§2. Матричное уравнение Ляпунова

(1) $\frac{dy}{dt} = Ay \quad (t>0) \qquad A - n\times n$

(2) $HA + A^*H = -C$, $C - n\times n$, $H - n\times n - ?$ — матричное ур-е Ляпунова

T1 Пусть $C = C^* > 0$. Предположим, что $\exists$ реш. (2) $H = H^* > 0 \Rightarrow$ н.р. (1) асимпт. устойч.

Д-во: $\nexists$ н.р. не асимп. уст. $\Rightarrow \exists \lambda_k : Re\lambda_k \geq 0$. Сопоставим $\lambda_k \to v_k$ - с.в., т.е. $Av_k = \lambda_k v_k$

$((HA + A^*H)v_k, v_k) = -(Cv_k, v_k) < 0$, т.к. $C>0$

$\overset{\shortparallel}{(HAv_k, v_k) + (A^*Hv_k, v_k)} = \lambda_k (Hv_k, v_k) + (Hv_k, Av_k) = \lambda_k (Hv_k, v_k) + \bar{\lambda}_k (Hv_k, v_k) =$

$= 2Re\overset{\geq 0}{\lambda_k} \overset{>0}{(Hv_k, v_k)} \geq 0 \Rightarrow$ противоречие

T2 $C = E$, $\exists H = H^* > 0$ — реш (2) $\Rightarrow \forall$ реш. (1) $\|y(t)\| \leq e^{-\frac{t}{2\|H\|}} \cdot \sqrt{\underset{\text{число обусл-ти}}{\nu(H)}} \times \|y(0)\|$ (4) оценка Крейна $\qquad \nu(H) = \|H\|\cdot\|H^{-1}\| = \frac{\lambda_{max}}{\lambda_{min}}$

Д-во: $(Hy(t), y(t)) \geq 0$ (0 при $y(t) = 0$)

$\frac{d}{dt}(Hy(t), y(t)) = (Hy'(t), y(t)) + (Hy(t), y'(t)) = (HAy(t), y(t)) + (Hy(t), Ay(t)) = ((\overset{-E}{\overset{\shortparallel}{HA + A^*H}})y(t), y(t)) =$

$= -\|y(t)\|^2$

(5) $\frac{d}{dt}(Hy(t), y(t)) + \|y(t)\|^2 = 0$

$(Hy(t), y(t)) \leq \|H\| \cdot \|y(t)\|^2 \qquad \frac{1}{\|H\|}(Hy(t), y(t)) \leq \|y(t)\|^2$

$\frac{d}{dt}(Hy(t), y(t)) + \frac{1}{\|H\|}(Hy(t), y(t)) \leq 0 \mid \cdot e^{\frac{t}{\|H\|}}$

$\frac{d}{dt}\left(e^{\frac{t}{\|H\|}}(Hy(t), y(t))\right) \leq 0 \Rightarrow$ невозр. $\quad e^{\frac{t}{\|H\|}}(Hy(t), y(t)) \leq (Hy(0), y(0))$

$\Rightarrow \lambda_{min}\|y(t)\|^2 \leq (Hy(t), y(t)) \leq e^{-\frac{t}{\|H\|}}(Hy(0), y(0)) \leq e^{-\frac{t}{\|H\|}} \cdot \underbrace{\|H\|}_{\lambda_{max}} \cdot \|y(0)\|^2$

$\Rightarrow \|y(t)\|^2 \leq e^{-\frac{t}{\|H\|}} \cdot \frac{\lambda_{max}}{\lambda_{min}} \|y(0)\|^2$

T3. $C = E$, $H = H^* > 0$ — реш. (2), $\lambda_1, \dots, \lambda_n - SpA \Rightarrow Re\lambda_j \leq -\frac{1}{2\|H\|}$

Д-во: $((HA + A^*H)v_j, v_j) = -\|v_j\|^2 \qquad \lambda_j \mapsto v_j : \quad Av_j = \lambda_j v_j$

$(HAv_j, v_j) + (Hv_j, Av_j) = (\lambda_j + \bar{\lambda}_j)(Hv_j, v_j) = -\|v_j\|^2$

$2Re\lambda_j (Hv_j, v_j) = -\|v_j\|^2 \leq \frac{(Hv_j, v_j)}{\|H\|}$

$(Hv_j, v_j) \leq \|H\| \cdot \|v_j\|^2 \qquad \frac{(Hv_j, v_j)}{\|H\|} \geq -\|v_j\|^2$

$2Re v_j \leq -\frac{1}{\|H\|} \Rightarrow Re v_j \leq \frac{1}{2\|H\|}$