Assignment 1: CS 754, Spring 2024-25

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Declaration: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

1. Consider a $m \times n$ sensing matrix $\boldsymbol{A}$ ($m < n$) with order-s restricted isometry constant (RIC) of δ_s. Let $\mathcal{S}$ be a subset of up to s elements from $\{1, 2, ..., n\}$. Let $\boldsymbol{A}_{\mathcal{S}}$ be a $m \times |S|$ sub-matrix of $\boldsymbol{A}$ with columns corresponding to indices in $\mathcal{S}$. Let λ_{max} be the maximum of the maximal eigenvalue of any matrix $\boldsymbol{A}_{\mathcal{S}}^T \boldsymbol{A}_{\mathcal{S}}$ (i.e. the maximum is taken across all possible subsets of size up to s). Let λ_{min} be the minimum of the minimal eigenvalue of any matrix $\boldsymbol{A}_{\mathcal{S}}^T \boldsymbol{A}_{\mathcal{S}}$ (i.e. the minimum is taken across all possible subsets of size up to s). Then prove that $\delta_s = \max(1 - \lambda_{min}, \lambda_{max} - 1)$. [15 points]

Soln:

A_{MxN} is a sensing matrix which obeys RIP of order-s with RIC of δ_s.
$\implies$

$$(1 - \delta_s) \, ||\theta||^2 \leq ||A\theta||^2 \leq (1 + \delta_s) \, ||\theta||^2$$

where θ is a s-sparse vector of dimension $n \times 1$.

Now, we can extend this to include any sub-matrix A_S of A of size $m \times |S|$ where S is a subset of $\{1, 2, ..., n\}$ of cardinality at-most s.

$$(1 - \delta_s) \, ||\theta||^2 \leq ||A_S\theta||^2 \leq (1 + \delta_s) \, ||\theta||^2$$

$\implies$

$$(1 - \delta_s) \; \leq \frac{||A_S\theta||^2}{||\theta||^2} \leq (1 + \delta_s) \tag{1}$$

We have seen in class that for a symmetric matrix $B^T B$, its maximum and minimum eigen-value λ_{max} is given by:

$$\lambda_{max} = \max_x \frac{||Bx||^2}{||x||^2}$$

$$\lambda_{min} = \min_x \frac{||Bx||^2}{||x||^2}$$

where x is a column vector of compatible dimensions.

Now, looking at the left-side inequality of equation (1), we obtain

$$(1 - \delta_s) \; \leq \frac{||A_S\theta||^2}{||\theta||^2}$$

For this inequality to hold for all subsets S, $(1 - \delta_s)$ must be smaller than $\min\limits_{S,\theta} \frac{||A_S\theta||^2}{||\theta||^2}$.

But this quantity is equal to λ_{min} (according to the definition of minimal eigen-value).

$$\therefore (1 - \delta_s) \leq \lambda_{min} \tag{2}$$

$$\implies \delta_s \geq 1 - \lambda_{min} \tag{3}$$

Similarly, extending this argument to the right-side inequality of equation (1), we get:

$$(1 + \delta_s) \geq \lambda_{max} \tag{4}$$

$$\implies \delta_s \geq \lambda_{max} - 1 \tag{5}$$

Finally, From equations (3) and (5), we get:

$$\delta_s \geq max(1 - \lambda_{min}, \lambda_{max} - 1) \tag{6}$$

Since, by definition, the RIC δ_s is taken to be the smallest possible constant satisfying the RIP property, we can replace the $\geq$ sign by an $=$
Therefore,

$$\delta_s = max(1 - \lambda_{min}, \lambda_{max} - 1) \tag{7}$$

(proved)