Generality & Existence I

Quantification & Free Logic

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To analyse the *quantifiers*

My Aim

To analyse the *quantifiers*
(including their interactions with *modals*)

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(including their interactions with *modals*)
using the tools of *proof theory*

To analyse the *quantifiers*
(including their interactions with *modals*)
using the tools of *proof theory*
in order to better understand
quantification, *existence* and *identity*.

Understanding the quantifier rules.

Sequents & Defining Rules

Generality & Classical Quantifiers

Quantifiers & Non-Denoting Terms

Derivations & Systems

Positions & Models

Completeness & Cut

Consequences & Questions

SEQUENTS & DEFINING RULES

Sequents

$$\Gamma \succ \Delta$$

Don't assert each element of Γ
and deny each element of Δ.

Identity: $A \succ A$

Structural Rules

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Weakening: $\dfrac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta} \quad \dfrac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}$

Structural Rules

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Weakening: $$\frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta} \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ A, \Delta}$$

Contraction: $$\frac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta} \qquad \frac{\Gamma \succ A, A, \Delta}{\Gamma \succ A, \Delta}$$

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Cut: $\dfrac{\Gamma \succ A, \Delta \quad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}$

Structural Rules

Identity: $A \succ A$

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Contraction: $$\frac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta} \qquad \frac{\Gamma \succ A, A, \Delta}{\Gamma \succ A, \Delta}$$

Cut: $$\frac{\Gamma \succ A, \Delta \quad \Gamma, A \succ \Delta}{\Gamma \succ \Delta}$$

Structural rules govern declarative sentences *as such*.

With Left/Right rules?

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta}\,[\land L] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \land B, \Delta}\,[\land R]$$

With Left/Right rules?

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta}\,[\land L] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \land B, \Delta}\,[\land R]$$

$$\frac{\Gamma, B \succ \Delta}{\Gamma, A \mathbin{\mathit{tonk}} B \succ \Delta}\,[\mathit{tonk}L] \qquad \frac{\Gamma \succ A, \Delta}{\Gamma \succ A \mathbin{\mathit{tonk}} B, \Delta}\,[\mathit{tonk}R]$$

What is involved in going from $\mathcal{L}$ to $\mathcal{L}'$?

Use $\succ_{\mathcal{L}}$ to *define* $\succ_{\mathcal{L}'}$.

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Desideratum #1: $\succ_{\mathcal{L}'}$ is *conservative:* $(\succ_{\mathcal{L}'})|_{\mathcal{L}}$ is $\succ_{\mathcal{L}}$.

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Desideratum #1: $\succ_{\mathcal{L}'}$ is *conservative:* $(\succ_{\mathcal{L}'})|_{\mathcal{L}}$ is $\succ_{\mathcal{L}}$.

Desideratum #2: Concepts are defined *uniquely*.

A Defining Rule

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta}\;[\land Df]$$

(The rule is printed with a double line between premise and conclusion.)

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Fully specifies norms governing conjunctions on the *left* in terms of simpler vocabulary.

A Defining Rule

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\ [\wedge Df]$$

Fully specifies norms governing conjunctions on the *left* in terms of simpler vocabulary.

Identity and *Cut* determine the behaviour of conjunctions on the *right*.

From $[\land Df]$ to $[\land L/R]$

$$
\dfrac{\Gamma \succ A, \Delta \qquad \dfrac{\Gamma \succ B, \Delta \qquad \dfrac{\dfrac{}{A \land B \succ A \land B}\,[Id]}{A, B \succ A \land B}\,[\land Df]}{\Gamma, A \succ A \land B, \Delta}\,[Cut]}{\Gamma \succ A \land B, \Delta}\,[Cut]
$$

From [$\land Df$] to [$\land L/R$]

$$
\begin{prooftree}
\AxiomC{$\Gamma \succ A, \Delta$}
\AxiomC{$\Gamma \succ B, \Delta$}
\AxiomC{}
\RightLabel{$[Id]$}
\UnaryInfC{$A \land B \succ A \land B$}
\RightLabel{$[\land Df]$}
\UnaryInfC{$A, B \succ A \land B$}
\RightLabel{$[Cut]$}
\BinaryInfC{$\Gamma, A \succ A \land B, \Delta$}
\RightLabel{$[Cut]$}
\BinaryInfC{$\Gamma \succ A \land B, \Delta$}
\end{prooftree}
$$

From [$\land Df$] to [$\land L/R$]

$$\dfrac{\Gamma \succ A, \Delta \qquad \dfrac{\Gamma \succ B, \Delta \qquad \dfrac{\dfrac{}{A \land B \succ A \land B}\,[Id]}{A, B \succ A \land B}\,[\land Df]}{\Gamma, A \succ A \land B, \Delta}\,[Cut]}{\Gamma \succ A \land B, \Delta}\,[Cut]$$

From [$\wedge Df$] to [$\wedge L/R$]

$$
\dfrac{\Gamma \succ A, \Delta \qquad \dfrac{\Gamma \succ B, \Delta \qquad \dfrac{\dfrac{}{A \wedge B \succ A \wedge B}\,[Id]}{A, B \succ A \wedge B}\,[\wedge Df]}{\Gamma, A \succ A \wedge B, \Delta}\,[Cut]}{\Gamma \succ A \wedge B, \Delta}\,[Cut]
$$

From $[\wedge Df]$ to $[\wedge L/R]$

$$\dfrac{\Gamma \succ A, \Delta \qquad \dfrac{\Gamma \succ B, \Delta \qquad \dfrac{\dfrac{}{A \wedge B \succ A \wedge B}\,[Id]}{A, B \succ A \wedge B}\,[\wedge Df]}{\Gamma, A \succ A \wedge B, \Delta}\,[Cut]}{\Gamma \succ A \wedge B, \Delta}\,[Cut]$$

$$\dfrac{\Gamma \succ A, \Delta \qquad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta}\,[\wedge R]$$

And Back

$$\dfrac{\dfrac{A \succ A \quad B \succ B}{A, B \succ A \land B}\,[\land R] \qquad \Gamma, A \land B \succ \Delta}{\Gamma, A, B \succ \Delta}\,[Cut]$$

Equivalence

$$\mathcal{L}[\wedge Df, Cut] = \mathcal{L}[\wedge L/R, Cut]$$

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This *generalises*: $\wedge$, $\vee$, $\supset$, $\neg$ work in the same way.

Equivalence

$$\mathcal{L}[\wedge Df, Cut] = \mathcal{L}[\wedge L/R, Cut] = \mathcal{L}[\wedge L/R]$$

This *generalises*: $\wedge$, $\vee$, $\supset$, $\neg$ work in the same way.

I want to see how this works for quantifiers.

GENERALITY & CLASSICAL QUANTIFIERS

The Rules

$$\dfrac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta}\ [\forall Df] \qquad \dfrac{\Gamma, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta}\ [\exists Df]$$

(where n is not present in the bottom sequent of both rules)

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(where n is not present in the bottom sequent of both rules)

For this to work as expected, n must be *deductively general*.

$$\frac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta}\ [\forall Df] \qquad \frac{\Gamma, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta}\ [\exists Df]$$

(where n is not present in the bottom sequent of both rules)

For this to work as expected, n must be *deductively general*.

Function terms are not deductively general:
$(\forall x)(0 \neq x') \succ 0 \neq 1$, but $(\forall x)(0 \neq x') \not\succ (\forall x)(0 \neq x)$.

Generality and Specification

A term $\mathfrak{n}$ is *deductively general* for the category $\mathfrak{T}$ iff the rule of *specification* is admissible for each term $\mathfrak{t}$ of category $\mathfrak{T}$.

$$\frac{\Gamma \succ \Delta}{\Gamma[\mathfrak{n} := \mathfrak{t}] \succ \Delta[\mathfrak{n} := \mathfrak{t}]}\ [Spec^{\mathfrak{n}}_{\mathfrak{t}}]$$

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In classical first order predicate logic, names are deductively general.

$[\forall Df]$ requires $[Spec]$

$$\frac{(\forall x)Fx \succ (\forall x)Fx}{(\forall x)Fx \succ Fn}\,[\forall Df]$$

[$\forall Df$] requires [*Spec*]

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How can we derive $(\forall x)Fx \succ Ft$?

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How can we derive $(\forall x)Fx \succ Ft$?

We must make explicit use of *specification*.

$$\dfrac{\dfrac{(\forall x)Fx \succ (\forall x)Fx}{(\forall x)Fx \succ Fn}\,[\forall Df]}{(\forall x)Fx \succ Ft}\,[Spec^n_t]$$

From [$\forall Df$] to [$\forall L$]

$$\dfrac{\dfrac{\overline{(\forall x)A(x) \succ (\forall x)A(x)}\ [Id]}{(\forall x)A(x) \succ A(n)}\ [\forall Df] \qquad \Gamma, A(n) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\ [Cut]$$

From $[\forall Df]$ to $[\forall L]$

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The rule that results no longer has the side condition for n, because the premise sequent $\Gamma, A(n) \succ \Delta$ is arbitrary.

$$\dfrac{\Gamma, A(n) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\,[\forall L\text{: for names}]$$

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However, it applies only to names, not terms.

From [$\forall Df$] to [$\forall L$], *cont.*

$$\dfrac{\dfrac{\dfrac{\overline{(\forall x)A(x) \succ (\forall x)A(x)}\ [Id]}{(\forall x)A(x) \succ A(n)}\ [\forall Df]}{(\forall x)A(x) \succ A(t)}\ [Spec^n_t] \qquad \Gamma, A(t) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\ [Cut]$$

From [$\forall Df$] to [$\forall L$], *cont.*

$$\dfrac{\dfrac{\dfrac{\overline{(\forall x)A(x) \succ (\forall x)A(x)}\ [Id]}{(\forall x)A(x) \succ A(n)}\ [\forall Df]}{(\forall x)A(x) \succ A(t)}\ [Spec^n_t] \qquad \Gamma, A(t) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\ [Cut]$$

$$\dfrac{\Gamma, A(t) \succ \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\ [\forall L]$$

Equivalence

$$\mathcal{L}[\forall Df, Spec, Cut] = \mathcal{L}[\forall L/R, Spec, Cut]$$

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QUANTIFIERS & NON-DENOTING TERMS

Non-Denoting Terms

$$\frac{1}{0} \qquad \lim_{x \to 0} \frac{\sin x}{x} \qquad \sum_{n=0}^{\infty} f(n)$$

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It is difficult to eliminate non-denoting terms as a matter of *syntax*.

$$(\forall x)(x < 0 \lor x = 0 \lor x > 0) \not\vdash (\tfrac{1}{0} < 0 \lor \tfrac{1}{0} = 0 \lor \tfrac{1}{0} > 0)$$

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It is difficult to eliminate non-denoting terms as a matter of *syntax*.

$$(\forall x)(x < 0 \lor x = 0 \lor x > 0) \not\succ (\tfrac{1}{0} < 0 \lor \tfrac{1}{0} = 0 \lor \tfrac{1}{0} > 0)$$

How can we modify the quantifier rules
to allow for non-denoting terms?

Pro and *Con* attitudes to Terms

To rule a term *in* is to take it as suitable
to substitute into a quantifier,
i.e., to take the term to *denote*.

To rule a term *out* is to take it as unsuitable
to substitute into a quantifier,
i.e., to take the term to *not denote*.

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to substitute into a quantifier,
i.e., to take the term to *not denote*.

We add terms to the LHS and RHS of sequents $\Gamma \succ \Delta$.

Structural Rules remain as before

$$\textit{Identity:}\quad X \succ X$$

$$\textit{Weakening:}\quad \frac{\Gamma \succ \Delta}{\Gamma, X \succ \Delta} \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ X, \Delta}$$

$$\textit{Contraction:}\quad \frac{\Gamma, X, X \succ \Delta}{\Gamma, X \succ \Delta} \qquad \frac{\Gamma \succ X, X, \Delta}{\Gamma \succ X, \Delta}$$

$$\textit{Cut:}\quad \frac{\Gamma \succ X, \Delta \qquad \Gamma, X \succ \Delta}{\Gamma \succ \Delta}$$

Here X is either a sentence or a term.

Quantifier Rules, allowing for non-denoting terms

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta}\ [\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta}\ [\exists Df]$$

From $[\forall Df]$ to $[\forall L]$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{(\forall x)A(x) \succ (\forall x)A(x)}\,[Id]}{(\forall x)A(x), n \succ A(n)}\,[\forall Df]}{(\forall x)A(x), t \succ A(t)}\,[Spec^n_t] \quad \Gamma, A(t) \succ \Delta}{\Gamma, (\forall x)A(x), t \succ \Delta}\,[Cut] \quad \Gamma \succ t, \Delta
$$
$$
\dfrac{\Gamma, (\forall x)A(x), t \succ \Delta \qquad \Gamma \succ t, \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\,[Cut]
$$

From [$\forall Df$] to [$\forall L$]

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\forall x)A(x) \succ (\forall x)A(x)}\,[Id]}{(\forall x)A(x), n \succ A(n)}\,[\forall Df]}{(\forall x)A(x), t \succ A(t)}\,[Spec^n_t] \quad \Gamma, A(t) \succ \Delta}{\dfrac{\Gamma, (\forall x)A(x), t \succ \Delta \qquad \Gamma \succ t, \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\,[Cut]}\,[Cut]$$

This results in a two-premise rule:

$$\dfrac{\Gamma, A(t) \succ \Delta \qquad \Gamma \succ t, \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\,[\forall L]$$

From $[\exists Df]$ to $[\exists R]$

$$\dfrac{\Gamma, t \succ \Delta \qquad \dfrac{\Gamma \succ A(t), \Delta \qquad \dfrac{\dfrac{\dfrac{}{(\exists x)A(x) \succ (\exists x)A(x)}[Id]}{A(n), n \succ (\exists x)A(x)}[\exists Df]}{A(t), t \succ (\exists x)A(x)}[Spec^n_t]}{\Gamma, t \succ (\exists x)A(x), \Delta}[Cut]}{\Gamma \succ (\exists x)A(x), \Delta}[Cut]$$

From [$\exists Df$] to [$\exists R$]

$$
\dfrac{\Gamma, t \succ \Delta \qquad \dfrac{\Gamma \succ A(t), \Delta \qquad \dfrac{\dfrac{\dfrac{}{(\exists x)A(x) \succ (\exists x)A(x)}\,[Id]}{A(n), n \succ (\exists x)A(x)}\,[\exists Df]}{A(t), t \succ (\exists x)A(x)}\,[Spec^{n}_{t}]}{\Gamma, t \succ (\exists x)A(x), \Delta}\,[Cut]}{\Gamma \succ (\exists x)A(x), \Delta}\,[Cut]
$$

This gives a two-premise [$\exists R$] rule:

$$
\dfrac{\Gamma \succ t, \Delta \qquad \Gamma \succ A(t), \Delta}{\Gamma \succ (\exists x)A(x), \Delta}\,[\exists R]
$$

Making Denotation Explicit

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t\!\downarrow\, \succ \Delta}\ [{\downarrow}Df]$$

Making Denotation Explicit

$$\frac{\Gamma, \mathsf{t} \succ \Delta}{\Gamma, \mathsf{t}{\downarrow} \succ \Delta}\ [{\downarrow}Df]$$

(The $[{\downarrow}Df]$ rule is printed with a double line.)

This results in the obvious $[{\downarrow}R]$ rule.

$$\dfrac{\Gamma \succ \mathsf{t}, \Delta \qquad \dfrac{\dfrac{}{\mathsf{t}{\downarrow} \succ \mathsf{t}{\downarrow}}\ [Id]}{\mathsf{t} \succ \mathsf{t}{\downarrow}}\ [{\downarrow}Df]}{\Gamma \succ \mathsf{t}{\downarrow}, \Delta}\ [Cut] \qquad \dfrac{\Gamma \succ \mathsf{t}, \Delta}{\Gamma \succ \mathsf{t}{\downarrow}, \Delta}\ [{\downarrow}R]$$

SOLOMON FEFERMAN*

DEFINEDNESS

ABSTRACT. Questions of definedness are ubiquitous in mathematics. Informally, these involve reasoning about expressions which may or may not have a value. This paper surveys work on logics in which such reasoning can be carried out directly, especially in computational contexts. It begins with a general logic of "partial terms", continues with partial combinatory and lambda calculi, and concludes with an expressively rich theory of partial functions and polymorphic types, where termination of functional programs can be established in a natural way.

Erkenntnis 43: 295–320, 1995.

Definedness, function terms and predicates

$$\frac{t_i, \Gamma \succ \Delta}{f(t_1, \ldots, t_n), \Gamma \succ \Delta}\,[fL] \qquad \frac{t_i, \Gamma \succ \Delta}{Ft_1 \cdots t_n, \Gamma \succ \Delta}\,[FL]$$

DERIVATIONS & SYSTEMS

Structural Rules

Identity: $X \succ X$

Weakening: $\dfrac{\Gamma \succ \Delta}{\Gamma, X \succ \Delta} \qquad \dfrac{\Gamma \succ \Delta}{\Gamma \succ X, \Delta}$

Contraction: $\dfrac{\Gamma, X, X \succ \Delta}{\Gamma, X \succ \Delta} \qquad \dfrac{\Gamma \succ X, X, \Delta}{\Gamma \succ X, \Delta}$

Cut: $\dfrac{\Gamma \succ X, \Delta \quad \Gamma, X \succ \Delta}{\Gamma \succ \Delta}$

Predicate and Function Rules

$$\frac{t_i, \Gamma \succ \Delta}{f(t_1, \ldots, t_n), \Gamma \succ \Delta}\,[fL] \qquad \frac{t_i, \Gamma \succ \Delta}{Ft_1 \cdots t_n, \Gamma \succ \Delta}\,[FL]$$

Specification

$$\frac{\Gamma \succ \Delta}{\Gamma[n := t] \succ \Delta[n := t]} \; [Spec^{n}_{t}]$$

Defining Rules

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \land B \succ \Delta}\ [\land Df] \qquad \frac{\Gamma \succ A, B, \Delta}{\Gamma \succ A \lor B, \Delta}\ [\lor Df]$$

$$\frac{\Gamma, A \succ B, \Delta}{\Gamma \succ A \supset B, \Delta}\ [\supset Df] \qquad \frac{\Gamma \succ A, \Delta}{\Gamma, \neg A \succ \Delta}\ [\neg Df]$$

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta}\ [\forall Df] \qquad \frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta}\ [\exists Df]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t{\downarrow} \succ \Delta}\ [{\downarrow} Df]$$

The System

DL[*Df*, *Cut*, *Spec*]

Example Derivation

$$
\dfrac{\dfrac{\dfrac{(\forall x)(Fx \supset Gx) \succ (\forall x)(Fx \supset Gx)}{(\forall x)(Fx \supset Gx), n \succ Fn \supset Gn}\,[\forall Df] \quad \dfrac{\dfrac{Fn \supset Gn \succ Fn \supset Gn}{Fn \supset Gn, Fn \succ Gn}\,[\supset Df] \quad \dfrac{(\exists x)Gx \succ (\exists x)Gx}{n, Gn \succ (\exists x)Gx}\,[\exists Df]}{Fn \supset Gn, n, Fn \succ (\exists x)Gx}\,[\mathit{Cut}]}{\dfrac{(\forall x)(Fx \supset Gx), n, Fn \succ (\exists x)Gx}{(\forall x)(Fx \supset Gx), (\exists x)Fx \succ (\exists x)Gx}\,[\exists Df]}\,[\mathit{Cut}]}{(\forall x)(Fx \supset Gx) \succ (\exists x)Fx \supset (\exists x)Gx}\,[\supset Df]
$$

Replace the quantifier rules by these *generalised* defining rules:

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta}\ [\forall Df\!\downarrow] \qquad \frac{\Gamma \succ (\forall x)A(x), \Delta}{\Gamma, t \succ A(t), \Delta}\ [\forall Df\!\Uparrow]$$

$$\frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta}\ [\exists Df\!\downarrow] \qquad \frac{\Gamma, (\exists x)A(x) \succ \Delta}{\Gamma, t, A(t) \succ \Delta}\ [\exists Df\!\Uparrow]$$

Replace the quantifier rules by these *generalised* defining rules:

$$\frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta}\ [\forall Df{\downarrow}] \qquad \frac{\Gamma \succ (\forall x)A(x), \Delta}{\Gamma, t \succ A(t), \Delta}\ [\forall Df{\Uparrow}]$$

$$\frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta}\ [\exists Df{\downarrow}] \qquad \frac{\Gamma, (\exists x)A(x) \succ \Delta}{\Gamma, t, A(t) \succ \Delta}\ [\exists Df{\Uparrow}]$$

DL[*GDf*, *Cut*]

Theorem

A derivation of a sequent $\Gamma \succ \Delta$ *in* DL[*Df, Cut, Spec*] *can be systematically transformed into a derivation of that sequent in* DL[*GDf, Cut*], *and vice versa.*

Proof.

All of the rules in DL[*GDf, Cut*], are closed under specification. Take a derivation in DL[*Df, Cut, Spec*], and systematically replace each derivation leading up to the first use of a $Spec^{n}_{t}$ rule by transforming that derivation by replacing n by t throughout.

Conversely, the *GDf* rules are a composition of Df rules and *Spec*, so a DL[*GDf, Cut*] derivation can be transformed into a DL[*Df, Cut,Spec*] derivation. □

Left/Right Rules for Connectives

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\,[\wedge L] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta}\,[\wedge R]$$

$$\frac{\Gamma, A \succ \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\,[\vee L] \qquad \frac{\Gamma \succ A, B, \Delta}{\Gamma \succ A \vee B, \Delta}\,[\vee R]$$

$$\frac{\Gamma \succ A, \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \supset B \succ \Delta}\,[\supset L] \qquad \frac{\Gamma, A \succ B, \Delta}{\Gamma \succ A \supset B, \Delta}\,[\supset R]$$

$$\frac{\Gamma \succ A, \Delta}{\Gamma, \neg A, \succ \Delta}\,[\neg L] \qquad \frac{\Gamma, A \succ \Delta}{\Gamma \succ \neg A, \Delta}\,[\neg R]$$

Left/Right Rules for Quantifiers and Definedness

$$\frac{\Gamma, A(t) \succ \Delta \quad \Gamma \succ t, \Delta}{\Gamma, (\forall x)A(x) \succ \Delta}\,[\forall L] \qquad \frac{\Gamma, n \succ A(n), \Delta}{\Gamma \succ (\forall x)A(x), \Delta}\,[\forall R]$$

$$\frac{\Gamma, n, A(n) \succ \Delta}{\Gamma, (\exists x)A(x) \succ \Delta}\,[\exists L] \qquad \frac{\Gamma \succ t, \Delta \qquad \Gamma \succ A(t), \Delta}{\Gamma \succ (\exists x)A(x), \Delta}\,[\exists R]$$

$$\frac{\Gamma, t \succ \Delta}{\Gamma, t{\downarrow} \succ \Delta}\,[{\downarrow} L] \qquad \frac{\Gamma \succ t, \Delta}{\Gamma \succ t{\downarrow}, \Delta}\,[{\downarrow} R]$$

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$$\frac{\Gamma, t \succ \Delta}{\Gamma, t\!\downarrow \succ \Delta}\,[\downarrow L] \qquad \frac{\Gamma \succ t, \Delta}{\Gamma \succ t\!\downarrow, \Delta}\,[\downarrow R]$$

$$\mathrm{DL}[L/R, \mathit{Cut}]$$

Theorem

A derivation of a sequent $\Gamma \succ \Delta$ in DL[*GDf*, *Cut*] *can be systematically transformed into a derivation of that sequent in* DL[*L/R*, *Cut*], *and vice versa.*

Proof.

Using *Cut* and *Id*, each (generalised) defining rule can mimic a Left/Right pair, and *vice versa*. □

These systems satisfy *Desideratum* #2

$$\dfrac{\dfrac{(\forall x)A(x) \succ (\forall x)A(x)}{(\forall x)A(x), n \succ A(n)}\,[\forall Df]}{(\forall x)A(x) \succ (\forall' x)A(x)}\,[\forall' Df] \qquad \dfrac{\dfrac{(\forall' x)A(x) \succ (\forall' x)A(x)}{(\forall' x)A(x), n \succ A(n)}\,[\forall' Df]}{(\forall' x)A(x) \succ (\forall x)A(x)}\,[\forall Df]$$

For *Desideratum* #1 we eliminate *Cut*

To show that L/R rules are conservative additions, we eliminate *Cut*, since the other rules do not introduce new connectives, quantifiers or predicates.

Then, any derivation of a sequent $\Gamma \succ \Delta$ in a system will use only the rules involving the connectives, quantifiers and predicates in that sequent.

POSITIONS & MODELS

Positions

$$[\Gamma : \Delta]$$

A pair of *sets*, Γ and Δ where for no $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ do we have $\Gamma' \succ \Delta'$.

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$$[\Gamma : \Delta]$$

A pair of *sets*, Γ and Δ where for no $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ do we have $\Gamma' \succ \Delta'$.

Here Γ and Δ can be infinite, unlike sequents.

Refinement

$[\Gamma_2 : \Delta_2]$ is a REFINEMENT of $[\Gamma_1 : \Delta_1]$ iff $\Gamma_1 \subseteq \Gamma_2$ and $\Delta_1 \subseteq \Delta_2$.

Refinement for Conjunction

$$\dfrac{\dfrac{\Gamma, A \land B, A, B \succ \Delta}{\Gamma, A \land B, A \land B \succ \Delta}\,[\land Df]}{\Gamma, A \land B \succ \Delta}\,[W]$$

Refinement for Conjunction

$$\dfrac{\dfrac{\Gamma, A \land B, A, B \succ \Delta}{\Gamma, A \land B, A \land B \succ \Delta}[\land Df]}{\Gamma, A \land B \succ \Delta}[W]$$

If $[\Gamma, A \land B : \Delta]$ is a position, so is $[\Gamma, A \land B, A, B : \Delta]$.

Refinement for Conjunction

$$\dfrac{\dfrac{\Gamma, A \land B, A, B \succ \Delta}{\Gamma, A \land B, A \land B \succ \Delta}\,[\land Df]}{\Gamma, A \land B \succ \Delta}\,[W] \qquad \dfrac{\dfrac{\Gamma \succ A, A \land B, \Delta \quad \Gamma \succ B, A \land B, \Delta}{\Gamma \succ A \land B, A \land B, \Delta}\,[\land R]}{\Gamma \succ A \land B, \Delta}\,[W]$$

If $[\Gamma, A \land B : \Delta]$ is a position, so is $[\Gamma, A \land B, A, B : \Delta]$.

Refinement for Conjunction

$$\dfrac{\dfrac{\Gamma, A \wedge B, A, B \succ \Delta}{\Gamma, A \wedge B, A \wedge B \succ \Delta}\,[\wedge Df]}{\Gamma, A \wedge B \succ \Delta}\,[W] \qquad \dfrac{\dfrac{\Gamma \succ A, A \wedge B, \Delta \quad \Gamma \succ B, A \wedge B, \Delta}{\Gamma \succ A \wedge B, A \wedge B, \Delta}\,[\wedge R]}{\Gamma \succ A \wedge B, \Delta}\,[W]$$

If $[\Gamma, A \wedge B : \Delta]$ is a position, so is $[\Gamma, A \wedge B, A, B : \Delta]$.

If $[\Gamma : A \wedge B, \Delta]$ is a position, so is (at least) one of $[\Gamma : A, A \wedge B, \Delta]$ and $[\Gamma : B, A \wedge B, \Delta]$.

Refinement for Connectives

POSITION	REFINEMENTS
$[\Gamma, A \land B : \Delta]$	$[\Gamma, A \land B, A, B : \Delta]$
$[\Gamma : A \land B, \Delta]$	One of $[\Gamma : A, A \land B, \Delta]$ and $[\Gamma : B, A \land B, \Delta]$

Refinement for Connectives

POSITION	REFINEMENTS
$[\Gamma, A \wedge B : \Delta]$	$[\Gamma, A \wedge B, A, B : \Delta]$
$[\Gamma : A \wedge B, \Delta]$	One of $[\Gamma : A, A \wedge B, \Delta]$ and $[\Gamma : B, A \wedge B, \Delta]$
$[\Gamma, A \vee B : \Delta]$	One of $[\Gamma, A, A \vee B : \Delta]$ and $[\Gamma, B, A \vee B : \Delta]$
$[\Gamma : A \vee B, \Delta]$	$[\Gamma : A, B, A \vee B, \Delta]$

Refinement for Connectives

POSITION	REFINEMENTS
$[\Gamma, A \wedge B : \Delta]$	$[\Gamma, A \wedge B, A, B : \Delta]$
$[\Gamma : A \wedge B, \Delta]$	One of $[\Gamma : A, A \wedge B, \Delta]$ and $[\Gamma : B, A \wedge B, \Delta]$
$[\Gamma, A \vee B : \Delta]$	One of $[\Gamma, A, A \vee B : \Delta]$ and $[\Gamma, B, A \vee B : \Delta]$
$[\Gamma : A \vee B, \Delta]$	$[\Gamma : A, B, A \vee B, \Delta]$
$[\Gamma, A \supset B : \Delta]$	One of $[\Gamma, A \supset B : A, \Delta]$ and $[\Gamma, B, A \supset B : \Delta]$
$[\Gamma : A \supset B, \Delta]$	$[\Gamma, A : B, A \vee B, \Delta]$

Refinement for Connectives

POSITION	REFINEMENTS
$[\Gamma, A \wedge B : \Delta]$	$[\Gamma, A \wedge B, A, B : \Delta]$
$[\Gamma : A \wedge B, \Delta]$	One of $[\Gamma : A, A \wedge B, \Delta]$ and $[\Gamma : B, A \wedge B, \Delta]$
$[\Gamma, A \vee B : \Delta]$	One of $[\Gamma, A, A \vee B : \Delta]$ and $[\Gamma, B, A \vee B : \Delta]$
$[\Gamma : A \vee B, \Delta]$	$[\Gamma : A, B, A \vee B, \Delta]$
$[\Gamma, A \supset B : \Delta]$	One of $[\Gamma, A \supset B : A, \Delta]$ and $[\Gamma, B, A \supset B : \Delta]$
$[\Gamma : A \supset B, \Delta]$	$[\Gamma, A : B, A \vee B, \Delta]$
$[\Gamma, \neg A : \Delta]$	$[\Gamma, \neg A : A, \Delta]$
$[\Gamma : \neg A, \Delta]$	$[\Gamma, A : \neg A, \Delta]$

Refinement for Quantifiers, Predicates and Functions

POSITION	REFINEMENTS
$[\Gamma, Ft_1 \cdots t_n : \Delta]$	$[\Gamma, Ft_1 \cdots t_n, t_1, \ldots, t_n : \Delta]$
$[\Gamma, f(t_1, \ldots, t_n) : \Delta]$	$[\Gamma, f(t_1, \ldots, t_n), t_1, \ldots, t_n : \Delta]$
$[\Gamma, (\forall x)A(x) : \Delta]$	One of $[\Gamma, (\forall x)A(x), A(t) : \Delta]$, $[\Gamma, (\forall x)A(x) : t, \Delta]$ for each term t in $[\Gamma, (\forall x)A(x) : \Delta]$.
$[\Gamma : (\forall x)A(x), \Delta]$	$[\Gamma, n : A(n), (\forall x)A(x), \Delta]$, for some n.
$[\Gamma, (\exists x)A(x) : \Delta]$	$[\Gamma, (\exists x)A(x), A(n), n : \Delta]$, for some n.
$[\Gamma : (\exists x)A(x), \Delta]$	One of $[\Gamma : A(t), (\exists x)A(x), \Delta]$, $[\Gamma : t, (\exists x)A(x), \Delta]$ for each term t in $[\Gamma : (\exists x)A(x), \Delta]$.
$[\Gamma, t{\downarrow} : \Delta]$	$[\Gamma, t{\downarrow}, t : A, \Delta]$
$[\Gamma : t{\downarrow}, \Delta]$	$[\Gamma, A : t, t{\downarrow}, \Delta]$

Fully Refined Positions

A position $[\Gamma : \Delta]$ is FULLY REFINED
when it is closed under each of these conditions.

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Theorem

Any DL[*LR*] *position* $[\Gamma : \Delta]$ *is extended by some fully refined position.*

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Theorem

Any DL[*LR*] *position* $[\Gamma : \Delta]$ *is extended by some fully refined position.*

Proof.

Use the usual tableaux method. □

A MODEL for the logic DL is a structure $\mathfrak{M}$ consisting of

1. A *domain* D.
2. An n-ary predicate F is interpreted as a subset $F^{\mathfrak{M}}$ of D^n (as usual).
3. An n-ary function symbol f is interpreted as a *partial function* $f^{\mathfrak{M}} : D^n \rightharpoonup D$.

Assigning Values

- α is a (partial) assignment of values to variables.
- $[\![x]\!]_{\mathfrak{M},\alpha} = \alpha(x)$
- $[\![f(t_1, \ldots, t_n)]\!]_{\mathfrak{M},\alpha} = f^{\mathfrak{M}}([\![t_1]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha})$ if each $[\![t_i]\!]_{\mathfrak{M},\alpha}$ is defined, and $f^{\mathfrak{M}}$ is defined on the inputs $[\![t_1]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha}$.

Interpreting a Language

- $\mathfrak{M} \vDash_\alpha t\downarrow$ iff $[\![t]\!]_{\mathfrak{M},\alpha}$ is defined.
- $\mathfrak{M} \vDash_\alpha Ft_1 \cdots t_n$ iff for each i, the value $[\![t_i]\!]_{\mathfrak{M},\alpha}$ is defined, and the n-tuple $\langle [\![t_n]\!]_{\mathfrak{M},\alpha}, \ldots, [\![t_n]\!]_{\mathfrak{M},\alpha} \rangle \in F^{\mathfrak{M}}$
- $\mathfrak{M} \vDash_\alpha A \wedge B$ iff $\mathfrak{M} \vDash_\alpha A$ and $\mathfrak{M} \vDash_\alpha B$.
- $\mathfrak{M} \vDash_\alpha A \vee B$ iff $\mathfrak{M} \vDash_\alpha A$ or $\mathfrak{M} \vDash_\alpha B$.
- $\mathfrak{M} \vDash_\alpha A \supset B$ iff $\mathfrak{M} \nvDash_\alpha A$ or $\mathfrak{M} \vDash_\alpha B$.
- $\mathfrak{M} \vDash_\alpha \neg A$ iff $\mathfrak{M} \nvDash_\alpha A$.
- $\mathfrak{M} \vDash_\alpha (\forall x)A(x)$ iff $\mathfrak{M} \vDash_{\alpha[x:=d]} A(x)$ for every d in D.
- $\mathfrak{M} \vDash_\alpha (\exists x)A(x)$ iff $\mathfrak{M} \vDash_{\alpha[x:=d]} A(x)$ for some d in D.

Models *of* Positions

$\mathfrak{M}$ is a MODEL OF THE POSITION $[\Gamma : \Delta]$ iff
every sentence in Γ is true in $\mathfrak{M}$,
every term in Γ is defined in $\mathfrak{M}$,
every sentence in Δ is false in $\mathfrak{M}$
and every term in Δ is undefined in $\mathfrak{M}$.

Models *from* Positions

For any fully refined position $[\Gamma : \Delta]$ the model where

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(3) the n-ary function symbol f is interpreted by setting $f(t_1, \ldots, t_n)$ to be defined iff it is in Γ, and then it takes *itself* as its value

is said to be the model *from* $[\Gamma : \Delta]$.

COMPLETENESS & CUT

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The model from *a fully refined position is a model* for *that position.*

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Inspect the conditions for satisfaction in a model. □

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Corollary

Each position has some model.

Completeness

Theorem

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Proof.

Inspect the conditions for satisfaction in a model. □

Corollary

Each position has some model.

Proof.

Extend $[\Gamma : \Delta]$ into a fully refined position. Take the model from that position. It is a model for $[\Gamma : \Delta]$. □

Admissibility of Cut

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If $\Gamma \succ \Delta$ *is derivable in* DL[*LR*, *Cut*] *then* $[\Gamma : \Delta]$ *has no model.*

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Proof.

Induction on the length of the derivation. The special case is *Cut*: If $[\Gamma : X, \Delta]$ and $[\Gamma, X : \Delta]$ have no model, then neither does $[\Gamma : \Delta]$. $\square$

Admissibility of Cut

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If $\Gamma \succ \Delta$ *is derivable in* $\mathrm{DL}[LR, Cut]$ *then* $[\Gamma : \Delta]$ *has no model.*

Proof.

Induction on the length of the derivation. The special case is *Cut*: If $[\Gamma : X, \Delta]$ and $[\Gamma, X : \Delta]$ have no model, then neither does $[\Gamma : \Delta]$. □

Corollary

If $\Gamma \succ \Delta$ *derivable in* $\mathrm{DL}[LR, Cut]$, *it derivable in* $\mathrm{DL}[LR]$ *too.*

Admissibility of Cut

Theorem

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Corollary

If $\Gamma \succ \Delta$ *derivable in* DL[*LR*, *Cut*], *it derivable in* DL[*LR*] *too.*

Proof.

If $\Gamma \succ \Delta$ is not derivable in DL[*LR*], then $[\Gamma : \Delta]$ has a model. So it is not derivable in DL[*LR*, *Cut*] either. □

CONSEQUENCES & QUESTIONS

Defining Rules, with *Generality*,
give insight into the quantifiers.

How wide is the category of terms?

If we allow for 'non-denoting' terms, defining rules for free logic are straightforward...

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If we allow for 'non-denoting' terms,
defining rules for free logic are straightforward...

...and they have a ready interpretation
in terms of rules governing our vocabulary
without taking models as *primary*.

Wider Quantifiers?

These are *also* defining rules:

$$\frac{\Gamma \succ A(n), \Delta}{\Gamma \succ (\Pi x)A(x), \Delta}\,[\Pi Df] \qquad \frac{\Gamma, A(n) \succ \Delta}{\Gamma, (\Sigma x)A(x) \succ \Delta}\,[\Sigma Df]$$

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Are they meaningful?

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Are they meaningful?

$$\dfrac{\dfrac{(\Sigma x)\neg x{\downarrow} \succ (\Sigma x)\neg x{\downarrow}}{\neg n{\downarrow} \succ (\Sigma x)\neg x{\downarrow}}\,[\Sigma Df]}{\neg 1/0{\downarrow} \succ (\Sigma x)\neg x{\downarrow}}\,[Spec^{n}_{1/0}]$$

Modality

Modality

Identity

THANK YOU!

http://consequently.org/presentation/2015/generality-and-existence-1

@consequently on Twitter