Elliptic Filter Design

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Till Now

We have seen the design of IIR filters and FIR filters. In the former, we had seen the design of

- Butterworth: Monotonic stop band and pass band
- Chebyshev: Monotonic stop band and equiripple pass band

In the latter, we saw many window types

- Hann window
- Hamming window
- Kaiser window

to list some.
For our analysis, we will take the normalized frequency as in the lectures as $\Omega_{p_L} = 1$.

Presentation

In the presentation, we will

1. motivate and look at the background theory required for Elliptic filters
2. move on to define the transfer function of the filter
3. analyze the transfer function, just as we did for Butterworth and Chebyshev filters
4. list down the steps required for the design of an Elliptic filter

Recap of Complex Functions (1)

- Say we have an open domain $\mathcal{D}$ in the complex plane. A function $f : \mathcal{D} \to \mathbb{C}$ is called holomorphic (or analytic) if the following limit exists for all $z_0 \in \mathcal{D}$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) \tag{1}$$

  We interpret it as f being differentiable in the complex sense in $\mathcal{D}$.
- All functions which are analytic apart from the appearance of isolated singularities (called poles) are meromorphic functions.
- A meromorphic function admits a Laurent series expansion in the neighborhood of a pole z_0 and the series converges uniformly on any annulus surrounding z_0 where $f(\cdot)$ is analytic.

$$f(z) = \sum_{z=-\infty}^{\infty} a_n (z - z_0)^n \tag{2}$$

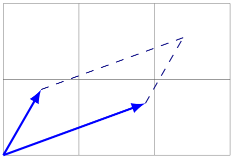
- Defining $z = \infty$ with the set $\{\zeta \in \mathbb{C} | |\zeta| \geq R \; \forall \; R > 0\}$, the compactified complex plane is $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and we will consider this in general for meromorphic functions.

Recap of Complex Functions (2)

- A complex function f is called periodic with period $\rho \in \mathbb{C} \setminus \{0\}$ if for any regular point z, we have
$$f(z+\rho) = f(z) \tag{3}$$
- We extend this as follows - a function is called multiply periodic if it has more than one primitive period, say $\rho_1, \cdots, \rho_n$ with each ρ_i independent. In this case, we denote $\Lambda = \left\{ \sum_i m_i \rho_i | m_i \in \mathbb{Z} \right\}$ as the period lattice, and the function satisfies
$$f(z + m_1\rho_1 + \cdots + m_n\rho_n) = f(z) \tag{4}$$
- **Theorem 1:** There does not exist a non-constant function with greater than 3 primitive periods.
- **Theorem 2:** There exists non-constant functions with $n = 2$ given primitive periods *if and only if* the ratio of the periods is not real.

Elliptic Functions

- A complex function $f(z)$ is called an elliptic function if it is meromorphic and doubly periodic.
- Theorem 2 implies that the ratio of the periods cannot be real. Hence, we can represent the two periods as the non-parallel edges of a parallelogram (which tesselates the complex plane) as follows:



- **Liouville's Theorem**
  1. Elliptic functions are fully characterized up to a constant multiplicative factor by their poles, zeros and their periods
  2. The sum of the residues w.r.t all the poles inside a single parallelogram of the period lattice is zero
  3. There does not exists a non-constant elliptic function that is regular in a period parallelogram
  4. The number of poles of an elliptic function in a period parallelogram (counting multiplicity) cannot be less than two

Elliptic Integrals (1)

- The elliptic integrals came about when the problem to find the arc length of an ellipse was being looked at. A very general definition of elliptic integrals is as follows.
- Let $s^2(t)$ by a cubic/quartic polynomial in t with simple zeros, and let $r(s,t)$ be a rational in s and t with at least one odd power of s. Then

$$\int r(s,t)dt \tag{5}$$

  is called an elliptic integral.
- Consider $1-\sin^2\phi \in \mathbb{C}\setminus(-\infty,0]$ and $1-k^2\sin^2\phi \in \mathbb{C}\setminus(-\infty,0]$ with the exception that at most one of them may be 0. Also, let $1-\alpha^2\sin^2\phi \in \mathbb{C}\setminus\{0\}$. Then, we define the incomplete elliptic integrals (Legendre canonical forms) as follows
- Incomplete Elliptic integral of the first kind

$$F(\phi,k)=\int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\int_0^{\sin\phi}\frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} \tag{6}$$

Elliptic Integrals (2)

- Incomplete Elliptic integral of the second kind

$$E(\phi, k) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \theta} d\theta = \int_0^{\sin \phi} \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}} dt \tag{7}$$

- Incomplete Elliptic integral of the third kind

$$\Pi(\phi, \alpha^2, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}(1 - \alpha^2 \sin^2 \theta)} = \int_0^{\sin \phi} \frac{dt}{\sqrt{1 - k^2 t^2}\sqrt{1 - t^2}(1 - \alpha^2 t^2)} dt \tag{8}$$

- Their corresponding complete integrals are given as follows

$$K(k) = F\left(\frac{\pi}{2}, k\right) \tag{9}$$

$$E(k) = E\left(\frac{\pi}{2}, k\right) \tag{10}$$

$$\Pi(\alpha^2, k) = \Pi\left(\frac{\pi}{2}, \alpha^2, k\right) \tag{11}$$

$$\tag{12}$$

Elliptic Integrals (3)

- Denoting $k' = \sqrt{1-k^2}$, we use the following notation

$$K'(k) = K(k') \quad E'(k) = E(k') \tag{13}$$

- In general, the integrals cannot be expressed in terms of elementary functions (with a few exceptions), but every elliptic integral can be brought to a form involving integrals over rational functions and the Legendre canonical forms.
- Elliptic functions, in fact, were discovered as inverse functions of elliptic integrals!

Theta Functions

- Denote $\tau \in \mathbb{C}$ such that $\Im(\tau) > 0$ as the lattice parameter, and $q = e^{j\pi\tau}$ to be the nome. It is clear that $0 < |q| < 1$.
- With this, we write the theta functions as

$$\theta_1(z, q) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+\frac{1}{2})^2}\sin((2n+1)z) \tag{14}$$

$$\theta_2(z, q) = 2\sum_{n=0}^{\infty} q^{(n+\frac{1}{2})^2}\cos((2n+1)z) \tag{15}$$

$$\theta_3(z, q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz) \tag{16}$$

$$\theta_4(z, q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2nz) \tag{17}$$

Jacobi Elliptic Functions (1)

- Having defined theta functions, we can utilize those to define a set of 12 functions, collectively known as the Jacobi elliptic functions.
- First, we write

$$q = e^{-\pi \frac{K'(k)}{K(k)}} \tag{18}$$

This allows us to write

$$k = \frac{\theta_2^2(0,q)}{\theta_3^2(0,q)} \tag{19}$$

$$k' = \frac{\theta_2^4(0,q)}{\theta_3^2(0,q)} \tag{20}$$

$$K(k) = \frac{\pi}{2}\theta_3^2(0,q) \tag{21}$$

Jacobi Elliptic Functions (2)

- Denoting $\zeta = \dfrac{\pi z}{2K(k)}$, we define the three basic Jacobi elliptic functions as

$$\mathrm{sn}(z,k) = \frac{\theta_3(0,q)}{\theta_2(0,q)}\frac{\theta_1(\zeta,q)}{\theta_4(\zeta,q)} \tag{22}$$

$$\mathrm{cn}(z,k) = \frac{\theta_4(0,q)}{\theta_2(0,q)}\frac{\theta_2(\zeta,q)}{\theta_4(\zeta,q)} \tag{23}$$

$$\mathrm{dn}(z,k) = \frac{\theta_4(0,q)}{\theta_3(0,q)}\frac{\theta_3(\zeta,q)}{\theta_4(\zeta,q)} \tag{24}$$

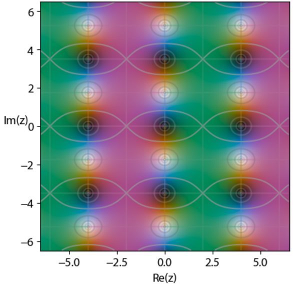
- A general Jacobi elliptic function is denoted as $\mathrm{pq}(z,k)$ where $\mathrm{p}, \mathrm{q} \in \{\mathrm{s}, \mathrm{c}, \mathrm{d}, \mathrm{n}\}$ and $\mathrm{pp}(z,k) = \mathrm{qq}(z,k) = 1$. To find any other function, we just use the following two relations

$$\frac{\mathrm{pr}(z,k)}{\mathrm{qr}(z,k)} = \mathrm{pq}(z,k) \tag{25}$$

$$\frac{1}{\mathrm{qp}(z,k)} = \mathrm{pq}(z,k) \tag{26}$$

Jacobi Elliptic Functions (3)

- The above definition may seem a bit abstract, and hence we state two more definitions for the same functions.
- The Jacobi elliptic functions are doubly periodic, meromorphic functions satisfying the following properties -
  - There is a simple zero at the corner p and simple pole at the corner q
  - The complex number p−q is equal to half the period of the function pq(z, k), i.e pq(z, k) is periodic in the direction *pq* with the period being 2(p−q) and it is also periodic in the the directions pp′ and pq′ such that periods p−p′ and p−q′ are quarter periods.
- The plot of sn(z, k) (using domain coloring method) is

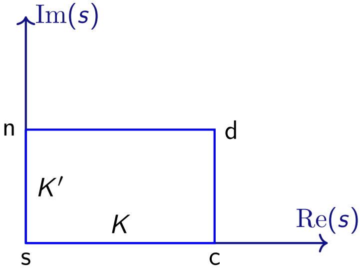


Jacobi Elliptic Functions (4)

- The other definition defines them as inverses of $F(\phi, k)$. Let
$$z = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{27}$$
Denote the Jacobi amplitude as
$$\mathrm{am}(z, k) = \phi \tag{28}$$
- Now, we define the functions as
$$\mathrm{sn}(z, k) = \sin(\mathrm{am}(z, k)) \tag{29}$$
$$\mathrm{cn}(z, k) = \cos(\mathrm{am}(z, k)) \tag{30}$$
$$\mathrm{dn}(z, k) = \frac{d}{dz}\mathrm{am}(z, k) \tag{31}$$
- Since they are doubly periodic, they must have periods such that their ratio is not real. In fact, one of the periods is real and the other is purely imaginary!
- Just like trigonometric functions, they have a real-valued period $2K(k)$ for dn, and $4K(k)$ for cn and sn.
- Just like hyperbolic functions, they have a purely imaginary period $j2K'(k)$ for sn, and $j4K'(k)$ for cn and dn.

Jacobi Elliptic Functions (5)

- We call the rectangle formed by the quarter periods $K(k)$ and $K'(k)$ as the fundamental rectangle, and is shown as follows -



- We define the function $U_N(\Omega_L)$ in an implicit manner as follows: call u to be that value which satisfies

$$\Omega_L = \mathrm{cd}(u \cdot K(k), k) \tag{32}$$

Then we have

$$U_N(\Omega_L) = \mathrm{cd}(N \cdot u \cdot K_1, k_1) \tag{33}$$

Elliptic Filter

- Having built the background, we can finally write the expression for the filter as

$$H_{LPF}(s_L)H_{LPF}(-s_L) = \frac{1}{1+\epsilon^2 U_N^2(\frac{s_L}{j})} \tag{34}$$

  This infact looks very similar to the Chebyshev filter, where U_N was replaced with the Chebyshev polynomial.
- The parameters above are given as

$$\epsilon = \sqrt{\frac{2\delta_1 - \delta_1^2}{1-2\delta_1+\delta_1^2}} \tag{35}$$

$$k_1 = \frac{\epsilon}{\sqrt{\frac{1}{\delta_2^2}-1}} \quad k_1' = \sqrt{1-k_1^2} \tag{36}$$

$$k = \frac{1}{\Omega_{s_L}} \quad k' = \sqrt{1-k^2} \tag{37}$$

$$N = \left\lceil \frac{K(k)K'(k_1)}{K'(k)K(k_1)} \right\rceil \tag{38}$$

Poles and Zeros of the Elliptic Filter (1)

- Now let's state the expression for the poles and zeros of $H_{LPF}(s_L)$.
- Set $N = 2L + r$ where $r \in \{0, 1\}$ and $\zeta_i = \mathrm{cd}(u_i \cdot K(k), k)$ where $u_i = \frac{2i-1}{N}$ for $i = 1, \cdots, L$.
- **Theorem:** The elliptic rational function $U_N(\Omega)$ can be expressed as

$$U_N(\Omega) = \Omega^r \prod_{i=1}^{L} \left[\left(\frac{\Omega^2 - \zeta_i^2}{1 - \Omega^2 k^2 \zeta_i^2} \right) \left(\frac{1 - k^2 \zeta_i^2}{1 - \zeta_i^2} \right) \right] \tag{39}$$

- Now, the zeros of the transfer function are the poles of $U_N(\Omega)$. Hence, those correspond to $\Omega_i = \frac{1}{k\zeta_i}$. Thus,

$$z_i = j\Omega_i = \frac{j}{k\zeta_i} \text{ for } i = 1, \cdots, L \tag{40}$$

Poles and Zeros of the Elliptic Filter (2)

- The poles of $H_{LPF}(s_L)$ are the solutions of

$$1 + \epsilon^2 U_N^2(\Omega) = 0 \implies U_N(\Omega) = \pm\frac{j}{\epsilon} \tag{41}$$

- Denoting $\nu_0 \in \mathbb{R}$ to be the solution of

$$\mathrm{sn}(j\nu_0 \cdot N \cdot K(k_1), k_1) = \frac{j}{\epsilon} \implies \nu_0 = -\frac{j}{N \cdot K(k_1)}\mathrm{sn}^{-1}\left(\frac{j}{\epsilon}, k_1\right) \tag{42}$$

- We write the solutions of Equation 41 as

$$p_i = j\mathrm{cd}((u_i - j\nu_0) \cdot K(k), k) \text{ for } i = 1, \cdots, L \tag{43}$$

If N is odd, there is an additional pole p_0 on the negative real-axis given as

$$p_0 = j\mathrm{cd}((1 - j\nu_0) \cdot K(k), k) = j\mathrm{sn}(j\nu_0 \cdot K(k), k) \tag{44}$$

Exercise

Check that the poles p_i stated in the above equation indeed satisfy

$$U_N(p_i) = \pm\frac{j}{\epsilon} \tag{45}$$

Hint: Use the identity

$$\mathrm{cd}(z + (2i-1)\cdot K(k), k) = (-1)^i \mathrm{sn}(z, k) \tag{46}$$

and the definition of U_N in Equations 32 and 33.

- There are some other identities too, which are useful

$$\mathrm{cd}(z + 2i \cdot K(k), k) = (-1)^i \mathrm{cd}(z, k) \tag{47}$$

$$\mathrm{cd}(jz, k) = \frac{1}{k\mathrm{dn}(z, k')} \text{ for } z \in \mathbb{R} \tag{48}$$

$$\mathrm{cd}(z + j\cdot K(k'), k) = \frac{1}{k\mathrm{cd}(z, k)} \tag{49}$$

Elliptic Filter

- Finally, having found the poles and zeros, we write

$$H_{LPF}(s_L) = \mathcal{A}\frac{\prod_i (s_L - z_i)}{\prod_i (s_L - p_i)} \tag{50}$$

  where $\mathcal{A}$ is the normalizer such that the maximum value does not exceed 1.
- Once we have the transfer function, the rest of the steps are same as the Chebyshev and Butterworth designs!

Comparison of the Filters

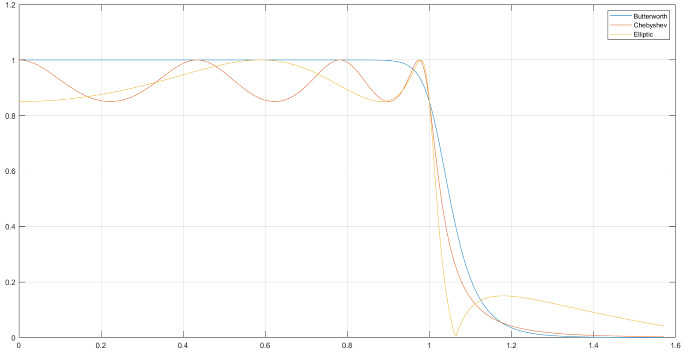


Figure: $\delta_1 = \delta_2 = 0.15$, $\Omega_{s_L} = 1.125$

- In the above figure, order of Butterworth filter is 21, of Chebyshev is 7 and of Elliptic is 4.

Phase Responses

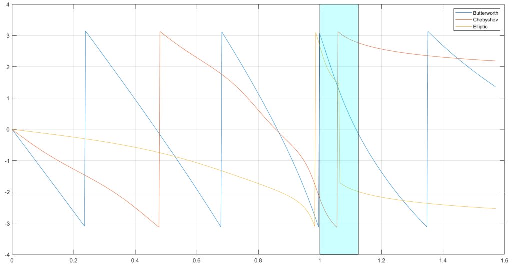


Figure: $\delta_1 = \delta_2 = 0.15$, $\Omega_{s_L} = 1.125$

Above, the cyan region denotes the transition band. Notice that the phase response of the elliptic filter is the most non-linear in the passband. Butterworth is the most linear while Chebyshev is in the middle. It clearly shows that *there is no free lunch*. We have a trade-off: Lower order of elliptic filter and good magnitude response but worst phase response.

References

- A. Antoniou, Digital Filters, 2nd ed., McGraw-Hill, New York, 1993.
- P. F. Byrd and M. D. Friedman, Handbook of Elliptic Integrals for Engineers and Scientists, Springer-Verlag, New York, 1971.
- Thomas W.. Parks, TW Parks, CS Burrus, Digital Filter Design, Wiley, 1987.
- Sophocles J. Orfanidis, Lecture Notes on Elliptic Filter Design, 2006
- Analog Elliptic Filter Design, PTC MathCAD, 2013