Negation, Conservativeness and the Justification of Deduction

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1. *Tertium non datur and Dummett's Master Argument*

The proof of *tertium non datur* $A \vee \sim A$ in the relevant logics **R** or **E** proceeds essentially as in classical logic. We have the following negation and disjunction introduction rules:

~I	from a proof of $\sim A_a$ on the hypothesis $A_{\{k\}}$, to infer $\sim A_{a-\{k\}}$, provided k is in a
Contrap	from B_a and a proof of $\sim B_b$ on the hypothesis $A_{\{k\}}$, where k is in b, to infer $\sim A_{(a \cup b)-\{k\}}$
~~E	from $\sim\sim A_a$ to infer A_a
∨I	from A_a to infer $A \vee B_a$, from B_a to infer $A \vee B_a$

First we prove ~~I

1	$\vdash$	A_a	hyp
2	$\mid\vdash$	$\sim A_1$	hyp
3	$\mid\mid$	$\sim A_1$	rep 2
4	$\mid$	$\sim\sim A_a$	Contrap 1, 2-3

Then we can prove $A \vee \sim A$

1	$\vdash$	$\sim(A \vee \sim A)_1$	hyp
2	$\mid\vdash$	A_2	hyp
3	$\mid\mid$	$A \vee \sim A_2$	∨I 2
4	$\mid\mid$	$\sim\sim(A \vee \sim A)_2$	~~I 3
5	$\mid$	$\sim A_1$	Contrap 1, 2-4
6	$\mid$	$A \vee \sim A_1$	∨I 5
7	$\mid$	$\sim\sim(A \vee \sim A)_1$	~~I 6
8		$\sim\sim(A \vee \sim A)$	~I 1, 7
9		$A \vee \sim A$	~~E 8

* I am particularly grateful to Prof. Ingolf Max and the research group 349 of the Deutsche Forschungsgemeinschaft 'Sprachtheoretische Grundlagen der Kognitionswissenschaft' (Linguistic Foundations of Cognitive Sciences) at the University of Leipzig, who made the talk at the conference 'Inference and Meaning 2004' possible in funding the journey to Melbourne as well as the research for this paper.

If Dummett has an argument against the proof of *tertium non datur* in *classical logic*, which would merely drop the relevant indices, then he has an argument against this proof, too. I shall call it his "Master Argument", for if it works, it is a very strong one, as would hold against *anything* worth being called a logic that contains *tertium non datur* as a logical law.[1]

The background of the Master Argument is the proof-theoretic justification of logical laws in a theory of meaning with its assumption that the meanings of the logical constants are completely determined by the introduction rules governing their use in a system of natural deduction.[2] The idea is that given the introduction rules for a logical constant satisfies certain constraints they may count as self-justifying and the meaning of the constant is determined by the moves in a derivation that these rules license. The proof of A∨~A violates what Dummett calls the *fundamental assumption*: namely that, for every valid deductive argument for a complex statement A, one can construct a valid argument for A which finishes with an application of one of the introduction rules governing its principle operator.[3] (*The Logical Basis of Metaphysics* (LBM) 254) The fundamental assumption is mandatory, if the introduction rules for logical constants are to determine their meanings completely, for then 'the introduction rules for a logical constant **c** represent the direct or canonical means of establishing the truth of a sentence with principal operator **c**' (LBM 252, *cf*. 257); everything else constitutes an only *indirect* means of verification. The fundamental assumption then says that, for every indirect verification of A, one can construct a direct one, which proceeds according to the composition of A.

A further requirement is that the canonical verification of a sentence must not proceed via more complex sentences than itself. *Compositionality* or *molecularity* of the language as a whole has to be secured. For Dummett, 'the principle of compositionality is not the mere truism, which even a holist must acknowledge, that the meaning of a sentence is determined by its composition. Its bite comes from the thesis that the understanding of a word consists in the ability to understand characteristic members of a particular range of sentences containing that word.' (LBM 225) Compositionality of the language is mirrored in a partial ordering that the theory of meaning imposes on the expressions and sentences of the language. This partial ordering exhibits how the language is learnable in a step by step

[1] But *cf*. footnote 9.

[2] One might also chose the elimination rules as determining the meaning of a constant, which arguably is the more natural thing to do in the case of ∨ and ∀. But this does not effect what is at issue here.

[3] Although Dummett formulates the fundamental assumption as a prerequisite for the proof-theoretic justification of logical laws, he lists arguments that leave it 'very shaky' (LBM 277). Suffice it to say that these are connected to the admittance of empirical, non-deductive modes of inference, *i.e.* to the question in how far the logical constants of a system of natural deduction reflect the meanings of 'or', 'all', 'if-then' in natural language. As long as one does not intend to account for this, one may ignore this complication.

process, from simple to complex, by making explicit the dependence of meanings or speakers' understanding[4] of more complex sentences on less complex sentences and the latter's independence of the former.[5] The purpose of this requirement is easily seen. Suppose C is more complex than B, *i.e.* by compositionality C depends on B, and suppose that B can only be verified by an argument that uses C. Such an argument would constitute the canonical verification of B, no other verification being available. Hence B would depend, for its meaning, on C. Due to this circular dependence of meaning, neither C nor B could have a stable meaning. Of course, as the example was put alternatively one might conclude that C and B are of the same complexity after all and can only be learnt together, so the point must be that this situation is not the case for all the expressions (or a large part) of the language as a whole, *i.e.* that there are some expressions which are of different complexity. Another very clear example, which shows nicely the kind of work compositionality is supposed to do, is an argument against double negation elimination or classical *reductio ad absurdum* using compositionality. By compositionality the meaning of ~~A is dependent on the meaning of A and negation. If A is verifiable only via its double negation, or, equivalently, via a *reductio ad absurdum* of its negation, then the move from the assumption ~A to A would determine the meaning of A; this would be its canonical verification, no other verification being available. The canonical verification of A would proceed via the more complex negation or double negation, but the latter in turn must depend, for its meaning, on the meaning of negation and the meaning of the sentence as already given. Due to this a circular dependence A could not have a stable meaning. A speaker could not break into the circle and learn the meaning of A.

If it is to count as self-justifying, in order to secure compositionality an introduction rule has to satisfy the *complexity condition*, namely that 'its form be such as to guarantee that, in any application of it, the conclusion will be of higher logical complexity than any of the premises and than any discharged hypothesis.' (LBM 258) The corresponding elimination rules for the logical constants are chosen as being *in harmony* with the introduction rules, *i.e.* such that it is never necessary to derive a sentence with an introduction rule for its principal logical constant which is then the major premise of an elimination rule for the same constant: it is required that any such moves in a proof may be avoided and maximal occurrences of formulas be removed. Furthermore, the constant in question must produce

[4] For Dummett, a theory of meaning is best understood as a theory of understanding.
[5] As the ordering is partial, there may be sentences which are incomparable with respect to complexity.

a conservative extension of a language to which it is added. (LBM 217)[6] The motivation for harmonious inference rules might be put somewhat like this: if we have grounds for asserting a complex sentence, then what we deduce from it should be no more than we already had. Logic does not tell us anything new. This is *intrinsic harmony*. (LBM 250) At the same time, it is a reasonable demand that, given a complex sentence, we should be able to deduce *everything* we may justifiably assert. Logic does not conceal anything either. Both demands together are what *stability* amount to: assuming the introduction rules of a constant as given, we can justify the elimination rules, and assuming the elimination rules as given, we can justify the introduction rules. (LBM 287ff)

The relation between the fundamental assumption and the complexity condition is this. Given a derivation of A from a set of assumptions Γ, then by the fundamental assumption, there always is a derivation of A that proceeds according to its composition in the truistic sense and ends with an application of an introduction rule for its principal operator. If the rules governing the operators of the logic satisfy the complexity condition, there is one that proceeds according to its composition in the substantial sense: the canonical argument for A from Γ need not invoke sentences of higher complexity than A in any of its subarguments.[7]

The proof at the beginning of this paper proceeds via the more complex formulas ~(A∨~A) and ~~(A∨~A) and concludes with the more simple formula A∨~A and thus violates both conditions on canonical proofs. According to Dummett's argument, A∨~A cannot have a stable meaning and this part of **R**, say, cannot be justified proof-theoretically even though its negation *is* conservative over its positive fragment. The problem is precisely that conservativeness of a constant over the positive fragment of a logic is not sufficient for Dummett, contrary to what sometimes appears to be assumed in the literature.[8] The complexity condition has to be met as well. This condition cannot be met, if we want to follow Dummett's account of the meanings of the logical constants and want to end up with a quasi-classical negation that validates *tertium non datur*. For how should a proof of A∨~A proceed that respects the conditions on canonical proofs? To satisfy the fundamental assumption, it would have to come from A or from ~A. Whichever it is, somehow it must come from assumptions that are

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[6] Explicating harmony as the possibility of levelling of local peaks is not quite enough, as an example of Dummett's shows (LBM 290). He conjectures that 'intrinsic harmony implies total harmony in a context where stability prevails.' (LBM 290)

[7] This of course ignores the definitions of 'valid (canonical) argument', *cf.* LBM 259ff.

[8] Hence someone who followed Peter Milne's suggestion of viewing classical *reductio ad absurdum* 'from Γ, ~A⊢A to infer Γ⊢A' as an introduction rule for A would still not satisfy Dummett's constraints, as noted in CLASSICAL HARMONY: RULES OF INFERENCE AND THE MEANINGS OF THE LOGICAL CONSTANTS. He also concludes, as I shall below, that the meaning of negation cannot be given proof-theoretically.

discharged in the process of the argument. It cannot be A, for this may be an atomic sentence and no atomic sentence follows from no premises at all. But it couldn't be ~A either, as the same consideration shows, for, if A is atomic, neither does ~A follow from no premises at all. For anything worth being called a logic, there is no chance of its meeting Dummett's objection as well as having A∨~A as a theorem.[9] Note that from Dummett's perspective the complexity condition is sufficient to show that the proof above is unacceptable, as it may be used to argue against the elimination rules used in the proof; the fundamental assumption then insures that there is no other proof of A∨~A.

Dummett's Master Argument against A∨~A from the proof-theoretic justification of logical laws is immensely strong, if it works. On the face of it, the account of compositionality above appears to presuppose a certain priority of proof over truth. But this is not so. The point is quite independent of how we conceive of the meaning of sentences as being given. All that is needed to get the argument going is that meaning is use. These uses of sentences as the conclusions of a certain deductive arguments confer particular meanings on them: double negation elimination or classical *reductio ad absurdum* and the derivation of B from C in the other example would, if employed, justify certain uses of A and B that otherwise could not be made and hence confer upon them a certain meaning. Whatever you may think about Dummett's other famous argument against the Principle of Bivalence, this argument still stands: it is independent of the argument from manifestability. Its force comes from the fact that, if viable, it establishes that compositionality suffices for a rejection of *tertium non datur*. And the compositionality of language is a very weak assumption indeed and a very plausible one at that.

2. *The Fundamental Assumption*

Reflecting on the fundamental assumption, something springs to mind: can it be a fair objection to a logic be that there is no derivation of A∨~A that ends with an application of ∨I? There simply cannot be a proof-theoretic justification of A∨~A as a logical law that proceeds along Dummett's line, and if there was one, the considerations just voiced show that we'd better reject the 'logic' that made this possible. An intuitionist might reply: 'Yes, that certainly is just what is the case. Nothing can justify A∨~A as a logical law, as it isn't!' But somehow one feels that, in order for this to be in any sense a deep result, there should at least be some conceivable chance of justifying A∨~A.[10]

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[9] There is, however, at least one view according to which this is possible. Leibniz apparently thought that every truth is analytic, *i.e.* a logical truth.

[10] Dummett actually expresses something like this when he says that the meaning of negation is relative to the language in which it occurs: this is to admit that there is no proof theoretic justification of negation (not even of intuitionist negation, but

What is so puzzling about Dummett's argument against A∨~A is that A∨~A contains negation and is a theorem as much about ~ as it is about ∨. What could it matter that the proof of A∨~A proceeds via the more complex ~(A∨~A) and ends with an application of a ~ rule, as once I understood negation, it is hard to see why I shouldn't use the rules governing ~ as often as I please in a proof. What could there be that a speaker needs to understand in order to understand ~~(A∨~A) or ~(A∨~A) that he does not need to understand in order to understand A∨~A? This is obscure and I should suggest there really is nothing in the one that is not in the other. To understand one of A∨~A, ~(A∨~A) and ~~(A∨~A), I need to understand ~, ∨ and that A stands for a sentence; nothing else is needed in the proof. A logical law is true only in virtue of the logical constants occurring in it and in virtue of the rules stipulated for them and a contradiction false for the same reason. Of course the structure or composition of the sentences play a role, too, but I understand this composition if I understand the workings of the constants. If I understand the operator and can apply it in one case (*e.g.* ~A), I can also be expected to apply it in any case (*e.g.* ~(A∨~A)), given I understand the rest of the context. This is just the point of explaining logical constants as operations on sentence: their meaning can be given in a completely general way, independently of the meanings of any sentences in which they might occur. The rules governing a constant tell us how to proceed when the constant applies to any sentences whatsoever and not, so to speak, piecemeal. In the case under consideration *ex hypothesi* the rest of the context is understood, as ∨ is understood. Thus it is hard to see what the alleged violation of the complexity condition in the proof of A∨~A consists in. And for the same reason, as A∨~A can be viewed as showing something about disjunction as well as about negation, in the light of this theorem the fundamental assumption reveals itself as an excessive requirement.

Now the question must be: where are we to put A∨~A in the partial ordering? The apparently *less* complex A∨~A cannot depend on the apparently *more* complex ~(A∨~A) or ~~(A∨~A) for its meaning, although its proof does proceed via them. The fact that the canonical proof of A∨~A has this property must not entail that A∨~A has an unstable meaning or that the rules governing ~ somehow disturb the meaning of ∨, as determined by those rules. The solution suggested by the preceding discussion is to *deny* that A∨~A, ~(A∨~A) and ~~(A∨~A) are of different complexity, as only ∨ and ~ need to be understood in order to understand them.

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that is not essential to the question concerning A∨~A). The way out that Dummett sketches is to leave proof theory and try it in semantics.

However, this approach works only for certain logically true or false sentences where *all* the speaker needs to understand are the logical constants occurring in them. Even if we drop the fundamental assumption in the generality Dummett puts forward and adopt an account of complexity in accordance with the argument of the last paragraphs, there still remains the problem that, adopting classical logic, it may happen that a *non-logical*, *contingent* sentence A can only be verified through verifying its double negation first. A could still not have a fixed place in the partial ordering. The above account does not preclude that A could not have a stable meaning, because the meaning of A has to be known and to be determined somehow beforehand. In the case of a primitive sentence A, Dummett can provide for a simple answer to the question what a speaker has to understand in order to understand ~~A or ~A that he does not need to understand in order to understand A: negation needs to be understood. And similarly for any other contingent sentence not containing negation. The real trouble with classical negation is that its non-conservativeness does not only affect logical sentences, but also *non-logical* ones, *i.e.* the non-conservativeness affects not only the logical vocabulary of a language, but also the non-logical vocabulary. Given A∨~A, it follows that ~~A⊃A, by the intuitionistically valid disjunctive syllogism. And it is the *latter* that, as a law, causes the real trouble. A similar problem occurs for certain logical sentences, *e.g.* (A⊃B)∨(B⊃A), A∨(A⊃B) *etc.*; *these* are the problematic cases, not A∨~A.

3. *Eadem est scientia oppositorum*

The first step towards a solution lies in a view on negation that Peter Geach has put forward. A point Geach stresses in THE LAW OF EXCLUDED MIDDLE and *MENTAL ACTS* is that a sentence and its negation are of the *same* complexity: *eadem est scientia oppositorum*. In the present context, the truth in this scholastic maxim, which is to say that I cannot understand *Fa* without understanding *~Fa*, is that there is an intrinsic difficulty in justifying negation proof-theoretically or determining its meaning completely by inference rules governing ~. The reason is that these rules must already involve negation or some notion closely akin to it, if they really are to count as negation rules. Dummett points out that the only way of formulating intuitionistic logic with harmonious introduction and elimination rules for negation is to use the *falsum* constant ⊥, the meaning of which is given by *ex falso quodlibet*. (LBM 295) If we understand ⊥ as the conjunction of *all* propositions, *ex falso quodlibet* says something like: ‘if you say *this*, you may as well say anything’. But if “anything” were equivalent only to the conjunction of all *positive* sentences, no absurdity follows. Any set of sentences (of classical and intuitionistic logic, anyway) containing only $\supset$, $\vee$ and &, but not negation, has a model: assign the truth value "true" to every propositional variable in the language; for intuitionistic logic, do so in a world that accesses only itself. So for $\perp$ to do the job of a constantly false sentence, the "anything" it stands for must cover some sentences containing negation and hence cannot be used in this way in determining the meaning of negation. Nevertheless Dummett *does* restrict $\perp$ to being equivalent to the conjunction of all atomic sentences. Hence Dummett's atomic sentences cannot be atomic in the sense of the *TRACTATUS* and as atomic sentences are explained in formal logic, namely as being *independent* of each other, in the sense that no conjunction of atomic sentences can be a contradiction and no disjunction of atomic sentences a tautology. If $\perp$ is to do the job it is thought to do in defining negation, amongst Dummett's atomic sentences there must be some that *exclude* each other. But isn't that fine, one might ask: surely there are mutually exclusive atomic sentences that cannot be true together, like '*a* is red' and '*a* is green'. Isn't it already sufficient in order to be justified in asserting ~A, that, instead of $\perp$, we can derive from A two such mutually exclusive sentences? Reflection shows that things are not so easy. We need the *right kind of exclusiveness*: it wouldn't suffice for logic that we can derive some unacceptable sentences, like 'Beetroots are delicious' and 'Scotch is disgusting'. Certainly those two cannot be true together, for both are false, but that would merely result in a logic for my personal prejudice (as minimal logic has been characterized). The reason why '*a* is green and *a* is red' would constitute the kind of exclusiveness we need in logic is just that what is green cannot be red and conversely: I need to know, that is, that if something is red, it is not green, hence if something is red as well as green, it is green as well as not green. And this is just to say that in defining negation proof-theoretically, in order to define a logical symbol that really *does* the job of negation, we already needed negation in the first place and the attempt at defining negation is circular.

According to Dummett, in order to understand the concept 'red', not only do I need to know which things are red, but I also need to know some other colour words. And now Geach's point is that to understand what it means that something is red or green or blue *etc*., I also need to understand that what is green is not red *etc*.. Otherwise I cannot be said to understand the concepts in question. Understanding that something is red is inseparable from an understanding of something's not being red: *eadem est scientia oppositorum*. Saying what something is, is also saying what it is not, or as Spinoza says: *omnis determinatio est negatio*. Grasping the application of a concept, I already grasp negation and hence understanding a sentence I understand its negation. Geach employs the picture of a closed line on a surface to represent a predicate: objects which lie inside the line have the property, those lying outside don't. Then 'the predicate and its negation will clearly be represented by one and the same line […] We must *a fortiori* reject the view that a negative predication needs to be backed by an affirmative one – that we are not justified in predicating the negation of P unless we can predicate some Q which is positive and incompatible with P.' (THE LAW OF EXCLUDED MIDDLE 79) This is just what is being done if philosophers require the meaning of negation to be determined by the verification conditions for a sentence with ~ as principal operator. According to Dummett, we are entitled to assert $\sim Fa$ if and only if we can show that the assumption of Fa entails $\perp$ or some contradiction, which is just to say that we are entitled to assert $\sim Fa$ just in case we can assert the predicate $F\xi\supset\perp$ of a. The argument of the last paragraph showed that such an attempt to achieve a proof theoretic explanation of the meaning of negation is a misconception and entirely out of place. There is no need for introduction and elimination rules for negation that determine the meaning of the symbol ~, or rather, it is not even possible to determine the meaning of ~ by rules governing it, for, in order to understand the point of these rules, I already have to understand negation.[11]

If the sense of an expression is something the speaker has to know about the expression in order to be able to use it, then a theory of negation along Geach's lines would have to specify simultaneously the sense of any predicate $F\xi$ and its negation $\sim F\xi$.[12] Obviously there must be *some* difference between the predicates $F\xi$ and $\sim F\xi$. No one would dispute that ~ adds to the content of a predicate or a sentence. Even if one cannot understand Fa without understanding $\sim Fa$, this does not mean that one cannot exhibit a *uniform contribution* that ~ makes to a sentence in which it occurs and that $\sim Fa$ is composed of ~ and Fa, or, of course, accepting classical logic, as Geach certainly does, that Fa is composed of

[11] This is actually Prior's point in CONJUNCTION AND CONTONKTION REVISITED restricted to negation. Once one accepts the so restricted point, one may well ask whether it does not also hold for all the other connectives, as Prior argued, and whether the whole approach of determining the meanings of logical constants via rules governing them doesn't break down. How could there be a special logical constant? Consider the kind example that Prior argued shows that any sentence tells us something about everything (*cf.* FACTS, PROPOSITIONS AND ENTAILMENT). 'Brisbane is in Queensland', *e.g.*, says about everything, apart from Brisbane, that it is not both, Brisbane and not in Queensland. Hence I can only be said to understand 'Brisbane is in Queensland', if I know, for everything that is not Brisbane, *e.g.* Mauritius Elector, the fellow whose sculpture graces the façade of the Bibliotheca Albertina of Leipzig University opposite my office, that it is not both, Brisbane and not in Queensland. This is to say that in order to understand *any* predication, I need to understand at least conjunction and negation, if not in addition quantification and possibly even equality. So we are easily lead to the *TRACTATUS* view on logic that there is only *one* logical constant and in order to understand *anything at all*, I need to understand *it* and with it the whole of logic (5.47, 5.472, 6.124f). This casts some serious doubts on the whole approach.

[12] Dummett and, according to his interpretation, Frege would call $\sim F\xi$ the negation of the *predicate* $F\xi$ only in a derivative way of speaking. Strictly speaking, there is no predicate negation according to Frege/Dummett. Negation is a function and functions always have *objects* as values, whereas predicate negation would have to be a function taking functions as values. $\sim F\xi$ is constructed, not from $F\xi$ by applying negation, but as is every predicate: from a sentence by omitting some occurrences of a name: from the *sentence* Fa we omit the name a to get the predicate $F\xi$, we apply negation to the *sentence* Fa to get $\sim Fa$ and drop a from *it* to get the predicate $\sim F\xi$.

$\sim Fa$ and ~. Geach's point is only that, whatever contribution ~ makes to a sentence, it is absolutely fundamental and you cannot be said to understand anything if you don't understand *it*. The relation between understanding Fa and $\sim Fa$ is one thing, rejecting compositionality quite another thing. One can still see negation as an operation on sentences: ~ creates the *opposite content* of $F\xi$. The rules that govern negation in a logical calculus exploit the meaning of ~, but cannot determine it. The possibility of treating $\sim F\xi$ as $F\xi \supset \perp$ is at best a *consequence* of our understanding ~. We may view the rules as an *explication*, though not as an *explanation* of the concept of negation and the contribution ~ makes to $\sim F\xi$. The rules for ~ must be of a different nature than the rules for the other connectives, if their meanings are to be given by self-justifying rules of inference.

Accepting *eadem est scientia oppositorum*, we may adopt classical negation rules without fear of destroying compositionality, as negation does not add to the complexity of sentences. If negation cannot be defined proof-theoretically, then from the perspective of self-justifying inference rules, there is no need for the negation rules to be in harmony either.[13] This part of Dummett's Master Argument looses its force. However, this is only a first step towards a solution to the problem at the end of Section 2. There is not yet a strategy to deal with the fundamental assumption: even if *eadem est scientia oppositorum*, so far there are still sentences A*B not containing negation that may be verified only via the assumption ~(A*B), and although this is not problem for compositionality anymore, the fundamental assumption is still violated. To put it differently, it would be a cheap victory over the Master Argument if, adopting the maxim, we could not make sense of the proof theoretic justification of logical laws of the positive fragment of logic anymore. (In the following, I shall call the logical constants of the positive fragment the *positive* logical constants.) It would be merely trivial if an adoption of the maxim would preclude Dummett's criteria for self-justifying rules from being applicable even in the case of the positive logical constants, as then this is simply to reject the proof theoretic justification of logical laws entirely. So the next task is to discuss, from the perspective of Geach's maxim, how the meanings of the positive logical constants are to be determined by rules governing them and whether sense can be made of Dummett's criteria for them. Let me note that in this paper I am not trying to argue that the meanings of the positive logical constants must be given by the

[13] Note that due to the conservativeness of negation in relevance logics, total harmony is not equal to the possibility of levelling local peaks. In a sense, this is not surprising: of course I can introduce a 'logical constant' δ, say, subject to the sole elimination rule 'from δA to infer A' and with no introduction rule (and δ does not occur in any other rule). This would quite trivially produce a conservative extension of any language it is added to, but also trivially the rules governing it are not harmonious.

rules governing them: rather, I am assuming this to be the case: I wish to save as much of Dummett's programme as possible, given Geach is right.[14]

4. *Logical Constants*

On Geach's account, it would be very natural to claim that, for a logical constant *, the meaning of A*B and the meaning of ~(A*B) are also essentially connected, just as are *Fa* and ~*Fa*, and that they too can only be grasped together. This would lead to formulating rules for when * is the principal operator as well as for the case when * is inside the scope of ~. Similarly for one-place logical operations, which, if *eadem est scientia oppositorum* holds for every predicate, is in nay case compulsory for quantifiers, if they are higher level predicates, as Frege says. The most obvious, harmonious, 'negative' rules for ~(A⊃B), ~(A∨B), ~(A&B), ~∀xFx, ~∃xFx[15] are the following:

$$\frac{A \quad \sim B}{\sim (A \supset B)} \qquad \frac{\sim (A \supset B)}{A} \qquad \frac{\sim (A \supset B)}{\sim B}$$

$$\frac{\sim A \quad \sim B}{\sim (A \vee B)} \qquad \frac{\sim (A \vee B)}{\sim A} \qquad \frac{\sim (A \vee B)}{\sim B}$$

$$\frac{\sim A}{\sim (A \,\&\, B)} \qquad \frac{\sim B}{\sim (A \,\&\, B)} \qquad \frac{\sim (A \,\&\, B) \quad \begin{array}{c}\overline{\sim A}^{(i)} \\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{\sim B}^{(i)} \\ \vdots \\ C\end{array}}{C}\,(i)$$

$$\frac{\sim Fa}{\sim \exists x Fx} \qquad \qquad \frac{\sim \exists x Fx}{\sim Ft}$$

a not in any assumption on which ~*Fa* depends

[14] *Cf.* footnote 11.

[15] Although I include rules for the quantifiers, in this paper I am concerned mainly with propositional logic, and confine myself to cursory remarks about first order logic.

$$\frac{\sim Ft}{\sim \forall xFx} \qquad\qquad \frac{\sim \forall xFx \qquad \begin{array}{c}\overline{\sim Fa}^{\,(i)}\\ \vdots \\ B\end{array}}{B}{\scriptstyle (i)}$$

a not free in B and in no assumption other than *Fa* on which the upper *B* depends

Natural as it is, it is not difficult to see that this move is entirely pointless. For now the question arises whether the negative rules *extend* the meaning of negation, as it applies to primitive predicates, to complex ones or leaves it as it already is: do these rules ass something to the meaning of negation or do they too exploit only what is already there? The motivation for the first option is that one might come to think, with view to compositionality, that at a first stage negation is learnt together with the primitive predicates and that at a further stage when positive logical constants are learnt, the negative rules extend the meaning of negation to complex predicates. This way one would extend the maxim *eadem est scientia oppositorum* from primitive to complex predicates. One only has to state this position in order to see that it is not viable. *Ex hypothesi* the meanings of the positive constants are determined by the positive *and* the negative rules. So to understand a logical constant, I would need to understand the positive *and* the negative rules. But to understand how the negative rules extend the meaning of negation, I would obviously have to understand the constants already. This strategy would have to assume that the logical constants can be taken for granted in the negative rules, so it either flouts compositionality or the motivation to introduce the negative rules. The less obviously hopeless strategy is of course to claim that the meaning of negation is assumed to be given in the negative rules. But a little reflection shows that this is not a strategy to be taken seriously either. For in that case the meaning of negation must be given for every case – primitive and complex – and the rules for it in a calculus, which we undoubtedly need, must therefore hold generally. Now on the Geachean account of negation, there is no reason not to adopt classical negation rules anymore. But then all the negative rules are derivable rules, given the positive ones. Hence they are superfluous: the negative rules could not tell us anything the positive rules alone do not already tell us.

The result is that negative rules can *not* add to the meanings of the positive logical constants.[16] We may conclude that *even if eadem est scientia oppositorum* and negation is absolutely fundamental to

[16] What about the possibility that they do *both*, extend the meaning of negation *and* determine the meanings of the logical constants? I suspect that this cannot be a viable position either, as then, *e.g.*, having learnt ∨, and afterwards ⊃ would change the meaning of ∨, as learning the meaning of ⊃ would imply learning the rules for ~(A⊃B), which would extend the

understanding, the meanings of those constants are completely determined by the positive rules governing them. Any stipulation of negative rules would be pointless or even counterproductive. Even on the Geachean account all we can say about the meanings of the positive logical constants *are* in fact their (positive) introduction and elimination rules and thus Dummett's criteria for self-justifying rules of inference can still be made sense of and are applicable without modification. It can be demanded that the rules governing the positive logical constants satisfy them.

5. *A Dilemma*

The argument of the last section supports the view that logical constants are of a different nature than predicates; for them, it is not true that *eadem est scientia oppositorum*.[17] The logical constants stand for *operations* on sentences. Contrary to predicates, or what they are true of, logical constants stand for nothing or do not apply to anything "in the world" (*Tractatus* 4.0312). You understand the constant if you understand the operation governing it. The *meaning* of a logical constant *is* the operation, its behaviour in sentences and in particular and most purely in logical laws. If, to understand a logical constant, I have to understand the operation it expresses or the rules governing it, then this should be *all* I need to know about it. Obviously there is some interplay between logical constants, if more than one figures in a sentence, but how should negation "enter" an operation somehow and contribute in conferring a certain meaning on it? Imagine two mechanisms: the one can influence the other, but how should one "enter" the other? Of course this is not a philosophical argument, but it serves well to illustrate what has been established in the last section. Even if negation is absolutely fundamental to understanding, the rules for the positive constants should be "self-sufficient" and tell us everything that needs to be known about the constants: their meaning should be completely and exclusively determined by the rules governing them, *i.e.* that those rules ought to be self-justifying in Dummett's sense.

This, however, is a curious result. Logically true sentences are true simply in virtue of the logical constants that occur in them and do not have any content apart from that; this is why they don't really

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meaning of $\sim$ to this case, but negation also occurs in $\sim(A\vee B)$, hence here $\sim$ would now have a different meaning and accordingly adding $\supset$ to the language would change the meaning of $\vee$. Accordingly meaning of a logical constant could only be given by the set of rules for all the constants of the calculus. It would then be difficult to say what would count as a canonical proof of, *e.g.*, $A\supset B$. Would it have to employ the other constants as well? If not, then how could the proof be said to proceed in accordance with how the meaning of $A\supset B$ is determined? If yes, what sentences should the other constants connect? Only A and B or also others? In the latter case it is hard to see how to avoid holism, as then a proof of $A\supset B$ might employ arbitrarily complex sentences, and the former looks like a rather artificial restriction, as the meaning of a constant must be given for every case. Choosing this option is rather to deny that the meaning of a logical constant is completely determined by the rules governing it.

[17] Actually, if quantifiers are predicates, the argument also establishes that Geach's account fails for higher level predicates.

say anything. Now in classical logic, ⊃ and ∨ apparently *are* governed by self-justifying, harmonious introduction and elimination rules, hence their meanings ought to be completely determined by those rules. However, A∨A⊃B, A⊃B⊃A⊃A and the like are theorems which cannot be proved by appealing to the rules for ⊃ and ∨ only and thus the rules for ⊃ and ∨ can *not* tell us everything about these constants. What calls for an explanation is that on the one hand one cannot say *more* about the logical constants than what can be said through the rules governing them, but on the other hand one is obliged to say more about the constants in order to account for the existence of certain theorems.

Now trivially, on the account of negation put forward here, I have to understand negation in order to understand any operation on sentences, as I have to understand negation if I am to understand anything at all. Furthermore, for a classical logician, a proof by classical *reductio ad absurdum* is as good a verification as anything and of course this manoeuvre might contribute to the meaning of any other expression of the language; this is just one way of putting the principle *eadem est scientia oppositorum*.[18] So it might appear that it can hardly matter that, in view of such theorems as A⊃B⊃A⊃A and (A⊃B)∨(B⊃A), to understand ⊃ and ∨ I need to understand negation. But this is just to repeat one horn of the dilemma and the point remains untouched that on the one hand, all there is to determine the meanings of the positive logical constants are the usual, positive rules governing them, but on the other hand in view of certain theorems these rules cannot be sufficient to tell us everything about a constant and negation does male a contribution to their meaning. If a double negation does not add to the content of a sentence, then employing double negation elimination or some equivalent rule in a derivation should not add to the content of the derived sentence either.

6. *The Significance of Conservativeness*

Presumably it is in any case incoherent to accept all of a) a theorem is true only in virtue of the logical constants occurring in it, b) the meaning of a logical constant is determined completely by the usual positive rules governing it, c) to accept A⊃B⊃A⊃A, (A⊃B)∨(B⊃A) and A∨(A⊃B) *etc*. as theorems. To argue that these must be theorems because A is either true or false, is again to deny that the meanings of the positive logical constants are determined completely by the rules governing them; it amounts to using A∨~A in the proof. From this perspective, the dilemma does not come as a surprise. But had the account of the logical constants in Section 4 worked, the incoherence would not have

[18] For this reason a classical logician is likely to hold anyway that the rules of a constant do not completely determine its meaning, *i.e*. the classical logician is likely reject the proof theoretic justification of logical laws. *Cf*. footnote 11.

occurred, as then there would also have been the negative rules to determine the meanings of the logical constants. What is less obviously incoherent is the account of the meanings of complex empirical sentences given *eadem est scientia oppositorum* while at the same time holding that the meanings of the logical constants are determined only by the rules governing them plus the whole of classical logic. But incoherent it is. The problem is, that we still need an account of direct and indirect verification; of direct, canonical verifications that proceed according to the meaning of A*B and other, indirect verifications that establish their conclusions as true only because they could in principle be turned into a canonical verification. The canonical verification of a sentence proceeds in accordance with how the meaning of the sentence is to be specified in a theory of meaning. *Ex hypothesi*, the meanings of the positive logical constants are given by their introduction rules. On Dummett's original account, this meant that the canonical verification had to end with an application of one of the introduction rules governing its principal operator. In Section 2 I argued that the fundamental assumption is exorbitant in the generality Dummett wishes it to have and that there is hardly any reason not to end the verification of a sentence containing negation with a negation rule. In fact, as negation is the only constant we have considered so far whose introduction and elimination rules are not intrinsically harmonious, it is the only constant that may have to be appealed to in a proof of a theorem which uses rules for constants that do not occur in the theorem, such that this use of the constant cannot be avoided through the procedure of levelling local peak. Using Geach's principle in Section 3 explained why it is not mysterious that sentences *containing negation* or *not containing logical constants* can be verified by refuting their negation without violating compositionality. No question arises concerning where to put A*B in the partial ordering, if it can only be verified by verifying its double negation. The question now is rather how it could have a meaning at all that is determined by the rules governing *. If we are to give the meanings of the logical constants proof-theoretically and are to justify the logical laws governing them accordingly, we are bound to accept that canonical proofs or verifications of sentences not containing negation ought not to use rules for negation and ought to end with an application of an introduction rule for their principal connectives. Because the account of section 4 failed and *eadem est scientia oppositorum* does not hold for the positive logical constants, all there is to know about them *are* the rules governing them. Hence the refutation of the negation of A*B must count as an indirect verification. The appeal to negation rules in a verification is unproblematic only in as much as negation is fundamental to understanding, but not any further; only in as much as *eadem est scientia oppositorum*. The argument of Sections 4 showed that for *complex* sentence even on

Geach's account classical *reductio ad absurdum* is only an equally good method of verification as anything if the sentence verified contains negation. Geach's principle does *not* hold for the logical constants. The incoherence lies in the fact that it has not been secured that any indirect verification may in principle be turned into a canonical one. What is mysterious about complex contingent sentences not containing negation that can be verified only by verifying their double negation, is that they cannot have a canonical verification that proceeds in accordance with how their meaning is to be determined and hence could not have a stable meaning.

The only cases in which the proof of a theorem or the verification of a sentence does not end with an application of an introduction rule for the principal constant ought to be those where negation occurs in the theorem or sentence. I ought to be able, that is, in the proof of a theorem or the verification of an empirical sentence to appeal only to rules governing the constants that occur in it. And that is just to say that negation ought to be a conservative extension over the positive fragment of a logic. And this is not trivial, even if nobody would ever accept a)-c) above, that even if negation is fundamental to understanding, we an still demand the rules governing it to be conservative over the positive fragment of logic. The dilemma of section 5 is precisely due to the fact that classical negation is not a conservative extension over the positive fragment of intuitionistic logic. Adding classical negation to the positive intuitionistic logic enables one to prove theorems not containing negation that have not been provable before, and it thus disturbs the meanings of the positive connectives, as given by the rules governing them, as on the one hand, the new theorems are, qua theoremhood, true only in virtue of the constants occurring them, but at the same time are not provable with reference exclusively to these rules. The problem with a non-conservative negation is that employing it in a calculus some *other* logical constants may cease to be self-sufficient and may cease to be completely determined, for their meaning, by the rules governing them: a non-conservative negation influences the behaviour of other operations. Thus the demand for total harmony that Dummett puts on logical constants holds even in the case of negation, which is fundamental to understanding.

7. $\supset$

The solution to the dilemma of Section 5 and the incoherence mentioned in Section 6 lies readily at hand: rather than blaming ~, as does Dummett, the problem may be located in classical $\supset$.[19] $\vee$ appears to be problematic not because of $A\vee{\sim}A$, but rather because of $(A\supset B)\vee A$ and $(A\supset B)\vee(B\supset A)$, but *all*

[19] Given we want to stick to classical negation rules. *Cf.* Section 9.

theorems of the propositional calculus that are problematic in this way contain ⊃: if a sentence contains at most the constants &, ∨, it cannot be a theorem. It must either contain ~ or ⊃. As negation is unproblematic and its meaning cannot be given proof-theoretically, we can blame ⊃. The other constants are, as they are, perfectly in order and can be viewed as completely determined by their positive rules and negation is – quite uninterestingly – conservative over the ∨, & fragment of classical logic. Contrary to that, classical ⊃ cannot have an independent meaning: the meaning of A⊃B cannot be computed with reference to A, B and ⊃ alone, but reference to ~ is needed in addition. It is not the *operation on sentences* alone, as specified by the rules for it in a system of natural deduction, that gives this symbol its meaning: the meaning of ⊃ cannot be given proof-theoretically. The conclusion is that classical ⊃ cannot be a genuine logical constant.

A synthesis of Geach and Dummett does not justify all the classical logician wants to have justified. What falls out as unjustified is precisely the implication connective of classical logic (and possibly disjunction, depending on how we conceive of the meaning of ∨ as being determined). If implication is the heart of logic, then the whole classical approach is responsible for the predicament. Of course it is not *per se* a problem that the meaning of ⊃ cannot be given completely by the rules governing it, but negation is needed in addition: we can introduce expressions into the language that depend, for their meaning, on expressions already there, *e.g.* by explicit definition. We can make explicit the way in which negation enters the meaning of ⊃, namely in viewing it as defined as ~(A&~B) or ~A∨B. But this is of no real use in the case of the logical constants, as they are fundamental in the sense that they do *not* depend on any other expressions, not even negation, as argued. Viewing A⊃B as so defined is to admit that it is not a genuine logical constant. Of course, in classical logic there also is nothing in ~(~A&~B), *e.g.*, that is not in A∨B, but the point here is that one can learn the meaning of ∨ and the meaning of & through learning the rules for them, and then *discover* that certain equivalences hold, whereas you can't learn the meaning of ⊃ without learning its interaction with ~, and it is this interaction that shows that really ⊃ is something else in disguise.[20]

[20] If you find the point about & and ∨ weak, I am happy to claim that classical logic is only about ~ and &. I simply haven't made up my mind, as I am not sure whether the meaning of ∨ is to be determined by introduction or by elimination rules and if by the elimination rule, how this is to be achieved. There are proofs of A∨(A⊃B) and (A⊃B)∨(B⊃A) that end with an application of the elimination rule for ∨, given A∨~A. If we agreed that the meaning of ∨ is given accordingly, these theorems are in a similar way problematic as is A⊃B⊃A⊃A, as then ~ enters the meaning of ∨. Adopting this position, only & is a proper two-place connective of classical propositional logic.

8. *Classical Logic, Negation and Relevance*

The classical logician could say that every logical law contains negation, because they are all equivalent to ones using only conjunction/disjunction and negation.[21] It would be more suitable to drop $\supset$ completely and formulate the logic only with &/∨ and ~ (and for quantificational logic, the universal quantifier). Then classical logic satisfies all of Dummett's criteria. This does not come as a surprise, of course. Hardly anyone would quarrel with that, as at least classical logic with only & and ~ can be interpreted in, *e.g.*, intuitionistic and relevant logics. The point here would be that classical logic is, as it is, perfectly in order as long as we are aware what its theorems state. Nevertheless, if it's *that* that classical logic does, then we might well ask whether it is all there is to logic. The problem now is less whether classical logic is all right, but rather whether classical logic tells us *all there is* about logic. And certainly, by Dummett's criteria, the answer has to be no: classical logic is not all of logic. In particular, if implication tells us something different from ~(A&~B) or ~A∨B, then classical $\supset$ is not an implication. For a relevant implication, sentences corresponding to the problematic theorems of classical logic – $(A\rightarrow B)\vee(B\rightarrow A)$, $A\rightarrow B\rightarrow A\rightarrow A$, $A\vee A\rightarrow B$ – are not theorems. This suggests the $\rightarrow$ of **R** as a better implication connective than classical $\supset$ and indeed the $\rightarrow$ of **R** can be shown to be governed by harmonious rules of inference. There are two possibilities for a local peak with $\rightarrow$.

1. The minor premise of the elimination rule of the local peak is a hypothesis of a subproof.

i	$\mid\vdash A_{\{k\}}$	hyp
i+n	$\mid\mid\ B_a$	derivation of B from A
i+n+1	$\mid\ A\rightarrow B_{a-\{k\}}$	$\rightarrow$I i, i+n
i+n+2	$\mid\vdash A_{\{l\}}$	hyp
i+n+3	$\mid\mid\ A\rightarrow B_{a-\{k\}}$	reit i+n+1
i+n+4	$\mid\mid\ B_{(a-\{k\})\cup\{l\}}$	$\rightarrow$E i+n+2, i+n+3

There may be steps before i and after i+n+4. l and k are new subscripts, as new hypotheses get new subscripts. $k\in a$, as otherwise i+n+1 wouldn't be a legitimate step. Thus the local peak affects the interchange of k and l in the relevant indices of B. The local peak can be levelled thus: Delete i+n+1 to i+n+4 and substitute k for l throughout the rest of the proof and change the comments on the left of the

[21] Let's leave disjunction out for simplicities sake. *Cf.* footnote 20.

formulas according to the deletion and substitution. It is easily seen that the list of formulas thus modified is still a derivation in **R**.

2. The minor premise of the local peak is derived from some hypotheses.

i	$\mid A_b$	derivation of A
i+1	$\mid \vdash A_{\{k\}}$	hyp
i+n	$\mid \mid \;\; B_a$	derivation of B from A
i+n+1	$\mid \;\; A \rightarrow B_{a-\{k\}}$	→I i+1, 1+n
i+n+2	$\mid \;\; B_{(a-\{k\}) \cup b}$	→E i, i+n+1

There are steps before i and there may be some after i+n+2. To level the local peak, delete i, i+n+1 and i+l+2, lower the rank of the subproof i+1 to i+n by 1 and replace the elements of b for k throughout the derivation. $A_{\{k\}}$ now becomes A_b: substitute 'hyp' right of it by whatever stands right of A_b in i of the original proof. Change 'reit' to the right of steps i to i+l to 'rep' and change all the other comments according to the deletion and substitution. Again it is easily seen that the resulting list is a derivation in **R**. *Ex hypothesi*, k∈a, as otherwise i+n+1 would be illegitimate. Hence the result reached in step i+n is a derivation of $B_{(a-\{k\}) \cup b}$ from whatever A_b is derived from.

This reduction procedure works for **R** as well as for **E**, so it cannot be that the meanings of both arrows are given exclusively by the rules →I and →E. Now **E** results from a certain restriction on the rule reit, namely that only formulas of the form A→B may be reiterated, whereas in **R** reit is a necessity due to the way proofs are set up. We may conclude that the meaning of the arrow of **E** is not given exclusively by the rules →E and →I, but also by the restriction on reit. The → of **R**, *e.g.*, is governed by self-justifying rules of inference, by Dummett's criteria, hence the relevant implication of **R** must count as a logical constant. Contrary to ⊃, as in **R** negation is conservative over the positive fragment, the meaning of → has a coherent meaning that is completely determined by the rules governing it. By similar considerations as those just given, it may be shown that the other connectives of $\mathbf{R}^+$ too are governed by self-justifying rules of inference.[22] If one ignores the problem, which needs to be

[22] There may be a problem with distribution. On the one hand, it can be shown be justified proof-theoretically, but on the other hand, dropping it would appear to change the meaning of & and v, and hence their meanings cannot be completely

addressed for any logic, of how to apply the concepts of logic to ordinary language, in particular of how to apply the concept of conservative extension to real languages, then in relevance logic, it is true generally that any sentence derived by employing negation rules will itself contain negation. The conservative extension of $\mathbf{R}^+$ by ~ secures that if ~~A is verified, there could in principle have been a verification of A that avoids the detour through ~. A proof of ~~A may in itself count as a proof of A. The relevantist can claim that sentences not containing negation have a stable meaning, which is not affected by ~~A→A.

What I should propose, then, is that relevance logic suggests itself as a justified logic precisely at a point where other logicians and philosophers most often problems with relevance logic, namely negation. We actually can, in building our logic, start with the implication connective, as do Anderson and Belnap in their book, and note that the implication connective of **R** is justified by Dummett's criteria and its meaning determined completely by the rules stipulated for it. We now observe that if you want to have a decent implication connective, you've got to have one that is not disturbed by negation, *i.e*. negation needs to be conservative with respect to implication (and, of course, all the other connectives, too). But if, in the negation rule Contrap, we drop the requirement that k∈b and hence make *ex contradictione quodlibet* a special case of it, the negation rules are *not* conservative over the implication fragment, for then A→.B→A is provable, which wasn't provable before:

1	$\vdash\ A_1$	hyp
2	$\mid\vdash\ B_2$	hyp
3	$\mid\mid\mid\ {\sim}A_3$	hyp
4	$\mid\mid\mid\ \ A_1$	reit 1
5	$\mid\mid\mid\ {\sim}B_{1,3}$	"Contrap" (*ex contradictione quodlibet*) 3, 4[23]
6	$\mid\mid\ A_{1,2}$	Contrap 2, 3-5,
7	$\mid B{\rightarrow}A_1$	→I 2, 6
8	$A{\rightarrow}.B{\rightarrow}A$	→I 1, 7

For similar reasons, of course, in a full system A&~A→B and B→Av~A must not be theorems (the equivalence of A&~A with ~(Av~A) resulting from the negation rules plus the rules for v and &).

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governed by their introduction and elimination rules. A more detailed argument for **R** would certainly have to discuss this problem. Suffice it to say that **R** may still not be an ideal logic.

[23] Dropping the requirement that k be in a amounts to allowing vacuous discharge of assumptions.

Hence, if to account for this behaviour of negation in a semantics we need some non-standard treatment of negation, then be it: that's a simple consequence of the justified treatment of implication. What is significant about this approach is that once the underlying logic is changed from classical to relevant logic, what is essentially the same connective (~) – from the point of view of its interpretation: 'creating the opposite content' as mirrored in theorems like ~~A→A and A∨~A – changes from being non-conservative to being conservative.

9. *Metaphysics or Pluralism?*

The situation we now face is this. The point of adopting Geach's argument was to reject a certain *motivation* for intuitionistic negation rules. Geachs line on a surface, however, is quite independent of any logic: the classical logician would in addition make the assumption that every object is on either or other side but never on both, never in neither, but if we do not assume that, the picture does not preclude objects from lying exactly on the line – those which both are and are not *F* – or from being nowhere – those that are neither *F* nor *~F*, or from not being determinately on either side, those for which *tertium non datur* can neither be asserted nor denied. The picture is pointless in a philosophical discussion on the validity of logical laws, as it presupposes certain set theoretic principles and hence a logic. It's content is less to argue for a certain logic, but rather the point that a predicate and its negation are of the same complexity and that the meaning of negation cannot be given proof-theoretically. The question remaining to be answered is this: which is the metaphysically illuminating part of logic? There was no direct argument against intuitionistic logic and indeed intuitionistic logic must be accepted as all right. For, given the main point about the logical constants is that they be governed exclusively by the rules for them in a calculus, clearly the positive intuitionistic rules qualify as forming a coherent positive logic. But to keep it intact, we need rules for negation different from the classical ones. We ended up with a pluralism of logics, contrary to what one would guess from Dummett's writings, who hardly ever envisages the possibility of there being more than One True Logic.[24] Double negation elimination and *tertium non datur* are valid for classical logic and relevance logic, but not for intuitionistic logic is the lapidary conclusion. We have not found a way of deciding the metaphysical issue from the proof-theoretic stance.[25]

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[24] The possibility of logical pluralism is included in Dummett's characterisation of the Realism/Anti-Realism Debate relative to certain parts of language; *cf.*, *e.g.*, LBM 14.

[25] We must accept at genuine pluralism, where intuitionistic and classical logic coexist side by side. There cannot be a super logic combining both logics: Let ⊃ be intuitionistic implication. Let's define ¬A as intuitionistic negation: A⊃⊥, where ⊥ is

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governed by the sole rule *ex falso quodlibet* and ~A as classical negation governed by the rules *ex contradictione quodlibet* and 'from Γ, ~A⊢A to infer Γ⊢A'. Then we can proof ¬¬A⊢A:

$$\dfrac{\dfrac{\dfrac{(A\supset\bot)\supset\bot \qquad \dfrac{\dfrac{\overline{\sim A}^{\,2} \quad \overline{A}^{\,1}}{\bot}}{A\supset\bot}{}^{1}}{\bot}}{A}}{A}{}^{2}$$