CSE 103: Discrete Mathematics

Predicate
Rules of Inference

Predicates - more examples

$L(x)$ = "x is a lion."
$F(x)$ = "x is fierce."
$C(x)$ = "x drinks coffee."

Universe of discourse is all creatures.

All lions are fierce. $\forall x\,(L(x) \rightarrow F(x))$

Some lions don't drink coffee. $\exists x\,(L(x) \wedge \neg C(x))$

Some fierce creatures don't drink coffee.

$\exists x\,(F(x) \wedge \neg C(x))$

Predicates - quantifier negation

So, $\neg\forall x\, P(x)$ is the same as $\exists x\, \neg P(x)$.

So, $\neg\exists x\, P(x)$ is the same as $\forall x\, \neg P(x)$.

General rule: to negate a quantifier, move negation to the right, changing quantifiers as you go.

Predicates - quantifier negation

No large birds live on honey.

$\neg\exists x\,(L(x) \wedge H(x))$	$\equiv \forall x\, \neg(L(x) \wedge H(x))$	Negation rule
	$\equiv \forall x\, (\neg L(x) \vee \neg H(x))$	DeMorgan's
	$\equiv \forall x\, (L(x) \rightarrow \neg H(x))$	Subst for $\rightarrow$

What's wrong with this proof?

Predicates - free and bound variables

A variable is bound if it is known or quantified. Otherwise, it is free.

Examples:

$P(x)$	x is free
$P(5)$	x is bound to 5
$\forall x\ P(x)$	x is bound by quantifier

Reminder: in a proposition, all variables must be bound.

Predicates - multiple quantifiers

To bind many variables, use many quantifiers!

Example: $P(x,y)$ = "$x > y$"

- $\forall x\, P(x,y)$ c)
- $\forall x \forall y\, P(x,y)$ b)
- $\forall x \exists y\, P(x,y)$ a)
- $\forall x\, P(x,3)$ b)

a) True proposition

b) False proposition

c) Not a proposition

d) No clue

Predicates - the meaning of multiple quantifiers

- $\forall x \forall y\, P(x,y)$

  P(x,y) true for all x, y pairs.

- $\exists x \exists y\, P(x,y)$

  P(x,y) true for at least one x, y pair.

- $\forall x \exists y\, P(x,y)$

  For every value of x we can find a (possibly different) y so that P(x,y) is true.

- $\exists x \forall y\, P(x,y)$

  There is at least one x for which P(x,y) is always true.

Suppose P(x,y) = “x’s favorite class is y.”

quantification order is not commutative.

Predicates - the meaning of multiple quantifiers

$N(x,y)$ = "x is sitting by y"

- $\forall x \forall y\ N(x,y)$ False
- $\exists x \exists y\ N(x,y)$ True
- $\forall x \exists y\ N(x,y)$ True?
- $\exists x \forall y\ N(x,y)$ False

Proofs - how do you know?

The following statements are true:

- If I am Mila, then I am a great swimmer.
- I am Mila.

What do we know to be true?

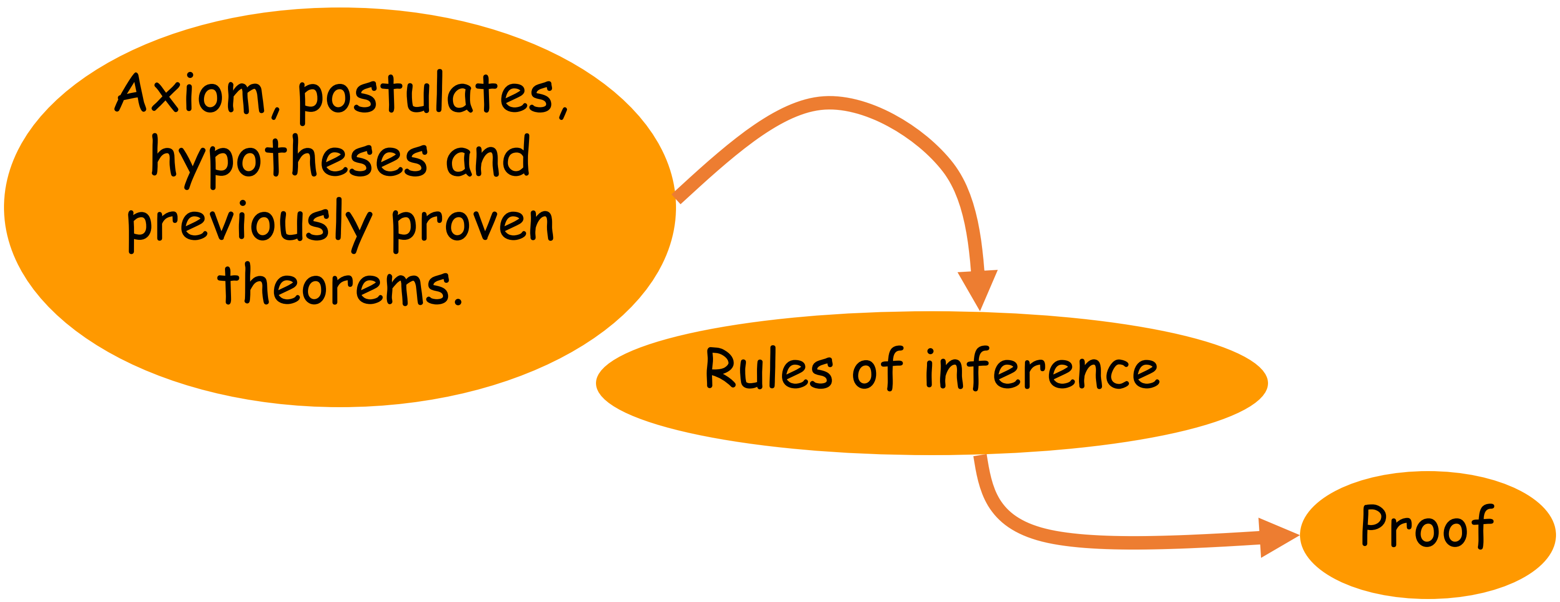
- I am a great swimmer!

How do we know it?

Proofs - how do you know?

A *theorem* is a statement that can be shown to be true.

A *proof* is the means of doing so.



Proofs - how do you know?

The following statements are true:

- If I am Mila, then I am a great swimmer.
- I am Mila.

What do we know to be true?

- I am a great swimmer!

What rule of inference can we use to justify it?

Proofs - Modus Ponens

I am Mila.
If I am Mila, then I am a great swimmer.

$\therefore$ I am a great swimmer!

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

Inference Rule:

Modus Ponens

Proofs - Modus Tollens

I am not a great skater.
If I am Erik, then I am a great skater.

$\therefore$ I am not Erik!

$$\neg q$$
$$p \rightarrow q$$
$$\therefore \neg p$$

Tautology:

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

Inference Rule:

Modus Tollens

Proofs - Addition

I am not a great skater.

$\therefore$ I am not a great skater or I am tall.

$$\frac{p}{\therefore\ p \vee q}$$

Tautology:

$p \rightarrow (p \vee q)$

Inference Rule:

Addition

Proofs - Simplification

I am not a great skater and you are sleepy.

$\therefore$ you are sleepy.

$$\frac{p \wedge q}{\therefore\ p}$$

Tautology:

$(p \wedge q) \rightarrow p$

Inference Rule:

Simplification

Proofs - Disjunctive Syllogism

I am a great eater or I am a great skater.
I am not a great skater.

$\therefore$ I am a great eater!

$$p \vee q$$

$$\neg q$$

$$\therefore\ p$$

Tautology:

$$((p \vee q) \wedge \neg q) \rightarrow p$$

Inference Rule:

Disjunctive Syllogism

Proofs - Hypothetical Syllogism

If you are an athlete, you are always hungry.
If you are always hungry, you have a snickers in your backpack.

$\therefore$ If you are an athlete, you have a snickers in your backpack.

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

Tautology:

$$((p \to q) \land (q \to r)) \to (p \to r)$$

Inference Rule:

Hypothetical Syllogism

Proofs - A little quiz...

Amy is a computer science major.

Addition

∴ Amy is a math major or a computer science major.

If Ernie is a math major then Ernie is geeky.
Ernie is not geeky!

∴ Ernie is not a math major.

Modus Tollens

Proofs - A little proof...

Here's what you know:

Ellen is a math major or a CS major.

If Ellen does not like discrete math, she is not a CS major.

If Ellen likes discrete math, she is smart.

Ellen is not a math major.

Can you conclude Ellen is smart?

$$M \vee C$$
$$\neg D \rightarrow \neg C$$
$$D \rightarrow S$$
$$\neg M$$

Proofs - A little proof…

1. $M \vee C$	Given
2. $\neg D \rightarrow \neg C$	Given
3. $D \rightarrow S$	Given
4. $\neg M$	Given
5. C	DS (1,4)
6. D	MT (2,5)
7. S	MP (3,6)

Ellen is smart!

Proofs - A little proof...

1. $M \vee C$	Given
2. $\neg D \rightarrow \neg C$	Given
3. $D \rightarrow S$	Given
4. $\neg M$	Given
5. C	Disjunctive Syllogism (1,4)
6. $C \rightarrow D$	Contrapositive of 2
7. $C \rightarrow S$	Hypothetical Syllogism (6,3)
8. S	Modus Ponens (5,7)

Ellen is smart!

Show that the premises "It is not sunny this afternoon and it is colder than yesterday," "We will go swimming only if it is sunny," "If we do not go swimming, then we will take a canoe trip," and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset."

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset." Then the premises become $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s$, and $s \rightarrow t$. The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t.

We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	**Reason**
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Show that the premises "If you send me an e-mail message, then I will finish writing the program," "If you do not send me an e-mail message, then I will go to sleep early," and "If I go to sleep early, then I will wake up feeling refreshed" lead to the conclusion "If I do not finish writing the program, then I will wake up feeling refreshed."

Solution: Let p be the proposition "You send me an e-mail message," q the proposition "I will finish writing the program," r the proposition "I will go to sleep early," and s the proposition "I will wake up feeling refreshed." Then the premises are $p \to q$, $\neg p \to r$, and $r \to s$. The desired conclusion is $\neg q \to s$. We need to give a valid argument with premises $p \to q$, $\neg p \to r$, and $r \to s$ and conclusion $\neg q \to s$.

This argument form shows that the premises lead to the desired conclusion.

Step	**Reason**
1. $p \to q$	Premise
2. $\neg q \to \neg p$	Contrapositive of (1)
3. $\neg p \to r$	Premise
4. $\neg q \to r$	Hypothetical syllogism using (2) and (3)
5. $r \to s$	Premise
6. $\neg q \to s$	Hypothetical syllogism using (4) and (5)

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution: Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \to C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.

The following steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\forall x(D(x) \to C(x))$	Premise
2. $D(\text{Marla}) \to C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

Proof Techniques - direct proofs

A totally different example:

Prove that if $n = 3 \bmod 4$, then $n^2 = 1 \bmod 4$.

HUH?

$7 = 3 \bmod 4$

$7 = 111 \bmod 4$

$37 = 1 \bmod 4$

$37 = 61 \bmod 4$

$94 = 2 \bmod 4$

$94 = 6 \bmod 4$

$16 = 0 \bmod 4$

$16 = 1024 \bmod 4$

Proof Techniques - direct proofs

A totally different example:

Prove that if $n = 3 \bmod 4$, then $n^2 = 1 \bmod 4$.

If $n = 3 \bmod 4$, then $n = 4k + 3$ for some int k.

But then,

$$
\begin{aligned}
n^2 &= (4k + 3)(4k + 3) \\
&= 16k^2 + 24k + 9 \\
&= 16k^2 + 24k + 8 + 1 \\
&= 4(4k^2 + 6k + 2) + 1 \\
&= 4j + 1 \text{ for some int } j \\
&= 1 \bmod 4.
\end{aligned}
$$

Proofs - Fallacies

Rules of inference, appropriately applied give *valid* arguments.

Mistakes in applying rules of inference are called *fallacies*.

Proofs - valid arg or fallacy?

If I am Bonnie Blair, then I skate fast

I skate fast!

∴ I am Bonnie Blair

I'm Eric Heiden

Affirming the conclusion.

$((p \rightarrow q) \wedge q) \rightarrow p$

Not a tautology.

If you don't give me $10, I bite your ear.

I bite your ear!

∴ You didn't give me $10.

I'm just mean.

Proofs - valid arg or fallacy?

If it rains then it is cloudy.

It does not rain.

∴ It is not cloudy

February!

Denying the hypothesis.

$((p \to q) \wedge \neg p) \to \neg q$

Not a tautology.

If it is a car, then it has 4 wheels.

It is not a car.

∴ It doesn't have 4 wheels.

ATV