Mathematical Reasoning

Methods of Proof

This Lecture

Now we have learnt the basics in logic.

We are going to apply the logical rules in proving mathematical theorems.

- Direct proof
- Contrapositive
- Proof by contradiction
- Proof by cases

“Proof:" We use these steps, where a and b are two equal positive integers.

Step	Reason
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a2 - b2 = ab - b2$	Subtract b2 from both sides of (2)
4. $(a-b)(a+b) = b(a-b)$	Factor both sides of (3)
5. $a+b = b$	Divide both sides of (4) by $a-b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2=1$	Divide both sides of (6) by b

What is wrong with this “proof?”

Basic Definitions

An integer n is an **even** number
if there exists an integer k such that $n = 2k$.

An integer n is an **odd** number
if there exists an integer k such that $n = 2k+1$.

Proving an Implication

Goal: If P, then Q. (P implies Q)

Method 1: Write assume P, then show that Q logically follows.

The sum of two even numbers is even.

Proof $x = 2m, y = 2n$

$$x+y = 2m+2n$$

$$= 2(m+n)$$

Direct Proofs

If n is an odd integer, then n^2 is odd.

Proof $n = 2k + 1$

$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$

By the definition of an odd integer, we can conclude that n^2 is an odd integer

if m and n are both perfect squares, then nm is also a perfect square .

Proof $m = a^2$ and $n = b^2$ for some integers a and b

Then $mn = a^2 b^2$

$= (a\, a)\,(b\, b) \quad = (ab)(ab)$

$= (ab)^2$

So mn is a perfect square.

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Proving an Implication

Goal: If P, then Q. (P implies Q)

Method 1: Write assume P, then show that Q logically follows.

Claim: If r is irrational, then √r is irrational.

How to begin with?

What if I prove "If √r is rational, then r is rational", is it equivalent?

Yes, this is equivalent, because it is the **contrapositive** of the statement, so proving "if P, then Q" is equivalent to proving "if not Q, then not P".

So, $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$.

Rational Number

R is rational ⇔ there are integers a and b such that

numerator

$$r = \frac{a}{b}$$

denominator

and b ≠ 0.

Is 0.281 a rational number? Yes, 281/1000

Is 0 a rational number? Yes, 0/1

If m and n are non-zero integers, is (m+n)/mn a rational number? Yes

Is the sum of two rational numbers a rational number? Yes, a/b+c/d=(ad+bc)/bd

Is x=0.12121212…… a rational number? Note that 100x-x=12, and so x=12/99.

Proving the Contrapositive

Goal: If P, then Q. (P implies Q)

Method 2: Prove the ***contrapositive***, i.e. prove "not Q implies not P".

Claim: If r is irrational, then $\sqrt{r}$ is irrational.

Proof:

We shall prove the contrapositive –

"if $\sqrt{r}$ is rational, then r is rational."

Since $\sqrt{r}$ is rational, $\sqrt{r} = a/b$ for some integers a,b.

So $r = a^2/b^2$. Since a,b are integers, a^2,b^2 are integers.

Therefore, r is rational. □ Q.E.D.

(Q.E.D.) "which was to be demonstrated", or "quite easily done". ☺

Proving an "if and only if"

Goal: Prove that two statements P and Q are "**logically equivalent**", that is, one holds if and only if the other holds.

Example: For an integer n, n is even if and only if n^2 is even.

Method 1a: Prove P implies Q **and** Q implies P.

Method 1b: Prove P implies Q **and** not P implies not Q.

Method 2: Construct a chain of if and only if statement.

Proof the Contrapositive

For an integer n, n is even if and only if n^2 is even.

Method 1a: Prove P implies Q **and** Q implies P.

Statement: If n is even, then n^2 is even

Proof: $n = 2k$

$n^2 = 4k^2$

Statement: If n^2 is even, then n is even

Proof: $n^2 = 2k$

$n = \sqrt{(2k)}$

??

Proof the Contrapositive

For an integer n, n is even if and only if n^2 is even.

Method 1b: Prove P implies Q **and** not Q implies not p.

Statement: If n^2 is even, then n is even

Contrapositive: If n is odd, then n^2 is odd.

Proof (the contrapositive):

Since n is an odd number, $n = 2k+1$ for some integer k.

So $n^2 = (2k+1)^2$

$$= (2k)^2 + 2(2k) + 1 \quad = 2(2k^2 + 2k) + 1$$

So n^2 is an odd number.

Prove that if n is an integer and 3n+2 is odd, then n is odd

Proof by contrapositive:
"If 3n+2 is odd, then n is odd" is false; namely,

Contrapositive: If n is even, then 3n+2 is even.

assume that n is even.
Then, by the definition of an even integer, $n = 2k$ for some integer k.

Substituting 2k for n, we find that
$3n+2 = 3(2k)+2 = 6k + 2 = 2(3k+1)$.

This tells us that 3n+2 is even (because it is a multiple of 2), and therefore not odd.

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Proof by Contradiction

$$\frac{\overline{P} \rightarrow \mathrm{F}}{P}$$

To prove P, you prove that not P would lead to ridiculous result, and so P must be true.

Proof by Contradiction, Example

- Prove that if n is an integer and n^3+5 is odd, then n is even
- Rephrased: If n^3+5 is odd, then n is even

- Assume p is true and q is false, Assume that n^3+5 is odd, and n is odd

$n=2k+1$ for some integer k (definition of odd numbers)

$n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3+6k^2+3k+3)$

As $2(4k^3+6k^2+3k+3)$ is 2 times an integer, it must be even

Contradiction!

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

$\sqrt{2}$

Solution: Let p be the proposition "√2 is irrational."
we suppose that ¬p is true. So, ¬p = √2 is rational

If √2 is rational, there exist integers a and b with

√2 = a/b,

where b != 0 and a and b have no common factors

$$2 = \frac{a^2}{b^2}$$ so, $2b^2 = a^2$

By the definition of an even integer it follows that a^2 is even.

if the square of an integer is even, then the integer itself must be even.

a^2 is even, so a must also be even,

by the definition of an even integer, $a = 2c$ for some integer c.

$2b^2 = a^2$

Thus, $2b^2 = 4c^2$. Dividing both sides of this equation by 2

$b^2 = 2c^2$.

By the definition of even, this means that b^2 is even. So, b must be even as well.

assumption of ¬p leads to the equation $\sqrt{2} = a/b$, where a and b have no common factors, that is, 2 divides both a and b. So $\sqrt{2}$ is rational that is false.

So, $\sqrt{2}$ is irrational.

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Proof by Cases

$$p \vee q$$
$$p \rightarrow r$$
$$q \rightarrow r$$
$$\therefore \; r$$

e.g. want to prove a nonzero number always has a positive square.

x is positive or x is negative

if x is positive, then $x^2 > 0$.

if x is negative, then $x^2 > 0$.

$\therefore$ $x^2 > 0$.

Proof by Cases

- Show a statement is true by showing all possible cases are true

- Thus, you are showing a statement of the form:

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \to q$$

is true by showing that:

$$[(p_1 \vee p_2 \vee \ldots \vee p_n) \to q] \leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \ldots \wedge (p_n \to q)]$$

Summary

We have learnt different techniques to prove mathematical statements.

- Direct proof
- Contrapositive
- Proof by contradiction
- Proof by cases

Next time we will focus on a very important technique, proof by induction.