Spring 2020 Semester Final Answer Sheet

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Intake: 39(1)

course title: Numerical Analysis

Course code: CSE 223

Answer to the question: 1(a)

Soln: Given,

Equations:

$$3x_1 + 2x_2 + x_3 = 10 \longrightarrow (1)$$
$$2x_1 + 3x_2 + 2x_3 = 14 \longrightarrow (2)$$
$$x_1 + 2x_2 + 3x_3 = 14 \longrightarrow (3)$$

Step: 1

eliminate x_1 from equ 2 & 3

$2\times(1) - 3\times(2)$ by doing this we get,

$$6x_1 + 4x_2 + 2x_3 - 6x_1 - 9x_2 - 6x_3 = 20 - 42$$

$$\therefore -5x_2 - 4x_3 = -22 \longrightarrow (4)$$

Again, $(1) - 3\times(3)$ by doing this we get,

$$3x_1 + 2x_2 + x_3 - 3x_1 - 6x_2 - 9x_3 = 10 - 42$$

$$\therefore -4x_2 - 8x_3 = -32 \longrightarrow (5)$$

Equation (1) remains same.

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So, we have:

$$3x_1 + 2x_2 + x_3 = 10 \longrightarrow (1)$$

$$-5x_2 - 4x_3 = -22 \longrightarrow (4)$$

$$-4x_2 - 8x_3 = -32 \longrightarrow (5)$$

Step: 2 $4\times(4) - 5\times(5)$ by doing this we can eliminate x_2 from (2) & (3) no equation.

$$-20x_2 - 16x_3 = -88$$

$$(-)\ -20x_2 - 40x_3 = -160$$

$$24x_3 = 72$$

$$\therefore x_3 = 3 \longrightarrow (6)$$

Now, we have these equations:

$$3x_1 + 2x_2 + x_3 = 10 \longrightarrow (1)$$

$$-5x_2 - 4x_3 = -22 \longrightarrow (4)$$

$$x_3 = 3 \longrightarrow (6)$$

By putting the value of x_3 in equation (4) we get.

$$-5x_2 - 4\times 3 = -22$$

$$\Rightarrow -5x_2 = -22 + 12$$

$$\therefore x_2 = 2$$

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By putting the value of x_2 & x_3 we get,

$$3x_1 + 2x_2 + x_3 = 10$$

$$\Rightarrow 3x_1 + 2\times 2 + 3 = 10$$

$$\Rightarrow 3x_1 = 10 - 7$$

$$\therefore x_1 = 1$$

So, the solution is: $x_1 = 1$

$x_2 = 2$

$x_3 = 3$

Ans:

Answer to the question: 1(b)

Soln: Given equations,

$$3x_1 - 6x_2 + 2x_3 = 15$$
$$4x_1 - x_2 + x_3 = 2$$
$$x_1 - 3x_2 + 7x_3 = 22$$

First, solving the equations for unknowns on the diagonal, that is -

$$x_1 = \frac{15 + 6x_2 - 2x_3}{3}$$

$$x_2 = -2 + 4x_1 + x_3$$

$$x_3 = \frac{22 - x_1 + 3x_2}{7}$$

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If we assume the initial values of x_1, x_2 and x_3 to be zero.

Then we get,

$$x_1^{(1)} = \frac{15+0-0}{3} = 5$$

$$x_2^{(1)} = -2+0+0 = -2$$

$$x_3^{(1)} = \frac{22-0+0}{7} = 3.143$$

For second iteration, we have -

$$x_1^{(2)} = \frac{15+6\times(-2)-2\times(3.143)}{3} = -1.095$$

$$x_2^{(2)} = -2+4\times(5)+3.143 = 21.143$$

$$x_3^{(2)} = \frac{22-5+3\times(-2)}{7} = 1.571$$

For the third iteration, we have, -

$$x_1^{(3)} = \frac{15+6\times 21.143 - 2\times 1.571}{3} = 46.239$$

$$x_2^{(3)} = -2+4\times(-1.095)+1.571 = -4.809$$

$$x_3^{(3)} = \frac{22-(-1.095)+3\times 21.143}{7}$$

$$= 12.361$$

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From Forth iteration, we have-

$$x_1^{(4)} = \frac{15+6\times(-4.809)-2\times12.361}{3} = -12.859$$

$$x_2^{(4)} = -2+4\times46.239+12.361 = 195.317$$

$$x_3^{(4)} = \frac{22-46.239+3\times(-4.809)}{7} = -5.524$$

From ~~Fith~~ fifth iteration, we have-

$$x_1^{(5)} = \frac{15+6\times(195.317)-2\times(-5.524)}{7}$$

$$= 171.136$$

$$x_2^{(5)} = -2+4\times(-12.859)+(-5.524)$$

$$= -58.96$$

$$x_3^{(5)} = \frac{22-(-12.859)+3\times195.317}{7}$$

$$= 88.687$$

So, After 5^{th} iteration values are: $x_1 = 171.136$

$x_2 = -58.96$

$x_3 = 88.687$

Ans:

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Answer to the question: 2(a)

Soln: Given table,

	x_0	x_1	x_2	x_3
i	0	1	2	3
x_i	2	4	5	8
$f(x)$	0.693	1.386	1.609	2.079
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

and, $x = 6$

Second order polynomial require only three data points.
So, we used the first 3 points.

here, $a_0 = f[x_0] = 0.693$

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1.386 - 0.693}{4-2}$$

$$= 0.3465$$

$$a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

here, $$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{1.609 - 1.386}{5-4}$$

$$= 0.223$$

and earlier we found $f[x_0, x_1] = 0.3465$

We know, $$a_2 = \frac{[(f_2 - f_1)/(x_2 - x_1)] - [(f_1 - f_0)/(x_1 - x_0)]}{x_2 - x_0}$$

Therefore, $$a_2 = \frac{0.223 - 0.3465}{5-2} = -0.0412$$

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Now, We know,

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$= 0.693 + 0.3465(x-2) + (-0.0412)(x-2)(x-4)$$

$$\therefore P_2(6) = 0.693 + 0.3465 \times (6-2) - 0.0412(6-2)(6-4)$$

$$= 0.693 + 0.3465 \times 4 - 0.0412 \times 4 \times 2$$

$$= 1.7494$$

So, the value of $f(x)$ when $x=6$ is: 1.7494

Ans:

Answer to the question: 2(b)

Soln: Given two points, (X_1, Y_1) and (X_2, Y_2)

~~Thi~~ The simplest form of interpolation is to approximate two data points by a straight line. These given two points can be connected linearly as ~~Known~~ shown in the Figure-1. Using the concept of similar triangles, we can show that

$$\frac{Y - Y_1}{X - X_1} = \frac{Y_2 - Y_1}{X_2 - X_1}$$

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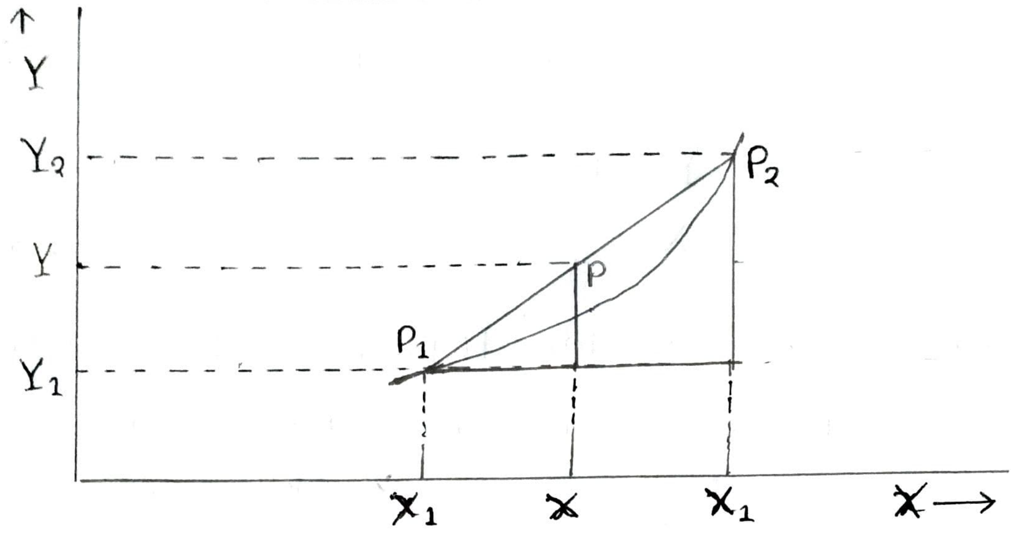

Figure-1: Graphical representation of linear interpolation.

Here, we will estimate what y value we could get for some x value that is between x_1 and x_2. Call this y value estimate - an interpolated value. To fit a linear curve that passes through the two data points given, we simply need to find the equation of the straight line that passes through the two points. We can do this by using the two-point from of the equation of straight line.

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\Rightarrow y = y_1 + \left(\frac{y_2-y_1}{x_2-x_1}\right) \times (x-x_1)$$

So, we can write it as,

$$\boxed{Y = Y_1 + (X-X_1)\left(\frac{Y_2-Y_1}{X_2-X_1}\right)} \longrightarrow (1)$$

linear interpolation formula

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Equation (1) is known as linear interpolation formula. ~~The~~ Note that the term,

$$\frac{Y_2 - Y_1}{X_2 - X_1}$$

represents slop of the line. Further, note the similarity of equations (1) with the Newton form of polynomial of first order.

$$e_1 = X_1$$
$$a_0 = Y_1$$
$$a_1 = \frac{Y_2 - Y_1}{X_2 - X_1}$$

We also can write it as,

$$C_1 = x_1$$
$$a_0 = f(x_1)$$
$$a_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The coefficient a_1 represents the first derivation of the function.

Answer to the question: 3(a)

Soln: Given table,

x	1.0	1.1	1.2
cos(x)	0.5403	0.4536	0.3624

Now, have to estimate the value : cos (1.15)

here, we are using the second order lagrange interpolation polynomial to find the value of cos (1.15)

Let us consider the following three points:

$x_0 = 1.0$ $\quad$ $x_1 = 1.1$ $\quad$ $x_2 = 1.2$

$f_0 = 0.5403$ $\quad$ $f_1 = 0.4536$ $\quad$ $f_2 = 0.3624$

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For $x = 1.15$ we have,

$$l_0(1.15) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(1.15-1.0)(1.15-1.2)}{(1.0-1.1)(1.0-1.2)}$$

$$= -0.375$$

$$l_1(1.15) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(1.15-1.0)(1.15-1.2)}{(1.1-1.0)(1.1-1.2)}$$

$$= 0.75$$

$$l_2(1.15) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(1.15-1.0)(1.15-1.1)}{(1.2-1.0)(1.2-1.1)}$$

$$= 0.375$$

We know,

$$P_2(x) = f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x)$$

$$\therefore P_2(1.15) = 0.5403 \times (-0.375) + 0.4536 \times (0.75) + 0.3624 \times (0.375)$$

$$= 0.2735$$

So, the value in $= 0.2735$

Ans:

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Answer to the question: 3(b)

Soln: constructing trapezoidal rule using the first two terms of Newton-Gregory Forward formula:

The Newton-Gregory formula is,

$$P_n(s) = f_0 + \Delta f_0 s + \frac{\Delta^2 f_0}{2!} s(s-1) + \frac{\Delta^3 f_0}{3!} s(s-1)(s-2) + \cdots \quad (1)$$

$$= T_0 + T_1 + T_2 + \cdots + T_n$$

where, $s = (x - x_0)/h$

and $h = x_{i+1} - x_i$

The trapezoidal rule is the first and the simplest of the Newton-coles formula. Since, it is two point formula, it uses the first order interpolation polynomial $P_1(x)$ for approximating the function $f(x)$ and assumes $x_0 = a$ and $x_1 = b$. This is illustrated in figure-2.

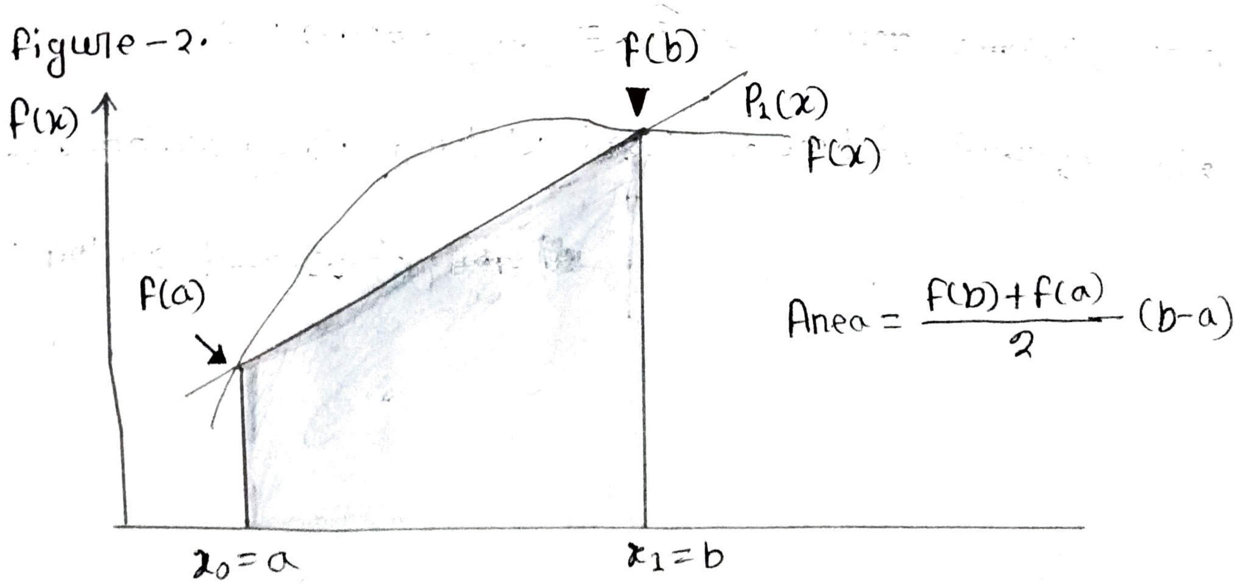

Figure 2: Representation of trapezoidal rule

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According to equation ① i.e. Newton-Gregory equation, $P_1(x)$ consists of the first two terms T_0 and T_1. Therefore, the integral for trapezoidal rule is given by.

$$I_t = \int_a^b (T_0 + T_1)\, dx$$

$$= \int_a^b T_0\, dx + \int_a^b T_1\, dx$$

$$= I_{t_1} + I_{t_2}$$

Since, T_i are expressed in terms of s, we need to use the following transformation:

$$dx = h \times ds$$

$$x_0 = a, \quad x_1 = b \quad \text{and} \quad h = b - a$$

$$\text{At} \quad x = a, \quad s = (a - x_0)/h = 0$$

$$\text{At} \quad x = b, \quad s = (b - x_0)/h = 1$$

$$I_{t_1} = \int_a^b T_0\, dx = \int_0^1 h f_0\, ds = h f_0$$

$$\text{H.} \; I_{t_2} = \int_a^b T_1\, dx = \int_0^1 \Delta f_0\, sh\, ds = h \frac{\Delta f_0}{2}$$

Therefore,

$$I_t = h\left[f_0 + \frac{\Delta f_0}{2}\right] = h\left[\frac{f_0 + f_1}{2}\right]$$

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Since, ~~we have~~ $f_0 = f(a)$ and $f_1 = f(b)$, We have

$$I_t = h \frac{f(a)+f(b)}{2}$$

$$= (b-a) \frac{f(a)+f(b)}{2}$$

Note that the area is the product of width of the segment (b-a) and average height of the points f(a) and f(b).

Ans:

Anwer to the question: 4(a)

Soln: In solving systems of equations, we are interested in identifying values of the variables that satisfy all equations in the system simultaneously. There are four possible solution:

1) System has a unique solution.
2) System has no solution.
3) System has a solution but not a unique one (i.e., it has infinite solutions).
4) System is ill conditioned.

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1) Unique Solution

There will be only one value of x and y. No other pair of values of x and y ~~wo~~ could satisfy the equation.

Ex: $\left.\begin{aligned} x + 2y &= 9 \\ 2x - 3y &= 4 \end{aligned}\right\}$ Ans: $x = 5$ and $y = 2$

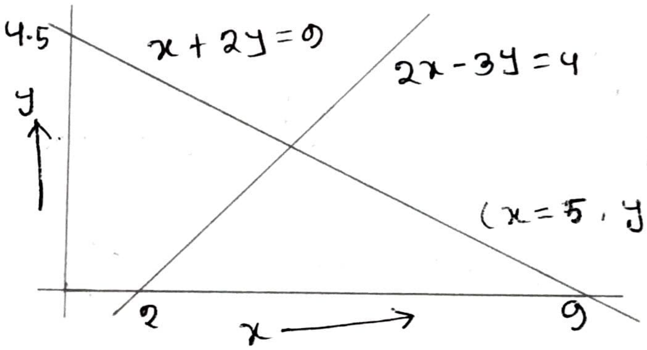

Fig: System with unique solution

2) No Solution

There will be no solution.

Ex: $\left.\begin{aligned} 2x - y &= 5 \\ 3x - 3/2y &= 4 \end{aligned}\right\}$ ~~E~~ No solution for this equation. There two lines are parallel. Therefore, they never meet. Such equations are called inconsistent equations.

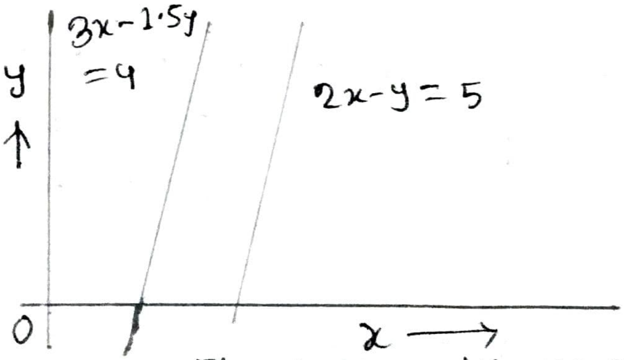

Fig: System with no solution.

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3) No Unique Solution

~~It has~~ In this case there will be many different solution.

The system,

$$-2x+3y=6$$
$$4x-6y=-12$$

has many different solutions. These two are two different forms of the same equation and, therefore they represented the same line.

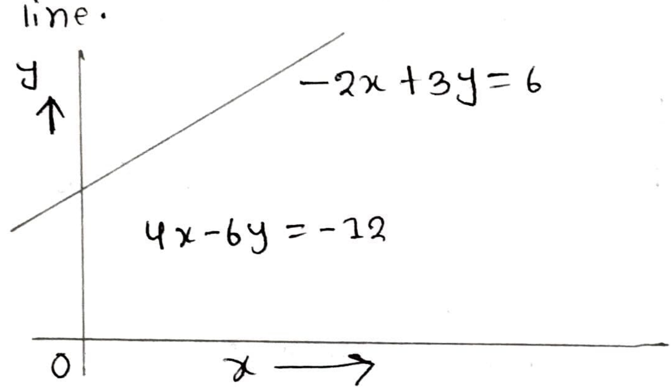

Fig: System with infinite solutions.

4) Ill-conditioned System

There ~~mayme~~ maybe a situation where the system has a solution but it is very close to being singular.

Ex: $x-2y=-2$

$0.45x-0.91y=1$

It has a solution. But it is very difficult to ~~understand~~ ~~identfy~~ identify the exact point at which the lines intersect.

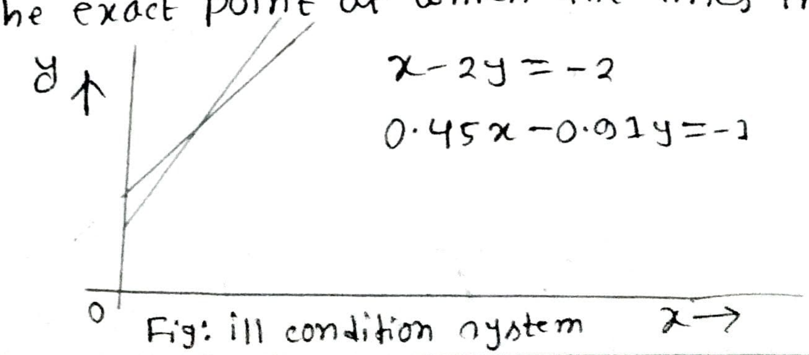

Fig: ill condition system

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Answer to the question: 4(b)

Soln:

(i)

Given,

$$I = \int_{1}^{2} (x^5 + 1)\, dx$$

here, $h = \frac{b-a}{3} = \frac{2-1}{3} = \frac{1}{3}$

$x_1 = a + h = 1 + \frac{1}{3} = \frac{4}{3}$

$x_2 = a + 2h = 1 + 2 \times 1/3 = \frac{5}{3}$

We know, $I_{S_2} = \frac{3h}{8} \left[f(a) + 3f(x_1) + 3f(x_2) + f(b) \right]$

$= \frac{3 \times \frac{1}{3}}{8} \left[f(1) + 3f(4/3) + 3f(5/3) + f(2) \right]$

$= \frac{1}{8} (2 + 3 \times 5.21 + 3 \times 13.86 + 33)$

$= \frac{1}{8} \times 92.21$

$= 11.53$ Ans:

(ii)

Soln: Given, $I = \int_{0}^{\pi/2} \sqrt{\cos(x)}\, dx$

here, $h = \frac{b-a}{3} = \frac{\frac{\pi}{2} - 0}{3} = \frac{\pi}{6}$

$x_1 = a + h = 0 + \frac{\pi}{6} = \frac{\pi}{6}$

$x_2 = a + 2h = 0 + 2 \times \pi/6 = \frac{\pi}{3}$

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We know, $I_{S_2} = \frac{3h}{8}\left[f(a) + 3f(x_1) + 3f(x_2) + f(b)\right]$

$= \frac{3 \times \frac{\pi}{6}}{8}\left[f(0) + 3f(\pi/6) + 3f(\pi/3) + f(\pi/2)\right]$

$= \frac{\pi}{16}\left[1 + 3 \times 0.93 + 3 \times 0.71 + 0\right]$

$= 1.162392$

Ans: