Take Home Quiz 1: EE 338, Spring 2024-25

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Obtain and neatly sketch the poles of the analog Butterworth lowpass filter magnitude system function, for the case N = 3 and N = 4.
Soln:

With reference to lecture conducted on *February 3, 2025*, the magnitude squared function of ButterWorth Filter in Fourier domain can be written as:

$$|H_{\text{analog, LPF}}(j\Omega)|^2 = \frac{1}{1 + \left(\frac{j\Omega}{j\Omega_c}\right)^{2N}}$$

for Ω being the angular frequency; Ω_c as the cutoff frequency; and N is the order of the filter.

This can be converted to Laplace domain as:

$$s = j\Omega_c e^{j(2k+1)\cdot\pi/2N}$$

However, since we are interested in the stable poles (those in the left-half plane), we only consider the poles for $k = 0, 1, \ldots, N-1$.
Rest of the calculation and plotting is done on the next page

For $N = 3$, the poles are calculated as follows:

$$s_k = j \cdot \Omega_c \cdot e^{j\frac{(2k+1)\pi}{6}}$$

$$s_k = \Omega_c \cdot e^{j\left(\frac{\pi}{2} + \frac{(2k+1)\pi}{6}\right)}$$

where k ranges from 0 to N-1, i.e. k $\in [0, 2]$

1. **k = 0:**

$$s_0 = \Omega_c \cdot e^{j\left(\frac{\pi}{2} + \frac{\pi}{6}\right)} = \Omega_c \cdot e^{j\left(\frac{2\pi}{3}\right)} = \Omega_c \cdot \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

This spans an angle of 120° anti-clockwise from the Re axis in z-plane on a circle of radius Ω_c

2. **k = 1:**

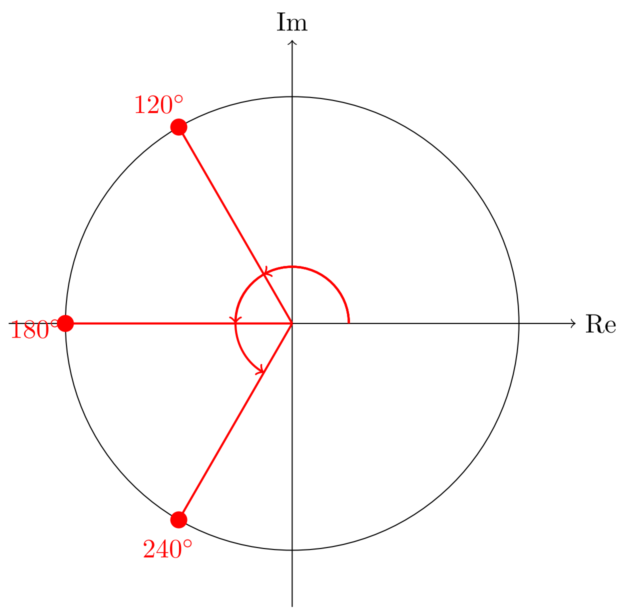
$$s_1 = \Omega_c \cdot e^{j\left(\frac{\pi}{2} + \frac{3\pi}{6}\right)} = \Omega_c \cdot e^{j(\pi)} = \Omega_c \cdot (-1)$$

This spans an angle of 180° anti-clockwise from the Re axis in z-plane on a circle of radius Ω_c

3. **k = 2:**

$$s_2 = \Omega_c \cdot e^{j\left(\frac{\pi}{2} + \frac{5\pi}{6}\right)} = \Omega_c \cdot e^{j\left(\frac{4\pi}{3}\right)} = \Omega_c \cdot \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

This spans an angle of 240° anti-clockwise from the Re axis in z-plane on a circle of radius Ω_c



For $N = 4$, the poles are calculated as follows:

$$s_k = j \cdot \Omega_c \cdot e^{j\frac{(2k+1)\pi}{8}}$$

$$s_k = \Omega_c \cdot e^{j\left(\frac{\pi}{2} + \frac{(2k+1)\pi}{8}\right)}$$

where k ranges from 0 to N-1, i.e. k $\in [0, 3]$

1. **k = 0:**

$$s_0 = \Omega_c \cdot e^{j\left(\frac{\pi}{2} + \frac{\pi}{8}\right)} = \Omega_c \cdot e^{j\left(\frac{5\pi}{8}\right)} = \Omega_c \cdot (-0.3827 + j0.9239)$$

This spans an angle of 112.5° anti-clockwise from the Re axis in z-plane on a circle of radius Ω_c

2. **k = 1:**

$$s_1 = \Omega_c \cdot e^{j\left(\frac{\pi}{2} + \frac{3\pi}{8}\right)} = \Omega_c \cdot e^{j\left(\frac{7\pi}{8}\right)} = \Omega_c \cdot (-0.9239 + j0.3827)$$

This spans an angle of 157.5° anti-clockwise from the Re axis in z-plane on a circle of radius Ω_c

3. **k = 2:**

$$s_2 = \Omega_c \cdot e^{j\left(\frac{\pi}{2} + \frac{5\pi}{8}\right)} = \Omega_c \cdot e^{j\left(\frac{9\pi}{8}\right)} = \Omega_c \cdot (-0.3827 - j0.9239)$$

This spans an angle of 202.5° anti-clockwise from the Re axis in z-plane on a circle of radius Ω_c

4. **k = 3:**

$$s_3 = \Omega_c \cdot e^{j\left(\frac{\pi}{2} + \frac{7\pi}{8}\right)} = \Omega_c \cdot e^{j\left(\frac{11\pi}{8}\right)} = \Omega_c \cdot (-0.9239 - j0.3827)$$

This spans an angle of 247.5° anti-clockwise from the Re axis in z-plane on a circle of radius Ω_c