Contingent Existence & Modal Definedness

GREG RESTALL • restall@unimelb.edu.au • AUSTRALASIAN ASSOCIATION FOR LOGIC CONFERENCE • 2 JULY 2015

0. MY TOPIC

I understand $(\forall x)$ & $(\exists x)$, the standard quantifiers in first order predicate logic. (Classical first order predicate logic seems fair enough for these purposes.)

I understand $\Box$ and $\Diamond$, the modal operators of possibility and necessity.

(The classical modal logic S5 seems to work well for these purposes.)

It is hard to understand how modal operators and quantifiers interact.

I have argued elsewhere (2005, 2012) that paying attention to rules of inference, and the normative significance connecting these rules to the speech acts of *assertion*, *denial* and *supposition* can shed light on logic and meaning. I will look here at *quantifiers* with *modal operators*.

1. INTRODUCTION

There is a *metaphysical* issue, as well as a *logical* or *semantic* issue.

> *Necessitism*: Necessarily, all things necessarily exist.
> *Contingentism*: The negation of necessitism.

The fact that necessitism and contingentism can be stated in the language of first order modal logic does not mean that logic and metaphysics *overlap*—logic may not decide between necessitism and contingentism, while a metaphysical view may well decide this.

It *does* mean that metaphysics and logic come close.

Some (Linsky and Zalta; Cresswell; Williamson) think that there is a logical derivation of necessitism. Here is a derivation of the Barcan inference (from "everything is necessarily *F*" to "necessarily, everything is *F*", and important plank of necessitism.)

$$\dfrac{\dfrac{\dfrac{\dfrac{Fa \vdash Fa}{\Box Fa \vdash \ \mid\ \vdash Fa}\,(\Box L)}{\forall x \Box Fx \vdash \ \mid\ \vdash Fa}\,(\forall L)}{\forall x \Box Fx \vdash \ \mid\ \vdash \forall x Fx}\,(\forall R)^*}{\forall x \Box Fx \vdash \Box \forall x Fx}\,(\Box R)$$

Is *contingentism* so easy to refute? Is *necessitism* just a matter of logic? To answer this, it's important to understand the behaviour of the *quantifiers* and the *modal operators*. We'll start with the *quantifiers*.

2. FREE LOGIC

Semantic analyses of contingentism start with *Free Logic*.

Free Logic: singular terms may be free of existential import.
Fa need not entail $(\exists x)Fx$.

There are metaphysically heavyweight (semantics with quantifcation over nonexistent objects) and metaphysically lightweight (the rest) variants of Free Logic.

I'll focus on a free logic well suited to a proof-theoretical interpretation, and hopefully suited to interaction with modal concepts. It's due to Solomon Feferman (1995), it's the free logic of *definedness*.

Motivated by applications in mathematics, Feferman's semantics allows for partially defined function terms, and statements like these:

> For every x and y, if $x \geq 0$ and $y > 0$ then $(x/y){\downarrow}$, and $x/y > 0$
>
> If $x \geq 0$ and $y = 0$ then $\neg(x/y{\downarrow})$.

Given a singular term t, the sentence "$t{\downarrow}$" is to be understood as "t is defined."

The logic is classical (two-valued) but singular terms need not always be defined. Quantifiers range over objects, with a standard Tarskian semantics, but the inferences from $(\forall x)Fx$ to Ft, and from Ft to $(\exists x)Fx$ can fail, since t may be undefined. (We can validly step from $(\forall x)Fx$ and $t{\downarrow}$ to Ft, and from Ft and $t{\downarrow}$ to $(\exists x)Fx$.)

[For every x, either $x \geq 0$ or $x < 0$. ($1/0 > 0$ or $1/0 < 0$) needn't follow. Similarly, $\neg(1/0{\downarrow})$, but it doesn't follow that there is an x where $\neg(x{\downarrow})$.]

This logic has a perspicuous explanation in terms of accepting and rejecting *statements* (asserting and denying) and *terms* (taking them to be defined or undefined).

$$D; X \vdash Y; U$$

This sequent states that there is a *clash* involved in accepting the terms in *D*, the statements in *X* and rejecting the statements in *Y* and the terms in *U*. Or: if the terms in *D* are defined and the sentences in *X* are true then one of the statements in *Y* is true or one of the terms in *U* is undefined.

The inference rules for definedness are straightforward.

$$\dfrac{t, D; \Gamma \vdash \Delta; U}{D; t{\downarrow}, \Gamma \vdash \Delta; U}\,({\downarrow}L) \qquad \dfrac{D; \Gamma \vdash \Delta; U, t}{D; \Gamma \vdash \Delta, t{\downarrow}; U}\,({\downarrow}R)$$

The standard rules for the quantifiers (used in the context where all terms are defined) are given a natural variation where we allow for undefined terms:

$$\dfrac{D; X \vdash Y; U, t \qquad D; A(t), X \vdash Y; U}{D; \forall x\, A(x), X \vdash Y; U}\,(\forall L)$$

$$\dfrac{D; X \vdash A(y), Y; U}{D; X \vdash \forall x\, A(x), Y; U}\,(\forall R)^*$$

This is natural and, I take it, a metaphysically lightweight and unobtrusive expansion of classical predicate logic, in its spirit.

We do not need an "outer domain" of nonexistent objects to make sense of this kind of free logic.

3. MODAL LOGIC

I have argued elsewhere that *hypersequents* for modal logics like S5 can explicate our grasp of modal concepts.

$$\Box p \vdash \ \mid\ \vdash p$$

There is a clash involved in asserting $\Box p$ in one discourse context and in denying p in another.

The rules for modal operators connect an assertion of the necessity of *A* to an assertion of *A*, and the denial of *A* to the denial that *A* is necessary. The rules underwrite straightforward derivations.

$$\dfrac{X' \vdash Y' \mid X, A \vdash Y \mid \Delta}{X', \Box A \vdash Y' \mid X \vdash Y \mid \Delta}\,(\Box L) \qquad \dfrac{\vdash A \mid X \vdash Y \mid \Delta}{X \vdash \Box A, Y \mid \Delta}\,(\Box R)$$

These rules can be generalised to other modal operators. In a simple setting where discourse contexts are all alternatives to each other, the result is an interpretation of the modal logic S5 setting where discourse contexts are all alternatives to each other, the result is an interpretation of the modal logic S5.

$$\dfrac{\dfrac{\dfrac{\dfrac{p \vdash p}{\Box p \vdash \;\mid\; \vdash p}\,(\Box L)}{\Box p \wedge \Box q \vdash \;\mid\; \vdash p}\,(\wedge L) \qquad \dfrac{\dfrac{q \vdash q}{\Box q \vdash \;\mid\; \vdash q}\,(\Box L)}{\Box p \wedge \Box q \vdash \;\mid\; \vdash q}\,(\wedge L)}{\Box p \wedge \Box q \vdash \;\mid\; \vdash p \wedge q}\,(\wedge R)}{\Box p \wedge \Box q \vdash \Box(p \wedge q)}\,(\Box R)$$

A more nuanced understanding of alternativeness gives us other normal modal logics, including 2D modal logics, etc.

We can agree on the interpretation of the box as "necessary that" without agreeing on what is necessary —by coordinating in our use of context shifts in hypothetical reasoning— in the same way that we can agree on the interpretation of the universal quantifier as meaning "every" —by coordinating in our use of substituting terms for the quantifier— even though we may disagree on what there is.

Modal reasoning has the structure it does (explicable in terms of "worlds"), not because of any independent access to worlds, but because these other 'possible worlds' stand to different positions in a discourse in the same way that this world (construed as the totality of facts—as how things are, and not just a big thing) is an "hypostasiation" of our 'home' position in a discourse (Restall 2009, 2012). Unlike other 'ersatz' accounts of modality, we have an *explanation* of why modal concepts have the logical structure they do.

Now, let's put the modal logic and the free logic together…

4. FREE MODAL LOGIC

…and you see that it *doesn't help at all*.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \;;x \;\mid\; \vdash Fx}\,(x\text{-}Id) \qquad \dfrac{Fx \vdash Fx}{\Box Fx \vdash \;\mid\; \vdash Fx}\,(\Box L)}{\forall x \Box Fx \vdash \;\mid\; \vdash Fx}\,(\forall L)}{\forall x \Box Fx \vdash \;\mid\; \vdash \forall x Fx}\,(\forall R)^*}{\forall x \Box Fx \vdash \Box \forall x Fx}\,(\Box R)$$

The Barcan formula is *still* derivable. We've just combined the standard rules for the modal operators and the quantifiers in the logic of definedness. The issue is the axiom (*x-Id*): $\vdash \;;x$

But *is* it incoherent to deny that *x* exists?

Not in modal reasoning. Consider some object *x*. Now consider—*had things gone differently*—that *x* doesn't exist.

This seems *totally* coherent —at least for the contingentist— this is the heart of contingentism.

The rules for quantifiers in the modal context take into account the possibility that variables needn't be defined everywhere.

$$\dfrac{D;X \vdash Y;U,t \mid \Delta \qquad D;A(t),X \vdash Y;U \mid \Delta}{D;\forall x\, A(x),X \vdash Y;U \mid \Delta}\,(\forall L)$$

$$\dfrac{y,D;X \vdash A(y),Y;U \mid \Delta}{D;X \vdash \forall x\, A(x),Y;U \mid \Delta}\,(\forall R)^*$$

The quantifiers range over only what exists (what *else* is there for them to range over?). There is no clash in asserting $(\forall x)Fx$ and denying Ft, if you deny that *t* denotes. There is no clash in asserting $(\forall x)Fx$ and denying *Fy*, provided that you deny *of the object y* that it exists.

You can derive that necessarily everything exists, not that everything necessarily exists.

$$\dfrac{\dfrac{\dfrac{\vdash \;\mid\; x;\; \vdash \;;x}{\vdash \;\mid\; x;\; \vdash x{\downarrow}}\,({\downarrow}R)}{\vdash \;\mid\; \vdash \forall x\, x{\downarrow}}\,(\forall R)^*}{\vdash \Box \forall x\, x{\downarrow}}\,(\Box R)$$

The corresponding proof where the quantifier is outside the scope of the modal operator, is unavailable.

Similarly, the Barcan sequent is underivable.

$$\forall x \Box Fx \vdash^{?} \Box \forall x\, Fx$$

We can find a counterexample out of a conversational context with two discourse "zones"

$$[x\,;\, Fx \;:\; Fy\,;\, y] \;\mid\; [x,y\,;\, Fx \;:\; Fy\,;\,]$$

one in which *x* and *Fx* are ruled in and *Fy* and *y* ruled out, and the other where *x*, *y* and *Fy* are ruled in, and *Fy* ruled out. There is no clash in this scenario, and, if you like, it can be filled out into a model in which there is a world (corresponding to the left discourse zone) where $\forall x \Box Fx$ holds and $\Box \forall x Fx$ fails.

5. THEOREMS

This is perfectly general. You can define the notion of a *model* as the limit of a process of filling out a discourse structure. The limit of a zone is a world in a structure, and the collection of those terms *ruled in* (*modulo* identities taken to be true) is the *domain* of that world.

This corresponds to a simple model theory for a variable domain free logic with a modal operator governed by the usual S5 evaluation rules.

Soundness and Completeness Theorem: A hypersequent is derivable if and only if it holds in every model.

[*Actually*, every derivable hypersequent holds in every model; and every hypersequent that cannot be derived *in the system without the Cut rule* can be shown to fail in some model. This shows not only that the Soundness and the Completeness Theorem holds, but a *Cut Admissibility Theorem* holds too. If a hypersequent is derivable, it can be derived without the use of the Cut rule. For if a hypersequent is not derived without Cut, there is some model in which it fails, and hence, the hypersequent is not derivable with Cut either.]

However, the model theory arises *out of* the rules for use of the concepts, not the other way around. The original rules make no use of talk of nonexistent objects. We have everyday terms, and we treat them *as if* they are defined or undefined.

6. THE UPSHOT (?)

I will end with two CLAIMS (one *measured*, and one more *extreme*), and one WORRYING THOUGHT.

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