Fixed Point Models for Theories of Properties and Classes

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THE UNIVERSITY OF MELBOURNE

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Our Target

Model Construction

Classifying Class Theories

Order and Continuity

Order Models

OUR TARGET

Class Abstraction

$$a \in \{x : \phi(x)\} \text{ iff } \phi(a)$$

$$a \mathrel{\varepsilon} \lambda x.\phi(x) \text{ iff } \phi(a)$$

Russell's Paradox

$$\{x : x \notin x\} \in \{x : x \notin x\} \text{ iff } \{x : x \notin x\} \notin \{x : x \notin x\}$$

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In general,

$$\{x : F(x \in x)\} \in \{x : F(x \in x)\} \text{ iff}$$
$$F\big(\{x : F(x \in x)\} \in \{x : F(x \in x)\}\big)$$

The Heterological Paradox

$$\lambda x.(x \not\varepsilon x)\ \varepsilon\ \lambda x.(x \not\varepsilon x)\ \text{ iff }\ \lambda x.(x \not\varepsilon x) \not\varepsilon \lambda x.(x \not\varepsilon x)$$

$$\lambda x.(x \not\varepsilon x) \;\varepsilon\; \lambda x.(x \not\varepsilon x) \text{ iff } \lambda x.(x \not\varepsilon x) \not\varepsilon \lambda x.(x \not\varepsilon x)$$

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Extensionality

If a and b have the same members, then $a = b$.

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(Extensionality will not play a significant role in what follows.)

MODEL CONSTRUCTION

Defining validity.

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Providing *counterexamples*,
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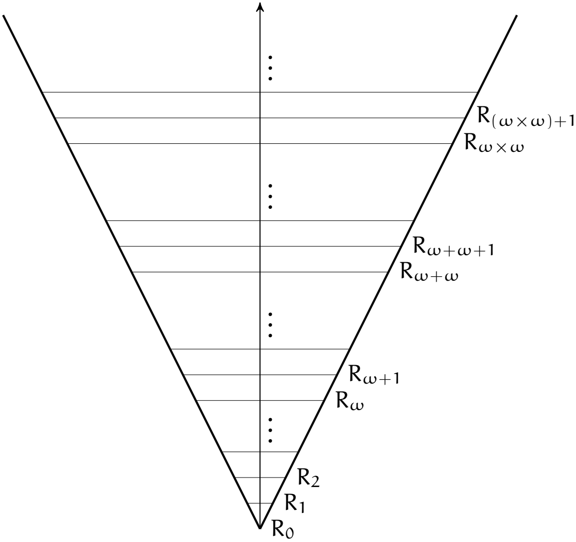
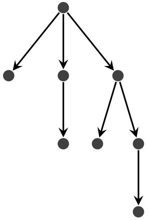
ZFC and its Cousins: The Iterative Conception of Set



ZFC and its Cousins: Anti-Foundation



$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

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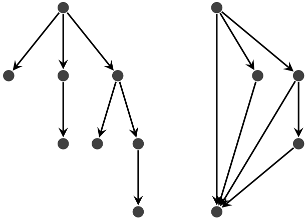

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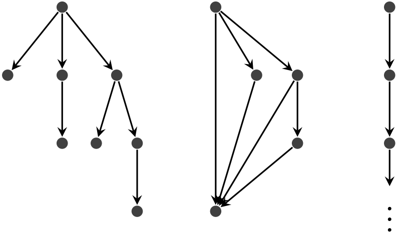

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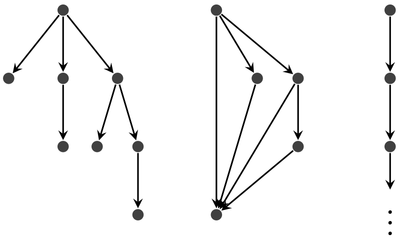

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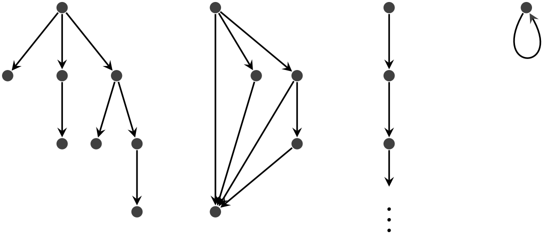

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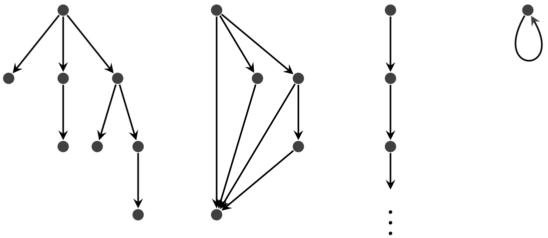

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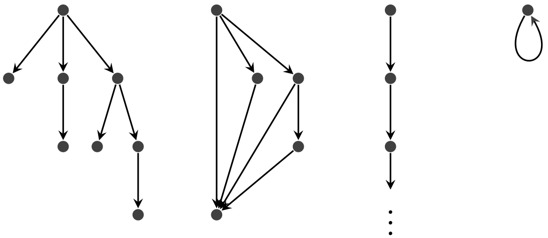

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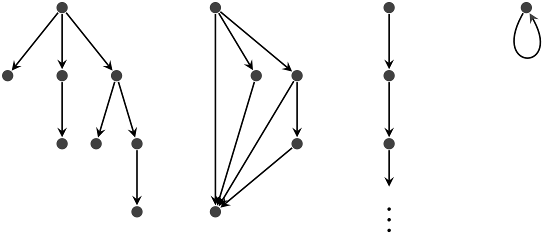

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These models are good for (1) *relating* ZFC to AFA, (2) motivating a choice of the anti-foundation axiom, and (3) explaining what the theory could be *about*.

If x is a *variable* and M is a *term*, $\lambda x.M$ is a *term*.

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$$(\lambda x.M)N = M[x := N].$$

Models of the Untyped λ Calculus

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$$D \qquad D \to D$$

$$D \cong D \to D$$

You bump up against *Cantor's Theorem*.

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The Scott Construction

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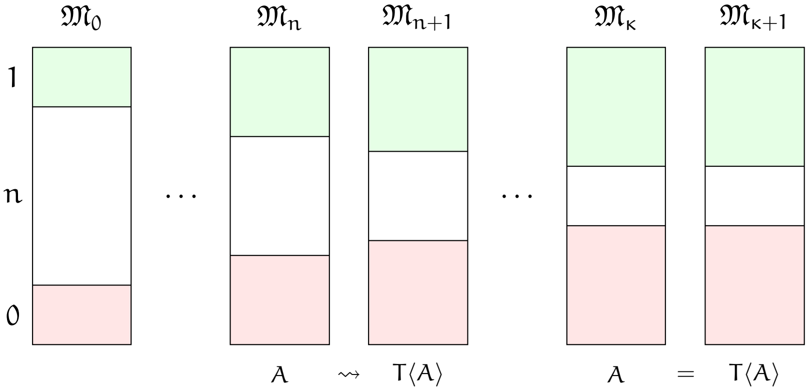
Embed D_i into $[D_i \to D_i] = D_{i+1}$
(Use the constant functions.)

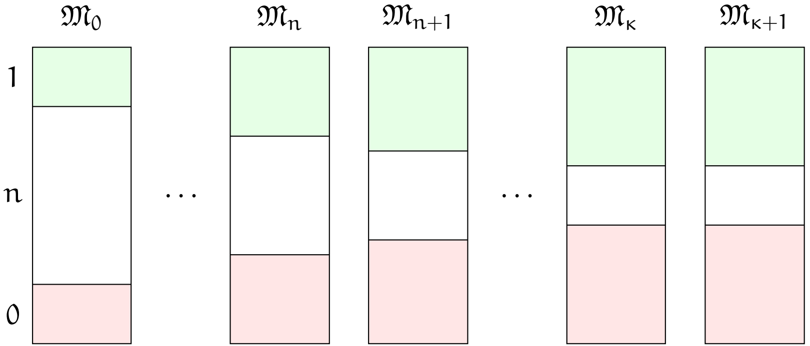
Let D_∞ be the limit: $D_\infty \cong [D_\infty \to D_\infty]$.

This is a model of the untyped λ calculus.

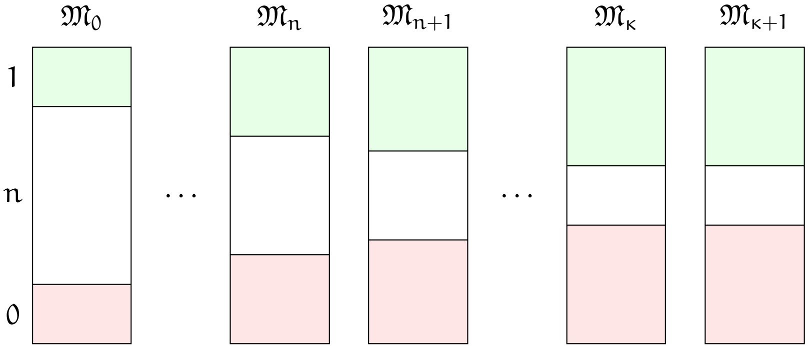
Truth Theories: Kripke, Woodruff, Gilmore, Brady



Class Theories



$\phi(a) \rightsquigarrow a \in \{x : \phi(x)\}$ $\qquad$ $\phi(a) = a \in \{x : \phi(x)\}$



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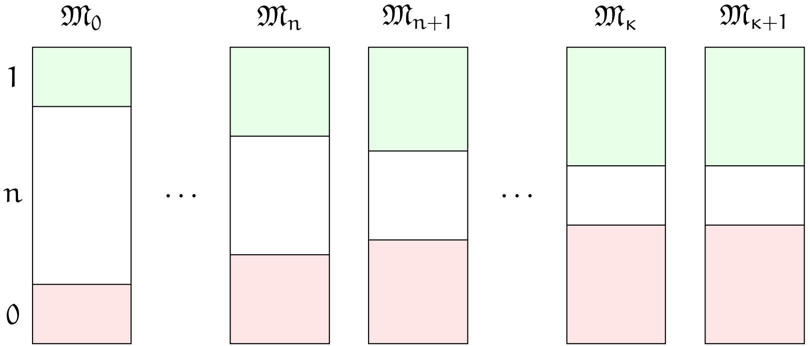

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This shows what the theory is *about* in only a very weak sense.

CLASSIFYING CLASS THEORIES

Gaps or *Gluts*?

Underlying Logic: Negation

Gaps or *Gluts*?

Paraconsistent or *Paracomplete*?

Do we have a conditional in the language?

Underlying Logic: The Conditional

Do we have a conditional in the language?

And if so, what is it like?

Underlying Logic: Not *that* important

These decisions are not *that* important.

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The logic must allow for *fixed points*.

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The logic must allow for *fixed points*.

For *any* sentence context $F(-)$, we need to allow for some p to be *equivalent to* $F(p)$.
If $c =_{df} \{x : F(x \in x)\}$, then $c \in c$ *iff* $F(c \in c)$

$$D$$

- D: the *ordinary* domain.

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$$D \qquad \Omega$$

- D: the *ordinary* domain.
- Ω: truth values.

$$\color{red}{C} \qquad\qquad D \to \Omega$$

- D: the *ordinary* domain.
- Ω: truth values.
- C: the classes

Semantic Values

$$C \qquad (C \cup D) \to \Omega$$

- D: the *ordinary* domain.
- Ω: truth values.
- C: the classes

$$C \cong (C \cup D) \to \Omega$$

- D: the *ordinary* domain.
- Ω: truth values.
- C: the classes

$$C \cong [(C \cup D) \to \Omega]$$

- D: the *ordinary* domain.
- Ω: truth values.
- C: the classes

Extensionality

We won't focus on extensionality here.

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But we'll *identify* classes by their extensions as much as possible.

Sharpening our Target

$$C \cong [C \cup D \to \Omega]$$

$$C \stackrel{\sim}{=} [C \cup D \to \Omega]$$

$\phi(x)$ gives a function $[C \cup D \to \Omega]$.
So we can find a class C to *match*.

$a \in \{x : \phi(x)\}$ has the *same* value in Ω as $\phi(a)$.

ORDER AND CONTINUITY

Underlying Logic: Preservation

$$\sqsubseteq$$

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(If $x \sqsubseteq x'$ and $y \sqsubseteq y'$ then $x \sharp y \sqsubseteq x' \sharp y'$, etc.)

Preservation on candidates for Ω

$$\begin{tikzcd} 0 & & 1 \\ & * \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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In Ł$_3$, $* \rightarrow *$ is 1; but $1 \rightarrow 0$ is 0

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$$\begin{tikzcd}
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\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& \star & \\
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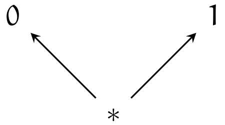

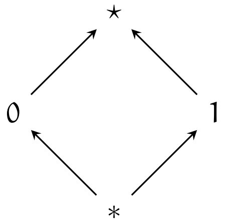

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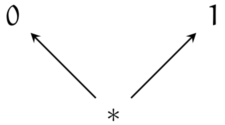

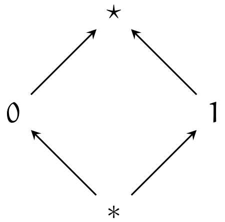

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Similar behaviour here.

Many other choices for Ω are possible.

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Even $\{0, 1\}$ can be ordered: $0 \sqsubseteq 1$. Then $\wedge, \vee, 0, 1$ are order preserving, but $\neg$ and $\supset$ are *not* order preserving.

3: our choice of Ω

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(I *really* don't care if you think of $*$ as *true*, or as *untrue*.)

ORDER MODELS

Defining Order Models

Given an order algebra Ω, and a domain D of urelements

$\langle C, \sqsubseteq, \Uparrow, \Downarrow\rangle$ is a $\langle D, \Omega\rangle$ *order model* iff

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– Write '$\Uparrow(c)$' as '$c_\Uparrow$' and '$\Downarrow(f)$' as '$f_\Downarrow$.' So $c_{\Uparrow\Downarrow} = c$ and $f_{\Downarrow\Uparrow} = f$.

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– If $b \in C \cup D$ and $c \in C$, then $c_\Uparrow(b)$ tells you whether b is in c.

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$$\text{If } x \sqsubseteq x' \text{ and } y \sqsubseteq y' \text{ then } x_{\Uparrow}(y) \sqsubseteq x'_{\Uparrow}(y').$$

$$\begin{array}{rcl} x_{\Uparrow}(y) \sqsubseteq x_{\Uparrow}(y') & — & y \sqsubseteq y' \text{ and } x_{\Uparrow} \text{ is order preserving.} \\ x_{\Uparrow} \sqsubseteq x'_{\Uparrow} & — & x \sqsubseteq x' \text{ and } \Uparrow \text{ is order preserving.} \end{array}$$

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$x_{\Uparrow}(y') \sqsubseteq x'_{\Uparrow}(y')$	—	by the definition of $\sqsubseteq$ for functions.

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- (Connectives and quantifiers are order preserving functions on 3 or $[C \cup D \to 3]$.)

Extending the Language with Terms

$$\{x : \phi(x)\}$$

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Since $[\![\phi(x)]\!]_{\mathfrak{M},\alpha[x:=v]}$ is order preserving in v we can use that function, in $[C \cup D \rightarrow 3]$, to select the extension of $\{x : \phi(x)\}$.

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$$[\![\{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha} = (\lambda v.[\![\phi(x)]\!]_{\mathfrak{M},\alpha[x:=v]})_{\Downarrow}$$

Strong Comprehension

$$[\![t \in \{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha}$$

Strong Comprehension

$$[\![t \in \{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha} \quad = \quad [\![\{x : \phi(x)\}]\!]_{\alpha_{\Uparrow}}([\![t]\!]_{\alpha})$$

(I've dropped reference to $\mathfrak{M}$ as it is constant throughout.)

Strong Comprehension

$$
\begin{aligned}
[\![t \in \{x : \phi(x)\}]\!]_{\mathfrak{M},\alpha} &= [\![\{x : \phi(x)\}]\!]_{\alpha_{\Uparrow}}([\![t]\!]_{\alpha}) \\
&= (\lambda v.[\![\phi(x)]\!]_{\alpha[x:=v]})_{\Downarrow\Uparrow}([\![t]\!]_{\alpha})
\end{aligned}
$$

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Λ, V and $V\!\!\!\Lambda$

$$\Lambda \quad = \quad \{x : 0\}$$

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Ordering the Classes

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In fact, $[\![\mathbb{X}]\!] \sqsubseteq c$ for every class $c \in C$.

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In fact, $[\![\mathbb{X}]\!] \sqsubseteq c$ for every class $c \in C$.

From now, we'll use '$\emptyset$', 'V' and '$\mathbb{X}$' as both the *class terms* in the language, and as their denotations, names for objects in C.

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for each object b in $C \cup D$
$c_{\Uparrow}(b)$ takes the value 0 or 1

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$\mathbb{X}$ is *not* sharp.

Almost No Classes are *Sharp*

If $c_{\Uparrow}(b) = 1$ and $c_{\Uparrow}(b') = 0$, then $c_{\Uparrow}(\mathfrak{X}) = *$.

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It follows that $c_{\Uparrow}(\mathbb{X}) = *$

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Once a class *includes* something
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It follows that there are no *crisp singletons*:
objects $\{a\}$ for which $[\![a \in \{x\}]\!] = 1$
and $[\![b \in \{x\}]\!] = 0$ *for all other* b.

Singletons and Anti-Signetons: $\{t\}$ and $\}t\{$

- $[\![\{t\}]\!]_\alpha$: (the class representative of) the function that
  - assigns 1 to x iff $[\![t]\!]_\alpha \sqsubseteq x$,
  - and 0 to x iff there is no z where $x \sqsubseteq z$ and $[\![t]\!]_\alpha \sqsubseteq z$,
  - and $*$ otherwise.
- $[\![\}t\{]\!]_\alpha$: (the class representative of) the function that
  - assigns 0 to x iff $[\![t]\!]_\alpha \sqsubseteq x$, and
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  - and $*$ otherwise.

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  … and *impure* order models.
- Find perspicuous ways to *construct* order models.
- Relate these constructions to other known model constructions.
- *Axiomatise* the logic of order models.
- Examine different *motivations* of order models.

THANK YOU!