Assignment 1: CS 754, Spring 2024-25

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Declaration: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

1. Consider positive integers s and t such that $s < t$. Argue which of the following statements is true: (i) $\delta_s < \delta_t$, (ii) $\delta_t < \delta_s$, (iii) $\delta_s = \delta_t$, (iv) It is not possible to establish a precise equality/inequality between δ_s and δ_t. [10 points]

Soln:

Let us define sets $\mathcal{S}$ and $\mathcal{T}$ in the following way:

- $\mathcal{S}$: subset of $\{1, 2, ..., n\}$ with $|\mathcal{S}| \leq s$
- $\mathcal{T}$: subset of $\{1, 2, ..., n\}$ with $|\mathcal{T}| \leq t$

Since $s < t$, number of possible such subsets $\mathcal{S}$ is less than that of $\mathcal{T}$.

Lets denote the maximal and minimal eigen-values in for both the subsets:

$$\lambda_{min,S} = \min_{S,\theta} \frac{\|A_S\theta\|^2}{\|\theta\|^2} \tag{1}$$

$$\lambda_{max,S} = \max_{S,\theta} \frac{\|A_S\theta\|^2}{\|\theta\|^2} \tag{2}$$

$$\lambda_{min,T} = \min_{T,\theta} \frac{\|A_T\theta\|^2}{\|\theta\|^2} \tag{3}$$

$$\lambda_{max,T} = \max_{T,\theta} \frac{\|A_T\theta\|^2}{\|\theta\|^2} \tag{4}$$

Since the number of possible sub-matrices covered by $\mathcal{S}$ is smaller than that covered by $\mathcal{T}$ and the one covered by the latter already contains the earlier one, i.e., $S \subset T$, we get:

$$\lambda_{min,S} \geq \lambda_{min,T}$$

and,

$$\lambda_{max,S} \leq \lambda_{max,T}$$

$\implies$

$$1 - \lambda_{min,S} \leq \lambda_{min,T}$$

$$\lambda_{max,S} - 1 \leq \lambda_{max,T} - 1$$

$\implies$

$$max(1 - \lambda_{min,S}, \lambda_{max,S} - 1)) \leq max(1 - \lambda_{min,T}, \lambda_{max,T} - 1))$$

Also, since $\delta_s = max(1 - \lambda_{min}, \lambda_{max} - 1))$, we get: $\delta_s \leq \delta_t$.

Therefore, statement (i) is always true while (iii) may or may not hold (it may happen that the minimal and maximal eigen-values are same for both s and t. In that case, RIC would be equal).