16.1 (2)

$$\begin{cases} \Delta u = 0 \\ u|_{|x|=1} = \sin^3\varphi \end{cases}$$

$u(x_1, x_2) = u(\rho\cos\varphi, \rho\sin\varphi) = \tilde{u}(\rho, \varphi)$

$x_1 = \rho\cos\varphi \qquad x_2 = \rho\sin\varphi \qquad u(x_1, x_2) \equiv \tilde{u}(\rho, \varphi)$

$$\frac{1}{\rho^2}\left[\rho^2 \tilde{u}''_{\rho\rho} + \rho\tilde{u}'_\rho + \tilde{u}_{\varphi\varphi}\right] = 0$$

$\tilde{u}(\rho, \varphi) = \Psi(\varphi)\cdot R(\rho)$

$\Psi\rho^2 R'' + \Psi\rho R' + R\Psi'' = 0$

$$\frac{\rho^2 R''}{R} + \frac{\rho R'}{R} = \frac{\Psi''}{\Psi} = -\lambda$$

- $\Psi'' + \lambda\Psi = 0$

$\tilde{u}(\rho, \varphi) - \tilde{u}(\rho, \varphi + 2\pi) \equiv 0$

$R(\rho)[\Psi(\varphi) - \Psi(\varphi + 2\pi)] \equiv 0$

$\Psi(\varphi) - \Psi(\varphi + 2\pi) \equiv 0$

$$\begin{cases} \Psi'' + \lambda\Psi = 0 \\ \Psi(0) = \Psi(2\pi) \\ \Psi'(0) = \Psi'(2\pi) \end{cases}, \quad 0 \le \varphi \le 2\pi$$

$\sqrt{\lambda_n} = n, \quad n = 0, 1, \ldots : \quad \Psi_n(\varphi) = A_n\cos n\varphi + B_n\sin n\varphi$

$$\tilde{u} = \sum_{n=0}^{\infty} R_n(\rho)\Psi_n(\varphi)$$

$\rho^2 R_n'' + \rho R_n' - n^2 R_n = 0$

$\rho = e^r \Rightarrow r = \ln\rho$

$R_n(\rho) = R_n(e^r) = \tilde{R}_n(r)$

$\rho^2 R_n'' = \tilde{R}_n'' - \tilde{R}_n' \qquad \rho R_n' = \tilde{R}_n'$

$\tilde{R}_n'' - n^2\tilde{R}_n = 0$

$$\tilde R_0 = c_0 + c_1 r$$

$$\tilde R_n = c_{1n} e^{nr} + c_{2n} e^{-nr}$$

$$R_0(\rho) = c_0 + c_1 \ln\rho$$

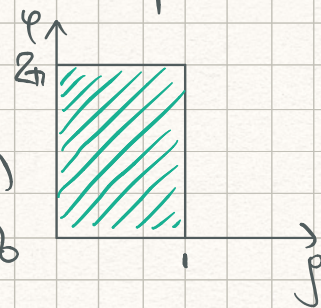
$$R_n(\rho) = c_{1n}\rho^n + c_{2n}\rho^{-n}$$

$$\tilde u(\rho,\varphi) = A_0 c_0 + A_0 c_1 \ln\rho + \sum_{n=1}^{\infty} c_{1n}\rho^n (A_n\cos n\varphi + B_n\sin n\varphi) + \sum_{n=1}^{\infty} c_{2n}\rho^{-n}(A_n\cos n\varphi + B_n\sin n\varphi)$$

$\tilde u|_{\rho=1} = \sin^3\varphi$ $\qquad$ $\tilde u \in C^2$

$\tilde u \in C^2[\Pi] \cap C(\bar\Pi)$

непр. вплоть до границ



т.е.
если задача в круге $\Rightarrow$ $c_1 = 0$, $c_{2n} = 0$

$$\begin{cases} \tilde u(\rho,\varphi) = \tilde c_0 + \sum_{n=1}^{\infty} \rho^n (\tilde A_n\cos n\varphi + \tilde B_n \sin n\varphi) \\ \tilde u|_{\rho=1} = \sin^3\varphi \end{cases}$$

задачу Дирихле получили

$$\tilde u|_{\rho=1} = \tilde c_0 + \sum_{n=1}^{\infty}(\tilde A_n\cos n\varphi + \tilde B_n\cos n\varphi) = \sin^3\varphi = \frac34\sin\varphi - \frac14\sin 3\varphi$$

$\tilde c_0 = 0$ $\quad$ $\tilde B_1 = \frac34$ $\quad$ $\tilde B_3 = -\frac14$ $\quad$ $\tilde B_n = 0$, $n \neq 1, 3$

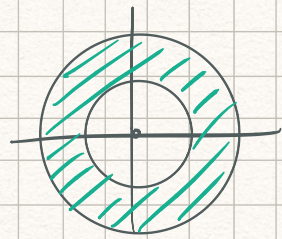
$\tilde A_n = 0$ $\quad \forall n \in \mathbb{N}$

$$\tilde u(\rho,\varphi) = \frac34\rho\sin\varphi - \frac14\rho^3 \underbrace{\sin 3\varphi}_{-4\sin^3\varphi + 3\sin\varphi}$$

$$u(x_1,x_2) = \frac34 x_2 + x_2^3 - \frac34(x_1^2 + x_2^2)\cdot x_2 = \frac34 x_2 - \frac34 x_1^2 x_2 + \frac14 x_2^3$$

16.4. ур-е Лапласа

$$\begin{cases} \Delta u = 0 \\ u|_{|x|=1} = 1 + \cos^2\varphi \\ u|_{|x|=2} = \sin^2\varphi \end{cases}$$



$$u(x_1, x_2) = \tilde{u}(\rho, \varphi)$$

$$\tilde{u}(\rho, \varphi) = c_0 + c_1 \ln\rho + \sum_{n=1}^{\infty} \rho^n (\tilde{A}_n \cos n\varphi + \tilde{B}_n \sin n\varphi) + \sum_{n=1}^{\infty} \rho^{-n} (\tilde{\tilde{A}}_n \cos n\varphi + \tilde{\tilde{B}}_n \sin n\varphi)$$

$$\tilde{u}|_{\rho=1} = c_0 + \sum_{n=1}^{\infty} (\tilde{A}_n + \tilde{\tilde{A}}_n)\cos n\varphi + (\tilde{B}_n + \tilde{\tilde{B}}_n)\sin n\varphi = 1 + \cos^2\varphi = 1 + \frac{1+\cos 2\varphi}{2} = \frac{3}{2} + \frac{1}{2}\cos 2\varphi$$

$$\Rightarrow c_0 = \frac{3}{2} \qquad \tilde{A}_2 + \tilde{\tilde{A}}_2 = \frac{1}{2} \qquad \tilde{A}_n + \tilde{\tilde{A}}_n = 0,\ n \neq 2$$

$$\tilde{B}_n + \tilde{\tilde{B}}_n = 0 \quad \forall n \in \mathbb{N}$$

$$\tilde{u}|_{\rho=2} = \frac{3}{2} + c_1 \ln 2 + \sum_{n=1}^{\infty} 2^n (\tilde{A}_n \cos n\varphi + \tilde{B}_n \sin n\varphi) + \sum_{n=1}^{\infty} \frac{1}{2^n} (\tilde{\tilde{A}}_n \cos n\varphi + \tilde{\tilde{B}}_n \sin n\varphi) = \sin^2\varphi = \frac{1-\cos 2\varphi}{2}$$

- $\frac{3}{2} + c_1 \ln 2 = \frac{1}{2}$
- $4\tilde{A}_2 + \frac{1}{4}\tilde{\tilde{A}}_2 = -\frac{1}{2}$

  $\tilde{A}_2 + \tilde{\tilde{A}}_2 = \frac{1}{2}$

- $2^n \tilde{A}_n + \frac{1}{2^n}\tilde{\tilde{A}}_n = 0,\ n \neq 2$

  $\tilde{A}_n + \tilde{\tilde{A}}_n = 0,\ n \neq 2$

  $\Rightarrow \tilde{A}_n = \tilde{\tilde{A}}_n = 0,\ n \neq 2$

- $2^n \tilde{B}_n + \frac{1}{2^n}\tilde{\tilde{B}}_n = 0$

  $\tilde{B}_n + \tilde{\tilde{B}}_n = 0$

  $\Rightarrow \tilde{B}_n = \tilde{\tilde{B}}_n = 0$

16.6. ур-е Пуассона

$$\begin{cases} \Delta u = -Axy, & 0 < x^2+y^2 < R \\ u\big|_{\sqrt{x^2+y^2}=R} = 0 & A - \text{const} \end{cases}$$

Можно решать так:

$$\frac{1}{\rho^2}\left(\rho^2 \tilde{u}_{\rho\rho} + \rho \tilde{u}_\rho + \tilde{u}_{\varphi\varphi}\right) = -A\rho^2 \sin\varphi\cos\varphi$$

$$\tilde{u} = R\Phi$$

А можно сдвиг сделать, чтобы справа 0 был.

$$u_{xx} + u_{yy} = -Axy$$

$$(u-v_0)_{xx} + (u-v_0)_{yy} = 0 \qquad v_{0xx} + v_{0yy} = -Axy$$

Возьмём: $v_0 = -\frac{A}{6}x^3y$ вот так и будет

или там $v_0 = -\frac{A}{12}x^3y - \frac{A}{12}xy^3 = -\frac{A}{12}xy(x^2+y^2)$

вот эту берём

$$u + \frac{A}{12}xy(x^2+y^2) = \tilde{u}$$

$$\begin{cases} \Delta\tilde{u} = 0 \\ \tilde{u}\big|_{\sqrt{x^2+y^2}=R} = \frac{A}{12}R^2 \cdot xy\big|_{\sqrt{x^2+y^2}=R} \end{cases}$$

$$\tilde{u}(x_1, x_2) = \tilde{u}(\rho\cos\varphi, \rho\sin\varphi) = v(\rho, \varphi)$$

$$\begin{cases} \rho^2 v_{\rho\rho} + \rho v_\rho + v_{\varphi\varphi} = 0 \\ v\big|_{\rho=R} = \frac{A}{12}R^2 \cdot R^2 \sin\varphi\cos\varphi = \frac{A}{24}R^4 \sin 2\varphi \end{cases}$$

ну а такое мы уже делали