HOW TO PROVE IT: CHAPTER 3

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These are the exercises for Chapter 3 from the third edition of *How to Prove It* by Daniel J. Velleman. They are numbered (Chapter).(Section).(Exercise).

3.1.1. Consider the following theorem. (This theorem was proven in the introduction.)

Theorem 1. *Suppose n is an integer larger than 1 and n is not prime. Then $2^n - 1$ is not prime.*

(1) Identify the hypotheses and the conclusion of the theorem. Are the hypotheses true when $n = 6$? What does the theorem tell you in this instance? Is it right?
(2) What can you conclude from the theorem in the case $n = 15$? Check directly that this conclusion is correct.
(3) What can you conclude from the theorem in the case $n = 11$?

Proof.

(1) This theorem has three hypotheses: n is an integer, $n > 1$, and n is not prime. The conclusion of the theorem is that $2^n - 1$ is not prime. In the case when $n = 6 = 2 \times 3$, all of the hypotheses are satisfied, so the theorem tells us that $2^6 - 1$ is not prime. We can directly check that $2^6 - 1 = 63 = 3^2 \times 7$ is not prime.
(2) In the case when $n = 15 = 3 \times 5$, all of the hypotheses are satisfied. This means that the theorem tells us that $2^{15} - 1$ is not prime. As discussed in Part (1) of Exercise I.1, $2^{15} - 1 = 32767 = 31 \times 1057$.
(3) In the case when $n = 11$, not all of the hypotheses are satisfied. In particular, 11 is prime. Because not all of the hypotheses of the theorem are satisfied, we cannot draw any conclusions from it. In particular, the theorem does not tell us anything about the primality of $2^{11} - 1$.

□

3.1.2. Consider the following theorem. (The theorem is correct, but we will not ask you to prove it here.)

Theorem 2. *Suppose that $b^2 > 4ac$. Then the quadratic equation $ax^2 + bx + c = 0$ has exactly two real solutions.*

(1) Identify the hypotheses and conclusion of the theorem.
(2) To give an instance of the theorem, you must specify values for a, b, and c, but not x. Why?
(3) What can you conclude from the theorem in the case $a = 2$, $b = -5$, $c = 3$? Check directly that this conclusion is correct.
(4) What can you conclude from the theorem in the case $a = 2$, $b = 4$, $c = 3$?

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Proof.

(1) This theorem has one implicit hypothesis, that a, b, c are all real numbers, and one explicit hypothesis, that $b^2 > 4ac$. The conclusion of the theorem is that the quadratic equation $ax^2 + bx + c = 0$ has exactly two real solutions.

(2) To give an instance of the theorem, we only need to specify values for a, b, and c since those are the only variables listed in the hypothesis. The values of x associated with a specific instance of the theorem are determined by the given values of a, b, c, since the x values are the two real solutions to the quadratic equation $ax^2 + bx + c = 0$. In other words, x is a dummy variable for the set $S = \{x \in \mathbb{R} \mid ax^2 + bx + c = 0\}$ for given values of a, b, c. When phrased this way, the conclusion of theorem states that S contains exactly two distinct elements.

(3) In the case $a = 2, b = -5, c = 3$, we have $b^2 = (-5)^2 = 25$ and $4ac = 4(2)(3) = 24$. Thus, the hypotheses $b^2 > 4ac$ is satisfied, and the theorem applies. We can then conclude that the quadratic equation $2x^2 - 5x + 3 = 0$ has two real solutions. Factoring the quadratic as $(2x - 3)(x - 1) = 0$, we can directly check that the two real solutions are $x = 3/2$ and $x = 1$.

(4) In the case $a = 2, b = 4, c = 3$, we have $b^2 = 4^2 = 16$ and $4ac = 4(2)(3) = 24$. In other words, $b^2 \not> 4ac$. Since not all of the hypotheses of the theorem are satisfied, we cannot draw any conclusions from it. In particular, the theorem does not tell us anything about the solution set of the quadratic equation $2x^2 + 4x + 3 = 0$.

□

3.1.3. Consider the following incorrect theorem.

Theorem 3. *Suppose n is a natural number larger than 2, and n is not a prime number. Then $2n + 13$ is not a prime number.*

What are the hypotheses and conclusion of this theorem? Show that the theorem is incorrect by finding a counterexample.

Proof. The theorem has three hypotheses: n is a natural number, $n > 2$, and n is not prime. The conclusion of the theorem is that $2n + 13$ is not a prime number. To see that this theorem is incorrect, consider the case of $n = 8$. This value of n is a natural number greater than 2 that is not prime, so it satisfies all of the hypotheses of the theorem. However, $2n + 13 = 2(8) + 13 = 29$ is a prime number. Since we could find a instance of the theorem where all of the hypotheses are satisfied but an incorrect conclusion is drawn, the theorem itself is incorrect. □

3.1.4. Complete the following alternative proof of the theorem in Example 3.1.2.

Theorem 4. *Suppose a and b are real numbers. If $0 < a < b$ then $a^2 < b^2$.*

Proof. Suppose $0 < a < b$. Then $b - a > 0$. [*Fill in a proof of $b^2 - a^2 > 0$ here.*] Since $b^2 - a^2 > 0$, it follows that $a^2 < b^2$. Therefore, if $0 < a < b$ then $a^2 < b^2$. □

Proof. Suppose $0 < a < b$. Then $b - a > 0$. In addition, since $0 < a < b$, we also know that $b + a > 0$. We can then multiply both sides of the inequality $b - a > 0$ by $b + a$ to get $(b - a)(b + a) = b^2 - a^2 > 0$. Since $b^2 - a^2 > 0$, it follows that $a^2 < b^2$. Therefore, if $0 < a < b$ then $a^2 < b^2$.

□

3.1.5. Suppose a and b are real numbers. Prove that if $a < b < 0$ then $a^2 > b^2$.

Proof. Suppose $a < b < 0$. Then $a - b < 0$ and $a + b < 0$. We can then multiply both sides of the inequality $a - b < 0$ by $a + b$ to get $(a - b)(a + b) = a^2 - b^2 > 0$. Since $a^2 - b^2 > 0$, it follows that $a^2 > b^2$. Therefore, if $a < b < 0$ then $a^2 > b^2$.

Note that an alternative strategy would be to mimic the proof of the theorem in Example 3.1.2 and instead multiply the given inequality by the negative numbers a and b, respectively. This would then result in the chain of inequalities $a^2 > ab > b^2 > 0$, which also gives the desired conclusion. □

3.1.6. Suppose a and b are real numbers. Prove that if $0 < a < b$ then $1/b < 1/a$.

Proof. Suppose $0 < a < b$. Dividing both sides of the inequality $a < b$ by the positive number a gives us $1 < b/a$. We can then divide both sides of $1 < b/a$ by the positive number b to conclude $1/b < 1/a$. Therefore, if $0 < a < b$ then $1/b < 1/a$. □

3.1.7. Suppose a is a real number. Prove that if $a^3 > a$ then $a^5 > a$. (*Hint: One approach is to start by completing the following equation: $a^5 - a = (a^3 - a) \cdot \underline{?}$.*)

Proof. Suppose $a^3 > a$, which we can rewrite as $a^3 - a > 0$. Following the hint, we note that we can factor $a^5 - a$ as $(a^3 - a)(a^2 + 1)$. Since $a^2 \geq 0$ for all $a \in \mathbb{R}$, we know that $a^2 + 1 > 0$. Thus, multiplying the inequality $a^3 - a > 0$ by the positive number $a^2 + 1$ gives us $0 < (a^3 - a)(a^2 + 1) = a^5 - a$. In other words, we conclude that $a^5 > a$. Therefore, if $a^3 > a$ then $a^5 > a$. □

3.1.8. Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

Proof. We prove the contrapositive statement: if $x \notin B$ then $x \in D$. Suppose $x \notin B$. Then, since we also know that $x \in A$, it follows that $x \in A \setminus B$. As $A \setminus B \subseteq C \cap D$, it then follows that $x \in C \cap D$. In other words, $x \in C$ and $x \in D$. The last inclusion, that $x \in D$, is what we wanted to prove to complete the proof of the contrapositive statement. Therefore, if $x \notin D$ then $x \in B$. □

3.1.9. Suppose $A \cap B \subseteq C \setminus D$. Prove that if $x \in A$, then if $x \in D$ then $x \notin B$.

Proof. Suppose $x \in A$. We prove the contrapositve statement: if $x \in B$ then $x \notin D$. Now suppose $x \in B$. Then, since $x \in A$ as well, it follows that $x \in A \cap B$. As $A \cap B \subseteq C \setminus D$, it then follows that $x \in C \setminus D$. From this, we conclude that $x \notin D$, which completes our proof of the contrapositive statement. Therefore, if $x \in A$, then if $x \in D$ then $x \notin B$. □

3.1.10. Suppose a and b are real numbers. Prove that if $a < b$ then $(a + b)/2 < b$.

Proof. Suppose $a < b$. Adding b to both sides of the inequality then gives us $a + b < 2b$. Next we divide both sides by 2 to obtain the desired inequality: $(a + b)/2 < b$. Therefore, if $a < b$ then $(a + b)/2 < b$. □

3.1.11. Suppose x is a real number and $x \neq 0$. Prove that if $(\sqrt[3]{x} + 5)/(x^2 + 6) = 1/x$ then $x \neq 8$.

Proof. We prove the contrapositive statement: if $x = 8$ then $(\sqrt[3]{x}+5)/(x^2+6) \neq 1/x$. Suppose $x = 8$. Then $1/x = 1/8$ and $(\sqrt[3]{x}+5)/(x^2+6) = (\sqrt[3]{8}+5)/(8^2+6) = 1/10$. Since $1/8 \neq 1/10$, this completes the proof of the contrapositive statement. Therefore, we conclude that $(\sqrt[3]{x}+5)/(x^2+6) = 1/x$ then $x \neq 8$. □

3.1.12. Suppose a, b, c, and d are real numbers, $0 < a < b$, and $d > 0$. Prove that if $ac \geq bd$ then $c > d$.

Proof. Suppose $ac \geq bd$. Since $b > a$ and $d > 0$, we can multiply both sides of the first inequality by d to obtain $bd > ad$. Then, since $ac \geq bd$, we have $ac \geq bd > ad$. It then follows that $ac > ad$. Dividing both sides of the inequality by the positive number a, we see that $c > d$. Thus, we conclude that if $ac \geq bd$ then $c > d$.

Note that we could also approach this proof by proving the contrapositive statement: if $c \leq d$ then $ac < bd$. □

3.1.13. Suppose x and y are real numbers, and that $3x + 2y \leq 5$. Prove that if $x > 1$ then $y < 1$.

Proof. Suppose $x > 1$. Then, $3x + 2y > 3 + 2y$. Combining this with the given inequality, $3x + 2y \leq 5$, we have $3 + 2y < 3x + 2y \leq 5$. In particular, $3 + 2y < 5$. After isolating y in the inequality, we end up with the desired result: $y < 1$. Therefore, we conclude that if $x > 1$ then $y < 1$.

Note that we could also approach this proof by proving the contrapositive statement: if $y \geq 1$ then $x \leq 1$. □

3.1.14. Suppose x and y are real numbers. Prove that if $x^2 + y = -3$ and $2x - y = 2$ then $x = -1$.

Proof. Suppose that $x^2 + y = -3$ and $2x - y = 2$. We can add the two equations together to obtain $x^2 + 2x = -1$. After rearranging, this becomes $x^2 + 2x + 1 = 0$, which we can factor as $(x + 1)^2 = 0$. Therefore, it follows that $x = -1$. Thus, we conclude that if $x^2 + y = -3$ and $2x - y = 2$ then $x = -1$. □

3.1.15. Prove the first theorem in Example 3.1.1.

Theorem 5. *Suppose $x > 3$ and $y < 2$. Then $x^2 - 2y > 5$.*

(*Hint: You might find it useful to apply the theorem from Example 3.1.2, which stated that if a and b are real numbers such that $0 < a < b$, then $a^2 < b^2$.*)

Proof. We follow the hint and apply the fact that if a and b are real numbers such that $0 < a < b$ then $a^2 < b^2$ with $a = 3$ and $b = x$ to conclude that $x^2 > 9$. Next, since $y < 2$, it follows that $-2y > -4$. Adding the inequalities $x^2 > 9$ and $-2y > -4$ together gives us the desired inequality: $x^2 - 2y > 5$. □

3.1.16. Consider the following theorem.

Theorem 6. *Suppose x is a real number and $x \neq 4$. If $(2x - 5)/(x - 4) = 3$ then $x = 7$.*

(1) What is wrong with the following proof of the theorem?

Proof. Suppose $x = 7$. Then $(2x - 5)/(x - 4) = (2 \cdot 7 - 5)/(7 - 4) = 9/3 = 3$. Therefore if $(2x - 5)/(x - 4) = 3$ then $x = 7$. □

(2) Give a correct proof of the theorem.

Proof.

(1) The problem with the incorrect proof of the theorem is that it starts by assuming that the conclusion, that $x = 7$, is true when that is the ultimate fact that we are trying to prove. In other words, it is instead a proof of the converse statement: if $x = 7$ then $(2x - 5)/(x - 4) = 3$, which is is a completely different statement.

(2) Suppose $(2x-5)/(x-4) = 3$. We can then clear the denominator of the left-hand side by multiplying the equation by $x - 4$. This gives us the equation $2x - 5 = 3(x - 4)$, or in other words $2x - 5 = 3x - 12$. We subtract $2x$ from both sides and then add 12 to both sides to see that $x = 7$. Therefore, we conclude that if $(2x - 5)/(x - 4) = 3$ then $x = 7$.

□

3.1.17. Consider the following incorrect theorem.

Theorem 7. *Suppose that x and y are real numbers and $x \neq 3$. If $x^2y = 9y$ then $y = 0$.*

(1) What's wrong with the following proof of the theorem?

Proof. Suppose that $x^2y = 9y$. Then $(x^2-9)y = 0$. Since $x \neq 3$, $x^2 \neq 9$, so $x^2-9 \neq 0$. Therefore we can divide both sides of the equation $(x^2 - 9)y = 0$ by $x^2 - 9$, which leads to the conclusion that $y = 0$. Thus, if $x^2y = 9y$ then $y = 0$. □

(2) Show that the theorem is incorrect by finding a counterexample.

Proof.

(1) The attempted proof makes a mistake in the statement "Since $x \neq 3$, $x^2 \neq 9$, so $x^2 - 9 \neq 0$." We only know that $x \neq 3$, but otherwise x could be any other real number. In particular, we could have $x = -3$, which would then result in $x^2 = 9$. Since the rest of the proof hinges on the assumption that $x^2 \neq 9$, it is rendered invalid.

(2) Consider $x = -3$ and $y = 1$. In this case, $x^2y = (-3)^2(1) = 9$ and $9y = 9(1) = 9$, but $y \neq 0$. Therefore, this counter example shows that the theorem is correct, since we were able to find values of x and y which satisfy the hypotheses but then lead to an incorrect conclusion.

□

3.2.1. This problem could be solved by using truth tables, but don't do it that way. Instead, use the methods for writing proofs discussed so far in this chapter.

(1) Suppose $P \to Q$ and $Q \to R$ are both true. Prove that $P \to R$ is true.
(2) Suppose $\neg R \to (P \to \neg Q)$ is true. Prove that $P \to (Q \to R)$ is true.

Proof.

(1) Suppose P is true. Then, since by the assumption $P \to Q$ is also true, we apply modus ponens to conclude that Q is true. Now we apply modus ponens again, this time the fact that Q is true and the assumption $Q \to R$ is true to conclude that R is true. Therefore, $P \to R$ is true.

(2) Suppose P is true. Now we want to show that $Q \to R$ is true. This is equivalent to proving the contrapositive statement, $\neg R \to \neg Q$. To prove $\neg R \to \neg Q$, suppose $\neg R$ is true. Then, we can combine this assumption with the given assumption that $\neg R \to (P \to \neg Q)$ is true to conclude that $P \to \neg Q$ is true by modus ponens. Now we apply modus ponens again with P and $P \to \neg Q$ to conclude that $\neg Q$ is true. Therefore, $\neg R \to \neg Q$ is true, so we conclude that $Q \to R$ is true. Finally, we conclude that $P \to (Q \to R)$ is true.

□

3.2.2. This problem could be solved using truth tables, but don't do it that way. Instead, use the methods for writing proofs discussed so far in this chapter.

(1) Suppose $P \to Q$ and $R \to \neg Q$ are both true. Prove that $P \to \neg R$ is true.
(2) Suppose that P is true. Prove that $Q \to \neg(Q \to \neg P)$ is true.

Proof.

(1) Suppose P is true. Then, since P and $P \to Q$ are both true, we conclude that Q is true by modus ponens. Next, since Q and $R \to \neg Q$ are both true, we conclude that $\neg R$ is true by modus tollens. Therefore, we conclude that $P \to \neg R$ is true.

(2) We prove the equivalent contrapositive statement: that $(Q \to \neg P) \to \neg Q$ is true. Suppose $Q \to \neg P$ is true. Then, since P is true by assumption, we use modus tollens to conclude that $\neg Q$ is true. Thus, $(Q \to \neg P) \to \neg Q$ is true. In other words, we conclude that $Q \to \neg(Q \to \neg P)$ is true.

□

3.2.3. Suppose $A \subseteq C$, and B and C are disjoint. Prove that if $x \in A$ then $x \notin B$.

Proof. Suppose $x \in A$. Then, since $A \subseteq C$, we also know that $x \in C$. Since B and C are disjoint, the fact that $x \in C$ means that $x \notin B$. Therefore, we conclude that if $x \in A$ then $x \notin B$. □

3.2.4. Suppose that $A \setminus B$ is disjoint from C and $x \in A$. Prove that if $x \in C$ then $x \in B$.

Proof. We prove the equivalent contrapositive statement: if $x \notin B$ then $x \notin C$. Suppose $x \notin B$. Then, since by assumption $x \in A$, $x \in A \setminus B$. Since $A \setminus B$ is disjoint from C, it follows that $x \notin C$. Thus, if $x \notin B$ then $x \notin C$. In other words, we conclude that if $x \in C$ then $x \in B$. □

3.2.5. Prove that it cannot be the case that $x \in A \setminus B$ and $x \in B \setminus C$.

Proof. Suppose $x \in A \setminus B$ and $x \in B \setminus C$. Since $x \in A \setminus B$, it follows that $x \in A$ and $x \notin B$. Similarly, since $x \in B \setminus C$, it follows that $x \in B$ and $x \in C$. However, having $x \notin B$ and $x \in B$ is a contradiction. Therefore, we conclude that it cannot be the case that $x \in A \setminus B$ and $x \in B \setminus C$. □

3.2.6. Use the method of proof by contradiction to prove the theorem in Example 3.2.1.

Theorem 8. *Suppose $A \cap C \subseteq B$ and $a \in C$. Prove that $a \notin A \setminus B$.*

Proof. Suppose, towards a contradiction, that $a \in A \setminus B$. In other words, $a \in A$ and $a \notin B$. Since also $a \in C$, we have $a \in A \cap C \subseteq B$. Therefore, $a \in B$. But this contradicts that $a \notin B$. Thus, we conclude that $a \notin A \setminus B$. □

3.2.7. Use the method of proof by contradiction to prove the theorem in Example 3.2.5.

Theorem 9. *Suppose $A \subseteq B$, $a \in A$, and $a \notin B \setminus C$. Prove that $a \in C$.*

Proof. Suppose, towards a contradiction, that $a \notin C$. Since $a \in A$ and $A \subseteq B$, it follows that $a \in B$. Then, since $a \in B$ and $a \notin C$, $a \in B \setminus C$. This, however, contradicts the fact that $a \notin B \setminus C$. Therefore, we conclude that $a \in C$. □

3.2.8. Suppose that $y + x = 2y - x$, and x and y are not both zero. Prove that $y \neq 0$.

Proof. Suppose, towards a contradiction, that $y = 0$. Substituting $y = 0$ into the equation $y + x = 2y - x$ gives us $x = -x$, which implies that $x = 0$. However, having $y = 0$ and $x = 0$ contradicts the fact that x and y are not both zero. Thus, we conclude that $y \neq 0$. □

3.2.9. Suppose that a and b are nonzero real numbers. Prove that if $a < 1/a < b < 1/b$ then $a < -1$.

Proof. Suppose $a < 1/a < b < 1/b$. We then break the proof into two steps: proving that $a < 0$ and then proving that $a < -1$.

Lemma 10. $a < 0$

Proof. Suppose, towards a contradiction, that $a \geq 0$. Since a is nonzero, we can assume, without loss of generality, that $a > 0$. Then, since $0 < a < b$, we can apply Exercise 3.1.6 to conclude that $1/b < 1/a$. This, however, contradicts the fact that $1/a < 1/b$. Thus, we conclude that $a < 0$. □

Now that we have proved that $a < 0$ in the above lemma, we move on to showing that $a < -1$. We once again use proof by contradiction. Suppose, towards a contradiction, that $a \geq -1$. In other words, $-1 \leq a < 0$. Since a is negative, dividing both sides of the inequality $a \geq -1$ gives us $1 \leq -1/a$. Rearranging this inequality results in $-1 \geq 1/a$. Thus, we have $a \geq -1 \geq 1/a$, or most importantly, $a \geq 1/a$. This however, contradicts the fact that $a < 1/a$, so we must instead have $a < -1$, as desired. Therefore, we conclude that if $a < 1/a < b < 1/b$ then $a < -1$. □

3.2.10. Suppose that x and y are real numbers. Prove that if $x^2 y = 2x + y$, then if $y \neq 0$ then $x \neq 0$.

Proof. Suppose $x^2 y = 2x + y$. Now our goal is to prove that if $y \neq 0$ then $x \neq 0$. We do this by proving the contrapositive statement: if $x = 0$ then $y = 0$. Now suppose that $x = 0$. Plugging this value for x into $x^2 y = 2x + y$ gives us $0 = y$, which is exactly what we wanted to show. Thus, if $x = 0$ then $y = 0$. In other words, if $y \neq 0$ then $x \neq 0$. We then conclude that if $x^2 y = 2x + y$, then if $y \neq 0$ then $x \neq 0$. □

3.2.11. Suppose that x and y are real numbers. Prove that if $x \neq 0$, then if $y = (3x^2 + 2y)/(x^2 + 2)$ then $y = 3$.

Proof. Suppose $x \neq 0$, and then suppose $y = (3x^2 + 2y)/(x^2 + 2)$. Multiplying both sides of the equation by $x^2 + 2$ gives us $x^2y + 2y = 3x^2 + 2y$. After subtracting $2y$ from both sides, we are left with $x^2y = 3x^2$. Since $x \neq 0$, $x^2 \neq 0$ as well, so we can divide both sides by x^2 to conclude that $y = 3$. Therefore, we conclude that if $x \neq 0$, then if $y = (3x^2 + 2y)/(x^2 + 2)$ then $y = 3$. □

3.2.12. Consider the following incorrect theorem.

Theorem 11. *Suppose x and y are real numbers and $x + y = 10$. Then $x \neq 3$ and $y \neq 8$.*

(1) What's wrong with the following proof of the theorem?

Proof. Suppose the conclusion of the theorem is false. Then $x = 3$ and $y = 8$. But then $x + y = 11$, which contradicts the given information that $x + y = 10$. Therefore, the conclusion must be true. □

(2) Show that the theorem is incorrect by finding a counterexample.

Proof.

(1) The proof incorrectly negates the conclusion when attempting to begin the proof by contradiction. The conclusion, that $x \neq 3$ and $y \neq 8$, has the logical form $(x \neq 3) \wedge (y \neq 8)$, so its negation is $(x = 3) \vee (y = 8)$, by DeMorgan's laws. Thus, the correct negation of the conclusion is that $x = 3$ **or** $y = 8$.
(2) One counterexample is the case of $x = 3$ and $y = 7$. These are two real numbers such that $x + y = 10$, but they do not satisfy the conclusion that $x \neq 3$ **and** $y \neq 8$, as only $y \neq 8$.

□

3.2.13. Consider the following incorrect theorem.

Theorem 12. *Suppose that $A \subseteq C$, $B \subseteq C$, and $x \in A$. Then $x \in B$.*

(1) What's wrong with the following proof of the theorem?

Proof. Suppose that $x \notin B$. Since $x \in A$ and $A \subseteq C$, $x \in C$. Since $x \notin B$ and $B \subseteq C$, $x \notin C$. But now we have proven both $x \in C$ and $x \notin C$, so we have reached a contradiction. Therefore $x \in B$. □

(2) Show that the theorem is incorrect by finding a counterexample.

Proof.

(1) The proof incorrectly concludes that since $x \notin B$ and $B \subseteq C$, $x \notin C$. This is an incorrect conclusion because B may be a proper subset of C, or in other words $C \setminus B$ may be nonempty. We can also confirm the incorrectness of this conclusion by examining the logical form of the statement $B \subseteq C$, which is $\forall x(x \in B \to x \in C)$. This is equivalent to $\forall x(x \notin B \vee x \in C)$, which is still true in the case when there are elements x such that $x \notin B$ and $x \in C$.
(2) Consider the case where $A = \{1\}$, $B = \{2\}$, $C = \{1, 2\}$, and $x = 1$. Clearly $x \in A$, $A \subseteq C$, and $B \subseteq C$, but $x \notin B$.

□

3.2.14. Use truth tables so show that modus tollens is a valid rule of inference.

Proof. Recall that modus tollens says that if $P \to Q$ is true and Q is false, then you can conclude that P must also be false. We now construct a truth table to evaluate the validity of this form of argument.

P	Q	$P \to Q$	$\neg Q$	$\neg P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

This form of argument is valid because whenever all of the premises, $P \to Q$ and $\neg Q$, are true, the conclusion, $\neg P$, is also true, as indicated by the blue highlighted row. □

3.2.15. Use truth tables to check the correctness of the theorem in Example 3.2.4.

Theorem 13. *Suppose $P \to (Q \to R)$. Prove that $\neg R \to (P \to \neg Q)$.*

Proof.

P	Q	R	$P \to (Q \to R)$	$\neg R \to (P \to \neg Q)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

This theorem is correct because whenever the premise, $P \to (Q \to R)$ is true, the conclusion $\neg R \to (P \to \neg Q)$ is also true, as indicated by the blue highlighted rows.

We can also see that the theorem is correct by comparing the logical form of the premise with the logical form of the conclusion. The premise, $P \to (Q \to R)$, is equivalent to $\neg P \vee (\neg Q \vee R)$, or simply $\neg P \vee \neg Q \vee R$ by associativity of disjunction. Similarly, the conclusion, $\neg R \to (P \to \neg Q)$ is equivalent to $R \vee (\neg P \vee \neg Q)$, which is equivalent to $\neg P \vee \neg Q \vee R$ by associativity and commutativity of disjunction. Since the premise and conclusion are equivalent logical statements, the conclusion is true whenever the premise is true. □

3.2.16. Use truth tables to check the correctness of the statements in Exercise 3.2.1.

Proof.

(1) Recall that the statement was "Suppose $P \to Q$ and $Q \to R$ are both true. Prove that $P \to R$ is true."

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

This statement is correct because whenever the premises, $P \to Q$ and $Q \to R$ are both true, the conclusion $P \to R$ is also true, as indicated by the blue highlighted rows.

(2) Recall that the statement was "Suppose $\neg R \to (P \to \neg Q)$ is true. Prove that $P \to (Q \to R)$ is true."

P	Q	R	$\neg R \to (P \to \neg Q)$	$P \to (Q \to R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

This statement is correct because whenever the premise, $\neg R \to (P \to \neg Q)$ is true, the conclusion $P \to (Q \to R)$ is also true, as indicated by the blue highlighted rows.

Our analysis from Exercise 3.2.15 also provides another perspective for the correctness of the statement, because the premise, $\neg R \to (P \to \neg Q)$, and the conclusion, $P \to (Q \to R)$, have equivalent logical forms.

$\square$

3.2.17. Use truth tables to check the correctness of the statements in Exercise 3.2.2.

Proof.

(1) Recall that the statement was "Suppose $P \to Q$ and $R \to \neg Q$ are both true. Prove that $P \to \neg R$ is true."

P	Q	R	$P \to Q$	$R \to \neg Q$	$P \to \neg R$
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	F	T	T
F	T	T	T	F	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

This statement is correct because whenever the premises, $P \to Q$ and $R \to \neg Q$ are both true, the conclusion $P \to \neg R$ is also true, as indicated by the blue highlighted rows.

(2) Recall that the statement was "Suppose that P is true. Prove that $Q \to \neg(Q \to \neg P)$ is true."

P	Q	P	$Q \to \neg(Q \to \neg P)$
T	T	T	T
T	F	T	T
F	T	F	F
F	F	F	T

The statement is correct because whenever the premise P is true, the conclusion $Q \to \neg(Q \to \neg P)$ is also true, as indicated by the blue highlighted rows.

Note that we can more easily fill out the truth table by observing that the conclusion is logically equivalent to the simpler statement $\neg Q \lor P$.

□

3.2.18. Can the proof in Example 3.2.2 be modified to prove that if $x^2 + y = 13$ and $x \neq 3$ then $y \neq 4$? Explain. Below is the theorem from Example 3.2.2 and its proof.

Theorem 14. *If $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$.*

Proof. Suppose $x^2 + y = 13$ and $y \neq 4$. Suppose $x = 3$. Substituting this into the equation $x^2 + y = 13$, we get $9 + y = 13$, so $y = 4$. But this contradicts the fact that $y \neq 4$. Therefore $x \neq 3$. Thus, if $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$. □

Proof. The proof in Example 3.2.2 cannot be modified to prove that if $x^2 + y = 13$ and $x \neq 3$ then $y \neq 4$? This is because the statement "If $x^2 + y = 13$ and $x \neq 3$ then $y \neq 4$," is false. To see why, consider the case when $x = -3$, which satisfies the hypothesis that $x \neq 3$. In this situation, when also $x^2 + y = 13$, we can plug in $x = -3$ to obtain the equation $9 + y = 13$. From the equation $9 + y = 13$ it follows that $y = 4$, which does not satisfy the conclusion $y \neq 4$. Since we have found a situation where the premises of the statement are true, but the conclusion is false, the statement itself is false. □

3.3.1. In Exercise 2.2.7 you used logical equivalences to show that $\exists x(P(x) \to Q(x))$ is equivalent to $\forall x P(x) \to \exists x Q(x)$. Now use the methods of this section to prove that if

$\exists x(P(x) \to Q(x))$ is true, then $\forall x P(x) \to \exists x Q(x)$ is true. (*Note: The other direction of the equivalence is quite a bit harder to prove. See Exercise 3.5.30.*)

Proof. Suppose that $\exists x(P(x) \to Q(x))$ is true. Now our goal is to prove that the statement $\forall x P(x) \to \exists x Q(x)$ is true. We then suppose that $\forall x P(x)$ is true and update our goal to proving that $\exists x Q(x)$ is true. Since $\exists x(P(x) \to Q(x))$ is true, there exists $x = x_0$ such that $P(x_0) \to Q(x_0)$ is true. Then, since $\forall x P(x)$ is true, the statement $P(x_0)$, in particular, is true. Because $P(x_0) \to Q(x_0)$ is true and $P(x_0)$ is true, we conclude that $Q(x_0)$ is true as well. In other words, we have found that there exists x such that $Q(x)$ is true. Thus, the statement $\forall x P(x) \to \exists x Q(x)$ is true, and we ultimately conclude that if $\exists x(P(x) \to Q(x))$ is true, then $\forall x P(x) \to \exists x Q(x)$ is true as well. This completes the proof. $\square$

3.3.2. Prove that if A and $B \setminus C$ are disjoint, then $A \cap B \subseteq C$.

Proof. Suppose A and $B \setminus C$ are disjoint. Let $x \in A \cap B$ be arbitrary. We now want to show that $x \in C$. Suppose, towards a contradiction, that $x \notin C$. Since $x \in A \cap B$, $x \in A$ and $x \in B$. Then, since $x \in B$ and $x \notin C$, $x \in B \setminus C$. It then follows that $x \in A \cap (B \setminus C)$, but this contradicts the assumption that A and $B \setminus C$ are disjoint. Thus, $x \in C$, so $A \cap B \subseteq C$. Therefore, we conclude that if A and $B \setminus C$ are disjoint, then $A \cap B \subseteq C$. $\square$

3.3.3. Prove that if $A \subseteq B \setminus C$ then A and C are disjoint.

Proof. Suppose $A \subseteq B \setminus C$ and suppose, towards a contradiction that A and C are not disjoint. In other words, there is $x \in A \cap C$. Since $x \in A$ and $A \subseteq B \setminus C$, it follows that $x \in B \setminus C$. In particular, $x \notin C$. However, this contradicts the fact that $x \in A \cap C$ means that $x \in C$. Thus, we conclude that A and C are disjoint. Therefore, if $A \subseteq B \setminus C$ then A and C are disjoint. $\square$

3.3.4. Suppose $A \subseteq \mathscr{P}(A)$. Prove that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$.

Proof. Let $X \in \mathscr{P}(A)$ be arbitrary. By the definition of the power set of a set A, $X \subseteq A$. Then, since $A \subseteq \mathscr{P}(A)$, it follows that $X \subseteq \mathscr{P}(A)$. In other words, X is an element of the power set of $\mathscr{P}(A)$. Since X was arbitrary, we conclude that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. $\square$

3.3.5. The hypothesis of the theorem in Exercise 3.3.4 is $A \subseteq \mathscr{P}(A)$.

(1) Can you think of a set A for which this hypothesis is true?
(2) Can you think of another?

Proof.

(1) Since the empty set $\varnothing$ is a subset of any set X, it is a set for which the hypothesis $A \subseteq \mathscr{P}(A)$ is true.
(2) Once we observe that $A = \varnothing$ satisfies the hypothesis $A \subseteq \mathscr{P}(A)$, the theorem tells us that $\mathscr{P}(A) \subseteq \mathscr{P}(\mathscr{P}(A))$. In other words, $\mathscr{P}(A)$ is another set which satisfies the hypothesis. We can check this directly by noting that $\mathscr{P}(\varnothing) = \{\varnothing\}$, $\mathscr{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$, and $\{\varnothing\} \subseteq \{\varnothing, \{\varnothing\}\}$.

$\square$

3.3.6. Suppose x is a real number.

(1) Prove that if $x \neq 1$ then there is a real number y such that $(y + 1)/(y - 2) = x$.

(2) Prove that if there is a real number y such that $(y+1)/(y-2) = x$, then $x \neq 1$.

Proof.

(1) Suppose $x \neq 1$. Consider $y = (2x+1)/(x-1)$, which is a well-defined real number since $x \neq 1$. We then plug this value of y into the expression $(y+1)/(y-2)$.

$$\frac{y+1}{y-2} = \frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2} = \frac{2x+1+(x-1)}{2x+1-2(x-1)} = \frac{3x}{3} = x$$

Thus, we have found a real number y such that $(y+1)/(y-2) = x$. We conclude that that if $x \neq 1$ then there is a real number y such that $(y+1)/(y-2) = x$.

(2) Suppose there is a real number y such that $(y+1)/(y-2) = x$. Since $y+1 \neq y-2$ for all real numbers y, it follows that $x = (y+1)/(y-2) \neq 1$. Therefore, we conclude that if there is a real number y such that $(y+1)/(y-2) = x$, then $x \neq 1$.

□

3.3.7. Prove that for every real number x, if $x > 2$ then there is a real number y such that $y + 1/y = x$.

Proof. Let $x > 2$ be an arbitrary real number. Consider $y = (x + \sqrt{x^2-4})/2$, which is a real number since $x > 2$. We then plug this value of y into the expression $y + 1/y$.

$$\begin{aligned} y + \frac{1}{y} &= \frac{x+\sqrt{x^2-4}}{2} + \frac{2}{x+\sqrt{x^2-4}} \\ &= \frac{x^2+2x\sqrt{x^2-4}+(x^2-4)}{2(x+\sqrt{x^2-4})} + \frac{4}{2(x+\sqrt{x^2-4})} \\ &= \frac{2x^2+2x\sqrt{x^2-4}}{2(x+\sqrt{x^2-4})} \\ &= \frac{2x(x+\sqrt{x^2-4})}{2(x+\sqrt{x^2-4})} \\ &= x \end{aligned}$$

Thus, we have found a real number y such that $y + 1/y = x$. Therefore, we conclude that for every real number x, if $x > 2$ then there is a real number y such that $y + 1/y = x$. □

3.3.8. Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \bigcup \mathcal{F}$.

Proof. Suppose $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$. Let $a \in A$ be arbitrary. We want to show that $a \in \bigcup \mathcal{F}$. Since $\bigcup \mathcal{F} = \{x \mid \exists X \in \mathcal{F}(x \in X)\}$, showing that $a \in \bigcup \mathcal{F}$ is equivalent to showing that there is $X \in \mathcal{F}$ such that $a \in X$. Let $X = A$. By assumption, $A \in \mathcal{F}$ and $a \in A$. Thus, we have found $X \in \mathcal{F}$ such that $a \in X$, so $a \in \bigcup \mathcal{F}$. Since $a \in A$ was arbitrary, we conclude that $A \subseteq \bigcup \mathcal{F}$. Therefore, if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $A \subseteq \bigcup \mathcal{F}$. □

3.3.9. Prove that if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $\bigcap \mathcal{F} \subseteq A$.

Proof. Suppose $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$. Let $x \in \bigcap \mathcal{F}$ be arbitrary. Recall that $\bigcap \mathcal{F} = \{x \mid \forall X \in \mathcal{F}(x \in X)\}$. In particular, since $A \in \mathcal{F}$, it follows that $x \in A$. Since $x \in \bigcap \mathcal{F}$ was arbitrary, we conclude that $\bigcap \mathcal{F} \subseteq A$. Therefore, if $\mathcal{F}$ is a family of sets and $A \in \mathcal{F}$, then $\bigcap \mathcal{F} \subseteq A$. $\square$

3.3.10. Suppose that $\mathcal{F}$ is a nonempty family of sets, B is a set, and for all $A \in \mathcal{F}$ that $B \subseteq A$. Prove that $B \subseteq \bigcap \mathcal{F}$.

Proof. Let $b \in B$ be arbitrary. We want to show that $b \in \bigcap \mathcal{F}$. Since $\bigcap \mathcal{F} = \{x \mid \forall A \in \mathcal{F}(x \in A)\}$, showing that $b \in \bigcap \mathcal{F}$ is equivalent to showing that $b \in A$ for all $A \in \mathcal{F}$. Because $B \subseteq A$ for all $A \in \mathcal{F}$ and $b \in B$, it follows that $b \in A$ for all $A \in \mathcal{F}$. In other words, $b \in \bigcap \mathcal{F}$. Since $b \in B$ was arbitrary, we conclude that $B \subseteq \bigcap \mathcal{F}$. This completes the proof. $\square$

3.3.11. Suppose that $\mathcal{F}$ is a family of sets. Prove that if $\varnothing \in \mathcal{F}$ then $\bigcap \mathcal{F} = \varnothing$.

Proof. Suppose $\varnothing \in \mathcal{F}$, and suppose, towards a contradiction, that $\bigcap \mathcal{F} \neq \varnothing$. In other words, there exists $x \in \bigcap \mathcal{F}$. By the definition of $\bigcap \mathcal{F}$, this means that $x \in X$ for all $X \in \mathcal{F}$. In particular, since $X = \varnothing \in \mathcal{F}$, this means that $x \in \varnothing$. This, however, is a contradiction against the definition of the empty set. We therefore conclude that $\bigcap \mathcal{F} = \varnothing$. Thus, if $\varnothing \in \mathcal{F}$ then $\bigcap \mathcal{F} = \varnothing$. $\square$

3.3.12. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$. *Note: In my digital copy of the book this exercise has a typo and asks for a proof of the incorrect statement that if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcup \mathcal{F} \subseteq \mathcal{G}$.*

Proof. Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let $x \in \bigcup \mathcal{F}$ be arbitrary. By the definition of $\bigcup \mathcal{F}$, this means that there is $X \in \mathcal{F}$ such that $x \in X$. Then, since $\mathcal{F} \subseteq \mathcal{G}$, it follows that $X \in \mathcal{G}$ as well. Since $x \in X$ and $X \in \mathcal{G}$, $x \in \bigcup \mathcal{G}$ by the definition of $\bigcup \mathcal{G}$. Thus, since $x \in \bigcup \mathcal{F}$ was arbitrary, we conclude that $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$. Therefore, if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcup \mathcal{F} \subseteq \bigcup \mathcal{G}$. $\square$

3.3.13. Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets. Prove that if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$.

Proof. Suppose $\mathcal{F} \subseteq \mathcal{F}$. Let $x \in \bigcap \mathcal{G}$ be arbitrary. By the definition of $\bigcap \mathcal{G}$, this means that for all $X \in \mathcal{G}$, $x \in X$. Note that the assumption that $\mathcal{G}$ is nonempty makes this a non-vacuous statement. Now, let $Y \in \mathcal{F}$ be arbitrary. Note that such a Y exists by the assumption that $\mathcal{F}$ is nonempty. Since $\mathcal{F} \subseteq \mathcal{G}$, $Y \in \mathcal{G}$. Because $Y \in \mathcal{G}$ and for all $X \in \mathcal{G}$, $x \in X$, it follows that $x \in Y$. Therefore, since $Y \in \mathcal{F}$ was arbitrary, $x \in Y$ for all $Y \in \mathcal{F}$. In other words, $x \in \bigcap \mathcal{F}$ by the definition of $\bigcap \mathcal{F}$. Since $x \in \bigcap \mathcal{G}$ was arbitrary, we conclude that $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$. Thus, if $\mathcal{F} \subseteq \mathcal{G}$ then $\bigcap \mathcal{G} \subseteq \bigcap \mathcal{F}$. $\square$

3.3.14. Suppose that $\{A_i \mid i \in I\}$ is an indexed family of sets. Prove that $\bigcup_{i \in I} \mathscr{P}(A_i) \subseteq \mathscr{P}\left(\bigcup_{i \in I} A_i\right)$. (*Hint: First make sure you know what all the notation means!*)

Proof. Let $X \in \bigcup_{i \in I} \mathscr{P}(A_i)$ be arbitrary. By the definition of $\bigcup_{i \in I} \mathscr{P}(A_i)$, this means that there is $i \in I$ such that $X \in \mathscr{P}(A_i)$. Further unraveling the definition of being an element of the power set of a set A, this means that there is $i \in I$ such that $X \subseteq A_i$. Since $X \subseteq A_i$ and $A_i \subseteq \bigcup_{i \in I} A_i$, it follows that $X \subseteq \bigcup_{i \in I} A_i$. In other words, $X \in \mathscr{P}\left(\bigcup_{i \in I} A_i\right)$. Since $X \in \bigcup_{i \in I} \mathscr{P}(A_i)$ was arbitrary, we conclude that $\bigcup_{i \in I} \mathscr{P}(A_i) \subseteq \mathscr{P}\left(\bigcup_{i \in I} A_i\right)$. This completes the proof. $\square$

3.3.15. Suppose $\{A_i \mid i \in I\}$ is an indexed family of sets and $I \neq \varnothing$. Prove that $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$. *Note: In my digital copy of the book this exercise has a typo and says that $I = \varnothing$.*

Proof. We start by unraveling the definitions of what it means to be an element of $\bigcap_{i\in I} \mathscr{P}(A_i)$. The first layer of definitions states that $X \in \bigcap_{i\in I} \mathscr{P}(A_i)$ if and only if $X \in \mathscr{P}(A_i)$ for all $i \in I$. Unraveling one step further, this is equivalent to $X \subseteq A_i$ for all $i \in I$, by the definition of the power set of a set A. In other words, proving that $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$ is equivalent to proving that $\bigcap_{i\in I} A_i \subseteq A_i$ for all $i \in I$. This follows directly from the definition of $\bigcap_{i\in I} A_i$. Letting $x \in \bigcap_{i\in I} A_i$ be arbitrary, $x \in A_i$ for all $i \in I$. Therefore, $\bigcap_{i\in I} A_i \subseteq A_i$ for all $i \in I$. This completes the proof that $\bigcap_{i\in I} A_i \in \bigcap_{i\in I} \mathscr{P}(A_i)$. □

3.3.16. Prove the converse of the statement proven in Example 3.3.5. In other words, prove that if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\bigcup \mathcal{F} \subseteq B$.

Proof. Suppose $\mathcal{F} \subseteq \mathscr{P}(B)$. Let $x \in \bigcup \mathcal{F}$ be arbitrary. By the definition of $\bigcup \mathcal{F}$, this means that there is $X \in \mathcal{F}$ such that $x \in X$. Then, since $\mathcal{F} \subseteq \mathscr{P}(B)$, $X \in \mathscr{P}(B)$. In other words, by the definition of the power set of B, $X \subseteq B$. It then follows that $x \in B$. Since $x \in \bigcup \mathcal{F}$ was arbitrary, we conclude that $\bigcup \mathcal{F} \subseteq B$. Therefore, we conclude that if $\mathcal{F} \subseteq \mathscr{P}(B)$ then $\bigcup \mathcal{F} \subseteq B$. □

3.3.17. Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets, and every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$. Prove that $\bigcup \mathcal{F} \subseteq \bigcap \mathcal{G}$.

Proof. Let $x \in \bigcup \mathcal{F}$ be arbitrary. By the definition of $\bigcup \mathcal{F}$, this means that there is $X \in \mathcal{F}$ such that $x \in X$. Then, by the assumption that every element of $\mathcal{F}$ is a subset of every element of $\mathcal{G}$, we know that for all $Y \in \mathcal{G}$, $X \subseteq Y$. It follows that $x \in Y$ for all $Y \in \mathcal{G}$. In other words, by the definition of $\bigcap \mathcal{G}$, $x \in \bigcap \mathcal{G}$. Hence, since $x \in \bigcup \mathcal{F}$ was arbitrary, we conclude that $\bigcup \mathcal{F} \subseteq \bigcap \mathcal{G}$. □

3.3.18. In this problem all variables range over $\mathbb{Z}$, the set of all integers.

(1) Prove that if a divides b and a divides c, then a divides $b + c$.
(2) Prove that if ac divides bc and $c \neq 0$, then a divides b.

Proof.

(1) Suppose a divides b and a divides c. Since a divides b, there is $m \in \mathbb{Z}$ such that $ma = b$. Similarly, since a divides c, there is $n \in \mathbb{Z}$ such that $na = c$. It then follows that $b + c = ma + na = (m + n)a$. Since $m + n$ is an integer, we conclude that a divides $b + c$. Therefore, if a divides b and a divides c, then a divides $b + c$.
(2) Suppose ac divides bc and $c \neq 0$. Since ac divides bc, there is $m \in \mathbb{Z}$ such that $mac = bc$. Then, since $c \neq 0$, we can divide both sides of the equation by c to conclude that $ma = b$. In other words, a divides b. Therefore, we conclude that if ac divides bc and $c \neq 0$, then a divides b.

□

3.3.19.

(1) Prove that for all real numbers x and y there is a real number z such that $x+z = y-z$.

(2) Would the statement in Part (1) be correct if "real number" were changed to "integer"? Justify your answer.

Proof.

(1) Let $x, y \in \mathbb{R}$ be arbitrary. Consider the real number $z = (y-x)/2$. We plug this value for z into the expressions $x+z$ and $y-z$ to check that they are equal.

$$x + z = x + \frac{y-x}{2} = \frac{x+y}{2}$$
$$y - z = y - \frac{y-x}{2} = \frac{x+y}{2}$$

As we can see, $x + z = (x+y)/2 = y - z$. This completes the proof.

(2) The statement in Part (1) would be incorrect if "real number" were changed to "integer". This comes from the way in which z is defined by isolating z in the equation $x + z = y - z$ to get $z = (y-x)/2$. The difference of two real numbers is not guaranteed to be an integer, let alone an integer that is divisible by 2. For example, consider the case of $x = 0$ and $y = 1$. In this case, the proof gives us a value of $z = 1/2$, which is not an integer.

$\square$

3.3.20. Consider the following theorem:

Theorem 15. *For every real number x, $x^2 \geq 0$.*

What's wrong with the following proof of the theorem?

Proof. Suppose not. Then for every for every real number x, $x^2 < 0$. In particular, plugging in $x = 3$ we would get $9 < 0$, which is clearly false. This contradiction shows that for every number x, $x^2 \geq 0$. $\square$

Proof. The issue with the given attempted proof is in the negation of the statement "For every real number x, $x^2 \geq 0$," to try and reach a contradiction. Since the statement has the logical form $\forall x P(x)$, the correct negation has the logical form $\exists x \neg P(x)$. In other words, the correct negation is "There exists a real number x such that $x^2 < 0$," instead of "For every real number x, $x^2 < 0$." $\square$

3.3.21. Consider the following incorrect theorem:

Theorem 16. *If $\forall x \in A(x \neq 0)$ and $A \subseteq B$ then $\forall x \in B(x \neq 0)$.*

(1) What's wrong with the following proof of the theorem?

Proof. Suppose that $\forall x \in A(x \neq 0)$ and $A \subseteq B$. Let x be an arbitrary element of A. Since $\forall x \in A(x \neq 0)$, we can conclude that $x \neq 0$. Also, since $A \subseteq B$, $x \in B$. Since $x \in B$, $x \neq 0$, and x was arbitrary, we can conclude that $\forall x \in B(x \neq 0)$. $\square$

(2) Find a counterexample to the theorem. In other words, find an example of sets A and B for which the hypotheses of the theorem are true but the conclusion is false.

Proof.

(1) The issue with the attempted proof is in the statement "Let x be an arbitrary element of A". The goal of the proof is to prove that for all $x \in B$, $x \neq 0$, so we must instead start with an arbitrary element of B.
(2) One counterexample to the theorem is when $A = \{1, 2, 3\}$ and $B = \{0, 1, 2, 3\}$. In this case, all elements of A are nonzero and $A \subseteq B$, but not all elements of B are nonzero.

□

3.3.22. Consider the following incorrect theorem:

Theorem 17. $\exists x \in \mathbb{R} \forall y \in \mathbb{R}(xy^2 = y - x)$.

What's wrong with the following proof of the theorem?

Proof. Let $x = y/(y^2 + 1)$. Then

$$y - x = y - \frac{y}{y^2+1} = \frac{y^3}{y^2+1} = \frac{y}{y^2+1} \cdot y^2 = xy^2.$$

□

Proof. The issue in the proof is that this value of x depends on the value of y. This is different from what the theorem states: that there is a single value of x, **independent** of the value of y, such that $xy^2 = y - x$ for all values of y. In terms of proof structure, we need to start by defining the real number x, which at this point cannot be defined in terms of y since y hasn't been introduced in the proof. Only after x is defined, would we take an arbitrary $y \in \mathbb{R}$ and attempt to show that $xy^2 = y - x$. □

3.3.23. Consider the following incorrect theorem:

Theorem 18. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. If $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint, then so are $\mathcal{F}$ and $\mathcal{G}$.*

(1) What's wrong with the following proof of the theorem?

Proof. Suppose $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint. Suppose $\mathcal{F}$ and $\mathcal{G}$ are not disjoint. Then we can choose some set A such that $A \in \mathcal{F}$ and $A \in \mathcal{G}$. Since $A \in \mathcal{F}$, by Exercise 3.3.8, $A \subseteq \bigcup \mathcal{F}$, so every element of A is in $\bigcup \mathcal{F}$. Similarly, since $A \in \mathcal{G}$, every element of A is in $\bigcup \mathcal{G}$. But then every element of A is in both $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$, and this is impossible since $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint. Thus, we have reached a contradiction, so $\mathcal{F}$ and $\mathcal{G}$ must be disjoint. □

(2) Find a counterexample to the theorem.

Proof.
(1) The issue with the attempted proof lies in the statement "Then we can choose some set A such that $A \in \mathcal{F}$ and $A \in \mathcal{G}$". We do not know that A is nonempty since there are no assumptions as to whether or not $\varnothing \in \mathcal{F} \cap \mathcal{G}$. If $A = \varnothing$, then there is no contradiction to having $A \in \mathcal{F} \cap \mathcal{G}$ and having $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ being disjoint, since A contains no elements.

(2) One counterexample to the theorem is the case where $\mathcal{F} = \{\varnothing, \{1\}\}$ and $\mathcal{G} = \{\varnothing, \{2\}\}$. In this case, $\bigcup \mathcal{F} = \{1\}$ and $\bigcup \mathcal{G} = \{2\}$, which are clearly disjoint, but $\mathcal{F} \cap \mathcal{G} = \{\varnothing\}$. As discussed previously, the set containing the empty set is **not** empty.

□

3.3.24. Consider the following putative theorem:

Theorem 19. *For all real numbers x and y, $x^2 + xy - 2y^2 = 0$.*

(1) What's wrong with the following proof of the theorem?

Proof. Let x and y be equal to some arbitrary real number r. Then

$$x^2 + xy - 2y^2 = r^2 + r \cdot r - 2r^2 = 0.$$

Since x and y were both arbitrary, this shows that for all real number x and y, $x^2 + xy - 2y^2 = 0$. □

(2) Is the theorem correct? Justify your answer with either a proof or a counterexample.

Proof.

(1) The issue with the attempted proof lies in the statement "Let x and y be equal to some arbitrary real number r", since there is no assumption in the theorem statement that $x = y$. Instead, we must assume that x is an arbitrary real number and that y is an arbitrary real number, while not making any further assumptions about the relationship between x and y.
(2) The theorem is incorrect. Consider the case where $x = 1$ and $y = 0$. In this case, $x^2 + xy - 2y^2 = 1^2 + (1)(0) - 2(0)^2 = 1 \neq 0$.

□

3.3.25. Prove that for every real number x there is a real number y such that for every real number z, $yz = (x + z)^2 - (x^2 + z^2)$.

Proof. Let $x \in \mathbb{R}$ be arbitrary. Let $y = 2x$. Now let $z \in \mathbb{R}$ be arbitrary. We then have $yz = (2x)z = 2xz$. We also have $(x+z)^2 - (x^2+z^2) = x^2 + 2xz + z^2 - x^2 - z^2 = 2xz$. Thus, $yz = 2xz = (x + z)^2 - (x^2 + z^2)$. This completes the proof.

Note that in order to properly prove the statement, we must be careful about the order in which we perform the setup. We first have to let $x \in \mathbb{R}$ be arbitrary. Once we do that, we then define y. When defining y, the only variable we can use is x, since that is the only other variable that has been introduced at this point. Only after we have defined y can we let $z \in \mathbb{R}$ be arbitrary. One way we can think about the order of introducing variables is that while the value of y might depend on the choice of x, once the values of x and y have been set, then the rest of the statement has to hold true for any value of z. □

3.3.26.

(1) Comparing the various rules for dealing with quantifiers in proofs, you should see a similarity between the rules for goals of the form $\forall x P(x)$ and givens of the form $\exists x P(x)$. What is this similarity? What about the rules for goals of the form $\exists x P(x)$ and givens of the form $\forall x P(x)$?

(2) Can you think of a reason why these similarities might be expected? (*Hint: Think about how a proof by contradiction works when the goal starts with a quantifier.*)

Proof.

(1) The similarity between the rules for goals of the form $\forall x P(x)$ and givens of the form $\exists x P(x)$ is that both involve introduce a variable x. When working toward a goal of the form $\forall x P(x)$, we introduce an arbitrary x and try to prove $P(x)$, while when working with a given $\exists x P(x)$, we introduce an arbitrary x that satisfies $P(x)$. For goals of the form $\exists x P(x)$ and givens of the form $\forall x P(x)$, we instead try to find or construct a particular value a to plug in for x. When working toward a goal of the form $\exists x P(x)$, we construct a particular value a such that $P(a)$ is true, while when working with a given $\forall x P(x)$, we try to find a relevant value a to then immediately plug into the statement P and conclude that $P(a)$ is true.

(2) One reason why these similarities might be expected comes from the quantifier negation laws: $\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$ and $\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$. If we are performing proof by contradiction when the goal starts with a quantifier, we use those negation laws when we add the negated goal as a given and then attempt to work toward a contradiction. In other words, a goal of the form $\forall x P(x)$ would turn into a given of the form $\exists x \neg P(x)$, while a goal of the form $\exists x P(x)$ would turn into a given of the form $\forall x \neg P(x)$.

□

3.4.1. Use the methods of this chapter to prove that $\forall x(P(x) \wedge Q(x))$ is equivalent to $\forall x P(x) \wedge \forall x Q(x)$.

Proof. We first prove the forward direction: $\forall x(P(x) \wedge Q(x)) \to \forall x P(x) \wedge \forall x Q(x)$. Suppose $\forall x(P(x) \wedge Q(x))$ is true. Now let x_1 and x_2 be arbitrary. By the assumption, $P(x_1) \wedge Q(x_1)$ is true, so in particular $P(x_1)$ is true. Since x_1 was arbitrary, we conclude that $\forall x P(x)$ is true. Using the same argument with x_2, we also conclude that $\forall x Q(x)$ is true. In other words, $\forall x P(x) \wedge \forall x Q(x)$ is true. This completes the proof of the forward direction.

Now we prove the other direction: $\forall x P(x) \wedge \forall x Q(x) \to \forall x(P(x) \wedge Q(x))$. Suppose $\forall x P(x) \wedge \forall x Q(x)$ is true. Let x_1 be arbitrary. By the assumption, $P(x_1)$ is true, and $Q(x_1)$ is true as well. In other words, $P(x_1) \wedge Q(x_1)$ is true. Since x_1 was arbitrary, we conclude that $\forall x(P(x) \wedge Q(x))$ is true. This completes the proof of the reverse direction. □

3.4.2. Prove that if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

Proof. Suppose $A \subseteq B$ and $A \subseteq C$. Let $x \in A$ be arbitrary. By the assumptions that $A \subseteq B$, it follows that $x \in B$. Similarly, $x \in C$ as well. In other words, $x \in B \cap C$. Since our choice of $x \in A$ was arbitrary, we conclude that $A \subseteq B \cap C$. Therefore, if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$. □

3.4.3. Suppose $A \subseteq B$. Prove that for every set C, $C \setminus B \subseteq C \setminus A$.

Proof. Let C be an arbitrary set. Now let $x \in C \setminus B$ be arbitrary. By the definition of $C \setminus B$, $x \in C$ and $x \notin B$. Since $x \notin B$ and $A \subseteq B$, it follows that $x \notin A$. Then, since $x \in C$ and $x \notin A$, $x \in C \setminus A$. Because $x \in C \setminus B$ was arbitrary, we conclude that $C \setminus B \subseteq C \setminus A$. Finally, since C was an arbitrary set, we conclude that for every set C, $C \setminus B \subseteq C \setminus A$. □

3.4.4. Prove that if $A \subseteq B$ and $A \nsubseteq C$ then $B \nsubseteq C$.

Proof. Suppose $A \subseteq B$ and $A \nsubseteq C$. Since $A \nsubseteq C$, there is $x \in A$ such that $x \notin C$. Then, since $A \subseteq B$, it follows that $x \in B$. Therefore, since $x \in B$ and $x \notin C$, we conclude that $B \nsubseteq C$. Thus, if $A \subseteq B$ and $A \nsubseteq C$ then $B \nsubseteq C$. □

3.4.5. Prove that if $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \nsubseteq C$.

Proof. Suppose $A \subseteq B \setminus C$ and $A \neq \varnothing$. Since $A \neq \varnothing$, there is $x \in A$. Then, since $A \subseteq B \setminus C$, it follows that $x \in B \setminus C$. By the definition of $B \setminus C$, this means that $x \in B$ and $x \notin C$. In other words, $B \nsubseteq C$. Thus, we conclude that if $A \subseteq B \setminus C$ and $A \neq \varnothing$ then $B \nsubseteq C$. □

3.4.6. Prove that for any sets A, B, and C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$, by finding a string of equivalences starting with $x \in A \setminus (B \cap C)$ and ending with $x \in (A \setminus B) \cup (A \setminus C)$.

Proof. Let A, B, and C be arbitrary sets. First, $x \in A \setminus (B \cap C)$ is equivalent to $(x \in A) \wedge (x \notin B \cap C)$. Since $x \notin B \cap C$ is equivalent to $(x \notin B) \vee (x \notin C)$, $(x \in A) \wedge (x \notin B \cap C)$ is equivalent to $(x \in A) \wedge [(x \notin B) \vee (x \notin C)]$. This is then equivalent to $[(x \in A) \wedge (x \notin B)] \vee [(x \in A) \wedge (x \notin C)]$. Finally, this is equivalent to $(x \in A \setminus B) \vee (x \in A \setminus C)$, or in other words, $x \in (A \setminus B) \cup (A \setminus C)$. Since A, B, and C were arbitrary sets, we conclude that for any sets A, B, and C, $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$. □

3.4.7. Use the methods of this chapter to prove that for any sets A and B, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.

Proof. Let A and B be arbitrary sets. We start by showing $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$. Let $X \in \mathscr{P}(A \cap B)$ be arbitrary. By the definition of $\mathscr{P}(A \cap B)$, $X \subseteq A \cap B$. Then, since $A \cap B \subseteq A$, $X \subseteq A$. In other words, $X \in \mathscr{P}(A)$. Similarly, since $A \cap B \subseteq B$, $X \in \mathscr{P}(B)$. Thus, $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$, and since $X \in \mathscr{P}(A \cap B)$ was arbitrary we conclude that $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$.

Next we show $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$. Let $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$ be arbitrary. It follows that $X \subseteq A$ and $X \subseteq B$, so $X \subseteq A \cap B$. In other words, $X \in \mathscr{P}(A \cap B)$. Thus, since $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$ was arbitrary, we conclude that $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$. Now that we have checked both containments, we conclude that $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$. Since A and B were arbitrary sets, this is true for any sets A and B. □

3.4.8. Prove that $A \subseteq B$ if and only if $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

Proof. First, suppose $A \subseteq B$. For this part of the proof, our goal is to show $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. Let $X \in \mathscr{P}(A)$ be arbitrary. By the definition of $\mathscr{P}(A)$, $X \subseteq A$. Then, since $A \subseteq B$, $X \subseteq B$. In other words, $X \in \mathscr{P}(B)$. Thus, $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

Now suppose $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. Our goal for this part of the proof is to show $A \subseteq B$. There are two cases to consider: $A = \varnothing$ and $A \neq \varnothing$. If $A = \varnothing$, then it is vacuously true that $A \subseteq B$. Thus, we can assume, without loss of generality, that $A \neq \varnothing$. Let $x \in A$ be arbitrary. Then the singleton set $\{x\}$ is a subset of A, and $\{x\} \in \mathscr{P}(A)$. Since $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, $\{x\} \in \mathscr{P}(B)$ as well. In other words, $\{x\} \subseteq B$, so $x \in B$. Hence, $A \subseteq B$. This completes the proof. □

3.4.9. Prove that if x and y are odd integers, then xy is odd.

Proof. Suppose x and y are odd integers. Since x and y are odd, there are $m, n \in \mathbb{Z}$ such that $x = 2m + 1$ and $y = 2n + 1$. Multiplying x and y together, we get

$$xy = (2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1.$$

In other words, $xy = 2k + 1$ where $k = 2mn + m + n$ is an integer. Thus, we conclude that xy is also odd. □

3.4.10. Prove that if x and y are odd integers, then $x - y$ is even.

Proof. Suppose x and y are odd integers. Since x and y are odd, there are $m, n \in \mathbb{Z}$ such that $x = 2m + 1$ and $y = 2n + 1$. Subtracting y from x, we get

$$x - y = (2m + 1) - (2n + 1) = 2m - 2n = 2(m - n).$$

In other words, $x - y = 2k$ where $k = m - n$ is an integer. Thus we conclude that $x - y$ is even. □

3.4.11. Prove that for every integer n, n^3 is even if and only if n is even.

Proof. Let $n \in \mathbb{Z}$ be arbitrary. To prove the forward direction, we instead check the contrapositive statement: if n is odd then n^3 is odd. Suppose n is odd, so there is $k \in \mathbb{Z}$ such that $n = 2k + 1$. We now compute n^3.

$$n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$$

In other words, $n^3 = 2j + 1$ where $j = 4k^3 + 6k^2 + 3k$ is an integer. Thus we conclude that n^3 is even. Hence, if n^3 is even, then n is even as well.

Now we check the backward direction: if n is even then n^3 is even. Suppose n is even, so there is $k \in \mathbb{Z}$ such that $n = 2k$. Computing n^3 gives us $n^3 = (2k)^3 = 8k^3 = 2(4k^3)$. In other words, $n^3 = 2j$ where $j = 4k^3$ is an integer. Thus we conclude that n^3 is even. Hence, if n is even, then n^3 is even as well. This completes the proof that for every integer n, n^3 is even if and only if n is even. □

3.4.12. Consider the following putative theorem:

Theorem 20. *Suppose m is an even integer and n is an odd integer. Then $n^2 - m^2 = n + m$.*

(1) What is wrong with the following proof of the theorem?

Proof. Since m is even, we can choose some integer k such that $m = 2k$. Similarly, since n is odd we have $n = 2k + 1$. Therefore

$$\begin{aligned} n^2 - m^2 &= (2k+1)^2 - (2k)^2 \\ &= 4k^2 + 4k + 1 - 4k^2 \\ &= 4k + 1 \\ &= (2k+1) + (2k) \\ &= n + m. \end{aligned}$$

□

(2) Is the theorem correct? Justify your answer with either a proof or a counterexample.

Proof.

(1) The issue with the attempted proof is in unraveling the definitions of m being even and n being odd. The proof incorrectly assumes that there is a single integer k such that $m = 2k$ and $n = 2k + 1$, but that is not necessarily the case. Instead, the proof should have said that there is some integer k such that $m = 2k$ and some integer l such that $n = 2l$.

(2) The theorem is incorrect. Consider the case where $m = 2$ and $n = 1$. In this case, $n^2 - m^2 = 1^2 - 2^2 = -3$, while $n + m = 1 + 2 = 3$. Thus, we have found values for m and n that satisfy the hypotheses of the theorem, but do not satisfy the conclusion.

□

3.4.13. Prove that $\forall x \in \mathbb{R}[\exists y \in \mathbb{R}(x + y = xy) \leftrightarrow x \neq 1]$. *Note: This exercise in my copy of the book has a typo, where instead of $x \neq 1$ it says $x = 1$.*

Proof. Translating the statement from logical symbols to mathematical English, we wish to prove that for all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x + y = xy$ if and only if $x \neq 1$. First, let $x \in \mathbb{R}$ be arbitrary. Now our goal is to prove that there exists $y \in \mathbb{R}$ such that $x+y = xy$ if and only if $x \neq 1$.

We start by proving the forward direction. Suppose there exists $y \in \mathbb{R}$ such that $x+y = xy$. Next suppose, towards a contradiction, that $x = 1$. Plugging this into our equation gives $1 + y = y$, which is a contradiction. Thus, we conclude that $x \neq 1$.

Next we prove the backward direction. Suppose $x \neq 1$. Consider $y = x/(x - 1)$, which is a real number since $x \neq 1$. Plugging in this value for y into the expression $x + y$ gives us

$$x + \frac{x}{x-1} = \frac{x^2 - x}{x-1} + \frac{x-1}{x-1} = \frac{x^2}{x-1} = xy.$$

Thus, we have found a value of y such that $x + y = xy$. Since $x \in \mathbb{R}$ was arbitrary and we have proved both directions, this completes the proof. □

3.4.14. Prove that $\exists z \in \mathbb{R}\forall x \in \mathbb{R}^+[\exists y \in \mathbb{R}(y - x = y/x) \leftrightarrow x \neq z]$. *Note: This exercise in my copy of the book has a typo, where instead of $x \neq z$ it says $x = z$.*

Proof. Translating the statement from logical symbols to mathematical English, we wish to prove that there exists $z \in \mathbb{R}$, such that for all $x \in \mathbb{R}^+$, there exists $y \in \mathbb{R}$ such that $y - x = y/x$ if and only if $x \neq z$. To start, let $z = 1$ and let $x \in \mathbb{R}^+$ be arbitrary. Now our goal is to prove that there exists $y \in \mathbb{R}$ such that $y - x = y/x$ if and only if $x \neq 1$.

We start by proving the forward direction. Suppose there exists $y \in \mathbb{R}$ such that $y - x = y/x$. Next suppose, towards a contradiction, that $x = 1$. Plugging this into our equation gives us $y - 1 = y$, which is a contradiction. Thus, we conclude that $x \neq 1$.

Next we prove the backward direction. Suppose $x \neq 1$. Consider $y = x^2/(x - 1)$, which is a real number since $x \neq 1$. Plugging this value for y into the expression $y - x$ gives us

$$\frac{x^2}{x-1} - x = \frac{x^2}{x-1} - \frac{x^2 - x}{x-1} = \frac{x}{x-1} = \frac{y}{x}.$$

Thus, we have found a value of y such that $y - x = y/x$.

Since $x \in \mathbb{R}^+$ was arbitrary and we have proved both directions, this completes the proof. *Note that the proof actually holds for all nonzero $x \in \mathbb{R}$.* □

3.4.15. Suppose B is a set and $\mathcal{F}$ is a family of sets. Prove that $\bigcup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \bigcup(\mathcal{F} \setminus \mathscr{P}(B))$.

Proof. Since the empty set is always a subset of any set S, we may assume, without loss of generality, that $\bigcup\{A \setminus B \mid A \in \mathcal{F}\}$ is nonempty. Let $x \in \bigcup\{A \setminus B \mid A \in \mathcal{F}\}$ be arbitrary. Our goal is to show that there exists a set $A \in \mathcal{F} \setminus \mathscr{P}(B)$ such that $x \in A$. By the definition of $\bigcup\{A \setminus B \mid A \in \mathcal{F}\}$, there is $A \in \mathcal{F}$ such that $x \in A \setminus B$. In particular, this means that $A \nsubseteq B$, or in other words, $A \notin \mathscr{P}(B)$. Since $A \notin \mathscr{P}(B)$, we have $A \in \mathcal{F} \setminus \mathscr{P}(B)$, as desired. Thus, $x \in \bigcup(\mathcal{F} \setminus \mathscr{P}(B))$. Since $x \in \bigcup\{A \setminus B \mid A \in \mathcal{F}\}$ was arbitrary, this completes the proof that $\bigcup\{A \setminus B \mid A \in \mathcal{F}\} \subseteq \bigcup(\mathcal{F} \setminus \mathscr{P}(B))$. $\square$

3.4.16. Suppose $\mathcal{F}$ and $\mathcal{G}$ are nonempty families of sets and every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$. Prove that $\bigcup \mathcal{F}$ and $\bigcap \mathcal{G}$ are disjoint.

Proof. Suppose, towards a contradiction, that $\bigcup \mathcal{F}$ and $\bigcap \mathcal{G}$ are not disjoint. Then there is $x \in (\bigcup \mathcal{F}) \cap (\bigcap \mathcal{G})$. Since $x \in \bigcup \mathcal{F}$, there is $A \in \mathcal{F}$ such that $x \in A$. Then, by the assumption that every element of $\mathcal{F}$ is disjoint from some element of $\mathcal{G}$, there is a set $B_A \in \mathcal{G}$ such that $A \cap B_A = \varnothing$. However, since $x \in \bigcap \mathcal{G}$, $x \in B$ for all $B \in \mathcal{G}$. In particular, this means $x \in B_A$, so $x \in A \cap B_A$. This, however, is a contradiction to the prior assumption that $A \cap B_A = \varnothing$. Thus, we conclude that $\bigcup \mathcal{F}$ and $\bigcap \mathcal{G}$ are disjoint. $\square$

3.4.17. Prove that for any set A, $A = \bigcup \mathscr{P}(A)$.

Proof. Let A be an arbitrary set. First note that if A is empty, then $\bigcup \mathscr{P}(A) = \bigcup\{\varnothing\} = \varnothing$, so $A = \bigcup \mathscr{P}(A)$ in this case. Now we assume that A is nonempty. We start by proving that $A \subseteq \bigcup \mathscr{P}(A)$. Let $x \in A$ be arbitrary. Since $x \in A$, the singleton set $S_x = \{x\}$ is a subset of A. In other words, $S_x \in \mathscr{P}(A)$ and $x \in S_x$, so $x \in \bigcup \mathscr{P}(A)$. Since $x \in A$ was arbitrary, we conclude that $A \subseteq \bigcup \mathscr{P}(A)$.

Next we prove that $\bigcup \mathscr{P}(A) \subseteq A$. Let $x \in \bigcup \mathscr{P}(A)$ be arbitrary. By the definition of $\bigcup \mathscr{P}(A)$, there is $S \in \mathscr{P}(A)$ such that $x \in S$. Then, by the definition of $\mathscr{P}(A)$, $S \subseteq A$, so it immediately follows that $x \in A$. Since $x \in \bigcup \mathscr{P}(A)$ was arbitrary, we conclude that $\bigcup \mathscr{P}(A) \subseteq A$. Hence, we conclude that $A = \bigcup \mathscr{P}(A)$. Since A was an arbitrary set, this holds for any set A. $\square$

3.4.18. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

(1) Prove that $\bigcup(\mathcal{F} \cap \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$.
(2) What is wrong with the following proof that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$?

Proof. Suppose $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$. This means that $x \in \bigcup \mathcal{F}$ and $x \in \bigcup \mathcal{G}$, so there exists $A \in \mathcal{F}$ such that $x \in A$, and there exists $A \in \mathcal{G}$ such that $x \in A$. Thus, we can choose a set A such that $A \in \mathcal{F}$, $A \in \mathcal{G}$, and $x \in A$. Since $A \in \mathcal{F}$ and $A \in \mathcal{G}$, $A \in \mathcal{F} \cap \mathcal{G}$. Therefore there exists $A \in \mathcal{F} \cap \mathcal{G}$ such that $x \in A$., so $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$. Since x was arbitrary, we can conclude that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. $\square$

(3) Find an example of families of sets $\mathcal{F}$ and $\mathcal{G}$ for which $\bigcup(\mathcal{F} \cap \mathcal{G}) \neq (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$.

Proof.

(1) Let $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$ be arbitrary. Then, by the definition of $\bigcup(\mathcal{F} \cap \mathcal{G})$, there is $X \in \mathcal{F} \cap \mathcal{G}$ such that $x \in X$. Since $X \in \mathcal{F}$, it follows that $x \in \bigcup \mathcal{F}$. Similarly, since $X \in \mathcal{G}$, it follows that $x \in \bigcup \mathcal{G}$. In other words, $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$. Since $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$ was arbitrary, we conclude that $\bigcup(\mathcal{F} \cap \mathcal{G}) \subseteq (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$.
(2) The error in the proof occurs when the existence statement in the definitions of $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ is incorrectly interpreted to mean the set $A \in \mathcal{F}$ containing x and the set $A \in \mathcal{G}$ containing x are the same set. They could be different sets, and a correct proof would recognize this by using the definition to assert the existence of a set $A \in \mathcal{F}$ containing x and a set $B \in \mathcal{G}$ containing x. Here giving the two sets different variable names emphasizes the fact that they need not be equal.
(3) Consider $\mathcal{F} = \{\{1, 2\}, \{3\}\}$ and $\mathcal{G} = \{\{1\}, \{2, 3\}\}$. In this case, $\bigcup \mathcal{F} = \bigcup \mathcal{G} = \{1, 2, 3\}$, so $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) = \{1, 2, 3\}$. However, $\mathcal{F} \cap \mathcal{G} = \varnothing$, so $\bigcup(\mathcal{F} \cap \mathcal{G}) = \varnothing$.

□

3.4.19. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$ if and only if for all $A \in \mathcal{F}$ and for all $B \in \mathcal{G}$, $A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$.

Proof. We start by proving the forward direction. Suppose $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. Let $A \in \mathcal{F}$ and $B \in \mathcal{G}$ be arbitrary. Without loss of generality, we may assume that $A \cap B \neq \varnothing$, since otherwise there is nothing to check, as the empty set is a subset of any set. Since A and B are not disjoint, there is $x \in A \cap B$. Next, since $x \in A$ and $A \in \mathcal{F}$, it follows that $x \in \bigcup \mathcal{F}$. Similarly, $x \in \bigcup \mathcal{G}$ as well. In other words, $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$, so it immediately follows that $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$ from the assumption that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. This allows us to conclude that $A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$, and since A and B were arbitrary, this holds for all $A \in \mathcal{F}$ and all $B \in \mathcal{G}$. This completes the proof of the forward direction.

Now suppose for all $A \in \mathcal{F}$ and for all $B \in \mathcal{G}$, $A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. First, without loss of generality, we may assume that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$ is nonempty, since once again there would be nothing to prove otherwise. Let $x \in (\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$ be arbitrary. Unraveling the definitions of $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G})$, this means that there is $A \in \mathcal{F}$ such that $x \in A$, and there is $B \in \mathcal{G}$ such that $x \in B$. By the assumption, $A \cap B \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$, so since $x \in A \cap B$, it follows that $x \in \bigcup(\mathcal{F} \cap \mathcal{G})$. Since x was arbitrary, this completes the proof that $(\bigcup \mathcal{F}) \cap (\bigcup \mathcal{G}) \subseteq \bigcup(\mathcal{F} \cap \mathcal{G})$. □

3.4.20. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are disjoint if and only if for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$, A and B are disjoint.

Proof. We instead prove the equivalent statement that $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are not disjoint if and only if there exist $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that A and B are not disjoint. Note that this equivalent statement comes from breaking the original biconditional statement into the individual directions and then taking the contrapositive of each direction. First, suppose $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are not disjoint. This means that there is x in the intersection of these two sets. In other words, there exists $A \in \mathcal{F}$ such that $x \in A$ and also there exists $B \in \mathcal{G}$ such that $x \in B$. Since $x \in A \cap B$, we have found two sets $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that A and B are not disjoint.

Next, suppose there exist $A \in \mathcal{F}$ and $B \in \mathcal{G}$ such that A and B are not disjoint. Since A and B are not disjoint, there exists $x \in A \cap B$. Then, since $x \in A$ and $A \in \mathcal{F}$, we know that $x \in \bigcup \mathcal{F}$. Similarly, $x \in \bigcup \mathcal{G}$ as well, so we conclude that $\bigcup \mathcal{F}$ and $\bigcup \mathcal{G}$ are not disjoint. This completes the proof. □

3.4.21. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets.

(1) Prove that $(\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G}) \subseteq \bigcup(\mathcal{F} \setminus \mathcal{G})$.
(2) What's wrong with the following proof that $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$?

Proof. Suppose $x \in \bigcup(\mathcal{F} \setminus \mathcal{G})$. Then we can choose some $A \in \mathcal{F} \setminus \mathcal{G}$ such that $x \in A$. Since $A \in \mathcal{F} \setminus \mathcal{G}$, $A \in \mathcal{F}$ and $A \notin \mathcal{G}$. Since $A \in \mathcal{F}$, $x \in \bigcup\mathcal{F}$. Since $x \in A$ and $A \notin \mathcal{G}$, $x \notin \bigcup\mathcal{G}$. Therefore $x \in (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$. $\square$

(3) Prove that $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$ if and only if for all $A \in \mathcal{F} \setminus \mathcal{G}$ and for all $B \in \mathcal{G}$, $A \cap B = \varnothing$.
(4) Find an example of families of sets $\mathcal{F}$ and $\mathcal{G}$ for which $\bigcup(\mathcal{F} \setminus \mathcal{G}) \neq (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$

Proof.

(1) Let $x \in (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$ be arbitrary. In other words, there is $A \in \mathcal{F}$ such that $x \in A$, and there does not exist $B \in \mathcal{G}$ such that $x \in B$, by the definitions of $x \in \bigcup\mathcal{F}$ and $x \notin \bigcup\mathcal{G}$, respectively. To show that $x \in \bigcup(\mathcal{F} \setminus \mathcal{G})$, we check that $A \in \mathcal{F} \setminus \mathcal{G}$. In particular, we show that $A \notin \mathcal{G}$. If $A \in \mathcal{G}$, then we would have a set in $\mathcal{G}$ containing x, which would contradict the prior assumption that no such set exists in $\mathcal{G}$. Thus, we conclude that $A \notin \mathcal{G}$, so it then follows that as $A \in \mathcal{F} \setminus \mathcal{G}$, $x \in \bigcup(\mathcal{F} \setminus \mathcal{G})$. Since x was arbitrary, we conclude that $(\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G}) \subseteq \bigcup(\mathcal{F} \setminus \mathcal{G})$.
(2) The issue with the proof lies in the fact that there might be another set $B \in \mathcal{G}$ such that $x \in B$. In order for x to be an element of $(\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$, there must not be *any* set $B \in \mathcal{G}$ containing x.
(3) First, suppose that for all $A \in \mathcal{F} \setminus \mathcal{G}$ and for all $B \in \mathcal{G}$, $A \cap B = \varnothing$. Let $x \in \bigcup(\mathcal{F} \setminus \mathcal{G})$ be arbitrary. Then there is some $A \in \mathcal{F} \setminus \mathcal{G}$ such that $x \in A$. Since for all $A \in \mathcal{F} \setminus \mathcal{G}$ and for all $B \in \mathcal{G}$, $A \cap B = \varnothing$, there is no $B \in \mathcal{G}$ such that $x \in B$. Otherwise, if such a B existed, then $x \in A \cap B$ would be a contradiction. Since there is no $B \in \mathcal{G}$ such that $x \in B$, it follows that $x \notin \bigcup\mathcal{G}$. As $x \in A$ and $A \in \mathcal{F}$ means $x \in \bigcup\mathcal{F}$, we conclude that $x \in (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$. Hence, $\bigcup(\mathcal{F} \setminus \mathcal{G}) \subseteq (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$.

   To prove the other direction, we work with the contrapositive statement: if there exist $A \in \mathcal{F} \setminus \mathcal{G}$ and $B \in \mathcal{G}$ such that $A \cap B \neq \varnothing$, then $\bigcup(\mathcal{F} \setminus \mathcal{G}) \nsubseteq (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$. Suppose there is $A \in \mathcal{F} \setminus \mathcal{G}$ and $B \in \mathcal{G}$ such that $A \cap B \neq \varnothing$. Then, since $A \cap B \neq \varnothing$, there exists $x \in A \cap B$. Since $x \in A$ and $A \in \mathcal{F} \setminus \mathcal{G}$, $x \in \bigcup(\mathcal{F} \setminus \mathcal{G})$. In addition, since A is in particular an element of $\mathcal{F}$, $x \in \bigcup\mathcal{F}$. On the other hand, since $x \in B$ and $B \in \mathcal{G}$, $x \in \bigcup\mathcal{G}$ as well. This means that $x \notin (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$. In other words, this proves that $\bigcup(\mathcal{F} \setminus \mathcal{G}) \nsubseteq (\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G})$. This completes the proof.
(4) Consider $\mathcal{F}\{\{1\}, \{1,2\}\}$ and $\mathcal{G} = \{\{1\}, \{2\}\}$. In this case, $\mathcal{F} \setminus \mathcal{G} = \{\{1,2\}\}$, so $\bigcup(\mathcal{F} \setminus \mathcal{G}) = \{1,2\}$. On the other hand, $\bigcup\mathcal{F} = \bigcup\mathcal{G} = \{1,2\}$, so $(\bigcup\mathcal{F}) \setminus (\bigcup\mathcal{G}) = \varnothing$.

$\square$

3.4.22. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets. Prove that if $\bigcup\mathcal{F} \nsubseteq \bigcup\mathcal{G}$, then there is some $A \in \mathcal{F}$ such that for all $B \in \mathcal{G}$, $A \nsubseteq B$.

Proof. Suppose $\bigcup\mathcal{F} \nsubseteq \bigcup\mathcal{G}$. This means that there is $x \in \bigcup\mathcal{F}$ such that $x \notin \bigcup\mathcal{G}$. Since $x \in \bigcup\mathcal{F}$, there is $A \in \mathcal{F}$ such that $x \in A$. Similarly, since $x \notin \bigcup\mathcal{G}$, we know that for all $B \in \mathcal{G}$, $x \notin B$. In particular, this means that $A \nsubseteq B$ for all $B \in \mathcal{G}$. Thus, we have found a set $A \in \mathcal{F}$ such that for all $B \in \mathcal{G}$, $A \nsubseteq B$. This completes the proof. $\square$

3.4.23. Suppose B is a set, $\{A_i \mid i \in I\}$ is an indexed family of sets, and $I \neq \varnothing$.

(1) What proof strategies are used in the following proof of the equation $B \cap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(B \cap A_i)$?

Proof. Let x be arbitrary. Suppose $x \in B \cap \left(\bigcup_{i\in I} A_i\right)$. Then $x \in B$ and $x \in \bigcup_{i\in I} A_i$, so we can choose some $i_0 \in I$ such that $x \in A_{i_0}$. Since $x \in B$ and $x \in A_{i_0}$, $x \in B \cap A_{i_0}$. Therefore, $x \in \bigcup_{i\in I}(B \cap A_i)$.

Now suppose $x \in \bigcup_{i\in I}(B \cap A_i)$. Then we can choose some $i_0 \in I$ such that $x \in B \cap A_{i_0}$. Therefore $x \in B$ and $x \in A_{i_0}$. Since $x \in A_{i_0}$, $x \in \bigcup_{i\in I} A_i$. Since $x \in B$ and $x \in \bigcup_{i\in I} A_i$, $x \in B \cap \left(\bigcup_{i\in I} A_i\right)$.

Since x was arbitrary, we have shown that for all x, $x \in B \cap \left(\bigcup_{i\in I} A_i\right)$ if and only if $x \in \bigcup_{i\in I}(B \cap A_i)$, so $B \cap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(B \cap A_i)$. □

(2) Prove that $B \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I}(B \setminus A_i)$.

Note: This exercise in my copy of the book has a typo, where it asks for us to prove that $B \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(B \setminus A_i)$. This statement is incorrect. To see why, consider the example of $I = \{1, 2\}$, $A_i = \{i, i+1\}$, and $B = \{1, 2, 3\}$. In this case, since $\bigcup_{i\in I} A_i = \{1, 2, 3\}$, $B \setminus \left(\bigcup_{i\in I} A_i\right) = \varnothing$, while $B \setminus A_1 = \{3\}$ and $B \setminus A_2 = \{1\}$ means that $\bigcup_{i\in I}(B \setminus A_i) = \{1, 3\}$.

(3) Can you discover and prove a similar theorem about $B \setminus \left(\bigcap_{i\in I} A_i\right)$? (*Hint: Try to guess the theorem, and then try to prove it. If you can't finish the proof, it might be because your guess was wrong. Change your guess and try again.*)

Note: I think this exercise in my copy of the book has a typo, since it asks for us to discover and prove a theorem about $B \setminus \left(\bigcup_{i\in I} A_i\right)$, which we already did in Part (2).

Proof.

(1) The proof is broken up into two parts: one part to check that $x \in B \cap \left(\bigcup_{i\in I} A_i\right) \to x \in \bigcup_{i\in I}(B \cap A_i)$ and one part to check that $x \in \bigcup_{i\in I}(B \cap A_i)B \to x \in \cap \left(\bigcup_{i\in I} A_i\right)$. Both parts start by taking an arbitrary x and then unraveling the definitions as to what it means to be an element of the starting set for each part. For the first part of the proof, this involves noting that $x \in B$ and using existential instantiation to choose some $i_0 \in I$ such that $x \in A_{i_0}$. For the second part of the proof, existential instantiation is used to choose some $i_0 \in I$ such that $x \in B \cap A_{i_0}$. In both parts of the proof, we then check that this i_0 satisfies the desired property to conclude that $x \in \bigcup_{i\in I}(B \cap A_i)$ in the first part and that $x \in B \cap \left(\bigcup_{i\in I} A_i\right)$ in the second part.

(2) Suppose $x \in B \setminus \left(\bigcup_{i\in I} A_i\right)$. Then $x \in B$ and $x \notin \bigcup_{i\in I} A_i$. Since $x \notin \bigcup_{i\in I} A_i$, $x \notin A_i$ for all $i \in I$. In other words, $x \in B \setminus A_i$ for all $i \in I$. This means that $x \in \bigcap_{i\in I}(B \setminus A_i)$, so $B \setminus \left(\bigcup_{i\in I} A_i\right) \subseteq \bigcap_{i\in I}(B \setminus A_i)$.

Now suppose $x \in \bigcap_{i\in I}(B \setminus A_i)$. This means that $x \in B \setminus A_i$ for all $i \in I$, or in other words, $x \in B$ and $x \notin A_i$ for all $i \in I$. Since $x \notin A_i$ for all $i \in I$, $x \notin \bigcup_{i\in I} A_i$. This means that $x \in B \setminus \left(\bigcup_{i\in I} A_i\right)$. In other words, $\bigcap_{i\in I}(B \setminus A_i) \subseteq B \setminus \left(\bigcup_{i\in I} A_i\right)$. This completes the proof that $B \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I}(B \setminus A_i)$.

(3) We will prove that the following theorem.

Theorem 21. $B \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I}(B \setminus A_i)$.

Suppose $x \in B \setminus \left(\bigcap_{i\in I} A_i\right)$. Then $x \in B$ and $x \notin \bigcap_{i\in I} A_i$. Since $x \notin \bigcap_{i\in I} A_i$, there exists $i \in I$ such that $x \notin A_i$. In other words, $x \in B \setminus A_i$ for that value of i. This means that $x \in \bigcup_{i\in I}(B \setminus A_i)$, so $B \setminus \left(\bigcap_{i\in I} A_i\right) \subseteq \bigcup_{i\in I}(B \setminus A_i)$.

Now suppose $x \in \bigcup_{i\in I}(B \setminus A_i)$. This means that there exists $i \in I$ such that $x \in B \setminus A_i$, or in other words $x \in B$ and $x \notin A_i$ for that value of i. Since there exists $i \in I$ such that $x \notin A_i$, $x \notin \bigcap_{i\in I} A_i$. This means that $x \in B \setminus \left(\bigcap_{i\in I} A_i\right)$. In other words, $\bigcup_{i\in I}(B \setminus A_i) \subseteq B \setminus \left(\bigcap_{i\in I} A_i\right)$. This completes the proof that $B \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I}(B \setminus A_i)$.

□

3.4.24. Suppose $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ are indexed families of sets and $I \neq \varnothing$.

(1) Prove that $\bigcup_{i\in I}(A_i \setminus B_i) \subseteq \left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcap_{i\in I} B_i\right)$.

Note: This exercise in my copy of the book has a typo, where it asks for us to prove that $\bigcup_{i\in I}(A_i \setminus B_i) \subseteq \left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcup_{i\in I} B_i\right)$. This statement is incorrect. To see why, consider the example of $I = \{1, 2\}$, $A_i = \{1, 2, 3\}$, and $B_i = \{i, i+1\}$. In this case, since $\bigcup_{i\in I} B_i = \{1, 2, 3\}$, $\left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcup_{i\in I} B_i\right) = \varnothing$, while $A_1 \setminus B_1 = \{3\}$ and $A_2 \setminus B_2 = \{1\}$ means that $\bigcup_{i\in I}(A_i \setminus B_i) = \{1, 3\}$.

(2) Find an example for which $\bigcup_{i\in I}(A_i \setminus B_i) \neq \left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcap_{i\in I} B_i\right)$.

Note: Since Part (1) had the typo of considering $\left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcup_{i\in I} B_i\right)$ instead of $\left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcap_{i\in I} B_i\right)$, I assume that Part (2) also had the same typo and have adjusted accordingly.

Proof.

(1) Suppose $x \in \bigcup_{i\in I}(A_i \setminus B_i)$. This means that there exists $i \in I$ such that $x \in A_i \setminus B_i$, or in other words there exists $i \in I$ such that $x \in A_i$ and $x \notin B_i$. Since $x \in A_i$, $x \in \bigcup_{i\in I} A_i$. Since $x \notin B_i$, $x \notin \bigcap_{i\in I} B_i$. Hence, $x \in \left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcap_{i\in I} B_i\right)$, so we conclude that $\bigcup_{i\in I}(A_i \setminus B_i) \subseteq \left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcap_{i\in I} B_i\right)$.

(2) Consider $I = \{1, 2\}$, $A_i = \{i, i+1\}$, and $B_i = \{i\}$. In this case, since $A_i \setminus B_i = \{i+1\}$, we have $\bigcup_{i\in I}(A_i \setminus B_i) = \{2, 3\}$. On the other hand, since $\bigcup_{i\in I} A_i = \{1, 2, 3\}$ and $\bigcap_{i\in I} B_i = \varnothing$, $\left(\bigcup_{i\in I} A_i\right) \setminus \left(\bigcap_{i\in I} B_i\right) = \{1, 2, 3\}$.

□

3.4.25. Suppose $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ are indexed families of sets.

(1) Prove that $\bigcup_{i\in I}(A_i \cap B_i) \subseteq \left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{i\in I} B_i\right)$.

(2) Find an example for which $\bigcup_{i\in I}(A_i \cap B_i) \neq \left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{i\in I} B_i\right)$.

Proof.

(1) Suppose $x \in \bigcup_{i\in I}(A_i \cap B_i)$. This means there is $i \in I$ such that $x \in A_i \cap B_i$. In other words, there is $i \in I$ such that $x \in A_i$ and $x \in B_i$. Since $x \in A_i$, it follows that $x \in \bigcup_{i\in I} A_i$. Similarly, $x \in B_i$ means that $x \in \bigcup_{i\in I} B_i$. Thus, $x \in \left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{i\in I} B_i\right)$. We conclude that $\bigcup_{i\in I}(A_i \cap B_i) \subseteq \left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{i\in I} B_i\right)$.

(2) Consider $A_1 = B_2 = \{1\}$ and $A_2 = B_1 = \{2\}$. In this case, $A_1 \cap B_1 = A_2 \cap B_2 = \varnothing$, so $\bigcup_{i\in I}(A_i \cap B_i) = \varnothing$. On the other hand, $\bigcup_{i\in I} A_i = \bigcup_{i\in I} B_i = \{1, 2\}$, so $\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{i\in I} B_i\right) = \{1, 2\}$.

□

3.4.26. Prove that for all integers a and b there is an integer c such that a divides c and b divides c.

Proof. Let a and b be arbitrary integers. Consider the integer $c = ab$. Since b is an integer and $c = ab$, we know that a divides c. Similarly, since a is an integer and $c = ab$, we know that b divides c as well. Thus, we have found an integer c such that a divides c and b divides c. Since a and b were arbitrary, this holds for all integers a and b. □

3.4.27.

(1) Prove that for every integer n, 15 divides n if and only if 3 divides n and 5 divides n.
(2) Prove that it is *not* true that for every integer n, 60 divides n if and only if 6 divides n and 10 divides n.

Proof.

(1) Let n be an arbitrary integer. First, suppose 15 divides n. In other words, there is an integer k such that $n = 15k$. Since $15 = 3 \times 5$, we have $n = 3(5k) = 5(3k)$. As $l = 5k$ and $m = 3k$ are both integers, we can write $n = 3l = 5m$, which means 3 and 5 both divide n.

Now suppose 3 and 5 each divide n. In other words, there is an integer k such that $n = 3k$ and there is an integer l such that $n = 5l$. To show that 15 divides n, we need to find an integer m such that $n = 15m$. Consider $m = 2l - k$. Plugging this value of m into the expression $15m$, we have

$$15m = 15(2l - k) = 30l - 15k = 6(5l) - 5(3k) = 6n - 5n = n.$$

Thus, since $n = 15m$, we conclude that 15 divides n. Since n was arbitrary, this completes the proof that for every integer n, 15 divides n if and only if 3 divides n and 5 divides n.

(2) Consider the case where $n = 90$. Since $90 = 15 \times 6 = 9 \times 10$, both 6 and 10 divide n, but 60 does not divide n. The key point that allowed us to conclude that n being divisible by 3 and 5 implies that n is also divisible by 15 in Part (1) is the fact that 3 and 5 are relatively prime.

□

3.5.1. Suppose A, B, and C are sets. Prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof. Suppose $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$. Since $x \in B \cup C$, there are two cases two consider: $x \in B$ and $x \in C$. If $x \in C$, then we are done, since this automatically means $x \in (A \cap B) \cup C$. If $x \in B$, then $x \in A \cap B$, and once again this means $x \in (A \cap B) \cup C$. Thus, since in either case $x \in (A \cap B) \cup C$, we conclude that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$. □

3.5.2. Suppose A, B, and C are sets. Prove that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$.

Proof. Suppose $x \in (A \cup B) \setminus C$. Then $x \in A \cup B$ and $x \notin C$. Since $x \in A \cup B$, there are two cases to consider: $x \in A$ and $x \in B$. If $x \in A$, then we are done, since this automatically means $x \in A \cup (B \setminus C)$. If $x \in B$, then $x \in B \setminus C$, and once again this means $x \in A \cup (B \setminus C)$. Thus, since in either case $x \in A \cup (B \setminus C)$, we conclude that $(A \cup B) \setminus C \subseteq A \cup (B \setminus C)$. □

3.5.3. Suppose A and B are sets. Prove that $A \setminus (A \setminus B) = A \cap B$.

Proof. We first check that $A \cap B \subseteq A \setminus (A \setminus B)$. Suppose $x \in A \cap B$. This means $x \in A$ and $x \in B$. We also note that this means $x \notin A \setminus B$. Hence, as $x \in A$ and $x \notin A \setminus B$, we conclude that $x \in A \setminus (A \setminus B)$. Thus, $A \cap B \subseteq A \setminus (A \setminus B)$.

Now we check that $A \setminus (A \setminus B) \subseteq A \cap B$. Suppose $x \in A \setminus (A \setminus B)$. This means $x \in A$ and $x \notin A \setminus B$. Since $x \in A \setminus B$ means that $x \in A$ and $x \notin B$, $x \notin A \setminus B$ means that $x \notin A$ or $x \in B$. We already know that $x \in A$, so we conclude that $x \in B$. In other words, since $x \in A$ and $x \in B$, $x \in A \cap B$. Thus, we conclude that $A \setminus (A \setminus B) \subseteq A \cap B$. This completes the proof that $A \setminus (A \setminus B) = A \cap B$. □

3.5.4. Suppose A, B, and C are sets. Prove that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.

Proof. We first check that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$. Suppose $x \in A \setminus (B \setminus C)$. This means $x \in A$ and $x \notin B \setminus C$. Since $x \notin B \setminus C$, we have two cases to consider: $x \notin B$ and $x \in C$. If $x \notin B$, then that $x \in A \setminus B$. On the other hand, if $x \in C$, then $x \in A \cap C$. In either case $x \in (A \setminus B) \cup (A \cap C)$, so we conclude that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup (A \cap C)$.

Now we check that $(A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$. Suppose $x \in (A \setminus B) \cup (A \cap C)$. We have two cases to consider: $x \in A \setminus B$ and $x \in A \cap C$. If $x \in A \setminus B$, then $x \in A$ and $x \notin B$. Since $x \notin B$, then $x \notin B \setminus C$ as well, so $x \in A \setminus (B \setminus C)$. In the other case, if $x \in A \cap C$, then $x \in A$ and $x \in C$. Since $x \in C$, then $x \notin B \setminus C$ as well, so once again $x \in A \setminus (B \setminus C)$. Since in either case $x \in A \setminus (B \setminus C)$, we conclude that $(A \setminus B) \cup (A \cap C) \subseteq A \setminus (B \setminus C)$. This completes the proof that $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$. □

3.5.5. Suppose $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$. Prove that $A \subseteq B$.

Proof. Suppose $x \in A$. Since $x \in A$, $x \in A \cup C$ as well. Then, since $x \in A \cup C$ and $A \cup C \subseteq B \cup C$, $x \in B \cup C$. This gives us two cases to consider: $x \in B$ and $x \in C$. If $x \in B$, then we have nothing left to check, so now suppose $x \in C$. Since $x \in A$ and $x \in C$, $x \in A \cap C$, and since $A \cap C \subseteq B \cap C$, $x \in B \cap C$. In particular, $x \in B$. Since in either case $x \in B$, we conclude that $A \subseteq B$. □

3.5.6. Recall from Section 1.4 that the symmetric difference of two sets A and B is the set $A \triangle B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$. Prove that if $A \triangle B \subseteq A$ then $B \subseteq A$.

Proof. Suppose $A \triangle B \subseteq A$. Now suppose $x \in B$ is arbitrary. There are two cases to consider: $x \in A$ and $x \notin A$. If $x \in A$, then there is nothing left to check and we are done, so we focus on the case $x \notin A$. If $x \notin A$, then since $x \in B$, $x \in B \setminus A$. This means that $x \in A \triangle B = (A \setminus B) \cup (B \setminus A)$. Since $A \triangle B \subseteq A$ by assumption, this means $x \in A$. But this contradicts the prior assumption that $x \notin A$. Thus, we conclude that we can only have $x \in A$. Since x was arbitrary, we conclude that $B \subseteq A$. This completes the proof that if $A \triangle B \subseteq A$ then $B \subseteq A$. □

3.5.7. Suppose A, B, and C are arbitrary sets. Prove that $A \cup C \subseteq B \cup C$ if and only if $A \setminus C \subseteq B \setminus C$.

Proof. First, suppose $A \cup C \subseteq B \cup C$. Suppose $x \in A \setminus C$ is arbitrary. Since $x \in A \setminus C$, we know $x \in A$ and $x \notin C$. To show that $x \in B \setminus C$, all we need to check is that $x \in B$. Since $x \in A$, $x \in A \cup C \subseteq B \cup C$. Then, since $x \in B \cup C$ and $x \notin C$, it follows that $x \in B$. Thus, we conclude that $x \in B \setminus C$. Since x was arbitrary, this completes the proof that $A \setminus C \subseteq B \setminus C$.

Now suppose $A \setminus C \subseteq B \setminus C$. Suppose $x \in A \cup C$ is arbitrary. If $x \in C$, then automatically $x \in B \cup C$ and we are done. If $x \notin C$, then since $x \in A \cup C$ it follows that $x \in A$. Since $x \in A$ and $x \notin C$, $x \in A \setminus C \subseteq B \setminus C$, so $x \in B \setminus C$. In particular, $x \in B$, so $x \in B \cup C$. Thus, since x was arbitrary, this completes the proof that $A \cup C \subseteq B \cup C$. □

3.5.8. Prove that for any sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$.

Proof. Let A and B be arbitrary sets. Suppose $X \in \mathscr{P}(A) \cup \mathscr{P}(B)$ is arbitrary. If $X \in \mathscr{P}(A)$, this means $X \subseteq A$, and since $A \subseteq A \cup B$, it follows that $X \subseteq A \cup B$. In other words, $X \in \mathscr{P}(A \cup B)$. Similarly, if $X \in \mathscr{P}(B)$, $X \subseteq B$ and it follows that $X \in \mathscr{P}(A \cup B)$. Since $X \in \mathscr{P}(A) \cup \mathscr{P}(B)$ was arbitrary and in either case $X \in \mathscr{P}(A \cup B)$, we conclude that $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$. Finally, since A and B were arbitrary sets, this holds for any sets A and B. This completes the proof. □

3.5.9. Prove that for any sets A and B, if $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$.

Proof. Let A and B be arbitrary sets. We will prove the equivalent contrapositive statement: if $A \nsubseteq B$ and $B \nsubseteq A$ then $\mathscr{P}(A) \cup \mathscr{P}(B) \neq \mathscr{P}(A \cup B)$. Since we showed that for any sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$ in Exercise 3.5.8 above, we only need to show that $\mathscr{P}(A \cup B) \nsubseteq \mathscr{P}(A) \cup \mathscr{P}(B)$. In other words, we need to find $X \in \mathscr{P}(A \cup B)$ such that $X \notin \mathscr{P}(A) \cup \mathscr{P}(B)$.

Suppose $A \nsubseteq B$ and $B \nsubseteq A$. Since $A \nsubseteq B$, there is $a \in A \setminus B$. Similarly, since $B \nsubseteq A$, there is $b \in B \setminus A$. Consider $X = \{a, b\}$, which is an element of $\mathscr{P}(A \cup B)$. Since $b \notin A$, $X \nsubseteq A$, and since $a \notin B$, $X \nsubseteq B$ as well. This means $X \notin \mathscr{P}(A)$ and $X \notin \mathscr{P}(B)$. In other words, $X \notin \mathscr{P}(A) \cup \mathscr{P}(B)$. Thus, we have found $X \in \mathscr{P}(A \cup B)$ such that $X \notin \mathscr{P}(A) \cup \mathscr{P}(B)$. Since A and B were arbitrary sets, this completes the proof that for any sets A and B, if $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$. □

3.5.10. Suppose x and y are real numbers and $x \neq 0$. Prove that $y + 1/x = 1 + y/x$ if and only if $x = 1$ or $y = 1$. *Note: My copy of the book has a typo where it says $x = 0$, which is incorrect.*

Proof. We start with the equation $y + 1/x = 1 + y/x$. Multiplying both sides by x, we have $xy + 1 = x + y$, which we can rearrange to get the equation $xy - x - y + 1 = 0$. Factoring the left hand side of the equation gives us $(x - 1)(y - 1) = 0$. In other words, $y + 1/x = 1 + y/x$ if and only if $(x - 1)(y - 1) = 0$. Since the product ab of two real numbers a and b is zero if and only if $a = 0$ or $b = 0$, we conclude that $x - 1 = 0$ or $y - 1 = 0$. In other words, $(x - 1)(y - 1) = 0$ if and only if $x = 1$ or $y = 1$. Thus, this chain of equivalences completes the proof that $y + 1/x = 1 + y/x$ if and only if $x = 1$ or $y = 1$. □

3.5.11. Prove that for every real number x, if $|x - 3| > 3$ then $x^2 > 6x$.

Hint: According to the definition of $|x - 3|$, if $x - 3 \geq 0$ then $|x - 3| = x - 3$, and if $x - 3 < 0$ then $|x - 3| = 3 - x$. The easiest way to use this fact is to break your proof into cases. Assume that $x - 3 \geq 0$ in the first case, and $x - 3 < 0$ in the second case.

Proof. Let $x \in \mathbb{R}$ be arbitrary. Suppose $|x - 3| > 3$. Following the hint, we break the proof into two cases: $x - 3 \geq 0$ and $x - 3 < 0$. First we consider the case when $x - 3 \geq 0$. In this case, $|x - 3| = x - 3$, so the inequality $|x - 3| > 3$ becomes $x - 3 > 3$. Rearranging this inequality gives us $x > 6$, so multiplying both sides by x results in the inequality $x^2 > 6x$, as desired. Next we consider the case when $x - 3 < 0$. In this case, $|x - 3| = 3 - x$, so the inequality $|x - 3| > 3$ becomes $3 - x > 3$. Rearranging this inequality gives us $x < 0$. While $x^2 > 0$ for all real numbers, since $x < 0$ when know that $6x < 0$ as well. Hence, $x^2 > 6x$ once again in this case. Now that we have checked both cases, we conclude that $x^2 > 6x$. Since $x \in \mathbb{R}$ was arbitrary, this holds for all real numbers. $\square$

3.5.12. Prove that for every real number x, $|2x - 6| > x$ if and only if $|x - 4| > 2$.
Hint: Read the hint for Exercise 3.5.11.

Proof. Let $x \in \mathbb{R}$ be arbitrary.

Suppose $|2x - 6| > x$. There are two cases to consider: $2x - 6 \geq 0$ and $2x - 6 < 0$. First we consider the case when $2x - 6 \geq 0$, so $|2x - 6| = 2x - 6$. This means the inequality $|2x - 6| > x$ can be rewritten as $2x - 6 > x$. Isolating x on one side of the inequality, we have $x > 6$. Since $x > 6$, $x - 4 > 2$, and in particular, $x - 4$ is positive. Thus, $|x - 4| = x - 4 > 2$, as desired.

Next we consider the case when $2x - 6 < 0$, so $|2x - 6| = 6 - 2x$. This means the inequality $|2x - 6| > x$ can be rewritten as $6 - 2x > x$. Isolating x on one side of the inequality, we have $x < 2$. Since $x < 2$, $x - 4 < -2$, or in other words $x - 4$ is negative and $4 - x > 2$. Thus, $|x - 4| = 4 - x > 2$, as desired. As we can see, in both cases we obtained the desired inequality $|x - 4| > 2$.

Now suppose $|x - 4| > 2$. Once again there are two cases to consider: $x - 4 \geq 0$ and $x - 4 < 0$. First we consider the case when $x - 4 \geq 0$, so $|x - 4| = x - 4$. This means the inequality $|x - 4| > 2$ can be rewritten as $x - 4 > 2$. Isolating x on one side of the inequality, we have $x > 6$. We can then add x to both sides of the inequality to get $2x > 6 + x$, which we can rewrite as $2x - 6 > x$. Since $x > 6$, it follows that $2x - 6 > 0$, so then $|2x - 6| = 2x - 6 > x$, as desired.

Next we consider the case when $x - 4 < 0$, so $|x - 4| = 4 - x$. This means the inequality $|x - 4| > 2$ can be rewritten as $4 - x > 2$. Isolating x on one side of the inequality, we have $x < 2$. We can then multiply both sides by 3 to get $3x < 6$. Subtracting $2x$ from both sides then gives us $6 - 2x > x$. Since $x < 2$, it follows that $2x - 6 < -2$, and in particular $2x - 6 < 0$, so then $|2x - 6| = 6 - 2x > x$, as desired. As we can see, in both cases we obtained the desired inequality $|2x - 6| > x$.

Since $x \in \mathbb{R}$ was arbitrary, we conclude that for every real number x, $|2x - 6| > x$ if and only if $|x - 4| > 2$. $\square$

3.5.13.

(1) Prove that for all real numbers a and b, $|a| \leq b$ if and only if $-b \leq a \leq b$.
(2) Prove that for any real number x, $-|x| \leq x \leq |x|$. *Hint: Use Part (1).*
(3) Prove that for all real numbers x and y, $|x + y| \leq |x| + |y|$. *This is called the* triangle inequality. *One way to prove this is to combine Parts (1) and (2), but you can also do it by considering a number of cases.*
(4) Prove that for all real numbers x and y, $|x + y| \geq |x| - |y|$. *Hint: Start with the equation $|x| = |(x + y) + (-y)|$ and then apply the triangle inequality to the right-hand side.*

Proof.

(1) Let a and b be arbitrary real numbers. Suppose $|a| \leq b$. First note that this means $b \geq 0$, since $|a| \geq 0$ by the definition of absolute value. There are two cases to consider: $a \geq 0$ and $a < 0$. In the case when $a \geq 0$, $|a| = a$, so $a = |a| \leq b$. In addition, since $b \geq 0$, $-b \leq 0$, and it follows that $a \geq -b$ as well. In other words, $-b \leq a \leq b$, as desired. Now we consider the case when $a < 0$. In this case, it automatically follows that $a \leq b$, since $b \geq 0$. Moreover, $a < 0$ means $|a| = -a$, so $-a = |a| \leq b$, which means $a \geq -b$. Thus, we once again conclude that $-b \leq a \leq b$. Hence, we conclude that if $|a| \leq b$, then $-b \leq a \leq b$.

    Now suppose $-b \leq a \leq b$. Once again this means $b \geq 0$, since $-b \leq b$. Again, we consider the cases $a \geq 0$ and $a < 0$. In the case when $a \geq 0$, $|a| = a$, so $|a| = a \leq b$ and we are done. In the case when $a < 0$, $|a| = -a$, so since $a \geq -b$ it follows that $|a| = -a \leq b$, and once again we are done. Thus, we conclude that $|a| \leq b$.

    Since a and b were arbitrary, this completes the proof that for all real numbers a and b, $|a| \leq b$ if and only if $-b \leq a \leq b$.

(2) Let x be an arbitrary real number. Following the hint to use Part (1), to show that $-|x| \leq x \leq |x|$ it suffices to check that $|a| \leq b$, where $a = x$ and $b = |x|$. In other words, it suffices to check that $|x| \leq |x|$, which is automatically true as any real number is equal to itself. Thus, by Part (1), since $|x| \leq |x|$, we conclude that $-|x| \leq x \leq |x|$. Since x was an arbitrary real number, this compound inequality holds for all real numbers x.

(3) Let x and y be arbitrary real numbers. To show that $|x+y| \leq |x| + |y|$, we will show that $-(|x|+|y|) \leq x+y \leq |x|+|y|$ and apply Part (1). By Part (2), we know that $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these two inequalities together then gives us the desired inequality: $-(|x|+|y|) \leq x+y \leq |x|+|y|$. This allows us to apply Part (1) with $a = x+y$ and $b = |x|+|y|$ to conclude that $|x+y| \leq |x|+|y|$. Since x and y were arbitrary real numbers, this inequality holds for all real numbers. This completes the proof.

(4) Let x and y be arbitrary real numbers. Following the hint, we start with the equation $|x| = |(x+y) + (-y)|$. Applying the triangle inequality to the right-hand side gives us $|x| = |(x+y)+(-y)| \leq |x+y| + |-y|$. In other words, $|x| \leq |x+y| + |y|$. Rearranging gives us the desired inequality $|x+y| \geq |x| - |y|$. Since x and y were arbitrary, this inequality holds for all real numbers.

$\square$

3.5.14. Prove for every integer x, $x^2 + x$ is even.

Proof. Let x be an arbitrary integer. There are two cases to consider: x is even and x is odd. First we consider the case when x is even, so $x = 2n$ for some integer n. In this case, $x^2+x = 4n^2+2n = 2(2n^2+n)$. Since $k = 2n^2+n$ is an integer, we conclude that $x^2+x = 2k$, or in other words x^2+x is even. Next we consider the case when x is odd, so $x = 2n+1$ for some integer n. In this case, $x^2+x = (4n^2+4n+1)+(2n+1) = 4n^2+6n+2 = 2(2n^2+3n+1)$. Since $k = 2n^2+3n+1$ is an integer, we conclude that $x^2+x = 2k$, or in other words x^2+x is once again even. Since x was an arbitrary integer and we saw that x^2+x was even in both cases, we conclude that x^2+x is even for all integers x. $\square$

3.5.15. Prove that for every $x \in \mathbb{Z}$, the remainder when x^4 is divided by 8 is either 0 or 1.

Proof. Let x be an arbitrary integer. There are two cases to consider: x is even and x is odd. First we consider the case when x is even, so $x = 2n$ for some integer n. In this case, $x^4 = (2n)^4 = 8(2n^4)$, and since $2n^4$ is an integer, the remainder when $x^4 = 8(2n^4)$ is divided by 8 is 0. Next we consider the case when x is odd, so $x = 2n + 1$ for some integer n. Computing $x^4 = (2n + 1)^4$, we have

$$\begin{aligned} x^4 &= (2n+1)^4 \\ &= (2n)^4 + 4(2n^3) + 6(2n)^2 + 4(2n) + 1 \\ &= 8(2n^4) + 8(4n^3) + 8(3n^2) + 8n + 1 \\ &= 8(2n^4 + 4n^3 + 3n^2 + n) + 1. \end{aligned}$$

Since $2n^4 + 4n^3 + 3n^2 + n$ is an integer, the remainder then $x^4 = 8(2n^4 + 4n^3 + 3n^2 + n) + 1$ is divided by 8 is 1. Since x was an arbitrary integer, we conclude that for every integer x, the remainder when x^4 is divided by 8 is either 0 (when x is even) or 1 (when x is odd). □

3.5.16.

Proof. □

3.5.17.

Proof. □

3.5.18.

Proof. □

3.5.19.

Proof. □

3.5.20.

Proof. □

3.5.21.

Proof. □

3.5.22.

Proof. □

3.5.23.

Proof. □

3.5.24.

Proof. □

3.5.25.

Proof. □

3.5.26.

Proof. □

3.5.27.

Proof. □

3.5.28.

Proof. □

3.5.29.

Proof. □

3.5.30.

Proof. □

3.5.31.

Proof. □

3.5.32.

Proof. □

3.5.33.

Proof. □

3.6.1.

Proof. □

3.6.2.

Proof. □

3.6.3.

Proof. □

3.6.4.

Proof. □

3.6.5.

Proof. □

3.6.6.

Proof. □

3.6.7.

Proof. □

3.6.8.

Proof. ☐

3.6.9.

Proof. ☐

3.6.10.

Proof. ☐

3.6.11.

Proof. ☐

3.6.12.

Proof. ☐

3.6.13.

Proof. ☐

3.7.1.

Proof. ☐

3.7.2.

Proof. ☐

3.7.3.

Proof. ☐

3.7.4.

Proof. ☐

3.7.5.

Proof. ☐

3.7.6.

Proof. ☐

3.7.7.

Proof. ☐

3.7.8.

Proof. ☐

3.7.9.

Proof. ☐

3.7.10.

Proof. □