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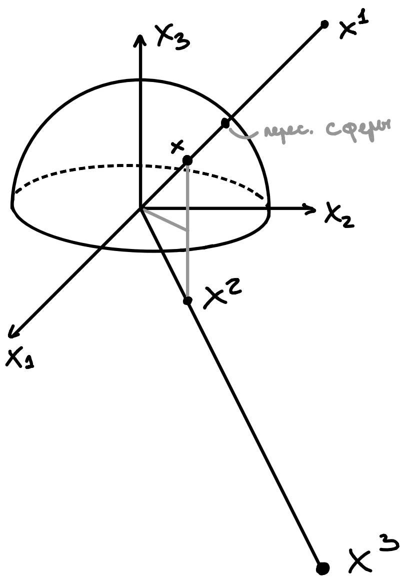


$$\begin{cases} \Delta u = 0, & |x| < R, \quad x_3 > 0 \\ u|_{|x|=R} = \varphi_1(x) \\ u|_{x_3=0} = \varphi_2(x_1, x_2) \end{cases}$$

$$x_j^1 = \frac{x_j}{|x|^2} R^2 \qquad x^2 = (x_1, x_2, -x_3)$$

$$x^3 = (x_1^1, x_2^1, -x_3^1)$$

$$|x^1 - y|\Big|_{|y|=R} = \frac{R}{|x|}|x-y|$$

$$|x^2 - y|\Big|_{y_3=0} = |x-y|$$

$$|x^3 - y|\Big|_{|y|=R} = \frac{R}{|x|}|x^2-y|$$

$$|x^3 - y|\Big|_{y_3=0} = |x^1-y|$$

$$g(y,x) = \sum_{i=1}^{3} \frac{\alpha_i(x)}{4\pi|x^i - y|} + \frac{1}{4\pi|x-y|}$$

$$g(y,x)\Big|_{|y|=R} = 0, \quad g(y,x)\Big|_{y_3=0}$$

$$0 = g(y,x)\Big|_{|y|=R} = \frac{1}{4\pi}\left(\frac{\alpha_1}{|x^1-y|} + \frac{\alpha_2}{|x^2-y|} + \frac{\alpha_3}{|x^3-y|} + \frac{1}{|x-y|}\right)\Bigg|_{|y|=R} =$$

$$= \frac{1}{4\pi}\left(\frac{|x|\alpha_1}{R|x-y|} + \frac{\alpha_2}{|x^2-y|} + \frac{|x|\alpha_3}{R|x^2-y|} + \frac{1}{|x-y|}\right)$$

Т.о. $\alpha_1 = -\frac{R}{|x|}$, $\alpha_2 = -\frac{|x|}{R}\alpha_3$

$$0 = g(y,x)\Big|_{y_3=0} = \frac{1}{4\pi}\left(-\frac{R}{|x||x^1-y|} - \frac{|x|\alpha_3}{R|x^2-y|} + \frac{\alpha_3}{|x^3-y|} + \frac{1}{|x-y|}\right)\Bigg|_{y_3=0} =$$

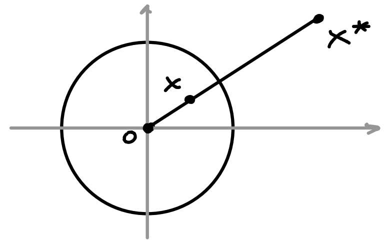
$$= \frac{1}{4\pi}\left(-\frac{R}{|x||x^3-y|} - \frac{|x|\alpha_3}{R|x-y|} + \frac{\alpha_3}{|x^3-y|} + \frac{1}{|x-y|}\right)$$

Т.о. $\alpha_3 = \frac{R}{|x|} \Rightarrow \alpha_2 = -1$

Итого $$g(y,x) = \frac{1}{4\pi}\left(-\frac{R}{|x||x^1-y|} - \frac{1}{|x^2-y|} + \frac{R}{|x||x^3-y|} + \frac{1}{|x-y|}\right)$$

и ещё подставить выражение x^1, x^2, x^3 и в ф-лу Грина.

$$n=2: \quad \begin{cases} \Delta u = 0, & \sqrt{x_1^2+x_2^2} < R \\ u\big|_{|x|=R} = \varphi(x') \end{cases}$$



$$\mathcal{E}_2(x) = \frac{1}{2\pi}\ln|x|$$ — фунд. реш.

$$g(y,x) = \tilde{g}(y,x) - \frac{1}{2\pi}\ln|x-y|$$

$$\Delta_y \tilde{g} = 0 \quad , \quad \Delta_y \ln|x^*-y| = 0 \ , \quad y \neq x \ , \quad x^* \notin G$$

$$\tilde{g}(y,x) = \frac{1}{2\pi}\ln|x^*-y|$$

$$\left(\frac{1}{2\pi}\ln|x^*-y| - \frac{1}{2\pi}\ln|x-y|\right)\Bigg|_{|y|=R} = 0$$

$$|x^*-y|\Big|_{|y|=R} = \frac{R}{|x|}|x-y| \qquad \text{т.к.} \qquad |x||x^*| = R^2$$

Значит, $$\ln\frac{R}{|x|}|x-y| - \ln|x-y|\Big|_{|y|=R} \neq 0$$

Забыли поставить α:

$$\tilde{g} = \frac{1}{2\pi}\ln \alpha(x)|x^*-y| \quad , \quad \alpha(x) = \frac{|x|}{R}$$

$$g(y,x) = \frac{1}{2\pi}\ln\frac{|x|}{R}|x^*-y| - \frac{1}{2\pi}\ln|x-y| \quad , \quad \text{а} \quad x_j^* = \frac{x_j}{|x|^2}R^2$$

$$u(x) = -\int_{\partial G}\frac{\partial}{\partial\nu}g(y,x)\,\varphi(y)\,ds$$ (можем выражать, т.к. по y есть непр-ть вплоть до гр) =

$$= -\int_{\partial G}\frac{1}{2\pi}\frac{\partial}{\partial\nu}\left(\ln\frac{|x|}{R}|x^*-y| - \ln|x-y|\right)\varphi(y)\,ds$$

$$\frac{\partial}{\partial\nu}g = \langle\nabla_y g, \vec{n}\rangle = \Big/\ \vec{n} = \begin{bmatrix} y_1/R \\ y_2/R \end{bmatrix}\Big/ = g_{y_1}\frac{y_1}{R} + g_{y_2}\frac{y_2}{R}$$

$$\frac{\partial}{\partial y_j}\ln|c-y| = \frac{1}{|c-y|}\cdot\frac{\partial}{\partial y_j}\left(\sqrt{(c_1-y_1)^2+(c_2-y_2)^2}\right) = \frac{1}{|c-y|}\cdot\frac{1}{2|c-y|}\cdot 2(c_j-y_j)$$

$$= \frac{c_j-y_j}{|c-y|^2}$$

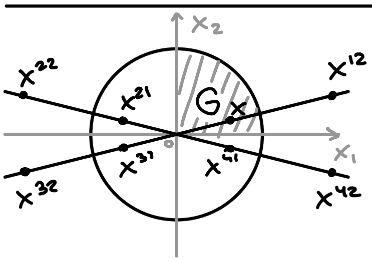
$$\frac{\partial}{\partial\nu}g = \frac{1}{2\pi}\left(\frac{y_1}{R}\left(\frac{y_1-x_1^*}{\underbrace{|x^*-y|^2}_{\frac{R^2}{|x|^2}|x-y|^2}} - \frac{y_1-x_1}{|x-y|^2}\right) + \frac{y_2}{R}\left(\frac{y_2-x_2^*}{\underbrace{|x^*-y|^2}_{\frac{R^2}{|x|^2}|x-y|^2}} - \frac{y_2-x_2}{|x-y|^2}\right)\right) =$$

$$= \frac{1}{2\pi|x-y|^2R}\left(y_1\left(\frac{|x|^2}{R^2}\left(y_1 - x_1\cdot\frac{R^2}{|x|^2}\right) - (y_1-x_1)\right) + y_2\left(\frac{|x|^2}{R^2}\left(y_2 - x_2\frac{R^2}{|x|^2}\right) - (y_2-x_2)\right)\right)$$

$$= \frac{1}{2\pi R|x-y|^2}\left(y_1\left(\frac{|x|^2}{R^2}y_1 - x_1 - y_1 + x_1\right) + y_2\left(\frac{|x|^2}{R^2}y_2 - x_2 - y_2 + x_2\right)\right) =$$

$$= \frac{1}{2\pi R|x-y|^2}\underbrace{\left(y_1^2 + y_2^2\right)}_{R^2}\left(\frac{|x|^2}{R^2} - 1\right) = \frac{|x|^2 - R^2}{2\pi R|x-y|^2}$$

Т.о. $u(x) = \dfrac{1}{2\pi R}\displaystyle\int_{|y|=R}\left(R^2 - |x|^2\right)\frac{1}{|x-y|^2}\varphi(y)\,ds$



$$\begin{cases} \Delta u = 0, & |x| < R,\ x_1 > 0,\ x_2 > 0 \\ u\big|_{|x|=R} = \varphi_1(x'), & |x'| = R \\ u\big|_{x_1=0} = \varphi_2(x_2) \\ u\big|_{x_2=0} = \varphi_3(x_1) \end{cases}$$

$$g(y,x) = \tilde{g}(y,x) - \frac{1}{2\pi}\ln|x-y|$$

$$\left(\tilde{g}(y,x) - \frac{1}{2\pi}\ln|x-y|\right)\Big|_{|y|=R} = 0$$

$$\left(\tilde{g}(y,x) - \frac{1}{2\pi}\ln|x-y|\right)\Big|_{y_1=0} = 0$$

$$\left(\tilde{g}(y,x) - \frac{1}{2\pi}\ln|x-y|\right)\Big|_{y_2=0} = 0$$

$$2\pi g(y,x) = \sum_{(i,j)\in I}\ln\left(\alpha_{ij}(x)|x_{ij} - y|\right) - \ln|x-y| = \ln\frac{\prod_{i,j\in I}\alpha_{ij}(x)|x_{ij}-y|}{|x-y|} \quad *$$

$$\left.\begin{aligned} &|x_{12} - y| = \frac{R}{|x|}|x-y| \\ &|x_{i2} - y| = \frac{R}{|x|}|x_{i1} - y|, \quad i = 2,3,4 \end{aligned}\right|\ |y| = R$$

$$\left.\begin{aligned} &|x_{41} - y| = |x-y| \\ &|x_{21} - y| = |x_{31} - y| \\ &|x_{12} - y| = |x_{42} - y| \\ &|x_{22} - y| = |x_{32} - y| \end{aligned}\right|\ y_1 = 0 \qquad \left.\begin{aligned} &|x_{21} - y| = |x-y| \\ &|x_{31} - y| = |x_{41} - y| \\ &|x_{12} - y| = |x_{22} - y| \\ &|x_{32} - y| = |x_{42} - y| \end{aligned}\right|\ y_2 = 0$$

Так получим $\ln\alpha_{41}|x_{41} - y| + \ln\alpha_{42}|x_{42} - y| =$

$$\ln\alpha_{41}|x_{41} - y| + \ln\alpha_{42}\frac{R}{|x|}|x_{41} - y| \neq 0 \quad ☹$$

Будем брать в итоге $\beta_{i1} = -\beta_{i2} \in \{\pm 1\}$

$$\ln\alpha_{41}|x_{41} - y|^{\beta_{41}} + \ln\alpha_{42}\frac{R}{|x|}|x_{41} - y|^{\beta_{42}} = 0 \quad ☺$$

Под $\ln$ в ⊛:

$$\frac{\alpha_{12} R}{|x|} \frac{|x-y|^{\beta_{12}}}{|x-y|} \cdot \alpha_{21} |x_{21}-y|^{\beta_{21}} \cdot \alpha_{22} \frac{R}{|x|} |x_{21}-y|^{\beta_{22}} \cdot$$

$$\cdot \alpha_{31} |x_{31}-y|^{\beta_{31}} \cdot \alpha_{32} \frac{R}{|x|} |x_{31}-y|^{\beta_{32}} \cdot \alpha_{41} |x_{41}-y|^{\beta_{41}} \cdot$$

$$\cdot \alpha_{42} \frac{R}{|x|} |x_{41}-y|^{\beta_{42}} = 1$$ (т.к. $\ln 1 = 0$)

$$\alpha_{12} = \frac{|x|}{R}, \quad \beta_{12} = 1$$

$$\alpha_{i1} = \frac{|x|}{2\alpha_{i2}}, \quad \beta_{i1} = -\beta_{i2}, \quad i = 2,3,4$$

9/3 доделать это внеш. з. Дирихле