Assignment 2: CS 663, Fall 2024

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1. Prove that the Laplacian operator is rotationally invariant. For this consider a rotation of the coordinate system from (x, y) to $u = x\cos\theta - y\sin\theta, v = x\sin\theta + y\cos\theta$, and show that $I_{xx} + I_{yy} = I_{uu} + I_{vv}$ for any image I. Prove that the second directional derivative of an image $I(x, y)$ in the direction of its gradient vector (*i.e.* in the direction $(\frac{I_x}{\sqrt{I_x^2 + I_y^2}}, \frac{I_y}{\sqrt{I_x^2 + I_y^2}}))$ is given by $\frac{I_x^2 I_{xx} + 2I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2}$. Note that $I_x = \frac{\partial I}{\partial x}, I_{xx} = \frac{\partial^2 I}{\partial x^2}$. Using this information, write down the expression for the second directional derivative of $I(x, y)$ in the direction **perpendicular** to its gradient vector and justify your answer. Note that the first directional derivative of $I(x, y)$ in a direction v is given by $\nabla I(x, y) \cdot v$. [6+6+3 = 15 points]

Soln:

(a)Laplacian operator is rotationally invariant.

Below are the mathematical equations to convert from (x, y) to (u, v) and vice-versa.

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u\cos\theta + v\sin\theta \\ -u\sin\theta + v\cos\theta \end{bmatrix}$$

For the Laplacian's Operator to be rotationally invariant, $I_{xx} + I_{yy} = I_{uu} + I_{vv}$

$$I_{uu} + I_{vv} = (\frac{\partial^2 I}{\partial u^2} + \frac{\partial^2 I}{\partial v^2}) = (\frac{\partial}{\partial u} \cdot (\frac{\partial I}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial I}{\partial y} \cdot \frac{\partial y}{\partial u})) + (\frac{\partial}{\partial v} \cdot (\frac{\partial I}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial I}{\partial y} \cdot \frac{\partial y}{\partial v}))$$

$$= \left(\frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial u}\right) \cdot \left(\frac{\partial I}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial I}{\partial y} \cdot \frac{\partial y}{\partial u}\right) + \left(\frac{\partial}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial}{\partial y} \cdot \frac{\partial y}{\partial v}\right) \cdot \left(\frac{\partial I}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial I}{\partial y} \cdot \frac{\partial y}{\partial v}\right)$$

$$= \frac{\partial^2 I}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial I}{\partial x}\right) \cdot \frac{\partial^2 x}{\partial u^2} + \left(\frac{\partial}{\partial x} \cdot \frac{\partial I}{\partial y}\right)\left(\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial u}\right) + \left(\frac{\partial}{\partial y} \cdot \frac{\partial I}{\partial x}\right)\left(\frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial u}\right) + \frac{\partial^2 I}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial I}{\partial y}\right) \cdot \frac{\partial^2 y}{\partial u^2}$$
$$+ \frac{\partial^2 I}{\partial x^2}\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial I}{\partial x}\right) \cdot \frac{\partial^2 x}{\partial v^2} + \left(\frac{\partial}{\partial x} \cdot \frac{\partial I}{\partial y}\right)\left(\frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial v}\right) + \left(\frac{\partial}{\partial y} \cdot \frac{\partial I}{\partial x}\right)\left(\frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial v}\right) + \frac{\partial^2 I}{\partial y^2}\left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial I}{\partial y}\right) \cdot \frac{\partial^2 y}{\partial v^2}$$

Substituting the values, we get

$$= I_{xx}\left(\cos\theta\right)^2 + I_x \cdot 0 + I_{xy}\left(-\cos\theta \cdot \sin\theta\right) + I_{yx}\left(-\cos\theta \cdot \sin\theta\right) + I_{yy}\left(-\sin\theta\right)^2 + I_y \cdot 0$$
$$+ I_{xx}\left(\sin\theta\right)^2 + I_x \cdot 0 + I_{xy}\left(\cos\theta \cdot \sin\theta\right) + I_{yx}\left(\cos\theta \cdot \sin\theta\right) + I_{yy}\left(\cos\theta\right)^2 + I_y \cdot 0$$
$$= I_{xx}(\cos^2\theta + \sin^2\theta) + I_{yy}(\cos^2\theta + \sin^2\theta) = I_{xx} + I_{yy}$$

$$I_{uu} + I_{vv} = I_{xx} + I_{yy}$$

Hence, the Laplacian Operator is rotationally invariant.

(b) Compute directional second derivative Prove that the second directional derivative of an image $I(x, y)$ in the direction of its gradient is given by:

$$D^2_{\mathbf{v}} I = \frac{I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2}$$

The unit vector in the direction of the gradient is given by:

$$\mathbf{v} = \left(\frac{I_x}{\sqrt{I_x^2 + I_y^2}}, \frac{I_y}{\sqrt{I_x^2 + I_y^2}} \right)$$

The first directional derivative in the direction of the gradient vector can be obtained by:

$$D_{\mathbf{v}} I = \nabla I \cdot \mathbf{v} = (I_x, I_y) \cdot \left(\frac{I_x}{\sqrt{I_x^2 + I_y^2}}, \frac{I_y}{\sqrt{I_x^2 + I_y^2}} \right) = \frac{I_x^2 + I_y^2}{\sqrt{I_x^2 + I_y^2}} = \sqrt{I_x^2 + I_y^2}$$

For the second directional derivative, we apply the gradient operator again along the direction of $\mathbf{v}$:

$$D^2_{\mathbf{v}} I = \mathbf{v}^T \cdot H(I) \cdot \mathbf{v}$$

$$H(I) = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix}$$

where, H(I) is the Hessian Matrix of Partial Derivatives.

Substituting $\mathbf{v}$ into the above formula, we obtain,

$$D^2_{\mathbf{v}} I = \left(\frac{I_x}{\sqrt{I_x^2 + I_y^2}}, \frac{I_y}{\sqrt{I_x^2 + I_y^2}} \right)^T \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix} \left(\frac{I_x}{\sqrt{I_x^2 + I_y^2}}, \frac{I_y}{\sqrt{I_x^2 + I_y^2}} \right)$$

$$H(I)\mathbf{v} = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix} \begin{pmatrix} \frac{I_x}{\sqrt{I_x^2 + I_y^2}} \\ \frac{I_y}{\sqrt{I_x^2 + I_y^2}} \end{pmatrix} = \begin{pmatrix} \frac{I_x I_{xx} + I_y I_{xy}}{\sqrt{I_x^2 + I_y^2}} \\ \frac{I_x I_{xy} + I_y I_{yy}}{\sqrt{I_x^2 + I_y^2}} \end{pmatrix}$$

$$D^2_{\mathbf{v}} I = \left(\frac{I_x}{\sqrt{I_x^2 + I_y^2}}, \frac{I_y}{\sqrt{I_x^2 + I_y^2}} \right) \cdot \begin{pmatrix} \frac{I_x I_{xx} + I_y I_{xy}}{\sqrt{I_x^2 + I_y^2}} \\ \frac{I_x I_{xy} + I_y I_{yy}}{\sqrt{I_x^2 + I_y^2}} \end{pmatrix}$$

This reduces to:

$$D^2_{\mathbf{v}} I = \frac{I_x(I_x I_{xx} + I_y I_{xy}) + I_y(I_x I_{xy} + I_y I_{yy})}{I_x^2 + I_y^2}$$

Expanding the terms in the numerator, we get:

$$D^2_{\mathbf{v}} I = \frac{I_x^2 I_{xx} + 2 I_x I_y I_{xy} + I_y^2 I_{yy}}{I_x^2 + I_y^2}$$

Hence, proved.

(c) Compute Second Derivative Directional Vector in a Perpendicular Direction to the Gradient Vector
The vector, perpendicular to the Gradient Vector is given by,

$$\mathbf{v}_\perp = \left(\frac{-I_y}{\sqrt{I_x^2 + I_y^2}}, \frac{I_x}{\sqrt{I_x^2 + I_y^2}} \right)$$

The second directional derivative in the direction of $\mathbf{v}_\perp$:

$$D^2_{\mathbf{v}_\perp} I = \mathbf{v}_\perp^T H(I) \mathbf{v}_\perp$$

Substituting $\mathbf{v}$ into the formula, we obtain:

$$D^2_{\mathbf{v}_\perp} I = \left(\frac{-I_y}{\sqrt{I_x^2 + I_y^2}}, \frac{I_x}{\sqrt{I_x^2 + I_y^2}} \right)^T \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix} \left(\frac{-I_y}{\sqrt{I_x^2 + I_y^2}}, \frac{I_x}{\sqrt{I_x^2 + I_y^2}} \right)$$

$$H(I)\mathbf{v}_\perp = \begin{pmatrix} I_{xx} & I_{xy} \\ I_{xy} & I_{yy} \end{pmatrix} \begin{pmatrix} \frac{-I_y}{\sqrt{I_x^2+I_y^2}} \\ \frac{I_x}{\sqrt{I_x^2+I_y^2}} \end{pmatrix} = \begin{pmatrix} \frac{-I_y I_{xx} + I_x I_{xy}}{\sqrt{I_x^2+I_y^2}} \\ \frac{-I_y I_{xy} + I_x I_{yy}}{\sqrt{I_x^2+I_y^2}} \end{pmatrix}$$

$$D^2_{\mathbf{v}_\perp} I = \left(\frac{-I_y}{\sqrt{I_x^2 + I_y^2}}, \frac{I_x}{\sqrt{I_x^2 + I_y^2}} \right) \cdot \begin{pmatrix} \frac{-I_y I_{xx} + I_x I_{xy}}{\sqrt{I_x^2+I_y^2}} \\ \frac{-I_y I_{xy} + I_x I_{yy}}{\sqrt{I_x^2+I_y^2}} \end{pmatrix}$$

Thus, the second derivative along the direction orthogonal to the gradient is given by:

$$D^2_{\mathbf{v}_\perp} I = \frac{I_y^2 I_{xx} - 2 I_x I_y I_{xy} + I_x^2 I_{yy}}{I_x^2 + I_y^2}$$

The second directional derivative provides useful information about ridges or valleys in the image.