FROM MODAL DISCOURSE TO POSSIBLE WORLDS

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The possible-worlds semantics for modality says that a sentence of the form ***possibly*** α is true in a world w iff α itself is true in some w' accessible from w. One area of debate is this. You may be happy to accept that talk about what is possible is *equivalent* to talk about what happens in a possible world in just the same way as talk about the past or future is equivalent to talk about what happens at other times. But you may be concerned about the *status* of possible worlds. What are they? Do they really exist? Are they concrete or abstract? Are they *linguistic* entities? Is knowing the meaning of a sentence knowing which worlds it is true in? Is the real world different and special from the merely possible worlds? You can ask these questions in just the same way as you can ask about the status of times, as in the dispute between those who think that only the present is real, and those who think that all times are equally real. If you are happy to put the debate about modality in terms of the status of possible worlds this paper is not addressed to you. This paper is addressed to those who think that modal discourse is in order as it is, and that any reference to worlds need not even arise in the first place. I'm imagining the sort of person who says: "Look, when I say that Auckland *might* have been the capital of New Zealand, I mean just that. I *don't* mean that there is a possible world in which Auckland *is* the capital of New Zealand. I don't talk of worlds at all. I just mean that Auckland might have been the capital of New Zealand, and that's an end to the matter!"[1]

The *structural response* to such modal primitivism is that, like it or not, modal sentences of natural language can be shewn to have at least the *structure* of quantification over worlds, so that the debate about their status and nature is one that we need to have given that we accept that modal sentences have truth values.[2] The structural response cites the completeness results in modal logic. Given a notion of the consistency of a set of sentences, and given classical propositional logic, one can easily prove that every consistent set of sentences can be embedded in a 'maximal consistent' set in which for every sentence α either α is in the set or ***not*** α is in the set. This carries no commitment to the nature of worlds, and in the standard completeness proofs they are maximal consistent sets of wff of a formal language. From this it follows that provided the truth of ***possibly*** α (meaning that α is possible) requires that α be consistent then α is true in at least one possible world.

My aim in this paper is to defend the structural response, but in so doing it will turn out that the principles required have to be stronger than those assumed in standard modal logic. In fact they have to be akin to those used in what is called *infinitary* logic.[3] Recall an important point about the construction of maximal consistent sets. The rules of proof in standard systems of logic are *finitary*. Any particular derivation (of a contradiction or of any other formula) involves only finitely many premises. For this reason the criterion of consistency in standard logic is that a set of wff is *consistent* (in the logic in question) iff every one of its finite subsets is, where a finite set is

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[1]Davidson 1967 suggests that the study of meaning need involve no reference to *entities*. And those who advocate adverbial theories of this that or the other seem to feel that the use of adverbs excuses them from doing any further semantical analysis. (Adverbial theories of ***possibly*** may be less common because of the influence of modal logic!)

[2]Some ways of interpreting Quine's views might suggest that he thinks that there is no such thing as modality at least in respectable literal 'scientific' discourse — but that seems a hard doctrine to maintain. It seems to say that we never really debate about whether or not we ought to make it the case that α or that ***not***-α. Quine is difficult to pin down on precisely this question. On p. 143 of Quine 1953 he speaks of the "so called *modal* contexts" (Quine's italics), and further down he speaks of the 'strict' modalities as based on "the putative notion of analyticity." See Lycan 1994, pp. 172-176, for an illustration of the ubiquity of modal locutions in ordinary language.

[3]Some remarks about infinitary logic are made in the appendix below.

consistent with respect to that system of logic iff you cannot derive a contradiction from its members. A *maximal*-consistent set is a consistent set in which for every wff α, either α or ***not*** α is in it. (This paper assumes classical logic.) The key principle used in the construction of maximal consistent sets is the principle that if Λ is consistent then so is either $\Lambda \cup \{\alpha\}$ or $\Lambda \cup \{$***not*** $\alpha\}$, for any wff α, and this is ensured by noting that if neither were consistent you could prove, for some conjunction β of wff in Λ, that both

(1) $\vdash$ ***not*** (β ***and*** α)

and

(2) $\vdash$ ***not*** (β ***and not*** α).

But then, by ordinary propositional logic,

(3) $\vdash$ ***not*** β

contradicting Λ's consistency. The finitary feature of the logic ensures that when you form a sequence of Λs by beginning with a set assumed consistent, and adding either α or ***not*** α for each wff α in turn, the consistency of the union of this sequence follows from the consistency of each of its finite subsets; and the consistency of each finite subset follows from the fact that (by construction) each finite subset is contained in one of the (consistent) Λs.

I now want to consider a corresponding construction, but based on an intuitive *semantic* notion of possibility. Here we assume that there is a word ***possibly*** which is intended to mean that a proposition is *possibly* true, in some intuitive sense of possibility. Also assume that the language has the words ***not***, where ***not*** α is true iff α is false, and ***and***, where α ***and*** β is true iff both α and β are. (These words have the meaning they have in classical logic.) We want to shew that any sentence α for which ***possibly*** α is true can be extended to a set which is maximally possible in the sense that all its members can be true together, in whatever sense of possibility is in question, and, for each sentence β, either β is in it or ***not*** β is in it. No other assumptions are made about the language except that it is an interpreted language in which every sentence has a meaning, and is either true or false. If you ask where this latter assumption comes from the answer is that I am taking the claim that modal discourse is in order as it is to entail that sentences of the form ***possibly*** α have truth values in a language in which all sentences have truth values. The paper is *not* addressing the problem of how to get to truth and falsity in the first place. Notice that it is crucial to the enterprise that we do *not* talk of worlds in constructing a maximally possible set of sentences, since the set is supposed to be something which, in a sense *is* a world. (I say 'in a sense' because, once we have *got* such a set of sentences we can ask whether it *is* a world, or merely represents a world.)

The version of the claim that modal discourse has the structure of quantification over worlds which I intend to establish in this paper is this:

1. There is a collection W of sets of sentences such that each set w in this collection is possible in that it is possible for all its members to be true together in some sense of ***possibly***, and has the property that for every sentence α, either α is in w or ***not*** α is in w.

2. For each sense of ***possible*** there is a relation of accessibility between members of W such that w' is accessible from w iff for every sentence α in w', ***possibly*** α is in w.

3. ***possibly*** α is in w iff there is some w' accessible from w which contains α.

A debate about what possible worlds are is a debate about what precisely it is to claim that all the members of a maximally possible set might be true together. One might say that a *possible world* is *whatever it is* which would make all the members of a maximally possible set of sentences true if they were true. The aim of the present paper is not to advance the debate about what worlds are, but to argue that it is a debate that semantics needs.

What follows is parallel to the standard construction of maximal consistent sets. However, both because there may be those who are not familiar with this construction, and because the justification of the steps is semantic and not proof-theoretical, I will present it in detail. Assume *per impossibile* that an omniscient speaker is presented with a sentence of the form ***possibly*** α. Mark this special α as α^*, and assume that our omniscient speaker agrees that

(4) ***possibly*** α^*

is true. We don't say just *what* sense of possibility this is. All that matters is that it is the *same* sense all the way through. Now further assume that we can order the sentences of the language in such a way that each will eventually appear. Imagine the sentences appearing in order in a kind of infinite 'sentence dictionary'. We gradually add sentences to a 'shopping cart' in the following way. We begin by putting α^* in the cart. Now consider the first sentence β_1 in the dictionary. Ask the speaker which of

(5) ***possibly***(α^* ***and*** β_1)

or

(6) ***possibly***(α^* ***and not*** β_1)

is true. At least one must be. Why? Suppose it might have been raining and take β_1 to be a sentence meaning

(7) It is over 15ºC.

Suppose it were not possible to have rain when it *is* over 15ºC, and not possible to have rain when it is *not* over 15ºC. The only way this could be would be if, in that same sense of possible, it is not possible to have rain at all. If rain *is* possible, then either rain is possible with β_1, or rain is possible without β_1. Notice that it does not matter that we were not specific about just what sense of possibility we had in mind. This is because the ability to add either β_1 or ***not*** β_1 partly *defines* what counts as the same sense of possibility. If the speaker says that (5) is true, then add β_1 to the shopping cart. Otherwise add ***not*** β_1 to the shopping cart. If *both* (5) and (6) are true the choice is arbitrary, but we know that at least one of them has to be.

Now assume that we have put n sentences in the cart. This means that of the first n-sentences in the dictionary, each is in the cart or its negation is. Let us write γ_n for β_n if β_n is in the cart, and for ***not*** β_n if ***not*** β_n is in the cart. So γ_n is either β_n or ***not*** β_n, whichever of these two is in the cart. For the case of β_1 what we have shewn is that we can add either β_1 or ***not*** β_1 to the shopping cart in such a way as to guarantee that *everything in the cart* could have been true together. What we need to shew therefore is that for any sentence β we can either add β or ***not*** β and still preserve this useful feature. Assume that α^*, γ_1, γ_2,... ,γ_n are all in the cart, and therefore could all be true together. Let β_{n+1} be the next sentence in the dictionary. Ask the speaker whether

(8) ***possibly***(α^* ***and*** γ_1 ***and*** ... ***and*** γ_n ***and*** β_{n+1})

or

(9) ***possibly***(α^* ***and*** γ_1 ***and*** ... ***and*** γ_n ***and not*** β_{n+1})

is true. We know that it is possible that α^* ***and*** γ_1 ***and*** ... ***and*** γ_n could be true, and so, for just the same reasons as in the case of β_1 we know that either α* ***and*** γ_1 ***and*** ... ***and*** γ_n could be true *with* β_{n+1} or α* ***and*** γ_1 ***and*** ... ***and*** γ_n ***and*** β_{n+1} could be true *without* β_{n+1} , and so either

(10) ***possibly***(α* ***and*** γ_1 ***and*** ... ***and*** γ_n ***and*** β_{n+1})

or

(11) ***possibly***(α* ***and*** γ_1 ***and*** ... ***and*** γ_n ***and not*** β_{n+1})

is true. If (10) is true then add β_{n+1} to the cart. If not add ***not*** β_{n+1} to the cart. Since (11) is true if (10) is not it is *still* possible that all members of the cart are true.

It is at this point that the parallel with the completeness proofs breaks down. While it is clear from the construction that the union of all these sets is a set of sentences which has the property that *every* sentence of the language is either in it, or its negation is in it; and while it is clear that *at each stage* in the construction of the cart the contents are all possible together, it does not follow that the union of them all is possible. For consistency we know that a set is consistent iff each of its finite subsets is consistent. But that assumption cannot be used with an intuitive notion of possibility. Here is the problem. Fred is an absolute wizard at counting numbers. Name any finite number you like, and it is possible for Fred to count that far. But even Fred cannot count infinitely far. So let α be the sentence

(12) Fred has not counted infinitely many numbers

and let each δ_n be the sentence

(13) Fred has counted up to n.

It is not difficult to see that it cannot be possible that α and *every* δ_n are true together. But any finite number of them, under the assumption of Fred's prodigious ability, could all be true together. For let n* be the highest δ_n in a given finite set. Then Fred *could* have counted up to n* and then stopped. And if he did every δ_n up to δ_{n*} would be true, and so would α. So that

(14) ***possibly***(α ***and*** δ_1 ***and*** ... ***and*** δ_{n*})

is true. So it would seem that we could fill the shopping cart with α and all the δ_ns. Clearly such a cart would not represent a possible world. The whole set of (12) and each (13) is not possible, for then Fred would have counted infinitely far, contradicting (12). An analogous, though importantly different, problem arises in the completeness theorem for first-order logic when we are given that not everything satisfies some predicate φ. In logical symbols we have $\sim\forall x\varphi x$. If this is so there must be something of which φ fails. We can call this something *a*, and add $\sim\varphi a$ as what is called a *witness* to the falsity of $\forall x\varphi x$. Provided that we begin with a consistent (i.e. possible) sentence, and provided we choose a name that has not previously been used, it is possible to prove that the resulting set is also consistent.

Suppose (12) is true. Then Fred has only counted finitely far, say as far as n−1. This means that there will be a sentence, the sentence we have called δ_n, which is false. Since we know that α and *all* the δs cannot be true we know that we can find a sentence ***not*** δ_n which can be added to the cart before we begin filling it with each sentence or its negation. The solution to the problem in first-order logic relies on a syntactic connection between $\sim\forall x\varphi x$ and $\sim\varphi a$, but there does not seem any *syntactic* way of obtaining the ***not*** δs from (12). Consider for instance a temporal counterpart of Fred's problem.[4] Suppose that time stretches back infinitely in the past but in discrete moments, and that ***will*** α means that α is true at some later moment. If we play along with an identification of times with days we can think of tomorrow, the day after tomorrow, the day after that and so on. Suppose that on Wednesday a great event takes place. The lost works of Democritus are discovered in a time-capsule buried in the sand on Waitarere Beach. Call this event **D**. Now consider the sentences

(15) $\mathbf{D_1}$ ***will*** **D**

$\mathbf{D_2}$ ***will will*** **D**

....

$\mathbf{D_n}$ ***will*** ... (n-times) ... ***will*** **D**

On Tuesday $\mathbf{D_1}$ is true, but none of the others. On Monday $\mathbf{D_1}$ and $\mathbf{D_2}$ are both true, but none of the others, ... and n-days ago $\mathbf{D_1}$ up to $\mathbf{D_n}$ are true, but none of the others. No one knew this of course, except for our omniscient speaker, but they were true nonetheless. Now although time stretches back infinitely far from the present no day is infinitely far from the present, so *no* day is a day on which all members of (15) are true. Yet consider any finite subset of (15). There will be a greatest n for which $\mathbf{D_n}$ is true, and for every m < n, $\mathbf{D_m}$ will also be true, and so

(16) ***will*** ($\mathbf{D_1}$ ***and*** ... ***and*** $\mathbf{D_n}$)

was once true. So if we arranged the dictionary so that every $\mathbf{D_n}$ preceded every ***not*** $\mathbf{D_n}$ we could ensure that there is a cart which contains them all. But there is no time at which they are all true. There doesn't seem anything comparable with a name to act as a witness, for the witnesses are the repetitions of ***will***. But in any case, even if we *could* fiddle the syntax that would not address what is in fact a *semantic* problem. What we want is a solution which makes no reference to the syntactic form of α, except for the ability to form ***not*** α, and which deals automatically and simultaneously with every Fred-like situation which might arise in the language.[5]

First an observation. The problem of Fred arises because we want to talk about the joint possibility of an infinite number of sentences — i.e. the possibility that all members of an infinite cartful of sentences are true together. If we were not allowed to say whether an infinite set of sentences is jointly possible without saying this of a single sentence — a finite set of sentences is jointly possible iff the single sentence which is its conjunction is — then we could not state Fred's problem in the first place. What we require therefore is a principle about the joint possibility of all the sentences in an infinite set which parallels that used in logic for finite sets.

Call the principle we have used in going from (4) to either (5) or (6) above the *Principle of finite addition*. It goes like this. Say that a finite set A of sentences is *jointly possible* iff, where its members are α_1, ..., α_n,

(17) ***possibly***(α_1 ***and*** ... ***and*** α_n)

is true. (Assume an arbitrary but fixed sense of ***possibly***.) The principle then says:

[4]Compare the remarks about a similar problem in what Segerberg 1994, p. 341f, calls the 'logic of common knowledge'. (For more on Segerberg's paper see the appendix below).

[5]What we are seeking is closer to the 'witness property' on p. 346 of Segerberg 1994.

PFA If a finite set A of sentences is jointly possible, and δ is any sentence whatever then either A together with δ is jointly possible, or A together with ***not*** δ is jointly possible.

This follows from the fact that if α is possible, then α is also possible together with whatever follows from α. And δ may be said to follow from α if it is not possible to have both α and ***not*** δ. This principle could be stated by saying that

(18) ***possibly*** α implies ***possibly*** (α ***and*** δ) ***or possibly*** (α ***and not*** δ).

In the infinite case you cannot define joint possibility as the possibility of the conjunction of all the members of the set since there are infinitely many of them. Yet the problem of Fred arises because we do feel that there is a notion of joint possibility that applies to infinite sets in the same way as it applies to finite sets. Certain non-standard logics allow infinite conjunctions, and the joint possibility of a set Δ of sentences could be represented in such a language as ***possibly*** α, where α is the infinite conjunction of all members of Δ. Even in a language without such infinitary operators their consideration might allow a way of extracting an infinite notion of joint possibility on the basis of the ***possibly*** α sentences in a world. The idea would be that a set Δ of sentences is jointly possible iff, were α a sentence which is equivalent to the infinite conjunction of all the members of Δ then ***possibly*** α would be true. Certainly, without infinite conjunctions in the language the right sense of 'equivalent' would need making clear but as before the reply would be that if there were no such sense Fred's problem could not arise.[6] What can be said at least is that if α is equivalent to the (infinite) conjunction of all the members of Δ then there is *no* sense of ***possible*** in which it is possible for α to be true without all the members of Δ being true. So that if α is *false* then we may automatically conclude that however the facts may turn out to be, at least one δ in Δ must be false.

Using this we can already motivate an extension to PFA whereby A may be allowed to be infinite. Suppose Δ is any set of sentences

PFA$^+$ If a set Δ (finite or infinite) of sentences is jointly possible, and δ is any sentence whatever then either Δ together with δ is jointly possible, or Δ together with ***not*** δ is jointly possible.

PFA$^+$ may be seen to be plausible if, in place of the conjunction (α_1 ***and*** ... ***and*** α_n) mentioned in (17) we imagine a sentence α equivalent to the infinite conjunction of the members of Δ. (Note that I am not claiming that the language really *does* contain such a sentence, but am only pointing out what would happen of it did.) The generalised *Principle of Addition* has to take the joint possibility of all the sentences of an infinite set as an intuitively given notion, satisfying of course the constraint that if any sentence α is in a jointly possible set then ***possibly*** α is true. The Principle of Addition then says, where Δ and Λ can be infinite sets of sentences:

PA If all members of a set Δ of sentences are jointly possible and there is a family Θ of sets of sentences such that for every Λ in Θ, Δ together with Λ is *not* jointly possible,

[6] In this paper I am not concerned with infinitary logic in the sense of a logic containing infinite formulae, usually infinite conjunctions and disjunctions. Some of the dangers inherent in such languages are discussed in Glanzberg 2001. Glanzberg, p.423f, notes that such languages enable the representation of possible worlds. His principal worry is that they allow the representation of arbitrary sets of individuals. While his worries are connected with mine they are not quite the same. Glanzberg's diagnosis (see p. 427) is that infinitary logic smuggles in set theory where we supposed there was only logic. I have no scruples about set theory, and am happy to allow that what I am shewing is that ordinary modal discourse with set theory easily yields possible worlds.

then there is a set Γ^* such that, for each Λ in Θ, Γ^* contains ***not*** δ for some δ in Λ, and Δ together with Γ^* *is* jointly possible.

Why should PA be true? Well, given that all the members of Δ are possible together suppose that α were a sentence equivalent to all its members. Since α is possible suppose it were true. Now $\Delta \cup \Lambda$ (for Λ in Θ) is not possible, and so under the supposition that α is true, where θ is the conjunction of all the sentences in Λ, α ***and*** θ is false. But then, given that α is true, there must be at least one conjunct δ in θ which is false. So, under the same supposition, ***not*** δ is true. So, under the supposition that every sentence in Δ is true, at least one δ in each Λ in Θ would be false, and so in such a case there would be one true ***not*** δ for some δ in every Λ. Let Γ^* be the set of just those ***not*** δs. This guarantees that Δ together with Γ^* is jointly possible.[7] Each ***not*** δ in Γ^* is the 'witness' which rules out the Λ which contains δ.

It is certainly easy to see why PA holds if ***possibly*** α *does* mean that α is true in at least one accessible world, so that we can have a guarantee that PA cannot lead to unintended trouble. Here is the reason: Suppose that what we are intending to establish is true; that propositions are the semantic values of sentences and are sets of worlds; and that a proposition is possible in a world iff it is true in at least one world accessible from the original world. If Δ is a set of sentences these will be jointly possible in a world w^* iff there is at least one world w, of the worlds accessible from w^* in the relevant sense of ***possibly***, such every member of Δ is true in w. Now take any Λ in Θ. It cannot be that every member of Λ is true in w, for then Δ together with Λ would be jointly possible. So there is at least one δ in Λ such that δ is not true in w. But then ***not*** δ *is* true in w, and therefore all the members of Δ, together with ***not*** δ are true in w. So for each Λ in Θ there is a δ in Λ with ***not*** δ true in w. So, where Γ^* is the set of all such ***not*** δs, all members of Γ^* are true in w. But so are all members of Δ, and so Δ will be jointly possible with Γ^*.

PA is a powerful principle, and leads directly to what we may call the *extension result*:

ER If Δ is jointly possible then there is a maximally possible set Γ containing Δ (i.e. $\Delta \subseteq \Gamma$).[8]

The proof of ER from PA is trivial. Let Θ be the set of all sets of the form $\{\beta$, ***not*** $\beta\}$ for every

[7] One might consider a simpler version which refers to one Λ at a time.

PA′ If all members of a set Δ of sentences are jointly possible and there is a set Λ such that Δ together with Λ is *not* jointly possible, then there is some δ in Λ such that all the members of Δ together with ***not*** δ, *are* jointly possible.

PA′ is just PA with Θ as $\{\Lambda\}$, but it has the advantage that it corresponds with a version of the cut rule in infinitary logic. (See the appendix to this paper.) I am grateful to Rob Goldblatt for pointing out to me that PA may not be derivable from PA′, and thus saving me from an embarrassing mistake in an earlier version of the paper. If the paper does not convince you that PA is reasonable then you must either reject PA′ or shew that there is a sense of ***possibly*** for which PA′ is reasonable but PA is not. If you reject PA′ then you must either reject PFA or shew that there is a sense of ***possibly*** for which PFA is reasonable but PA′ is not. If you reject PFA I don't know how to talk to you. At the very least, what the paper does is shew the *cost* of rejecting the use of possible worlds in semantics.

[8] The construction described earlier assumed that we began with a set $\{\alpha^*\}$, where ***possibly*** α was assumed to be true. As ER shews we could equally begin with an infinite set Δ assumed to be jointly One interesting comment here is that in the case of first-order logic there is actually the curious fact that (12) and all the (13)s are *consistent*, since in first-order logic any contradiction derivable from an infinite set is derivable from some finite part of it. In beginning with an infinite set in first-order logic the completeness proof needs to extend the language with infinitely many new individual symbols (variables or constants). In the construction described here that would be undesirable, for it would allow Fred's counting to be curtailed only by the fact that he has not counted the non-standard numbers!

sentence β. Then, by PA, there will be some Γ^* such that for every $\{\beta$, ***not*** $\beta\}$ in Θ either ***not*** β or ***not not*** β is in Γ^*, and $\Delta \cup \Gamma^*$ is jointly possible. Now, whatever joint possibility is it will need to satisfy the principle that any jointly possible set remains jointly possible when the sentences in it are replaced by logical equivalents — in this case where ***not not*** β is replaced by β, for each β. $\Delta \cup \Gamma^*$ so changed, call it $\Gamma^\dagger$, is maximally possible in that all its members can be true together, and for every sentence β either β is in $\Gamma^\dagger$ or ***not*** β is. $\Gamma^\dagger$ gives a decision on every β, making it either true or false, but not both. So either $\Gamma^\dagger$ *is* a world, or it represents a world, and the debate can continue as a debate about the nature and status of worlds. PA is an easy consequence of ER. For suppose Δ is jointly possible. Then by ER there is a maximally possible Γ^* with $\Delta \subseteq \Gamma^*$. Take any $\Lambda \in \Theta$. Since $\Delta \cup \Lambda$ is not possible $\Lambda \not\subseteq \Gamma^*$. So there is some $\delta \in \Lambda$ with $\delta \notin \Gamma^*$. Since Γ^* is maximal ***not*** δ is in Γ^*, and since $\Delta \subseteq \Gamma^*$ and Γ^* is possible then $\Delta \cup \Gamma^*$ is possible.

The construction so far has assumed a single though arbitrary sense of ***possibly***. It is a commonplace that in natural language ***possibly*** can be used in a number of senses. We could label these by indices, but for the present let us simply say that it is at least a necessary condition for a simple or complex adverbial phrase to count as a possibility operator that it can be interpreted in such a way that PA holds of it. (Complex adverbial phrases allow iteration, so that ***possibly$_1$ possibly$_2$*** also counts as a possibility operator, and so on.) We then say that a set A of sentences is jointly possible* iff it is jointly possible in terms of at least one of the possibility operators in the language. Possibility* is the weakest sense of possible — the sense sometimes called *logical* or sometimes *metaphysical* possibility.[9] Notice that there may be no operator in the language which expresses possibility*. This may reflect the fact that logical possibility is a philosopher's term of art.[10]

Despite this PA holds for possibility*, since if Δ is jointly possible* it is jointly possible in some sense expressible in the language, call it possibility$_n$, where PA holds for possibility$_n$. So Δ is jointly possible$_n$. If Δ together with every Λ in some Θ is not jointly possible* then it is not jointly possible$_n$, and so, given that PA holds for possibility$_n$ there is some Γ^* such that, for every Λ in Θ, there is some δ in Λ with ***not*** δ in Γ^*, and Δ together with Γ^* is jointly possible$_n$. But then Δ together with Γ^* is jointly possible*. Let the 'worlds' be the sets of sentences which are maximally possible*. Possibility* enables us to define the set of all possible worlds — or rather construct the set of sets of sentences which are maximally possible*. These sets of sentences are those which are possible in *any* sense of ***possible***. This construction is implicitly 'actualist' in that non actual 'worlds' exist only as sets of sentences, defined in terms of what is actually possible. For possible* this doesn't matter since what is possible* does not vary from world to world. But for senses of ***possible*** other than possibility* it is customary to speak of certain world as *accessible* from this world. Accessibility is needed because a shopping cart might itself contain a sentence of the form

(19) ***possibly*** α.

Consider the case with times, and suppose there was rain yesterday. Then

(20) ***past rain***

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[9]I prefer the phrase 'logical possibility/necessity' to the fashionable term 'metaphysical possibility/necessity'. This kind of necessity imposes no restrictions on the possible worlds. (On this issue see p. 202f of Stalnaker 2003.) There is of course a notion of logical truth which applies most properly to a schema. Thus $p \supset p$ is logically true in the sense that it remains true when p is replaced by any proposition whatsoever, whether true or false.

[10]See Lycan 1994 for an argument that logical possibility and necessity in this sense are not standard ordinary language modalities.

will be true, but

(21) ***past past rain***

will not. So that beginning the cart from yesterday we have to interpret the question of whether something *was* true, as something about what was previous to *that* time, not previous to *this* time. Thus the fact the there was rain yesterday cannot make ***past rain*** true then, because, although the rain is in today's past it is not in yesterday's past. Similarly with worlds, since possibilities may change, as when we say that today it is not possible to visit Mars or have a cure for cancer, but one day it might become possible. In the standard modal completeness proofs one defines w' to be accessible from w iff

(22) For every sentence α, if α is in w' then ***possibly*** α is in w.

One then wants to prove

(23) Where accessibility$_n$ is defined for ***possibly$_n$*** in accordance with (22), ***possibly$_n$*** α is in w iff α is in some w' accessible from w.

(23) may be established as follows. Obviously, if w' is accessible from w and α is in w' then ***possibly$_n$*** α is in w. For the converse, if ***possibly$_n$*** α is in w we need a world containing α and all the ***not*** βs with ***not possibly$_n$*** β in w. If there is no such world at all then the combination of ***possibly$_n$*** α and all the ***not possibly$_n$*** βs is not jointly possible*. In ordinary (finitary) modal logic one derives the needed world from the principle that

(24) (***possibly*** α ***and not possibly*** β) implies ***possibly*** (α ***and not*** β)

(for any sense of ***possibly***.) (24) says that if you can have α but can't have β, then you can have α without β. Its infinitary generalisation is the *modal principle of addition*

MPA If ***possibly*** α together with a family of sentences of the form ***not possibly*** β is jointly possible* then so is α together with all these ***not*** βs.[11]

The ***possibly*** in MPA is intended to hold for any sense of ***possibly***. Where Δ is α together with every ***not*** β such that ***not possibly*** β is in w, then MPA ensures that Δ is jointly possible*, and ER for possibility* ensures that $\Delta \subseteq w'$ for some w'. If β is in w' then ***not*** β is not in w', and so ***not possibly*** β is not in w, and so ***possibly*** β *is* in w. So w' is accessible from w, and w' contains α. This gives us (23).

In terms of possible-worlds semantics the maximally possible* sets of sentences have a number of noteworthy features. In the first place if worlds are *identified* with maximally possible* sets we have a form of what Lewis 1986 p. 142 calls *linguistic ersatzism*. The purpose of the present paper is not to take sides on the merits or otherwise of linguistic ersatzism. My aim has been to shew that the appropriate semantic debate is precisely about such issues as whether sets of sentences constructed in this way are what possible worlds are. A second feature of maximally possible* sets is that they are dictated by *actual* joint possibility, and thus may appear to signal a commitment to actualism. Nevertheless, as a set of sentences the existence, even of the set of those actually true, has to be abstract, since it is not restricted to sentences which are actually uttered. No

[11]See the appendix for a connection between MPA and 'Scott's Rule' on p. 344 of Segerberg 1994.

set of sentences is more real than any other, and so, as before, the debate turns on their status. Are they worlds themselves or do they describe worlds?

More interesting perhaps are questions to do with expressive capacity. Consider for instance MPA. MPA has been stated in terms of joint possibility*. But we need to be assured that constructing worlds using possibility* rules out Fred-like situations. Here's how a Fred-like situation might arise. Suppose we had ignored (12) and simply considered all the (13)s. But suppose we were using a sense of ***possibly*** (***possibly$_n$***) in which all the (13)s *on their own* are not jointly possible$_n$, though every finite subset of them is. Such a sense of ***possibly$_n$*** is not unreasonable, yet, we may assume, the set of all the (13)s on their own *is* jointly possible*. Let w' be this set. Then (a) w' is possible*, and (b) w' is accessible$_n$, since for every α in w', ***possibly$_n$*** α is true. Yet w' is not jointly possible$_n$.

One way of addressing this is to make an assumption about the expressive capacity of the language. Given that all the (13)s are not possible$_n$ we can always add a sentence like (12) which *says* this. Adding (12) ensures that a set of sentences which is not jointly possible in some restricted sense, but *is* jointly possible*, is replaced by a set which is not after all jointly possible*. In general, whenever you have a family B of sentences which are not jointly possible in some restricted sense you may use the construction of the present paper provided there is a sentence α whose meaning is simply that the members of B are not all true. Then B together with α is not jointly possible*. The assumption could be supported by claiming that if there were no such α the language would be incapable of setting up a Fred-like situation in the first place.

That is one response to the MPA problem. Here is another. Begin by noting that for worlds accessible$_n$ from the actual world we may define accessibility$_n$ in terms of joint possibility$_n$. The problem of the previous paragraphs only arises for worlds accessible from non-actual worlds. In defining possibility* a possibility operator could be an iterated expression. This means that there will be sufficient maximally possible* sets to account for any level of embedding of possibility operators. Up to now, although a single sentence ***possibly$_n$*** α can be a member of many different maximally possible sets of sentences, joint possibility has been defined only in an absolute way, as possibility as things actually are. For possibility* this does not matter, but for possibility$_n$ what we seem to need is to be able to speak of relative joint possibility$_n$ — that is to say we need to be able to say that, where w is any maximally possible* set of sentences it makes sense to ask whether any set Δ would or would not be jointly possible$_n$ were all then members of w to be true. If they could say that Δ is jointly possible$_n$ at w. If one could *not* ask this then MPA will do as it stands. If one *can* suppose that the notion of relative joint possibility makes sense then the replacement for MPA would be:

MPA$^+$ If w is maximally possible* and ***possibly$_n$*** α is in w then there is a family Δ of sentences which is jointly possible$_n$ at w, and contains α and every β such that ***not possibly$_n$*** β is in w.

MPA$^+$ is a generalisation of MPA in the same way as PA is a generalisation of PFA, and if it is to be rejected one will need either to produce an account of possibility which satisfies MPA but not MPA$^+$, or one will need an argument against MPA itself.

Could a language fail to discriminate between two distinct worlds? Easily, and even in the finite case. Take the language of propositional logic, and take there to be just two worlds w_1 and w_2. Assume that each atomic sentence is either true in both or true in neither. Then no sentence can distinguish between w_1 and w_2.[12] Again this raises the debate about whether worlds are linguistic. It may be thought that the infinitary features of semantics raise cardinality concerns. I suggested that

[12] Compare the case of 'normal' interpretations in LPC with identity.

understanding the joint possibility of an infinite set A of sentences is achieved by assuming a sentence α whose truth is equivalent to the joint truth of all members of A. Since there are non-denumerably many As there cannot be an α for each of them. In the case of languages with infinite conjunctions and disjunctions there could appear to be more sentences than there are, which can be troubling, as Glanzberg 2001 (pp. 431-434) observes. The languages of the present paper do not contain such sentences and so are not subject to *that* problem. Still it might seem that we need to say more things than we can. Cardinality problems are known to arise within possible worlds semantics. David Kaplan raises the following one. Suppose propositions are sets of worlds. Suppose that any proposition can be the *only* proposition a person queries in a given world.[13] Then there must be as many worlds as there are sets of worlds. Kaplan's argument has not met with ready acceptance, and here too it is not my intention to take sides. What it does shew is that there are hard cardinality problems in semantics whether you begin with possible worlds, or are led to them by the semantics of modal discourse.

The construction of worlds described in this paper is based solely on the possibility operators in a given language. It is true that the set of such operators may be open-ended in that complex adverbial phrases can count, provided they satisfy PA and MPA. But that might still be held to be not quite enough. Imagine a language with an adjective ***brittle***. Presumably if something is brittle it might break if dropped. So it would seem that if a sentence like

(25) This plate is brittle

is true there should be a possible world in which it is dropped and breaks. But what if the language contains no possibility operator which covers this sense? If so this world would not be even possible* according to the construction used here. The answer would probably go along something like the following lines: If the language does lack the relevant sense of ***possibly*** then the modal meaning of ***brittle*** might not be expressible, and there may no way of telling whether ***brittle*** does or does not have this implication. Still it does put the meaning of ***brittle***, and indeed the meaning of *any* sentences of the language at the mercy of its expressive capabilities in terms of possibility operators.[14]

There is much unfinished business here, given the aim of shewing that all semantics can be based on truth-conditions, where these are understood as sets of possible worlds. And even more generally one would need parallel arguments for times and other indices, to say nothing of the semantics of quantification. My aim however, has been the limited one of exhibiting similarities between intuitive notions of possibility and features of infinitary logic. It has not been to make any advances in infinitary logic itself, but rather to shew that considerations similar to those which can motivate infinitary logic can be seen to have considerable *semantic* importance in the justification of the use of possible worlds.

APPENDIX: INFINITARY LOGIC

This paper has not been about infinitary logic as such, but it might be helpful to conclude by noting

[13]See Kaplan 1995, p. 43. Kaplan mounts the argument on the basis that any language could contain an operator Q such that in any world w there is just one proposition p such that Qp is true. He gives 'queried' as a possible example, though all he requires is that there could be such an operator. (See also his note 11 on p. 49f.) Comments on Kaplan's claim are referred to in his note 1 on p. 48f.

[14]Maybe this is not a bad thing. On David Lewis's semantics for counterfactuals (Lewis 1973, p. 16), knowing that this would break if dropped involves knowing that there is a world in which it is dropped and breaks. If you could not articulate this then maybe you could not express the meaning of ***brittle***.

what the correlate of the argument I have presented is in such a logic. The completeness of modal systems of infinitary logic has been studied in Segerberg 1994[15]. On p. 343 Segerberg lists the following principles for the consequence relation $\vdash$ between sets of wff. He uses Γ and Ω, and I shall follow him in what follows, except for continuing with lower case Greek letters for sentences:

(RX)	$\{\alpha\} \vdash \alpha$
(MN)	If $\Gamma \vdash \alpha$ then $\Gamma \cup \Delta \vdash \alpha$
(CT)	If $\Gamma \vdash \gamma$ for every $\gamma \in \Omega$, and $\Omega \vdash \alpha$, then $\Gamma \vdash \alpha$
(SB)	If $\Gamma \vdash \alpha$ then $s\Gamma \vdash s\alpha$ for every substitution function s

I shall not have anything to say about the substitution function. In this appendix I shall follow Segerberg and use standard logical symbols, and in particular a 'falsum' $\perp$ as primitive For particular operators I shall only need the following (my labels):

(D)	If $\Gamma \cup \{\alpha\} \vdash \gamma$ and $\Gamma \cup \{\beta\} \vdash \gamma$ then $\Gamma \cup \{\alpha \vee \beta\} \vdash \gamma$
(LEM)	$\vdash \alpha \vee {\sim}\alpha$
(EFQ)	$\perp \vdash \alpha$

Although $\vdash$ can be thought of as a syntactical (or proof-theoretical) consequence operator in an infinitary 'logic', one may also think of it in terms of an intuitive notion of ***possibly***. One might then define $\Gamma \vdash \beta$ to hold iff it is not possible, in the assumed sense of ***possible*** to have all of Γ true together without having β true also. Γ will be *jointly possible* iff $\Gamma \dashv \perp$. The principle PA′, mentioned in note 7, then becomes the principle that if $\Gamma \cup \Omega$ is not jointly possible then there is some $\beta \in \Omega$ such that $\Gamma \cup \{{\sim}\beta\}$ is jointly possible. PA′ may be seen to follow from Segerberg's rules.[16] First observe that if $\Gamma \cup \{{\sim}\beta\} \vdash \perp$, i.e. if $\Gamma \cup \{{\sim}\beta\}$ is not jointly possible then $\Gamma \vdash \beta$.[17] Then proceed as follows: Suppose $\Gamma \cup \{{\sim}\beta\}$ were not jointly possible for any $\beta \in \Omega$. Then $\Gamma \vdash \beta$ for every $\beta \in \Omega$. Since $\Gamma \vdash \alpha$ for every $\alpha \in \Gamma$ we have $\Gamma \vdash \beta$ for every $\beta \in \Gamma \cup \Omega$. But $\Gamma \cup \Omega \vdash \perp$, and so, by CT $\Gamma \vdash \perp$. But this contradicts Γ's possibility.

CT is a natural rule in infinitary logic. A rule to provide PA is more complex. The following is a contraposed form of PA with an arbitrary α in place of $\perp$:

(CT^+) Suppose there is a Θ such that

(i) for every Λ in Θ, $\Lambda \vdash \alpha$

and

[15]See also Chapters 8 and 9 of Goldblatt ****

[16]Segerberg by contrast with the present paper *is* concerned to establish the completeness of an infinitary logic, and assumes on p. 345 that the rules of this logic, which will be a set of pairs of the form (Γ,α), are at most denumerable. This condition (Condition P) allows him to use a construction based on this enumerability. (See lemma 2.7 on p. 349.) I am not able to rely on such a condition, and for that reason need the principle PA rather than PA′.

[17]The proof of this is as follows. Suppose Γ U $\{{\sim}\beta\} \vdash \perp$. Then

$\Gamma, {\sim}\beta \vdash \perp$	hypothesis
$\perp \vdash \beta$	EFQ
$\Gamma, {\sim}\beta \vdash \beta$	CT
$\Gamma, \beta \vdash \beta$	RX, MN
$\Gamma, \beta \vee {\sim}\beta \vdash \beta$	D
$\vdash \beta \vee {\sim}\beta$	LEM
$\Gamma \vdash \beta$	CT

(ii) for every Γ, if (ii^a) for every Λ in Θ, there is some δ in Λ such that $\Gamma \cup \{\delta\} \vdash \alpha$, then (ii^b) $\Delta \cup \Gamma \vdash \alpha$.

Then $\Delta \vdash \alpha$.

CT follows from CT$^+$ in the following way. Assuming that $\Delta \vdash \gamma$ for all $\gamma \in \Omega$, put $\Theta = \{\Omega\}$. Then if $\Omega \vdash \alpha$, condition (i) of CT$^+$ holds. Let Γ be any set for which $\Gamma \cup \{\gamma\} \vdash \alpha$ for some $\gamma \in \Omega$. Since $\Delta \vdash \gamma$ then $\Delta \cup \Gamma \vdash \alpha$, and so condition (ii) of CT$^+$ holds. So, by CT$^+$, $\Delta \vdash \alpha$. PA may be derived from CT$^+$ in the following way. Suppose that Δ is jointly possible — that is to say that $\Delta \dashv \bot$. Suppose also that for every Λ in Θ, $\Delta \cup \Lambda \vdash \bot$. Let Θ^* be the set of all Λ^* such that $\Lambda^* = \Delta \cup \Lambda$ for some $\Lambda \in \Theta$. Then $\Lambda^* \vdash \bot$ for every $\Lambda^* \in \Theta^*$. For the case where α is $\bot$, and assuming that CT$^+$ holds, we therefore have the falsity of condition (ii) — that is to say there is some $\Gamma^\dagger$ such that

(a) for every Λ in Θ^* there is some δ in Λ with $\Gamma \cup \{\delta\} \vdash \bot$

and

(b) $\Delta \cup \Gamma^\dagger \dashv \bot$.

It is sufficient to shew that where Γ^* is precisely $\sim\delta$ for each of the δs in (a) then $\Delta \cup \Gamma^* \dashv \bot$. And to shew the latter it is sufficient to shew that if $\Delta \cup \Gamma^* \vdash \bot$ then $\Delta \cup \Gamma^\dagger \vdash \bot$ (since by (b) $\Delta \cup \Gamma^\dagger \dashv \bot$). Take any β in $\Delta \cup \Gamma^*$. If $\beta \in \Delta$ then $\Delta \cup \Gamma^\dagger \vdash \beta$; and if $\beta \in \Gamma^*$ then β is $\sim\delta$ for some δ such that $\Gamma^\dagger \cup \{\delta\} \vdash \bot$, and so here too, $\Delta \cup \Gamma^\dagger \vdash \beta$. So by CT, $\Delta \cup \Gamma^\dagger \vdash \bot$, contradicting (b).[18]

It would be pleasant to have a more intuitive rule, and even more pleasant to compare the power of CT$^+$ with CT. Such an investigation would make clearer just what those who dislike PA but like PA′ might be able to say in support of such a position.[19]

In footnote 11 I mentioned what Segerberg (p. 344) calls ‘Scott’s Rule’. With $\Box$ for ***not possibly not*** Scott’s Rule is:

SR If $\Gamma \vdash \beta$ then $\{\Box\gamma : \gamma \in \Gamma\} \vdash \Box\beta$

The infinitary logic correlate of MPA states

MPA′ If $\{$***possibly*** $\alpha\} \cup B \dashv \bot$ then $\{\alpha\} \cup \{$***not*** β : ***not possibly*** $\beta \in B\} \dashv \bot$.

[18]It is possible to give a semantic ‘soundness’ argument for CT$^+$ in the style in which one was given for PA in the text, at least if we assume that the logic contains negation, and interpret $\Delta \vdash \alpha$ to mean that there is no possible world in which all members of Δ are true but α is not. For suppose $\Delta \dashv \alpha$. Then there is a world *w* such that every $\beta \in \Delta$ is true in *w* but α is false in *w*. Consider any $\Lambda \in \Theta$. From (i) at least one δ in Λ must be false in *w*. Let Γ^* be the set of $\sim\delta$ for every such δ. Then $\Gamma^* \cup \{\delta\} \vdash \alpha$, but every $\sim\delta$ is true in *w*, and so $\Delta \cup \Gamma^* \dashv \alpha$, contradicting (ii^b). The argument also works for a language without ~ provided it has $\supset$, for then let Γ^* be $\delta \supset \alpha$, for each of the appropriate δs.

[19]One place to begin might be an investigation of entities like the ‘possibilities’ studied in Humberstone 1981.

We prove MPA′ contrapositively using SR. Suppose $\{\alpha\} \cup \{$***not*** β : ***not possibly*** $\beta\} \vdash \bot$. Then $\{\sim\beta : \Box\sim\beta \in B\} \vdash \sim\alpha$. So, by SR $\{\Box\sim\beta : \Box\sim\beta \in B\} \vdash \Box\sim\alpha$. But $\{\Box\sim\beta : \Box\sim\beta \in B\} \subseteq B$, and $\Box\sim\alpha \vdash$ ***not possibly*** α, and so $B \vdash$ ***not possibly*** α. So $\{$***possibly*** $\alpha\} \cup \beta \vdash \bot$.

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