$u_t - u_{xx} = 0$ — решаем д/з

Автомодельное решение $\quad u(t,x) = t^{\alpha} f\left(\frac{x}{t^{\beta}}\right)$

$$u_t = \alpha t^{\alpha-1} f\left(\frac{x}{t^\beta}\right) - \frac{t^\alpha x}{t^{\beta+1}} \beta f'\left(\frac{x}{t^\beta}\right)$$

$$u_{xx} = \frac{t^\alpha}{t^{2\beta}} f''\left(\frac{x}{t^\beta}\right)$$

$$\alpha t^{\alpha-1} f\left(\frac{x}{t^\beta}\right) - \frac{x t^\alpha}{t^{\beta+1}} \beta f'\left(\frac{x}{t^\beta}\right) - \frac{t^\alpha}{t^{2\beta}} f''\left(\frac{x}{t^\beta}\right) = 0 \quad \Big| : t^{\alpha-1}$$

$$\alpha f\left(\frac{x}{t^\beta}\right) - \frac{\beta x}{t^\beta} f'\left(\frac{x}{t^\beta}\right) - \frac{1}{t^{2\beta-1}} f''\left(\frac{x}{t^\beta}\right) = 0$$

$$\alpha f(\xi) - \beta \xi f'(\xi) - \frac{1}{t^{2\beta-1}} f''(\xi) = 0$$

$$\beta := \frac{1}{2} \quad \Rightarrow \quad \alpha f - \frac{1}{2}\xi f - f'' = 0$$

- $\alpha = 0 \quad \Rightarrow \quad \frac{\xi}{2} f' + f'' = 0 \quad \rightsquigarrow \; / f' := y / \; \rightsquigarrow \quad \frac{\xi}{2} y + y' = 0$

$$y' = -\frac{\xi}{2} y \quad \Rightarrow \quad \ln|y| = -\frac{\xi^2}{4} + c \quad \Rightarrow \quad y = ce^{-\frac{\xi^2}{4}}$$

$$f' = ce^{-\frac{\xi^2}{4}}$$

$$f = c\int_0^{\xi} e^{-\frac{s^2}{4}} ds + c_1$$

$$u(t,x) = c\int_0^{x/t^{1/2}} e^{-\frac{s^2}{4}} ds + c_1$$

- $\alpha = -\frac{1}{2}$

$$-\frac{1}{2} f - \frac{\xi}{2} f' - f'' = 0$$

$$\frac{1}{2}(f\xi)' + f'' = 0$$

$$\left(\frac{1}{2}(f\xi) + f'\right)' = 0$$

$$\frac{1}{2}(f\xi) + f' = c$$

$$f' = -\frac{1}{2} f\xi + c \quad \Big| e^{-\frac{\xi^2}{4}}$$

$$f' e^{\xi^2/4} = -\frac{1}{2} f\xi e^{\xi^2/4} + ce^{\xi^2/4}$$

$$\left(fe^{\xi^2/4}\right)' = -ce^{\xi^2/4}$$

$$fe^{\xi^2/4} = -c\int_0^{\xi} e^{s^2/4} ds + c_1, \quad \xi = x/t^\beta$$

$$u(t,x) = t^{-\frac{1}{2}} e^{-\frac{1}{4}\left(\frac{x}{t^{1/2}}\right)^2} \left(\tilde{c}\int_0^{x/t^{1/2}} e^{s^2/4} ds + c_1\right)$$

- $\alpha = \frac{1}{2}$

$$\frac{1}{2} f - \frac{1}{2}\xi f' - f'' = 0 \quad \Big| \frac{d}{d\xi}$$

$$\frac{1}{2} f' - \frac{1}{2} f' - \frac{1}{2}\xi f'' - f''' = 0$$

$$-\frac{1}{2}\xi f'' - f''' = 0$$

$$y := f'', \quad y' = -\frac{1}{2}\xi y$$

$$y = c e^{-\xi^2/4}$$

$f'' = c e^{-\xi^2/4}$ – но тут есть лишние реш из-за

Подставим в исх: $\frac{1}{2} f - \frac{1}{2}\xi f' - c e^{-\xi^2/4} = 0 \quad \Big| \frac{2}{\xi}$

$$f' - \frac{1}{\xi} f = \frac{\tilde{c}}{\xi} e^{-\xi^2/4} \quad \Big| \frac{1}{\xi}$$

$$\frac{1}{\xi} f' - \frac{1}{\xi^2} f = \frac{\tilde{c}}{\xi^2} e^{-\xi^2/4}$$

$$\left(\frac{f}{\xi}\right)' = \frac{1}{\xi^2} \tilde{c} e^{-\xi^2/2}$$

$$f = \xi\left(\tilde{c}\int_0^{\xi} \frac{1}{s^2} e^{-s^2/4} ds + c_1\right)$$

$$u(t,x) = x\left(c \int_0^{x/t^{1/2}} \frac{1}{s^2} e^{-\frac{s^2}{4}} ds + c_1\right)$$

Т.о. реш $\begin{cases} u_t - u_{xx} = 0, & t > 0, \quad x > 0 \\ u|_{t=0} = 0 \\ u|_{x=0} = 1 \end{cases}$

$$u(t,x) = c \int_0^{x/t^{1/2}} e^{-\frac{s^2}{4}} ds + c_1$$

$x = 0 \Rightarrow u(t,0) = c_1 = 1$

$t = 0 \Rightarrow c \int_0^{+\infty} e^{-\frac{s^2}{4}} ds + 1 = /u = \frac{s}{\sqrt{2}}/ = \sqrt{2}\int_0^{+\infty} e^{-\xi^2/2} d\xi = \sqrt{2}\sqrt{2\pi} = 2\sqrt{\pi}$

т.е. $2\sqrt{\pi} c + 1 = 0$

$$c = -\frac{1}{2\sqrt{\pi}}$$

$$u = -\frac{1}{2\sqrt{\pi}} \int_0^{x/t^{1/2}} e^{-s^2/4} ds + 1 = \frac{1}{2\sqrt{\pi}} \int_{x/t^{1/2}}^{+\infty} e^{-s^2/4} ds$$

Теперь смотрим $\begin{cases} u_t - u_{xx} = 0 \\ u|_{t=0} = 0 \\ u|_{x=0} = \mu(t) \end{cases}$ $\mu(0) = 0$

$u(t,x) = \int\limits_0^t v(t-s,x)\, D_s\mu(s)\, ds$, где $v(t,x)$ - реш $\begin{cases} v_t - v_{xx} = 0 \\ v|_{t=0} = 0 \\ v|_{x=0} = 1 \end{cases}$

Проверим, что это решение

$$u|_{t=0} = \int\limits_0^0 \cdots ds = 0$$

$$u|_{x=0} = \int\limits_0^t v(t-s,0)\, D_s\mu(s)\, ds = \int\limits_0^t 1 \cdot \frac{d}{ds}\mu(s)\, ds = \mu(t)$$

$$u_t = v(0,x)\, D_s\mu_s + \int\limits_0^t v_t(t-s,x)\, D_s\mu(s)\, ds = \int\limits_0^t v_t(t-s,x)\, D_s\mu(s)\, ds$$

$$u_{xx} = \int\limits_0^t v_{xx}(t-s,x)\, D_s\mu(s)\, ds$$

$$u_t - u_{xx} = \int\limits_0^t \big(v_t(t-s,x) - v_{xx}(t-s,x)\big)\, D_s\mu(s)\, ds = \int\limits_0^t 0\, ds = 0$$

А если не дано, что $\mu(0) = 0$?

$$\begin{cases} u_t - u_{xx} = 0 \\ u|_{t=0} = 0 \\ u|_{x=0} = \mu(t) \end{cases}$$

$$u := w + \mu(t)$$

$$\begin{cases} w_t - w_{xx} = -\mu'(t) \\ w|_{t=0} = -\mu(0) \\ w|_{x=0} = 0 \end{cases}$$

Разобьем на 2 задачи $w = v + y$

$$\begin{cases} v_t - v_{xx} = 0 \\ v|_{t=0} = -\mu(0) \\ v|_{x=0} = 0 \end{cases} \qquad \begin{cases} y_t - y_{xx} = -\mu'(t) \\ y|_{t=0} = 0 \\ y|_{x=0} = 0 \end{cases}$$

$y(t,x) = \int\limits_0^t z(t-s,x,s)\, ds$, где z - реш $\begin{cases} z_t - z_{xx} = 0 \\ z|_{t=0} = -\mu'(0) \\ z|_{x=0} = 0 \end{cases}$ это было тут?

вспом. задача

$\begin{cases} u_t - u_{xx} = 0, & t>0, \ x>0 \\ u|_{t=0} = 0 \\ u_x|_{x=0} = 0 \end{cases}$ $u_x \in M^+$ Похожа на I кр. з. $\begin{cases} u_t - u_{xx} = 0 \\ u|_{t=0} = 0 \\ u|_{x=0} = 0 \end{cases}$ $u \in M^+$

$w = u_x$, продиф-ем I. кр. задачу

$$\begin{cases} w_t - w_{xx} = 0 \\ w|_{t=0} = 0 \\ w|_{x=0} = 0 \end{cases} \Rightarrow w \equiv 0 \Rightarrow u_x \equiv 0 \Rightarrow u = c(t) \to u = 0$$

$$\begin{cases} c'(t) = 0 \\ c(0) = 0 \end{cases} \Rightarrow c(t) \equiv 0$$

Т.о. д-ли единств-то реш-я задачи $\begin{cases} u_t - u_{xx} = 0 & t>0, \quad x>0 \\ u|_{t=0} = \varphi(x) \\ u_x|_{x=0} = 0 \end{cases}$

$$\overline{\varphi}(x) = \begin{cases} \varphi(x), & x \geqslant 0 \\ \mu(x), & x<0 \end{cases} \Rightarrow \begin{cases} u_t - u_{xx} = 0 & t>0, \quad x \in \mathbb{R} \\ u|_{t=0} = \overline{\varphi}(x) \\ u_x|_{x=0} = 0 \end{cases}$$

$$u(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} \overline{\varphi}(y)\, dy$$

$$u_x\Big|_{x=0} = \frac{1}{\sqrt{4\pi t}} \underline{\int_{\mathbb{R}} e^{-\frac{y^2}{4t}} \frac{y}{2t} \overline{\varphi}(y)\, dy} = 0$$

$$\to \int_0^{+\infty} e^{-y^2/4t} \frac{y}{2t} \varphi(y)\, dy + \int_{-\infty}^{0} e^{-y^2/4t} \frac{y}{2t} \mu(y)\, dy =$$

$$= \int_0^{+\infty} e^{-y^2/4t} \frac{y}{2t} \varphi(y)\, dy - \int_0^{+\infty} e^{-z^2/4t} \frac{z}{2t} \mu(-z)\, dz = 0$$

$$\text{Т.о.} \int_0^{+\infty} e^{-y^2/4t} \frac{y}{2t} \varphi(y)\, dy = \int_0^{+\infty} e^{-z^2/4t} \frac{z}{2t} \mu(-z)\, dz$$

$$\varphi(y) \equiv \mu(-y)$$

Д/З $Au_t + Bu_y = 0$, $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ Найти реш-е и характеристики

__№1__ $A = \begin{bmatrix} 2 & -6 \\ -1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 8 \\ 3 & 4 \end{bmatrix}$

__№2__ $A = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix}$

Тут матрицы не симметр, но полож. опр
Обратная есть – подсказка

__№3__ $A = \begin{bmatrix} 1 & -5 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 2 \\ 0 & -4 \end{bmatrix}$

__№4__ $\begin{cases} u_{tt} + v_x = 0 \\ v_{tt} - u_x = 0 \end{cases}$ Показать, что з. Коши для этой сист. некорректна (построить пример Адамара)