29.10.19

Analytic Function

Chapter : 02

Lec-07

→ Limit
→ continuity
→ Differentiability

→ Ordinary Differentiation
→ Partial Differentiation

Differentiation → Rate of change (পরিবর্তনের হার)

Ex: Ice melting → temparature → invironment

Variable
- → Dependent (D)
- → Independent (I)

i) (D) ← $y = x$ → (I)
(fixed) (Kinus)

$dy = d \cdot x$

$\frac{dy}{dx}$

ii) 2s ⟶ 50 Km/h
2.5s ⟶ 70 km/h

time (I) Velocity (D)

$t = t_2 - t_1 = 0.5$

$v = v_2 - v_1 = 20$

$\frac{v_2 - v_1}{t_2 - t_1} \rightarrow \frac{dv}{dt}$ differentiation

$\frac{20}{0.5}$ (decreasing)

Ordinary Differentiation:

The rate of change of dependent variable with respect to an independent variable is called differentiation.

1) Ordinary Differentiation

→ 1 dependent 1 independent (D) $(y = x \rightarrow (I))$
fixed vinus

2) Partial Differentiation

→ 1 dependent more that 1 (many) independent

$$y = V + R + S$$

Fever ← y → Sun burn

V → Virus, R → Rain

$$\frac{\partial y}{\partial V}$$

$$\frac{\partial y}{\partial R}$$

$$\frac{\partial y}{\partial S}$$

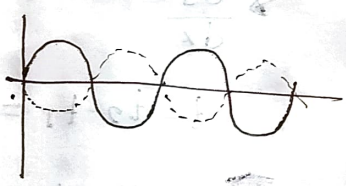
Periodic Function:-

$\sin x, \cos x, \tan x$ etc. will not decrease

But changes direction (দিক পরিবর্তন হয়।)

Non-Periodic Function:-

Ex: x^{10}, x, $y = 2x$, etc will decrease

$\sin x =$ 

$y = 10$

$$= \frac{10 - 10}{2019 - 1863}$$

$$= 0$$

Rate of change is 0

Constant = 0 in differentiation

$y_1 = f(x)$

$y_2 = f(x+h)$

Independent

Dependent

$$\frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$h \to (x + h - x) = h$

30.10.19

Lec-08

Analytic Function: A complex function $f(z)$ is to be analytic at point z_0. If its derivation exist not only at z_0 but also each point z in some neighbourhood z_0.

→ If a function is not continuous then there will be no derivation/derivative for it.

$z = x + iy$

$\lim_{x \to 2}$ $\lim_{z \to 2}$ $\lim_{z \to z_0}$ $\lim_{x \to 2}$ → for real num.

Exercise

$\boxed{0.1}$ # Write the function $f(z) = z^3 + z + 1$ in the form

$F(z) = u(x,y) + iv(x,y)$ → independent differentiation

Soln: let, $F(z) = x + iy$

Given, $f(z) = z^3 + z + 1$

$= (x+iy)^3 + (x+iy) + 1$

$= (x)^3 + 3.(x)^2.iy + 3.x.(iy)^2 + (iy)^3 + x + iy + 1$

$= x^3 + 3ix^2y - 3xy^2 - iy^3 + x + iy + 1$

$= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$

$\therefore F(z) = u(x,y) + iv(x,y)$

where,
$u(x,y) = x^3 - 3xy^2 + x + 1$
$v(x,y) = 3x^2y - y^3 + y$

Ans:

Q.2 # Show that $\frac{d}{dz}(\bar{z})$ does not exist anywhere.

Soln: From the defination of original differentiation

we know,

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \text{(For complex)}$$

$$\boxed{\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{(For real)}}$$

Now, $$\frac{d}{dz}(\bar{z}) = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Along real axis $y = 0$ $\quad \Delta y = 0$.

$$= \lim_{\Delta x \to 0} \frac{\Delta x - i \cdot 0}{\Delta x + i \cdot 0}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$

$$= 1$$

Ex:

$$\lim_{\Delta x \to 0} \frac{24x+2}{3}$$

$$= \lim_{\Delta x \to 0} \frac{24 \cdot 0 + 2}{3}$$

$$= \lim_{\Delta x \to 0} \frac{2}{3}$$

$$= \frac{2}{3}$$

Now, along imaginary axis $x=0$; $\Delta x=0$

$$= \lim_{\Delta y \to 0} \frac{0-iy}{0+iy}$$

$$= \lim_{\Delta y \to 0} (-1)$$

$$= -1$$

Since, The derivative real axis and imaginary axis is not same. so the ~~derivation~~ derivative $\frac{d}{dz}(\bar{z})$ does n't exist.

$\boxed{\text{N.B}}$ w

$f(x) = x^2$ (single var function) $f(x,y) = x^2+y$ (double var function) $f(x,y,w) = x^2+y+w$ (many var function)	Not continuous ↓ Differentiation does n't exist.	$\Delta z = \Delta x + i\Delta y$ ↓ if it decrease ↓ it will not decrease This two will decrease. ↓ cause "i" related to axis

$\boxed{0.3}$ # Prove that $\lim_{z \to 0} \frac{\bar{z}}{z}$ does not exist.

Soln: Given,

$$\lim_{z \to 0} \frac{\bar{z}}{z}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x - iy}{x + iy}$$

along real axis $y = 0$

$$\lim_{x \to 0} \frac{x - 0}{x + 0}$$

$$= \lim_{x \to 0} \frac{x}{x}$$

$$= \lim_{x \to 0} (1)$$

$$= 1$$

along imaginary axis $x = 0$

$$= \lim_{y \to 0} \frac{0 - iy}{0 + iy}$$

$$= \lim_{y \to 0} \frac{-iy}{iy}$$

$$= \lim_{y \to 0} (-1)$$

$$= -1$$

Since, the limiting value is not same for real axis and imaginary axis so the $\lim_{z \to 0} \frac{\bar{z}}{z}$ doesn't exist.

0.4 # Show that the function $f(z) = \bar{z}$ is non-analytic.

$\boxed{\frac{d}{dz}(\bar{z})}$ → does n't exist

Soln: We know,

$$f(z) = x + iy$$

Given, $f(z) = \bar{z}$

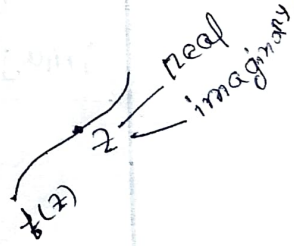
From the definition of differentiation we know,

$$\Rightarrow \frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

Here, $\frac{d}{dz} F(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta x - i\Delta y)}{\Delta x + i\Delta y}$



Now, along real axis $y = 0$; $\Delta y = 0$

$$= \lim_{\Delta x \to 0} \frac{\Delta x - i \cdot 0}{\Delta x + i \cdot 0}$$

$$= \lim_{\Delta x \to 0} (1) = 1$$

along imaginary axis $x = 0$; $\Delta x = 0$ $(y \neq 0)$

$$= \lim_{\Delta y \to 0} \frac{0 - iy}{0 + iy}$$

$$= \lim_{\Delta y \to 0} (-1) = -1$$

Since the derivative along real and imaginary axis is not same. So, it is non-analytic.

05.11.19

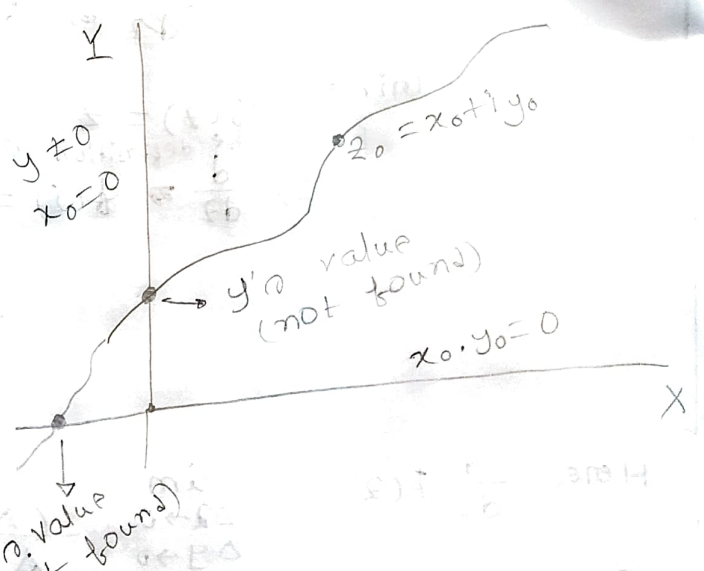
Lec-09

~~poi~~ points
surroundings
z_0 ~~z~~ z_0
also differentiable
is analytic.

N.B:

If real axis = imaginary axis

then the function is analytic. And then we have to ~~finf~~ find the value of "z" point along with real and imaginary axis



$F(z) = u(x,y) + iv(x,y)$

$$\frac{\partial u}{\partial x} \qquad \frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial v} \qquad \frac{\partial v}{\partial y}$$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

Cauchi Raymond

Formula

C-R Equation

Q.1 # For the function, $f(z)$ defined by

$$F(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$$

→ Piecewise function

→ Prove for different ~~Values~~ Values

Show that the C-R equations are satisfied at origin but the function is not differentiable at origin.

1st part

Soln: Given,

$$f(z) = \frac{(\bar{z})^2}{z} \quad ; z \neq 0$$

$$\Rightarrow F(z) = \frac{(x-iy)^2}{(x+iy)}$$

$$\Rightarrow F(z) = \frac{(x-iy)^3}{(x+iy)(x-iy)}$$ [Multiply with the conjugate $(x+iy)$].

$$\Rightarrow F(z) = \frac{(x)^3 - 3\cdot x^2\cdot(iy) + 3\cdot(x)\cdot(iy)^2 - (iy)^3}{x^2 - i^2y^2}$$

$$\Rightarrow F(z) = \frac{x^3 - i3x^2y - 3xy^2 + iy^3}{x^2+y^2}$$

$$\Rightarrow F(z) = \frac{x^3 - 3xy^2}{x^2+y^2} + i\,\frac{y^3 - 3x^2y}{x^2+y^2}$$

Now, let, $f(z) = u + iv = \dfrac{x^3 - 3xy^2}{x^2+y^2} + \dfrac{y^3 - 3x^2y}{x^2+y^2}$

Equating the connesponding terms.

$$u = \frac{x^3 - 3xy^2}{x^2+y^2} \quad ; \quad v = \frac{y^3 - 3x^2y}{x^2+y^2}$$

or $u(x,y)$

$u(0,0) = 0$

$v(0,0) = 0$

$= \boxed{\text{at origin}} =$

Now,

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$\boxed{\begin{aligned} &y = f(x) \\ &\frac{d}{dx} y = \lim_{h \to 0} \frac{f(x+\Delta x) - f(x)}{h} \end{aligned}}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(0+\Delta x, 0) - u(0,0)}{\Delta x}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0)}{\Delta x}$$

$$\boxed{\begin{aligned} u(x,y) &= \frac{x^3 - 3xy^2}{x^2+y^2} \\ u(\Delta x, 0) &= \frac{\Delta x^3}{\Delta x^2} \end{aligned}}$$ Path

$$\Rightarrow \frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{(\Delta x)^3 - 3\cdot 0}{(\Delta x)^2 + 0}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{(\Delta x)^3}{(\Delta x)^2}\right]$$

$$= \lim_{\Delta x \to 0} (1)$$

$$= 1 = \boxed{\text{at origin}}$$

(4 part)

1 2

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

3 $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}$ 4

Again, $$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{v(0, \Delta y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y}\left[\frac{(\Delta y)^3}{(\Delta y)^2}\right]$$

$$= \lim_{\Delta y \to 0} (1)$$

$$= 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Again $\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$

$$= \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - 0}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \left[\frac{0 - 3 \cdot 0}{(\Delta y)^2} \right]$$

$$= 0$$

Again $\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(0,0)}{\Delta x}$$

$$= \lim_{\Delta y \to} \frac{1}{\Delta x} [0] \quad \left[\frac{y^3 - 3x^2 y}{x^2 + y^2} = \frac{0-0}{x^2+0} \right]$$

$$= 0$$

Here, $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$

$0 = 0 \times (-1)$

$0 = (-1) \times 0$

So, we can write

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad [\because 0 = (-1) \times 0]$$

So, C-R equation is satisfied.

$\boxed{2^{nd} \text{ part}}$

From the definition of differentiation.

$$\frac{d}{dz} F(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - F(z)}{\Delta z}$$

at origin (0, 0)

$$\frac{d}{dz} F(0) = \lim_{\Delta z \to 0} \frac{F(0, \Delta z) - f(0,0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{f(\Delta z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} F(\Delta z)$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \cdot \frac{(\Delta \bar{z})^2}{(\Delta z)}$$

$$= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \cdot \frac{(\Delta x - i\,\Delta y)^2}{(\Delta x + i\Delta y)}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{1}{(\Delta x + i\Delta y)} \cdot \frac{(\Delta x - i\Delta y)^2}{(\Delta x + i\Delta y)}$$

Now, along real axis, $x \neq 0$, $y = 0$

$$\therefore \frac{d}{dz} F(0) = \lim_{\substack{\Delta x \to 0 \\ y = 0}} \frac{1}{(\Delta x + 0)} \left[\frac{(\Delta x - 0)^2}{(\Delta x + i0)} \right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{(\Delta x)^2}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} (1) = 1$$

along imaginary axis,

$x = 0$
$y \neq 0$

$$\therefore \frac{d}{dz} f(0) = \lim_{\substack{\Delta x = 0 \\ \Delta y \to 0}} \frac{1}{(0 + i\Delta y)} \cdot \left[\frac{(0 - i\Delta y)^2}{(0 + i\Delta y)} \right]$$

$$= \lim_{\Delta y \to 0} \frac{1}{i\Delta y} \cdot \frac{i^2 (\Delta y)^2}{i(\Delta y)}$$

$$= \lim_{\Delta y \to 0} (1) = 1$$

now along $y = x$

where $y \neq 0$
$x \neq 0$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{1}{(\Delta x + i\Delta x)} \cdot \left[\frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)} \right]$$

$[\because y = x]$

connect

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{1}{(\Delta x + i\Delta x)} \cdot \frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y = \Delta x}} \frac{(\Delta x - i\Delta x)^2}{(\Delta x + i\Delta x)^2}$$

connect

0×0

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y = \Delta x}} \frac{\{\Delta x(1-i)\}^2}{\{\Delta x(1+i)\}^2}$$

$$= \lim_{\Delta x \to 0} \frac{(1-i)^2}{(1+i)^2}$$

$$= \frac{(1-i)^2}{(1+i)^2}$$

$$= \frac{1^2 - 2\cdot 1\cdot i + (i)^2}{1^2 + 2\cdot 1\cdot i + (i)^2}$$

$$= \frac{1 - 2i - 1}{1 + 2i - 1}$$

$$= \frac{-2i}{2i}$$

$$= -1$$

Since the derivative along Real axis, imaginary axis and along $y=x$ is not same at origin so the function $f(z)$ is not differentiable at origin.

$\boxed{0.2}$ # Show that the function

$$F(z) = \begin{cases} \dfrac{x^3 y^4 (x+iy)}{x^6+y^8} & ; \text{ when } z \neq 0 \\ 0 & ; \text{ when } z = 0 \end{cases}$$

the C-R equation are satisfied at origin but not analytic there.

Soln:

$$F(x) = \begin{cases} x^2 & ; \text{ when } 0 \le x < 3 \\ x^3+1 & ; \text{ when } 2 \le x \le 3 \\ x^4+2 & ; \text{ when } 3 \le x \le 10 \end{cases}$$

$f(2) \rightarrow$ then 2 no equ.

$\boxed{\text{1'st part}}$

Soln: Given,

$$F(z) = \frac{x^3 y^4 (x+iy)}{x^6+y^8} \quad ; \text{ when } z \neq 0$$

$$= \frac{x^4 . y^4 + i x^3 y^5}{x^6+y^8}$$

$$F(z) = \frac{x^4 y^4}{x^6+y^8} + i \frac{x^3 y^5}{x^6+y^8}$$

$$\Rightarrow f(z) = u(x,y) + i v(x,y)$$

Where $u(x,y) = \dfrac{x^4 y^4}{x^6+y^8}$

and $u(0,0) = 0$

$v(x,y) = \dfrac{x^3 y^5}{x^6+y^8}$

$v(0,0) = 0$

at origin (0,0)

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[u(\Delta x, 0) \right]$$

$$u(\Delta x, 0) = \frac{(\Delta x)^4 \cdot 0}{(\Delta x)^6 + 0} = 0$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} [0]$$

$$= 0$$

Again at origin (0,0)

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{v(0, \Delta y)}{\Delta y}$$

$$v(x,y) = \frac{x^3 y^5}{x^6 + y^8}$$

$$= \frac{0 \cdot (\Delta y)^5}{0 + (\Delta y)^8} = 0$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} [0]$$

$$= 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Again at origin $(0,0)$

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} [u(0, \Delta y)]$$

$$= \lim_{\Delta y \to 0} [0]$$

$$= 0$$

$$u(x,y) = \frac{x^4 y^4}{x^6 + y^8}$$

$$u(0, \Delta y) = 0$$

Again at origin $(0,0)$

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(0,0)}{\Delta x}$$

$$= 0$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ C-R equation is satisfied.

2nd part

From the definition of differentiation we know,

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

N.B

$$f(z) = \frac{x^3 y^4 (x+iy)}{x^6+y^8}$$

$$f(x+iy) = \frac{x^3 y^4 (x+iy)}{x^6+y^8}$$

$$f(\Delta x + i\Delta y) = \frac{(\Delta x)^3 \cdot (\Delta y)^4 (\Delta x + i\Delta y)}{(\Delta x)^6 + (\Delta y)^8}$$

$f(\Delta z)$

at origin $(0,0)$

$$\frac{d}{dz} f(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

$$\Rightarrow \frac{d}{dz} f(0) = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \left[f(\Delta z) \right]$$

$\Delta z = \Delta x + i\Delta y$

$$\Rightarrow \frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta x)^3 (\Delta y)^4}{(\Delta x)^6 + (\Delta y)^8}$$

Now along real axis $x \neq 0, y = 0$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} (0) = 0$$

$$\frac{0 \cdot 0\,(0+0)}{0+0} = 0$$

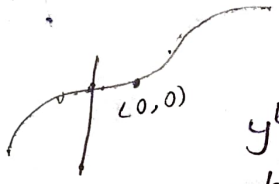
Now along imaginary axis $y \neq 0, x = 0$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} (0) = 0$$

Now, along $y = x$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = \Delta x}} \frac{(\Delta x)^3 \cdot (\Delta x)^4}{(\Delta x)^6 + (\Delta x)^8}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^7}{\Delta x^6 \{1+(\Delta x)^2\}}$$



this function will not analytic top

$y^6 = x^8$ $y^6 = x^8$

$$\frac{y^6 x^8 (x+iy)}{x^2+y^2}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{1+(\Delta x)^2} = 0$$

Again, ~~lim~~ along $y^4 = x^3$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^4 = \Delta x^3}} \frac{(\Delta x)^3 \cdot (\Delta x)^3}{(\Delta x)^6 + (\Delta x^3)^2}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y^4 = \Delta x^3}} \frac{(\Delta x)^6}{2(\Delta x)^6}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y^4 = \Delta x^3}} \left(\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Since, the derivation along real axis, imaginary axis ~~and~~, along $y=x$ and along $y^4 = x^3$ is not same

So, the function $F(z)$ is not analytic at origin.

A.W-6

Q.1 Show that the function

$$f(z) = \begin{cases} \dfrac{(1+i)x^3-(1-i)y^3}{x^2+y^2} & ; \text{ when } z \neq 0 \\ 0 & ; \text{ when } z = 0 \end{cases}$$

is continuous and that the cauchy-Riemann equation are satisfied at the origin yet $f'(0)$ does n't exist.

1st part

Soln:

Cauchy - Riemann Equation:

Given, $f(z) = \dfrac{(1+i)x^3-(1-i)y^3}{x^2+y^2}$

$$= \frac{x^3+ix^3-y^3+iy^3}{x^2+y^2}$$

$$\Rightarrow f(z) = \frac{x^3-y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}$$

$$\Rightarrow f(z) = u(x,y) + i\,v(x,y)$$

where,

$u(x,y) = \dfrac{x^3-y^3}{x^2+y^2}$

and $u(0,0) = 0$

again, $v(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

and $v(0,0) = 0$

at origin (0,0),

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \frac{(\Delta x)^3 - 0}{(\Delta x)^2 - 0}$$

$$= \lim_{\Delta x \to 0} (1)$$

$$= 1$$

Again,
at origin (0,0),

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{v(0, \Delta y) - v(0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \cdot \frac{0 + (\Delta y)^3}{0 + (\Delta y)^2}$$

$$= \lim_{\Delta y \to 0} (1)$$

$$= 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Again at origin (0,0),

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \cdot \frac{0 - (\Delta y)^3}{0 + (\Delta y)^2}$$

$$= \lim_{\Delta y \to 0} (-1)$$

$$= -1$$

Again at origin (0,0),

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \frac{(\Delta x)^3 + 0}{(\Delta x)^2 + 0}$$

$$= \lim_{\Delta x \to 0} (1)$$

$$= 1$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ C-R equation is satisfied.

$\boxed{2^{nd} \text{ part}}$

From the definition of differentiation we know,

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

at origin (0,0)

$$\frac{d}{dz} f(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

$$\Rightarrow \frac{d}{dz} f(0) = \lim_{\Delta z \to 0} \frac{1}{\Delta z} \cdot [f(\Delta z)]$$

$$\Rightarrow \frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta x^3 - \Delta y^3) + i(\Delta x^3 + \Delta y^3)}{(\Delta x)^2 + (\Delta y)^2} \times \frac{1}{\Delta x + i\Delta y}$$

Now along real axis $x \neq 0$, $y = 0$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{(0-0)+0}{0}$$

$$= \lim_{\Delta x \to 0} (0)$$

$$= 0$$

Now along imaginary axis, $y \neq 0$, $x = 0$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} \frac{(0 - \Delta y^3) + i(0 + \Delta y^3)}{0 + (\Delta y)^2} \times \frac{1}{0 + i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{0+0}{0}$$

$$= \lim_{\Delta y \to 0} (0)$$

$$= 0$$

Now along $y = x$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = \Delta x}} \frac{(\Delta x^3 - \Delta x^3) + i(\Delta x^3 + \Delta x^3)}{(\Delta x)^2 + (\Delta x)^2} \times \frac{1}{\Delta x + i\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0 + i\, 2\Delta x^3}{2(\Delta x)^2 \cdot \Delta x(1+i)}$$

$$= \lim_{\Delta x \to 0} \frac{i \cdot 2\Delta x^3}{(1+i) \cdot 2\Delta x^3}$$

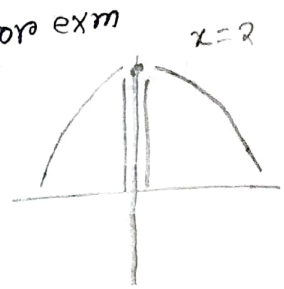
$$= \lim_{\Delta x \to 0} \left(\frac{1}{1+i}\right)$$

$$= \frac{1}{1+i}$$

Since, the derivation along real axis, imaginary axis and along $y = x$ is not same. so, $f(0)$ does n't exist.

Different math

$\boxed{\text{3rd part}}$ → for exm



continuity

Given, $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$

$$\frac{d}{dz} \cdot f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy^2(x+iy)}{x^2+y^4}$$

Along real axis $x\neq 0, y=0$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x\cdot 0(x+0)}{x^2+0}$$

$= 0$

Along Imaginary axis $x=0, y\neq 0$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{0\cdot y^2(0+iy)}{0+y^4}$$

$= 0$

Along $y=x$

$$= \lim_{\substack{x\to 0\\ y=x}} \frac{x\cdot x^2(x+ix)}{x^2+(x)^4}$$

$$= \lim_{x\to 0} \frac{x^3\cdot x(1+i)}{x^2+x^4}$$

$$= \lim_{x\to 0} \frac{x^4(1+i)}{x^4(1+1/x^2)}$$

$$= \frac{1+i}{1+1/0}$$

$$= \frac{1+i}{1+\infty}$$

$= 0$

$\frac{1}{0} = \infty$

$\frac{1}{\infty} = 0$

So, this function is continuous.