# Proof Theory: Logical and Philosophical Aspects

## Class 1: Foundations


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To introduce *proof theory*, with a focus on its applications in philosophy, linguistics and computer science.

Introduce the basics of sequent systems and Gentzen's *Cut Elimination Theorem*.

Sequents

Left and Right Rules

Structural Rules

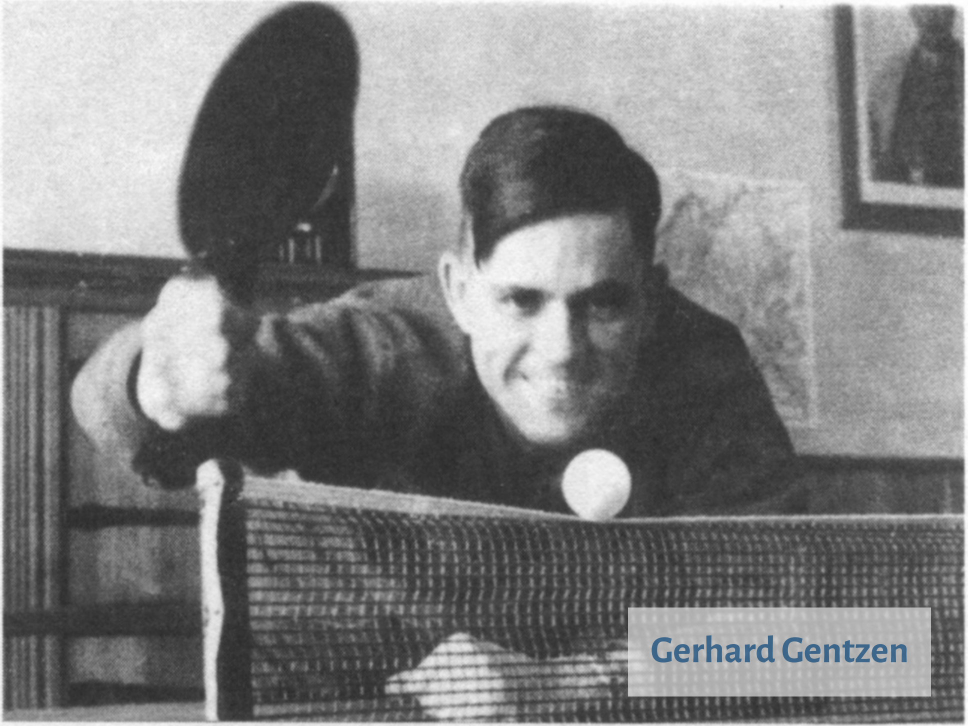
Cut Elimination

Consequences

Onward to Classical Logic

Another approach to Cut Elimination

# SEQUENTS



Gerhard Gentzen

# Natural deduction to sequents

$$
\dfrac{\dfrac{\dfrac{A \to (B \to C) \qquad A^{[1]}}{B \to C}\,[\to E] \qquad B}{C}\,[\to E]}{A \to C}\,[\to I]\ 1
$$

- $A \to (B \to C), A \vdash B \to C$
- $A \to (B \to C), A, B \vdash C$
- $A \to (B \to C), B \vdash A \to C$

Sequents record consequences of premises

Lay out relations explicitly

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# Natural deduction to sequents

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- $A \to (B \to C), B \vdash A \to C$

Sequents record consequences of premises

Lay out relations explicitly

# Sequents

$$X \vdash A$$

$X$ is a *sequence*
Could also use *sets*, *multisets*, or more *general structures*

# Sequent proofs

Rather than introduction and elimination rules,
sequent systems use *left* and *right* introduction rules

Proofs are trees built up by rules.

There are two sorts of rules: *Connective rules* and *structural rules*

# LEFT AND RIGHT RULES

# Left and right rules

$$\frac{X, A, Y \vdash C}{X, A \land B, Y \vdash C}\,[\land L_1] \qquad \frac{X, A, Y \vdash C \quad U, B, V \vdash C}{X, U, A \lor B, Y, V \vdash C}\,[\lor L]$$

$$\frac{X, B, Y \vdash C}{X, A \land B, Y \vdash C}\,[\land L_2] \qquad \frac{X \vdash A}{X \vdash A \lor B}\,[\lor R_1]$$

$$\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \land B}\,[\land R] \qquad \frac{X \vdash B}{X \vdash A \lor B}\,[\lor R_2]$$

# Left and right rules

$$\frac{X \vdash A}{X, \neg A \vdash}\ [\neg L]$$

$$\frac{X \vdash A \qquad Y, B, Z \vdash C}{Y, X, A \to B, Z \vdash C}\ [\to L]$$

$$\frac{X, A \vdash}{X \vdash \neg A}\ [\neg R]$$

$$\frac{X, A \vdash B}{X \vdash A \to B}\ [\to R]$$

$$\dfrac{\dfrac{\dfrac{p \vdash p}{p \land r \vdash p}\,[\land L_1]}{p \land r \vdash p \lor q}\,[\lor R_1] \qquad \dfrac{q \vdash q}{q \vdash p \lor q}\,[\lor R_2]}{\dfrac{(p \land r) \lor q \vdash p \lor q \qquad\qquad s \vdash s}{(p \land r) \lor q, s \vdash (p \lor q) \land s}}$$

$$\dfrac{\dfrac{\dfrac{p \vdash p}{p, \neg p \vdash}\,[\neg L]}{p \vdash \neg\neg p}\,[\neg R]}{\vdash p \to \neg\neg p}\,[\to R]$$

# STRUCTURAL RULES

# Identity axiom

$$p \vdash p$$

What about arbitrary formulas in the axioms?

Either prove a theorem or take generalizations as axioms

$$A \vdash A$$

# Weakening

$$\frac{X, Y \vdash C}{X, A, Y \vdash C}\,[KL] \qquad \frac{X \vdash}{X \vdash A}\,[KR]$$

# Contraction

$$\frac{X, A, A, Z \vdash C}{X, A, Z \vdash C} \; [WL]$$

# Permutation

$$\frac{X, A, B, Z \vdash C}{X, B, A, Z \vdash C}\ [\mathrm{CL}]$$

# Cut

$$\frac{X \vdash A \qquad Y, A, Z \vdash B}{Y, X, Z \vdash B}\,[\text{Cut}]$$

The system with all the connective rules, the axiom rule, and the structural rules [KL], [KR], [CL], [*W*L] will be LJ

LJ+Cut will be LJ with the addition of [Cut]

# Sequent Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{q, p \vdash p}\,[KL]}{p \wedge q, p \vdash p}\,[\wedge L_2]}{p, p \wedge q \vdash p}\,[CL]}{p \wedge q, p \wedge q \vdash p}\,[\wedge L_1]}{p \wedge q \vdash p}\,[WL]$$

$$\dfrac{\dfrac{p \vdash p}{p, \neg p \vdash}\,[\neg L]}{p, \neg p \vdash q}\,[KR]$$

Cut is the only rule in which formulas *disappear*
going from premiss to conclusion

A proof is *Cut-free* iff it does not contain an application of the Cut rule

If you know there is a Cut-free derivation of a sequent,
it can make finding a proof easier

# CUT ELIMINATION

# Hauptsatz

Gentzen called his Elimination Theorem the *Hauptsatz*

He showed that for sequent derivable with a Cut, there is a Cut-free derivation

# Admissibility and derivability

$$\frac{S_1, \ldots, S_n}{S}\,[R]$$

A rule [R] is *derivable* iff given derivations of $S_1, \ldots, S_n$, one can extend those derivations to obtain a derivation of $S$

A rule [R] is *admissible* iff if there are derivations of $S_1, \ldots, S_n$, then there is a derivation of $S$

## Admissibility and derivability

The rule

$$\frac{X, A, B \vdash C}{X, A \wedge B \vdash C}\ [\wedge L_3]$$

is *derivable*

The Elimination Theorem shows that Cut is *admissible*, even though it is not derivable

# Theorem

If there is a derivation of $X \vdash A$ in $\mathrm{LJ} + \mathrm{Cut}$,
then there is a Cut-free derivation of $X \vdash A$

# Auxiliary concepts

In the Cut rule,

$$\dfrac{(\mathrm{L})\quad X \vdash A \qquad Y, A, Z \vdash B \quad (\mathrm{R})}{(\mathrm{C})\quad Y, X, Z \vdash B}\,[\mathrm{Cut}]$$

the displayed A is the *cut formula*

There are two ways of measuring the complexity of a Cut:
*grade* and *rank* of cut formula

# Auxiliary concepts

The *grade*, $\gamma(A)$, of $A$ is the number of logical symbols in $A$.

The *left rank*, $\rho_L(A)$, of $A$ is the length of the longest path starting with $(L)$ containing $A$ in the succeedent

The *right rank*, $\rho_R(A)$, is the length of the longest path starting with $(R)$ containing $A$ in the antecedent

The *rank*, $\rho(A)$, is $\rho_L(A) + \rho_R(A)$

Double induction on grade and rank of a Cut

Outer induction is on grade, inner induction is on rank

# Proof strategy

Show how to move Cuts above rules,
lowering left rank, then right rank,
then lowering grade

*Parametric* Cuts are cuts in which
the Cut formula is not the one displayed in a rule,

and *principal* Cuts are ones in which
the Cut formula is the one displayed in a rule

If one premiss of a Cut comes via an axiom or a weakening step,
then the Cut can be eliminated entirely

# Eliminating Cuts: Parametric

$$\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X' \vdash A}}{X \vdash A}\,[\sharp] \qquad \overset{\vdots\,\pi_2}{A, Y \vdash C}}{X, Y \vdash C}\,[\mathrm{Cut}] \qquad\qquad \dfrac{\overset{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\overset{\vdots\,\pi_2}{A, Y' \vdash C}}{A, Y \vdash C}\,[\flat]}{X, Y \vdash C}\,[\mathrm{Cut}]$$

$$\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X' \vdash A} \qquad \overset{\vdots\,\pi_2}{A, Y \vdash C}}{X', Y \vdash C}\,[\mathrm{Cut}]}{X, Y \vdash C}\,[\sharp] \qquad\qquad \dfrac{\dfrac{\overset{\vdots\,\pi_1}{X \vdash A} \qquad \overset{\vdots\,\pi_2}{A, Y' \vdash C}}{X, Y' \vdash C}\,[\mathrm{Cut}]}{X, Y \vdash C}\,[\flat]$$

# Eliminating Cuts: Parametric

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ X, A \vdash C\end{matrix} \qquad \begin{matrix}\vdots\,\pi_2\\ Y, B \vdash C\end{matrix}}{X, Y, A \lor B \vdash C}\,[\lor L] \qquad \begin{matrix}\vdots\,\pi_3\\ C, Z \vdash D\end{matrix}}{X, Y, A \lor B, Z \vdash D}\,[Cut]$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ X, A \vdash C\end{matrix} \qquad \begin{matrix}\vdots\,\pi_3\\ C, Z \vdash D\end{matrix}}{X, A, Z \vdash D}\,[Cut] \qquad \dfrac{\begin{matrix}\vdots\,\pi_2\\ Y, B \vdash C\end{matrix} \qquad \begin{matrix}\vdots\,\pi_3\\ C, Z \vdash D\end{matrix}}{Y, B, Z \vdash D}\,[Cut]}{X, Y, A \lor B, Z, Z \vdash D}\,[\lor L]}{X, Y, A \lor B, Z \vdash D}\,[WL]$$

# Eliminating Cuts: Principal

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array}}{X \vdash A \lor B}\,[\lor R] \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ A, Y \vdash C\end{array} \quad \begin{array}{c}\vdots\,\pi_3\\ B, Z \vdash C\end{array}}{A \lor B, Y, Z \vdash C}\,[\lor L]}{X, Y, Z \vdash C}\,[\mathrm{Cut}]$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array} \quad \begin{array}{c}\vdots\,\pi_2\\ A, Y \vdash C\end{array}}{X, Y \vdash C}\,[\mathrm{Cut}]}{X, Y, Z \vdash C}\,[KL]$$

# Elmiinating Cuts: Principal

$$\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X, A \vdash B}}{X \vdash A \to B}\,[\to R] \qquad \dfrac{\overset{\vdots\,\pi_2}{U \vdash A} \qquad \overset{\vdots\,\pi_3}{Y, B, Z \vdash C}}{Y, U, A \to B, Z \vdash C}\,[\to L]}{Y, U, X, Z \vdash C}\,[\text{Cut}]$$

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_2}{U \vdash A} \qquad \overset{\vdots\,\pi_1}{X, A \vdash B}}{X, U \vdash B}\,[\text{Cut}] \qquad \overset{\vdots\,\pi_3}{Y, B, Z \vdash C}}{Y, X, U, Z \vdash C}\,[\text{Cut}]}{Y, U, X, Z \vdash C}\,[\text{CL}]$$

# Eliminating Cuts: Special Cases

$$\dfrac{\dfrac{\vdots\,\pi_1}{X \vdash p} \qquad p \vdash p}{X \vdash p}\,[\mathrm{Cut}] \qquad\qquad \dfrac{\dfrac{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\dfrac{\vdots\,\pi_2}{Y \vdash C}}{A, Y \vdash C}\,[\mathrm{KL}]}{X, Y \vdash C}\,[\mathrm{Cut}]$$

$$\dfrac{\vdots\,\pi_1}{X \vdash p} \qquad\qquad \dfrac{\dfrac{\vdots\,\pi_2}{Y \vdash C}}{X, Y \vdash C}\,[\mathrm{KL}]$$

# Contraction

Contraction causes some problems for this proof

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_2\\ A, A, Y \vdash C\end{array}}{A, Y \vdash C}\,[WL]}{X, Y \vdash C}\,[\text{Cut}]$$

$$\dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\,\pi_1\\ X \vdash A\end{array} \qquad \begin{array}{c}\vdots\,\pi_2\\ A, A, Y \vdash C\end{array}}{X, A, Y \vdash C}\,[\text{Cut}]}{X, X, Y \vdash C}\,[\text{Cut}]$$

# Solution

Use a stronger rule that removes *all* copies of the formula in one go

$$\frac{X \vdash A \qquad Y \vdash B}{X, Y^{-A} \vdash B}\,[\mathrm{Mix}]$$

$Y$ is required to contain at least one copy of $A$

We can extend the proof to cover contraction by proving that Mix is admissible

The admissibility of Mix has the admissibility of Cut as a corollary

# Mix cases

$$\dfrac{\dfrac{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\dfrac{\vdots\,\pi_2}{A, A, Y \vdash C}}{A, Y \vdash C}\,[WL]}{X, Y^{-A} \vdash C}\,[Mix]$$

$$\dfrac{\dfrac{\vdots\,\pi_1}{X \vdash A} \qquad \dfrac{\vdots\,\pi_2}{A, A, Y \vdash C}}{X, Y^{-A} \vdash C}\,[Mix]$$

# Eliminating Mix: Complications with rank

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ X, A \vdash\end{matrix}}{X \vdash \neg A}\,[\neg R] \qquad \dfrac{\begin{matrix}\vdots\,\pi_2\\ \neg A, Y \vdash A\end{matrix}}{\neg A, Y, \neg A \vdash}\,[\neg L]}{X, Y^{-\neg A} \vdash}\,[Mix]$$

$$\dfrac{\dfrac{X, X^{-\neg A}, Y^{-\neg A} \vdash}{}\,}{}$$

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ X, A \vdash\end{matrix}}{X \vdash \neg A}\,[\neg R] \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\,\pi_1\\ X, A \vdash\end{matrix}}{X \vdash \neg A}\,[\neg R] \qquad \begin{matrix}\vdots\,\pi_2\\ \neg A, Y \vdash A\end{matrix}}{X, Y^{-\neg A} \vdash A}\,[Mix]}{X, Y^{-\neg A}, \neg A \vdash}\,[\neg L]}{X, X^{-\neg A}, Y^{-\neg A} \vdash}\,[Mix]}{X, Y^{-\neg A} \vdash}\,[WL]$$

# Eliminating Mix: Complications with grade

$$\dfrac{\dfrac{\overset{\vdots\,\pi_1}{X, A \vdash}}{X \vdash \neg A}\,[\neg R] \qquad \dfrac{\overset{\vdots\,\pi_2}{Y \vdash A}}{Y, \neg A \vdash}\,[\neg L]}{X, Y \vdash}\,[Mix]$$

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_2}{Y \vdash A} \qquad \overset{\vdots\,\pi_1}{X, A \vdash}}{Y, X^{-A} \vdash}\,[Mix]}{Y, X \vdash}\,[KL]}{X, Y \vdash}\,[CL]$$

# CONSEQUENCES

# Subformula property

In rules besides Cut, all formulas
appearing in the premises appear in the conclusion

This is the *Subformula Property*

In Cut-free derivations, formulas not appearing in the end sequent don't appear in the rest of the proof, which makes proof search easier

# Conservative extension

One consequence relation $\vdash^+$ is a *conservative extension* of another consequence relation $\vdash$, just in case the language of $\vdash^+$ extends that of $\vdash$ and if $X \vdash^+ A$ then $X \vdash A$, when $X, A$ are in the language of $\vdash$

The Elimination Theorem yields conservative extension results via the Subformula Property

If $X$ and $A$ are all in the base language, then the Subformula Property guarantees that a proof of $X \vdash^+ A$ will not use any of the rules not available for $\vdash$.

# Consistency

In the presence of [KL] and [KR], $\emptyset \vdash \emptyset$ says everything implies everything.

The Elimination Theorem implies that that is not provable

Suppose that it is. There is then a Cut-free derivation.
All the axioms have formulas on both sides,
and no rules delete formulas.
So there is no derivation of $\emptyset \vdash \emptyset$.

# Unprovability results

Similar arguments can be used to show that $\vdash p \vee \neg p$ isn't derivable.

How would a Cut-free derivation go?
The last rule would have to be $[\vee R]$,
applied to either $\vdash p$ or $\vdash \neg p$,
neither of which is provable

# Disjunction property

Suppose that $\vdash A \lor B$ is derivable

There is a Cut-free derivation,
so the last rule has to be $[\lor R]$.
So either $\vdash A$ or $\vdash B$ is derivable.

# ONWARD TO CLASSICAL LOGIC

# A seemingly magical fact

LJ is complete for *intuitionistic logic*

A sequent system for classical logic, LK, can be obtained by allowing the succedent to contain more than one formula

$A_1, \ldots, A_k \vdash B_1, \ldots, B_n$ says that if all the $A_i$s hold, then one of the $B_j$s does too.

Ian Hacking remarked that this seemed magical, and it was explored in Peter Milne's paper "Harmony, Purity, Simplicity, and a 'Seemingly Magical Fact'"

# Left and right rules

$$\frac{X, A, Y \vdash Z}{X, A \wedge B, Y \vdash Z}\ [\wedge L_1] \qquad \frac{X, A, Y \vdash Z \qquad U, B, V \vdash W}{X, U, A \vee B, Y, V \vdash Z, W}\ [\vee L]$$

$$\frac{X, B, Y \vdash Z}{X, A \wedge B, Y \vdash Z}\ [\wedge L_2] \qquad \frac{X \vdash Y, A, Z}{X \vdash Y, A \vee B, Z}\ [\vee R]$$

$$\frac{X \vdash Y, A, Z \qquad U \vdash V, B, W}{X, U \vdash Y, V, A \wedge B, Z, W}\ [\wedge R] \qquad \frac{X \vdash Y, B, Z}{X \vdash Y, A \vee B, Z}\ [\vee R]$$

# Left and right rules

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\,[\neg L] \qquad \frac{X \vdash Y, A, Z \qquad U, B, V \vdash W}{U, X, A \to B, V \vdash Y, Z, W}\,[\to L]$$

$$\frac{X, A \vdash Y}{X \vdash \neg A, Y}\,[\neg R] \qquad \frac{X, A \vdash B, Y}{X \vdash A \to B, Y}\,[\to R]$$

# Weakening

$$\frac{X \vdash Y}{A, X \vdash Y} \text{[KL]}$$

$$\frac{X \vdash Y}{X \vdash Y, A} \text{[KR]}$$

# Contraction

$$\frac{X, A, A, Z \vdash Y}{X, A, Z \vdash Y}\ [WL]$$

$$\frac{X \vdash Y, A, A, Z}{X \vdash Y, A, Z}\ [WR]$$

# Permutation

$$\frac{X, A, B, Z \vdash Y}{X, B, A, Z \vdash Y}\ [CL]$$

$$\frac{X \vdash Y, A, B, Z}{X \vdash Y, B, A, Z}\ [CR]$$

# Classical proofs

$$
\begin{prooftree}
\AxiomC{$p \vdash p$}
\RightLabel{[$\neg R$]}
\UnaryInfC{$\vdash p, \neg p$}
\RightLabel{[$\vee R_1$]}
\UnaryInfC{$\vdash p \vee \neg p, \neg p$}
\RightLabel{[$\vee R_2$]}
\UnaryInfC{$\vdash p \vee \neg p, p \vee \neg p$}
\RightLabel{[$WR$]}
\UnaryInfC{$\vdash p \vee \neg p$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$q \vdash q$}
\RightLabel{[$\neg L$]}
\UnaryInfC{$q, \neg q \vdash$}
\RightLabel{[$\wedge L_1$]}
\UnaryInfC{$q \wedge \neg q, \neg q \vdash$}
\RightLabel{[$\wedge L_2$]}
\UnaryInfC{$q \wedge \neg q, q \wedge \neg q \vdash$}
\RightLabel{[$WL$]}
\UnaryInfC{$q \wedge \neg q \vdash$}
\end{prooftree}
$$

# Some features

An Elimination Theorem is provable for LK

Since LK can have multiple formulas on the right,
one can apply [$WR$] as well as the connective rules
as the final rule in a proof of $\vdash A$

Consequently, LK does not have the Disjunction Property

# ANOTHER APPROACH TO CUT ELIMINATION

# Alternatives

Different ways of setting up a sequent system may lead
to different ways to prove the Elimination Theorem

One way, explored by Dyckhoff, Negri and von Plato, originally due to Dragalin, is to *absorb* the structural rules into the connective rules

There are no structural rules in this system,
but their effects are implicit in the connective rules

Instead of sequences in the sequents, we will use multisets

# Rules

Identity axiom: $X, p \vdash p, Y$

$$\frac{A, B, X \vdash Y}{A \wedge B, X \vdash Y}\,[\wedge L]$$

$$\frac{X \vdash Y, A \qquad X \vdash Y, B}{X \vdash Y, A \wedge B}\,[\wedge R]$$

$$\frac{X \vdash Y, A, B}{X \vdash Y, A \vee B}\,[\vee R]$$

$$\frac{A, X \vdash Y \qquad B, X \vdash Y}{A \vee B, X \vdash Y}\,[\vee L]$$

*Weakening Admissibility*: If $X \vdash Y$ is provable in $n$ steps, then $X' \vdash Y'$ is provable in at most $n$ steps, where $X \subseteq X'$, $Y \subseteq Y'$

*Inversion Lemma*: If the conclusion of a rule is provable in $n$ steps, then the premiss of the rule is provable in at most $n$ steps

*Contraction Admissibility*: If $A, A, X \vdash Y$ is provable in $n$ steps, then $A, X \vdash Y$ is; and if $X \vdash Y, A, A$ is provable in at most $n$ steps, then $X \vdash Y, A$ is.

These are *height-preserving admissibility* lemmas

# Elimination Theorem

One can show Cut is admissible

Since there are no contraction rules, we do not have to use Mix

Since there are fewer rules, there are fewer cases to check

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## Next Class

Substructural Logics and their Proof Theory

# THANK YOU!

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