Proof Theory: Logical and Philosophical Aspects

Class 5: Semantics and beyond

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Our Aim

To introduce *proof theory*, with a focus in its applications in philosophy, linguistics and computer science.

Our Aim for Today

Examine the connections between proof theory and semantics, both formal *model theory*, and more general philosophical considerations concerning meaning.

Today's Plan

Speech Acts and Norms

Proofs and Models

Beyond

SPEECH ACTS AND NORMS

Normative Pragmatics

An idea found in Brandom's *Making It Explicit* is that the *meaning* of linguistic items should first be understood in terms of their *use*

The interest is not just in what one does, but also what one ought or ought not do

Logic lets us institute new concepts and norms

Rules as Definitions

The rules that govern a connective can be taken to *define* the new connective

This appears to make it really easy to introduce new logical concepts

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But, there's a problem

Tonk

Arthur Prior pointed out that if a set of rules is enough to define a connective, then *tonk* is legitimate

$$\frac{X,A \vdash C}{X,A \circledcirc B \vdash C} \ \ \frac{X \vdash B}{X \vdash A \circledcirc B} \ \ [tonkR]}{\frac{B \vdash B}{B \vdash A \circledcirc B}} \ \ \frac{A \vdash A}{A \circledcirc B \vdash A} \ \ [tonkL]}{[Cut]}$$

Responding to tonk

Nuel Belnap responded to Prior's article, saying that additional conditions needed to be satisfied in order to define a connective

Connectives aren't introduced out of thin air, there is a *context of deducibility*, e.g. the full set of Gentzen's structural rules

In order to be a definition, an extension has to be *conservative*, while tonk manifestly is *not*

This idea has been taken up and developed by Dummett and others in discussions of *harmony*

Assertion and Denial

Many philosophers and logicians take *assertion* to be the primary speech act, which is used to define others

Others argue that denial should be understood as a primitive act on its own

We take logic, in particular valid sequents, as presenting normative relations between assertions and denial

 $X \vdash Y$ tells us that one should not assert everything in X while denying everything in Y

Positions



Positions



Positions

[X : Y]

Invalid sequents can be viewed as positions in a discourse

What do the structural rules say in terms of assertion and denial?

 $A \vdash A$

Asserting A clashes with denying A

$$\frac{X, Y \vdash Z}{X, A, Y \vdash Z}^{[KL]}$$

$$\frac{X \vdash Y, Z}{X \vdash Y, A, Z}^{[KR]}$$

If asserting X, Y clashes with denying Z, then asserting more stuff still clashes

$$\frac{X, A, AY \vdash Z}{X, A, Y \vdash Z}_{[WL]}$$

$$\frac{X \vdash Y, A, A, Z}{X \vdash Y, A, Z}_{[WR]}$$

If asserting or denying A twice results in a clash, then asserting or denying A just once results in a clash

$$\frac{X, A, BY \vdash Z}{X, B, A, Y \vdash Z} [CL]$$

$$\frac{X \vdash Y, A, B, Z}{X \vdash Y, B, A, Z} [CR]$$

If some assertions and denials clash, then asserting and denying the same things in a different order still clashes

$$\frac{X \vdash Y, A \quad A, X \vdash Y}{X \vdash Y} [Cut]$$

If asserting X and denying A and Y clashes, and asserting X and A while denying Y clashes, then asserting X and denying Y

Contrapositively, if asserting X and denying Y does not clash, then either asserting X and A while denying Y does not clash or asserting X while denying Y and A does not clash

Declaratives are Not Enough

Belnap argued that a systematic logical treatment of language should give equal weight to imperatives and interrogatives

1. THE DECLARATIVE FALLACY

My thesis is simple: systematic theorists should not only stop neglecting interrogatives and imperatives, but should begin to give them equal weight with declaratives. A study of the grammar, semantics, and pragmatics of all three types of sentence is needed for every single serious program in philosophy that involves giving important attention to language.¹

Attempting to understand all linguistic behavior in terms of assertions commits the *Declarative Fallacy*

The hope is that the view of sequents and logic can be extended to other speech acts

PROOFS AND MODELS

Models as Ideal Positions

How might *truth* enter this picture?

Models are ways of systematically elaborating finite positions into ideal, infinite positions that settle every proposition

In the propositional case, valuations are generated by ideal positions

Positions to models

The members of X are *true* and the members of Y are *false*

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The members of X are *true* and the members of Y are *false* (relative to [X : Y]).

$$[p \lor q, r : \neg p]$$

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 $p \lor q, r$

$$[p \lor q, r : \neg p]$$

$$p \lor q, r$$
 true

$$[p \lor q, r : \neg p]$$

$$p \lor q, r$$
 true $\neg p$

$$[p \lor q, r : \neg p]$$

$$p \lor q, r$$
 true $\neg p$ false

$$[p \lor q, r : \neg p]$$

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```
p∨q, r true

¬p false

p ???
```

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 p true
 $p \land r$

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 $A \wedge B$ is true at [X : Y] iff A and B are true at [X : Y].

 $A \lor B$ is false at [X : Y] iff A and B are false at [X : Y].

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Similarly, r is neither true nor false at [p:q].

FACT: If A is neither true nor false in [X : Y]

then both [X, A : Y] and [X : A, Y] is invalid,

and each sequent settles A — one as *true* and the other as *false*.

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$$\frac{X \vdash Y, A \qquad A, X \vdash Y}{X \vdash Y} [Cut]$$

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FACT: If $X \not\vdash Y$, there's a maximal $[\mathcal{X} : \mathcal{Y}]$ extending [X : Y].

Models

Assign truth values relative to maximal positions.

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In a slogan, $truth\ value = location\ in\ a\ maximal\ position$,

Variations

The ideal position construction handles classical logic

With some small adjustments, it can be used to provide models for intuitionistic logic

The system LJ is single-conclusion, but there is another intuitionistic sequent systems that has multiple conclusions

The construction with these two systems yield Kripke models and Beth models



The hypersequent system for S5 can be used to give a similar construction

Each component of a hypersequent describes a possible world

S₅ hypersequents

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']} \stackrel{[\Box L]}{=}$$

$$\frac{\mathcal{H}[X \vdash Y \mid A \vdash]}{\mathcal{H}[\Diamond A, X \vdash Y]} {}_{[\Diamond L]}$$

$$\frac{\mathcal{H}[X \vdash Y \mid \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]} \,_{[\Box R]}$$

$$\frac{\mathcal{H}[X \vdash Y \mid X' \vdash A, Y']}{\mathcal{H}[X \vdash \lozenge A, Y \mid X' \vdash Y']} \ ^{[\lozenge R]}$$

Extending positions

Invalid sequents [X : Y]

Invalid hypersequents [[X : Y], [X' : Y'], ...]

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Invalid sequents [X : Y]

Invalid hypersequents $[[X : Y], [X' : Y'], \ldots]$

Say one set of pairs $\mathcal H$ extends another $\mathcal G$, $\mathcal G \preceq \mathcal H$, just in case for each component [X:Y] in $\mathcal G$, there is a component [U:V] in $\mathcal H$ such that $X\subseteq U$ and $Y\subseteq V$

Example: {[p : q], [s : r]} is extended by both {[p, s : r, q, t]} and by ${[p, t : q], [s : r, p]}$

Where are the truth values now?

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Maximal positions [X:Y]

Maximal modal positions $[[\mathcal{X}:\mathcal{Y}], [\mathcal{X}':\mathcal{Y}'], \ldots]$

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Maximal positions [X:Y]

Maximal modal positions $[[\mathcal{X}:\mathcal{Y}], [\mathcal{X}':\mathcal{Y}'], \ldots]$

A set of pairs $\mathcal H$ is a modal position iff there is no valid hypersequent $X_1 \vdash Y_1 \mid \cdots \mid X_n \vdash Y_n$ extended by $\mathcal H$

A modal position ${\mathcal H}$ is maximal iff there is no modal position ${\mathbb I}$ such that ${\mathcal H} \prec {\mathbb I}$

Building maximal modal positions

The process of expanding a modal position can add formulas to a *component* as well as adding *more components*

Some maximal modal positions, however, will contain finitely many components

The construction builds connected chunks of the S5 canonical model, taking the accessibility relation to be an equivalence relation rather than the universal relation

Building maximal modal positions

As in the classical case, the Cut rule adds new formulas to individual components

The modal rules can extend a position with new components

Building maximal modal positions

$$\frac{\mathcal{H}[X \vdash Y \mid X', A \vdash Y']}{\mathcal{H}[X, \Box A \vdash Y \mid X' \vdash Y']} \ ^{[\Box L]}$$

$$\frac{\mathcal{H}[X \vdash Y \mid \vdash A]}{\mathcal{H}[X \vdash \Box A, Y]} \,_{[\Box R]}$$

If $[[X: \Box A, Y], [X_i: Y_i]]$ isn't derivable, then $[[X:Y], [:A], [X_i: Y_i]]$ can't be either

If the latter were derivable then the former would be by $[\Box R]$

Similarly but for $[\Box L]$, if, e.g. $[[X, \Box A: Y], [X': Y'], [X_i: Y_i]]$ isn't derivable, then $[[X: Y], [X', A: Y'], [X_i: Y_i]]$ can't be

Necessity in maximal modal positions

```
For a maximal modal position \{[\mathcal{X}_i : \mathcal{Y}_i] : i \in I\}, \square A is true at [\mathcal{X}_i : \mathcal{Y}_i] iff A is true at each [\mathcal{X}_j : \mathcal{Y}_i], j \in I
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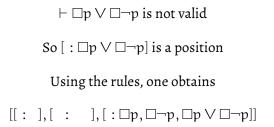
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(\Rightarrow) \text{ If } A \text{ is true at } [\mathcal{X}_i:\mathcal{Y}_i] \text{ and } A \text{ were not true at some component } [\mathcal{X}:\mathcal{Y}], \\ \text{ then since } [\mathcal{X}:\mathcal{Y}] \text{ is a maximal position, we would have } A \in \mathcal{Y} \text{ but } \\ \Box A \vdash | \vdash A \text{ is a valid sequent (by } [\Box L] \text{ from the axiom} \vdash | A \vdash A), \text{ so } \\ \{[\mathcal{X}_i:\mathcal{Y}_i], [[\mathcal{X}_j:\mathcal{Y}_j]]\} \text{ would not be a position, as } \Box A \in \mathcal{X}_i \text{ and } A \in \mathcal{Y}_j, \text{ so } \\ \{[\mathcal{X}_i:\mathcal{Y}_i]:i\in I\} \text{ isn't a position. As it is, whenever } \Box A \ [\mathcal{X}_i:\mathcal{Y}_i], A \text{ is true at every component.} \\ \end{cases}
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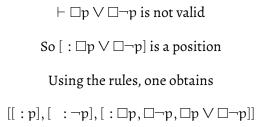
Necessity in maximal modal positions

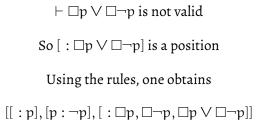
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```

```
(\Leftarrow) Suppose \Box A isn't true at [\mathcal{X}_i:\mathcal{Y}_i]. So we have \Box A \in \mathcal{Y}_i. Take \{[\mathcal{X}_i:\mathcal{Y}_i]:i\in I\}\cup [:A]. This is a position. Suppose that it is not. Then there is a derivable hypersequent \vdash A\mid X\vdash Y \Delta, where X\subseteq \mathcal{X}_i,Y\subseteq \mathcal{Y}_i and \Delta is extended by the other components of the modal position. If that were the case, then by [\Box R], we could derive X\vdash \Box A,Y\mid \Delta, but that is extended by the original modal position. It is, then, not be valid. So, \{[\mathcal{X}_i:\mathcal{Y}_i]:i\in I\}\cup [:A] is a position, so it is extended by a maximal modal position, which must be \{[\mathcal{X}_i:\mathcal{Y}_i]:i\in I\}, as that is not extended by any modal positions. Therefore, for some j\in I, A\in \mathcal{Y}_i.
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		- 🗆	p∨□¬r	is not valid
So [: $\Box p \lor \Box \neg p$] is a position				
Using the rules, one obtains				
[:],[:],[:	$\Box \mathfrak{p} \lor \Box \neg \mathfrak{p}]]$







Maximality Facts

Each modal position can be extended to a maximal modal position

Each component of a maximal modal position is a maximal position

Each maximal modal position corresponds to a simple Kripke model: $\Box A$ is true at $[\mathcal{X}; \mathcal{Y}]$ iff A is true in *every* position in the modal position

BEYOND

Further directions

There are many directions one could go from here

One could add quantifiers and predicates

One could add axioms to obtain theories

Quantifiers

$$\frac{A(a), X \vdash Y}{\forall x A(x), X \vdash Y} [\forall L]$$

$$\frac{X \vdash Y, A(y)}{X \vdash Y, \forall x A(x)} \, {}_{[\forall R]}$$

In $[\forall R]$, y cannot occur freely in the lower sequent

These rules, and their duals for the existential quantifier, permit an Elimination Theorem

Identity

$$\vdash a = a$$

$$\frac{A(s), X \vdash Y}{s = t, A(t), X \vdash Y} [= L]$$

$$\frac{X \vdash Y, A(s)}{s = t, X \vdash Y, A(t)} [= L]$$

Identity, alternative rules

Alternative rules make it easier to eliminate Cut

$$\frac{a = a, X \vdash Y}{X \vdash Y} [= R]$$

$$\frac{s = t, A(t), A(s), X \vdash Y}{s = t, A(t), X \vdash Y} [= L]$$

One can use the Dragalin-style proof to show that Cut can be eliminated

Truth

$$\frac{A, X \vdash Y}{T\langle A \rangle, X \vdash Y} [TL]$$

$$\frac{X \vdash Y, A}{X \vdash Y, T\langle A \rangle} [TR]$$

These rules are inconsistent in classical logic, so one will need to go non-classical to hang onto them

They also take complex formulas to atomic formulas

Arithmetic

Take a language with =, 0, ', +, \times

$$x' = y' \vdash x = y$$

$$\vdash x + 0 = x$$

$$0 = x' \vdash$$

$$\vdash x + y' = (x + y)'$$

$$\vdash x \times 0 = 0$$

$$X \vdash A(0), Y \quad X, A(x) \vdash A(x'), Y$$

$$X \vdash A(x), Y$$

$$X \vdash A(x), Y$$

$$X \vdash A(x), Y$$

$$X \vdash A(x), Y \quad A(0), X \vdash Y$$

$$X \vdash A(x), X \vdash Y$$

Positions and Models



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THANK YOU!

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