push(1);

    while(!Q.empty()){
        cout<<Q.top()<<endl; // 1 3 5
        Q.pop();
    }

}
```

By default, STL priority queue is a big root heap.

You can reload the < operator or define your own compare function to specify the comparison rules (be careful with the greater and smaller sign).

E.g.1 Expedition (POJ 2431)

You need to drive a car for a distance of L. Initially there are P units of petrol in the car. Travelling a unit distance takes i unit of petrol. The car can't move if there's no petrol left. There are N gas stations on the way, the ith station is A_i unit distance away from the starting point, can provide maximum of B_i unit of petrol. Suppose the car can carry infinite amount of petrol, determine if the car can reach the end point. If so, output the minimum number of times needed to add petrol, else output -1.

$1 \leq N \leq 10^4, 1 \leq L \leq 10^6, 1 \leq P \leq 10^6, 1 \leq A_i < L, 1 \leq B_i \leq 100$

Adding the same amount of petrol sooner or later does not affect the final outcome. Therefore, we can consider passing through a gas station as adding this gas station as a possible option in the queue that can later be chosen. We only add petrol when there is no petrol left to move forward to the next gas station. Every time, we add petrol from the gas station with the maximum petrol from the queue.

```
const int MAXN = 10005;
int L, P, N;
int A[MAX_N], B[MAX_N];
//A: gas station pos
//B: petrol amount

void solve(){
    // add end point as a gas station
    A[N] = L;
    B[N] = 0;
    N++;

    priority_queue<int> que;
    int ans=0, pos=0, tank=P;

    for(int i=0; i<N; i++){
        int d = A[i] - pos; //dist to go

        // keep adding gas until enough to reach next
        while(tank - d < 0){
            if(que.empty()){
                puts("-1");
                return;
            }

            tank += que.top();
            que.pop();
            ans++;
        }

        tank -= d;
        pos = A[i];
        que.push(B[i]);
    }

    printf("%d\n", ans);
}
```

5.2 Binary Search Tree

Example implementation of BST :

```
struct node{
    int val;
    node *lch, *rch;
```

```
};

node *insert(node *p, int x){
    // p: parent node
    if(p == NULL){
        node *q = new node;
        q->val = x;
        q->lch = q->rch = NULL;
        return q;
    }
    else{
        if(x < p->val) p->lch = insert(p->lch, x);
        else p->rch = insert(p->rch, x);
        return p;
    }
}

bool find(node *p, int x){
    if(p==NULL) return false;
    else if(x==p->val) return true;
    else if(x < p->val) return find(p->lch, x);
    else return find(p->rch, x);
}

node* remove(node *p, int x){
    if(p==NULL) return NULL;
    else if(x < p->val) p->lch = remove(p->lch, x);
    else if(x > p->val) p->rch = remove(p->rch, x);
    // remove current node
    else if(p->lch == NULL){
        node *q = p->rch;
        delete p;
        return q;
    }
    else if(p->lch->rch == NULL){
        node *q = p->lch;
        q->rch = p->rch;
        delete p;
        return q;
    }
    else{
        node *q;
        for(q=p->lch; q->rch->rch!=NULL; q=q->rch);
        node *r = q->rch; //predecessor
        q->rch = r->lch;
        r->lch = p->lch;
        r->rch = p->rch;
        delete p;
        return r;
```

```
    }
}
```

Self-balanced BST is more efficient. Examples are AVL, Red-Black, Splay, SBT, etc. (Will include some of them when I get time.)

We can directly use set or map from STL for balanced BST.

```
#include <cstdio>
#include <set>
using namespace std;

int main(){
    set<int> s;

    s.insert(1);
    s.insert(3);

    set<int>::iterator ite;

    ite = s.find(1);
    if(ite==s.end()) puts("not found");
    else puts("found");

    s.erase(3);

    if(s.count(3)!=0) puts("found");
    else puts("found");

    for(ite=s.begin(); ite!=s.end(); ++ite){
        printf("%d\n", *ite);
    }
}
```

```cpp
#include <cstdio>
#include <map>
#include <string>
using namespace std;

int main(){
    map<int, const char*> m;

    m.insert(make_pair(1, "ONE"));
    m.insert(make_pair(10, "TEN"));
    m[100] = "HUNDRED";

    map<int, const char*>::iterator ite;
    ite = m.find(1);
    if(ite==m.end()) puts("not found");
    else puts(ite->second);

    puts(m[10]);

    m.erase(10);

    for(ite=m.begin(); ite!=m.end(); ++ite){
        printf("%d: %s\n", ite->first, ite->second);
    }

    return 0;
}
```

set and map do not allow yu to store repeated elements, you can do so with multiset and multimap.

5.3 Disjoint Set (Union Find)

Disjoint set use tree structures to represent groupings. Initially every node's parent node is itself. If we want to merge two tree, we can just set one root to be the child of the other root. We can compare if two nodes are in the same group by comparing if they have the same root node. Two common tricks that can speed up the operations are path compression: connect nodes directly to the root node instead of passing through a lot of intermediate parent nodes; and merge by rank: set the shorted tree as the child of the higher tree when merging.

(From my own experience, disjoint set with path compression is usually fast enough.)

```
int par[MAX_N]; //parent
int rank[MAX_N]; //height

void init(int n){
    for(int i=0; i<n; i++){
        par[i] = i;
        rank[i] = 0;
    }
}

int find(int x){
    if(par[x]==x)
        return x;
    return par[x] = find(par[x]); //path compression
}

// merge
void unite(int x, int y){
    x = find(x);
    y = find(y);
    if(x==y) return;

    //merge by rank
    if(rank[x]<rank[y]){
        par[x] = y;
    }
    else{
        par[y] = x;
        if(rank[x]==rank[y]) rank[x]++;
    }
}

bool same(int x, int y){
    return find(x)==find(y);
}
```

E.g.1 Food Chain (POJ 1182)

There are N animals, indexed 1, 2, ..., N. Each animal belongs to one of A, B, C group. A eats B, B eats C, C eats A. Input K messages of two types: 1) x and y belong to the same group. 2) x eats y.

However, some messages may be wrong. For example, they provide indices that are out of range or messages in conflict with previous messages. Output the number of wrong messages.

$1 \leq N \leq 5 * 10^4, 0 \leq K \leq 10^5$

For each animal i, we create 3 elements: $i-A, i-B, i-C$ and construct disjoint set with these $3 \times N$ elements. $i-x$ means animal i belongs to group x. Each group in the disjoint set means that all elements in the group either all happen or all not happen.

For each message, we add all possibilities. I.e:

If x and y same group: merge $x-A \& y-A, x-B \& y-B, x-C \& y-C$.

If x eats y: merge $x-A \& y-B, x-B \& y-C, x-C \& y-A$.

```
int N, K;
int T[MAX_K], X[MAX_K], Y[MAX_K];
//T: message type

//disjoint set implementation omitted here
void solve(){
    init(N*3);

    int ans=0;
    for(int i=0; i<K; i++){
        int t = T[i];
        int x = X[i]-1, y= Y[i]-1;

        if(x<0 || x>=N || y<0 || y>=N){
            ans++;
            continue;
        }

        if(t==1){ //type1
            if(same(x, y+N)||same(x, y+2*N)){
                ans++;
            }
            else{
                unite(x, y);
                unite(x+N, y+N);
                unite(x+N*2, y+N*2);
            }
        }
        else{ //type2
            if(same(x, y)||same(x, y+2*N)){
                ans++;
            }
            else{
                unite(x, y+N);
                unite(x+N, y+2*N);
                unite(x+2*N, y);
            }
        }
    }

    printf("%d\n", ans);
}
```

E.g.2 Experimental Charges (2019 SG NOI Prelim Q3)

Particles can have either positive or negative charges. Particles of the same charge will repel each other, and particles of different charges will repel each other. Given the behaviour of some pairs of charges, determine if 2 charges will attract or repel, or cannot be determined from the given information.

This question is a simplified version of the above example. We only need to create two copies of each element $i - pos, i - neg$ to include all possibilities.

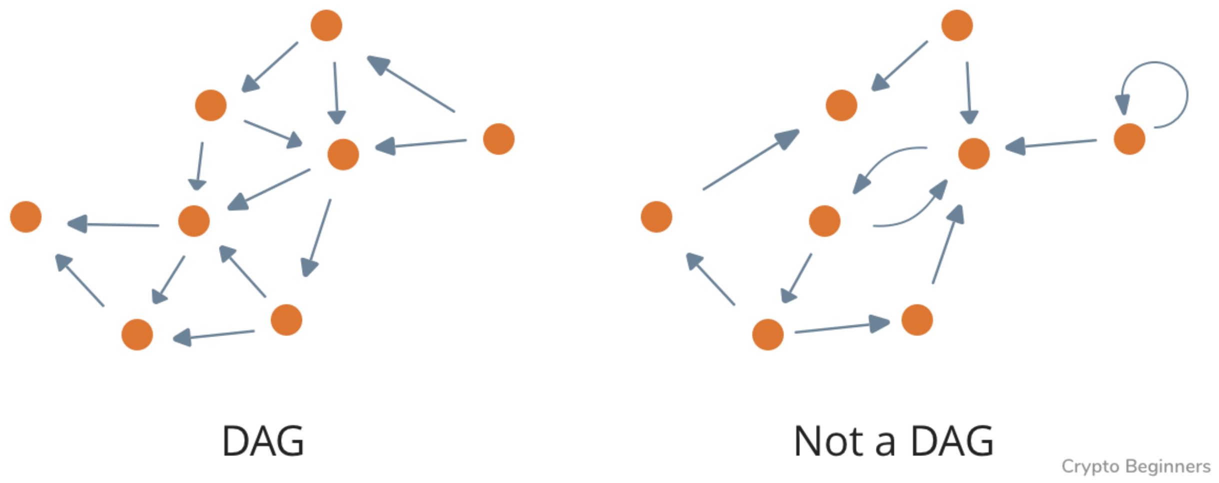
AC codes:

```
#include <iostream>
#include <cstring>
#include <vector>
using namespace std;

const int MAXN = 99999;
int father[MAXN*2];
int N, Q;

int find_father(int n){
    if(father[n]!=n){
        father[n]=find_father(father[n]);
    }
    return father[n];
}

void join(int a, int b){
    int f_a = find_father(a);
    int f_b = find_father(b);
    if(f_a!=f_b){
        father[f_a]=f_b;
    }
}

int main(){
    char cmd;
    int a, b;
    cin>>N>>Q;
    for(int i=1; i<=2*N; i++){
        father[i] = i;
    }

    for(int i=0; i<Q; i++){
        cin>>cmd>>a>>b;
        if(cmd=='Q'){
            int f_a = find_father(a);
            int f_b = find_father(b);
            int f_aN = find_father(a+N);
```

```
            if(f_a==f_b){
                cout<<'R'<<endl;
            }
            else if(f_aN==f_b){
                cout<<'A'<<endl;
            }
            else{
                cout<<'?'<<endl;
            }
        }
        else if(cmd=='R'){
            join(a, b);
            join(a+N, b+N);
        }
        else{
            join(a, b+N);
            join(a+N, b);
        }
    }
}
```

6. Graph

6.1 Representation and Search



A graph consists of vertices/nodes and edges. A graph can be wither directed or undirected (without directions). Directed graphs that do not contain cycles are called DAG (Directed Acyclic Graph).

Graphs can be represented by adjacent matrix or adjacent list. In an adjacent matrix, each number represents whether two nodes are connected or the distantce between two nodes.

In an adjacent list, each list stores all nodes (sometimes together with the distance) connected to one particular node.

E.g.1 Bipartite Graph

Given a graph with n nodes. Colour each node of the graph so that adjacent nodes have differnet colours. Decide if it is possible to only use two different colours to do so. Given that there is no repeated edges or self-cycles. (Such graphs are called bipartite graphs.)

$1 \leq n \leq 10^3$

Since only two different colours are allowed, once we know the colour of one node, we should know the colour of all the adjacent nodes of this node. Hence we just need to iterate through all nodes with DFS and complete the colouring.

```
vector<int> G[MAX_V];
int V;
int color[MAX_V]; // 1 or -1

bool dfs(int v, int c){
    color[v] = c;
    for(int i=0; i<G[v].size(); i++){
        if(color[G[v][i]]==c) return false;
        if(color[G[V][i]]==0 && !dfs(G[v][i], -c)) return false;
    }
    return true;
}

void solve(){
    for(int i=0; i<V; i++){
        if(color[i]==0){
            if(!dfs(i, 1)){
                printf("No\n");
                return;
            }
        }
    }
    printf("Yes\n");
}
```

O(|V|+|E|)

6.2 Shortest Path

1. Bellman-Ford

Suppose the minimum distance from starting point to node i is $d[i]$, we have:

$d[i] = \min\{d[j] + dist(j,i) | e = (j,i) \in E\}$

In other words, we just need to keep checking if passing through an edge can shorten the distance between the two nodes.

```
struct edge{ int from, to, dist; };
edge es[MAX_E];

int d[MAX_V];
int V, E;

void shortest_path(int s){
    for(int i=0; i<V; i++) d[i]=INF;
    d[s] = 0;
    while(true){
        bool update = false;
        for(int i=0; i<E; i++){
            edge e = es[i];
            if(d[e.from]!=INF && d[e.to]>d[e.from]+e.dist){
                d[e.to] = d[e.from] + e.dist;
                update = true;
            }
        }
        if(!update) break;
    }
}
```

The same vertex will be updated at most once, so the while loop will run at most $|V| - 1$ times. Hence the complexity is O($|V| \times |E|$). However, this does not hold if there are negative cycles because we can run over the negative cycles forever and keep reducing the distance. We can use this property to check for the existence of negative cycles:

```
bool find_negative_loop(){
    for(int i=0; i<V; i++) d[i]=INF;
    d[s] = 0;

    for(int i=0; i<V; i++){
        for(int j=0; j<E; j++){
            edge e = es[j];
            if(d[e.to] > d[e.from] + e.dist){
                d[e.to] = d[e.from] + e.dist;

                //if the Vth loop still updates
                if(i==V-1) return true;
            }
        }
    }
    return false;
}
```

The algorithm is easy to implement because it does not even require you to store the graph. You just need to store and iterate through the edges.

An improvement is to only take care of edges that have been updated with a queue. Note that now you need to store the graph in an adjacent list.

```cpp
int SPFA(int start, int target){
    queue<int> Q;
    for(int i=1; i<=n; i++){
        dis[i] = INF;
    }
    dis[start] = 0;
    memset(vis, false, sizeof(vis));
    Q.push(start);
    while(!Q.empty()){
        int u = Q.front();
        Q.pop();
        vis[u] = false; //out of Q
        if(++count[u]>=n){
            //count number of times being pushed
            cout<<"negative cycle!"<<endl;
            return -1;
        }
        for(auto edge: adjlist[u]){
            int v = edge.v;
            int w = edge.w;
            if(dis[v] > dis[u]+w){
                dis[v] = dis[u] + w;
                //push if v not in Q yet
                if(!vis[v]){
                    Q.push(v);
                    vis[v] = true;
                }
            }
        }
    }
    return dis[target];
}
```

Although there is some improvement, the complexity is still O($|V| \times |E|$).

2. Dijkstra

Now let's consider cases without negative cycles. In Bellman-Ford, it is a waste of time to update $d[i]$ from (j, i) if $d[j]$ itself is not yet the shortest distance because in that case the updated $d[i]$ still can't be the shortest distance anyway. Also, it is a waste of time to keep checking those points that are already updated to the shortest distance.

To avoid those cases, every time we choose the closest node from the starting point and save their shortest distance. This is like a greedy strategy.

```
int map[MAX_V][MAX_V], dist[MAX_V],  visited[MAX_V];

void dijkstra(){
    memset(dist, 0x3f, sizeof(dist));
    memset(visited, 0, sizeof(visited));
    int min_dist, min_vertex;

    dist[start] = 0;
    for(int i=0; i<V; i++){
        min_dist = INF;
        for(int j=0; j<V; j++){
            if(dist[j]<min_dist && !visited[j]){
                min_dist = dist[j];
                min_vertex = j;
            }
        }

        visited[min_vertex] = 1;

        for(int k=0; k<V; k++){
            if(map[min_vertex][k] < INF){
                dist[k] = min(dist[k], min_dist+map[min_vertex][k]);
            }
        }
    }
}
```

Complexity is: O(V^2).

Iterating through all nodes to find the closest vertex takes O(V). To improve this, we can use a priority queue.

```cpp
struct edge{ int to, dist; };
typedef pair<int, int> P;
//first: dist, second: vertex index
//p_q sorts the first value by default
int V;
vector<edge> G[MAX_V];
int d[MAX_V];

void dijkstra(int s){
    priority_queue<P, vector<P>, greater<P> > que;
    fill(d, d+V, INF);
    d[s] = 0;
    que.push(P(0, s));

    while(!que.empty()){
        P p = que.top();
        que.pop();
        int v = p.second;
        if(d[v] <= p.first) continue;
        for(int i=0; i<G[v].size(); i++){
            edge e = G[V][i];
            if(d[e.to] > d[v] + e.dist){
                d[e.to] = d[v] + e.dist;
                que.push(P(d[e.to], e.to));
            }
        }
    }
}
```

Compared to O($|E||V|$) of Bellman-Ford, the complexity of Dijkstra is O($|E|\log|V|$). However, note that Dijkstra can't deal with graphs negative edges.

3. Floyd-Warshall

Floyd-Warshall is used to calculate the shoartest distance between any pair of two points in a graph. It is essentially a dynamic programming algorithm.

Suppose $dp[k][i][j]$ represents the shortest distance between node i and j that only passes nodes 0 ~k on the way. Then $dp[0][i][j] = graph[i][j]$ because it cannot pass any other node than i, j themselves. When passing only nodes 0 ~k, the shortest path either passes the node k once or does not pass node k. If it does not pass node k, $dp[k][i][j] = dp[k-1][i][j]$. If it passes node k, $dp[k][i][j] = dp[k-1][i][k] + dp[k-1][k][j]$. Hence we get our transition equation: $dp[k][i][j] = min(dp[k-1][i][j], dp[k-1][i][k] + dp[k-1][k][j])$. This could be implemented with a rolling array: $dp[i][j] = min(dp[i][j], dp[i][k] + dp[k][j])$.

```
int d[MAX_V][MAX_V];
int V;

void floyd_warshall(){
    for(int k=0; k<V; k++){
        for(int i=0; i<V; i++){
            for(int j=0; j<V; j++){
                d[i][j] = min(d[i][j], d[i][k]+d[k][j]);
            }
        }
    }
}
```

The complexity is O(|V|^3). Like Bellman-Ford, it works on graphs with negative edges and to detect negative cycles we just need to check if any $d[i][i] = 0$ after the loops.

4. Path Reconstruction

To reconstruct the shortest path, we need to store the previous node of every node in the shortest path. We need to update it every time make update the shortest distance of a node. For example, in Dijkstra:

```
int prev[MAX_V];

void dijkstra(int s){
    fill(d, d+V, INF);
    fill(used, used+V, false);
    fill(prev, prev+V, -1);
    d[s] = 0;

    while(true){
        int v = -1;
        for(int u=0; u<V; u++){
            if(!used[u] && (v==-1 || d[u]<d[v])) v = u;
        }

        if(v==-1) break;
        used[v] = true;

        for(int u=0; u<V; u++){
            if(d[u] > d[v] + cost[v][u]){
                d[u] = d[v] + cost[v][u];
                prev[u] = v;
            }
        }
    }
}

vector<int> get_path(int t){
    vector<int> path;
    for(; t!=-1; t=prev[t]) path.push_back(t);
    reverse(path.begin(), path.end()); //from s to t
    return path;
}
```

This can be applied on Bellman-Ford and Floyd-Warshall similarly.

6.3 Minimum Spanning Tree

Given an undirected graph, if it has a subgraph where any two nodes in the subgraph are connected and the subgraph is a tree, then it is called a spanning tree. The one with miminum sum of edge costs is the minimum spanning tree (MST).

1. Prim

Prim is similar to Dijkstra, where we keep adding new edges that are closest to the current MST, which is like a greedy approach.

```
int cost[MAX_V][MAX_V]; //adj matrix
int mincost[MAX_V]; //from node to MST
bool used[MAX_V];
int V;

int prim(){
    for(int i=0; i<V; i++){
        mincost[i] = INF;
        used[i] = false;
    }
    //doesn't matter where to start
    mincost[0] = 0;
    int res = 0;

    while(true){
        int v = -1;
        for(int u=0; u<V; u++){
            if(!used[u] && (v==-1 || mincost[u]<mincost[v])) v=u;
        }

        if(v==-1) break;
        used[v] = true;
        res += mincost[v];

        //update dist from MST
        for(int u=0; u<V; u++){
            mincost[u] = min(mincost[u], cost[v][u]);
        }
    }
    return res;
}
```

This takes O(|V|^2) but similar to Dijkstra, can be imporved to O($|E|\log|V|$) by using a priority queue instead of iterating and choosing the closest point.

2. Kruskal

Kruskal is also a greedy approach where we add the shorst edge every time unless it forms cycles. To determine whethere the new edge forms cycles with already added edges, we can use disjoint set.

```
struct edge{ int u, v, cost; };

bool comp(const edge& e1, const edge& e2){
    return e1.cost < e2.cost;
}

edges es[MAX_E];
int V, E;

int kruskal(){
    sort(es, es+E, comp);
    init_union_find(V);
    int res = 0;
    for(int i=0; i<E; i++){
        edge e = es[i];
        if(!same_father(e.u, e.v)){
            unite(e.u, e.v);
            res += e.cost;
        }
    }
    return res;
}
```

The implementation of disjont set is omitted here for simplicity. The time complexity is O($|E|\log|V|$).

3. Applications

E.g.1 Roadblocks (POJ 3255)

A district has R roads and N crossings. All roads are bidirectional. Find the second shortest path length from crossing number 1 to number N. The same road can be passed many times.

$$1 \leq N \leq 5000, 1 \leq R \leq 10000$$

The second shortest path to some point v is either the shortest path to another point u plus the edge $u \rightarrow v$, or the second shortest path to u plus the edge $u \rightarrow v$. Hence, for every node, we need to store not only the shortest distance, but also the second shortest distance.

```
int N, R;
vector<edge> G[MAX_N];

int dist[MAX_N];
int dist2[MAX_N];

void solve(){
    priority_queue<P, vector<P>, greater<P> > que;
    fill(dist, dist+N, INF);
    fill(dist2, dist2+N, INF);
    dist[0] = 0;
    que.push(P(0, 0));

    while(!que.empty()){
        P p = que.top();
        que.pop();
        int v = p.second, d = p.first;
        if(d > dist2[v]) continue;
        for(int i=0; i<G[v].size(); i++){
            edge &e = G[v][i];
            int d2 = d + e.cost;
            //d2 may be shortest or second shortest
            if(d2 < dist[e.to]){
                swap(dist[e.to], d2);
                que.push(P(dist[e.to], e.to));
            }
            if(d2 < dist2[e.to] && d2 > dist[e.to]){
                dist2[e.to] = d2;
                que.push(P(dist2[e.to], e.to));
            }
        }
    }
    printf("%d\n", dist2[N-1]);
}
```

E.g.2 Conscription (POJ 3723)

We need to conscript N women and M men. Conscripting each person costs $10000. But if they are familiar with the conscripted people, the cost can be lower. Given the closeness (1~9999) of some people (R relationships), the cost of conscripting a new person is 10000 - (max closeness with a person among the conscripted people). Find the order of conscription that makes the total cost of conscription the lowest.

$$1 \leq N, M \leq 10000, 0 \leq R \leq 50000, 0 < d < 10000$$

[input (x, y, d): closeness between woman x and man y is d]

First of all, let's think of this undirected graph where the people are nodes and their closeness are edges. The graph cannot contain any cycles otherwise the order will have conflicts (i.e. first person cannot be the last conscripted at the same time). So it will actually be a tree. Since not all people are connected, the trees will form a forest. The problem now becomes to find the maximum edge cost forest, which can be solved by truning all edges costs to negative sign and find miminum spanning trees.

```
int N, M, R;
int x[MAX_R], y[MAX_R], d[MAX_R];

void solve(){
    V = N+M;
    E = R;
    for(int i=0; i<R; i++){
        es[i] = (edge){x[i], N+y[i], -d[i]};
    }
    printf("%d\n", 10000*(N+M)+kruskal());
}

```

E.g.3 Layout (POJ 3169)

FJ has N (2 <= N <= 1,000) cows numbered 1..N standing along a straight line waiting for feed. The cows are standing in the same order as they are numbered, and since they can be rather pushy, it is possible that two or more cows can line up at exactly the same location (that is, if we think of each cow as being located at some coordinate on a number line, then it is possible for two or more cows to share the same coordinate).

Some cows like each other and want to be within a certain distance of each other in line. Some really dislike each other and want to be separated by at least a certain distance. A list of ML (1 <= ML <= 10,000) constraints describes which cows like each other and the maximum distance by which they may be separated; a subsequent list of MD constraints (1 <= MD <= 10,000) tells which cows dislike each other and the minimum distance by which they must be separated.

Your job is to compute, if possible, the maximum possible distance between cow 1 and cow N that satisfies the distance constraints.

Analysis: First of all, the cows are ordered so $d[i] \leq d[i+1]$. For cows like each other, there is maximum distance constraint: $d[BL] \leq d[AL] + DL$, for cows dislike each other, there is minimum distance constraint: $d[BD] \geq d[AD] + DD$. The problem is then to find max value of $d[N] - d[1]$ while satisfying the above constraints. This is a linear programming algorithm and there are solutions like simplex algorithm. But we will use a simpler method here.

Actually, the shortest path problem can be expressed as a linear programming problem as well. If we denoate the shortest distance from s to v as $d(v)$, then for edge $e = (v, u)$ with cost w, we have $d(v) + w \geq d(u)$. Then, for d satisfying all constraints, the max value of $d(v) - d(s)$ is the shorstest distance from s to v. Note that it is the max value not min value (min value can be 0, i.e. shorter than the actual edge costs).

In this way, each constraint in the original problem can be thought of as an edge in the graph, and then we just need to find the shortest path. $d[i] \leq d[i+1]$ becomes $d[i+1] + 0 \geq d[i]$, so an edge from $i+1$ to i with weight 0; $d[AL] + DL \geq d[BL]$, so an edge from AL to BL weight DL; $d[BD] - DD \geq d[AD]$, so an edge from BD to AD with weight $(-DD)$. To find the max value of $d[N] - d[1]$, we find the shortest distance between node 1 and N. Since there are negative edges in the graph, we use Bellman-Ford instead of Dijkstra.

```
int N, ML, MD;
int AL[MAX_ML], BL[MAX_ML], DL[MAX_ML];
int AD[MAX_MD], BD[MAX_MD], DD[MAX_MD];

int d[MAX_N];

void solve(){
    fill(d, d+N, INF);
    d[0] = 0;

    // Bellman-Ford
    // run N iterations to detect neg cycles
    for(int k=0; k<N; k++){
        // i+1 to i: 0
        for(int i=0; i+1<N; i++){
            if(d[i+1] < INF) d[i]=min(d[i], d[i+1]);
        }
        // AL to BL: DL
        for(int i=0; i<ML; i++){
            if(d[AL[i]-1] < INF){
                d[BL[i]-1] = min(d[BL[i]-1], d[AL[i]-1]+DL[i]);
            }
        }
        // BD to AD: -DD
        for(int i=0; i<MD; i++){
            if(d[BD[i]-1] < INF){
                d[AD[i]-1] = min(d[AD[i]-1], d[BD[i]-1]-DD[i]);
            }
        }
    }

    int res = d[N-1];
    if(d[0] < 0){
        // has neg cycles, no solution
        res = -1;
    }
    else if(res==INF){
        // res can be INF large
        res = -2;
    }
    printf("%d\n", res);
}
```

Complexity: O($N(N+ML+MD)$)