Midsem: CS 754, Advanced Image Processing, 23/2/2024

Instructions: There are 120 minutes for this exam. This exam is worth 10% of your final grade. Attempt all questions. Write brief answers - **ideally, no more than 4 sentences per sub-question.** Wherever necessary, please write equations with the meaning of all terms clearly stated. You can quote results/theorems done in class directly without proving/justifying them. Each question carries 10 points.

Useful information:

1. Given a matrix $\boldsymbol{\Phi} \in \mathbb{R}^{m \times n}, m < n$, the s-order restricted isometry constant δ_s of $\boldsymbol{\Phi}$ is the smallest number for which the following is true for any s-sparse vector $\boldsymbol{f}$: $(1-\delta_s)\|\boldsymbol{f}\|_2^2 \leq \|\boldsymbol{\Phi f}\|_2^2 \leq (1+\delta_s)\|\boldsymbol{f}\|_2^2$. If $\delta_s \in [0,1)$, then $\boldsymbol{\Phi}$ is said to obey the restricted isometry property (RIP) of order s.

2. Let $\boldsymbol{\theta}^\star$ be the result of the following minimization problem: (P1) $\min\|\boldsymbol{\theta}\|_1$ such that $\|\boldsymbol{y} - \boldsymbol{\Phi\Psi\theta}\|_2 \leq \varepsilon$, where $\boldsymbol{y}$ is an m-element measurement vector of the form $\boldsymbol{y} = \boldsymbol{\Phi x} + \boldsymbol{\eta}$, $\boldsymbol{\Phi}$ is a $m \times n$ measurement matrix ($m < n$), $\boldsymbol{\Psi}$ is a $n \times n$ orthonormal basis in which n-element signal $\boldsymbol{x}$ has a sparse representation of the form $\boldsymbol{x} = \boldsymbol{\Psi\theta}$ for (sparse) coefficient vector $\boldsymbol{\theta}$. Note that ε is an upper bound on the magnitude of the noise vector $\boldsymbol{\eta}$. Theorem 3 we studied in class states the following: If $\boldsymbol{\Phi}$ obeys the restricted isometry property with isometry constant $\delta_{2s} < \sqrt{2}-1$, then we have $\|\boldsymbol{\theta} - \boldsymbol{\theta}^\star\|_2 \leq C_1 s^{-1/2}\|\boldsymbol{\theta} - \boldsymbol{\theta_s}\|_1 + C_2\varepsilon$ where C_1 and C_2 are functions of only δ_{2s} and where $\forall i \in \mathcal{S}, [\boldsymbol{\theta_s}]_i = \theta_i; \forall i \notin \mathcal{S}, [\boldsymbol{\theta_s}]_i = 0$. Here $\mathcal{S}$ is a set containing the s largest magnitude elements of $\boldsymbol{\theta}$.

Questions:

1. Consider the CASSI architecture for compressive acqusition of hyperspectral images. Also consider the technique of color filter arrays (CFAs) for acquisition of RGB images, which can be interpreted as a form of compressive imaging. Explain the basic difference between the forward models of image acquisition in CASSI and CFAs, with the help of appropriate equations. Do *not* merely state that one model is for hyperspectral and the other is for RGB images. What are the difficulties you would face in image reconstruction if the forward model of CFAs were to be used for hyperspectral image acquisition? Justify. [6+4 = 10 points]

2. In compressed sensing, we are given measurement vector $\boldsymbol{y} \in \mathbb{R}^m$ of an unknown sparse signal $\boldsymbol{x} \in \mathbb{R}^n$ of the form $\boldsymbol{y} = \boldsymbol{\Phi x}$, where $\boldsymbol{\Phi} \in \mathbb{R}^{m \times n}$ is the known sensing matrix ($m < n$). We know that $\boldsymbol{x}$ can be inferred accurately from $\boldsymbol{y}, \boldsymbol{\Phi}$ if the matrix $\boldsymbol{\Phi}$ meets some conditions such as RIP. A curious (and therefore, very good :-)) student asks: How will you estimate $\boldsymbol{x}$ if you knew in advance that most of the elements of $\boldsymbol{x}$ are not zero in value, but equal to say $\alpha \neq 0$ in value, in the case where α is known? Your job is to answer the student's question. Now if α is unknown, how would you estimate $\boldsymbol{x}$? Would the theoretical guarantees for estimating $\boldsymbol{x}$ in the sparse case also apply in these two cases (i.e. α known and unknown)? Justify. [4+4+1+1=10 points]

3. In the estimator (P1) defined above, explain how ε is to be set in practice, if you know that the noise distribution is: (a) Uniform$(-r, r)$ with known r; (b) $\mathcal{N}(0, \sigma^2)$ with known σ. Also, what is the motivation behind minimizing $\|\boldsymbol{x}\|_1$ instead of $\|\boldsymbol{x}\|_0$? [3+4+3 = 10 points]

4. Consider two s-sparse signals $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$, both in $\mathbb{R}^n$. You have access to their compressive measurement vectors $\boldsymbol{y_1} = \boldsymbol{\Phi f_1}, \boldsymbol{y_2} = \boldsymbol{\Phi f_2}$ (both in $\mathbb{R}^m, m < n$, $\boldsymbol{\Phi} \in \mathbb{R}^{m \times n}$) respectively, but you do not have access to the signals themselves. *Without* reconstructing these signals, how will you determine approximate values of the following quantities *directly* from $\boldsymbol{y_1}, \boldsymbol{y_2}$? Assume that $\boldsymbol{\Phi}$ obeys RIP of order $2s$. [3+5+2 = 10 points]

   (a) $\|\boldsymbol{f_1} - \boldsymbol{f_2}\|_2^2$

   (b) $\boldsymbol{f_1}^t\boldsymbol{f_2}$, if you knew that $\|\boldsymbol{f_1}\|_2^2 = \|\boldsymbol{f_2}\|_2^2 = 1$

(c) List any one practical application of being able to estimate $\|\boldsymbol{f_1} - \boldsymbol{f_2}\|_2^2$ from $\boldsymbol{y_1}, \boldsymbol{y_2}$ in machine learning or image retrieval.

5. Consider that you are given the Radon projections of an image $f(x, y)$ (defined on domain Ω), in directions $\theta_1, \theta_2, ..., \theta_K; K > 1$. *Without* reconstructing the image, state how you will infer the following properties of the image *directly* from the projections? [5+5=10 points]

   (a) $\sum_{(x,y)\in\Omega} f(x, y)$

   (b) A slice of the 2D Fourier transform of f in direction θ_1 in the frequency plane and passing through the origin of the frequency plane

6. Can a Gaussian random matrix be used as a pooling matrix for RTPCR group testing? Why (not)? Can a Gaussian random matrix be used as the sensing matrix for the snapshot-based video compressed sensing camera? Why (not)? [5+5=10 points]

7. Consider the OMP algorithm to estimate sparse signal $\boldsymbol{x}$ from its compressive measurements $\boldsymbol{y} = \boldsymbol{\Phi x}$ given a sensing matrix $\boldsymbol{\Phi}$. Why does OMP never re-select a column of $\boldsymbol{\Phi}$ that was selected in some previous iteration? [6 points] If you knew that $\boldsymbol{x}$ had exactly one non-zero element, how many measurements are *necessary* to uniquely compute $\boldsymbol{x}$ from $\boldsymbol{y}, \boldsymbol{\Phi}$, using the best possible method that exists? Assume that $\boldsymbol{\Phi}$ obeys RIP of appropriate order. [4 points]