Assignment 2: CS 754, Spring 2024-25

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1. A matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ where $m < n$ is said to satisfy the modified null space property (MNSP) relative to a set $S \subset [n] := \{1, 2, ..., n\}$ if for all $\boldsymbol{v} \in \text{nullspace}(\boldsymbol{A}) - \{\boldsymbol{0}\}$, we have $\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1$ where $\bar{S}$ stands for the complement of the set S. The matrix $\boldsymbol{A}$ is said to satisfy MNSP of order s if it satisfies the MNSP relative to any set $S \subset [n]$ where $|S| \leq s$. Now answer the following questions:

   (a) Consider a given matrix $\boldsymbol{A}$ and $\boldsymbol{v} \in \text{nullspace}(\boldsymbol{A}) - \{\boldsymbol{0}\}$. Suppose the condition $\|\boldsymbol{v}_S\|_1 \leq \|\boldsymbol{v}_{\bar{S}}\|_1$ is true for set S that contains the indices of the s largest absolute value entries of $\boldsymbol{v}$. Then is this condition also true for any other set S such that $|S| \leq s$? Why (not)?

   (b) Show that the MNSP implies that $\|\boldsymbol{v}\|_1 < 2\sigma_{s,1}(\boldsymbol{v})$ for $\boldsymbol{v} \in \text{nullspace}(\boldsymbol{A}) - \{\boldsymbol{0}\}$ where $\sigma_{s,1}(\boldsymbol{v}) := \inf_{\|\boldsymbol{w}\|_0 \leq s} \|\boldsymbol{v} - \boldsymbol{w}\|_1$.

   (c) Given a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ of size $m \times n$, any s-sparse vector $\boldsymbol{x} \in \mathbb{R}^n$ is a unique solution of the P1 problem with the constraint $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ if and only if $\boldsymbol{A}$ satisfies the MNSP of order s.

   [4+4+8=16 points]

   Soln:

   (a) Yes, this condition does necessarily hold for any other subset S with $|S| \leq s$. Let $\boldsymbol{v} \in \text{nullspace}(\boldsymbol{A}) - \{\boldsymbol{0}\}$ and define S^* as the set containing the indices of the s largest (in absolute value) entries of v. By the construction of S^*, for any other subset $S \subset \{1, 2, \ldots, n\}$ with $|S| \leq s$, we have

   $$\|v_S\|_1 \leq \|v_{S^*}\|_1.$$

   We are given that

   $$\|v_{S^*}\|_1 \leq \|v_{\overline{S^*}}\|_1,$$

   where $\overline{S^*}$ denotes the complement of S^* in $\{1, 2, \ldots, n\}$.

   Notice that for any such S with $|S| \leq s$, the complement $\overline{S}$ satisfies

   $$\|v_{\overline{S}}\|_1 \geq \|v_{\overline{S^*}}\|_1,$$

   since S^* consists of the indices corresponding to the largest absolute values of v, and hence $\overline{S}$ must contain at least as much of the remaining (smaller) contributions as $\overline{S^*}$ does.

   Combining the above inequalities, we obtain:

   $$\|v_S\|_1 \leq \|v_{S^*}\|_1 \leq \|v_{\overline{S^*}}\|_1 \leq \|v_{\overline{S}}\|_1.$$

   Thus, the condition

   $$\|v_S\|_1 \leq \|v_{\overline{S}}\|_1$$

   holds for any subset S with $|S| \leq s$.

   (b) We first recall that for a given vector $\boldsymbol{v}$, its best s-term approximation error in the ℓ_1-norm is defined as

   $$\sigma_{s,1}(\boldsymbol{v}) := \inf_{\|\boldsymbol{w}\|_0 \leq s} \|\boldsymbol{v} - \boldsymbol{w}\|_1.$$

A natural choice is to take $\boldsymbol{w}$ as the vector $\boldsymbol{v}_S$ obtained by retaining the s largest (in absolute value) entries of $\boldsymbol{v}$ and setting the rest to zero. If we denote by S the index set corresponding to these entries, then

$$\sigma_{s,1}(\boldsymbol{v}) = \|\boldsymbol{v}_{\bar{S}}\|_1,$$

where $\bar{S}$ denotes the complement of S.

Assume that $\boldsymbol{A}$ satisfies the Modified Null Space Property (MNSP) of order s; that is, for every set $S' \subset [n]$ with $|S'| \leq s$ and every nonzero $\boldsymbol{v} \in \text{nullspace}(\boldsymbol{A})$, we have

$$\|\boldsymbol{v}_{S'}\|_1 < \|\boldsymbol{v}_{\bar{S}'}\|_1.$$

Let $\boldsymbol{v} \in \text{nullspace}(\boldsymbol{A}) \setminus \{0\}$ and choose S to be the set of indices corresponding to the s largest (in absolute value) entries of $\boldsymbol{v}$. Then, by the MNSP,

$$\|\boldsymbol{v}_S\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1.$$

Thus, we can write

$$\|\boldsymbol{v}\|_1 = \|\boldsymbol{v}_S\|_1 + \|\boldsymbol{v}_{\bar{S}}\|_1 < \|\boldsymbol{v}_{\bar{S}}\|_1 + \|\boldsymbol{v}_{\bar{S}}\|_1 = 2\|\boldsymbol{v}_{\bar{S}}\|_1$$

So,

$$\|\boldsymbol{v}\|_1 < 2\|\boldsymbol{v}_{\bar{S}}\|_1$$

Since $\sigma_{s,1}(\boldsymbol{v}) = \|\boldsymbol{v}_{\bar{S}}\|_1$, it follows that

$$\|\boldsymbol{v}\|_1 < 2\,\sigma_{s,1}(\boldsymbol{v}).$$

Thus, the desired inequality holds.

(c) Let $A \in \mathbb{R}^{m \times n}$ and fix an integer $s \geq 1$. Then, every s-sparse vector $x \in \mathbb{R}^n$ is the unique solution of

$$\min_{z \in \mathbb{R}^n} \|z\|_1 \quad \text{subject to } Az = Ax \text{ and } y = Ax$$

if and only if A satisfies the MNSP of order s.

We prove the above statement by establishing both directions.

Suppose that A satisfies the MNSP of order s. Let $x \in \mathbb{R}^n$ be any s-sparse vector with support

$$S = \{i : x_i \neq 0\}, \quad |S| \leq s.$$

Assume that $z \in \mathbb{R}^n$ satisfies

$$Az = Ax.$$

Then we can write

$$z = x + v \quad \text{with } v \in \text{null}(A).$$

If $v = 0$, then $z = x$. If $v \neq 0$, note that

$$z_S = x_S + v_S \quad \text{and} \quad z_{\bar{S}} = v_{\bar{S}}.$$

By the triangle inequality we have

$$\|z_S\|_1 \geq \|x_S\|_1 - \|v_S\|_1.$$

Thus,

$$\|z\|_1 = \|z_S\|_1 + \|z_{\bar{S}}\|_1 \geq \|x_S\|_1 - \|v_S\|_1 + \|v_{\bar{S}}\|_1.$$

Since A satisfies the MNSP relative to S, we know that

$$\|v_S\|_1 < \|v_{\bar{S}}\|_1.$$

It follows that

$$\|z\|_1 > \|x_S\|_1 = \|x\|_1.$$

Hence, any feasible $z \neq x$ has a strictly larger ℓ_1 norm than x, so x is the unique minimizer.

Assume that any s-sparse vector $x \in \mathbb{R}^n$ is the unique solution of

$$\min_{z \in \mathbb{R}^n} \|z\|_1 \quad \text{subject to} \quad Az = Ax.$$

We will show that A must satisfy the MNSP of order s. We argue by contradiction.

Suppose that A does <u>not</u> satisfy the MNSP of order s. Then there exists a set $S \subset [n]$ with $|S| \leq s$ and a nonzero vector $v \in nullspace(A)$ such that

$$\|v_S\|_1 \geq \|v_{\bar{S}}\|_1.$$

Now, choose any s-sparse vector $x \in \mathbb{R}^n$ with support exactly S. Since $v \in nullspace(A)$, both x and z = x+v satisfy $Az = Ax$. We now compare their ℓ_1 norms.

Since x is supported on S, we have

$$\|x\|_1 = \|x_S\|_1.$$

For the vector z, note that

$$z_S = x_S + v_S \quad \text{and} \quad z_{\bar{S}} = v_{\bar{S}}.$$

Thus, by the triangle inequality,

$$\|z_S\|_1 \geq \|x_S\|_1 - \|v_S\|_1,$$

so that

$$\|z\|_1 = \|z_S\|_1 + \|z_{\bar{S}}\|_1 \geq \|x_S\|_1 - \|v_S\|_1 + \|v_{\bar{S}}\|_1$$

because

$$\|z_{\bar{S}}\|_1 = \|v_{\bar{S}}\|_1$$

Using the assumption $\|v_S\|_1 \geq \|v_{\bar{S}}\|_1$, it follows that

$$\|z\|_1 \geq \|x_S\|_1 = \|x\|_1.$$

Since $v \neq 0$, we have $z \neq x$. Hence, there exists a distinct feasible vector z with

$$\|z\|_1 \leq \|x\|_1,$$

which contradicts the uniqueness of x as the minimizer.

Thus, our assumption must be false, and A must satisfy the MNSP of order s.