Q1.) Clean image : $I(x,y)$.

Noise is added to this image.

At every pixel location (x,y), the noise η is sampled from a zero-mean Gaussian distribution. The sampling is done independently of both $I(x,y)$ and the pixel co-ordinates (x,y).

Given : Intensity values are continuous.

$-\infty < I(x,y) < \infty$

$\Rightarrow I(x,y) \in \mathbb{R}$ and

let $P_I(i)$ denote the P.D.F. of clean image $I(x,y)$

and $P_N(n)$ denote the P.D.F. of noise

since $N(x,y) \sim$ Normal Distribution, $\mathcal{N}(0,\sigma^2)$,

$$P_N(n) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-\eta^2}{2\sigma^2}\right)$$

Let $\tilde{I}(x,y) = I(x,y) + N(x,y)$ be the corrupted image

$$\Rightarrow p_{\tilde{I}}(\tilde{i}) = \int_{-\infty}^{\infty} P_{I,N}(\tilde{i}, \lambda - \tilde{i})\, d\lambda$$

[$\because \tilde{I}$ is a function of two random variables, I and N]

Hence we take the joint P.D.F.

Let $F_{\tilde{I}}(\tilde{i})$, $F_I(i)$ and $F_N(n)$ denote the C.D.F. of image $\tilde{I}$, I and noise N.

$\Rightarrow \tilde{I} = I + N$

$$\Rightarrow F_{\tilde{I}}(\tilde{i}) = F_{I+N}(\tilde{i}) = P\{I + N \leq \tilde{i}\}$$

$$= \iint_{i+n \leq \tilde{i}} P_I(i)\, P_N(n)\, di\, dn \quad [\because I \text{ and } N \text{ are independent}]$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\tilde{i}-n} P_I(i)\, di \right) p_N(n)\, dn$$

$$= \int_{-\infty}^{\infty} F_I(\tilde{i} - n)\, p_N(n)\, dn \quad \text{— (1)}$$

Differentiating equation ①, we get

$$P_{I+N}(\tilde{i}) = \frac{d}{d\tilde{i}} \int_{-\infty}^{\infty} F_I(\tilde{i}-n) P_N(n)\,dn$$

$$\Rightarrow \cancel{p_{\tilde{I}}(\tilde{i}) = P_I *} = \int_{-\infty}^{\infty} \frac{d}{d\tilde{i}} F_I(\tilde{i}-n) P_N(n)\,dn$$

$$= \int_{-\infty}^{\infty} P_I(\tilde{i}-n) P_N(n)\,dn$$

$$\Rightarrow \boxed{P_{I+N}(\tilde{i}) = P_{\tilde{I}}(\tilde{i}) = \int_{-\infty}^{\infty} P_I(\tilde{i}-n) P_N(n)\,dn}$$

Hence, this is same as convolution of the P.D.f.s of the original image and the noise

$$P_{\tilde{I}}(\tilde{i}) = (P_I * P_N)(\tilde{i}) = \frac{\int_{-\infty}^{\infty} P_I(n) \exp\left(-\frac{(\tilde{i}-n)^2}{2\sigma^2}\right) dn}{\sqrt{2\pi\sigma^2}}$$

For a uniformly distributed noise $U(x,y)$, uniformly distributed from $-r$ to r;

$$P_U(u) = \begin{cases} \frac{1}{2r} & ; -r \leq u \leq r \\ 0 & ; \text{otherwise} \end{cases}$$

$$p_{\tilde{I}}(\tilde{i}) = P_{I+U}(\tilde{i}) = \int_{-\infty}^{\infty} P_I(n) P_U(\tilde{i}-n)\,dn$$

$$= \int_{-\infty}^{\infty} p_I(\tilde{i}-n) p_U(n)\,dn$$ [By convolution property of convolution]

$$\boxed{P_{\tilde{I}}(\tilde{i}) = \frac{1}{2r}\int_{-r}^{r} p_I(\tilde{i}-n)\,dn}$$ ← P.D.F. of corrupted image in 2^{nd} case (uniform noise)