Question 4: Assignment 3: CS 663, Fall 2024

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1. Consider a 201 × 201 image whose pixels are all black except for the central column (i.e. column index 101 beginning from 1 to 201) in which all pixels have the value 255. Derive the Fourier transform of this image analytically, and also plot the logarithm of its Fourier magnitude using fft2 and fftshift in MATLAB. Use appropriate colorbars. [8+2=10 points]

Soln:

The image, where all pixel values are black except for the central column i.e. 101st column, whose values are all white.

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 255 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 255 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 255 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 255 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

The Fourier Transform of the image is given as:

$$F_d(u,v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{y=0}^{W_1-1} \sum_{x=0}^{W_2-1} f(x,y) \exp\left(-j2\pi\left(\frac{ux}{W_1} + \frac{vy}{W_2}\right)\right) \tag{1}$$

Substituting the image height and width, we get:

$$F_d(u,v) = \frac{1}{201} \sum_{y=0}^{200} \sum_{x=0}^{200} f(x,y) \exp\left(-j2\pi\left(\frac{ux}{201} + \frac{vy}{201}\right)\right) \tag{2}$$

Assuming, x represents the columns and y represents the rows. Now, if we're looping through the y^{th} row:

$$\sum_{x=0}^{200} f(x,y) \exp\left(-j2\pi\left(\frac{u\cdot x}{201} + \frac{v\cdot y}{201}\right)\right) = \sum_{x=0}^{99} 0 + 255 \cdot \exp\left(-j2\pi\left(\frac{u\cdot 100}{201} + \frac{v\cdot y}{201}\right)\right) + \sum_{x=101}^{200} 0 \tag{3}$$

Substituting the value of equation (3) in equation (2), we get:

$$F_d(u,v) = \frac{1}{201} \sum_{y=0}^{200} 255 \cdot \exp\left(-j2\pi\left(\frac{u\cdot 100}{201} + \frac{v\cdot y}{201}\right)\right) = \frac{255}{201} \exp\left(-j2\pi\left(\frac{u\cdot 100}{201}\right)\right) \sum_{y=0}^{200} \exp\left(-j2\pi\left(\frac{v\cdot y}{201}\right)\right) \tag{4}$$

The summation over the exponentials is like the sum of a Geometric Progression, where

$$a_0 = \exp\left(-j2\pi\left(\frac{v\cdot 0}{201}\right)\right) = 1 \qquad r = \exp\left(-j2\pi\left(\frac{v}{201}\right)\right) \tag{5}$$

And since $r < 1$, except for $r = 1$ at $v = 0$, the sum is given by,

$$\begin{aligned} S_n &= \frac{1 - \exp\left(-j2\pi\left(\frac{vn}{201}\right)\right)}{1 - \exp\left(-j2\pi\left(\frac{v}{201}\right)\right)}, \quad \text{for} \quad r \neq 1 \\ S_n &= a_0 \cdot n = n, \quad \text{for} \quad r = 1 \end{aligned} \tag{6}$$

Thus, the discrete Fourier transform of the image becomes:

$$\begin{aligned} F_d(u,v) &= \frac{255}{201} \exp\left(-j2\pi\left(\frac{u \cdot 100}{201}\right)\right) \frac{1 - \exp\left(-j2\pi\left(\frac{v}{1}\right)\right)}{1 - \exp\left(-j2\pi\left(\frac{v}{201}\right)\right)}, \quad \text{for} \quad v \neq 0 \\ F_d(u,v) &= \frac{255}{201} \exp\left(-j2\pi\left(\frac{u \cdot 101}{201}\right)\right) \cdot 201 = 255 \cdot \exp\left(-j2\pi\left(\frac{u \cdot 101}{201}\right)\right), \quad \text{for} \quad v = 0 \end{aligned} \tag{7}$$

Plot of the logarithm of the Fourier Magnitude of this image is given by:

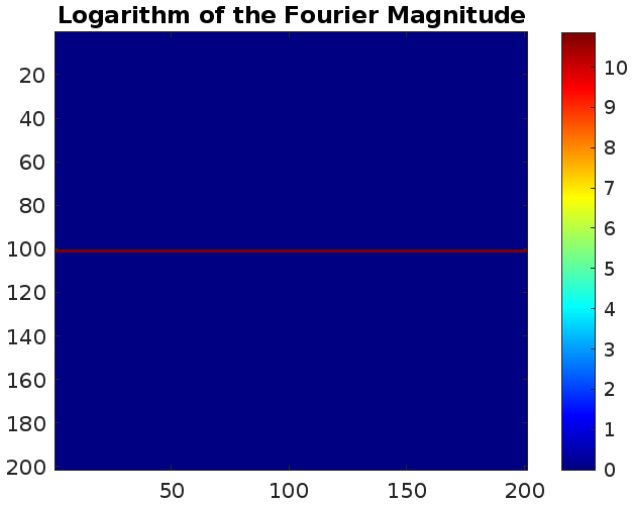

Figure 1: FT of given image

The **MATLAB code** for the Fourier transform of above image is attached as well as mentioned in the report below.

```
image = zeros(201, 201);
image(:, 101) = 255;

F = fft2(image);

F_shifted = fftshift(F);

log_magnitude = log(abs(F_shifted) + 1);

figure;
imagesc(log_magnitude);
colormap('jet');
colorbar;
title('Logarithm of the Fourier Magnitude');
```