Chebyshev Type II Filter Design
EE338:Digital Signal Processing

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Filter Design

[Chebyshev Type II]

Recall

In the class, we saw how to implement Chebyshev filter. In fact, that filter is none other than Chebyshev Type *I* Filter with equiripple pass-band and monotonic stop-band.
Can we talk about a filter which has monotonic pass-band and equiripple stop-band?
Answer is Chebyshev Type *II* Filter or Inverse Chebyshev Filter[1].

[1] I will go through all the material that I found about this beautiful filter from google and will try to explain, so please stay tuned.

Overview

Finding Magnitude Response

We have already seen in the class that the square magnitude response of Chebyshev Type I is given by following equation:

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 C_N^2(\frac{\Omega}{\Omega_P})} \tag{1}$$

Let $|F(j\Omega)|^2$ be obtained by following equation:

$$|F(j\Omega)|^2 = 1 - |H(j\Omega)|^2 = \frac{\epsilon^2 C_N^2(\frac{\Omega}{\Omega_P})}{1 + \epsilon^2 C_N^2(\frac{\Omega}{\Omega_P})} \tag{2}$$

Finally, the desired magnitude-squared response of Chebyshev Type II is obtained by replacing $\frac{\Omega}{\Omega_P}$ by $\frac{\Omega_S}{\Omega}$.

Magnitude-Squared Response

Definition of the magnitude-squared Chebyshev Type II response:

$$\boxed{|F(j\Omega)|^2 = \frac{\epsilon^2 C_N^2(\frac{\Omega_S}{\Omega})}{1 + \epsilon^2 C_N^2(\frac{\Omega_S}{\Omega})}} \quad (3)$$

where

$$\boxed{C_N(\frac{\Omega_S}{\Omega}) = \begin{cases} cos(Ncos^{-1}(\frac{\Omega_S}{\Omega})), & \text{if } |\frac{\Omega_S}{\Omega}| \leq 1. \\ cosh(Ncosh^{-1}(\frac{\Omega_S}{\Omega})), & \text{otherwise.} \end{cases}} \quad (4)$$

and Ω_S is a frequency scaling constant(can be seen as stop-band edge of Chebyshev low-pass filter), and ϵ is a constant that adjusts the influence of $C_N(\frac{\Omega_S}{\Omega})$ in the denominator of $|F(j\Omega)|^2$, and N is the order of Chebyshev polynomial.

Some Observations

Note that $0 \leq C_N^2(\frac{\Omega_S}{\Omega}) \leq 1$, for $|\Omega| \geq \Omega_S$, and $C_N^2(\frac{\Omega_S}{\Omega}) \geq 1$, for $0 \leq |\Omega| \leq \Omega_S$.

Therefore, $|\Omega| \geq \Omega_S$ defines the stopband, and $|F(j\Omega)|^2$ ripples within the stopband following the cosine function.

It is easy to see that

$$|F(j\Omega)|^2_{\Omega=0} = 1, \tag{5}$$

independent of N, and that

$$|F(j\Omega)|^2_{\Omega=\Omega_S} = \frac{\epsilon^2}{1+\epsilon^2}. \tag{6}$$

You can also note that in **Low-pass Inverse Chebyshev Filter** we use hyperbolic cosine for low frequencies (i.e. for $\Omega \leq \Omega_S$) , and the trigonometric cosine is used for high frequencies beyond Ω_S resulting in a rippling response of small magnitude.

Observations Continued....

Magnitude-squared-response is zero in the stop-band when $C_N^2(\frac{\Omega_S}{\Omega})$=0 , it happens for

$$N \times cos^{-1}(\frac{\Omega_S}{\Omega}) = \frac{(2n+1) \times \pi}{2}, \forall n \in \mathbb{N} \bigcup 0 \qquad (7)$$

Therefore the frequencies where the response is zero in stop-band are as follows,

$$\Omega_k^{zero} = \frac{\Omega_S}{cos[\frac{(2k-1)\pi}{2N}]}, k = 1, 2, ..., \mathbb{N}_P \qquad (8)$$

where $\mathbb{N}_P$=$\frac{N+1}{2}$ if N is odd, and $\mathbb{N}_P$=$\frac{N}{2}$ if N is even.
Note that if N is odd the highest frequency where the response is zero is infinity: there are only $\frac{N-1}{2}$ finite frequencies where the response is zero. If N is even there are $\frac{N}{2}$ finite frequencies where the response is zero.

Observation Continued....

Similarly, response achieves maximum value in stop-band when $C_N^2(\frac{\Omega_S}{\Omega})=1$
The frequencies for these points are as follows,

$$\Omega_k^{max} = \frac{\Omega_S}{cos[\frac{k\pi}{N}]}, k = 0, 1, 2, ..., \mathbb{N}_v \tag{9}$$

where $\mathbb{N}_v=\frac{N-1}{2}$ if N is odd, and $\mathbb{N}_P=\frac{N}{2}$ if N is even.
Note that if N is even the highest frequency where the stop-band response reaches its maximum value is infinity.
Note: N is the order of Chebyshev polynomial.

Overview

Filter Parameter Selections

Assume that δ_P and δ_S are passband and stopband tolerances.
Let's define some new parameters,

$$D_1 = \frac{1}{(1-\delta_P)^2} - 1 \tag{10}$$

$$D_2 = \frac{1}{\delta_S^2} - 1 \tag{11}$$

A_P is called the peak passband ripple .

$$A_P = -20log_{10}(1-\delta_P) \tag{12}$$

A_S is called the minimum stopband attenuation .

$$A_S = -20log_{10}(\delta_S) \tag{13}$$

Since response is monotonic in passband we can say that response takes it's minimum value at Ω=Ω_P.
Let's first compute ϵ and then Ω_P.

Filter Parameter Selections Continued....

Note that response can achieve maximum value tolerance δ_S in stopband. Thus,

$$\frac{\epsilon^2}{1+\epsilon^2} \leq \delta_S^2 \tag{14}$$

Ignoring the inequality and using equation 13 and 14 we got,

$$A_S = -10log_{10}(\frac{\epsilon^2}{1+\epsilon^2}) \tag{15}$$

Applying simple algebra we can find ϵ in terms of A_S,

$$\epsilon = \frac{1}{\sqrt{10^{(A_s/10)}-1}} = \frac{1}{\sqrt{D_2}} \tag{16}$$

Filter Parameter Selections Continued....

At Ω=Ω_P using tolerances condition we can say that,

$$\frac{\epsilon^2 C_N^2(\frac{\Omega_S}{\Omega_P})}{1+\epsilon^2 C_N^2(\frac{\Omega_S}{\Omega_P})} \geq (1-\delta_P)^2 \tag{17}$$

Thus ignoring inequalities and using equation 12 and 17 ,

$$A_P = -10 \times log_{10}(\frac{\epsilon^2 C_N^2(\frac{\Omega_S}{\Omega_P})}{1+\epsilon^2 C_N^2(\frac{\Omega_S}{\Omega_P})}) = 10 \times log_{10}(\frac{1+\epsilon^2 C_N^2(\frac{\Omega_S}{\Omega_P})}{\epsilon^2 C_N^2(\frac{\Omega_S}{\Omega_P})}) \tag{18}$$

and then solve Ω_P, for making use of the hyperbolic cosine form ,

$$\Omega_P = \frac{\Omega_S}{cosh[(\frac{1}{N}) \times cosh^{-1}(\frac{1}{\epsilon\sqrt{10^{A_P/10}-1}})]} \tag{19}$$

Also note that,

$$D_1 = 10^{A_P/10} - 1 \tag{20}$$

Determination of Order

We can find the order of Inverse Chebyshev Filter using boundary conditions at $\Omega = \Omega_P$.

$$-10 \times log_{10}(|F(j\Omega_P)|)^2 = 10 \times log_{10}\left(\frac{1 + \epsilon^2 C_N^2(\frac{\Omega_S}{\Omega_P})}{\epsilon^2 C_N^2(\frac{\Omega_S}{\Omega_P})}\right) \leq A_P \qquad (21)$$

$$\epsilon^2 C_N^2\left(\frac{\Omega_S}{\Omega_P}\right) \geq \frac{1}{10^{A_P/10} - 1} \qquad (22)$$

$$C_N\left(\frac{\Omega_S}{\Omega_P}\right) \geq \frac{1}{\sqrt{10^{A_P/10} - 1} \times \epsilon} \qquad (23)$$

Now using definition of Chebyshev polynomial for passband,

$$N_{min} \geq \frac{cosh^{-1}\left(\frac{1}{\sqrt{10^{A_P/10}-1}\times\epsilon}\right)}{cosh^{-1}\left(\frac{\Omega_S}{\Omega_P}\right)} \qquad (24)$$

Determination of Order Continued....

From equation 20 and 16 note that,

$$D_1 = 10^{A_P/10} - 1$$

$$\epsilon = \frac{1}{\sqrt{D_2}}$$

Using substitution we get,

$$N_{min} \geq = \frac{cosh^{-1}(\frac{\sqrt{D_2}}{\sqrt{D_1}})}{cosh^{-1}(\frac{\Omega_S}{\Omega_P})} = \frac{cosh^{-1}(\sqrt{\frac{D_2}{D_1}})}{cosh^{-1}(\frac{\Omega_S}{\Omega_P})} \tag{25}$$

Thus minimum order of Inverse Chebyshev Filter is given by,

$$N_{min} = \lceil \frac{cosh^{-1}(\sqrt{\frac{D_2}{D_1}})}{cosh^{-1}(\frac{\Omega_S}{\Omega_P})} \rceil \tag{26}$$

Overview

Zeros

Using magnitude-squared-response and replacing $j\Omega$ with s in equation we get,

$$F(s)F(-s) = \frac{\epsilon^2 \times C_N^2(\frac{j\Omega_S}{s})}{1 + \epsilon^2 \times C_N^2(\frac{j\Omega_S}{s})} \quad (27)$$

The zeros may be found by the setting,

$$C_N(\frac{j\Omega_S}{s_z}) = 0, \quad (28)$$

and solving for the values of s_z. The trigonometric cosine form of Chebyshev polynomial must be used since the inverse hyperbolic cosine of zero doesn't exist:

$$cos(Ncos^{-1}(\frac{j\Omega_S}{s_z})) = 0, \quad (29)$$

$$Ncos^{-1}(\frac{j\Omega_S}{s_z}) = \frac{(2z-1)\pi}{2}, z = 1, 2, ... \quad (30)$$

Zeros Continued....

Solving (30) for s_z results in

$$s_z = \frac{j\Omega_S}{cos[\frac{(2z-1)\pi}{2N}]}, \tag{31}$$

which, except for the j, is identical with (8). Therefore, the zeros of the transfer function of a Chebyshev Type II filter are as follows:

$$s_z = \pm j\Omega_z^{zero}, z = 1, 2, ..., \mathbb{N}_P \tag{32}$$

Poles

The poles of (27) may be found by setting,

$$C_N(\frac{j\Omega_S}{s_p}) = \pm\frac{j}{\epsilon}, \tag{33}$$

Since $\epsilon < 1$, then $|\frac{\pm j}{\epsilon}| > 1$, and the hyperbolic form of C_N is perhaps the more appropriate:

$$C_N(\frac{j\Omega_S}{s_p}) = cosh[Ncosh^{-1}(\frac{j\Omega_S}{s_p})] = \pm\frac{j}{\epsilon}, \tag{34}$$

Since $\pm\frac{j}{\epsilon}$ is complex, either form, the cosine or the hyperbolic, is equally valid. Either approach will yield the same equations for finding the poles. Since the approach here is identical to the approach used in class for Type-1 Chebyshev Filter, except that $\frac{s}{j\Omega_P}$ is replaced by $\frac{j\Omega_S}{s}$, it follows that the equivalent to class method would be as follows:

Poles Continued...

$$s_p = \frac{j\Omega_S}{cosh[\frac{sinh^{-1}(\frac{1}{\epsilon})}{N} + \frac{j(2p-1)\pi}{2N}]}, \tag{35}$$

For LHP Poles,

$$s_p = \frac{j\Omega_S}{cosh[\frac{sinh^{-1}(\frac{1}{\epsilon})}{N} + \frac{j(2p-1)\pi}{2N}]}, p = 1, 2, ..., N \tag{36}$$

- Finally, having found the poles and zeros, we write

$$H_{LPF}(s_L) = \mathbb{A} \times \frac{\prod_i (s - z_i)}{\prod_i (s - p_i)} \tag{37}$$

where $\mathbb{A}$ is the normalizer such that the maximum value does not exceed 1.

Here the theory part ends of Inverse Chebyshev Filter.
Thanks for sticking around.
Let's try to understand Inverse Chebyshev filter with the help of examples.

Overview

Example

Question: You have given some specifications of lowpass filter. Based on data compare Butterworth, Chebyshev, and Inverse Chebyshev filter for Ω_P=1, Ω_S=1.1155681386148188, and tolerances in both passband and stopband are 0.15.

Answer: Computed Parameters:

- For Butterworth Filter:
  - N_{min} is 22
  - Ω_P is 1
  - $D_2 \approx 43.44444444444444$
  - $D_1 \approx 0.3840830449826991$
  - $\Omega_S \approx 1.1155681386148188$
  - $\Omega_c \approx 1.023$
  - LHP poles are $s_p = \Omega_c \times e^{j(\frac{\pi}{2} + \frac{(2l-1)\pi}{2N})}$, for l=1,2,..,N (Note N=22)
- For Chebyshev Filter:
  - N_{min} is 7
  - Ω_P is 1
  - $D_2 \approx 43.44444444444444$
  - $D_1 \approx 0.3840830449826991$ and $\epsilon \approx 0.6197443384031024$

Example Continued....

- For Chebyshev Filter:
  - $\Omega_S \approx 1.1155681386148188$
  - LHP poles are given by $s_p {=} j\Omega_p \times cos(\frac{(2k-1)\pi}{2N} + j\frac{sinh^{-1}(\frac{1}{\epsilon})}{N})$, for k=1,2,..N
- For Inverse Chebyshev Filter:
  - N_{min} is 7
  - Ω_P is 1
  - $D_2 \approx 43.44444444444444$ and $\epsilon \approx 0.1517165212272521$
  - $D_1 \approx 0.3840830449826991$
  - $\Omega_S \approx 1.1155681386148188$
  - LHP poles are given by equation (36)

Clarification: The definition of ϵ is different for both Chebyshev and inverse Chebyshev filters.
Note that for same specifications Butterworth filter has order 22 while Chebyshev and Inverse Chebyshev Filter has 7.(Shows huge difference between these two filters).

Example Continued....

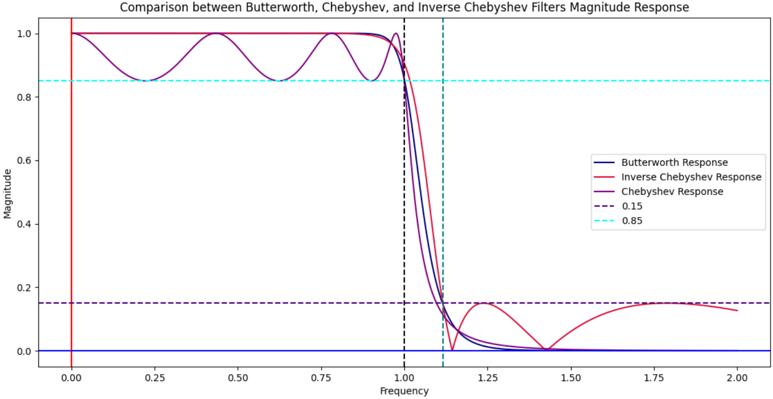

Figure: $\delta_P = \delta_S = 0.15, \Omega_{S_L} = 1.115568$

Example Continued....

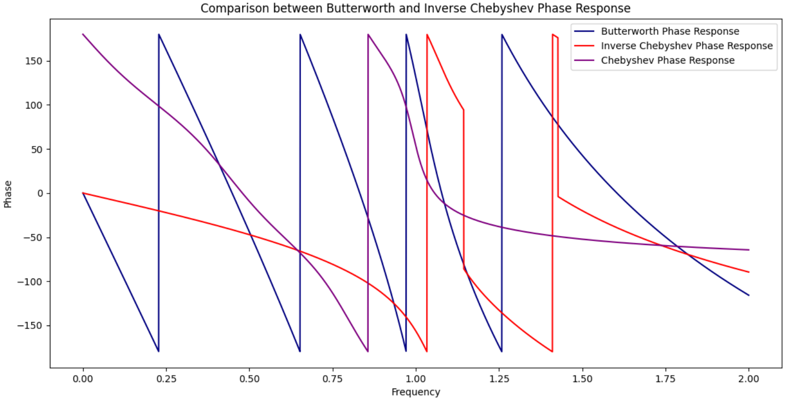

Figure: $\delta_P = \delta_S = 0.15, \Omega_{S_L} = 1.115568$

Example Continued....

- In the above figure, order of Butterworth filter is 22 , of Chebyshev is 7, and of Inverse Chebyshev is 7.
- Note that phase response of Chebyshev and Inverse Chebyshev filter is non-linear while Butterworth has mostly linear phase response.
- Here we have a trade-off:lower order of Chebyshev and Inverse Chebyshev filter and good magnitude response but worst phase response.
- We can note that there are many differences between Butterworth and Chebyshev filters for similar specifications.
- **Advantage of Chebyshev II filter over Chebyshev I filter:**Chebyshev II filters minimize peak error in the stopband instead of the passband. Minimizing peak error in the stopband instead of the passband is an advantage of Chebyshev II filters over Chebyshev filters.

References

- Design and Analysis of Inverse Chebyshev Filter, Springer
- Chebyshev Filter, Science-Direct
- A. Antoniou, Digital Filters, 2nd ed., McGraw-Hill, New York, 1993.