Q5) Let $G(k)$ denote the gaussian kernel with mean 0 and standard deviation σ. Then for a given image $I(x)$, if we apply the filter $G(k)$, the output $J(x)$ would be the convolution $(G * I)(x)$.

$$J(x) = (G * I)(x)$$

$$J(x) = \sum_{k=-\infty}^{+\infty} I(x-k)\, G(k)$$

$$= \sum_{k=-\infty}^{+\infty} (cx - ck + d)\, G(k)$$

$$= \sum_{k=-\infty}^{+\infty} cx\, G(k) - \sum_{k=-\infty}^{+\infty} ck\, G(k) + d \sum_{k=-\infty}^{+\infty} G(k)$$

$$= cx\left(\sum_{k=-\infty}^{+\infty} G(k)\right) - c\left(\sum_{k=-\infty}^{+\infty} k\, G(k)\right) + d\left(\sum_{k=-\infty}^{+\infty} G(k)\right)$$

Note that for any PMF, $\sum_{k=-\infty}^{+\infty} G(k) = 1$.

Also note that $\sum_{k=-\infty}^{+\infty} k\, G(k) = 0$ because $G(k) = +G(-k)$

So, we can say that $J(x) = cx(1) - 0 + d(1)$

$$\boxed{J(x) = cx + d}$$

Now, for a Bilateral Filter, we know that

$$BF[I]_p = \frac{1}{W_p} \sum_{q \in S} G_{\sigma_s}(\|p-q\|)\, G_{\sigma_r}(I_p - I_q)\, I_q$$

where $W_p = \sum_{q \in S} G_{\sigma_s}(\|p-q\|)\, G_{\sigma_r}(I_p - I_q)$

Note that the image is of infinite extent. So q goes from $-\infty$ to $+\infty$.

Let $q = k$ and $p = x$ for convenience.

So, k goes from $-\infty$ to $+\infty$

$$BF[I]_x = \frac{\sum_{k=-\infty}^{+\infty} I(k)\, e^{-\frac{(x-k)^2}{2\sigma_s^2}}\, e^{-\frac{(I(x)-I(k))^2}{2\sigma_r^2}}}{\sum_{k=-\infty}^{+\infty} e^{-\frac{(x-k)^2}{2\sigma_s^2}}\, e^{-\frac{(I(x)-I(k))^2}{2\sigma_r^2}}}$$

$$= \frac{\sum_{k=-\infty}^{+\infty} (ck+d)\, e^{-\frac{(x-k)^2}{2\sigma_s^2}}\, e^{-\frac{c^2(x-k)^2}{2\sigma_r^2}}}{\sum_{k=-\infty}^{+\infty} e^{-\frac{(x-k)^2}{2\sigma_s^2}}\, e^{-\frac{c^2(x-k)^2}{2\sigma_r^2}}}$$

$$= d + c\left(\frac{\sum_{k=-\infty}^{\infty} k\, e^{-\frac{(x-k)^2}{2\sigma_s^2} - \frac{c^2(x-k)^2}{2\sigma_r^2}}}{\sum_{k=-\infty}^{+\infty} e^{-\frac{(x-k)^2}{2\sigma_s^2}}\, e^{-\frac{c^2(x-k)^2}{2\sigma_r^2}}}\right)$$

$$= d + c\left(\frac{\sum_{k=-\infty}^{+\infty} k\, e^{-\left[\frac{(x-k)^2}{2}\right]\left[\frac{1}{\sigma_s^2} + \frac{c^2}{\sigma_r^2}\right]}}{\sum_{k=-\infty}^{+\infty} e^{-\left[\frac{(x-k)^2}{2}\right]\left[\frac{1}{\sigma_s^2} + \frac{c^2}{\sigma_r^2}\right]}}\right)$$

Let $\boxed{\frac{1}{\alpha^2} = \frac{1}{\sigma_s^2} + \frac{c^2}{\sigma_r^2}}$

$$BF[I]_x = d + c\left(\frac{\sum_{k=-\infty}^{+\infty} k\, e^{-\frac{(x-k)^2}{2\alpha^2}}}{\sum_{k=-\infty}^{+\infty} e^{-\frac{(x-k)^2}{2\alpha^2}}}\right)$$

$$= d + c\left(\frac{\sum_{k=-\infty}^{+\infty} k\, e^{-\frac{(k-x)^2}{2\alpha^2}}}{\sum_{k=-\infty}^{+\infty} e^{-\frac{(k-x)^2}{2\alpha^2}}}\right)$$

The denominator is the PDF of a gaussian with mean $= x$ and standard deviation α.

Since sum of PDF must be 1, the denominator is 1.

The numerator is convolution between $f(x) = x$ and $g(x) = e^{-x^2/2\alpha^2}$.

So, we can also write is as $\sum_{K=-\infty}^{+\infty} (x-K)\, e^{-K^2/2\alpha^2}$ because convolution is commutative.

$$\text{Numerator} = x \sum_{K=-\infty}^{+\infty} e^{-K^2/2\alpha^2} - \sum_{K=-\infty}^{+\infty} K e^{-K^2/2\alpha^2}$$

PDF sums to 1

odd function. So, sum from $-\infty$ to $+\infty$ will be zero.

$$= x(1) - 0$$

$$= x$$

$$\therefore \quad BF[I]_x = d + \frac{c(x)}{(1)}$$

$$\boxed{BF[I]_x = (x + d}$$