Question 5: Assignment 5: CS 663, Fall 2024

Amitesh Shekhar
IIT Bombay
22b0014@iitb.ac.in

Anupam Rawat
IIT Bombay
22b3982@iitb.ac.in

Toshan Achintya Golla
IIT Bombay
22b2234@iitb.ac.in

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1. In this exercise, we will study a nice application of the SVD which is widely used in computer vision, computer graphics and image processing. Consider we have a set of points $\boldsymbol{P}_1 \in \mathbb{R}^{2\times N}$ and another set of points $\boldsymbol{P}_2 \in \mathbb{R}^{2\times N}$ such that $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are related by an orthonormal transformation $\boldsymbol{R}$ such that $\boldsymbol{P}_1 = \boldsymbol{R}\boldsymbol{P}_2 + \boldsymbol{E}$ where $\boldsymbol{E} \in \mathbb{R}^{2\times N}$ is an error (or noise) matrix. The aim is to find $\boldsymbol{R}$ given $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ imposing the constraint that $\boldsymbol{R}$ is orthonormal. Answer the following questions: [30 points = 3 + 3 + 3 + 3 + 3 + (8 + 4 + 3)]

   (a) The standard least squares solution given by $\boldsymbol{R} = \boldsymbol{P_1}\boldsymbol{P_2}^T(\boldsymbol{P_2}\boldsymbol{P_2}^T)^{-1}$ will fail. Why?

   (b) To solve for $\boldsymbol{R}$ incorporating the important constraints, we seek to minimize the following quantity:

$$
\begin{aligned}
E(\boldsymbol{R}) &= \|\boldsymbol{P_1} - \boldsymbol{R}\boldsymbol{P_2}\|_F^2 && (1)\\
&= \text{trace}((\boldsymbol{P_1} - \boldsymbol{R}\boldsymbol{P_2})^T(\boldsymbol{P_1} - \boldsymbol{R}\boldsymbol{P_2})) && (2)\\
&= \text{trace}(\boldsymbol{P_1^T}\boldsymbol{P_1} + \boldsymbol{P_2^T}\boldsymbol{R^T}\boldsymbol{R}\boldsymbol{P_2} - \boldsymbol{P_2^T}\boldsymbol{R^T}\boldsymbol{P_1} - \boldsymbol{P_1^T}\boldsymbol{R}\boldsymbol{P_2}) && (3)\\
&= \text{trace}(\boldsymbol{P_1^T}\boldsymbol{P_1} + \boldsymbol{P_2^T}\boldsymbol{P_2} - \boldsymbol{P_2^T}\boldsymbol{R^T}\boldsymbol{P_1} - \boldsymbol{P_1^T}\boldsymbol{R}\boldsymbol{P_2}) \text{ (justify this step given the previous one)} && (4)\\
&= \text{trace}(\boldsymbol{P_1^T}\boldsymbol{P_1} + \boldsymbol{P_2^T}\boldsymbol{P_2}) - 2\text{trace}(\boldsymbol{P_1^T}\boldsymbol{R}\boldsymbol{P_2}) \text{ (justify this step given the previous one)} && (5)
\end{aligned}
$$

   (c) Why is minimizing $E(\boldsymbol{R})$ w.r.t. $\boldsymbol{R}$ is equivalent to maximizing trace($\boldsymbol{P_1^T}\boldsymbol{R}\boldsymbol{P_2}$) w.r.t. $\boldsymbol{R}$?

   (d) Now, we have

$$
\begin{aligned}
\text{trace}(\boldsymbol{P_1^T}\boldsymbol{R}\boldsymbol{P_2}) &= \text{trace}(\boldsymbol{R}\boldsymbol{P_2}\boldsymbol{P_1^T}) \text{ (justify this step)} && (6)\\
&= \text{trace}(\boldsymbol{R}\boldsymbol{U'}\boldsymbol{S'}\boldsymbol{V'^T}) \text{ using SVD of } \boldsymbol{P_2}\boldsymbol{P_1^T} = \boldsymbol{U'}\boldsymbol{S'}\boldsymbol{V'^T} && (7)\\
&= \text{trace}(\boldsymbol{S'}\boldsymbol{V'^T}\boldsymbol{R}\boldsymbol{U'}) = \text{trace}(\boldsymbol{S'}\boldsymbol{X}) \text{ where } \boldsymbol{X} = \boldsymbol{V'^T}\boldsymbol{R}\boldsymbol{U'} && (8)\\
& && (9)
\end{aligned}
$$

   (e) For what matrix $\boldsymbol{X}$ will the above expression be maximized? (Note that $\boldsymbol{S'}$ is a diagonal matrix.)

   (f) Given this $\boldsymbol{X}$, how will you determine $\boldsymbol{R}$?

   (g) If you had to impose the constraint that $\boldsymbol{R}$ is specifically a rotation matrix, what additional constraint would you need to impose?

Soln:

   (a) $\boldsymbol{R} = \boldsymbol{P_1}\boldsymbol{P_2}^T(\boldsymbol{P_2}\boldsymbol{P_2}^T)^{-1}$ is required to be an orthonormal matrix.

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I} \implies \boldsymbol{R}^T = \boldsymbol{R}^{-1}$$

$$\boldsymbol{R}^T = \left(\boldsymbol{P_1}\boldsymbol{P_2}^T(\boldsymbol{P_2}\boldsymbol{P_2}^T)^{-1}\right)^T = (\boldsymbol{P_2}\boldsymbol{P_2}^T)^{-1}\boldsymbol{P_2}\boldsymbol{P_1}^T$$

$$\boldsymbol{R}^T \cdot \boldsymbol{R} = (\boldsymbol{P_2}\boldsymbol{P_2}^T)^{-1}\boldsymbol{P_2}\boldsymbol{P_1}^T\boldsymbol{P_1}\boldsymbol{P_2}^T(\boldsymbol{P_2}\boldsymbol{P_2}^T)^{-1} \neq \boldsymbol{I}$$

   (b) Below is the pair of equations that we need to justify:

$$
\begin{aligned}
E(\boldsymbol{R}) &= \|\boldsymbol{P_1} - \boldsymbol{R}\boldsymbol{P_2}\|_F^2 && (10)\\
&= \text{trace}((\boldsymbol{P_1} - \boldsymbol{R}\boldsymbol{P_2})^T(\boldsymbol{P_1} - \boldsymbol{R}\boldsymbol{P_2})) && (11)\\
&= \text{trace}(\boldsymbol{P_1^T}\boldsymbol{P_1} + \boldsymbol{P_2^T}\boldsymbol{R^T}\boldsymbol{R}\boldsymbol{P_2} - \boldsymbol{P_2^T}\boldsymbol{R^T}\boldsymbol{P_1} - \boldsymbol{P_1^T}\boldsymbol{R}\boldsymbol{P_2}) && (12)\\
&= \text{trace}(\boldsymbol{P_1^T}\boldsymbol{P_1} + \boldsymbol{P_2^T}\boldsymbol{P_2} - \boldsymbol{P_2^T}\boldsymbol{R^T}\boldsymbol{P_1} - \boldsymbol{P_1^T}\boldsymbol{R}\boldsymbol{P_2}) \text{ (since } \boldsymbol{R} \text{ is a orthonormal matrix, } \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}) && (13)\\
&= \text{trace}(\boldsymbol{P_1^T}\boldsymbol{P_1} + \boldsymbol{P_2^T}\boldsymbol{P_2}) - 2\text{trace}(\boldsymbol{P_1^T}\boldsymbol{R}\boldsymbol{P_2}) \text{ (Justification provided in the below line)} && (14)
\end{aligned}
$$

For the above equation (14), we can clearly see that $\boldsymbol{P_2^T R^T P_1} = \boldsymbol{(P_1^T R P_2)}^T$ and we also know that if we have a matrix $\boldsymbol{A}$, then trace($\boldsymbol{A}$) = trace($\boldsymbol{A}^T$). Hence, we can write trace($\boldsymbol{P_1^T R P_2}$) = trace($\boldsymbol{P_2^T R^T P_1}$). Next, we also know that $trace(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{A}+\boldsymbol{B}$. Thus using the last two properties of matrices and their trace, we can clearly say that the equation (14) is justified, given the previous one.

(c) As it can be seen from the equation (14), that $E(\boldsymbol{R}) = \text{trace}(\boldsymbol{P_1^T P_1} + \boldsymbol{P_2^T P_2}) - 2\text{trace}(\boldsymbol{P_1^T R P_2})$. We can see that the first term is a constant. Thus the second term is the only one which can be controlled and varied. And also, since the second term varies inversely with $E(\boldsymbol{R})$, we can conclude, that to minimize $E(\boldsymbol{R})$, we need to maximize trace($\boldsymbol{P_1^T R P_2}$). Hence, minimizing $E(\boldsymbol{R})$ w.r.t. $\boldsymbol{R}$ is equivalent to maximizing trace($\boldsymbol{P_1^T R P_2}$) w.r.t. $\boldsymbol{R}$.

(d) The step, $(\boldsymbol{P_1^T R P_2}) = \text{trace}(\boldsymbol{R P_2 P_1^T})$ can be justified by the fact that, the trace of a product of matrices is invariant under cyclic permutations. This means that for any matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ where the products are defined: trace(ABC) = trace(BCA) = trace(CAB).

(e) $\boldsymbol{X} = \boldsymbol{V'^T R U'}$. And since, all of the $\boldsymbol{V'}^T$, $\boldsymbol{U}$ and $\boldsymbol{R}$ are orthonormal, we can easily conclude that $\boldsymbol{X}$ is also orthonormal.
Now, since $\boldsymbol{S'}$ is a diagonal matrix, trace($\boldsymbol{S'X}$) will be maximized when the diagonal entries of $\boldsymbol{X}$ is maximized. The condition that X is orthonormal and that diagonal entries of X are maximized, can be only satisfied when $\boldsymbol{X}$ is the identity matrix, $\boldsymbol{I}$.

(f) Once again, $\boldsymbol{X}$ is given by $\boldsymbol{V'^T R U'}$. And since $\boldsymbol{X}$ is the identity matrix, $\boldsymbol{V'^T R U'} = \boldsymbol{I}$. This implies that $\boldsymbol{R} = \boldsymbol{V' U'^T}$, leading to $\boldsymbol{X}$ being the identity matrix.

(g) To ensure that R is specifically a rotation matrix, not just any orthonormal matrix, we need to impose the following additional constraint:

$$\det(R) = 1$$

- A rotation matrix R is an orthonormal matrix with a determinant of +1. This ensures that R represents a proper rotation, preserving the orientation of the space.
- Orthogonal matrices can have determinants of either +1 or -1. A determinant of -1 would indicate a reflection or improper rotation (a combination of rotation and reflection), which is not what we want for a pure rotation matrix.

In order to impose this constraint, when computing $R = U'V'^T$, if $\det(R) = -1$, you need to adjust the sign of one of the columns in U' or V' to ensure $\det(R) = 1$. Specifically, you can modify U' by flipping the sign of the last column:

$$U'_{\text{adjusted}} = U'\text{diag}(1, 1, \ldots, -1)$$

This change will make $\det\left(U'_{\text{adjusted}} V'^T\right) = 1$ while preserving the orthogonality of R.
In conclusion, to ensure R is a rotation matrix, use:

$$R = U'_{\text{adjusted}} V'^T$$

where U'_{adjusted} is modified to satisfy $\det(R) = 1$.