$$\begin{cases} u_{tt} + \pi^2 u = f(t) \\ u|_{t=0} = 0 \\ u|_{t=1} = 0 \end{cases} \qquad f(t) = \begin{cases} 1, & t \in (0, \frac{1}{3}) \\ \alpha, & t \in (\frac{1}{3}, 1) \end{cases}$$

... пропустим

$$\int_0^1 (-v_t u_t + \pi^2 v u)\, dt + v u_t \Big|_0^1 = \int_0^1 f v\, dt$$

Если $v \in \overset{\circ}{W}{}^1_2(0,1)$, то $-\int_0^1 (v_t u_t + \pi^2 v u)\, dt = \int_0^1 f v\, dt$

Тогда $u \in \overset{\circ}{W}{}^1_2(0,1)$

Рассмотрим однородную задачу: $u_{tt} + \pi^2 u = 0$

$$\lambda^2 + \pi^2 = 0$$

$$\lambda = \pm i\pi$$

$$u_{одн} = e^{\pm i\pi t} = c_1 \cos \pi t + c_2 \sin \pi t$$

$$u_{одн}|_{t=0} = c_1 = 0 \implies u_{одн} = c_2 \sin \pi t$$

$$u_1 := \sin \pi t$$

Если мы хотим найти решение, то $(f, u_1)_{L_2} = 0$, т.е.

$$\int_0^1 f \cdot \sin \pi t\, dt = \int_0^{1/3} \sin \pi t\, dt + \alpha \int_{1/3}^1 \sin \pi t\, dt = -\frac{1}{\pi} \cos \pi t \Big|_0^{1/3} -$$

$$- \frac{\alpha}{\pi} \cos \pi t \Big|_{1/3}^1 = \frac{1}{\pi}\left(1 - \cos\frac{\pi}{3} + \alpha\left(\cos\frac{\pi}{3} - \cos \pi\right)\right) = \frac{1}{\pi}\left(1 - \frac{1}{2} + \alpha\left(\frac{1}{2} + 1\right)\right)$$

$$= \frac{1}{\pi}\left(\frac{1}{2} + \frac{3}{2}\alpha\right) = \frac{1}{2\pi}(1 + 3\alpha) := 0, \quad \alpha = -\frac{1}{3}$$

Эта задача имеет решение только при $\alpha = -\frac{1}{3}$

Разрешимость $\iff$ $f \perp u_{частн.}$

$(0, \frac{1}{3})$: $u_{tt} + \pi^2 u = 1$

частн. реш: $\frac{1}{\pi^2}$

реш: $u = \frac{1}{\pi^2} + c_1 \cos \pi t + c_2 \sin \pi t$

$$u|_{t=0} = \frac{1}{\pi^2} + c_1 = 0 \implies u = \frac{1}{\pi^2} - \frac{1}{\pi^2} \cos \pi t + c_2 \sin \pi t$$

$(\frac{1}{3}, 1)$: $u_{tt} + \pi^2 u = -\frac{1}{3}$

частн. реш: $-\frac{1}{3\pi^2}$

общ. реш: $u = -\frac{1}{3\pi^2} + \tilde{c}_1 \cos\pi t + \tilde{c}_2 \sin\pi t$

$$u\big|_{t=0} = -\frac{1}{3\pi^2} - \tilde{c}_1 = 0 \quad \Rightarrow \quad u = -\frac{1}{3\pi^2} - \frac{1}{3\pi^2}\cos\pi t + \tilde{c}_2 \sin\pi t$$

$$u \in \overset{\circ}{W}{}^1_2(0,1) \hookrightarrow C[0,1] \quad \Rightarrow \quad u \in C[0,1]$$

Склеим в $\frac{1}{3}$:

$$\frac{1}{\pi^2} - \frac{1}{\pi^2}\cos\frac{\pi}{3} + c_2 \sin\frac{\pi}{3} = -\frac{1}{3\pi^2} - \frac{1}{3\pi^2}\cos\frac{\pi}{3} + \tilde{c}_2 \sin\frac{\pi}{3}$$

$$\frac{1}{\pi^2} - \frac{1}{2\pi^2} + c_2\frac{\sqrt{3}}{3} = -\frac{1}{3\pi^2} - \frac{1}{6\pi^2} + \tilde{c}_2\frac{\sqrt{3}}{3}$$

$$\frac{1}{\pi^2} + c_2\frac{\sqrt{3}}{3} = \tilde{c}_2\frac{\sqrt{3}}{3}$$

$$\frac{\sqrt{3}}{\pi^2} + c_2 = \tilde{c}_2$$

$$u\Big|_{(\frac{1}{3},1)} = -\frac{1}{3\pi^2} - \frac{1}{3\pi^2}\cos\pi t + \left(\frac{\sqrt{3}}{\pi^2} + c_2\right)^{*} \sin\pi t$$

Решение

$$u = \begin{cases} \frac{1}{\pi^2} - \frac{1}{\pi^2}\cos\pi t + c_2 \sin\pi t, & t \in (0, \frac{1}{3}) \\ -\frac{1}{3\pi^2} - \frac{1}{3\pi^2}\cos\pi t + \left(\frac{\sqrt{3}}{\pi^2} + c_2\right)\sin\pi t, & t \in (\frac{1}{3}, 1) \end{cases}$$

$$u_t = \begin{cases} \frac{1}{\pi}\sin\pi t + \pi c_2 \cos\pi t, & t \in (0, \frac{1}{3}) \\ \pi\left(c_2 + \frac{\sqrt{3}}{\pi^2}\right)\cos\pi t + \frac{1}{3\pi}\sin\pi t, & t \in (\frac{1}{3}, 1) \end{cases}$$

$(*) = \left(c_2 + \frac{2\sqrt{3}}{3\pi^2}\right)$

У Валеры получилась другая константа (*)

$$\int_0^1 (-v_t u_t + \pi^2 v u)\,dt = \overset{\mathrm{I}}{\int_0^{1/3} (-v_t u_t + v - v\cos\pi t + \pi^2 c_2 v \sin\pi t)\,dt} +$$

$$+ \overset{\mathrm{II}}{\int_{1/3}^1 \left(-v_t u_t - \frac{v}{3} - \frac{v}{3}\cos\pi t + v(\sqrt{3} + \pi^2 c_2)\sin\pi t\right)dt} =$$

$$\mathrm{I}: \int_0^{1/3}\left(\left(-v c_2 \pi\cos\pi t\right)' - \left(\frac{v}{\pi}\sin\pi t\right)'\right)dt = \underline{-v\left(\tfrac{1}{3}\right)\cdot\frac{1}{2}\left(\pi c_2 + \frac{\sqrt{3}}{\pi}\right)} + \int_0^{1/3} v\,dt$$

$$\mathrm{II}: \int_{1/3}^1\left(\left(-v\left(\pi c_2 + \frac{2\sqrt{3}}{3\pi}\right)\cos\pi t\right)' + \left(-\tfrac{1}{3\pi} v \sin\pi t\right)' - \frac{1}{3}v\right)dt =$$

$$= \underline{v\left(\tfrac{1}{3}\right)\left(\frac{\pi}{2}c_2 + \frac{\sqrt{3}}{3\pi} + \frac{\sqrt{3}}{6\pi}\right)} - \frac{1}{3}\int_{1/3}^1 v\,dt$$

как раз $\int_0^1 f v\,dt$

сократится

У Валеры вроде правильно (*)

все класс

показали, что реш. неоднозн.

$$\begin{cases} u_{tt} + \pi^2 u = f(t) \\ u_t|_{t=0} = 0 \\ u_t|_{t=1} = 0 \end{cases} \qquad f(t) = \begin{cases} 1, & t\in(0,\frac{1}{3}) \\ \alpha, & t\in(\frac{1}{3},1) \end{cases}$$

говорят, почему док-во для з. Дирихле сработает и для з. Нейм.

$$\int_0^1 (v u_{tt} + \pi^2 v u)\, dt = \int_0^1 f v\, dt \qquad \forall v \in W_2^1$$

$$\|$$

$$\int_0^1 (v_t u_t - \pi^2 v u)\, dt + v u_t \Big|_{t=0}^{t=1} = -\int_0^1 f v\, dt$$

Однор: $u_{tt} + \pi^2 u = 0$

$c_1 \cos\pi t + c_2 \sin\pi t = u_{одн}$

$u_{одн_t} = -\pi c_1 \sin t + \pi c_2 \cos t$

$u_{одн_t}|_{t=0} = \pi c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow u_{одн} = c_1 \cos\pi t$

Ортогональность:

$$\int_0^1 f \cos\pi t\, dt = \frac{1}{\pi} \sin\pi t \Big|_0^{1/3} + \frac{\alpha}{\pi} \sin\pi t \Big|_{\pi t}^{1} = 0 \Rightarrow \sin\frac{\pi}{3}(1-\alpha) = 0 \Rightarrow \alpha = 1$$

$u_{частн} = \frac{1}{\pi^2} \Rightarrow u = c_1 \cos\pi t + c_2 \sin\pi t + \frac{1}{\pi^2} \quad t \in (0,1)$ т.к. $f \in C(0,1)$

$u_t = -\pi c_1 \sin\pi t + \pi c_2 \cos\pi t \qquad u_t|_{t=0} = \pi c_2 = 0 \Rightarrow c_2 = 0$

$u = \frac{1}{\pi^2} + c_1 \cos\pi t$ решение снова не единственно

$$\begin{cases} u_{tt} = f(t) \\ u_t|_{t=0} = 0 \\ u_t|_{t=1} = 0 \end{cases}, \qquad f(t) = \begin{cases} 1, & t\in(0,\frac{1}{3}) \\ 2, & t\in(\frac{1}{3},\frac{2}{3}) \\ \alpha, & t\in(\frac{2}{3},1) \end{cases}$$

Обобщ. реш: $u \in W_2^1$: $\int_0^1 u_{tt} v\, dt = \int_0^1 f v\, dt \quad \forall v \in W_2^1(0,1)$

$$\|$$

$$-\int_0^1 u_t v_t\, dt$$

– по частям, второго слаг. нет в силу кр. усл-ий

Однор: $u_{tt} = 0 \qquad \lambda^2 = 0 \qquad u = c_1 + c_2 t, \quad u_t = c_2 \qquad u_t|_{t=0} = c_2 = 0$

Т.о. $u = c_1 \qquad u_1 = 1$

$$\int_0^{1/3} dt + 2\int_{1/3}^{2/3} dt + \alpha \int_{2/3}^{1} dt = \frac{1}{3} + \frac{2}{3} + \alpha \frac{1}{3} = 0 \quad \Rightarrow \quad \alpha = -3$$

$$f(t) := \frac{d}{dt}\left(\int_0^t f(\xi)\, d\xi \right)$$

$$\int_0^1 f v\, dt = \int_0^1 v\, d\left(\int_0^t f(\xi) d\xi \right) = v \int_0^t f(\xi) d\xi \Big|_0^1 - \int_0^1 v_t \left(\int_0^t f(\xi)\, d\xi \right) dt =$$

$$= v(1) \int_0^1 f(\xi)\, d\xi - \int_0^1 v_t \left(\int_0^t f(\xi) d\xi \right) dt = - \int_0^1 v_t \left(\int_0^t f(\xi) d\xi \right) = - \int_0^1 v_t u_t\, dt$$

$$\text{Это} \quad \forall v \in W_2^1 \Rightarrow \int_0^t f(\xi)\, d\xi = u_t = \begin{cases} t, & t \in (0, 1/3) \\ \frac{1}{3} + 2\left(t - \frac{1}{3}\right) = 2t - \frac{1}{3}, & t \in (1/3, 2/3) \\ 1 - 3\left(t - \frac{2}{3}\right) = 3 - 3t, & t \in (2/3, 1) \end{cases}$$

$$u = \begin{cases} t^2/2 + c_1, & t \in (0, \frac{1}{3}) \\ t^2 - \frac{t}{3} + c_2, & t \in (\frac{1}{3}, \frac{2}{3}) \\ -\frac{3}{2}t^2 + 3t + c_3, & t \in (\frac{2}{3}, 1) \end{cases}$$

$$\frac{1}{18} + c_1 = \frac{1}{9} - \frac{1}{9} + c_2 = c_2 \quad \text{- в } \frac{1}{3}$$

$$\frac{4}{9} - \frac{2}{9} + c_2 = -\frac{3}{2} \cdot \frac{4}{9} + 2 + c_3 \quad \text{- в } \frac{2}{3}$$

$$\frac{2}{9} + c_2 = -\frac{4}{3} + c_3$$