Функция Грина

З. Дирихле: $(*)\begin{cases}\Delta u = f(x) \\ u|_{\partial G} = \varphi(x'), \quad x' \in \partial G\end{cases}$

фунд. реш. ур-е Лапласа

$g(y,x)$ – **ф-ия Грина** $(*)$, если $g(y,x) = \tilde{g}(y,x) - \mathcal{E}_n(x-y)$ и

- $\tilde{g}(y,x)$ – гармоническая по y ф-ия $\forall x \in G$, $\tilde{g} \in C^1(\overline{G})$

  т.е. $\Delta_y \tilde{g} = 0$

- $\forall x \in G \quad g|_{\partial G} = 0 \implies \left(\tilde{g}(y,x) - \mathcal{E}_n(x-y)\right)\Big|_{\partial G} = 0 \quad \forall x \in G$

$$\mathcal{E}_n(x) = \begin{cases}\frac{1}{2\pi}\ln|x|, & n=2 \\ \frac{1}{-(n-2)\,\sigma_n}\,\frac{1}{|x|^{n-2}}, & n\geq 3\end{cases}, \quad \Delta_x \mathcal{E}_n(x) = 0 \quad \forall x \in G\setminus\{0\}$$

мера единич. сферы

эта ф-ия реш ур-е Лапласа везде, кроме 0

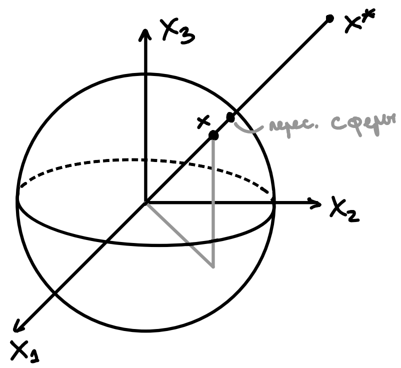
Но $g(y,x)$ – ок везде

Теорема о представл.
$f \in C(\overline{G})$, $\varphi \in C(\partial G)$
$\exists\, u \in C^2(G) \cap C^1(\overline{G})$ – класс реш з. Дирихле $(*)$
$\exists\, g(y,x) = \tilde{g}(y,x) - \mathcal{E}_n(x-y)$ – ф-ия Грина : $\tilde{g}_y \in C^2(G)\cap C^1(\overline{G})$

$$\implies \forall x \in G \quad u(x) = -\int_G g(y,x) f(y)\,dy - \int_{\partial G} \frac{\partial}{\partial \nu} g(y,x)\,\varphi(y)\,ds_y$$

только по y

Рассмотрим $\begin{cases}\Delta u = f(x), & |x| < r \\ u\big|_{|x|=r} = \varphi(x'), & x' \in \partial G\end{cases} \qquad n=3$



$$g(y,x) = \tilde{g}(y,x) + \underbrace{\frac{1}{4\pi}\cdot\frac{1}{|x-y|}}_{\mathcal{E}_3(x-y)}$$

$\Delta_y \tilde{g} = 0 \qquad \forall x \in G$

$$\left(\tilde{g}(y,x) + \frac{1}{4\pi}\cdot\frac{1}{|x-y|}\right)\Big|_{|y|=r} = 0$$

$$\tilde{g}\,\Big|_{|y|=r} = -\frac{1}{4\pi}\cdot\frac{1}{|x-y|}\,\Big|_{|y|=r}$$

а ещё $\Delta_y \frac{1}{|x-y|} = 0 \qquad \forall y \neq x$

$$g(y,x) := \frac{1}{4\pi}\,\frac{\alpha(x)}{|x^*-y|} + \frac{1}{4\pi}\cdot\frac{1}{|x-y|}, \quad x^* \notin G$$

тогда $\forall y \in G \quad y \neq x^*$

x^* подбирается таким образом, чтобы $\left.\frac{\alpha(x)}{4\pi|x^*-y|}\right|_{|y|=r} = -\left.\frac{1}{4\pi|x-y|}\right|_{|y|=r}$

Возьмем x^* - симметр. x отн. сферы:

$|x^*||x| = r^2 \quad - x$

$x_j^* = x_j \frac{r^2}{|x|^2}$, $x \in G$, $x^* \notin \overline{G}$

$$\left.|x^*-y|^2\right|_{|y|=R} = \sum_j \left(\frac{x_j R^2}{|x|^2} - y_j\right)^2 = \sum_j \left(\frac{x_j^2 R^4}{|x|^4} - 2x_j y_j \frac{R^2}{|x|^2} + y_j^2\right) =$$

$$= -\left(\sum_j 2x_j y_j \frac{r^2}{|x|^2}\right) + R^2 + \frac{R^2}{|x|^2} = \frac{R^2}{|x|^2}\left(-\sum_j 2x_j y_j + |x|^2 + R^2\right) \overset{|y|=R}{=}$$

$$= \frac{R^2}{|x|^2}\sum_j (x_j - y_j)^2 = \frac{R^2}{|x|^2}|x-y|^2$$

Т.о. $\left.|x^*-y|\right|_{|y|=R} = \left.\frac{R}{|x|}|x-y|\right|_{|y|=R}$

$$\left.\left(\frac{\alpha(x)}{4\pi R|x-y|} + \frac{1}{4\pi|x-y|}\right)\right|_{|y|=R} = 0$$

$\alpha(x) = -\frac{R}{|x|}$, $g(y,x) = -\frac{R}{|x| 4\pi |x^*-y|} + \frac{1}{4\pi|x-y|}$ — ф-ия Гр. для З. Дир. на сфере

$$\begin{cases} \Delta u = 0, & |x| < R \\ u|_{|x|=R} = \varphi(x'), & x' \in \partial G \end{cases}$$

$$u(x) = -\int_{|y|=R} \frac{\partial}{\partial \nu}\left(-\frac{R}{|x| 4\pi |x^*-y|} + \frac{1}{4\pi|x-y|}\right)\cdot \varphi(y)\, ds$$

$$\frac{\partial}{\partial y_j}\frac{1}{|x-y|} = \frac{\partial}{\partial y_j}\frac{1}{\sqrt{\sum (x_j-y_j)^2}} = \left(\frac{2(x_j-y_j)}{2|x-y|^3}\right)$$

$$\frac{\partial}{\partial \nu}\frac{1}{|x-y|} = \langle \nabla, \vec{n}\rangle, \quad \vec{n} = \begin{pmatrix} y_1/R \\ y_2/R \\ y_3/R \end{pmatrix}$$

$$\langle \nabla, \vec{n}\rangle = \sum_j \frac{(x_j-y_j)\, y_j}{R|x-y|^3}$$

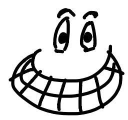
$$\frac{\partial}{\partial \nu}\frac{1}{|x^*-y|} = \sum_j \frac{y_j(x_j^*-y_j)}{R|x^*-y|^3} = \sum_j \frac{y_j(x_j^*-y_j)}{R|x-y|^3}\frac{|x|^3}{R^3} = \sum_j \frac{y_j\left(x_j \frac{R^2}{|x|^2} - y_j\right)}{R|x-y|^3}\cdot\frac{|x|^3}{R^3}$$

$$u(x) = \frac{1}{4\pi} \int\limits_{|y|=R} \sum_j \left(\frac{y_j\left(x_j \frac{R^2}{|x|^2} - y_j\right)|x|^2 - R^2(x_j - y_j)}{|x-y|^3 R^3} \right) \cdot \varphi(y)\, ds =$$

$$= \frac{1}{4\pi} \int\limits_{|y|=R} \sum_j \frac{y_j^2 (R^2 - |x|^2)}{|x-y|^3 R^3} \cdot \varphi(y)\, ds \overset{\sum y_j^2 = R^2}{=} \frac{1}{4\pi R} \int\limits_{|y|=R} \frac{R^2 - |x|^2}{|x-y|^3} \varphi(y)\, ds$$

формула Пуассона

9/3 $n=2$



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$$\begin{cases} \Delta u = 0 \\ u|_{|x|=R} = \varphi(x) \end{cases}$$

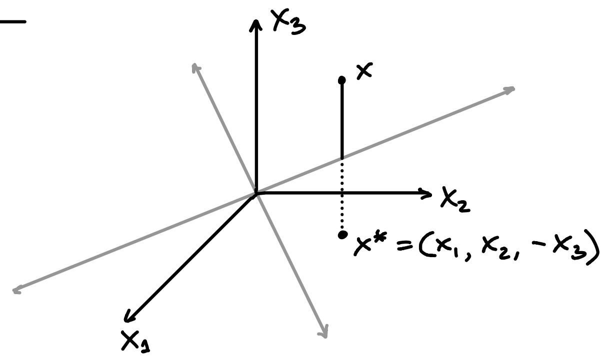
$x_1^2 + x_2^2 < R$

$\mathcal{E}_2 = \frac{1}{2\pi} \ln|x|$

$g(y,x)$ – ? (т.е. $\tilde{g}(y,x)$)

$x_1, x_2 \in \mathbb{R}, \quad x_3 > 0$

$$\begin{cases} \Delta u = 0 \\ u|_{x_3=0} = \varphi(x_1, x_2) \\ u \xrightarrow[|x|\to\infty]{} 0 \end{cases}$$



верхняя половина $\mathbb{R}^3$

$n=3$

$$g(y,x) = \tilde{g}(y,x) + \frac{1}{4\pi|x-y|} \quad ; \quad \tilde{g}(y,x) = \frac{\alpha(x)}{4\pi|x^*-y|}$$

$$|x^*-y|\Big|_{y_3=0} = |x-y|\Big|_{y_3=0}$$

$$\frac{\alpha(x)}{4\pi|x^*-y|}\Bigg|_{y_3=0} = -\frac{1}{4\pi|x-y|}\Bigg|_{y=3}$$

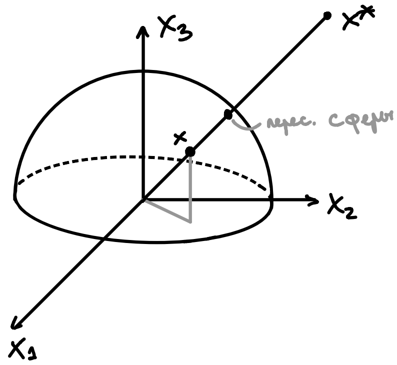
т.к. $g|_{\partial G} = 0 \Rightarrow \tilde{g} + \frac{1}{4\pi|x-y|}\Big|_{y_3=0} = 0$

$\alpha(x) = -1$

9/3 – доделать

$|x| < R$, $x_3 > 0$

$$\begin{cases} \Delta u = 0 \\ u|_{\partial G} = \varphi(x') \end{cases}$$

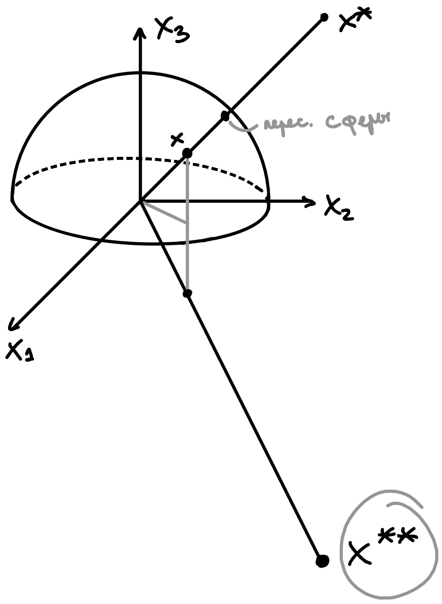


$$g = \frac{\alpha(x)}{4\pi |x^* - y|} + \frac{1}{4\pi |x - y|}$$

$g\Big|_{\substack{|y|=R \\ y_3>0}} = 0$, но ещё $g\Big|_{\substack{y_3=0 \\ |y|=R}} = 0$ – усложняет выбор α

тут было $\alpha = \frac{R}{|x|}$, а тут $\alpha = -1$

Т.е. x^* не подойдет



$$|x^* - y|\Big|_{|y|=R} = \frac{R}{|x|} \, |x - y|\Big|_{|y|=R}$$

$$|x^{**} - y|\Big|_{|y|=R} = \frac{R}{|x|} \, |x^{\perp} - y|\Big|_{|y|=R}$$

доел :)