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Homework-02 (CS754, Spring 2025)

(Q1)

Statement 1.0:- If $\delta_{2s} = 1$, then $2s$-columns of Φ may be linearly dependent in which case there exists a $2s$-sparse vector h obeying $\Phi h = 0$

ϕ follows RIP property with $\delta_{2s} = 1$ and order $2s$.

$$\Rightarrow \quad 0 \leq \|\phi h\|_2^2 \leq 2\|h\|_2^2 \quad \forall \text{ } 2s\text{-sparse vectors } h \quad -(i)$$

Assume $\|\phi h\|_2^2 > 0 \Rightarrow \|\phi h\|_2^2 = 0 + \Delta$.

If we now test the RIP criteria for a new $\tilde{\delta}_{2s} = 1 - \Delta$, we get

$$(1 - (1-\Delta))\|h\|_2^2 \leq \|\phi h\|_2^2 \leq (2-\Delta)\|h\|_2^2 \quad -(ii)$$

$$\Rightarrow \quad \Delta\|h\|_2^2 \leq \|\phi h\|_2^2 \leq (2-\Delta)\|h\|_2^2$$

clearly, (ii) holds whenever (i) holds.

This tells us that $\exists$ a smaller RIC than $\delta_{2s} = 1$.

This is not possible (by definition of RIP)

Hence, $\|\phi h\|_2^2 = 0$

$$\Rightarrow \|\phi h\| = 0$$

$$\Rightarrow \sum_{i=1}^{n} h_i \phi_i$$ where h_i is a scalar and ϕ_i is the ith column of $\phi_{m \times n}$ matrix

Since h is $2s$-sparse, $\exists$ at most $2s$ non-zero elements of h. Let they be indicated by index set T.

$$\Rightarrow \sum_{i \in T} h_i \phi_i = 0$$

$\therefore$ At most, $2s$ columns of Φ may be linearly independent (proved).

Statement 1.1

$\|\phi(x^* - x)\|_2 \leq \|\phi x^* - y\|_2 + \|y - \phi x\|_2 \leq 2\varepsilon$

x^* is the solution to P1 problem for observation with noise

i.e., if $y = \phi x + z$, we construct x^* as the solution to the convex optimization

$$\min_{\|\tilde{x}\|_1,\ \tilde{x} \in \mathbb{R}^n} \quad \text{such that } \|y - \phi \tilde{x}\|_2 \leq \varepsilon \quad \text{— (7)}$$

$\Rightarrow$ $\|y - \phi x\|_2 \leq \varepsilon$ and $\|y - \phi x^*\|_2 \leq \varepsilon$

($\because$ x is feasible for (7))

$\Rightarrow$ $\|y - \phi x\|_2 + \|y - \phi x^*\|_2 \leq 2\varepsilon$ — (i)

(Right Inequality)

For the left inequality, we use triangle inequality for vectors w and $v \in \mathbb{R}^n$

$\|w\|_2 + \|v\|_2 \geq \|w + v\|_2$

$\Rightarrow \|\phi x^* - y\|_2 + \|y - \phi x\|_2 \geq \|\phi x^* - \cancel{y} + \cancel{y} - \phi x\|_2$

$= \|\phi(x^* - x)\|_2$

$\therefore \|\phi x^* - y\|_2 + \|y - \phi x\|_2 \geq \|\phi(x^* - x)\|_2$ — (ii)

Results (i) and (ii) prove the given statement.

Statement (1.2)

For each $j \geq 2$,

$$\|h_{T_j}\|_2 \leqslant s^{1/2} \|h_{T_j}\|_{\ell_\infty} \leqslant s^{-1/2} \|h_{T_{j-1}}\|_{\ell_1}$$

First, let us see what is $\|h_{T_j}\|_{\ell_\infty}$

for any vector x, ~~$\|x\|_{\ell_\infty} = \lim_{n\to\infty} x_1^n + x_2^n + \dots + x$.~~

$$\|x\|_\infty = \lim_{N\to\infty} (x_1^N + x_2^N + \dots + x_n^N)^{\frac{1}{N}} \equiv \max_{1\leqslant i\leqslant n} |x_i|$$

Hence, the ℓ_∞-norm of a vector is just the maximum absolute value of the entries of x.

$$\Rightarrow \|h_{T_j}\|_\infty = \max_{1\leqslant i\leqslant n} |(h_{T_j})_i|$$

$$\Rightarrow (h_{T_j})_i^2 \leqslant \|h_{T_j}\|_\infty^2 \quad \forall\, i \in T_j$$

$$\Rightarrow \sum_{i\in T_j} (h_{T_j})_i^2 \leqslant \sum_{i\in T_j} \|h_{T_j}\|_\infty^2$$

entry at i^{th} index of h_{T_j}.

Since $|T_j| = s$, we have

$$\|h_{T_j}\|_2^2 \leqslant s \cdot \|h_{T_j}\|_\infty^2$$

$$\Rightarrow \boxed{\|h_{T_j}\|_2 \leqslant (s)^{1/2} \|h_{T_j}\|_\infty} \quad \text{—(i)}$$

Now, according to the way h_{T_j} is defined for $j \geqslant 1$,

$$|(h_{T_j})_p| \leqslant |(h_{T_{j-1}})_q| \quad \forall\, p \in T_j \text{ and } q \in T_{j-1}$$

p^{th} indexed entry of h_{T_j}

q^{th} indexed entry of $h_{T_{j-1}}$

$$\Rightarrow \|h_{T_j}\|_\infty^{\cancel{2}} \leqslant |(h_{T_{j-1}})_q|^{\cancel{2}}$$

$$\Rightarrow \sum_{q\in T_{j-1}} \|h_{T_j}\|_\infty^{\cancel{2}} \leqslant \sum_{q\in T_{j-1}} |(h_{T_{j-1}})_q|$$

$$\Rightarrow s\, \|h_{T_j}\|_\infty \leqslant \|h_{T_{j-1}}\|_1$$

$$\Rightarrow \boxed{s^{1/2} \|h_{T_j}\|_\infty \leqslant (s)^{-1/2} \|h_{T_{j-1}}\|_1} \quad \text{—(ii)}$$

From (i) and (ii),

$$\|h_{T_j}\|_2 \leqslant s^{1/2} \|h_{T_j}\|_{\ell_\infty} \leqslant s^{-1/2} \|h_{T_{j-1}}\|_{\ell_1}$$

(proved)

Statement 1·3

$$\sum_{j \geq 2} \|h_{T_j}\|_2 \leqslant s^{-1/2} \left(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \cdots \right)$$
$$\leqslant s^{-1/2} \|h_{T_0^c}\|_1$$

First, let us look at the left inequality

$$\sum_{j \geq 2} \|h_{T_j}\|_2 \leqslant s^{-1/2} \left(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \cdots \right)$$

From previous result (eqn 10 in paper), we know

$$\|h_{T_j}\|_2 \leqslant s^{-1/2} \|h_{T_{j-1}}\|_1 \quad \forall\, j \geq 2$$

$$\Rightarrow \sum_{j \geq 2} \|h_{T_j}\|_2 \leqslant s^{-1/2} \left(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \cdots \right) \quad \text{— (i)}$$

For the right sided inequality, note that

$$T_1 \cup T_2 \cup T_3 \cdots = T_0^c$$

and $\quad T_i \cap T_j = \phi \quad \forall \; i, j \geq 1$ and $i \neq j$

$$\Rightarrow \|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \cdots = \|h_{T_0^c}\|_1$$

(since all T_i's are disjoint, we can add the absolute values of entries of h_{T_i} without interference from other h_{T_j})

$$\Rightarrow s^{-1/2} \left(\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \cdots \right) = s^{-1/2} \|h_{T_0^c}\|_1 \quad \text{— (ii)}$$

From (i) and (ii), our statement is proved.

statement 1.4)

$$\| h_{(T_0 \cup T_1)^c} \|_2 = \| \sum_{j \geq 2} h_{T_j} \|_2 \leq \sum_{j \geq 2} \| h_{T_j} \|_2$$

Soln: we have,

$$T_0^c = T_1 \cup T_2 \cup T_3 \cup \dots$$

$$\Rightarrow T_0^c \cap T_1^c = \underbrace{(T_1^c \cap T_1)}_{\phi} \cup T_2 \cup T_3 \cup \dots$$

$$\Rightarrow (T_0 \cup T_1)^c = T_2 \cup T_3 \cup \dots$$

$$\Rightarrow h_{(T_0 \cup T_1)^c} = h_{T_2} + h_{T_3} + \dots$$

(since T_i and T_j are disjoint for $i \neq j$, $i \geq 1$, $j \geq 1$)

$$\Rightarrow \| h_{(T_0 \cup T_1)^c} \|_2 = \| \sum_{j \geq 2} h_{T_j} \|_2 \quad \text{——(i)}$$

Also, by repeatedly using triangle inequality, we have

$$\| \sum_{j \geq 2} h_{T_j} \|_2 \leq \sum_{j \geq 2} \| h_{T_j} \|_2 \quad \text{——(ii)}$$

From (i) and (ii),

$$\| h_{(T_0 \cup T_1)^c} \| = \| \sum_{j \geq 2} h_{T_j} \|_2 \leq \sum_{j \geq 2} \| h_{T_j} \|_2 \quad \text{(proved)}$$

statement 1.5) $\sum_{j \geq 2} \| h_{T_j} \|_{\ell_2} \leq s^{-1/2} \| h_{T_0^c} \|_{\ell_1}$

In result (10) (according to paper) $\nearrow$ statement (1.3), we have

$$\sum_{j \geq 2} \| h_{T_j} \|_{\ell_2} \leq s^{-1/2} \left(\| h_{T_1} \|_1 + \| h_{T_2} \|_1 + \dots \right) \leq s^{-1/2} \| h_{T_0^c} \|_1$$

$$\Rightarrow \boxed{\sum_{j \geq 2} \| h_{T_j} \|_2 \leq s^{-1/2} \| h_{T_0^c} \|_1} \quad \text{(proved)}$$

Statement 1.6)

$$\|x\|_{\ell_1} \geqslant \|x+h\|_{\ell_1} = \sum_{i\in T_0} |x_i+h_i| + \sum_{i\in T_0^C} |x_i+h_i| \geqslant$$

$$\|x_{T_0}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|$$

We have,

$$\|x+h\|_1 = \sum_{i\in T_0} |x_i+h_i| + \sum_{i\in T_0^c} |x_i+h_i| \qquad (\because T_0 \cup T_0^c = \{1,\dots,n\})$$

Now, $|x_i+h_i| = |x_i-(-h_i)| \geqslant \big|\,|x_i| - |-h_i|\,\big| = \big|\,|x_i| - |h_i|\,\big|$

(by reverse triangle inequality for real numbers)

$$\Rightarrow \quad |x_i+h_i| \geqslant \big|\,|x_i|-|h_i|\,\big| \geqslant |x_i|-|h_i| \quad \text{—(i)}$$

Similarily,

$$|x_i+h_i| \geqslant |h_i|-|x_i| \quad \text{—(ii)}$$

$$\Rightarrow \sum_{i\in T_0}|x_i+h_i| + \sum_{i\in T_0^C} |x_i+h_i| \geqslant \sum_{i\in T_0} |x_i| - \sum_{i\in T_0}|h_i| + \sum_{i\in T_0^c}|h_i| - \sum_{i\in T_0^c}|x_i|$$

(from (i) and (ii))

$$\Rightarrow \quad " \quad \geqslant \|x_{T_0}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1}$$

(proved)

Statement (1.7)

$\|h_{T_0^c}\| \leq \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}$

we have, $x^* = x + h$ where x^* is the optimal solution of the convex optimization problem.

$\Rightarrow$ x^* is the vector with minimum L1 norm which satisfies $\|y - \phi x\| \leq \varepsilon$

$\Rightarrow \quad \|x\|_{\ell_1} \geq \|x^*\|_{\ell_1}$

$\Rightarrow \quad \|x\|_{\ell_1} \geq \|x+h\|_{\ell_1}$

By previous result. (Statement 1.6) , we have

$$\|x+h\|_{\ell_1} \geq \|x_{T_0}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1}$$

$$\Rightarrow \|\hat{x}\|_{\ell_1} \geq \|x_{T_0}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1} \quad \text{— (i)}$$

But, $\|x_{T_0^c}\|_{\ell_1} = \|x - x_s\|_{\ell_1}$

$\Rightarrow \|x_{T_0^c}\|_{\ell_1} \geq \|x\|_{\ell_1} - \|x_s\|_{\ell_1}$ (By reverse triangle inequality)

But, $\|x_s\|_{\ell_1} = \|x_{T_0}\|_{\ell_1}$

$$\Rightarrow \|x\|_{\ell_1} \leq \|x_{T_0^c}\|_{\ell_1} + \|x_{T_0}\|_{\ell_1} \quad \text{— (ii)}$$

from (i) and (ii) , we can write

$$\|x_{T_0^c}\|_{\ell_1} + \cancel{\|x_{T_0}\|_{\ell_1}} \geq \cancel{\|x_{T_0}\|_{\ell_1}} - \|h_{T_0}\|_{\ell_1} + \|h_{T_0^c}\|_{\ell_1} - \|x_{T_0^c}\|_{\ell_1}$$

$$\Rightarrow \boxed{\|h_{T_0^c}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_1} + 2\|x_{T_0^c}\|_{\ell_1}}$$ (proved)

Statement (1.8)

$$\|h_{(T_0 \cup T_1)^c}\|_{\ell_2} \leq \|h_{T_0}\|_{\ell_2} + 2e_0 \ ; \quad e_0 \equiv s^{-1/2}\|x - x_s\|_{\ell_1}$$

From statement (1.4) and (1.5), we have

$$\sum \|h_{(T_0 \cup T_1)^c}\|_2 \leq s^{-1/2} \|h_{T_0^c}\|_{\ell_1} \quad -(i)$$

From statement (1.7), we have

$$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + 2\|x_{T_0^c}\|_1$$

$$= \|h_{T_0}\|_1 + 2\|x - x_s\|_1 \quad -(ii)$$

$\Rightarrow$ From (i) and (ii), we get

$$\|h_{(T_0 \cup T_1)^c}\|_2 \leq s^{-1/2}\|h_{T_0}\|_1 + 2s^{-1/2}\|x - x_s\|_1 \quad -(iii)$$

Now, $\|h_{T_0}\|_1 = |h_{T_0} \cdot \vec{1}| \leq \|h_{T_0}\|_2 \|\vec{1}\|_2$

(cauchy-schwarz) $= \sqrt{n}\|h_{T_0}\|_2$

we can reduce the upper bound of $\|h_{T_0}\|_1$ by using the fact that h_{T_0} is s-sparse

$\Rightarrow \|h_{T_0}\|_1 = |h_{T_0} \cdot \vec{P}| \leq \|h_{T_0}\|_2 \|\vec{P}\|_2 = \sqrt{s}\|h_{T_0}\|_2$

where $P_i = \begin{cases} \text{sign}(h_{T_0})_i & \text{if } i \in T_0 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore \|h_{T_0}\|_1 \leq \sqrt{s}\|h_{T_0}\|_2 \quad -(iv)$$

From (iii) and (iv), we get

$$\boxed{\|h_{(T_0 \cup T_1)^c}\|_2 \leq \|h_{T_0}\|_2 + 2e_0}$$ (proved)

Statement (1.9)

$$|\langle \phi h_{T_0 \cup T_1}, \phi h\rangle| \leq \|\phi h_{T_0\cup T_1}\|_2 \; \|\phi h\|_2 \leq$$
$$2\varepsilon\sqrt{1+\delta_{2s}} \; \|h_{T_0\cup T_1}\|_2$$

proof :-

From Cauchy- Schwarz inequality, we have

$$|\langle \phi h_{T_0\cup T_1}, \phi h\rangle| \leq \|\phi h_{T_0\cup T_1}\|_2 \, \|\phi h\|_2 \quad \text{—(i)}$$

From RIP property for ϕ (order 2s), we have

$$(1-\delta_{2s})\|h_{T_0\cup T_1}\|_2^2 \leq \|\phi h_{T_0\cup T_1}\|_2^2 \leq (1+\delta_{2s})\|h_{T_0\cup T_1}\|_2^2$$

(since $h_{T_0\cup T_1}$ is also a 2s-sparse vector, $T_0 \cap T_1 = \phi$)
and h_{T_0} and h_{T_1} are s-sparse

$$\Rightarrow \|\phi h_{T_0\cup T_1}\|_2 \leq \sqrt{1+\delta_{2s}} \; \|h_{T_0\cup T_1}\|_2 \quad \text{—(ii)}$$

Also, from statement (1.1),

$$\|\phi(x^* - x)\|_2 \leq 2\varepsilon$$

but $x^* = x + h \Rightarrow x^* - x = h$

$$\therefore \|\phi h\|_2 \leq 2\varepsilon \quad \text{—(iii)}$$

From (i), (ii) and (iii), we get

$$|\langle \phi h_{T_0\cup T_1}, \phi h\rangle| \leq \|\phi h_{T_0\cup T_1}\|_2 \, \|\phi h\|_2 \leq 2\varepsilon\sqrt{1+\delta_{2s}} \, \|h_{T_0\cup T_1}\|_2$$

(proved)

Statement (1.10)

$$|\langle \phi h_{T_0}, \phi h_{T_j} \rangle| \leq \delta_{2s} \|h_{T_0}\|_2 \|h_{T_j}\|_2$$

Lemma 2.1 of the paper stats

$|\langle \phi x, \phi x' \rangle| \leq \delta_{s+s'} \|x\|_2 \|x'\|$ where x and x' are supported on disjoint support T and T' where $|T| < s$ and $|T'| < s'$

Here also, ~~h_{T_j}~~ $T_j \subseteq T_0^C$

$\Rightarrow$ ~~h_{T_j}~~ $T_j \cap T_0^C = \phi$

and h_{T_0} and h_{T_j} are both s-sparse

$$\therefore \boxed{|\langle \phi h_{T_0}, \phi h_{T_j} \rangle| \leq \delta_{2s} \|h_{T_0}\|_2 \|h_{T_j}\|_2}$$

Statement (1.11), $\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2} \|h_{T_0 \cup T_1}\|_2$

$h_{T_0 \cup T_1} = h_{T_0} + h_{T_1}$ since T_0 and T_1 are disjoint

Square both sides of the statement

$\|h_{T_0}\|_2^2 + \|h_{T_1}\|_2^2 + 2\|h_{T_0}\|_2 \|h_{T_1}\|_2 \leq 2 \|h_{T_0} + h_{T_1}\|_2^2$

$\Rightarrow \|h_{T_0}\|_2^2 + \|h_{T_1}\|_2^2 + 2\underbrace{\langle h_{T_0}, h_{T_1} \rangle}_{0})$

$= 2(\|h_{T_0}\|_2^2 + \|h_{T_1}\|_2^2$

$\Rightarrow \|h_{T_0}\|_2^2 + \|h_{T_1}\|_2^2 - 2\|h_{T_0}\|_2 \|h_{T_1}\|_2 \geq 0$

$\Rightarrow \|h_{T_0} - h_{T_1}\|_2^2 \geq 0$

which is always true.

Hence, the original statement always holds (proved)

Statement (1.12)

$$(1-\delta_{2s})\|h_{T_0 \cup T_1}\|_2^2 \le \|\phi h_{T_0 \cup T_1}\|_2^2 \le \|h_{T_0 \cup T_1}\|_2 \left(2\varepsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s}\sum_{j\ge 2}\|h_{T_j}\|_2\right)$$

From lemma 2.1 (paper),

$$|\langle \phi h_{T_0}, \phi h_{T_j}\rangle| \le \delta_{2s}\|h_{T_0}\|_2\|h_{T_j}\|_2 \quad \text{and}$$

$$|\langle \phi h_{T_1}, \phi h_{T_j}\rangle| \le \delta_{2s}\|h_{T_1}\|_2\|h_{T_j}\|_2$$

$$\Rightarrow |\langle \phi h_{T_0}, \phi h_{T_j}\rangle| + |\langle \phi h_{T_1}, \phi h_{T_j}\rangle| \le \delta_{2s}\|h_{T_j}\|_2\left(\|h_{T_0}\|_2+\|h_{T_1}\|_2\right)$$

Using triangle inequality,

$$|\langle \phi h_{T_0}, \phi h_{T_j}\rangle| + |\langle \phi h_{T_1}, \phi h_{T_j}\rangle| \ge |\langle \underbrace{\phi h_{T_0}+\phi h_{T_1}}_{\phi h_{T_0 \cup T_1}}, \phi h_{T_j}\rangle|$$

$$\Rightarrow |\langle \phi h_{T_0 \cup T_1}, \phi h_{T_j}\rangle| \le \delta_{2s}\|h_{T_j}\|_2\left(\|h_{T_0}\|_2+\|h_{T_1}\|_2\right)$$

Sum over all $j \ge 2$

$$\Rightarrow \sum_{j\ge 2}|\langle \phi h_{T_0 \cup T_1}, \phi h_{T_j}\rangle| \le \delta_{2s}\left(\|h_{T_0}\|_2+\|h_{T_1}\|_2\right)\sum_{j\ge 2}\|h_{T_j}\|_2$$

Using repeated triangle inequality in the LHS, we can bring the summation inside

$$\Rightarrow |\langle \phi h_{T_0 \cup T_1}, \sum_{j\ge 2}\phi h_{T_j}\rangle| \le \delta_{2s}\left(\|h_{T_0}\|_2+\|h_{T_1}\|_2\right)\sum_{j\ge 2}\|h_{T_j}\|_2$$

Using statement (1.11), we further simplify

$$|\langle \phi h_{T_0 \cup T_1}, \sum_{j\ge 2}\phi h_{T_j}\rangle| \le \delta_{2s}\cdot\sqrt{2}\,\|h_{T_0 \cup T_1}\|\left(\sum_{j\ge 2}\|h_{T_j}\|_2\right) \quad \text{——— (i)}$$

Now, consider inequality from statement (1.2):-

$$|\langle \phi h_{T_0 \cup T_1}, \phi h \rangle| \leq 2\varepsilon \sqrt{1+\delta_{2s}}\, \|h_{T_0 \cup T_1}\|_2$$

$$\phi h = \phi h_{T_0 \cup T_1} + \sum_{j \geq 2} \phi h_{T_j}$$

$$\Rightarrow |\langle \phi h_{T_0 \cup T_1}, \phi h_{T_0 \cup T_1} + \sum_{j \geq 2} \phi h_{T_j} \rangle| \leq 2\varepsilon \sqrt{1+\delta_{2s}}\, \|h_{T_0 \cup T_1}\|_2$$

$$\Rightarrow \left| \|\phi h_{T_0 \cup T_1}\|_2^2 + \langle \phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \phi h_{T_j} \rangle \right| \leq 2\varepsilon \sqrt{1+\delta_{2s}}\, \|h_{T_0 \cup T_1}\|_2 \quad \text{—(ii)}$$

Add inequalities (i) and (ii)

$$\left| \|\phi h_{T_0 \cup T_1}\|_2^2 + \langle \phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \phi h_{T_j} \rangle \right| + |\langle \phi h_{T_0 \cup T_1}, \sum_{j \geq 2} \phi h_{T_j} \rangle| \leq 2\epsilon \sqrt{1+\delta_{2s}}\, \|h_{T_0 \cup T_1}\|_2 + \delta_{2s}\sqrt{2}\, \|h_{T_0 \cup T_1}\| \left(\sum_{j \geq 2} \|h_{T_j}\|_2 \right)$$

Use triangle inequality :-

$$\|\phi h_{T_0 \cup T_1}\|_2^2 \leq \left(2\epsilon \sqrt{1+\delta_{2s}} + \sqrt{2}\, \delta_{2s} \sum_{j \geq 2} \|h_{T_j}\|_2 \right) \|h_{T_0 \cup T_1}\|_2 \quad \text{—(iii)}$$

Also, from RIP property,

$$\|\phi h_{T_0 \cup T_1}\|_2^2 \geq (1-\delta_{2s})\, \|h_{T_0 \cup T_1}\|_2^2 \quad \text{—(iv)}$$

$$\cancel{\Rightarrow (1-\delta_{2s})\, \|h_{T_0 \cup T_1}\|_2^2 \leq \left(2\epsilon \sqrt{1+\delta_{2s}} + \sqrt{2}\, \delta_{2s} \sum_{j \geq 2} \|h_{T_j}\|_2 \right) \|h_{T_0 \cup T_1}\|}$$

From (iii) and (iv);

$$\boxed{(1-\delta_{2s})\, \|h_{T_0 \cup T_1}\|_2^2 \leq \|\phi h_{T_0 \cup T_1}\|_2^2 \leq \left(2\epsilon \sqrt{1+\delta_{2s}} + \sqrt{2}\, \delta_{2s} \sum_{j \geq 2} \|h_{T_j}\|_2 \right) \|h_{T_0 \cup T_1}\|}$$

statement (1.13)

$$\|h_{T_0 \cup T_1}\|_2 \leq \alpha \varepsilon + \rho s^{-1/2} \|h_{T_0^c}\|_1$$

where $\alpha \equiv \dfrac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}$, $\rho = \dfrac{\sqrt{2}\,\delta_{2s}}{1-\delta_{2s}}$

From statement (1.14), we have

$$\|h_{(T_0 \cup T_1)^c}\|_2 = \Big\|\sum_{j\geq 2} h_{T_j}\Big\|_2 \leq \sum_{j\geq 2} \|h_{T_j}\|_2 \leq s^{-1/2}\|h_{T_0^c}\|_1$$

And, from previous results,

$$(1-\delta_{2s})\|h_{T_0 \cup T_1}\|_2^{\cancel{2}} \leq 2\varepsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s}\sum_{j\geq 2}(\|h_{T_j}\|_2)) \cancel{\|h_{T_0 \cup T_1}\|}$$

$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \leq \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\cdot\varepsilon + \frac{\sqrt{2}\,\delta_{2s}}{1-\delta_{2s}}\cdot s^{-1/2}\|h_{T_0^c}\|_1$$

$$\therefore \boxed{\|h_{T_0 \cup T_1}\|_2 \leq \alpha\varepsilon + \rho s^{-1/2}\|h_{T_0^c}\|_1}$$ (proved)

statement (1.14)

$\cancel{\|h_{T_0 \cup T_1}\|_2 \leq \|h_{T_0 \cup T_1}\|_2 \; \|h_{(T_0 \cup T_1)}\|_2}$

$$\|h_{T_0 \cup T_1}\|_2 \leq \alpha\varepsilon + \rho\|h_{T_0 \cup T_1}\|_2 + 2\rho e_0 \quad \text{where } e_0 = s^{-1/2}\|x - x_s\|_1$$

from statement (1.13), $\|h_{T_0 \cup T_1}\|_2 \leq \alpha\varepsilon + \rho s^{-1/2}\|h_{T_0^c}\|_1$

from statement (1.7), we have

$$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + 2\|x_{T_0^c}\|_1$$

$$\Rightarrow \|h_{T_0 \cup T_1}\|_2 \leq \alpha\varepsilon + \rho s^{-1/2}\|h_{T_0}\|_1 + 2\rho\|x_{T_0^c}\|_1 s^{-1/2}$$

but $\|x_{T_0^c}\|_1 = \|x - x_s\|_1 = e s^{1/2}$

$$\Rightarrow \|h_{T_0 \cup T_1}\| \leq \alpha\varepsilon + 2\rho e_0 + \rho\|h_{T_0}\|_1$$

use $\|h_{T_0}\|_1 \leq \|h_{T_0}\|_2 \times \sqrt{s}$ and $\|h_{T_0}\|_2 \leq \|h_{T_0} + h_{T_1}\|_2 = \|h_{T_0 \cup T_1}\|_2$

to get the result.

Statement (1.15)

$$\|h\|_2 \leq \|h_{T_0 \cup T_1}\|_2 + \|h_{(T_0\cup T_1)^c}\|_2 \leq 2\|h_{(T_0\cup T_1)}\|_2 + 2e_0$$

$$\leq 2(1-\rho)^{-1}(\alpha\varepsilon + (1+\rho)e_0)$$

the first inequality

$\|h\|_2 \leq \|h_{T_0\cup T_1}\|_2 + \|h_{(T_0\cup T_1)^c}\|_2$ follows from — (i)

triangle inequality.

Now, from statement (1.8), we have

$$\|h_{(T_0\cup T_1)^c}\|_2 \leq \|h_{T_0}\|_2 + 2e_0 \leq \|h_{T_0\cup T_1}\|_2 + 2e_0$$

$$\Rightarrow \|h_{T_0\cup T_1}\|_2 + \|h_{(T_0\cup T_1)^c}\|_2 \leq 2\|h_{T_0\cup T_1}\|_2 + 2e_0 \quad \text{—(ii)}$$

from statement (1.14), we have

$$\|h_{T_0\cup T_1}\|_2 \leq \alpha\varepsilon + \rho\|h_{T_0\cup T_1}\|_2 + 2\rho e_0$$

$$\Rightarrow \|h_{T_0\cup T_1}\|_2 \leq (1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0) \quad \text{—(iii)}$$

From (ii) and (iii), we get

$$\|h\|_2 \leq 2\|h_{(T_0\cup T_1)}\|_2 + 2e_0 \leq 2(1-\rho)^{-1}(\alpha\varepsilon + 2\rho e_0)$$

Statement 1.16

$$\underbrace{\|h\|_1 = \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1}_{\substack{\text{from triangle inequality} \\ (h = h_{T_0} + h_{T_0^c})}} \leq 2(1+\rho)(1-\rho)^{-1}\|x_{T_0^c}\|_1$$

From lemma 2.2 (paper); $\|h_{T_0}\|_1 \leq \rho\|h_{T_0^c}\|_1$

$$\Rightarrow \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 \leq (\rho+1)\|h_{T_0^c}\|$$

$$\Rightarrow \|h\|_1 \leq (1+\rho)\|h_{T_0^c}\| \quad \text{—(i)}.$$

Using result from statement ~~ko~~ (1.7) i.e.,

$$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + 2\|x_{T_0^c}\|_1$$, we get

$$\|h_{T_0^c}\|_1 \leq \rho\,\|h_{T_0^c}\| + 2\|x_{T_0^c}\|_1$$

$$\Rightarrow \quad \|h_{T_0^c}\| \leq 2(1-\rho)^{-1}\|x_{T_0^c}\|_1 \quad \text{—(ii)}$$

from (i) and (ii), we get

$$\boxed{\|h\|_1 \leq 2(1+\rho)(1-\rho)^{-1}\|h_{T_0^c}\|}$$ (proved)