Fixed Point Models for Theories of Properties and Classes

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Today's Plan

Our Target

Model Construction

Classifying Class Theories

Order and Continuity

Order Models

OUR TARGET

Class Abstraction

$$a \in \{x : \phi(x)\} \text{ iff } \phi(a)$$

Property Abstraction

$$a \; \varepsilon \; \lambda x.\phi(x) \;\text{ iff }\; \phi(a)$$

Russell's Paradox

$$\{x : x \notin x\} \in \{x : x \notin x\} \text{ iff } \{x : x \notin x\} \notin \{x : x \notin x\}$$

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$$\{x : F(x \in x)\} \in \{x : F(x \in x)\} \text{ iff}$$
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The Heterological Paradox

$$\lambda x.(x \not\varepsilon x)\ \varepsilon\ \lambda x.(x \not\varepsilon x)\ \text{ iff }\ \lambda x.(x \not\varepsilon x) \not\varepsilon \lambda x.(x \not\varepsilon x)$$

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(Extensionality will not play a significant role in what follows.)

MODEL CONSTRUCTION

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Defining validity.

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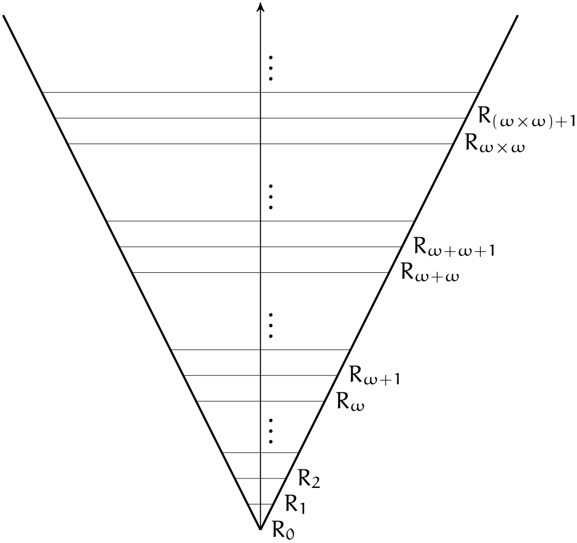
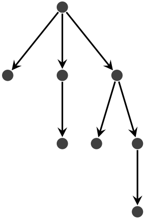
ZFC and its Cousins: The Iterative Conception of Set



ZFC and its Cousins: Anti-Foundation



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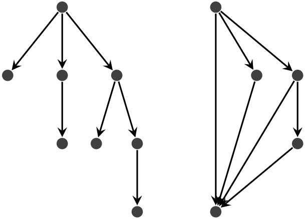

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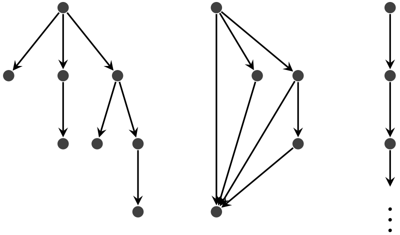

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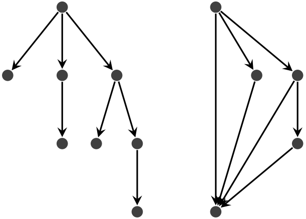

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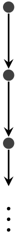

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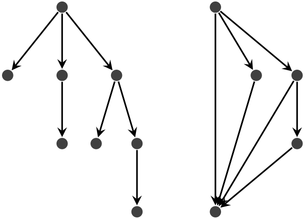

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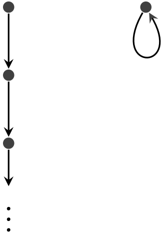

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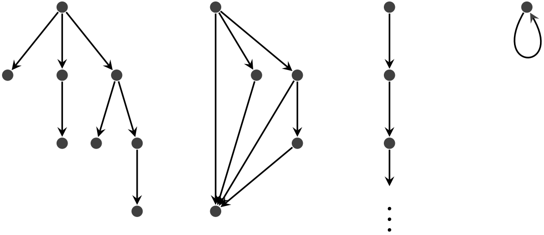

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These models are good for (1) *relating* ZFC to AFA,

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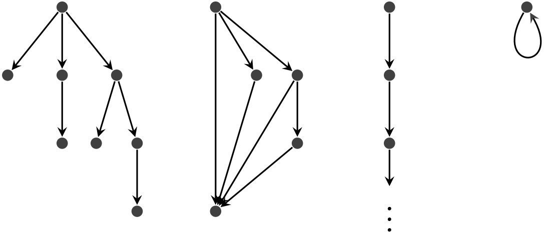

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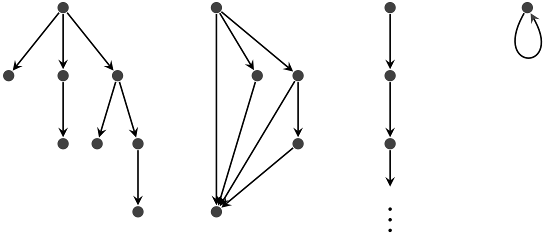

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These models are good for (1) *relating* ZFC to AFA, (2) motivating a choice of the anti-foundation axiom, and (3) explaining what the theory could be *about*.

Untyped λ Calculus

If x is a *variable* and M is a *term*, $\lambda x.M$ is a *term*.

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$$(\lambda x.M)N = M[x := N].$$

Models of the Untyped λ Calculus

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$$D \qquad D \to D$$

Models of the Untyped λ Calculus

$$\mathrm{D} \cong \mathrm{D} \to \mathrm{D}$$

You bump up against *Cantor's Theorem*.

Models of the Untyped λ Calculus

$$\mathrm{D} \cong [\mathrm{D} \to \mathrm{D}]$$

The Scott Construction

$[D \to E]$: the *order preserving functions* from $(D, \sqsubseteq)$ to $(E, \sqsubseteq)$.

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$[D \longrightarrow E]$: the *order preserving functions* from $(D, \sqsubseteq)$ to $(E, \sqsubseteq)$.

It's ordered too: $f \sqsubseteq g$ iff $(\forall x)(f(x) \sqsubseteq g(x))$.

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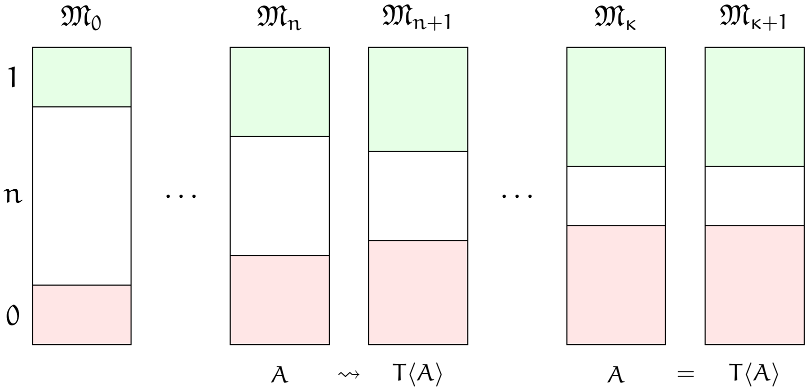
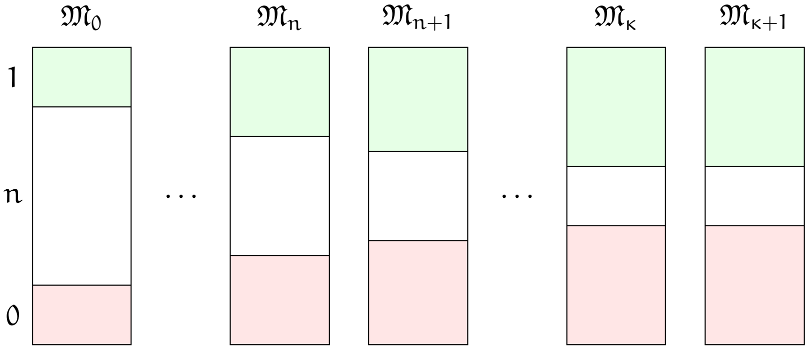
Let D_∞ be the limit: $D_\infty \cong [D_\infty \to D_\infty]$.

This is a model of the untyped λ calculus.

Truth Theories: Kripke, Woodruff, Gilmore, Brady



Class Theories



$\phi(a) \rightsquigarrow a \in \{x : \phi(x)\}$ $\qquad$ $\phi(a) = a \in \{x : \phi(x)\}$

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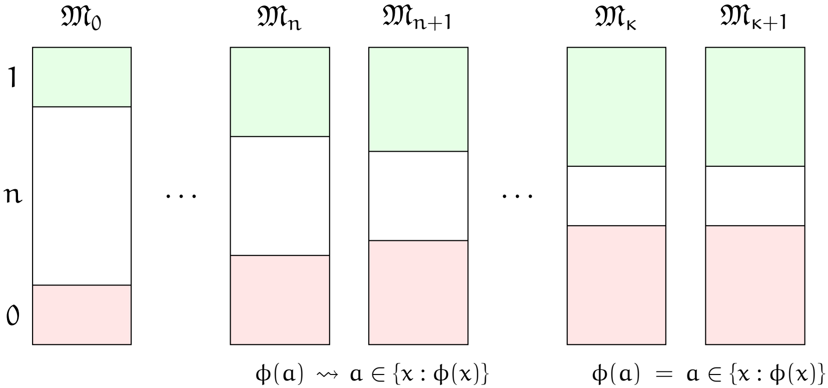
$\mathfrak{M}_0$
$\mathfrak{M}_n$
$\mathfrak{M}_{n+1}$
$\mathfrak{M}_\kappa$
$\mathfrak{M}_{\kappa+1}$
1
n
0
$\ldots$
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This is *not* like the other model constructions:
the domain is constant—the terms $\{x : \phi(x)\}$.

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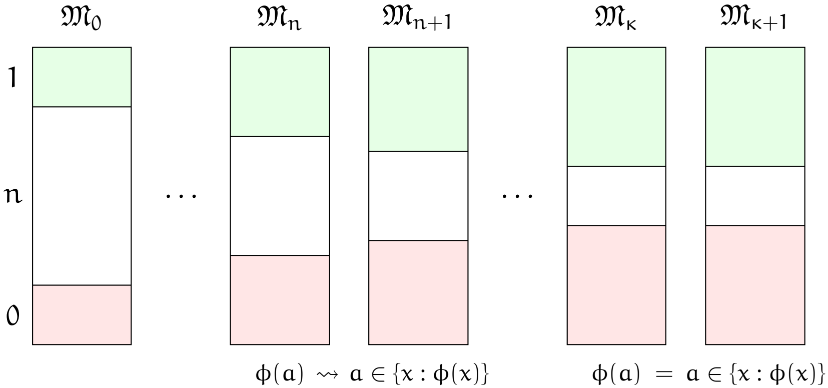
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the domain is constant—the terms $\{x : \phi(x)\}$.

This shows what the theory is *about* in only a very weak sense.

CLASSIFYING CLASS THEORIES

Underlying Logic: Negation

Gaps or *Gluts*?

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Gaps or *Gluts*?

Paraconsistent or *Paracomplete*?

Underlying Logic: The Conditional

Do we have a conditional in the language?

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Do we have a conditional in the language?

And if so, what is it like?

Underlying Logic: Not *that* important

These decisions are not *that* important.

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The logic must allow for *fixed points*.

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The logic must allow for *fixed points*.

For *any* sentence context $\mathsf{F}(-)$, we need to allow for some p to be *equivalent to* $\mathsf{F}(p)$.
If $c =_{df} \{x : \mathsf{F}(x \in x)\}$, then $c \in c$ *iff* $\mathsf{F}(c \in c)$

Semantic Values

$$D$$

- D: the *ordinary* domain.

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Semantic Values

$$D \qquad \Omega$$

- D: the *ordinary* domain.
- Ω: truth values.

Semantic Values

$$C \qquad\qquad D \to \Omega$$

- D: the *ordinary* domain.
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Extensionality

We won’t focus on extensionality here.

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But we’ll *identify* classes by their extensions as much as possible.

Sharpening our Target

$$C \cong [C \cup D \to \Omega]$$

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$$C \cong [C \cup D \to \Omega]$$

$\phi(x)$ gives a function $[C \cup D \to \Omega]$.
So we can find a class C to *match*.

$a \in \{x : \phi(x)\}$ has the *same* value in Ω as $\phi(a)$.

ORDER AND CONTINUITY

Underlying Logic: Preservation

$$\sqsubseteq$$

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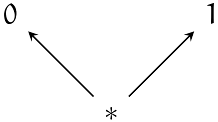
(If $x \sqsubseteq x'$ and $y \sqsubseteq y'$ then $x \sharp y \sqsubseteq x' \sharp y'$, etc.)

Preservation on candidates for Ω



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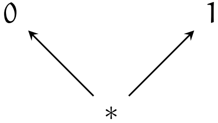
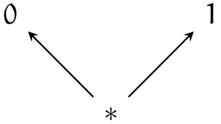


K3 or LP

Preservation on candidates for Ω



K$_3$ or LP, but not Ł$_3$

In Ł$_3$, $* \to *$ is 1; but $1 \to 0$ is 0

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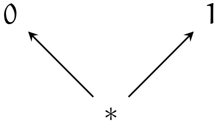

K_3 or LP, but not $Ł_3$ or RM_3

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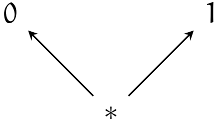

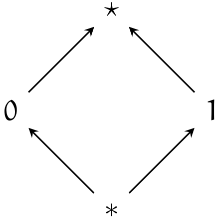

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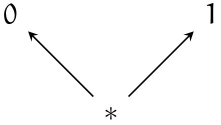

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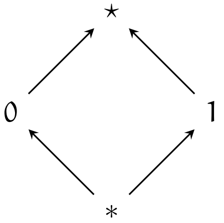

FDE, but no robust conditionals.

Preservation on candidates for Ω

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\end{tikzcd}$$

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Similar behaviour here.

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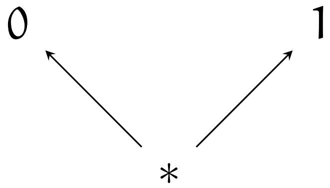
Even $\{0, 1\}$ can be ordered: $0 \sqsubseteq 1$. Then $\wedge, \vee, 0, 1$ are order preserving, but $\neg$ and $\supset$ are *not* order preserving.

3: our choice of Ω



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(I *really* don’t care if you think of $*$ as *true*, or as *untrue*.)

ORDER MODELS

Defining Order Models

Given an order algebra Ω, and a domain D of urelements

$\langle C, \sqsubseteq, \Uparrow, \Downarrow \rangle$ is a $\langle D, \Omega \rangle$ *order model* iff

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– Write '$\Uparrow(c)$' as '$c_{\Uparrow}$' and '$\Downarrow(f)$' as '$f_{\Downarrow}$.' So $c_{\Uparrow\Downarrow} = c$ and $f_{\Downarrow\Uparrow} = f$.

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- If $b \in C \cup D$ and $c \in C$, then $c_\Uparrow(b)$ tells you whether b is in c.

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$$\begin{aligned} x_{\Uparrow}(y) \sqsubseteq x_{\Uparrow}(y') &\quad — \quad y \sqsubseteq y' \text{ and } x_{\Uparrow} \text{ is order preserving.} \\ x_{\Uparrow} \sqsubseteq x'_{\Uparrow} &\quad — \quad x \sqsubseteq x' \text{ and } \Uparrow \text{ is order preserving.} \\ x_{\Uparrow}(y') \sqsubseteq x'_{\Uparrow}(y') &\quad — \quad \text{by the definition of } \sqsubseteq \text{ for functions.} \end{aligned}$$

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- An assignment α, takes variables to values in $C \cup D$.
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- (We abbreviate this $[\![x]\!]$ when $\mathfrak{M}$ and α is clear.)

Interpreting a Language

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- An assignment α, takes variables to values in $C \cup D$.
- $[\![x]\!]_{\mathfrak{M},\alpha} = \alpha(x)$ is the interpretation of the variable x.
- (We abbreviate this $[\![x]\!]$ when $\mathfrak{M}$ and α is clear.)
- $[\![s \in t]\!]_{\mathfrak{M},\alpha}$ is $[\![t]\!]_{\Uparrow}([\![s]\!])$ when $[\![t]\!] \in C$,
  and is 0 when $[\![t]\!] \in D$.

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– (Connectives and quantifiers are order preserving functions on 3 or $[C \cup D \to 3]$.)

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$$[\![\{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha} = (\lambda v.[\![\phi(x)]\!]_{\mathfrak{M},\alpha[x:=v]})_{\Downarrow}$$

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$$[\![t \in \{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha} \quad = \quad [\![\{x : \phi(x)\}]\!]_{\alpha_{\Uparrow}}([\![t]\!]_{\alpha})$$

(I’ve dropped reference to $\mathfrak{M}$ as it is constant throughout.)

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Logical Constants

$0 \qquad\qquad 1$

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$$0 \qquad * \qquad 1$$

Λ, V and $\rlap{V}\Lambda$

$$\Lambda \quad = \quad \{x : 0\}$$

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$x \in \Lambda$ is always false.

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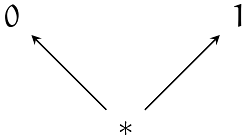
Ordering the Classes



Ordering the Classes

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In fact, $[\![\mathbb{X}]\!] \sqsubseteq c$ for every class $c \in C$.

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In fact, $[\![\mathbb{X}]\!] \sqsubseteq c$ for every class $c \in C$.

From now, we'll use '$\emptyset$', 'V' and '$\mathbb{X}$' as both the *class terms* in the language, and as their denotations, names for objects in C.

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In a model $\mathfrak{M}$, a class c is SHARP iff for each object b in $C \cup D$ $c_{\Uparrow}(b)$ takes the value 0 or 1

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Λ and V are sharp.

$\mathbb{X}$ is *not* sharp.

Almost No Classes are *Sharp*

If $c_{\Uparrow}(b) = 1$ and $c_{\Uparrow}(b') = 0$, then $c_{\Uparrow}(\mathbb{X}) = *$.

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It follows that $c_{\Uparrow}(\mathbb{X}) = *$

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Once a class *includes* something
and *excludes* something,
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it is *indecisive* about $\mathbb{X}$.

It follows that there are no *crisp singletons*:
objects $\{a\}$ for which $[\![a \in \{x\}]\!] = 1$
and $[\![b \in \{x\}]\!] = 0$ *for all other* b.

Singletons and Anti-Signetons: $\{t\}$ and $\}t\{$

- $[\![\{t\}]\!]_\alpha$: (the class representative of) the function that
  - assigns 1 to x iff $[\![t]\!]_\alpha \sqsubseteq x$,
  - and 0 to x iff there is no z where $x \sqsubseteq z$ and $[\![t]\!]_\alpha \sqsubseteq z$,
  - and $*$ otherwise.
- $[\![\}t\{]\!]_\alpha$: (the class representative of) the function that
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- Find perspicuous ways to *construct* order models.
- Relate these constructions to other known model constructions.
- *Axiomatise* the logic of order models.
- Examine different *motivations* of order models.

Model Construction: D_1 — $[* \to \Omega]$

$$\begin{tikzcd}
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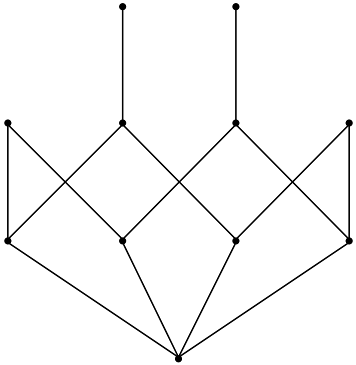
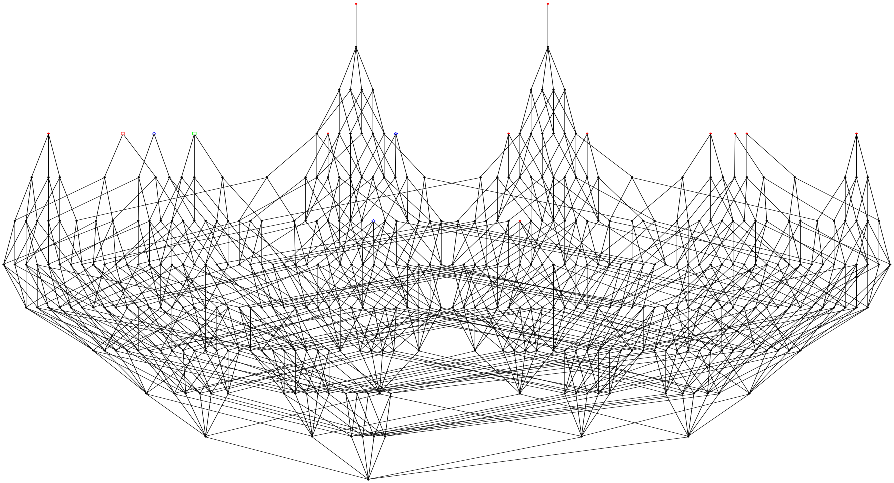
\end{tikzcd}$$

Model Construction: D_2 — $[D_1 \to \Omega]$



Model Construction: D_3 — $[D_2 \to \Omega]$



THANK YOU!

`http://consequently.org/presentation/2016/fixed-point-models-fnclmp-2016`

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