Terms for Classical Sequents

Proof Invariants & Strong Normalisation

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My Aim

To introduce a new *invariant*
for classical propositional proofs
and to show how they can be used.

Background

Preterms

Derivations

Terms

Eliminating Cuts

Strong Normalisation

Further Work

BACKGROUND

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \succ p}{p \succ p \lor q}\,{\lor R}}{p \land q \succ p \lor q}\,{\land L}
\qquad
\dfrac{\dfrac{p \land q}{p}\,{\land E}}{p \lor q}\,{\lor I}
\qquad
\dfrac{\dfrac{p \succ p}{p \land q \succ p}\,{\lor R}}{p \land q \succ p \lor q}\,{\land L}$$

When is π_1 the same proof as π_2?

$$\cfrac{\cfrac{p \succ p}{p \succ p \lor q}\,{\lor}R}{p \land q \succ p \lor q}\,{\land}L \qquad \cfrac{\cfrac{p \land q}{p}\,{\land}E}{p \lor q}\,{\lor}I \qquad \cfrac{\cfrac{p \succ p}{p \land q \succ p}\,{\lor}R}{p \land q \succ p \lor q}\,{\land}L$$

$$\cfrac{\cfrac{q \succ q}{q \succ p \lor q}\,{\lor}R}{p \land q \succ p \lor q}\,{\land}L \qquad \cfrac{\cfrac{p \land q}{q}\,{\land}E}{p \lor q}\,{\lor}I \qquad \cfrac{\cfrac{q \succ q}{p \land q \succ q}\,{\lor}R}{p \land q \succ p \lor q}\,{\land}L$$

When is π_1 the same proof as π_2?

$$\dfrac{\dfrac{p \lor q \qquad \dfrac{[p]^1}{q \lor p}\,{\lor}I \qquad \dfrac{[q]^1}{q \lor p}\,{\lor}I}{q \lor p}\,{\lor}E^1}{(q \lor p) \lor r}\,{\lor}I \qquad\qquad \dfrac{p \lor q \qquad \dfrac{\dfrac{[p]^1}{q \lor p}\,{\lor}I}{(q \lor p) \lor r}\,{\lor}I \qquad \dfrac{\dfrac{[q]^1}{q \lor p}\,{\lor}I}{(q \lor p) \lor r}\,{\lor}I}{(q \lor p) \lor r}\,{\lor}E^1$$

Are these *different proofs*, or *different ways of presenting the same proof*?

Natural deduction is a slightly paradoxical system: it is limited to the intuitionistic case (in the classical case it has no particularly good properties) but it is only satisfactory for the $(\wedge, \Rightarrow, \forall)$ fragment of the language: we shall defer consideration of $\vee$ and $\exists$ until chapter 10. Yet disjunction and existence are the two most *typically* intuitionistic connectors!

The basic idea of natural deduction is an asymmetry: a proof is a vaguely tree-like structure (this view is more a graphical illusion than a mathematical reality, but it is a pleasant illusion) with one or more hypotheses (possibly none) but a single conclusion. The deep symmetry of the calculus is shown by the *introduction* and *elimination* rules which match each other exactly. Observe, incidentally, that with a tree-like structure, one can always decide uniquely what was the *last* rule used, which is something we could not say if there were several conclusions.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x : p \supset (q \supset r)] \quad [z : p]}{xz : q \supset r}\supset E \quad \dfrac{[y : p \supset q] \quad [z : p]}{yz : q}\supset E}{(xz)(yz) : r}\supset E}{\lambda z\,(xz)(yz) : p \supset r}\supset I}{\lambda y \lambda z\,(xz)(yz) : (p \supset q) \supset (p \supset r)}\supset I}{\lambda x \lambda y \lambda z\,(xz)(yz) : (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))}\supset I
$$

Contraction and *weakening* are managed by variables

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\supset I$$

$$\dfrac{\dfrac{[x : p]}{\lambda y\,x : q \supset p}\;{\supset}I}{\lambda x\lambda y\,x : p \supset (q \supset p)}\;{\supset}I$$

$$\dfrac{\dfrac{\dfrac{x : p \supset (p \supset q) \qquad [y : p]}{xy : p \supset q}\;{\supset}E \qquad [y : p]}{(xy)y : q}\;{\supset}E}{\lambda y\,(xy)y : p \supset q}\;{\supset}I$$

Classical Sequent Derivations

$$\dfrac{\dfrac{p \succ p}{\succ p, \neg p}\,\neg R}{\succ p \vee \neg p}\,\vee R \qquad \dfrac{\dfrac{p \succ p}{p, \neg p \succ}\,\neg L}{p \wedge \neg p \succ}\,\wedge L$$

Classical Sequent Derivations

$$
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\qquad\qquad
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$$

$$
\dfrac{\dfrac{\dfrac{p \succ p \qquad \dfrac{q \succ q \quad r \succ r}{q \vee r \succ q, r}\ \vee L}{p, q \vee r \succ p \wedge q, r}\ \wedge R}{p \wedge (q \vee r) \succ p \wedge q, r}\ \wedge L}{p \wedge (q \vee r) \succ (p \wedge q) \vee r}\ \vee R
$$

$$X \succ Y \qquad X \succ A, Y \qquad X, A \succ Y$$

Where do you put the *variables*,
and where do you put the *terms*?

Our Choice

$$x_1 : A_1, \ldots, x_n : A_n \succ y_1 : B_1, \ldots, y_m : B_m$$

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Each premise and conclusion is decorated with variables.

Our Choice

$$\frac{\pi(x_1, \ldots, x_n)[y_1, \ldots, y_m]}{x_1 : A_1, \ldots, x_n : A_n \succ y_1 : B_1, \ldots, y_m : B_m}$$

Each premise and conclusion is decorated with variables.

The *sequent* gets the term, connecting inputs & outputs.

Example 1

$$
\dfrac{\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \dfrac{\overset{y\frown y}{y : q \succ y : q} \qquad \overset{z\frown z}{z : r \succ z : r}}{\overset{\mathsf{L}w\frown y \quad \mathsf{R}w\frown z}{w : q \vee r \succ y : q, z : r}}\,\vee L}{\overset{x\frown \mathsf{F}v \quad \mathsf{L}w\frown \mathsf{S}v \quad \mathsf{R}w\frown z}{x : p, w : q \vee r \succ v : p \wedge q, z : r}}\,\wedge R}{\overset{\mathsf{F}u\frown \mathsf{F}v \quad \mathsf{LS}u\frown \mathsf{S}v \quad \mathsf{RS}u\frown z}{u : p \wedge (q \vee r) \succ v : p \wedge q, z : r}}\,\wedge L}{\overset{\mathsf{F}u\frown \mathsf{FL}t \quad \mathsf{LS}u\frown \mathsf{SL}t \quad \mathsf{RS}u\frown \mathsf{R}t}{u : p \wedge (q \vee r) \succ t : (p \wedge q) \vee r}}\,\vee R
$$

Example 2

$$\dfrac{\dfrac{\overset{x\frown x}{x:p \succ x:p} \qquad \overset{x\frown x}{x:p \succ x:p}}{\overset{x\frown Fy \quad x\frown Sy}{x:p \succ y:p \land p}}\,{\land}R \qquad \dfrac{\overset{z\frown z}{z:p \succ z:p}}{\overset{Fw\frown z}{w:p \land p \succ z:p}}\,{\land}L}{\overset{x\frown F\bullet \quad x\frown S\bullet \quad F\bullet\frown z}{x:p \succ z:p}}\,\mathit{Cut}$$

PRETERMS

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– We use $x, y, z, u, v, w, \ldots; \bullet, \star, *, \sharp, \flat$ as schematic letters for variables and cut points, ommitting type superscripts where possible.

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- If n is an $A \supset B$ node, then An is an A node and Cn is a B node.
- If n is a $\neg A$ node, then Nn is an A node.
- For each complex node Ln, Rn, Fn, Sn, An, Cn and Nn, n is its IMMEDIATE subnode, and the subnodes of n are also subnodes of the original node.

Linkings, Inputs and Outputs

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  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *input position*, n is in *output position*.

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  - L, R, F, S and C each *preserve position*.
  - If $\mathsf{A}n$ or $\mathsf{N}n$ is in *input position*, n is in *output position*.
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    - – L, R, F, S and C each *preserve position*.
    - If An or Nn is in *input position*, n is in *output position*.
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    - – A and N *reverse position*.

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  - – L, R, F, S and C each *preserve position*.
  - If An or Nn is in *input position*, n is in *output position*.
  - If An or Nn is in *output position*, n is in *input position*.
  - – A and N *reverse position*.
- The INPUTS (OUTPUTS) of a linking are the *variables* in INPUT (OUTPUT) position of that linking.

Example Linkings

x of type $((p \supset q) \supset p) \supset p$

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x of type $((p \supset q) \supset p) \supset p$

$AAAx^\frown Cx$

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x of type $((p \supset q) \supset p) \supset p$

$\mathsf{AAA}x \frown \mathsf{C}x$

$\mathsf{CA}x \frown \mathsf{C}x$

Preterms

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- Its OUTPUTS are the outputs of its linkings.

DERIVATIONS

Annotating Derivations: Identity

$$\Sigma, x : A \overset{x \frown y}{\succ} y : A, \Delta$$

Annotating Derivations: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\underset{}{\overset{\pi(\mathsf{F}z,\mathsf{S}z)}{\Sigma, z : A \wedge B \succ \Delta}}}\wedge L$$

Annotating Derivations: Conjunction

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \wedge B \succ \Delta}} \wedge L \qquad \dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta'}} \wedge R$$

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\supset I$$

Excursus on *Weakening* and *Variables*

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\supset I$$

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz, Sz)}{\Sigma, z : A \wedge B \succ \Delta}}\wedge L \quad \text{can be} \quad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Fz)}{\Sigma, z : A \wedge B \succ \Delta}}\wedge L$$

Excursus on *Weakening* and *Variables*

$$\dfrac{\dfrac{[x : p]}{\lambda y\, x : q \supset p}\supset I}{\lambda x \lambda y\, x : p \supset (q \supset p)}\supset I$$

$$\dfrac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \wedge B \succ \Delta}}\wedge L \quad \text{can be} \quad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Fz)}{\Sigma, z : A \wedge B \succ \Delta}}\wedge L$$

In a premise $\pi(x, y)$ the indicated x and y display all of the x and y inputs to the proof term.

There might be *none*.

Annotating Derivations: Negation

$$\frac{\begin{array}{c}\pi[x]\\ \Sigma \succ x : A, \Delta\end{array}}{\begin{array}{c}\pi[\mathsf{N}z]\\ \Sigma, z : \neg A \succ \Delta\end{array}}\neg L \qquad \frac{\begin{array}{c}\pi(x)\\ \Sigma, x : A \succ \Delta\end{array}}{\begin{array}{c}\pi(\mathsf{N}z)\\ \Sigma \succ z : \neg A, \Delta\end{array}}\neg R$$

Annotating Derivations: Disjunction

$$\frac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz) \quad \pi'(Rz)}{\Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta'}} \vee L \qquad \frac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz, Rz]}{\Sigma \succ z : A \vee B, \Delta}} \vee R$$

Annotating Derivations: Conditional

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az] \quad \pi'(Lz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset L \qquad \dfrac{\overset{\pi(x)[y]}{\Sigma, x : A \succ y : B, \Delta}}{\overset{\pi(Az)[Cz]}{\Sigma \succ z : A \supset B, \Delta}} \supset R$$

Example Annotation

$$
\dfrac{
  \dfrac{
    \dfrac{
      \overset{x^\frown x}{x : p \succ x : p}
      \qquad
      \dfrac{
        \overset{y^\frown y}{y : q \succ y : q}
        \qquad
        \overset{z^\frown z}{z : r \succ z : r}
      }{
        \overset{\mathsf{L}w^\frown y \quad \mathsf{R}w^\frown z}{w : q \lor r \succ y : q, z : r}
      }\,{\lor L}
    }{
      \overset{x^\frown \mathsf{F}v \quad \mathsf{L}w^\frown \mathsf{S}v \quad \mathsf{R}w^\frown z}{x : p, w : q \lor r \succ v : p \land q, z : r}
    }\,{\land R}
  }{
    \overset{\mathsf{F}u^\frown \mathsf{F}v \quad \mathsf{LS}u^\frown \mathsf{S}v \quad \mathsf{RS}u^\frown z}{u : p \land (q \lor r) \succ v : p \land q, z : r}
  }\,{\land L}
}{
  \overset{\mathsf{F}u^\frown \mathsf{FL}t \quad \mathsf{LS}u^\frown \mathsf{SL}t \quad \mathsf{RS}u^\frown \mathsf{R}t}{u : p \land (q \lor r) \succ t : (p \land q) \lor r}
}\,{\lor R}
$$

Annotating Derivations: Cut

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : A \succ \Delta'}}{\overset{\pi[\bullet] \quad \pi'(\bullet)}{\Sigma, \Sigma' \succ \Delta, \Delta'}}\ \mathit{Cut}$$

Example Annotation, with *Cut*

$$
\dfrac{
\dfrac{
\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}
}{
\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}
}\,{\lor}L
\qquad
\dfrac{
\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}
}{
\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}
}\,{\land}R
}{
\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \lor p \succ z : p \land p}
}\,\mathit{Cut}
$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{z\frown Ly}{z : p \succ y : p \vee q}}\,\vee R}{\overset{Fx\frown Ly}{x : p \wedge q \succ y : p \vee q}}\,\wedge L
\qquad
\dfrac{\dfrac{p \wedge q}{p}\,\wedge E}{p \vee q}\,\vee I
\qquad
\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fx\frown z}{x : p \wedge q \succ z : p}}\,\vee R}{\overset{Fx\frown Ly}{x : p \wedge q \succ y : p \vee q}}\,\wedge L$$

$$\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{z\frown Ly}{z : p \succ y : p \lor q}}\,{\scriptstyle \lor R}}{\overset{Fx\frown Ly}{x : p \land q \succ y : p \lor q}}\,{\scriptstyle \land L}
\qquad
\dfrac{\dfrac{p \land q}{p}\,{\scriptstyle \land E}}{p \lor q}\,{\scriptstyle \lor I}
\qquad
\dfrac{\dfrac{\overset{z\frown z}{z : p \succ z : p}}{\overset{Fx\frown z}{x : p \land q \succ z : p}}\,{\scriptstyle \lor R}}{\overset{Fx\frown Ly}{x : p \land q \succ y : p \lor q}}\,{\scriptstyle \land L}$$

$$\dfrac{\dfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{w\frown Ry}{w : q \succ y : p \lor q}}\,{\scriptstyle \lor R}}{\overset{Sx\frown Ry}{x : p \land q \succ y : p \lor q}}\,{\scriptstyle \land L}
\qquad
\dfrac{\dfrac{p \land q}{q}\,{\scriptstyle \land E}}{p \lor q}\,{\scriptstyle \lor I}
\qquad
\dfrac{\dfrac{\overset{w\frown w}{w : q \succ w : q}}{\overset{Sx\frown w}{x : p \land q \succ w : q}}\,{\scriptstyle \lor R}}{\overset{Sx\frown Ry}{x : p \land q \succ y : p \lor q}}\,{\scriptstyle \land L}$$

When is π_1 the same proof as π_2 (revisited)?

$$
\dfrac{\dfrac{p \lor q \qquad \dfrac{[p]^1}{q \lor p}\,{\lor I} \qquad \dfrac{[q]^1}{q \lor p}\,{\lor I}}{q \lor p}\,{\lor E^1}}{(q \lor p) \lor r}\,{\lor I}
$$

$$
\dfrac{\dfrac{\dfrac{\overset{x \frown x}{x : p \succ x : p}}{\overset{x \frown Rz}{x : p \succ z : q \lor p}}\,{\lor R} \qquad \dfrac{\overset{y \frown y}{y : q \succ y : q}}{\overset{y \frown Lz}{y : q \succ z : q \lor p}}\,{\lor R}}{\overset{Lw \frown Rz \quad Rw \frown Lz}{w : p \lor q \succ z : q \lor p}}\,{\lor L}}{\overset{Lw \frown RLu \quad Rw \frown LLu}{w : p \lor q \succ u : (q \lor p) \lor r}}\,{\lor R}
$$

When is π_1 the same proof as π_2 (revisited)?

$$\dfrac{p \lor q \qquad \dfrac{\dfrac{[p]^1}{q \lor p}\,{\lor}I}{(q \lor p) \lor r}\,{\lor}I \qquad \dfrac{\dfrac{[q]^1}{q \lor p}\,{\lor}I}{(q \lor p) \lor r}\,{\lor}I}{(q \lor p) \lor r}\,{\lor}E^1$$

$$\dfrac{\dfrac{\dfrac{\overset{x \frown x}{x : p \succ x : p}}{\overset{x \frown Rz}{x : p \succ z : q \lor p}}\,{\lor}R}{\overset{x \frown RLu}{x : p \succ u : (q \lor p) \lor r}}\,{\lor}R \qquad \dfrac{\dfrac{\overset{y \frown y}{y : q \succ y : q}}{\overset{x \frown LLu}{y : q \succ z : q \lor p}}\,{\lor}R}{\overset{y \frown LLu}{y : q \succ u : (q \lor p) \lor r}}\,{\lor}R}{\overset{Lw \frown RLu \quad Rw \frown LLu}{w : p \lor q \succ u : (q \lor p) \lor r}}\,{\lor}L$$

A preterm is SEQUENTIALISABLE
iff it is the conclusion of some derivation.

TERMS

Nonsequentialisable Preterms

$$\overset{Lx\frown Fy \quad Rx\frown Sy}{x : p \vee q \succ y : p \wedge q}$$

This is connected, but it is not connected *enough*.

Switching Example

$$\overset{\color{darkred}{Lx \frown Fy}}{} \quad \overset{\color{darkred}{Rx \frown Sy}}{}$$

$$x : p \vee q \succ y : p \wedge q$$

Switching Example

$$\overset{Lx^\frown Fy \quad Rx^\frown Sy}{x : p \vee q \succ y : p \wedge q}$$

$$\overset{Lx^\frown Fy \quad \cancel{Rx}^\frown \cancel{Sy}}{x : p \vee - \succ y : p \wedge -}$$

$$\overset{Lx^\frown \cancel{Fy} \quad \cancel{Rx}^\frown Sy}{x : p \vee - \succ y : - \wedge q}$$

$$\overset{\cancel{Lx}^\frown Fy \quad Rx^\frown \cancel{Sy}}{x : - \vee q \succ y : p \wedge -}$$

$$\overset{\cancel{Lx}^\frown \cancel{Fy} \quad Rx^\frown Sy}{x : - \vee q \succ y : - \wedge q}$$

Switchings

- The SWITCHINGS of a preterm π are found by selecting for each pair of subterms $\mathsf{L}n$ and $\mathsf{R}n$ in *input* position; $\mathsf{F}n$ and $\mathsf{S}n$ in *output position*, $\mathsf{A}n$ in *output* position and $\mathsf{C}n$ in *input position*; or the cut point $\bullet$ (in both *input* and *output position*), one item of the pair to keep, and the other to DELETE.

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- A LINKING in a switching of a preterm π SURVIVES if and only if neither side of the link involves a deletion.

- A preterm is SPANNED if every switching has at least one surviving linking.

Example

$\mathsf{Fu}^\frown\mathsf{FLt} \quad \mathsf{LSu}^\frown\mathsf{SLt} \quad \mathsf{RSu}^\frown\mathsf{Rt}$

This has two pairs for switching:

$\mathsf{LSu}/\mathsf{RSu}$ in *input position*. $\mathsf{FLt}/\mathsf{SLt}$ in *output position*.

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This has two pairs for switching:

$\mathsf{LSu}/\mathsf{RSu}$ in *input position*. $\mathsf{FLt}/\mathsf{SLt}$ in *output position*.

$$\mathsf{Fu}\frown\cancel{\mathsf{FLt}}\quad \cancel{\mathsf{LSu}}\frown\mathsf{SLt}\quad \mathsf{RSu}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\mathsf{FLt}\quad \cancel{\mathsf{LSu}}\frown\cancel{\mathsf{SLt}}\quad \mathsf{RSu}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\cancel{\mathsf{FLt}}\quad \mathsf{LSu}\frown\mathsf{SLt}\quad \cancel{\mathsf{RSu}}\frown\mathsf{Rt}$$

$$\mathsf{Fu}\frown\mathsf{FLt}\quad \mathsf{LSu}\frown\cancel{\mathsf{SLt}}\quad \cancel{\mathsf{RSu}}\frown\mathsf{Rt}$$

Terms

A preterm π is a TERM when it is SPANNED.

By induction on the derivation sequentialising π.

Sequentialisable Preterms are Terms: Identity

$$\Sigma, x : A \overset{x \frown y}{\succ} y : A, \Delta$$

Sequentialisable Preterms are Terms: Conjunction

$$\frac{\overset{\pi(x,y)}{\Sigma, x : A, y : B \succ \Delta}}{\underset{}{\overset{\pi(Fz,Sz)}{\Sigma, z : A \wedge B \succ \Delta}}}\wedge L \qquad \frac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta'}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta'}}\wedge R$$

Sequentialisable Preterms are Terms: Negation

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta}}{\overset{\pi[Nz]}{\Sigma, z : \neg A \succ \Delta}}\,\neg L \qquad \dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta}}{\overset{\pi(Nz)}{\Sigma \succ z : \neg A, \Delta}}\,\neg R$$

Sequentialisable Preterms are Terms: Disjunction

$$\dfrac{\overset{\pi(x)}{\Sigma, x : A \succ \Delta} \qquad \overset{\pi'(y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi(Lz) \quad \pi'(Rz)}{\Sigma, \Sigma', z : A \vee B \succ \Delta, \Delta'}} \vee L \qquad \dfrac{\overset{\pi[x, y]}{\Sigma \succ x : A, y : B, \Delta}}{\overset{\pi[Lz, Rz]}{\Sigma \succ z : A \vee B, \Delta}} \vee R$$

Sequentialisable Preterms are Terms: Conditional

$$\dfrac{\Sigma \succ \overset{\pi[x]}{x} : A, \Delta \qquad \Sigma', \overset{\pi'(y)}{y} : B \succ \Delta'}{\Sigma, \Sigma', \overset{\pi[Az] \quad \pi'(Cz)}{z} : A \supset B \succ \Delta, \Delta'} \supset L \qquad \dfrac{\Sigma, x : A \succ \overset{\pi(x)[y]}{y} : B, \Delta}{\Sigma \succ \overset{\pi(Az)[Cz]}{z} : A \supset B, \Delta} \supset R$$

Sequentialisable Preterms are Terms: Cut

$$\dfrac{\Sigma \succ \overset{\pi[x]}{x} : A, \Delta \qquad \Sigma', \overset{\pi'(y)}{y} : A \succ \Delta'}{\Sigma, \Sigma' \succ \overset{\pi[\bullet] \quad \pi'(\bullet)}{\Delta, \Delta'}}\ \mathit{Cut}$$

By induction on the number of pairs for switching in π.

Except …

$$x \frown y \qquad u \frown v$$

ELIMINATING CUTS

Conjunction Cut Reduction

$$\dfrac{\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}}{\overset{\pi[\mathsf{F}z] \quad \pi'[\mathsf{S}z]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta}}{\scriptstyle \wedge R} \qquad \dfrac{\overset{\pi''(u, v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi''(\mathsf{F}w, \mathsf{S}w)}{\Sigma'', w : A \wedge B \succ \Delta''}}{\scriptstyle \wedge L}}{\overset{\pi[\mathsf{F}\bullet] \quad \pi'[\mathsf{S}\bullet] \quad \pi''(\mathsf{F}\bullet, \mathsf{S}\bullet)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}{\scriptstyle \mathit{Cut}}$$

Conjunction Cut Reduction

$$\dfrac{\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi'[y]}{\Sigma' \succ y : B, \Delta}}{\overset{\pi[Fz] \quad \pi'[Sz]}{\Sigma, \Sigma' \succ z : A \wedge B, \Delta, \Delta}}\wedge R \qquad \dfrac{\overset{\pi''(u,v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi''(Fw, Sw)}{\Sigma'', w : A \wedge B \succ \Delta''}}\wedge L}{\overset{\pi[F\bullet] \quad \pi'[S\bullet] \quad \pi''(F\bullet, S\bullet)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}\mathit{Cut}$$

reduces to

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \dfrac{\overset{\pi'[y]}{\Sigma' \succ y : B, \Delta} \qquad \overset{\pi''(u,v)}{\Sigma'', u : A, v : B \succ \Delta''}}{\overset{\pi'[\star] \quad \pi''(u, \star)}{\Sigma', \Sigma'', u : A \succ \Delta', \Delta''}}\mathit{Cut}}{\overset{\pi[*] \quad \pi'[\star] \quad \pi''(*, \star)}{\Sigma, \Sigma', \Sigma'' \succ \Delta, \Delta', \Delta''}}\mathit{Cut}$$

Identity Cut Reduction

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{y\frown z}{\Sigma', y : A \succ z : A, \Delta'}}{\overset{\pi[\bullet] \quad \bullet\frown z}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}}\, Cut$$

Identity Cut Reduction

$$\dfrac{\overset{\pi[x]}{\Sigma \succ x : A, \Delta} \qquad \overset{y^\frown z}{\Sigma', y : A \succ z : A, \Delta'}}{\overset{\pi[\bullet] \;\; \bullet^\frown z}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}}\ \mathit{Cut}$$

reduces to

$$\overset{\pi[z]}{\Sigma, \Sigma' \succ z : A, \Delta, \Delta'}$$

Difficult Cases: Contraction

$$
\dfrac{
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \lor p \succ x : p}}\,{\lor L}
\qquad
\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \land p}}\,{\land R}
}{
\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \lor p \succ z : p \land p}
}\,\mathit{Cut}
$$

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$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \vee p \succ x : p}}\,{\vee}L \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{x\frown Fz \quad x\frown Sz}{x : p \succ z : p \wedge p}}\,{\wedge}R}{\overset{Ly\frown\bullet \quad Ry\frown\bullet \quad \bullet\frown Fz \quad \bullet\frown Sz}{y : p \vee p \succ z : p \wedge p}}\,\mathit{Cut}$$

$$\dfrac{\dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \vee p \succ x : p}}\,{\vee}L \qquad \dfrac{\overset{x\frown x}{x : p \succ x : p} \qquad \overset{x\frown x}{x : p \succ x : p}}{\overset{Ly\frown x \quad Ry\frown x}{y : p \vee p \succ x : p}}\,{\vee}L}{\overset{Ly\frown Fz \quad Ry\frown Fz \quad Ly\frown Sz \quad Ry\frown Sz}{y : p \vee p \succ z : p \wedge p}}\,{\wedge}R$$

$$\dfrac{\dfrac{\overset{x\frown x}{x:p\succ x:p} \qquad \overset{x\frown x}{x:p\succ x:p}}{\overset{Ly\frown x\quad Ry\frown x}{y:p\lor p\succ x:p}}\,{\lor}L \qquad \dfrac{\overset{x\frown x}{x:p\succ x:p}\qquad \overset{x\frown x}{x:p\succ x:p}}{\overset{x\frown Fz\quad x\frown Sz}{x:p\succ z:p\land p}}\,{\land}R}{\overset{Ly\frown\bullet\quad Ry\frown\bullet\quad \bullet\frown Fz\quad \bullet\frown Sz}{y:p\lor p\succ z:p\land p}}\,\mathit{Cut}$$

$$\dfrac{\dfrac{\overset{x\frown x}{x:p\succ x:p} \qquad \overset{x\frown x}{x:p\succ x:p}}{\overset{x\frown Fz\quad x\frown Sz}{x:p\succ z:p\land p}}\,{\land}R \qquad \dfrac{\overset{x\frown x}{x:p\succ x:p}\qquad \overset{x\frown x}{x:p\succ x:p}}{\overset{x\frown Fz\quad x\frown Sz}{x:p\succ z:p\land p}}\,{\land}R}{\overset{Ly\frown Fz\quad Ry\frown Fz\quad Ly\frown Sz\quad Ry\frown Sz}{y:p\lor p\succ z:p\land p}}\,{\lor}L$$

$$\dfrac{\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}} \qquad \dfrac{\overset{\pi'}{\Sigma \succ \Delta}}{\overset{\pi'}{\Sigma, y : A \succ \Delta}}}{\overset{\pi \;\; \pi'}{\Sigma \succ \Delta}}\ \mathit{Cut}$$

Difficult Cases: Weakening

$$\dfrac{\dfrac{\overset{\pi}{\Sigma \succ \Delta}}{\overset{\pi}{\Sigma \succ x : A, \Delta}} \qquad \dfrac{\overset{\pi'}{\Sigma \succ \Delta}}{\overset{\pi'}{\Sigma, y : A \succ \Delta}}}{\overset{\pi \ \ \pi'}{\Sigma \succ \Delta}}\ Cut$$

$$\dfrac{\overset{\pi}{\Sigma \succ \Delta} \qquad \overset{\pi'}{\Sigma \succ \Delta}}{\overset{\pi \ \ \pi'}{\Sigma \succ \Delta}}\ Mix$$

Back to Sequentialisation

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$$\dfrac{\overset{x \frown y}{x : A \succ y : A} \qquad \overset{u \frown v}{u : B \succ v : B}}{\overset{x \frown y \quad u \frown v}{x : A, u : B \succ y : B, v : A}}\;\mathit{Mix}$$

Sequentialisation: Terms with *No* Switchings

The term contains no Ln, Rn, Cn and • in input position
or Fn, Sn, An and • in output position.

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It has a derivation using the linear rules
$\wedge L$, $\neg L$, $\neg R$, $\vee R$ and $\supset R$ and *mixes*.

Sequentialisation: Terms with *No* Switchings

The term contains no $\mathsf{L}n$, $\mathsf{R}n$, $\mathsf{C}n$ and $\bullet$ in input position or $\mathsf{F}n$, $\mathsf{S}n$, $\mathsf{A}n$ and $\bullet$ in output position.

It has a derivation using the linear rules $\wedge L$, $\neg L$, $\neg R$, $\vee R$ and $\supset R$ and *mixes*.

$$\mathsf{F}y\frown\mathsf{L}z \quad \mathsf{NR}z\frown\mathsf{L}z \quad \mathsf{S}y\frown\mathsf{R}z$$

$$y : p \wedge \neg p \succ z : p \vee \neg p$$

Terms with No Switchings: Example

$$
\begin{prooftree}
\AxiomC{$\stackrel{x\frown x}{x : p \succ x : p}$}
\RightLabel{$\vee R$}
\UnaryInfC{$\stackrel{x\frown \mathsf{L}z}{x : p \succ z : p \vee \neg p}$}
\RightLabel{$\wedge L$}
\UnaryInfC{$\stackrel{\mathsf{F}y\frown \mathsf{L}z}{y : p \wedge \neg p \succ z : p \vee \neg p}$}
\AxiomC{$\stackrel{u\frown u}{u : p \succ u : p}$}
\RightLabel{$\neg R$}
\UnaryInfC{$\stackrel{\mathsf{N}v\frown u}{\succ u : p, v : \neg p}$}
\RightLabel{$\vee R$}
\UnaryInfC{$\stackrel{\mathsf{NR}z\frown \mathsf{L}z}{\succ z : p \vee \neg p}$}
\AxiomC{$\stackrel{v\frown v}{v : \neg p \succ v : \neg p}$}
\RightLabel{$\vee R$}
\UnaryInfC{$\stackrel{v\frown \mathsf{R}z}{v : \neg p \succ z : p \vee \neg p}$}
\RightLabel{$\wedge L$}
\UnaryInfC{$\stackrel{\mathsf{S}y\frown \mathsf{R}z}{y : p \wedge \neg p \succ z : p \vee \neg p}$}
\RightLabel{Mix}
\BinaryInfC{$\stackrel{\mathsf{NR}z\frown \mathsf{L}z \quad \mathsf{S}y\frown \mathsf{R}z}{y : p \wedge \neg p \succ z : p \vee \neg p}$}
\RightLabel{Mix}
\BinaryInfC{$\stackrel{\mathsf{F}y\frown \mathsf{L}z \quad \mathsf{NR}z\frown \mathsf{L}z \quad \mathsf{S}y\frown \mathsf{R}z}{y : p \wedge \neg p \succ z : p \vee \neg p}$}
\end{prooftree}
$$

By induction on the number of switched pairs.

Take a switched pair at the *adjacent to variables* or *cut points* (peel away unswitched steps if there aren't any).

$$\dfrac{\overset{\pi[x](-)}{\Sigma \succ x : A, \Delta} \qquad \overset{\pi[-](y)}{\Sigma', y : B \succ \Delta'}}{\overset{\pi[Az](Lz)}{\Sigma, \Sigma', z : A \supset B \succ \Delta, \Delta'}} \supset L$$

$$
\dfrac{
\dfrac{\overset{\pi[x,u]}{\succ x : A \wedge B, u : A} \qquad \overset{\pi'[x,v]}{\succ x : A \wedge B, v : B}}{\overset{\pi[x,\mathsf{F}x] \quad \pi'[x,\mathsf{S}x]}{\succ x : A \wedge B}}\,{\wedge R}
\qquad
\dfrac{\overset{\pi''(y,z,x)}{y : A, z : B, x : A \wedge B \succ}}{\overset{\pi''(\mathsf{F}x,\mathsf{S}x,x)}{x : A \wedge B \succ}}\,{\wedge L}
}{
\overset{\pi[\bullet,\mathsf{F}\bullet] \quad \pi'[\bullet,\mathsf{S}\bullet] \quad \pi''(\mathsf{F}\bullet,\mathsf{S}\bullet,\bullet)}{\Sigma \succ \Delta}
}\,\mathit{Cut}
$$

Cut Reductions

Given a term $\pi(\bullet)[\bullet]$ and a cut-point $\bullet$, the $\bullet$-REDUCTION of π is found by:

- *atomic*: replace each pair $n\frown\bullet$ and $\bullet\frown m$ by $n\frown m$.

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- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

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$$Sz \frown F\bullet \quad Fz \frown S\bullet \quad F\bullet \frown Sx \quad S\bullet \frown Fx \quad Ny \frown \bullet \quad \bullet \frown v$$

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$$Sz\frown F\bullet \quad Fz\frown S\bullet \quad F\bullet\frown Sx \quad S\bullet\frown Fx \quad Ny\frown\bullet \quad \bullet\frown\nu$$

$Sz\frown\star$ $Fz\frown *$ $\star\frown Sx$ $*\frown Fx$

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

$$\mathsf{S}z\frown\star \quad \mathsf{F}z\frown* \quad \star\frown\mathsf{S}x \quad *\frown\mathsf{F}x \quad \mathsf{FN}y\frown\mathsf{S}x \quad \mathsf{SN}y\frown\mathsf{F}x \quad \mathsf{N}y\frown\nu$$

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$$\mathsf{S}z\frown\mathsf{F}\bullet \quad \mathsf{F}z\frown\mathsf{S}\bullet \quad \mathsf{F}\bullet\frown\mathsf{S}x \quad \mathsf{S}\bullet\frown\mathsf{F}x \quad \mathsf{N}y\frown\bullet \quad \bullet\frown\nu$$

$$\mathsf{S}z\frown\star \quad \mathsf{F}z\frown* \quad \star\frown\mathsf{S}x \quad *\frown\mathsf{F}x \quad \mathsf{FN}y\frown\mathsf{S}x \quad \mathsf{SN}y\frown\mathsf{F}x \quad \mathsf{N}y\frown\nu \quad \mathsf{S}z\frown\mathsf{F}\nu \quad \mathsf{F}z\frown\mathsf{S}\nu$$

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- *negation*: for each $\mathsf{N}\bullet$, add a new cut point $\star$. For any $\bullet\frown n$ add $l(n)$ for each link $l(\bullet)$ with n as input. For any $n\frown\bullet$ add $l[n]$ for each link $l[\bullet]$ with n as output.

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- *disjunction*: for each $\mathsf{L}\bullet/\mathsf{R}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

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- *conjunction*: for each $\mathsf{F}\bullet/\mathsf{S}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.
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- *conditional*: for each $\mathsf{A}\bullet/\mathsf{C}\bullet$, add new cut points $\star$ and $*$. For any $\bullet\frown\mathfrak{n}$ add $\mathfrak{l}(\mathfrak{n})$ for each link $\mathfrak{l}(\bullet)$ with $\mathfrak{n}$ as input. For any $\mathfrak{n}\frown\bullet$ add $\mathfrak{l}[\mathfrak{n}]$ for each link $\mathfrak{l}[\bullet]$ with $\mathfrak{n}$ as output.

STRONG NORMALISATION

Any reduction for π terminates in a unique* term π^*

- There is *some* terminating reduction process.

Any reduction for π terminates in a unique* term π^*

- There is *some* terminating reduction process.

- Proof reduction is confluent.

  – If $\pi \rightsquigarrow_\bullet \pi'$ and $\pi \rightsquigarrow_\star \pi''$ then there is a π''' where $\pi' \rightsquigarrow_\star \pi'''$ and $\pi'' \rightsquigarrow_\bullet \pi'''$.

FURTHER WORK

To Do List

- Are these genuine *invariants*? (Can we show that if two derivations have the same term, some set of permutations permute one to the other?)

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THANK YOU!

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