Question 7: Assignment 3: CS 663, Fall 2024

Amitesh Shekhar
IIT Bombay
22b0014@iitb.ac.in

Anupam Rawat
IIT Bombay
22b3982@iitb.ac.in

Toshan Achintya Golla
IIT Bombay
22b2234@iitb.ac.in

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1. Consider the partial differential equation $\frac{\partial I}{\partial t} = c\left(\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}\right)$ where c is some non-negative constant. This is the isotropic heat equation. Using the differentiation theorem in Fourier transforms, prove that running this PDE on an image I is equivalent to convolving it with a Gaussian of zero mean and appropriate standard deviation. What is the value of the standard deviation? You will also need to use the result that the Fourier transform of a Gaussian is also a Gaussian. [15 points] **[12+3=15 points]**

Soln:

The heat equation is given as:

$$\frac{\partial I}{\partial t} = c\left(\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2}\right)$$

We can consider the image to be a function of x, y and t i.e., I(x, y, t), something like a video clip. Let's assume the 2D Fourier Transform of the image to be I_F(u, v, t). We can ignore t, because it doesn't come in the equation of the Fourier Transform. The original equation therefore becomes:

$$\frac{\partial I_F(u,v,t)}{\partial t} = c\left(\mathcal{F}\left(\frac{\partial^2 I(x,y,t)}{\partial x^2}\right) + \mathcal{F}\left(\frac{\partial^2 I(x,y,t)}{\partial y^2}\right)\right)$$

Simplifying the Right Hand Side of the Equation by using Differentiation Theorem, we get:

$$\mathcal{F}\left(\frac{\partial^2 I(x,y,t)}{\partial x^2}\right) = -(2\pi u)^2 I_F(u,v,t) = -4\pi^2 u^2 I_F(u,v,t)$$

$$\mathcal{F}\left(\frac{\partial^2 I(x,y,t)}{\partial y^2}\right) = -(2\pi v)^2 I_F(u,v,t) = -4\pi^2 v^2 I_F(u,v,t)$$

Thus the modified heat equation in context of image becomes:

$$\frac{\partial I_F(u,v,t)}{\partial t} = -4\pi^2 c(u^2+v^2)\cdot I_F(u,v,t)$$

This PDE can be expressed in the solution form as below:

$$I_F(u,v,t) = I_F(u,v,0)\cdot \exp\{-4\pi^2 ct(u^2+v^2)\}$$

We can use another variable H(u, v, t) in place of the exponential term.

$$I_F(u,v,t) = I_F(u,v,0)\cdot H(u,v,t)$$

This can be expressed in the time domain as:

$$f(x,y,t) = f(x,y,0)\star h(x,y,t)$$

That is the solution in domain at time t is the convolution of solution at time (t = 0) and h(x, y, t), which is the inverse Fourier transform of H(u, v, t). Therefore, h(x, y, t) can be computed as:

$$\mathcal{F}^{-1}(H(u,v,t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(u,v,t)\exp\{2\pi j(ux+vy)\}\,du\,dv$$

$$\mathcal{F}^{-1}(H(u,v,t)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-4\pi^2 ct(u^2+v^2)\}\exp\{2\pi j(ux+vy)\}\,du\,dv$$

Separating the variables.
(From here on, we'll be taking inspiration from the proof of "Fourier of a Gaussian yields a Gaussian")

$$\mathcal{F}^{-1}(H(u,v,t)) = \left(\int_{-\infty}^{\infty} \exp\{-4\pi^2 ctu^2\} \exp\{2\pi jux\}\, du\right) \left(\int_{-\infty}^{\infty} \exp\{-4\pi^2 ctv^2\} \exp\{2\pi jvy\}\, dv\right)$$

$$\mathcal{F}^{-1}(H(u,v,t)) = \left(\int_{-\infty}^{\infty} \exp\left\{-4\pi^2 ct \left(u^2 - \frac{jux}{2\pi ct}\right)\right\} du\right) \left(\int_{-\infty}^{\infty} \exp\left\{-4\pi^2 ct \left(v^2 - \frac{jvy}{2\pi ct}\right)\right\} dv\right)$$

Completing the squares, we get:

$$\mathcal{F}^{-1}(H(u,v,t)) = \left(\int_{-\infty}^{\infty} \exp\left\{-4\pi^2 ct \left(\left(u - \frac{jx}{4\pi ct}\right)^2 - \left(\frac{jx}{4\pi ct}\right)^2\right)\right\} du\right) \times \left(\int_{-\infty}^{\infty} \exp\left\{-4\pi^2 ct \left(\left(v - \frac{jy}{4\pi ct}\right)^2 - \left(\frac{jy}{4\pi ct}\right)^2\right)\right\} dv\right)$$

$$\mathcal{F}^{-1}(H(u,v,t)) = \left(\int_{-\infty}^{\infty} \exp\left\{-4\pi^2 ct \cdot \left(u - \frac{jx}{4\pi ct}\right)^2 + 4\pi^2 ct \cdot \left(\frac{jx}{4\pi ct}\right)^2\right\} du\right) \times \left(\int_{-\infty}^{\infty} \exp\left\{-4\pi^2 ct \cdot \left(v - \frac{jy}{4\pi ct}\right)^2 + 4\pi^2 ct \cdot \left(\frac{jy}{4\pi ct}\right)^2\right\} dv\right)$$

$$\mathcal{F}^{-1}(H(u,v,t)) = \left(\int_{-\infty}^{\infty} \exp\left\{-\left(2\pi\sqrt{ct} \cdot u - 2\pi\sqrt{ct} \cdot \frac{jx}{4\pi ct}\right)^2 + \cdot \left(2\pi\sqrt{ct} \cdot \frac{jx}{4\pi ct}\right)^2\right\} du\right) \times \left(\int_{-\infty}^{\infty} \exp\left\{-\left(2\pi\sqrt{ct}v - 2\pi\sqrt{ct} \cdot \frac{jy}{4\pi ct}\right)^2 + \left(2\pi\sqrt{ct} \cdot \frac{jy}{4\pi ct}\right)^2\right\} dv\right)$$

$$\mathcal{F}^{-1}(H(u,v,t)) = \left(\int_{-\infty}^{\infty} \exp\left\{-\left(2\pi\sqrt{ct} \cdot u - \frac{jx}{2\sqrt{ct}}\right)^2 + \left(\frac{jx}{2\sqrt{ct}}\right)^2\right\} du\right) \times \left(\int_{-\infty}^{\infty} \exp\left\{-\left(2\pi\sqrt{ct} \cdot v - \frac{jy}{2\sqrt{ct}}\right)^2 + \left(\frac{jy}{2\sqrt{ct}}\right)^2\right\} dv\right)$$

$$\mathcal{F}^{-1}(H(u,v,t)) = \exp\left\{\left(\frac{jx}{2\sqrt{ct}}\right)^2\right\} \times \left(\int_{-\infty}^{\infty} \exp\left\{-\left(2\pi\sqrt{ct} \cdot u - \frac{jx}{2\sqrt{ct}}\right)^2\right\} du\right) \times \exp\left\{\left(\frac{jy}{2\sqrt{ct}}\right)^2\right\} \times \left(\int_{-\infty}^{\infty} \exp\left\{-\left(2\sqrt{ct} \cdot v - \frac{jy}{2\sqrt{ct}}\right)^2\right\} dv\right)$$

$$\mathcal{F}^{-1}(H(u,v,t)) = \exp\left\{-\frac{x^2 + y^2}{4ct}\right\} \times \left(\int_{-\infty}^{\infty} \exp\left\{-\left(2\pi\sqrt{ct} \cdot u - \frac{jx}{2\sqrt{ct}}\right)^2\right\} du\right) \times \left(\int_{-\infty}^{\infty} \exp\left\{-\left(2\pi\sqrt{ct} \cdot v - \frac{jy}{2\sqrt{ct}}\right)^2\right\} dv\right)$$

Substituting m $= 2\pi u\sqrt{ct} - \frac{jx}{2\sqrt{ct}}$ and n $= 2\pi v\sqrt{ct} - \frac{jy}{2\sqrt{ct}}$. We therefore have,

$$dm = 2\pi\sqrt{ct}\, du$$

$$dn = 2\pi\sqrt{ct}\, dv$$

Thus the above equation for computing h(x, y, t) becomes

$$\mathcal{F}^{-1}(H(u,v,t)) = \exp\left\{-\frac{x^2 + y^2}{4ct}\right\} \times \left(\int_{-\infty}^{\infty} \frac{\exp(-m^2)}{2\pi\sqrt{ct}} dm\right) \times \left(\int_{-\infty}^{\infty} \frac{\exp(-n^2)}{2\pi\sqrt{ct}} dn\right)$$

Since, we know that $\int_{-\infty}^{\infty} exp(-x^2)dx = \sqrt{\pi}$, we can resolve the above equation into:

$$\mathcal{F}^{-1}(H(u,v,t)) = \exp\left\{-\frac{x^2+y^2}{4ct}\right\} \times \frac{\sqrt{\pi}}{2\pi\sqrt{ct}} \times \frac{\sqrt{\pi}}{2\pi\sqrt{ct}} = \frac{1}{4\pi ct} \cdot \exp\left\{-\frac{x^2+y^2}{4ct}\right\} = h(x,y,t)$$

We know that equation of a standard gaussian in 2D is given as,

$$g(x,y) = \frac{1}{2\pi\sigma^2} \cdot \exp\left\{-\frac{(x-\mu_x)^2+(y-\mu_y)^2}{2\sigma^2}\right\}$$

Taking mean as zero, it reduces to,

$$g(x,y) = \frac{1}{2\pi\sigma^2} \cdot \exp\left\{-\frac{x^2+y^2}{2\sigma^2}\right\}$$

Hence proved that running this PDE on an image I is equivalent to convolving the value of I at time t = 0 with a Gaussian of zero mean and some standard deviation.

Comparing the above equation of gaussian in 2D with the obtained form for h(x, y, t), we can say that standard deviation, $\sigma = \sqrt{2ct}$ increases over time. Eventually, as $t \to \infty$, the image becomes a constant value, similar to the way heat distribution reaches a constant value over time.