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Сущ-е решения $\begin{cases} \Delta u = 0, & |x|>R \\ u\big|_{|x|=R} = \varphi(x') \\ u(x) \xrightarrow[|x|\to\infty]{} 0 \end{cases}$ $n=3$

$$u(x) = -\int\limits_{\partial G_y} \frac{\partial}{\partial \nu_y} g(y,x)\,\varphi(y)\,ds = \frac{1}{4\pi R}\int\limits_{|y|=R} \varphi(y)\,\frac{|x|^2-R^2}{|x-y|^3}\,ds$$

$\Delta u = 0$ – ✓ т.к. производная вносится под интегр.

$u\big|_{|x|=R} = \varphi(x')$: $u(x)-\varphi(x') \xrightarrow[x\to x']{} 0$ – докажем это

Если $\varphi(x') \equiv \frac{1}{R}$, то $u = \frac{1}{|x|} = \frac{1}{4\pi R}\int\limits_{|y|=R} \frac{1}{R}\,\frac{|x|^2-R^2}{|x-y|^3}\,ds$

$$u(x)-\varphi(x') = \frac{1}{4\pi R}\int\limits_{|y|=R}\varphi(y)\frac{|x|^2-R^2}{|x-y|^3}ds - \varphi(x')\,|x|\cdot\underbrace{\frac{1}{4\pi R}\int\limits_{|y|=R}\frac{|x|^2-R^2}{|x-y|^3}ds}_{=1} =$$

$$= \frac{1}{4\pi R}\int \underbrace{\left(\varphi(y)-\varphi(x')\frac{|x|}{R}\right)}_{\text{докажем, что оно мало}} \frac{|x|^2-R^2}{|x-y|^3}\,ds \quad *$$

$$\left|\varphi(y)-\varphi(x')+\left(-\frac{|x|}{R}+1\right)\varphi(x')\right| < \left|\varphi(y)-\varphi(x')\right| + \left|\left(1-\frac{|x|}{R}\right)\varphi(x')\right|$$

$$* \left|\frac{1}{4\pi R}\int\limits_{\Gamma}\left(\varphi(y)-\varphi(x)\right)\frac{|x|^2-R^2}{|x-y|^3}ds\right| \le \frac{\varepsilon}{4\pi R}\underbrace{\int\limits_{\Gamma}\frac{|x|^2-R^2}{|x-y|^3}ds}_{\frac{R}{|x|}} = \frac{\varepsilon}{4\pi|x|} \xrightarrow[x\to x']{} \frac{\varepsilon}{4\pi|x'|} = \frac{\varepsilon}{4\pi R} \xrightarrow[R\to 0]{} 0$$

Гиперболические уравнения

З. Коши для волнового уравнения

$$\begin{cases} u_{tt} - \Delta u = 0, & n\ge 2 \\ u\big|_{t=0} = \varphi_1(x) \\ u_t\big|_{t=0} = \varphi_2(x) \end{cases}$$

Рассмотрим ур-е : $u_{tt} - \Delta u = 0$, $n=3$

Такие решения, что $u(t,x) = u(t,|x|)$ – зависит только от $|x|$

$|x| := \rho$. Перепишем ур-е, чтобы была зависимость от t,ρ

$u_{x_1} = \frac{\partial f}{\partial \rho}\frac{\partial \rho}{\partial x_1} = \frac{x_1}{\rho}\cdot\frac{\partial u}{\partial \rho}$ т.к. $|x| = \sqrt{x_1^2+x_2^2+x_3^2}$, $\frac{\partial \rho}{\partial x_1} = \frac{2x_1}{2\sqrt{\rho^2}} = \frac{x_1}{\rho}$

$$u_{x_1x_1} = \left(\frac{x_1}{\rho}\right)^2 \cdot \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial f}{\partial \rho} x_1\left(-\frac{x_1}{\rho^3}\right) \qquad u(t,x) = f(t,|x|)$$

$u_{tt} \equiv f_{tt}$

$$f_{tt} - \sum_{j=1}^{3}\left(\left(\frac{x_j}{\rho}\right)^2 \frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial \rho} + \frac{\partial f}{\partial \rho}\frac{x_j^2}{\rho^3}\right) = 0$$

$$f_{tt} - \frac{\partial^2 f}{\partial \rho^2} - \frac{3}{\rho}\frac{\partial f}{\partial \rho} + \frac{1}{\rho}\frac{\partial f}{\partial \rho}$$

$$\rho f_{tt} - \underbrace{\rho f_{\rho\rho} - 2 f_\rho}_{= -(\rho f_\rho)_\rho - f_\rho} = 0$$

$$\rho f_{tt} - (\rho f_\rho - f)_\rho = 0$$

$(\rho f)_{tt} - (\rho f)_{\rho\rho} = 0$ — ур-е колебаний струны

$g := \rho f$

$g(t,\rho) =$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \rho}\right)\underbrace{\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \rho}\right) g}_{:= v} = v_t - v_\rho = 0$$

$\rho = -t + c \implies I\int: \rho + t = \mathrm{const} \implies v = \varphi(\rho + t)$

$g_t + g_\rho = \varphi(\rho + t)$

Решаем через вспом. ур-е: $w_t + w_\rho + \varphi(\rho+t)\, w_g = 0$

$$\frac{d\rho}{dt} = 1 \qquad \frac{dg}{dt} = \varphi(\rho + t) = \varphi(g_t + c_1)$$

$$g = c_2 + \int_0^t \varphi(2s + \rho t)\, ds$$

$$c_2 = g - \int_0^t \varphi(2s + \rho t)\, ds$$

$$G = \left(\rho - t,\ g - \int_0^t \varphi(2s + \rho t)\, ds\right) = \psi(\rho - t)$$

тут что-то было но я sdox

Т.о. $\rho f = g = \psi(\rho - t) + \varphi(\rho + t)$

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u|_{t=0} = \varphi_1(|x|) \\ u_t|_{t=0} = \varphi_2(|x|) \end{cases} \qquad n = 3$$

Если $u(t,x) = f(t,|x|)$, то

$$u = \frac{\psi(\rho - t) + \varphi(\rho + t)}{\rho}$$

$$\begin{cases} u|_{t=0} = \dfrac{\psi(\rho)+\varphi(\rho)}{\rho} = \varphi_1(\rho) \\ u_t|_{t=0} = \dfrac{\psi'_{\rho-t}(\rho-t)\cdot(\rho-t)_t + \varphi'_{\rho+t}(\rho+t)\cdot(\rho+t)_t}{\rho}\Bigg|_{t=0} = \dfrac{-\psi'_\rho(\rho)+\varphi'_\rho(\rho)}{\rho} = \varphi_2(\rho) \end{cases}$$

$$\big(\varphi(\rho)-\psi(\rho)\big)_\rho = \rho\,\varphi_2(\rho)$$

$$\varphi(\rho)-\psi(\rho) = \int_0^\rho s\,\varphi_2(s)\,ds + c$$

$$\varphi(\rho) = \frac{c+\rho(\varphi_1)+\int_0^\rho s\varphi_2(s)\,ds}{2} \qquad \psi(\rho) = \frac{\rho\varphi_1(\rho) - c - \int_0^\rho s\varphi_2(s)\,ds}{2}$$

<u>Замеч</u> $\rho>0$, $t>0$ — обл. опр-я задачи

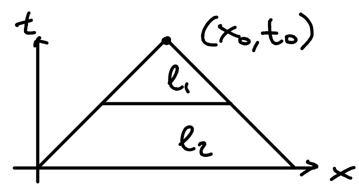
$\rho-t>0$, т.к. $\psi(\rho)$, а $\rho\geq 0$

А для $\rho-t<0$ $\quad\psi(-t) = -\varphi(t)$

Т.о. $$f = \begin{cases} \dfrac{\psi(\rho-t)+\varphi(\rho+t)}{\rho}, & \rho\geq t \\ \dfrac{-\varphi(t-\rho)+\varphi(\rho+t)}{\rho}, & \rho<t \end{cases}$$

д/з №1 $$\begin{cases} u_{tt}-u_{xx}-u_{yy}=0 \\ u|_{x=0} = \varphi_1(t,y) \\ u_x|_{x=0} = \varphi_2(t,y) \end{cases}$$ — З.К. некорректна, док-ть

№2



$(x-x_0)^2-(t-t_0)^2$

$$\int_{l_1}\left((u_x)^2+(u_t)^2\right)dx \leq \int_{l_2}\left((u_x)^2+(u_t)^2\right)dx$$

№3 $$\begin{cases} u_{tt}-\Delta u=0 \\ u|_{t=0}=0 \\ u_t|_{t=0} = \varphi(x) \end{cases} \qquad u(t,x) = \frac{t}{4\pi}\int_{|z|=1}\varphi(x+tz)\,ds \qquad x\in\mathbb{R}^3$$

A AAAAAA AAAA