Proofs, and what they're good for

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To explain the nature of *proof*, from the perspective of a *normative pragmatic account of meaning*, using the formal tools of *proof theory*.

Outline

Motivation

Background

What Proofs Are

How Proofs Work

MOTIVATION

Example Proof 1

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Example Proof 1 (the formal structure)

$$
\begin{prooftree}
\AxiomC{$Da \succ Da$}
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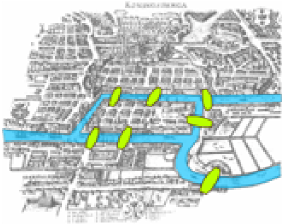
\AxiomC{$La \succ La$}
\RightLabel{$\vee L$}
\BinaryInfC{$Ba \vee La \succ Ba, La$}
\RightLabel{$\supset L$}
\BinaryInfC{$Da \supset (Ba \vee La), Da \succ Ba, La$}
\RightLabel{$\forall L$}
\UnaryInfC{$(\forall x)(Dx \supset (Bx \vee Lx)), Da \succ Ba, La$}
\AxiomC{$Da \succ Da$}
\RightLabel{$\wedge R$}
\BinaryInfC{$(\forall x)(Dx \supset (Bx \vee Lx)), Da \succ Ba, Da \wedge La$}
\RightLabel{$\exists R$}
\UnaryInfC{$(\forall x)(Dx \supset (Bx \vee Lx)), Da \succ Ba, (\exists x)(Dx \wedge Lx)$}
\RightLabel{$\supset R$}
\UnaryInfC{$(\forall x)(Dx \supset (Bx \vee Lx)) \succ Da \supset Ba, (\exists x)(Dx \wedge Lx)$}
\RightLabel{$\forall R$}
\UnaryInfC{$(\forall x)(Dx \supset (Bx \vee Lx)) \succ (\forall x)(Dx \supset Bx), (\exists x)(Dx \wedge Lx)$}
\RightLabel{$\vee R$}
\UnaryInfC{$(\forall x)(Dx \supset (Bx \vee Lx)) \succ (\forall x)(Dx \supset Bx) \vee (\exists x)(Dx \wedge Lx)$}
\end{prooftree}
$$

Example Proof 2 (The Bridges of Königsberg)



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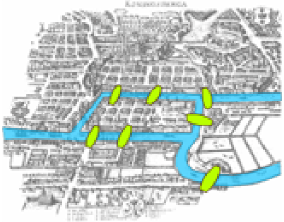

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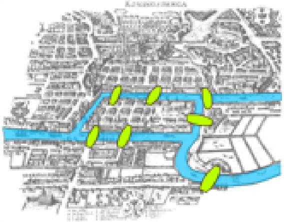

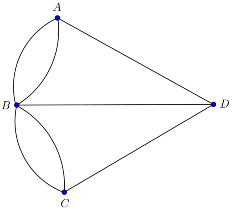

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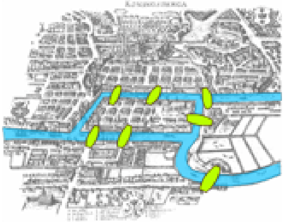

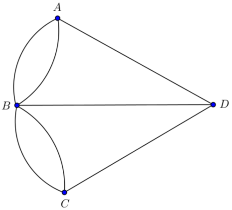

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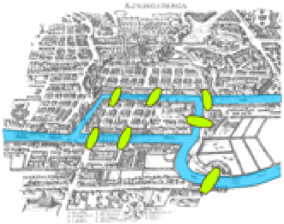

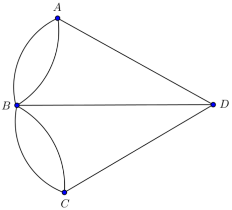

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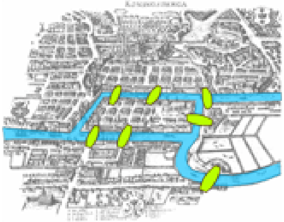

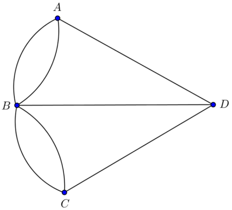

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But what I say here can be extended to proof relying on other concepts.

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- What *grounds* the necessity in the connection between the premises and the conclusion?
- (Notice that these are important questions for proofs in first order predicate logic, as much as for proof more generally.)

BACKGROUND

Assertions and *Denials*

$$[X : Y]$$

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They are connected to other speech acts, too, like imperatives, interrogatives, recognitives, observatives, *etc*.

Norms for Assertion and Denial

Assertions and denials take a *stand* (*pro* or *con*) on something.

DENIAL clashes with assertion.
ASSERTION clashes with denial.

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- A position that is OUT OF BOUNDS doesn't succeed in taking a stand.

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Logical concepts are similarly sharply delimited, but they cannot all be given explicit definitions.

Definition through a rule for use

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if and only if

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$$\frac{X, A, B \vdash Y}{X, A \land B \vdash Y}\ \land Df$$

What about when to *deny* a conjunction?

$$
\cfrac{X \vdash A, Y \qquad \cfrac{X \vdash B, Y \qquad \cfrac{\cfrac{}{A \land B \vdash A \land B}\,Id}{A, B \vdash A \land B}\,{\land}Df}{X, A \vdash A \land B, Y}\,Cut}{X \vdash A \land B, Y}\,Cut
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So, we have

$$\dfrac{X \vdash A, Y \qquad X \vdash B, Y}{X \vdash A \land B, Y}\,{\land}R$$

Definitions for other logical concepts

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y} \neg Df \qquad \frac{X, A \vdash B, Y}{X \vdash A \supset B, Y} \supset Df \qquad \frac{X \vdash A, B, Y}{X \vdash A \vee B, Y} \vee Df$$

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$$\frac{X \vdash Fa, Y}{X \vdash (\forall x)Fx, Y}\,\forall Df \qquad \frac{X, Fa \vdash Y}{X, (\exists x)Fx \vdash Y}\,\forall Df \qquad \frac{X, Gb \vdash Gc, Y}{X \vdash b = c, Y}\,\forall Df$$

(Where a and G are not present in X and Y.)

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- Are *subject matter neutral*. (They work wherever you assert and deny—and have singular terms and predicates.)
- In Brandom's terms, they *make explicit* some of what was implicit in the practice of assertion and denial.

WHAT PROOFS ARE

A Tiny Proof

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It's Friday.

Therefore, *I'm in Melbourne.*

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$$[\textit{It's Friday} \supset \textit{I'm in Melbourne}, \textit{It's Friday} : \textit{I'm in Melbourne}]$$

(This is out of bounds.)

The *Undeniable*

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it's Friday $\supset$ *I'm in Melbourne*
and I've asserted *it's Friday*,
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The *stance* (*pro* or *con*)
on *I'm in Melbourne* was already made.

A *proof* of $X \vdash Y$ shows that the position $[X : Y]$ is out of bounds, by way of the defining rules for the concepts involved in the proof.

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In this sense, proofs are *analytic*.

They apply, given the definitions, independently of the positions taken by those giving the proof.

A proof of $A, B \vdash C, D$ can be seen

as a *proof* of C from $[A, B : D]$,

and a *refutation* of A from $[B : C, D]$,

and *more*.

HOW PROOFS WORK

Observation 0: Proofs are *analytic*

These proofs are grounded in the *rules defining* the concepts used in them.

Observation 1: *Specification* outstrips *Recognition*

Our ability to *specify* concepts and consequence far outstrips our ability to *recognise* that consequence.

SUCCESSOR AXIOMS:

PA1: $\forall x \forall y(x' = y' \supset x = y)$;

PA2: $\forall x(0 \neq x')$.

ADDITION AXIOMS:

PA3: $\forall x(x + 0 = x)$;

PA4: $\forall x\big(x + y' = (x + y)'\big)$.

MULTIPLICATION AXIOMS:

PA5: $\forall x(x \times 0 = 0)$;

PA6: $\forall x \forall y(x \times y' = (x \times y) + x)$.

INDUCTION SCHEME:

PA7: $(\phi(0) \wedge \forall x(\phi(x) \supset \phi(x'))) \supset \forall x \phi(x)$.

GOLDBACH'S CONJECTURE:

GC: $\forall x \exists y \exists z(\text{Prime}\, y \wedge \text{Prime}\, z \wedge 0'' \times x = y + z)$

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Verifying a putative proof is straightforward.
Checking that something *has* a proof is not so easy.

Are we logically omniscient?

Suppose that PA $\vdash$ GC
(but we don't possess that proof)

and that we *know* PA.

Do we know GC?

In a weak sense of ‘know’, *yes*, we do know GC

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In a weak sense of ‘know’, *yes*, we do know GC

- It’s a logical consequence of what we know.
- It is implicitly present in what we already know.
- There is no epistemic possibility (no circumstance consistent with our knowledge) that leaves GC out.

In a not-so-weak sense, we don't know GC

- Do we *believe* GC?

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In a not-so-weak sense, we don't know GC

- Do we *believe* GC?
- If we believed it, do we believe it *in the right way*?
- There is evidence for GC (its proof from PA, for example), but if that evidence plays no role in our belief...

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- This *follows from* the concepts of consequence and truth.

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- Here, p transforms warrants for the premises into warrant for the conclusion.
- This works only for *categorical, conclusive* warrants (*grounds*), not for *defeasible* warrants.

A Caveat on Defeasible Warrants

Consider the “Lottery Paradox.”

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$$
\begin{aligned}
[\ &(\exists x)(Tx \wedge Wx),\\
&(\forall x)(Tx \equiv (x = t_1 \vee x = t_2 \vee \cdots \vee x = t_{1\,000\,000}))\\
&\qquad\qquad\qquad : \ Wt_1, Wt_2, \ldots, Wt_{1\,000\,000}\ ]
\end{aligned}
$$

Consider the "Lottery Paradox."

$$
\begin{aligned}
[\ & (\exists x)(Tx \wedge Wx), \\
& (\forall x)(Tx \equiv (x = t_1 \vee x = t_2 \vee \cdots \vee x = t_{1\,000\,000})) \\
& \qquad\qquad : \ Wt_1, Wt_2, \ldots, Wt_{1\,000\,000}\ ]
\end{aligned}
$$

We have a very high degree of confidence in each part.
Each component is highly probable.
But the whole position is out of bounds.

Observation 4: Achilles and the Tortoise

"Well, now, let's take a little bit of the argument in that First Proposition—just *two* steps, and the conclusion drawn from them. Kindly enter them in your note-book. And in order to refer to them conveniently, let's call them *A*, *B*, and *Z* :—

(*A*) Things that are equal to the same are equal to each other.

(*B*) The two sides of this Triangle are things that are equal to the same.

(*Z*) The two sides of this Triangle are equal to each other.

Readers of Euclid will grant, I suppose, that *Z* follows logically from *A* and *B*, so that any one who accepts *A* and *B* as true, *must* accept *Z* as true?"

"Undoubtedly! The youngest child in a High School—as soon as High Schools are invented, which will not be till some two thousand years later—will grant *that*."

"And if some reader had *not* yet accepted *A* and *B* as true, he might still accept the *sequence* as a *valid* one, I suppose?"

Observation 4: Achilles and the Tortoise

"No doubt such a reader might exist. He might say 'I accept as true the Hypothetical Proposition that, *if* A and B be true, Z must be true; but, I *don't* accept A and B as true.' Such a reader would do wisely in abandoning Euclid, and taking to football."

"And might there not *also* be some reader who would say 'I accept A and B as true, but I *don't* accept the Hypothetical'?"

"Certainly there might. *He*, also, had better take to football."

"And *neither* of these readers," the Tortoise continued, "is *as yet* under any logical necessity to accept Z as true?"

"Quite so," Achilles assented.

"Well, now, I want you to consider *me* as a reader of the *second* kind, and to force me, logically, to accept Z as true."

"A tortoise playing football would be—" Achilles was beginning

"—an anomaly, of course," the Tortoise hastily interrupted. "Don't wander from the point. Let's have Z first, and football afterwards!"

"I'm to force you to accept Z, am I?" Achilles said musingly. "And your present position is that you accept A and B, but you *don't* accept the Hypothetical—"

"Let's call it C," said the Tortoise.

"—but you *don't* accept

(C) If A and B are true, Z must be true."

"That is my present position," said the Tortoise.

"Then I must ask you to accept C."

Our Analysis

$$A, B \vdash Z$$

Our Analysis

$$A, B \vdash Z$$

or

$$A, A \supset Z \vdash Z$$

Our Analysis

$$A, B \vdash Z$$

or

$$A, A \supset Z \vdash Z$$

This *doesn't* mean when I accept A
and I accept $A \supset Z$,
I ought to also accept Z.

Our Analysis

$$A, B \vdash Z$$

or

$$A, A \supset Z \vdash Z$$

This *doesn't* mean when I accept A
and I accept $A \supset Z$,
I ought to also accept Z.

However, if I assert A and $A \supset Z$ then Z is *undeniable*.

If I assert A and *if* A *then* Z and *deny* Z, then I am using '*if* …*then*' in a way that deviates from the defining rule for $\supset$, or I am explicitly contradicting myself.

If I assert A and *if* A *then* Z and *deny* Z, then I am using '*if* …*then*' in a way that deviates from the defining rule for $\supset$, or I am explicitly contradicting myself.

$$\frac{A \supset B \vdash A \supset B}{A \supset B, A \vdash B}\ {\supset}Df$$

Upshot

An account of the logical concepts given in terms of defining rules governing assertions and denials

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helps explain how (*first order predicate logic*) proof works,

Upshot

An account of the logical concepts given in terms
of defining rules governing assertions and denials

helps explain how (*first order predicate logic*) proof works,

how possessing a proof can expand our knowledge,
while proofs make explicit what is implicit in what we know.

THANK YOU!

`https://consequently.org/presentation/2016/proofs-and-what-theyre-good-for-AAP/`

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