Proof Theory: Logical and Philosophical Aspects

Class 2: Substructural Logics

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To introduce *proof theory*, with a focus in its applications in philosophy, linguistics and computer science.

Examine the proof theory of substructural logics.

Structural Rules

The Case of Distribution

Different Systems and their Applications

Revisiting Cut Elimination

STRUCTURAL RULES

Weakening

$$\frac{X, Y \vdash Z}{X, A, Y \vdash Z}\,[KL]$$

$$\frac{X \vdash Y, Z}{X \vdash Y, A, Z}\,[KR]$$

Contraction

$$\frac{X, A, A, Y \vdash Z}{X, A, Y \vdash Z}\ [WL]$$

$$\frac{X \vdash Y, A, A, Z}{X \vdash Y, A, Z}\ [WR]$$

Permutation

$$\frac{X, A, B, Y \vdash Z}{X, B, A, Y \vdash Z}\,[CL]$$

$$\frac{X \vdash Y, A, B, Z}{X \vdash Y, B, A, Z}\,[CR]$$

Dropping rules

We can drop some (or all) of these rules to get different logics

Dropping rules also leads to some distinctions

THE CASE OF DISTRIBUTION

Two kinds of conjunction

Extensional, additive, context-sensitive, lattice-theoretic

$$\frac{X(A) \vdash Y}{X(A \wedge B) \vdash Y}\ [\wedge L_1] \qquad \frac{X(B) \vdash Y}{X(A \wedge B) \vdash Y}\ [\wedge L_2]$$

$$\frac{X \vdash Y(A) \qquad X \vdash Y(B)}{X \vdash Y(A \wedge B)}\ [\wedge R]$$

Intensional, multiplicative, context-free, group-theoretic

$$\frac{X(A, B) \vdash Y}{X(A \circ B) \vdash Y}\ [\circ L] \qquad \frac{X \vdash Y, A \qquad U \vdash B, V}{X, U \vdash Y, A \circ B, V}\ [\circ R]$$

Two kinds of disjunction

Extensional, additive, context-sensitive, lattice-theoretic

$$\frac{X \vdash Y(A)}{X \vdash Y(A \vee B)}\ [\vee R_1] \qquad \frac{X \vdash Y(B)}{X \vdash Y(A \vee B)}\ [\vee R_2]$$

$$\frac{X(A) \vdash Y \qquad X(B) \vdash Y}{X(A \vee B) \vdash Y}\ [\vee L]$$

Intensional, multiplicative, context-free, group-theoretic

$$\frac{X \vdash Y(A, B)}{X \vdash Y(A + B)}\ [+R] \qquad \frac{X, A \vdash Y \qquad B, U \vdash V}{X, A + B, U \vdash Y, V}\ [+L]$$

Difference

In the presence of weakening and contraction, $\wedge$ and $\circ$ are equivalent, as are $\vee$ and $+$

$$A \wedge B \dashv\vdash A \circ B \qquad A \vee B \dashv\vdash A + B$$

They are not equivalent without both of those structural rules

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,[KL]}{A \circ B \vdash A}\,[\circ L] \quad \dfrac{\dfrac{B \vdash B}{A, B \vdash B}\,[KL]}{A \circ B \vdash B}\,[\circ L]}{A \circ B \vdash A \wedge B}\,[\wedge R] \qquad \dfrac{\dfrac{\dfrac{A \vdash A}{A \wedge B \vdash A}\,[\wedge_1 L] \quad \dfrac{B \vdash B}{A \wedge B \vdash B}\,[\wedge_2 L]}{A \wedge B, A \wedge B \vdash A \circ B}\,[\circ R]}{A \wedge B \vdash A \circ B}\,[WL]$$

The issue with distribution

One of the distribution laws relating extensional conjunction and disjunction isn't derivable without weakening

$$A \wedge (B \vee C) \vdash (A \wedge B) \vee C$$

The intensional version is derivable, although some distribution laws aren't derivable without contraction

$$A \circ (B + C) \vdash (A \circ B) + C$$

Proof

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\dfrac{A \vdash A}{A, B \vdash A}[KL] \quad \dfrac{B \vdash B}{A, B \vdash B}[KL]}{A, B \vdash A \wedge B}[\wedge R]
      }{A, B \vdash (A \wedge B) \vee C}[\vee R_1]
      \quad
      \dfrac{\dfrac{C \vdash C}{A, C \vdash C}[KL]}{A, C \vdash (A \wedge B) \vee C}[\vee R_2]
    }{A, B \vee C \vdash (A \wedge B) \vee C}[\vee L]
  }{A \wedge (B \vee C), B \vee C \vdash (A \wedge B) \vee C}[\wedge L_1]
}{
  \dfrac{A \wedge (B \vee C), A \wedge (B \vee C) \vdash (A \wedge B) \vee C}{A \wedge (B \vee C) \vdash (A \wedge B) \vee C}[WL]
}[\wedge L_2]
$$

Proof

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\AxiomC{$B \vdash B$}
\AxiomC{$C \vdash C$}
\RightLabel{[$+$L]}
\BinaryInfC{$B + C \vdash B, C$}
\RightLabel{[$\circ$R]}
\BinaryInfC{$A, B + C \vdash A \circ B, C$}
\RightLabel{[$+$R]}
\UnaryInfC{$A, B + C \vdash (A \circ B) + C$}
\RightLabel{[$\circ$L]}
\UnaryInfC{$A \circ (B + C) \vdash (A \circ B) + C$}
\end{prooftree}
$$

Why distribution?

It seems like truth-functional conjunction and disjunction, $\wedge$ and $\vee$, should obey the distribution laws

DIFFERENT SYSTEMS AND THEIR APPLICATIONS

Applications

We will look at three substructural systems and their applications

- Relevance
- Resource-sensitivity, paradox
- Grammar, modality

Relevance

Classically, both $p \to (q \to p)$ and $q \to (p \to p)$ are valid, but what how does q *imply* $p \to p$?

These are two paradoxes of material implication, usually written with $\supset$, rather than $\to$

In *relevant logic*, valid conditionals indicate a connection of relevance or entailment

Paraconsistency

Classically, $A, \neg A \vdash B$, for any B whatsoever,

You might doubt that *contradictions* entail everything

How, after all, is an *arbitrary* B relevant to A?

A logic is *paraconsistent* iff contradictions don't entail every formula

A couple of proofs

$$\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A \rightarrow A}\,[\rightarrow R]}{B \vdash A \rightarrow A}\,[KL]}{\vdash B \rightarrow (A \rightarrow A)}\,[\rightarrow R]$$

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, B \vdash A}\,[KL]}{A \vdash B \rightarrow A}\,[\rightarrow R]}{\vdash A \rightarrow (B \rightarrow A)}\,[\rightarrow R]$$

$$\dfrac{\dfrac{A \vdash A}{\neg A, A \vdash}\,[\neg L]}{\neg A, A \vdash B}\,[KR]$$

Weakening

Rejecting the weakening rules is the way to obtain a relevant logic, and it is one way to obtain a paraconsistent logic

The arrow fragment with C and W is the logic R, of Anderson and Belnap.

Provable

What is provable in the arrow fragment of the logic with contraction and permutation?

- $A \to (A \to B) \vdash A \to B$
- $A \to (B \to C) \vdash B \to (A \to C)$
- $A \to B \vdash (C \to A) \to (C \to B)$
- $A \to B \vdash (B \to C) \to (A \to C)$

Unprovable

What is unprovable in the arrow fragment of the logic with contraction and permutation?

- $\vdash B \to (A \to A)$
- $A \vdash B \to A$
- $\vdash A \to (A \to A)$
- $(A \to B) \to A \vdash A$

Adding connectives

Relevant logics usually take the additive rules to govern conjunction and disjunction

Meyer showed that one gets R minus distribution by taking the additive connective rules with *mulitple conclusion sequents*

This system is cut-free and decidable, but it does not have distribution

Full R, with distribution, is *undecidable*, as shown by Urquhart

Conjunction and comma

Classically, the following are equivalent

- $A, B, C \vdash D$
- $\vdash (A \land B \land C) \to D$
- $\vdash (A \land B) \to (C \to D)$
- $\vdash A \to (B \to (C \to D))$

We can't have all four equivalent, and exclude the paradoxes of material implication

Substructural sequents

We want $A \wedge B \vdash A$

If $A, B \vdash C$ is derivable, then by $[\rightarrow R]$, $A \vdash B \rightarrow C$ is too

So A, B to the left of the turnstile can't be equivalent to $A \wedge B$

Solution: $A, B \vdash C$ is equivalent to $A \circ B \vdash C$

Distribution again

If we both adopt the additive rules for conjunction and disjunction and reject weakening, then there will be a problem proving distribution

This has lead to the introduction of a new structural connective – the semicolon

More structure

The parts of a sequent can be built up with comma and semicolon

The two structural connectives can obey different structural rules

In particular, have comma obey weakening, but have semicolon appear in the rules for $\circ$ and for $\rightarrow$.

$$\frac{X(A; B) \vdash C}{X(A \circ B) \vdash C}\,[\circ L] \qquad \frac{X; A \vdash B}{X \vdash A \rightarrow B}\,[\rightarrow R]$$

Consequences

The system with the extra structure is cut-free

And, with the extra structure one can prove distribution for $\wedge$ and $\vee$

Distribution again

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{[KL]}
\UnaryInfC{$A, B \vdash A$}
\AxiomC{$B \vdash B$}
\RightLabel{[KL]}
\UnaryInfC{$A, B \vdash B$}
\RightLabel{[$\wedge$R]}
\BinaryInfC{$A, B \vdash A \wedge B$}
\RightLabel{[$\vee$R$_1$]}
\UnaryInfC{$A, B \vdash (A \wedge B) \vee C$}
\AxiomC{$C \vdash C$}
\RightLabel{[KL]}
\UnaryInfC{$A, C \vdash C$}
\RightLabel{[$\vee$R$_2$]}
\UnaryInfC{$A, C \vdash (A \wedge B) \vee C$}
\RightLabel{[$\vee$L]}
\BinaryInfC{$A, B \vee C \vdash (A \wedge B) \vee C$}
\RightLabel{[$\wedge$L$_1$]}
\UnaryInfC{$A \wedge (B \vee C), B \vee C \vdash (A \wedge B) \vee C$}
\RightLabel{[$\wedge$L$_2$]}
\UnaryInfC{$A \wedge (B \vee C), A \wedge (B \vee C) \vdash (A \wedge B) \vee C$}
\RightLabel{[WL]}
\UnaryInfC{$A \wedge (B \vee C) \vdash (A \wedge B) \vee C$}
\end{prooftree}
$$

Consequences

With the extra structure one can prove distribution for $\wedge$ and $\vee$

We can prove $A, B \vdash A$, but cannot move to $A \vdash B \to A$ via $[\to R]$

That move would require $A; B \vdash A$, which we *cannot* prove

Consequences

A downside is that proof search complexity increases

The full (positive) system is undecidable

But, this idea of adding additional structure to a sequent is one we will see again

For more on relevant logic

See Dunn and Restall's "Relevance logic"
`https://consequently.org/papers/rle.pdf`

See also Anderson and Belnap's *Entailment*

For a different take on relevant logic, see Tennant's "Core Logic" papers

Resource-sensitivity

If contraction rules are in the system, then *one* copy of a formula is as good as *two*

If logic is concerned with *propositions*, then contraction may be motivated

If one considers the logic of *actions*, then contraction is less appealing

Actions

One can view formulas as resources, in which case how many you have matters

For example, let D stand for 'Shawn pays a dollar' and F for 'Shawn gets a flat white'.
The sequent $D, D, D \vdash F$ will be satisfied at the cafe while $D, D \vdash F$ won't be.

Dropping contraction permits the logic to be sensitive to these distinctions

Paradox

The naive set comprehension scheme is $t \in \{y : A(y)\} \leftrightarrow A(t)$

In terms of sequent rules, the biconditional is captured by

$$\frac{A(t), X \vdash Y}{t \in \{x : A(x)\}, X \vdash Y}\ [\in L] \qquad \frac{X \vdash Y, A(t)}{X \vdash Y, t \in \{x : A(x)\}}\ [\in R]$$

As is well-known, in classical and intuitionistic logic, it leads to paradox

Let $R = \{x : x \notin x\}$

$$
\begin{prooftree}
\AxiomC{$R \in R \vdash R \in R$}
\RightLabel{[$\neg R$]}
\UnaryInfC{$\vdash R \in R, R \notin R$}
\RightLabel{[$\in R$]}
\UnaryInfC{$\vdash R \in R, R \in R$}
\RightLabel{[WR]}
\UnaryInfC{$\vdash R \in R$}
\AxiomC{$R \in R \vdash R \in R$}
\RightLabel{[$\neg L$]}
\UnaryInfC{$R \notin R, R \in R \vdash$}
\RightLabel{[$\in L$]}
\UnaryInfC{$R \in R, R \in R \vdash$}
\RightLabel{[WL]}
\UnaryInfC{$R \in R \vdash$}
\RightLabel{[Cut]}
\BinaryInfC{$\vdash$}
\end{prooftree}
$$

Curry's paradox

$$\text{Let } C = \{x : x \in x \to p\}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \in C \vdash C \in C \qquad p \vdash p}{C \in C \to p, C \in C \vdash p}\,[\to L]}{C \in C, C \in C \vdash p}\,[\in L]}{C \in C \vdash p}\,[WL]}{\vdash C \in C \to p}\,[\to R]}{\vdash C \in C}\,[\in R] \qquad \dfrac{\dfrac{\dfrac{C \in C \vdash C \in C \qquad p \vdash p}{C \in C \to p, C \in C \vdash p}\,[\to L]}{C \in C, C \in C \vdash p}\,[\in L]}{C \in C \vdash p}\,[WL]}{\vdash p}\,[Cut]$$

Paradox and contraction

As observed by Haskell Curry, contraction is essentially involved in Curry's paradox

Dropping contraction, in all its forms, permits one to have the naive set comprehension rules, and biconditionals, non-trivially

The same goes for the full set of Tarski biconditionals: $T\langle A\rangle \leftrightarrow A$

Multiplicative, additive linear logic (MALL) is obtained by taking permutation as the only structural rule and using both the additive and multiplicative sets of rules

Some sequents provable in MALL

- $A \to (B \to C) \vdash B \to (A \to C)$
- $A \circ (B \vee C) \dashv\vdash (A \circ B) \vee (A \circ C)$
- $A + (B \wedge C) \dashv\vdash (A + B) \wedge (A + C)$
- $(A + B) \vee (A + C) \vdash A + (B \vee C)$

Unprovable

Some sequents unprovable in MALL

- $A \to (A \to B) \vdash A \to B$
- $A \circ (B + C) \vdash (A \circ B) + (A \circ C)$
- $A \vee (B \wedge C) \vdash (A \vee B) \wedge (A \vee C)$
- $(A \wedge B) \to C \vdash A \to (B \to C)$
- $A \circ B \vdash A$

One can *expand* the vocabulary to regain some of the structural rules

Girard did this with the exponentials of linear logic

Introduce two new unary connectives, ! and ?

Rules

$$\text{If } X \text{ is } A_1, \ldots, A_n, \ !X \text{ is } !A_1, \ldots, !A_n$$

$$\frac{X(A) \vdash Y}{X(!A) \vdash Y}\,[!L] \qquad \frac{!X \vdash A, ?Y}{!X \vdash !A, ?Y}\,[!R]$$

$$\frac{X \vdash Y}{!A, X \vdash Y}\,[K!L] \qquad \frac{X(!A, !A) \vdash Y}{X(!A) \vdash Y}\,[W!L]$$

$$\frac{X \vdash Y(A)}{X \vdash Y(?A)}\,[?R] \qquad \frac{!X, A \vdash ?Y}{!X, ?A \vdash ?Y}\,[?L]$$

$$\frac{X \vdash Y}{X \vdash Y, ?A}\,[K?R] \qquad \frac{X \vdash Y(?A, ?A)}{X \vdash Y(?A)}\,[W?R]$$

Idea

The exponentials let one ignore the resource sensitivity

$!A$ says that A may be used as a premiss as many times as you want

Similarly, $?A$ says A may be used as a conclusion as much as one wants

Provable

Some sequents provable in MALL with exponentials

- $!A \vdash A$
- $A \vdash\ !B \rightarrow A$
- $!A \rightarrow (!A \rightarrow B) \vdash\ !A \rightarrow B$
- $!(A \rightarrow B) \vdash\ !A \rightarrow B$

Embedding

One can define an *embedding* t of classical logic LK into linear logic with exponentials LLE so that the following are equivalent

- $t(X) \vdash t(Y)$ is derivable in LLE
- $X \vdash Y$ is derivable in LK

Linear logic with exponentials is an interesting system and, like the full logic R

Free choice

"You can have coffee or tea" seems to imply "you can have coffee" and "you can have tea"

This is the phenomenon of *free choice permission*

Barker has argued that the way to understand free choice is by using the connectives of linear logic

Permission is treated as a kind of resource, and it falls out naturally that the first entails each of the others, although it doesn't give both together.

For more

For more on linear logic, see Davoren's "A Lazy Logician's Guide to Linear Logic"
`https://blogs.unimelb.edu.au/logic/files/2015/11/Davoren-LLGLL-2cedcbe.pdf`

See also Restall's *Introduction to Substructural Logics*

Take two English noun phrase, *birds* and *spiders*, and an English verb, *eat*

The order in which these are combined *matters*

Compare: Birds eat spiders, and Spiders eat birds

Modality

Sometimes entailment, $\rightarrow$, is taken to have some kind of necessitating, *modal force*

Just because p happens to be the case, it is not correct to infer that q is entailed by the fact that p entails q

In that case, we don't want $A \vdash (A \rightarrow B) \rightarrow B$

$$
\cfrac{\cfrac{\cfrac{A \vdash A \qquad B \vdash B}{A \rightarrow B, A \vdash B}\,[\rightarrow L]}{A, A \rightarrow B \vdash B}\,[CL]}{A \vdash (A \rightarrow B) \rightarrow B}\,[\rightarrow R]
$$

Permutation

In both these applications, the *order* of the premises matter

Both of these applications motivate dropping the Permutation rules

Dropping Permutation lets us draw more distinctions

More arrows

The usual arrow rules are the following

$$\frac{X, A \vdash B}{X \vdash A \to B} \qquad \frac{X \vdash A \quad Y(B) \vdash C}{Y(A \to B, X) \vdash C}$$

We can add another arrow

$$\frac{A, X \vdash B}{X \vdash B \leftarrow A} \qquad \frac{X \vdash A \quad Y(B) \vdash C}{Y(X, B \leftarrow A) \vdash C}$$

Distinctions

In the presence of Permutation, this distinction collapses

$$A \to B \dashv\vdash B \leftarrow A$$

Without Permutation, the distinction stands

We can *also* add a second negation following the same pattern

Lambek calculus

The *Lambek calculus* is a proof system for categorial grammar

We take the rules for $\circ$, together with the rules for $\rightarrow$ and $\leftarrow$

We do not use any structural rules

This gives a basic categorial grammar

The atomic letters are treated as different lexical items, possibly typed, from a given lexicon

Derivable

The following are derivable

- $A \to B \vdash (C \to A) \to (C \to B)$
- $B \leftarrow A \vdash (B \leftarrow C) \leftarrow (A \leftarrow C)$
- $A \to (B \to C) \vdash (A \circ B) \to C$
- $(C \leftarrow B) \leftarrow A \vdash C \leftarrow (A \circ B)$
- $A \vdash B \leftarrow (A \to B)$

Underivable

The following are underivable

- $A \rightarrow B \vdash (B \rightarrow C) \rightarrow (A \rightarrow C)$
- $A \circ B \vdash B \circ A$
- $C \leftarrow B, B \vdash C$
- $A, A \rightarrow B \vdash B$

For more

For more on Lambek Calculus, see Morrill's *Categorial Grammar*, van Benthem's *Language in Action*, or Moot and Retoré's *Logic of Categorial Grammar*

For more on modal restrictions on permutation, see Anderson and Belnap's *Entailment*

REVISITING CUT ELIMINATION

Cut revisited

Here is the form of Cut appropriate to (single conclusion) substructural logic

$$\frac{X \vdash A \qquad Y(A) \vdash B}{Y(X) \vdash B}\ [\mathrm{Cut}]$$

Looking at the proof of Cut Elimination yesterday, it turns out that we used *lots* of Weakening, Contraction, and Permutation

In the substructural setting, we have to be a bit more careful

$$\frac{X \vdash A \qquad Y[A] \vdash B}{Y[X] \vdash B}\,[\mathrm{Mix}]$$

$Y[X]$ is obtained by replacing all copies of A in Y with X

Mix eliminates *all* the copies of A in Y

Mix helped us get around the problem with Contraction, but it would sometimes eliminate too many copies, which required weakening some back in

Dropping Weakening

Without Weakening, we cannot show Mix admissible

Rather than Mix, show that Multicut is admissible

$$\frac{X \vdash A \qquad Y[A] \vdash B}{Y[X] \vdash B}\,[\text{Multicut}]$$

In the Multicut rule, $Y[A]$ is Y with some $n \geq 1$ occurrences of A highlighted, and $Y[X]$ is obtained by replacing the highlighted occurrences of A in $Y[A]$ with X

The proof strategy proceeds much as with Mix

Dropping Contraction

If one drops contraction, then one does not need to show Mix admissible, going directly for Cut

Rather than use a *double induction*, one can instead use a simpler, single induction proof

This is because without Contraction, the elimination procedure will not double up any proof branches

So one can simply use the number of nodes above a Cut as the Cut complexity

Dropping Permutation

Without Permutation, we have to be careful about how exactly each rule is stated and how Cut is stated

We *cannot* use Mix without Permutation, so we had better drop Contraction as well

The proof of Cut Elimination can proceed directly, using a single induction on Cut complexity

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