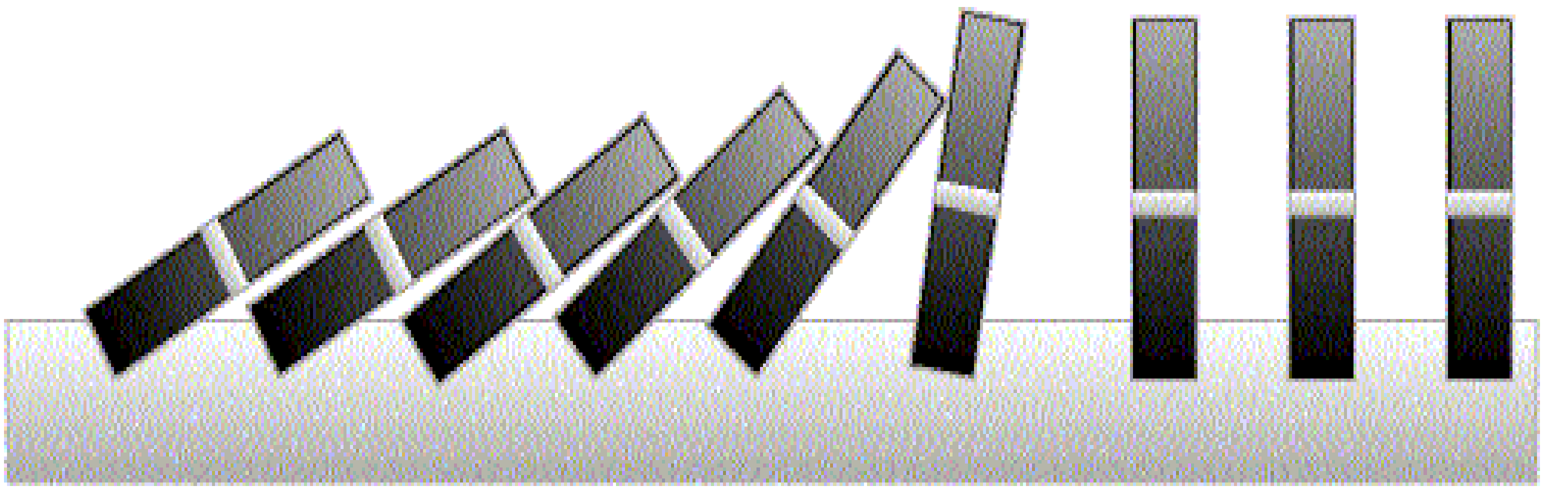
Mathematical Induction



- **Show that if n is a positive integer, then $1+2+\cdots+n = n(n+1)/2$**

Solution: Let P(n) be the proposition that the sum of the first n positive integers, $1+2+\cdots n = n(n+1)\ 2$, is $n(n+1)/2$. We must do two things to prove that P(n) is true for n = 1,2,3,....

BASIS STEP: $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$. (The left-hand side of this equation is 1 because 1 is the sum of the first positive integer. The right-hand side is found by substituting 1 for n in $n(n+1)/2$.)

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer k. That is, we assume that

$$1+2+\cdots+k = \frac{k(k+1)}{2}.$$

Under this assumption, it must be shown that $P(k+1)$ is true, namely, that

$$1+2+\cdots+k+(k+1) = \frac{(k+1)[(k+1)+1]}{2} = \frac{(k+1)(k+2)}{2}$$

is also true. When we add $k+1$ to both sides of the equation in $P(k)$, we obtain

$$\begin{aligned} 1+2+\cdots+k+(k+1) &\overset{\text{IH}}{=} \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

This last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n. That is, we have proven that $1+2+\cdots+n = n(n+1)/2$ for all positive integers n.

- **Use mathematical induction to show that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ for all nonnegative integers n.**

Solution: Let $P(n)$ be the proposition that $1+2+2^2+\cdots+2^n = 2^{n+1}-1$ for the integer n.

BASIS STEP: $P(0)$ is true because $2^0 = 1 = 2^1 - 1$. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis, we assume that $P(k)$ is true for an arbitrary nonnegative integer k. That is, we assume that

$$1+2+2^2+\cdots+2^k = 2^{k+1}-1.$$

To carry out the inductive step using this assumption, we must show that when we assume that $P(k)$ is true, then $P(k+1)$ is also true. That is, we must show that

$$1+2+2^2+\cdots+2^k+2^{k+1} = 2^{(k+1)+1}-1 = 2^{k+2}-1$$

assuming the inductive hypothesis $P(k)$. Under the assumption of $P(k)$, we see that

$$\begin{aligned} 1+2+2^2+\cdots+2^k+2^{k+1} &= (1+2+2^2+\cdots+2^k)+2^{k+1} \\ &\overset{\text{IH}}{=} (2^{k+1}-1)+2^{k+1} \\ &= 2\cdot 2^{k+1}-1 \\ &= 2^{k+2}-1. \end{aligned}$$

Note that we used the inductive hypothesis in the second equation in this string of equalities to replace $1+2+2^2+\cdots+2^k$ by $2^{k+1}-1$. We have completed the inductive step.

Because we have completed the basis step and the inductive step, by mathematical induction we know that $P(n)$ is true for all nonnegative integers n. That is, $1+2+\cdots+2^n = 2^{n+1}-1$ for all nonnegative integers n. ◀

Prime Factorization

- An integer *p* greater than 1 is called *prime* if the only positive factors of *p* are 1 and *p*.
- A positive integer that is greater than 1 and is not prime is called *composite*.

The integer 7 is prime because its only positive factors are 1 and 7, whereas the integer 9 is composite because it is divisible by 3

The prime factorizations of 100, 641, 999, and 1024 are given by

$100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2\, 5^2$
$641 = 641,$
$999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37,$
$1024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}$

Prime Factorization

- If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$

- **Show that 101 is prime**

Solution: The only primes not exceeding $\sqrt{101}$ are 2, 3, 5, and 7. Because 101 is not divisible by 2, 3, 5, or 7 (the quotient of 101 and each of these integers is not an integer),

it follows that 101 is prime

- **Find the prime factorization of 7007**

Solution:

$$7007 = 7 \cdot 1001 = 7 \cdot 7 \cdot 143 = 7 \cdot 7 \cdot 11 \cdot 13$$
$$= 7^2 \cdot 11 \cdot 13$$