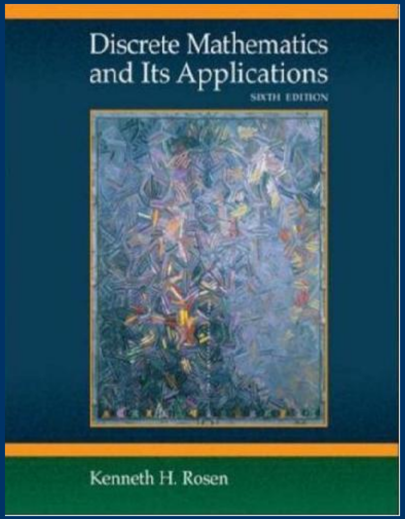
Chapter 9 (Part 1): Graphs

- Introduction to Graphs (9.1)
- Graph Terminology (9.2)



History

- Basic ideas were introduced in the eighteenth century by Leonard Euler (Swiss mathematician)

- Euler was interested in solving the Königsberg bridge problem (Town of Königsberg is in Kaliningrad, Republic of Russia)

- Graphs have several applications in many areas:
  - Study of the structure of the World Wide Web
  - Shortest path between 2 cities in a transportation network
  - Molecular chemistry

Introduction to Graphs (9.1)

- There are 5 main categories of graphs:

  – Simple graph
  – Multigraph
  – Pseudograph
  – Directed graph
  – Directed multigraph

Introduction to Graphs (9.1) (cont.)

– Definition 1

> A simple graph G = (V,E) consists of V, a nonempty set of vertices, and E, a set of unordered pairs of distinct elements of V called edges.

– Example: Telephone lines connecting computers in different cities.

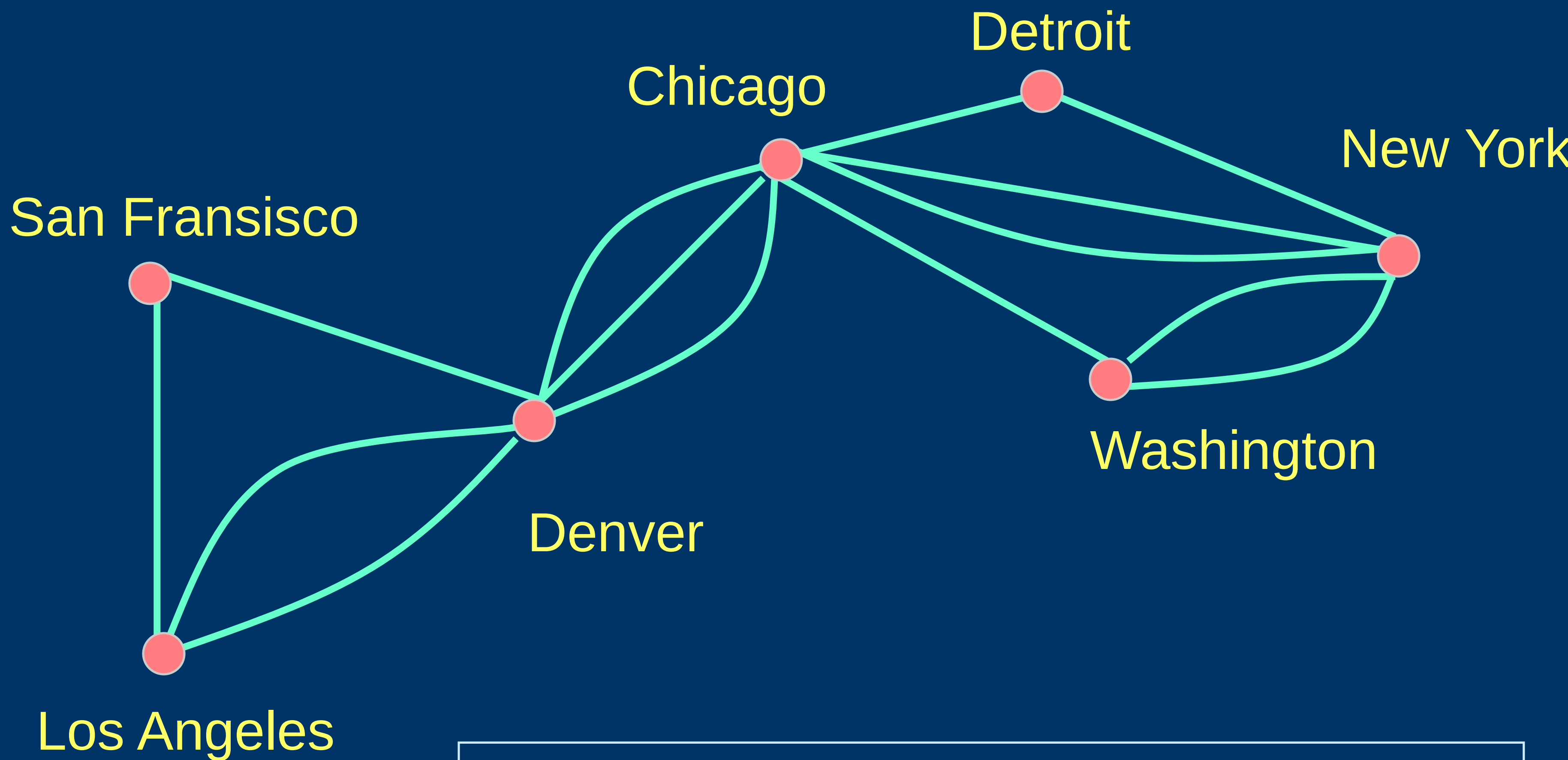
Introduction to Graphs (9.1) (cont.)

– Definition 2:

A multigraph $G = (V,E)$ consists of a set E of edges, and a function f from E to $\{\{u,v\} \mid u, v \in V, u \neq v\}$. The edges e_1 and e_2 are called multiple or parallel edges if $f(e_1) = f(e_2)$.

– Example: Multiple telephone lines connecting computers in different cities.

Introduction to Graphs (9.1) (cont.)



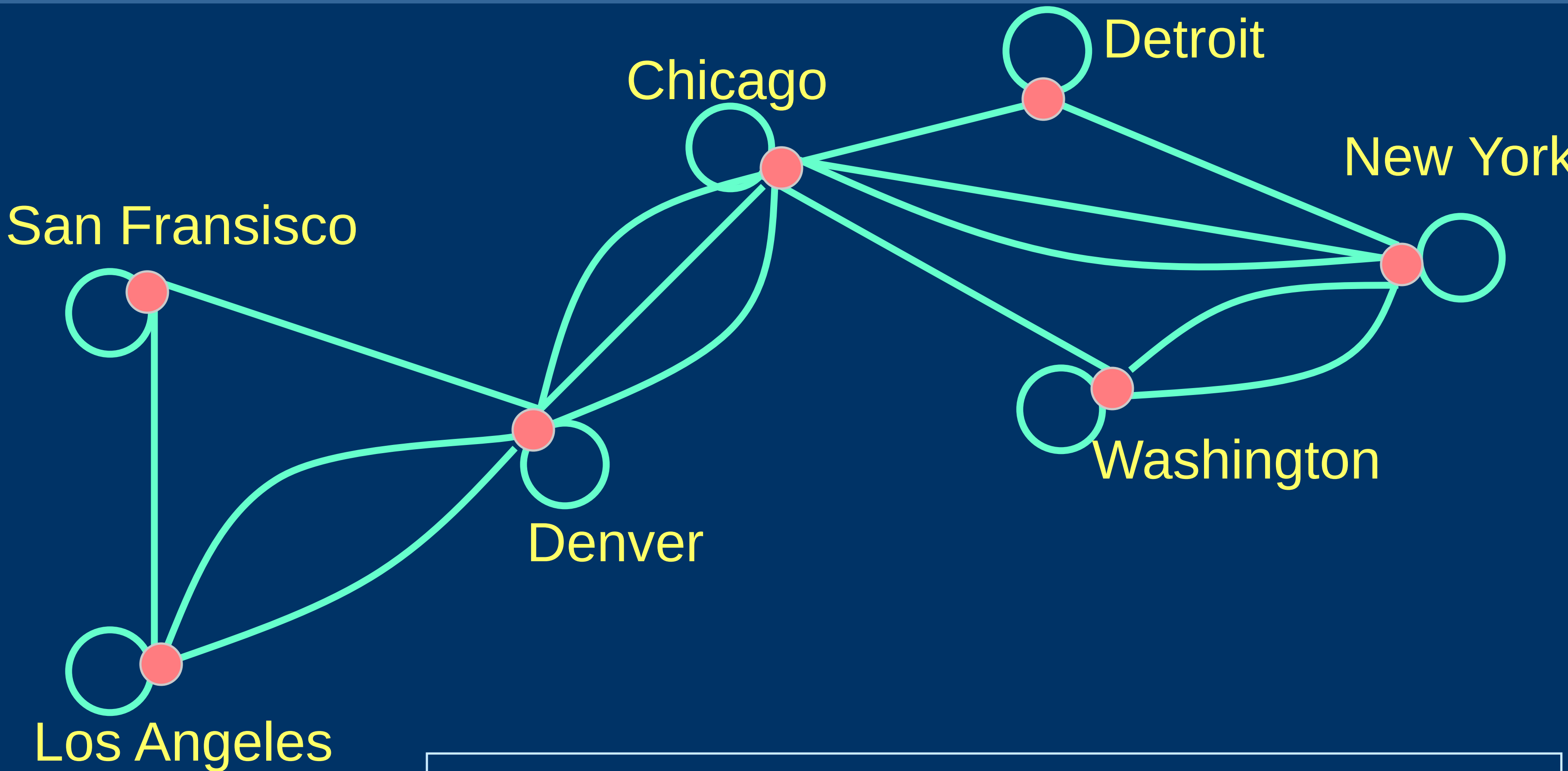
A Computer network with multiple lines

Introduction to Graphs (9.1) (cont.)

– Definition 3:

A pseudograph G = (V,E) consists of a set V of vertices, a set E of edges, and a function f from E to $\{\{u,v\} \mid u, v \in V\}$. An edge is a loop if $f(e) = \{u,u\} = \{u\}$ for some $u \in V$.

Introduction to Graphs (9.1) (cont.)



A Computer network with diagnostic lines

Introduction to Graphs (9.1) (cont.)

– Definition 4:

A directed graph (V,E) consists of a set of vertices V and a set of edges E that are ordered pairs of elements of V.

Introduction to Graphs (9.1) (cont.)

A Communication network with one-way telephone lines

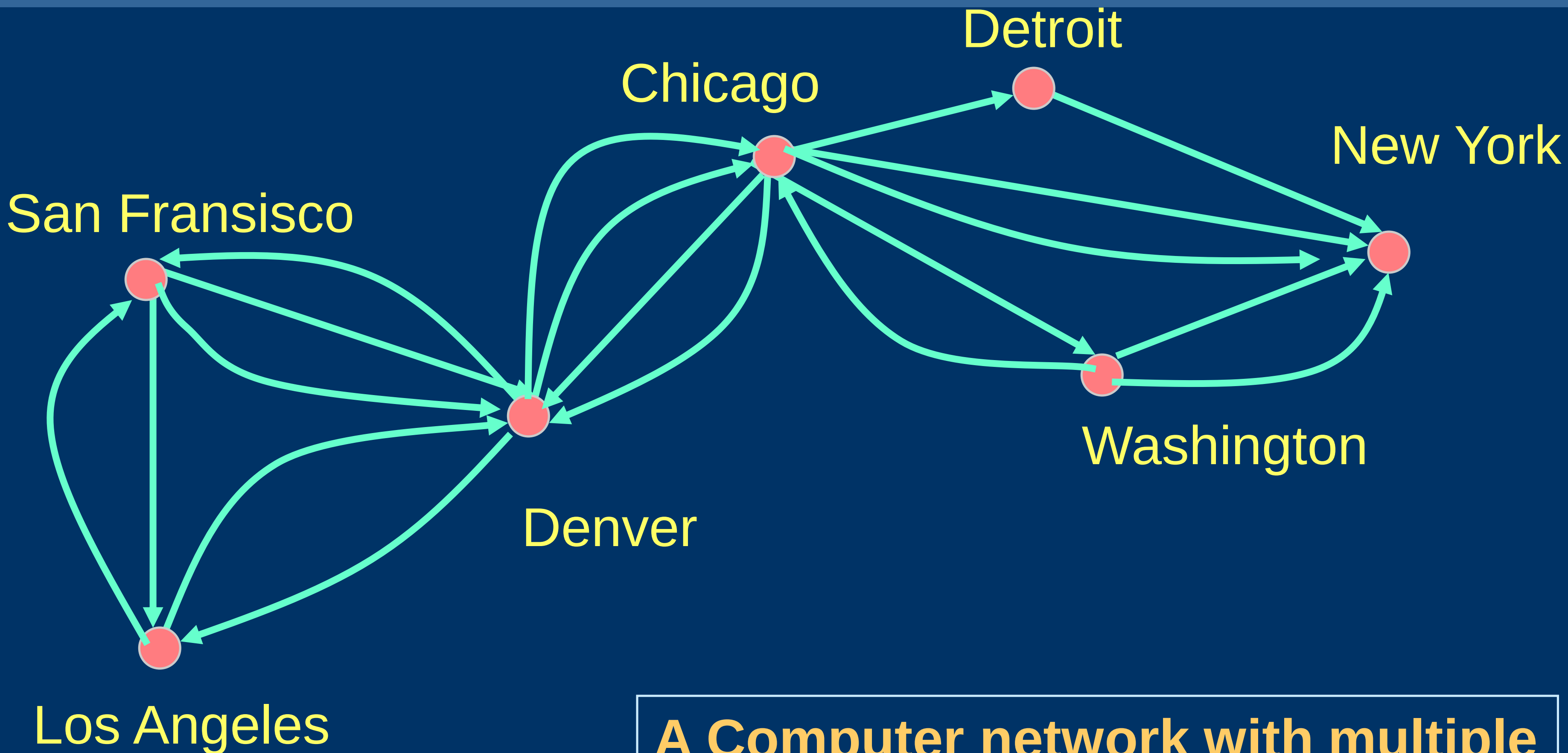
This example shows that the host computer <u>can only receive data</u> from other computer, it cannot emit

Introduction to Graphs (9.1) (cont.)

– Definition 5:

A directed multigraph $G = (V,E)$ consists of a set V of vertices, a set E of edges, and a function f from E to $\{\{u,v\} \mid u, v \in V\}$. The edges e_1 and e_2 are multiple edges if $f(e_1) = f(e_2)$.

Introduction to Graphs (9.1) (cont.)



A Computer network with multiple one-way telephone lines

Graph Terminology

Theorem: An undirected graph has an even number of vertices of odd degree.

- **Idea:** There are three possibilities for adding an edge to connect two vertices in the graph:

- **Before:** **After:**

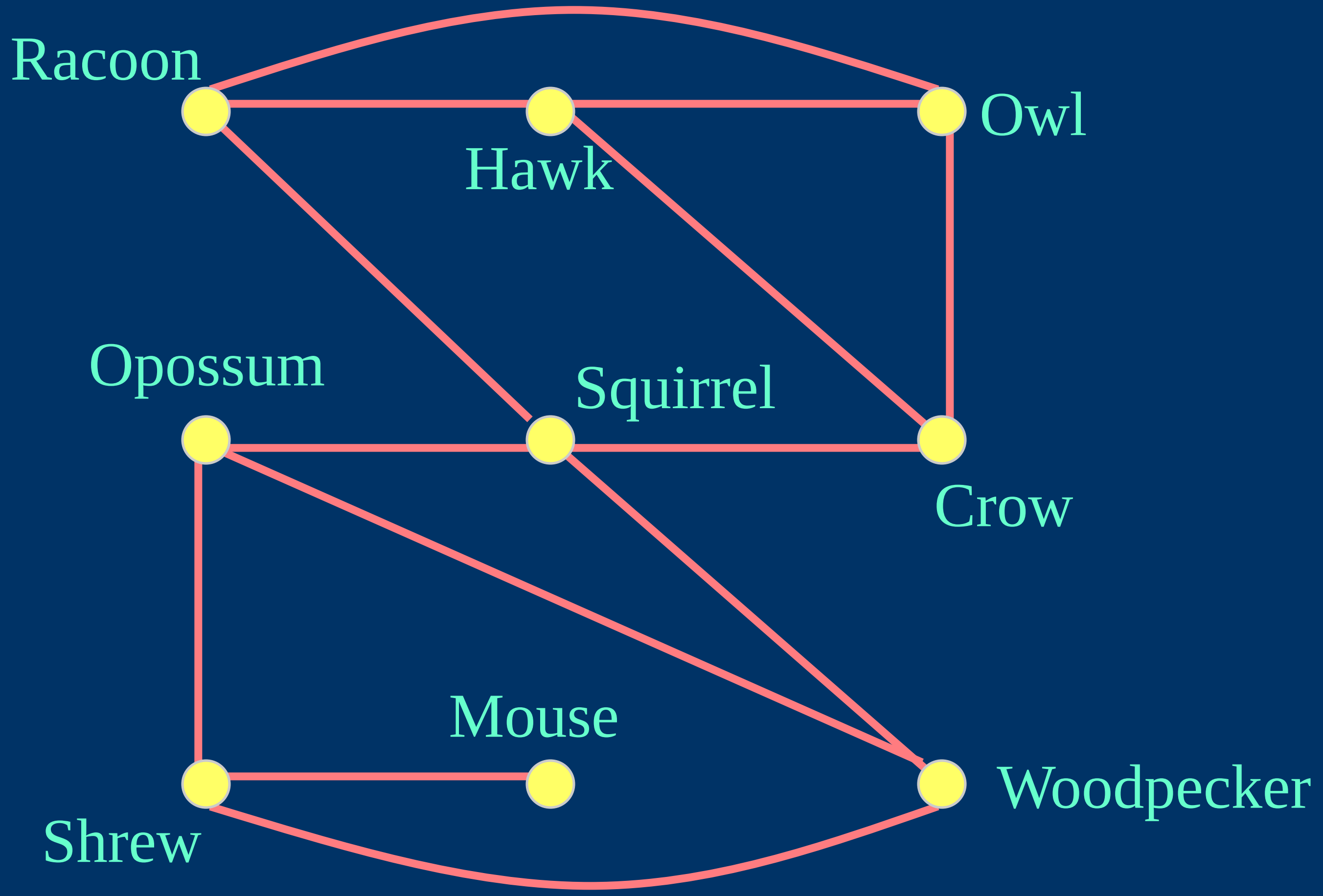
Before		After
Both vertices have even degree	⇨	Both vertices have odd degree
Both vertices have odd degree	⇨	Both vertices have even degree
One vertex has odd degree, the other even	⇨	One vertex has even degree, the other odd

Introduction to Graphs (9.1) (cont.)

- Modeling graphs
  - Example: Competition between species in an ecological system can be modeled using a niche overlap graph.

    An undirected edge connect two vertices if the two species represented by these vertices *compete for food*.

Introduction to Graphs (9.1) (cont.)



A niche overlap graph

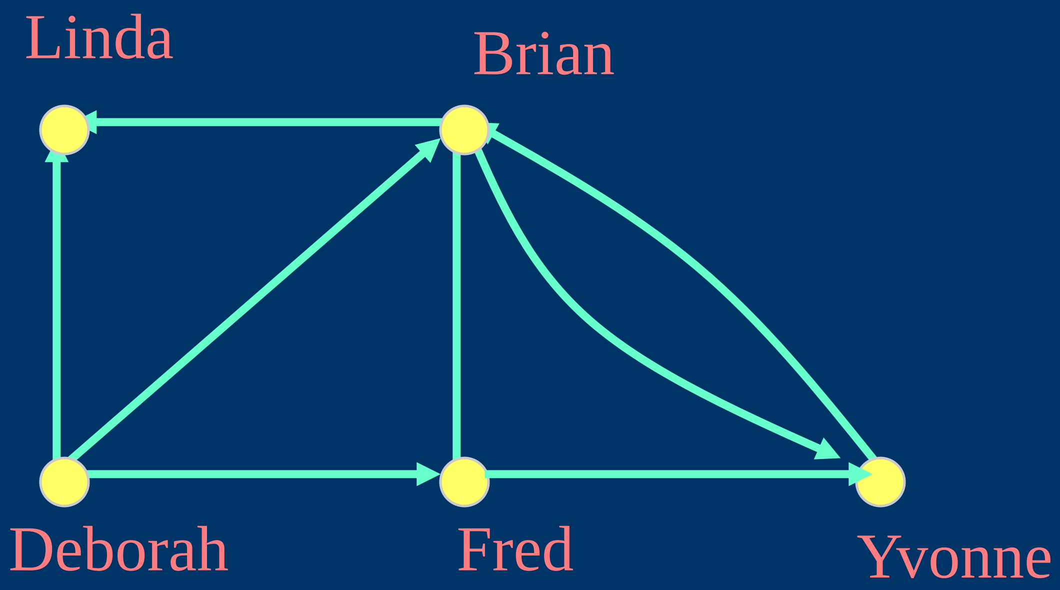
Introduction to Graphs (9.1) (cont.)

– Example: Influence of one person in society

- A directed graph called an influence graph is used to model this behavior

- There is a directed edge from vertex a to vertex b *if the person represented by a vertex a influences the person represented by vertex b*.

Introduction to Graphs (9.1) (cont.)



An influence graph

Introduction to Graphs (9.1) (cont.)

– Example:

The World Wide Web can be modeled as a directed graph where each web page is represented by a vertex and where an edge connects 2 web pages *if there is a link between the 2 pages*

Graph Terminology (9.2)

- Basic Terminology
  - **Goal:** Introduce graph terminology in order to further classify graphs
  - Definition 1:

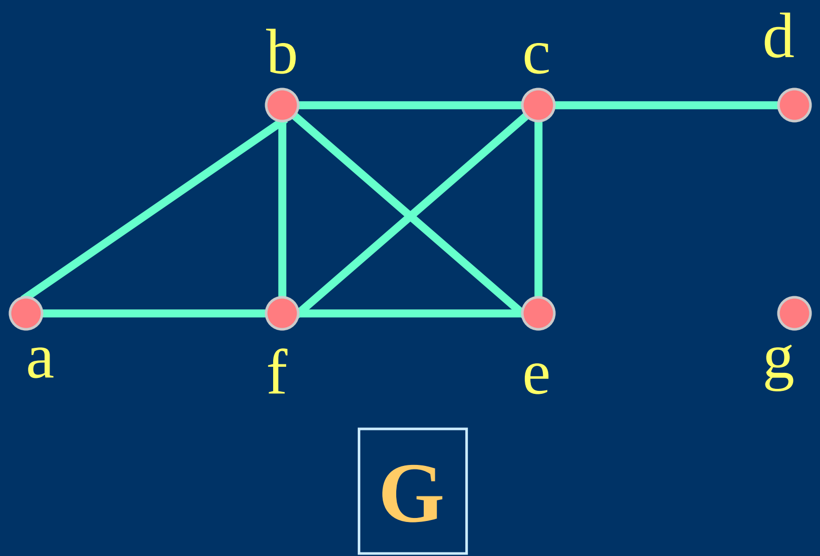
> Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if {u,v} is an edge of G. If e = {u,v}, the edge e is called incident with the vertices u and v. The edge e is also said to connect u and v. The vertices u and v are called endpoints of the edge {u,v}.

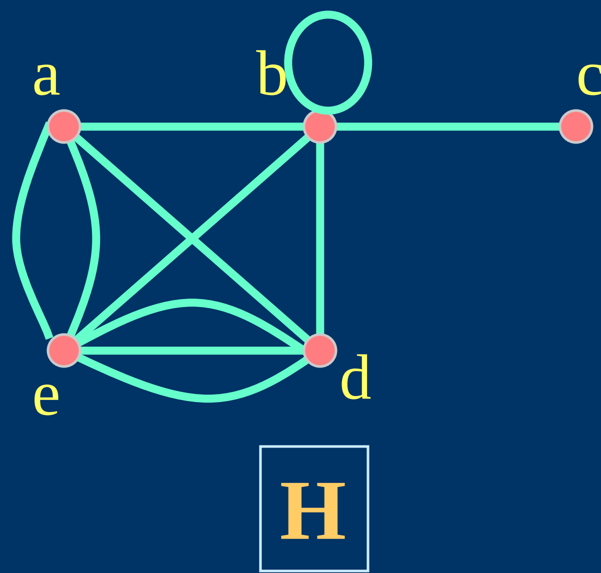
Graph Terminology (9.2) (cont.)

– Definition 2:

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

– Example: What are the degrees of the vertices in the graphs G and H?





Solution:

$$In\,G\begin{cases} deg(a)=2 \\ deg(b)=deg(c)=deg(f)=4 \\ deg(d)=1 \\ deg(e)=3 \\ deg(g)=0 \end{cases}$$

$$In\,H\begin{cases} deg(a)=4 \\ deg(b)=deg(e)=6 \\ deg(c)=1 \\ deg(d)=5 \end{cases}$$

Graph Terminology (9.2) (cont.)

– Theorem 1:

The handshaking theorem

Let G = (V,E) be an undirected graph with e edges. Then

$$2e = \sum_{v \in V} deg(v).$$

(Note that this applies even if multiple edges & loops are present.)

Graph Terminology (9.2) (cont.)

– Example: How many edges are there in a graph with ten vertices each of degree 6 ?

Solution: Since the sum of the degrees of the vertices is $6*10 = 60 \Rightarrow 2e = 60$. Therefore, $e = 30$

Prove that an undirected graph has an even number of vertices of odd degree.

Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph G = (V,E).

Then

$$2e = \sum_{v \in V} \deg(v) = \underbrace{\sum_{v \in V_1} \deg(v)} + \sum_{v \in V_2} \deg(v).$$

Since deg(v) is even for $v \in V_1$, <u>this term</u> is even.

Prove that an undirected graph has an even number of vertices of odd degree.

$$2e = \sum_{v \in V} \deg(v) = \underbrace{\sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v)}.$$

Furthermore, the sum of these two terms is even, since the sum is 2e. Hence, the second term in the sum is also even. (Why?) Since all the terms in the sum are odd, there must be an even number of such terms. (Why?) Thus there are an even number of vertices of odd degree.

What can we say about the vertices of even degree?

Graph Terminology (9.2) (cont.)

– Definition 3:

When (u,v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex of (u,v), and v is called the terminal or end vertex of (u,v). The initial vertex and terminal vertex of a loop are the same.

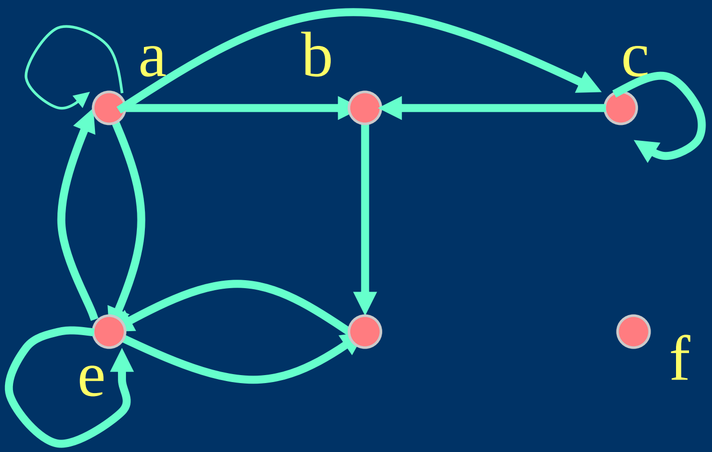
Graph Terminology (9.2) (cont.)

– Definition 4:

In a graph with directed edges the in-degree of a vertex v, denoted $deg^{-}(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $deg^{+}(v)$, is the number of edges with v as their initial vertex.
(Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex)

Graph Terminology (9.2) (cont.)

– Example: Find the in-degree and the out-degree of each vertex in the graph G



Solution: The in-degree of G are: $deg^-(a) = 2$, $deg^-(b) = 2$, $deg^-(c) = 3$, $deg^-(d) = 2$, $deg^-(e) = 3$, and $deg^-(f) = 0$.

The in-degree of G are: $deg^+(a) = 4$, $deg^+(b) = 1$, $deg^+(c) = 2$, $deg^+(d) = 2$, $deg^+(e) = 3$, and $deg^+(f) = 0$

Graph Terminology (9.2) (cont.)

– Theorem 3:

Let G = (V,E) be a graph with directed edges. Then

$$\sum_{v \in V} deg^-(v) = \sum_{v \in V} deg^+(v) = |E|.$$