Terms for Classical Sequents

Proof Invariants & Strong Normalisation

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To introduce a new *invariant*
for classical propositional proofs
and to show how they can be used.

Today's Plan

Background

Preterms

Derivations

Terms

Eliminating Cuts

Strong Normalisation

Further Work

BACKGROUND

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \vee q}\,\vee R}{p \wedge q \succ p \vee q}\,\wedge L$$

$$\dfrac{\dfrac{p \wedge q}{p}\,\wedge E}{p \vee q}\,\vee I$$

$$\dfrac{\dfrac{p \succ p}{p \wedge q \succ p}\,\wedge L}{p \wedge q \succ p \vee q}\,\vee R$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \lor q}\,{\lor R}}{p \land q \succ p \lor q}\,{\land L}$$

$$\dfrac{\dfrac{p \land q}{p}\,{\land E}}{p \lor q}\,{\lor I}$$

$$\dfrac{\dfrac{p \succ p}{p \land q \succ p}\,{\land L}}{p \land q \succ p \lor q}\,{\lor R}$$

$$\dfrac{\dfrac{q \succ q}{q \succ p \lor q}\,{\lor R}}{p \land q \succ p \lor q}\,{\land L}$$

$$\dfrac{\dfrac{p \land q}{q}\,{\land E}}{p \lor q}\,{\lor I}$$

$$\dfrac{\dfrac{q \succ q}{p \land q \succ q}\,{\land L}}{p \land q \succ p \lor q}\,{\lor R}$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \vee q \qquad \dfrac{[p]^1}{q \vee p}\,\vee I \qquad \dfrac{[q]^1}{q \vee p}\,\vee I}{q \vee p}\,\vee E^1}{(q \vee p) \vee r}\,\vee I$$

$$\dfrac{p \vee q \qquad \dfrac{\dfrac{[p]^1}{q \vee p}\,\vee I}{(q \vee p) \vee r}\,\vee I \qquad \dfrac{\dfrac{[q]^1}{q \vee p}\,\vee I}{(q \vee p) \vee r}\,\vee I}{(q \vee p) \vee r}\,\vee E^1$$

Are these *different proofs*, or *different ways of presenting the same proof*?

Natural deduction is a slightly paradoxical system: it is limited to the intuitionistic case (in the classical case it has no particularly good properties) but it is only satisfactory for the $(\wedge, \Rightarrow, \forall)$ fragment of the language: we shall defer consideration of $\vee$ and $\exists$ until chapter 10. Yet disjunction and existence are the two most *typically* intuitionistic connectors!

The basic idea of natural deduction is an asymmetry: a proof is a vaguely tree-like structure (this view is more a graphical illusion than a mathematical reality, but it is a pleasant illusion) with one or more hypotheses (possibly none) but a single conclusion. The deep symmetry of the calculus is shown by the *introduction* and *elimination* rules which match each other exactly. Observe, incidentally, that with a tree-like structure, one can always decide uniquely what was the *last* rule used, which is something we could not say if there were several conclusions.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{[x : p \supset (q \supset r)] \quad [z : p]}{xz : q \supset r}\supset E
      \qquad
      \dfrac{[y : p \supset q] \quad [z : p]}{yz : q}\supset E
    }{(xz)(yz) : r}\supset E
  }{\dfrac{\lambda z\,(xz)(yz) : p \supset r}{\lambda y \lambda z\,(xz)(yz) : (p \supset q) \supset (p \supset r)}\supset I}\supset I
}{\lambda x \lambda y \lambda z\,(xz)(yz) : (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))}\supset I
$$

Contraction and *weakening* are managed by variables

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\supset I$$

Contraction and *weakening* are managed by variables

$$
\begin{prooftree}
\AxiomC{$[x : p]$}
\RightLabel{$\supset I$}
\UnaryInfC{$\lambda y\, x : q \supset p$}
\RightLabel{$\supset I$}
\UnaryInfC{$\lambda x \lambda y\, x : p \supset (q \supset p)$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{$x : p \supset (p \supset q)$}
\AxiomC{$[y : p]$}
\RightLabel{$\supset E$}
\BinaryInfC{$xy : p \supset q$}
\AxiomC{$[y : p]$}
\RightLabel{$\supset E$}
\BinaryInfC{$(xy)y : q$}
\RightLabel{$\supset I$}
\UnaryInfC{$\lambda y\, (xy)y : p \supset q$}
\end{prooftree}
$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \vee \neg p}\,\vee R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \wedge \neg p \succ}\,\wedge L$$

Classical Sequent Derivations

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$$\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{q \succ q \quad r \succ r}{q \vee r \succ q, r}\,\vee L}{p, q \vee r \succ p \wedge q, r}\,\wedge R}{p \wedge (q \vee r) \succ p \wedge q, r}\,\wedge L}{p \wedge (q \vee r) \succ (p \wedge q) \vee r}\,\vee R$$

Sequents and Terms

$$X \succ Y \qquad X \succ A, Y \qquad X, A \succ Y$$

Where do you put the *variables*,
and where do you put the *terms*?

Our Choice

$$x_1 : A_1, \ldots, x_n : A_n \succ y_1 : B_1, \ldots, y_m : B_m$$

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Each premise and conclusion is decorated with variables.

Our Choice

$$x_1 : A_1, \ldots, x_n : A_n \overset{\pi(x_1,\ldots,x_n)[y_1,\ldots,y_m]}{\succ} y_1 : B_1, \ldots, y_m : B_m$$

Each premise and conclusion is decorated with variables.

The *sequent* gets the term, showing how inputs & outputs are connected, with as much parallelism as possible.

Example 1

$$
\dfrac{\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \dfrac{\overset{y\frown y}{y : q \succ y : q} \qquad \overset{z\frown z}{z : r \succ z : r}}{\overset{Lw\frown y \quad Rw\frown z}{w : q \vee r \succ y : q, z : r}}\,\vee L}{\overset{x\frown Fv \quad Lw\frown Sv \quad Rw\frown z}{x : p, w : q \vee r \succ v : p \wedge q, z : r}}\,\wedge R}{\overset{Fu\frown Fv \quad LSu\frown Sv \quad RSu\frown z}{u : p \wedge (q \vee r) \succ v : p \wedge q, z : r}}\,\wedge L}{\overset{Fu\frown FLt \quad LSu\frown SLt \quad RSu\frown Rt}{u : p \wedge (q \vee r) \succ t : (p \wedge q) \vee r}}\,\vee R
$$

Example 2

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fy \quad x\frown Sy}{x : p \succ y : p \wedge p}}\,{\wedge}R \qquad \dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fw\frown z}{w : p \wedge p \succ z : p}}\,{\wedge}L}{\overset{x\frown F\bullet \quad x\frown S\bullet \quad F\bullet\frown z}{x : p \succ z : p}}\,\mathit{Cut}$$

PRETERMS

- For each formula A, $x_1^A, x_2^A, \ldots$ are VARIABLES of type A.

Variables and Cut Points

- For each formula A, $x^A_1, x^A_2, \ldots$ are VARIABLES of type A.
- For each formula A, $\bullet^A_1, \bullet^A_2, \ldots$ are CUT POINTS of type A.

Variables and Cut Points

- For each formula A, $x_1^A, x_2^A, \ldots$ are VARIABLES of type A.
- For each formula A, $\bullet_1^A, \bullet_2^A, \ldots$ are CUT POINTS of type A.
  - We use $x, y, z, u, v, w, \ldots; \bullet, \star, *, \sharp, \flat$ as schematic letters for variables and cut points, ommitting type superscripts where possible.

Nodes and Subnodes

- A variable x of type A and a cut point $\bullet$ of type A are both A NODES.

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- If n is an $A \land B$ node, then $\mathsf{L}n$ is an A node and $\mathsf{R}n$ is a B node.
- If n is an $A \lor B$ node, then $\mathsf{F}n$ is an A node and $\mathsf{S}n$ is a B node.
- If n is an $A \supset B$ node, then $\mathsf{A}n$ is an A node and $\mathsf{C}n$ is a B node.
- If n is a $\neg A$ node, then $\mathsf{N}n$ is an A node.

Nodes and Subnodes

- A variable x of type A and a cut point $\bullet$ of type A are both A NODES.
- If n is an $A \wedge B$ node, then Ln is an A node and Rn is a B node.
- If n is an $A \vee B$ node, then Fn is an A node and Sn is a B node.
- If n is an $A \supset B$ node, then An is an A node and Cn is a B node.
- If n is a $\neg A$ node, then Nn is an A node.
- For each complex node Ln, Rn, Fn, Sn, An, Cn and Nn, n is its IMMEDIATE subnode, and the subnodes of n are also subnodes of the original node.

Linkings, Inputs and Outputs

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- A LINKING is a pair $n \frown m$ of nodes of the same type.
- In $n \frown m$, n is in INPUT POSITION, and m is in OUTPUT POSITION.
- Positions generalise to subnodes as follows:
  - If $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{F}n$, $\mathsf{S}n$ or $\mathsf{C}n$ are in *input position*, n is also in *input position*.
  - If $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{F}n$, $\mathsf{S}n$ or $\mathsf{C}n$ are in *output position*, n is also in *output position*.
  - – L, R, F, S and C each *preserve position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *input position*, n is in *output position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *output position*, n is in *input position*.
  - – A and N *reverse position*.

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  - L, R, F, S and C each *preserve position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *input position*, n is in *output position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *output position*, n is in *input position*.
  - A and N *reverse position*.
- The INPUTS (OUTPUTS) of a linking are the *variables* in INPUT (OUTPUT) position of that linking.

Example Linkings

x of type $((p \supset q) \supset p) \supset p$

$$AAAx \frown Cx$$

$$CAx \frown Cx$$

Preterms

- DEFINITION: A PRETERM is a finite set of linkings.

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- DEFINITION: A PRETERM is a finite set of linkings.
  - The INPUTS of a preterm are the inputs of its linkings.
  - Its OUTPUTS are the outputs of its linkings.

DERIVATIONS

Annotating Derivations: Identity

$$\Sigma, x : A \succ y : A, \Delta \quad x \frown y$$

Annotating Derivations: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \wedge B \succ \Delta}}\,{\wedge}L$$

Annotating Derivations: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(\mathsf{F}z, \mathsf{S}z)}{\Sigma, z : A \land B \succ \Delta}}\ {\land}L \qquad \dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[\mathsf{F}z] \quad \pi'[\mathsf{S}z]}{\Sigma, \Sigma' \succ z : A \land B, \Delta, \Delta'}}\ {\land}R$$

$$
\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\ {\supset}I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\ {\supset}I
$$

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\supset I$$

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz, Sz)}{\Sigma, z : A \wedge B \succ \Delta}}\wedge L \quad \text{can be} \quad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Fz)}{\Sigma, z : A \wedge B \succ \Delta}}\wedge L$$

Excursus on *Weakening* and *Variables*

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\,{\supset}I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\,{\supset}I$$

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \wedge B \succ \Delta}}\,{\wedge}L \qquad \text{can be} \qquad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Fz)}{\Sigma, z : A \wedge B \succ \Delta}}\,{\wedge}L$$

In a premise $\pi(x, y)$ the indicated x and y display all of the x and y inputs to the proof term.

There might be *none*.

Annotating Derivations: Negation

$$\frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\overset{\pi[\mathsf{N}z]}{\Sigma, z : \neg A \succ \Delta}}\neg L \qquad \frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(\mathsf{N}z)}{\Sigma \succ z : \neg A, \Delta}}\neg R$$

Annotating Derivations: Disjunction

$$\dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz) \quad \pi'(Rz)}{\Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta'}} \vee L \qquad \dfrac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz, Rz]}{\Sigma \succ z : A \vee B, \Delta}} \vee R$$

Annotating Derivations: Conditional

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az] \quad \pi'(Lz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(Az)[Cz]}{\Sigma \succ z : A \supset B, \Delta}} \supset R$$

Example Annotation

$$
\dfrac{\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \dfrac{\overset{y\frown y}{y : q \succ y : q} \qquad \overset{z\frown z}{z : r \succ z : r}}{\overset{Lw\frown y \quad Rw\frown z}{w : q \lor r \succ y : q, z : r}}\,{\lor}L}{\overset{x\frown Fv \quad Lw\frown Sv \quad Rw\frown z}{x : p, w : q \lor r \succ v : p \land q, z : r}}\,{\land}R}{\overset{Fu\frown Fv \quad LSu\frown Sv \quad RSu\frown z}{u : p \land (q \lor r) \succ v : p \land q, z : r}}\,{\land}L}{\overset{Fu\frown FLt \quad LSu\frown SLt \quad RSu\frown Rt}{u : p \land (q \lor r) \succ t : (p \land q) \lor r}}\,{\lor}R
$$

Annotating Derivations: Cut

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet] \quad \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}$$

Identify Terms up to α equivalence

If π can be transformed into π' by relabelling cut points we treat them as identical (they are α equivalent).

Example Annotation, with *Cut*

$$
\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \vee p \succ x : p}}\,\vee L \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \wedge p}}\,\wedge R}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \vee p \succ z : p \wedge p}}\,\mathit{Cut}
$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{z\frown Ly}{z : p \succ y : p \vee q}}\vee R}{\overset{Fx\frown Ly}{x : p \wedge q \succ y : p \vee q}}\wedge L$$

$$\dfrac{\dfrac{p \wedge q}{p}\wedge E}{p \vee q}\vee I$$

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fx\frown z}{x : p \wedge q \succ z : p}}\vee R}{\overset{Fx\frown Ly}{x : p \wedge q \succ y : p \vee q}}\wedge L$$

When is π_1 the same proof as π_2 (revisited)?

$$\cfrac{\cfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{z\frown \mathsf{L}y}{z : p \succ y : p \lor q}}\ \lor R}{\overset{\mathsf{F}x\frown \mathsf{L}y}{x : p \land q \succ y : p \lor q}}\ \land L$$

$$\cfrac{\cfrac{p \land q}{p}\ \land E}{p \lor q}\ \lor I$$

$$\cfrac{\cfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{\mathsf{F}x\frown z}{x : p \land q \succ z : p}}\ \lor R}{\overset{\mathsf{F}x\frown \mathsf{L}y}{x : p \land q \succ y : p \lor q}}\ \land L$$

$$\cfrac{\cfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{w\frown \mathsf{R}y}{w : q \succ y : p \lor q}}\ \lor R}{\overset{\mathsf{S}x\frown \mathsf{R}y}{x : p \land q \succ y : p \lor q}}\ \land L$$

$$\cfrac{\cfrac{p \land q}{q}\ \land E}{p \lor q}\ \lor I$$

$$\cfrac{\cfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{\mathsf{S}x\frown w}{x : p \land q \succ w : q}}\ \lor R}{\overset{\mathsf{S}x\frown \mathsf{R}y}{x : p \land q \succ y : p \lor q}}\ \land L$$

When is π_1 the same proof as π_2 (revisited)?

$$\cfrac{\cfrac{p \vee q \qquad \cfrac{[p]^1}{q \vee p}\,\vee I \qquad \cfrac{[q]^1}{q \vee p}\,\vee I}{q \vee p}\,\vee E^1}{(q \vee p) \vee r}\,\vee I$$

$$\cfrac{\cfrac{\overset{x \frown x}{x : p \succ x : p}}{\overset{x \frown Rz}{x : p \succ z : q \vee p}}\,\vee R \qquad \cfrac{\overset{y \frown y}{y : q \succ y : q}}{\overset{y \frown Lz}{y : q \succ z : q \vee p}}\,\vee R}{\cfrac{\overset{Lw \frown Rz \quad Rw \frown Lz}{w : p \vee q \succ z : q \vee p}}{\overset{Lw \frown RLu \quad Rw \frown LLu}{w : p \vee q \succ u : (q \vee p) \vee r}}\,\vee R}\,\vee L$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{p \vee q \qquad \dfrac{\dfrac{[p]^1}{q \vee p}\,\vee I}{(q \vee p) \vee r}\,\vee I \qquad \dfrac{\dfrac{[q]^1}{q \vee p}\,\vee I}{(q \vee p) \vee r}\,\vee I}{(q \vee p) \vee r}\,\vee E^1$$

$$\dfrac{\dfrac{\dfrac{\overset{x \frown x}{x : p \succ x : p}}{\overset{x \frown Rz}{x : p \succ z : q \vee p}}\,\vee R}{\overset{x \frown RLu}{x : p \succ u : (q \vee p) \vee r}}\,\vee R \qquad \dfrac{\dfrac{\overset{y \frown y}{y : q \succ y : q}}{\overset{x \frown LLu}{y : q \succ z : q \vee p}}\,\vee R}{\overset{y \frown LLu}{y : q \succ u : (q \vee p) \vee r}}\,\vee R}{\overset{Lw \frown RLu \quad Rw \frown LLu}{w : p \vee q \succ u : (q \vee p) \vee r}}\,\vee L$$

DEFINITION: A preterm is SEQUENTIALISABLE iff it is the conclusion of some derivation.

TERMS

Nonsequentialisable Preterms

$$\overset{Lx\frown Fy \quad Rx\frown Sy}{x : p \lor q \succ y : p \land q}$$

This is connected, but it is not connected *enough*.

Switching Example

$$\overset{Lx\frown Fy}{x : p \lor q} \succ \overset{Rx\frown Sy}{y : p \land q}$$

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$$\overset{Lx\frown Fy}{x : p \lor q} \succ \overset{Rx\frown Sy}{y : p \land q}$$

$$\overset{Lx\frown Fy}{x : p \lor -} \succ \overset{\cancel{Rx}\frown \cancel{Sy}}{y : p \land -}$$

$$\overset{Lx\frown \cancel{Fy}}{x : p \lor -} \succ \overset{\cancel{Rx}\frown Sy}{y : - \land q}$$

$$\overset{\cancel{Lx}\frown Fy}{x : - \lor q} \succ \overset{Rx\frown \cancel{Sy}}{y : p \land -}$$

$$\overset{\cancel{Lx}\frown \cancel{Fy}}{x : - \lor q} \succ \overset{Rx\frown Sy}{y : - \land q}$$

Switchings

- The SWITCHINGS of a preterm π are found by selecting for each pair of subterms $\mathsf{L}n$ and $\mathsf{R}n$ in *input* position; $\mathsf{F}n$ and $\mathsf{S}n$ in *output position*, $\mathsf{A}n$ in *output* position and $\mathsf{C}n$ in *input position*; or the cut point $\bullet$ (in both *input* and *output position*), one item of the pair to keep, and the other to DELETE.

- A LINKING in a switching of a preterm π SURVIVES if and only if neither side of the link involves a deletion.

- A preterm is SPANNED if every switching has at least one surviving linking.

Example

$$\mathsf{Fu}^\frown\mathsf{FLt} \quad \mathsf{LSu}^\frown\mathsf{SLt} \quad \mathsf{RSu}^\frown\mathsf{Rt}$$

This has two pairs for switching:

$\mathsf{LSu}/\mathsf{RSu}$ in *input position*. $\mathsf{FLt}/\mathsf{SLt}$ in *output position*.

Example

$$\mathsf{Fu}\frown\mathsf{FLt}\quad\mathsf{LSu}\frown\mathsf{SLt}\quad\mathsf{RSu}\frown\mathsf{Rt}$$

This has two pairs for switching:

$\mathsf{LSu}/\mathsf{RSu}$ in *input position*. $\mathsf{FLt}/\mathsf{SLt}$ in *output position*.

$$\mathsf{Fu}\frown\cancel{\mathsf{FLt}}\quad\cancel{\mathsf{LSu}}\frown\mathsf{SLt}\quad\mathsf{RSu}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\mathsf{FLt}\quad\cancel{\mathsf{LSu}}\frown\cancel{\mathsf{SLt}}\quad\mathsf{RSu}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\cancel{\mathsf{FLt}}\quad\mathsf{LSu}\frown\mathsf{SLt}\quad\cancel{\mathsf{RSu}}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\mathsf{FLt}\quad\mathsf{LSu}\frown\cancel{\mathsf{SLt}}\quad\cancel{\mathsf{RSu}}\frown\mathsf{Rt}$$

DEFINITION: A preterm π is a TERM when it is SPANNED.

Theorem: Sequentialisable Preterms are Terms

By induction on the derivation sequentialising π.

Sequentialisable Preterms are Terms: Identity

$$\Sigma, x : A \succ \overset{x \frown y}{} y : A, \Delta$$

Sequentialisable Preterms are Terms: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\underset{}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \wedge B \succ \Delta}}}\wedge L \qquad \dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta'}}\wedge R$$

Sequentialisable Preterms are Terms: Negation

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\underset{}{\overset{\pi[Nz]}{\Sigma, z : \neg A \succ \Delta}}}\ \neg L \qquad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Nz)}{\Sigma \succ z : \neg A, \Delta}}\ \neg R$$

Sequentialisable Preterms are Terms: Disjunction

$$\frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz) \quad \pi'(Rz)}{\Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta'}} \vee L \qquad \frac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz, Rz]}{\Sigma \succ z : A \vee B, \Delta}} \vee R$$

Sequentialisable Preterms are Terms: Conditional

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az] \quad \pi'(Cz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}}\,{\supset}L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(Az)[Cz]}{\Sigma \succ z : A \supset B, \Delta}}\,{\supset}R$$

Sequentialisable Preterms are Terms: Cut

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet] \quad \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}$$

By induction on the number of pairs for switching in π.

Except …

$$x \frown y \quad u \frown v$$

ELIMINATING CUTS

Conjunction Cut Reduction

$$
\cfrac{\cfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}}{\overset{\pi[Fz]\quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \land B, \Delta, \Delta}}\,{\wedge}R \qquad \cfrac{\overset{\pi''(u,v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi''(Fw, Sw)}{\Sigma'', w : A \land B \succ \Delta''}}\,{\wedge}L}{\overset{\pi[F\bullet]\quad \pi'[S\bullet]\quad \pi''(F\bullet, S\bullet)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}\,\mathit{Cut}
$$

Conjunction Cut Reduction

$$\dfrac{\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta}}\,\wedge R \qquad \dfrac{\overset{\pi''(u, v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi''(Fw, Sw)}{\Sigma'', w : A \wedge B \succ \Delta''}}\,\wedge L}{\overset{\pi[F\bullet] \quad \pi'[S\bullet] \quad \pi''(F\bullet, S\bullet)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}\,\mathit{Cut}$$

reduces to

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \dfrac{\overset{\pi'[y]}{\Sigma' \succ y : B, \Delta} \qquad \overset{\pi''(u, v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi'[\star] \quad \pi''(u, \star)}{\Sigma', \Sigma'', u : A \succ \Delta', \Delta''}}\,\mathit{Cut}}{\overset{\pi[*] \quad \pi'[\star] \quad \pi''(*, \star)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}\,\mathit{Cut}$$

Identity Cut Reduction

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{y\frown z}{\Sigma', y : A \succ z : A, \Delta'}}{\overset{\pi[\bullet] \quad \bullet\frown z}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}}\ Cut$$

Identity Cut Reduction

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{y^\frown z}{\Sigma', y : A \succ z : A, \Delta'}}{\overset{\pi[\bullet] \;\; \bullet^\frown z}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}}\;Cut$$

reduces to

$$\overset{\pi[z]}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}$$

Difficult Cases: Contraction

$$
\dfrac{
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{\mathsf{L}y\frown x \quad \mathsf{R}y\frown x}{y : p \lor p \succ x : p}}{\scriptstyle \lor L}
\qquad
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown \mathsf{F}z \quad x\frown \mathsf{S}z}{x : p \succ z : p \land p}}{\scriptstyle \land R}
}{\overset{\mathsf{L}y\frown\bullet \quad \mathsf{R}y\frown\bullet \quad \bullet\frown \mathsf{F}z \quad \bullet\frown \mathsf{S}z}{y : p \lor p \succ z : p \land p}}{\scriptstyle Cut}
$$

Difficult Cases: Contraction

$$
\dfrac{
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor}L
\qquad
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land}R
}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \lor p \succ z : p \land p}}\,\mathit{Cut}
$$

$$
\dfrac{
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor}L
\qquad
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor}L
}{\overset{Ly\frown Fz \quad Ry\frown Fz \quad Ly\frown Sz \quad Ry\frown Sz}{y : p \lor p \succ z : p \land p}}\,{\land}R
$$

Difficult Cases: Contraction

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor}L \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land}R}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \lor p \succ z : p \land p}}\,\mathit{Cut}$$

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land}R \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land}R}{\overset{Ly\frown Fz \quad Ry\frown Fz \quad Ly\frown Sz \quad Ry\frown Sz}{y : p \lor p \succ z : p \land p}}\,{\lor}L$$

Difficult Cases: Weakening

$$
\dfrac{\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}} \qquad \dfrac{\overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi'}{\Sigma', y : A \succ \Delta'}}}{\overset{\pi \;\; \pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}
$$

Difficult Cases: Weakening

$$\dfrac{\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}} \qquad \dfrac{\overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi'}{\Sigma', y : A \succ \Delta'}}}{\overset{\pi\ \ \pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}$$

$$\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Weak} \qquad \dfrac{\overset{\pi}{\Sigma \succ \Delta} \qquad \overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi\ \ \pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Mix} \qquad \dfrac{\overset{\pi'}{\Sigma' \succ \Delta'}}{\overset{\pi'}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Weak}$$

Back to Sequentialisation

$$x \frown y \quad u \frown v$$

$$\dfrac{\overset{x \frown y}{x : A \succ y : A} \qquad \overset{u \frown v}{u : B \succ v : B}}{\overset{x \frown y \quad u \frown v}{x : A, u : B \succ y : B, v : A}}\ \textit{Mix}$$

Sequentialisation: Terms with *No* Switchings

The term contains no Ln, Rn, Cn and • in input position
or Fn, Sn, An and • in output position.

It has a derivation using the linear rules
$\wedge L$, $\neg L$, $\neg R$, $\vee R$ and $\supset R$ and *mixes*.

The term contains no Ln, Rn, Cn and $\bullet$ in input position or Fn, Sn, An and $\bullet$ in output position.

It has a derivation using the linear rules
$\wedge L$, $\neg L$, $\neg R$, $\vee R$ and $\supset R$ and *mixes*.

$$\mathsf{Fy}{}^\frown\mathsf{Lz} \quad \mathsf{NRz}{}^\frown\mathsf{Lz} \quad \mathsf{Sy}{}^\frown\mathsf{Rz}$$

$$y : p \wedge \neg p \succ z : p \vee \neg p$$

Terms with No Switchings: Example

$$
\dfrac{
  \dfrac{
    \dfrac{\begin{array}{c} x\frown x \\ x : p \succ x : p \end{array}}{\begin{array}{c} x\frown Lz \\ x : p \succ z : p \vee \neg p \end{array}}\;\vee R
  }{\begin{array}{c} Fy\frown Lz \\ y : p \wedge \neg p \succ z : p \vee \neg p \end{array}}\;\wedge L
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\begin{array}{c} u\frown u \\ u : p \succ u : p \end{array}}{\begin{array}{c} Nv\frown u \\ \succ u : p, v : \neg p \end{array}}\;\neg R
    }{\begin{array}{c} NRz\frown Lz \\ \succ z : p \vee \neg p \end{array}}\;\vee R
    \qquad
    \dfrac{
      \dfrac{\begin{array}{c} v\frown v \\ v : \neg p \succ v : \neg p \end{array}}{\begin{array}{c} v\frown Rz \\ v : \neg p \succ z : p \vee \neg p \end{array}}\;\vee R
    }{\begin{array}{c} Sy\frown Rz \\ y : p \wedge \neg p \succ z : p \vee \neg p \end{array}}\;\wedge L
  }{\begin{array}{c} NRz\frown Lz \quad Sy\frown Rz \\ y : p \wedge \neg p \succ z : p \vee \neg p \end{array}}\;\mathit{Mix}
}{\begin{array}{c} Fy\frown Lz \quad NRz\frown Lz \quad Sy\frown Rz \\ y : p \wedge \neg p \succ z : p \vee \neg p \end{array}}\;\mathit{Mix}
$$

By induction on the number of switched pairs.

Take a switched pair at the *adjacent to variables* or *cut points* (peel away unswitched steps if there aren't any).

$$\dfrac{\overset{\pi[x](-)}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi[-](y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az](Lz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset\! L$$

Back to Eliminating Cuts: Cuts can be Complicated

$$
\dfrac{
\dfrac{\overset{\pi[x,u]}{\succ x : A \wedge B, u : A} \qquad \overset{\pi'[x,v]}{\succ x : A \wedge B, v : B}}{\overset{\pi[x,\mathsf{F}x] \quad \pi'[x,\mathsf{S}x]}{\succ x : A \wedge B}}\,\wedge R
\qquad
\dfrac{\overset{\pi''(y,z,x)}{y : A, z : B, x : A \wedge B \succ}}{\overset{\pi''(\mathsf{F}x,\mathsf{S}x,x)}{x : A \wedge B \succ}}\,\wedge L
}{\overset{\pi[\bullet,\mathsf{F}\bullet] \quad \pi'[\bullet,\mathsf{S}\bullet] \quad \pi''(\mathsf{F}\bullet,\mathsf{S}\bullet,\bullet)}{\Sigma \succ \Delta}}\,\mathit{Cut}
$$

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

- *atomic*: replace each pair $n \frown \bullet$ and $\bullet \frown m$ by $n \frown m$.

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- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

$$\mathsf{S}z\frown\star \quad \mathsf{F}z\frown* \quad \star\frown\mathsf{S}x \quad *\frown\mathsf{F}x$$

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- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

$$\mathsf{S}z\frown\star \quad \mathsf{F}z\frown* \quad \star\frown\mathsf{S}x \quad *\frown\mathsf{F}x \quad \mathsf{FN}y\frown\mathsf{S}x \quad \mathsf{SN}y\frown\mathsf{F}x \quad \mathsf{N}y\frown\nu$$

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

$$\mathsf{S}z\frown\star \quad \mathsf{F}z\frown* \quad \star\frown\mathsf{S}x \quad *\frown\mathsf{F}x \quad \mathsf{FN}y\frown\mathsf{S}x \quad \mathsf{SN}y\frown\mathsf{F}x \quad \mathsf{N}y\frown\nu \quad \mathsf{S}z\frown\mathsf{F}\nu \quad \mathsf{F}z\frown\mathsf{S}\nu$$

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- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.
- *negation*: for each $\mathsf{N}\bullet$, add a new cut point $\star$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

- *atomic*: replace each pair $n\frown\bullet$ and $\bullet\frown m$ by $n\frown m$.
- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.
- *negation*: for each $\mathsf{N}\bullet$, add a new cut point $\star$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.
- *disjunction*: for each $\mathsf{L}\bullet/\mathsf{R}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.

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- *conditional*: for each $\mathsf{A}\bullet/\mathsf{C}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

STRONG NORMALISATION

Any reduction for π terminates in a unique* term π^*

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- There is *some* terminating reduction process.

Any reduction for π terminates in a unique* term π^*

- There is *some* terminating reduction process.
- Proof reduction is confluent.
  - If $\pi \rightsquigarrow_{\bullet} \pi'$ and $\pi \rightsquigarrow_{\star} \pi''$ then there is a π''' where $\pi' \rightsquigarrow_{\star} \pi'''$ and $\pi'' \rightsquigarrow_{\bullet} \pi'''$. (This is where α equivalence is required.)

FURTHER WORK

To Do List

- Are these genuine *invariants*? (Can we show that if two derivations have the same term, some set of permutations permute one to the other?)
- Apply these terms to other kinds of proofs (Fitch, Lemmon, tableaux, Hilbert, resolution…)
- *Categories* (The class of *single input, single output* terms with composition by defined by *Cut + reduction* is a category. What are its properties?)
- Apply terms to theories of warrants.
- Extend beyond propositional logic.

THANK YOU!

`https://consequently.org/presentation/2016/terms-for-classical-sequents-logicmelb/`

@consequently on Twitter