EE338: Take-Home Midsemester Examination

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M = 104

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TASK

We are required to design an Infinite-Impulse-Response (IIR) filter with bands lying in two frequency ranges (Group I & II). The overall filter should be a multi-band pass filter and the pass-bands and stop-bands are required to be monotonic.

The filter type that satisfies these criterion is Butterworth filter, which would be used as baseline.

SPECIFICATIONS REQUIRED

- The analog signal is bandlimited to 280kHz, and it's ideally sampled with a sampling rate 630kHz.

  now, ~~bandwidth × 2~~

  2 × bandwidth < sampling rate;

  the sampling obeys Nyquist Criteria & can be reconstructed w/o loss.

- The tolerance for stopband & passband are $\delta = 0.15$ in magnitude
- $M = 104 \rightarrow Q = \text{floor}(M/11) = 9$, $R = M \pmod{11} = \cancel{4}5$
- Range of Group I frequency: $(40+5D)$ to $(70+5D)$; where $D = Q$

  lower edge $= 40 + 5\times 9 = 85$kHz

  upper edge $= 70 + 5\times 9 = 115$kHz

- Range of Group II frequency:

$$(170+5D) \text{ to } (200+5D) \quad \text{where } D=R=5.$$

$\underline{\text{Lower edge} = 170 + 5\times5 = 195 \text{ kHz}}$

$\underline{\text{Upper edge} = 200 + 5\times5 = 225 \text{ kHz}}$

- The transition on each side of the passbands is 5kHz. Group I & II ranges represent the two passband.

DESIRED FREQUENCY RESPONSE

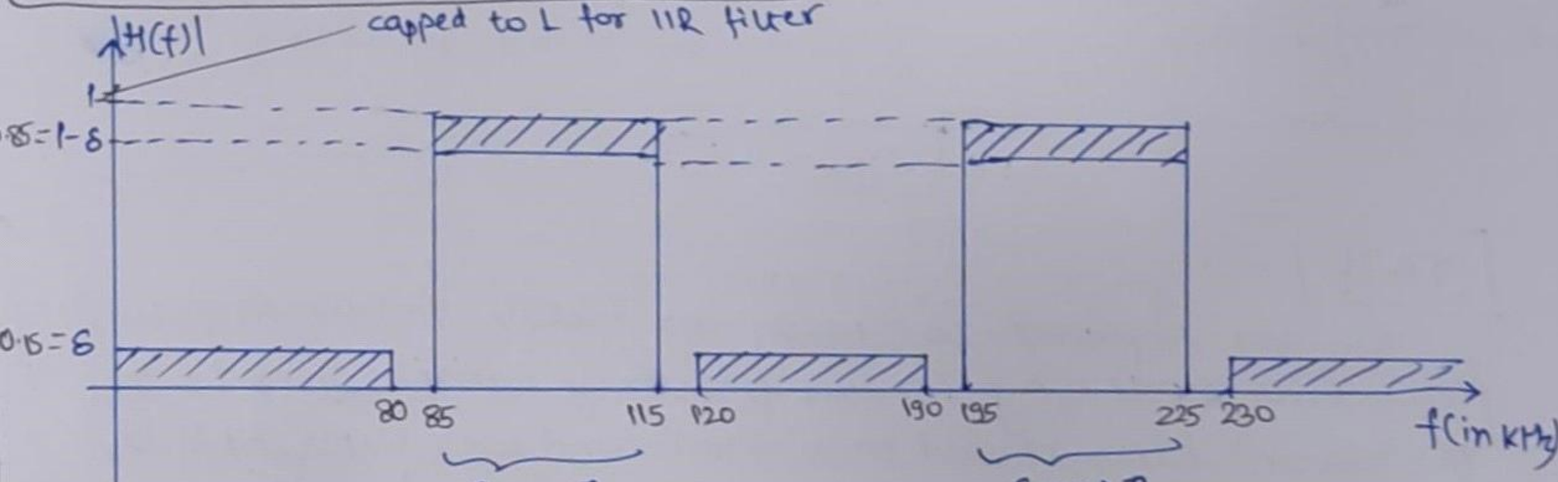


Figure-1

Realization of the filter response

The filter response can be realised cascading a Bandpass filter with Bandstop filter in series OR by adding two Bandpass filters in parallel. For the purpose of this assignment, we'll be using the two cascading filters, the details of both of them are plotted below.

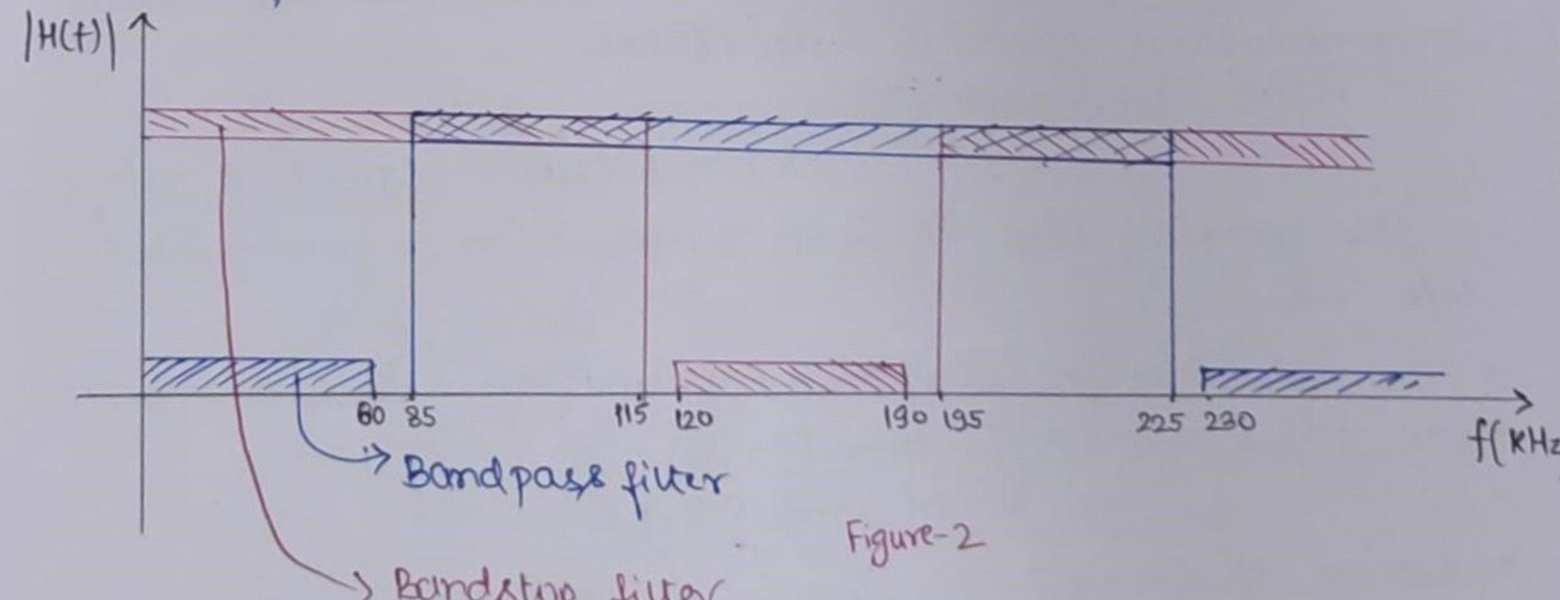


Figure-2

DESIGNING FILTER

We now require to design the Bandpass and Bandstop filter

- Bandpass filter
  - → passband range required: 85kHz − 225kHz
  - → passband tolerance: δ_1 → bandpass tolerance: 5kHz
  - → stopband tolerance: δ_2
- Bandstop filter
  - → stopband range required: 120kHz − 190kHz
  - → passband tolerance: δ_3 → stopband tolerance: 5kHz
  - → stopband tolerance: δ_4

After combining these two filters in cascade; following will be the value of tolerances & the respective f ⇒

$$
\begin{cases}
\delta_2(1-\delta_3) & 0^+ \le f < 80\,\text{kHz} \\
(1-\delta_1)(1-\delta_3) & 85\,\text{kHz} \le f < 115\,\text{kHz} \\
\delta_4(1-\delta_1) & 120\,\text{kHz} \le f < 190\,\text{kHz} \\
(1-\delta_1)(1-\delta_3) & 195\,\text{kHz} \le f < 225\,\text{kHz} \\
\delta_2(1-\delta_3) & 230\,\text{kHz} \le f < 315\,\text{kHz}
\end{cases}
$$

Equating these tolerance values to the required tolerance values as per figure-1

⇒ equating for $85\,\text{kHz} \le f < 115\,\text{kHz}$ and $195\,\text{kHz} \le f < 225\,\text{kHz}$

$$(1-\delta_1)(1-\delta_3) = 1-\delta = 0.85$$
$$1-\delta_1-\delta_3+\delta_1\delta_3 = 0.85$$
$$\boxed{\delta_1+\delta_3-\delta_1\delta_3 = 0.15}$$

⇒ equating for $0^+\,\text{kHz} \le f < 80\,\text{kHz}$ and $230\,\text{kHz} \le f < 315\,\text{kHz}$

$$\boxed{\delta_2(1-\delta_3) = \delta = 0.15}$$

But equating for $120\,\text{kHz} \le f < 190\,\text{kHz}$ gives us $\boxed{\delta_4(1-\delta_1) = \delta = 0.15}$

~~Since $1-\delta_3$ can't be zero;~~ $\boxed{\delta_2 = \delta_4}$ & $\boxed{\delta_2(1-\delta_3) = \delta_4(1-\delta_3) = 0.15}$

~~Assu By symmetry~~

Assuming symmetry; $\delta_1 = \delta_3$ & $\delta_2 = \delta_4$

$$\delta_1+\delta_3-\delta_1\delta_3 = \delta_1+\delta_1-\delta_1^2 = 0.15 \Rightarrow \delta_1^2 - 2\delta_1 + 0.15 = 0$$

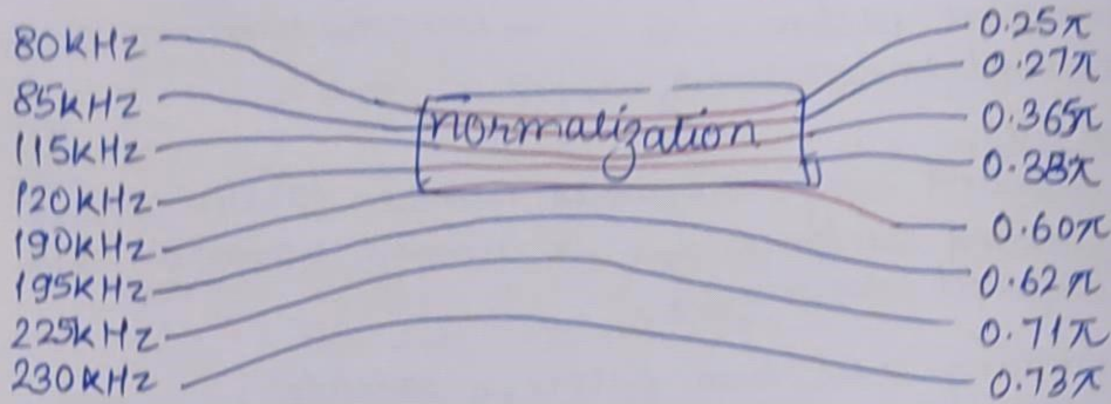
$$\delta_1 = \frac{+2 \pm \sqrt{4-0.60}}{2}$$
$$= \frac{2 \pm 1.84}{2} = 1 \pm 0.92$$
$$= 1.92 \text{ OR } 0.08$$

δ_1 can't be >1 ⇒ $\boxed{\boxed{\delta_1 = 0.08} = \delta_3}$

$$\delta_2(1-\delta_3) = 0.15 \Rightarrow \delta_2 = \frac{0.15}{0.92} = 0.16 \qquad \boxed{\delta_2 = \delta_4 = 0.16}$$

Normalizing the frequency

$$\omega = \frac{\Omega \times 2\pi}{\Omega_{sampling}} = \frac{(\Omega)2\pi}{315 \times 10^3}$$



Converting the values to analog

$$\Omega = \tan(\omega/2)$$

f(KHz)	80	85	115	120	190	195	225	230	
normalized frequency	0.25π	0.27π	0.365π	0.38π	0.60π	0.62π	0.71π	0.73π	
analog equivalent	0.41	0.45	0.645	0.68	1.38	1.47	2.04	2.21	

Calculating Bandpass filter Analog characteristics

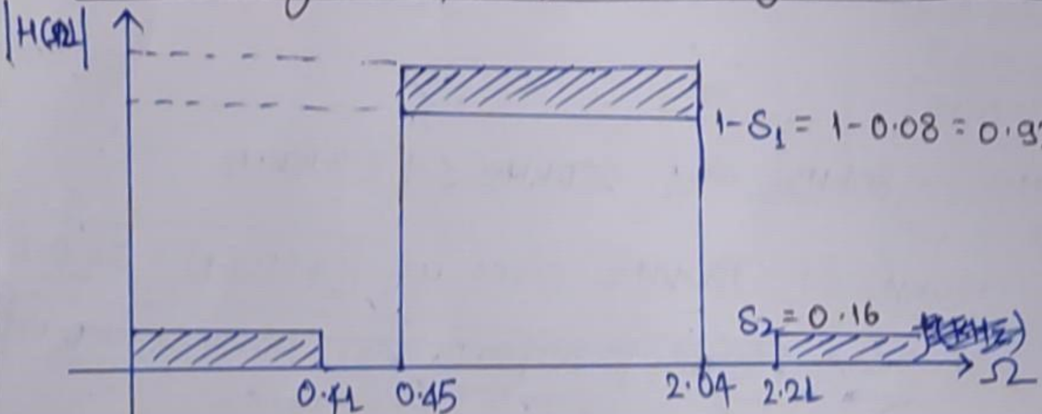
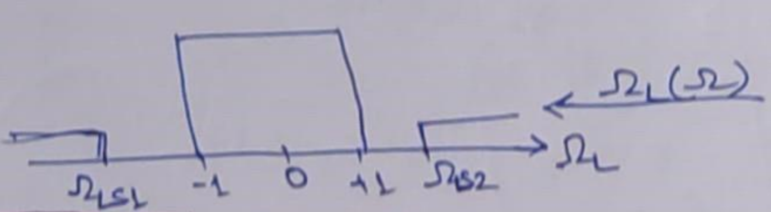


Figure-3

Using the result for converting a Bandpass filter to ButterWorth LPF; as taught in class on "10th February 2025 (Monday) (L16)":

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \quad \& \quad \Omega_0^2 = \Omega_{p1}\Omega_{p2} \quad \& \quad B = \Omega_{p2} - \Omega_{p1}$$

for



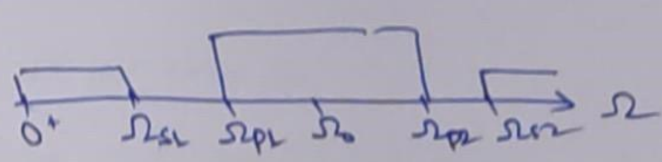


Figure - 4

Figure - 5

Equating the values of Ω by comparing the graphs in Fig-3 & Fig-4

$$\Omega_{s1} = 0.41 \qquad \Omega_{p1} = 0.45$$
$$\Omega_{s2} = 2.21 \qquad \Omega_{p2} = 2.04$$

substing the above values of Ω_{p1} & Ω_{p2} into the formulae of $\Omega_L \Rightarrow$

$$\Omega_L = \frac{\Omega^2 - (0.45)(2.04)}{\Omega(-0.45 + 2.04)} = \frac{\Omega^2 - 0.918}{1.59\Omega}$$

As per figure-4 & figure-5;

Ω_{p2} gets mapped to $+1$

$\Omega_{p1} \longrightarrow -1$

$\Omega_{s1} \longrightarrow \Omega_{Ls1}$

$\Omega_{s2} \longrightarrow \Omega_{Ls2}$

calculating the values of Ω_{Ls1} & Ω_{Ls2};

$$\Omega_{Ls1}(\Omega) = \Omega_{Ls1}(0.41) = \frac{(0.41)^2 - 0.918}{1.59 \times 0.41} = -1.15$$

$$\Omega_{Ls2}(\Omega) = \Omega_{Ls2}(02.21) = \frac{(2.21)^2 - 0.918}{1.59 \times 2.21} = 1.128 \sim 1.13$$

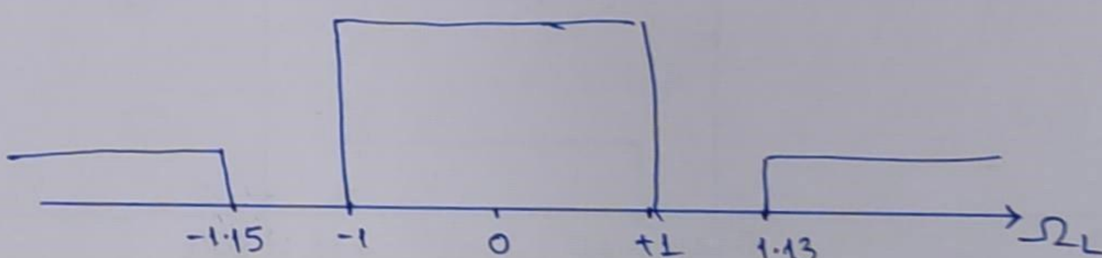


Figure - 6

<u>specifications of low pass filter</u>

Ω_p passband edge $= +1$

Ω_s stopband edge $= \min(|1.13|, |-1.15|) = 1.13$

$$D_1 = \frac{1}{(1-\delta_1)^2} - 1 = \frac{1}{(0.92)^2} - 1 = 0.18$$

$$D_2 = \frac{1}{\delta_2^2} - 1 = \frac{1}{(0.16)^2} - 1 = 38$$

Using the result for calculating the order of a ~~two~~ Butterworth filter; as obtained on '<u>04th February, 2025 (Tuesday) (L-14)</u>'

$$N \geq \left\lceil \frac{1}{2} \cdot \frac{\log(D_2/D_1)}{\log(\Omega_s/\Omega_p)} \right\rceil$$

substituting the values calculated to obtain order;

$$N \geq \left\lceil \frac{1}{2} \cdot \frac{\log\left(38/0.18\right)}{\log\left(1.13/1\right)} \right\rceil = \left\lceil \frac{1}{2} \cdot \frac{(2.3245)}{(0.053)} \right\rceil$$

$$N \geq \lceil 21.89 \rceil$$

$\boxed{N = 22}$ → N should be taken as minimum as possible for resource efficiency.

Values for Bandpass filter → Butterworth filter

$D_1 = 0.18$ $\quad \Omega_s = 1.13$ $\quad N = 22.$

$D_2 = 38$ $\quad \Omega_p = 1$

Calculating Bandstop filter Analog characteristics

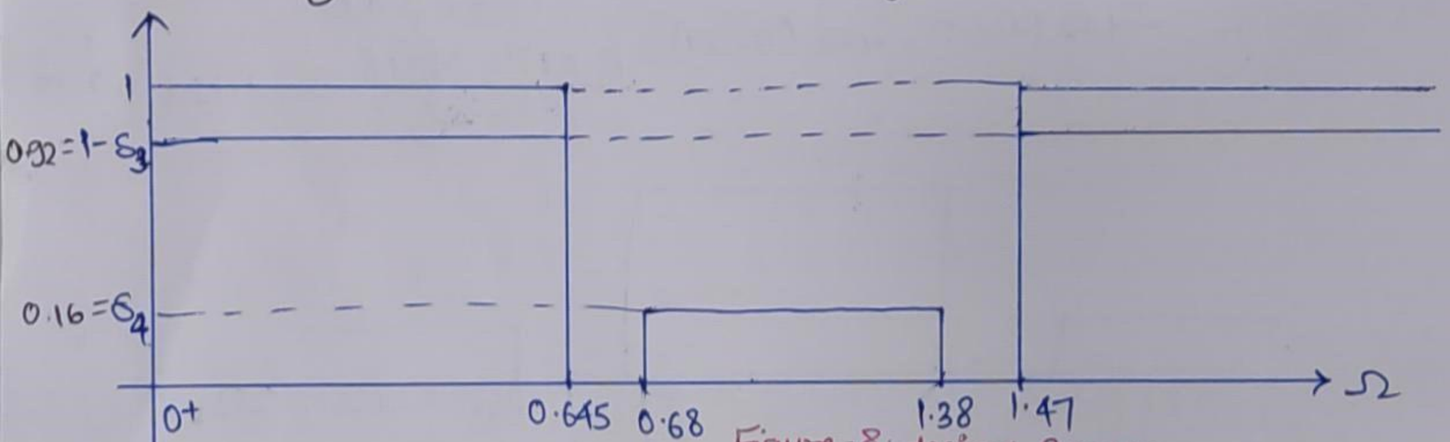


Figure-8: desired Bandstop response.

Using the result of Bandstop filter specifications as discussed in class on "10th February, 2025 (Monday) (L-16)".

$$\Omega_L = \frac{B\Omega}{\Omega_0^2 - \Omega^2} \quad \begin{cases} B = \Omega_{p2} - \Omega_{p1} \\ \Omega_0^2 = \Omega_{p1}\Omega_{p2} \end{cases}$$

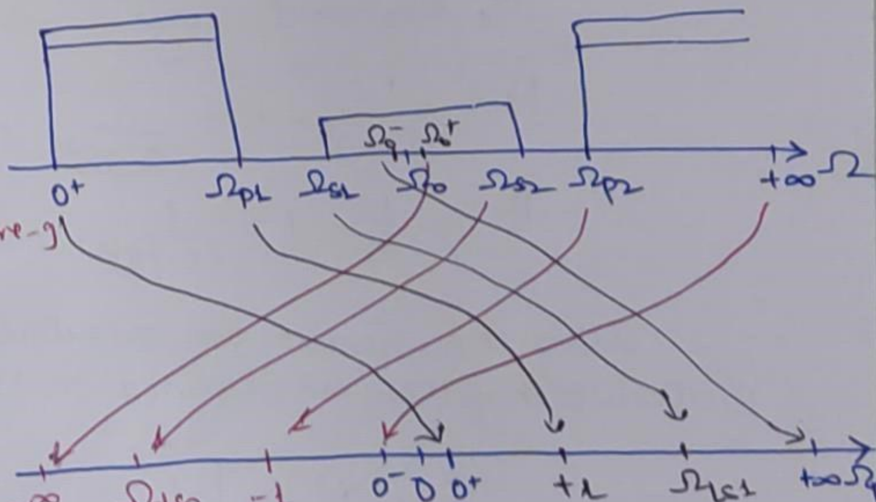


Figure-9

Figure-10.

comparing figure 8 & figure-9; we obtain :-

$$\Omega_{p1} = 0.645 \qquad \Omega_{s1} = 0.68$$
$$\Omega_{p2} = 1.47 \qquad \Omega_{s2} = 1.38$$

$$\Omega_0^2 = \Omega_{p1} \times \Omega_{p2} = 0.645 \times 1.47 = 0.94815$$
$$B = \Omega_{p2} - \Omega_{p1} = 1.47 - 0.645 = 0.825$$

$$\Omega_{LS1} = \frac{B\Omega_{s1}}{\Omega_0^2 - \Omega_{s1}^2} = \frac{(0.825)(0.68)}{(0.94815) - 0.68^2} = 1.154$$

$$\Omega_{LS2} = \frac{B\Omega_{s2}}{\Omega_0^2 - \Omega_{s2}^2} = \frac{(0.825)(1.38)}{0.94815 - 1.38^2} = -1.19$$

<u>for the lowpass filter</u>

(Ω_p) passband edge = 1

(Ω_s) stopband edge = $\min(|\Omega_{LS1}|, |\Omega_{LS2}|) = 1.154$

$$D_1 = \frac{1}{(1-\delta_3)^2} - 1 = \frac{1}{(1-0.08)^2} - 1 = 0.18$$

$$D_2 = \frac{1}{\delta_4^2} - 1 = \frac{1}{(0.16)^2} - 1 = 38$$

$$N \geq \left\lceil \frac{1}{2} \cdot \frac{\log(D_2/D_1)}{\log(\Omega_s/\Omega_p)} \right\rceil = \left\lceil \frac{1}{2} \cdot \frac{\log(38/0.18)}{\log(1.154/1)} \right\rceil$$

$$= \left\lceil \frac{1}{2} \cdot \frac{(2.3245)}{(0.062)} \right\rceil = \lceil 18.745 \rceil$$

once again; N should be chosen as less as possible to save on resources; $\boxed{N = 19}$

<u>Bandpass filter</u>

$$H_{analog, LPF_1}(s) = \frac{1}{1 + \left(\frac{s_L}{j\Omega_c}\right)^{2N}}$$

$$\frac{\Omega_p}{(D_1)^{1/2N}} \leq \Omega_c \leq \frac{\Omega_s}{(D_2)^{1/2N}}$$

$$\frac{1}{(0.18)^{1/44}} \leq \Omega_c \leq \frac{1.13}{(38)^{1/44}} \rightarrow 1.0397 \leq \Omega_c \leq 1.04$$

$1.0397 \leq \Omega_c \leq 1.0403 \rightarrow$ let's take $\boxed{\Omega_c = 1.04}$

$$H_{analog, filter 1}(s) = \frac{1}{1 + \left(\frac{s}{j(1.04)}\right)^{2 \times 22}}$$

<u>Bandstop filter</u>

$$\frac{\Omega_p}{(D_1)^{1/2N}} \leq \Omega_c \leq \frac{\Omega_s}{(D_2)^{1/2N}}$$

$$\frac{1}{(0.18)^{1/38}} \leq \Omega_c \leq \frac{1.154}{(38)^{1/38}}$$

$$1.04615 \leq \Omega_c \leq 1.04865$$

let's take $\boxed{\Omega_c = 1.047}$

$$H_{analog, filter 2}(s) = \frac{1}{1 + (s/j.1.047)^{2 \times 19}}$$

The poles "s_k" of a Butterworth filter are given by

$$s_k = j\Omega_c e^{-j\frac{(2k+1)}{2N}\pi}$$ where $k \in [0, N-1]$ & $k \in \mathbb{Z}$, $N \in \mathbb{Z}$

$$H_{filter}(s) = \frac{\prod_{k=1}^{N} s_k}{\prod_{k=1}^{N} (s - s_k)}$$

<u>for digital filter</u>

$$s \leftarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

now, since N is very large; we'll compute the poles using python.