***HOW TO PROVE IT*: CHAPTER 3**

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These are the exercises for Chapter 3 from the third edition of *How to Prove It* by Daniel J. Velleman. They are numbered (Chapter).(Section).(Exercise).

3.1.1. Consider the following theorem. (This theorem was proven in the introduction.)

Theorem 1. *Suppose n is an integer larger than 1 and n is not prime. Then $2^n - 1$ is not prime.*

(1) Identify the hypotheses and the conclusion of the theorem. Are the hypotheses true when $n = 6$? What does the theorem tell you in this instance? Is it right?
(2) What can you conclude from the theorem in the case $n = 15$? Check directly that this conclusion is correct.
(3) What can you conclude from the theorem in the case $n = 11$?

Proof.

(1) This theorem has three hypotheses: n is an integer, $n > 1$, and n is not prime. The conclusion of the theorem is that $2^n - 1$ is not prime. In the case when $n = 6 = 2 \times 3$, all of the hypotheses are satisfied, so the theorem tells us that $2^6 - 1$ is not prime. We can directly check that $2^6 - 1 = 63 = 3^2 \times 7$ is not prime.
(2) In the case when $n = 15 = 3 \times 5$, all of the hypotheses are satisfied. This means that the theorem tells us that $2^{15} - 1$ is not prime. As discussed in Part (1) of Exercise I.1, $2^{15} - 1 = 32767 = 31 \times 1057$.
(3) In the case when $n = 11$, not all of the hypotheses are satisfied. In particular, 11 is prime. Because not all of the hypotheses of the theorem are satisfied, we cannot draw any conclusions from it. In particular, the theorem does not tell us anything about the primality of $2^{11} - 1$.

$\square$

3.1.2. Consider the following theorem. (The theorem is correct, but we will not ask you to prove it here.)

Theorem 2. *Suppose that $b^2 > 4ac$. Then the quadratic equation $ax^2 + bx + c = 0$ has exactly two real solutions.*

(1) Identify the hypotheses and conclusion of the theorem.
(2) To give an instance of the theorem, you must specify values for a, b, and c, but not x. Why?
(3) What can you conclude from the theorem in the case $a = 2$, $b = -5$, $c = 3$? Check directly that this conclusion is correct.
(4) What can you conclude from the theorem in the case $a = 2$, $b = 4$, $c = 3$?

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Proof.

(1) This theorem has one implicit hypothesis, that a, b, c are all real numbers, and one explicit hypothesis, that $b^2 > 4ac$. The conclusion of the theorem is that the quadratic equation $ax^2 + bx + c = 0$ has exactly two real solutions.

(2) To give an instance of the theorem, we only need to specify values for a, b, and c since those are the only variables listed in the hypothesis. The values of x associated with a specific instance of the theorem are determined by the given values of a, b, c, since the x values are the two real solutions to the quadratic equation $ax^2 + bx + c = 0$. In other words, x is a dummy variable for the set $S = \{x \in \mathbb{R} \mid ax^2 + bx + c = 0\}$ for given values of a, b, c. When phrased this way, the conclusion of theorem states that S contains exactly two distinct elements.

(3) In the case $a = 2, b = -5, c = 3$, we have $b^2 = (-5)^2 = 25$ and $4ac = 4(2)(3) = 24$. Thus, the hypotheses $b^2 > 4ac$ is satisfied, and the theorem applies. We can then conclude that the quadratic equation $2x^2 - 5x + 3 = 0$ has two real solutions. Factoring the quadratic as $(2x-3)(x-1) = 0$, we can directly check that the two real solutions are $x = 3/2$ and $x = 1$.

(4) In the case $a = 2, b = 4, c = 3$, we have $b^2 = 4^2 = 16$ and $4ac = 4(2)(3) = 24$. In other words, $b^2 \not> 4ac$. Since not all of the hypotheses of the theorem are satisfied, we cannot draw any conclusions from it. In particular, the theorem does not tell us anything about the solution set of the quadratic equation $2x^2 + 4x + 3 = 0$.

$\square$

3.1.3. Consider the following incorrect theorem.

Theorem 3. *Suppose n is a natural number larger than 2, and n is not a prime number. Then $2n + 13$ is not a prime number.*

What are the hypotheses and conclusion of this theorem? Show that the theorem is incorrect by finding a counterexample.

Proof. $\square$

3.1.4. Complete the following alternative proof of the theorem in Example 3.1.2.

Theorem 4. *Suppose a and b are real numbers. If $0 < a < b$ then $a^2 < b^2$.*

Proof. Suppose $0 < a < b$. Then $b - a > 0$. [*Fill in a proof of $b^2 - a^2 > 0$ here.*] Since $b^2 - a^2 > 0$, it follows that $a^2 < b^2$. Therefore, if $0 < a < b$ then $a^2 < b^2$. $\square$

Proof. $\square$

3.1.5. Suppose a and b are real numbers. Prove that if $a < b < 0$ then $a^2 > b^2$.

Proof. $\square$

3.1.6. Suppose a and b are real numbers. Prove that if $0 < a < b$ then $1/b < 1/a$.

Proof. $\square$

3.1.7. Suppose a is a real number. Prove that if $a^3 > a$ then $a^5 > a$. (*Hint: One approach is to start by completing the following equation: $a^5 - a = (a^3 - a) \cdot \underline{?}$.*)

Proof. □

3.1.8. Suppose $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

Proof. □

3.1.9. Suppose $A \cap B \subseteq C \setminus D$. Prove that if $x \in A$, then if $x \in D$ then $x \notin B$.

Proof. □

3.1.10. Suppose a and b are real numbers. Prove that if $a < b$ then $(a + b)/2 < b$.

Proof. □

3.1.11. Suppose x is a real number and $x \neq 0$. Prove that if $(\sqrt[3]{x} + 5)/(x^2 + 6) = 1/x$ then $x \neq 8$.

Proof. □

3.1.12. Suppose a, b, c, and d are real numbers, $0 < a < b$, and $d > 0$. Prove that if $ac \geq bd$ then $c > d$.

Proof. □

3.1.13. Suppose x and y are real numbers, and that $3x + 2y \leq 5$. Prove that if $x > 1$ then $y < 1$.

Proof. □

3.1.14. Suppose x and y are real numbers. Prove that if $x^2 + y = -3$ and $2x - y = 2$ then $x = -1$.

Proof. □

3.1.15. Prove the first theorem in Example 3.1.1.

Theorem 5. *Suppose $x > 3$ and $y < 2$. Then $x^2 - 2y > 5$.*

(*Hint: You might find it useful to apply the theorem from Example 3.1.2, which stated that if a and b are real numbers such that $0 < a < b$, then $a^2 < b^2$.*)

Proof. □

3.1.16. Consider the following theorem.

Theorem 6. *Suppose x is a real number and $x \neq 4$. If $(2x - 5)/(x - 4) = 3$ then $x = 7$.*

(1) What is wrong with the following proof of the theorem?

Proof. Suppose $x = 7$. Then $(2x - 5)/(x - 4) = (2 \cdot 7 - 5)/(7 - 4) = 9/3 = 3$. Therefore if $(2x - 5)/(x - 4) = 3$ then $x = 7$. □

(2) Give a correct proof of the theorem.

Proof. □

3.1.17. Consider the following incorrect theorem.

Theorem 7. *Suppose that x and y are real numbers and $x \neq 3$. If $x^2y = 9y$ then $y = 0$.*

(1) What's wrong with the following proof of the theorem?

Proof. Suppose that $x^2y = 9y$. Then $(x^2-9)y = 0$. Since $x \neq 3$, $x^2 \neq 9$, so $x^2-9 \neq 0$. Therefore we can divide both sides of the equation $(x^2 - 9)y = 0$ by $x^2 - 9$, which leads to the conclusion that $y = 0$. Thus, if $x^2y = 9y$ then $y = 0$. □

(2) Show that the theorem is incorrect by finding a counterexample.

Proof. □

3.2.1.

Proof. □

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Proof. □

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