Question 5: Assignment 3: CS 663, Fall 2024

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1. If a function $f(x, y)$ is real, prove that its Discrete Fourier transform $F(u, v)$ satisfies $F^*(u, v) = F(-u, -v)$; if $f(x, y)$ is real and even, prove that $F(u, v)$ is also real and even, where $f(x, y) = f(-x, -y)$. [15 points]

Soln:

We know that the Discrete Fourier Transform of a discrete function f(x, y) is given by:

$$F_d(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x, y) \exp\left(-j2\pi\left(\frac{ux}{W_1} + \frac{vy}{W_2}\right)\right) \tag{1}$$

Now, if we were to take conjugate of $F_d(u, v)$, i.e. $F_d^*(u, v)$:

$$F_d^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f^*(x, y) \exp\left(j2\pi\left(\frac{ux}{W_1} + \frac{vy}{W_2}\right)\right) \tag{2}$$

But since f(x, y) is real, as per the question. We can write, f(x, y) = $f^*(x, y)$. Modified equation:

$$F_d^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x, y) \exp\left(j2\pi\left(\frac{ux}{W_1} + \frac{vy}{W_2}\right)\right) \tag{3}$$

$$F_d^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x, y) \exp\left(-j2\pi\left(\frac{-ux}{W_1} + \frac{-vy}{W_2}\right)\right) \tag{4}$$

R.H.S of the Equation (4) looks like, Equation with the signs of u and v reversed. Thus, we can say:

$$F_d^*(u, v) = F_d(-u, -v) \tag{5}$$

Next, we want to prove that $F_d(u, v)$ is real and even given that $f(x, y)$ is also real and even.
Firstly lets try to prove that $F_d(u, v)$ is real given that $f(x, y)$ is real and even. We know that the conjugate of $F_d(u, v)$ i.e. $F_d^*(u, v)$ is given by:

$$\mathcal{F}_d^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f^*(x, y) \exp\left(j2\pi\left(\frac{ux}{W_1} + \frac{vy}{W_2}\right)\right) \tag{6}$$

But since f(x, y) is real valued and even, we can replace f*(x, y) by f(x, y):

$$\mathcal{F}_d^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x, y) \exp\left(j2\pi\left(\frac{ux}{W_1} + \frac{vy}{W_2}\right)\right) \tag{7}$$

$$\mathcal{F}_d^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x, y) \exp\left(-j2\pi\left(\frac{-ux}{W_1} + \frac{-vy}{W_2}\right)\right) \tag{8}$$

But since, f(x, y) is a even function, f(x, y) = f(-x, -y). Substituting x by $\tilde{x}$ and y by $\tilde{y}$, where $\tilde{x}$ = (-x) and $\tilde{y}$ = (-y):

$$\mathcal{F}_d^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{\tilde{x}=0}^{-(W_1-1)} \sum_{\tilde{y}=0}^{-(W_2-1)} f(\tilde{x}, \tilde{y}) \exp\left(-j2\pi\left(\frac{u\tilde{x}}{W_1} + \frac{v\tilde{y}}{W_2}\right)\right) \tag{9}$$

$$\mathcal{F}_d^*(u, v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{\tilde{x}=0}^{W_1-1} \sum_{\tilde{y}=0}^{W_2-1} f(\tilde{x}, \tilde{y}) \exp\left(-j2\pi\left(\frac{u\tilde{x}}{W_1} + \frac{v\tilde{y}}{W_2}\right)\right) = \mathcal{F}_d(u, v) \tag{10}$$

Hence $\mathcal{F}_d(u, v)$ is real given f(x, y) is real and even.

Now, lets prove that $F_d(u,v)$ is even given that f(x, y) is real and even.

$$\mathcal{F}_d(-u,-v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{x=0}^{W_1-1} \sum_{y=0}^{W_2-1} f(x,y) \exp\left(-j2\pi\left(\frac{-ux}{W_1} + \frac{-vy}{W_2}\right)\right) \tag{11}$$

Substituting x by $\tilde{x}$ and y by $\tilde{y}$, where $\tilde{x}$ = (-x) and $\tilde{y}$ = (-y):

$$\mathcal{F}_d(-u,-v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{\tilde{x}=0}^{-(W_1-1)} \sum_{\tilde{y}=0}^{-(W_2-1)} f(-\tilde{x},-\tilde{y}) \exp\left(-j2\pi\left(\frac{u\tilde{x}}{W_1} + \frac{v\tilde{y}}{W_2}\right)\right) \tag{12}$$

But since f(x, y) is real:

$$\mathcal{F}_d(-u,-v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{\tilde{x}=0}^{-(W_1-1)} \sum_{\tilde{y}=0}^{-(W_2-1)} f(\tilde{x},\tilde{y}) \exp\left(-j2\pi\left(\frac{u\tilde{x}}{W_1} + \frac{v\tilde{y}}{W_2}\right)\right) \tag{13}$$

$$\mathcal{F}_d(-u,-v) = \frac{1}{\sqrt{W_1 W_2}} \sum_{\tilde{x}=0}^{W_1-1} \sum_{\tilde{y}=0}^{W_2-1} f(\tilde{x},\tilde{y}) \exp\left(-j2\pi\left(\frac{u\tilde{x}}{W_1} + \frac{v\tilde{y}}{W_2}\right)\right) = \mathcal{F}_d(u,v) \tag{14}$$

Hence proved that $\mathcal{F}_d(u,v)$ is even, given that f(x, y) is real and even. We can thereby conclude that $F_d(u,v)$ is real as well as even.
A simpler way to prove that $F_d(u,v)$ is real and even is using the fact that:

$$F_d^*(u,v) = F_d(-u,-v) \tag{15}$$

Also, we proved that $F_d(u,v)$ is real above, which means that $F_d(u,v) = F_d^*(u,v) = F_d(-u,-v)$. Clearly, $F_d(u,v)$ is real and even.