

Number Theory

ITT9131 Konkreetne Matemaatika

Chapter Four

Divisibility

Primes

Prime examples

Factorial Factors

Relative primality

'MOD': the Congruence Relation

Independent Residues

Additional Applications

Phi and Mu



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1 Prime and Composite Numbers

- Divisibility

2 Greatest Common Divisor

- Definition
- The Euclidean algorithm

3 Primes

- The Fundamental Theorem of Arithmetic
- Distribution of prime numbers



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Division (with remainder)

Definition

Let a and b be integers and $a > 0$. Then **division** of b by a is finding an integer **quotient** q and a **remainder** r satisfying the condition

$$b = aq + r, \text{ where } 0 \leq r < a.$$

Here

b	– dividend
a	– divider (=divisor) (=factor)
$q = \lfloor a/b \rfloor$	– quotient
$r = a \bmod b$	– remainder (=residue)

Example

If $a = 3$ and $b = 17$, then

$$17 = 3 \cdot 5 + 2.$$



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Negative dividend

- If the divisor is positive, then the remainder is always **non-negative**.

For example

If $a = 3$ ja $b = -17$, then

$$-17 = 3 \cdot (-6) + 1.$$

- Integer b can be always represented as $b = aq + r$ with $0 \leq r < a$ due to the fact that b either coincides with a term of the sequence

$$\dots, -3a, -2a, -a, 0, a, 2a, 3a, \dots$$

or lies between two succeeding figures.



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NB! Division by a negative integer yields a negative remainder

$$5 \bmod 3 = 5 - 3 \lfloor 5/3 \rfloor = 2$$

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Be careful!

Some computer languages use another definition.

We assume $a > 0$ in further slides!



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Divisibility

Definition

Let a and b be integers. We say that a divides b , or a is a divisor of b , or b is a multiple of a , if there exists an integer m such that $b = a \cdot m$.

Notations:

- $a|b$ a divides b
- $a \setminus b$ a divides b
- $b : a$ b is a multiple of a

For example

$$3|111$$

$$7|-91$$

$$-7|-91$$



Divisors

Definitioon

If $a|b$, then

- an integer a is called **divisor** or **factor** or **multiplier** of an integer b .

Properties

- Any integer b at least four divisors: $1, -1, b, -b$
- $a|0$ for any integer a ; reverse relation $0|a$ is valid only for $a = 0$. That means $0|0$.
- $1|b$ for any integer b , whereas $b|1$ is valid iff $b = 1$ or $b = -1$.



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More properties:

- 1 If $a|b$, then $\pm a|\pm b$.
- 2 If $a|b$ and $a|c$, for every m, n integer it is valid that $a|mb + nc$.
- 3 $a|b$ iff $ac|bc$ for every integer c .

The first property allows to restrict ourselves to study divisibility on positive integers.

It follows from the second property that if an integer a is a divisor of b and c , then it is the divisor their sum and difference.

Here a is called **common divisor** of b and c (as well as of $b + c$, $b - c$, $b + 2c$ etc.)



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Greatest Common Divisor

Definition

The **greatest common divisor** (*gcd*) of two or more non-zero integers is the largest positive integer that divides the numbers without a remainder.

Example

The common divisors of 36 and 60 are 1, 2, 3, 4, 6, 12.

The greatest common divisor $\gcd(36,60) = 12$.

- The greatest common divisor exists always because of the set of common divisors of the given integers is non-empty and finite.



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The Euclidean algorithm

The algorithm to compute $\gcd(a, b)$ for positive integers a and b

Input: Positive integers a and b , assume that $a > b$

Output: $\gcd(a, b)$

```
■ while  $b > 0$ 
  do
    1  $r := a \bmod b$ 
    2  $a := b$ 
    3  $b := r$ 
  od
■ return( $a$ )
```



Example: compute $\gcd(2322, 654)$

a	b
2322	654
654	360
360	294
294	66
66	30
30	6
6	0



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Important questions to answer:

- Does the algorithm terminate for every input?
- Is the result the *greatest* common divisor?
- How long does it take?



Termination of the Euclidean algorithm

- In any cycle, the pair of integers (a, b) is replaced by (b, r) , where r is the remainder of division of a by b .
- Hence $r < b$.
- The second number of the pair decreases, but remains non-negative, so the process cannot last infinitely long.



Correctness of the Euclidean algorithm

Theorem

If r is a remainder of division of a by b , then

$$\gcd(a, b) = \gcd(b, r)$$

Proof. It follows from the equality $a = bq + r$ that

- 1 if $d|a$ and $d|b$, then $d|r$
- 2 if $d|b$ and $d|r$, then $d|a$

In other words, the set of common divisors of a and b equals to the set of common divisors of b and r , recomputing of (b, r) does not change the greatest common divisor of the pair.

The number returned $r = \gcd(r, 0)$.

Q.E.D.



Complexity of the Euclidean algorithm

Theorem

The number of steps of the Euclidean algorithm applied to two positive integers a and b is at most

$$1 + \log_2 a + \log_2 b.$$

Proof. Let consider the step where the pair (a, b) is replaced by (b, r) . Then we have $r < b$ and $b + r \leq a$. Hence $2r < r + b \leq a$ or $br < ab/2$. This is that the product of the elements of the pair decreases at least 2 times.

If after k cycles the product is still positive, then $ab/2^k > 1$, that gives

$$k \leq \log_2(ab) = \log_2 a + \log_2 b$$

Q.E.D.



The numbers produced by the Euclidean algorithm

$$a = bq_1 + r_1$$

r_1 can be expressed in terms of b and a

$$b = r_1q_2 + r_2$$

r_2 can be expressed in terms of r_1 and b

$$r_1 = r_2q_3 + r_3$$

r_3 can be expressed in terms of r_2 and r_1

.....

.....

$$r_{k-3} = r_{k-2}q_{k-1} + r_{k-1} \quad r_{k-1} \text{ can be expressed in terms of } r_{k-2} \text{ and } r_{k-3}$$

$$r_{k-2} = r_{k-1}q_k + r_k \quad r_k \text{ can be expressed in terms of } r_{k-1} \text{ and } r_{k-2}$$

$$r_{k-1} = r_kq_{k+1}$$

Now, one can extract $r_k = \gcd(a, b)$ from the second last equality and substitute there step-by-step r_{k-1}, r_{k-2}, \dots using previous equations. We obtain finally that r_k equals to a linear combination of a and b with (not necessarily positive) integer coefficients.



GCD as a linear combination

Theorem (Bézout's identity)

Let $d = \gcd(a, b)$. Then d can be written in the form

$$d = as + bt$$

where s and t are integers. In addition,

$$\gcd(a, b) = \min\{n \geq 1 \mid \exists s, t \in \mathbb{Z} : n = as + bt\}.$$

For example: $a = 360$ and $b = 294$

$$\gcd(a, b) = 294 \cdot (-11) + 360 \cdot 9 = -11a + 9b$$



Application of EA: solving of linear Diophantine Equations

Corollary

Let a , b and c be positive integers. The equation

$$ax + by = c$$

has integer solutions if and only if c is a multiple of $\gcd(a, b)$.

The method: Making use of Euclidean algorithm, compute such coefficients s and t that $sa + tb = \gcd(a, b)$. Then

$$x = \frac{cs}{\gcd(a, b)}$$

$$y = \frac{ct}{\gcd(a, b)}$$



Linear Diophantine Equations (2)

Example: $92x + 17y = 3$

From EA:

a	b	Seos
92	17	
17	7	$92 = 5 \cdot 17 + 7$
7	3	$17 = 2 \cdot 7 + 3$
3	1	$7 = 2 \cdot 3 + 1$
1	0	

Transformations:

$$\begin{aligned}1 &= 7 - 2 \cdot 3 \\&= 7 - 2 \cdot (17 - 7 \cdot 2) = (-2) \cdot 17 + 5 \cdot 7 = \\&= (-2) \cdot 17 + 5 \cdot (92 - 5 \cdot 17) = 5 \cdot 92 + (-27) \cdot 17\end{aligned}$$

$\gcd(92, 17) | 3$ yields a solution

$$\begin{aligned}x &= \frac{3 \cdot 5}{\gcd(92, 17)} = 3 \cdot 5 = 15 \\y &= \frac{3 \cdot (-27)}{\gcd(92, 17)} = -3 \cdot 27 = -81\end{aligned}$$



Linear Diophantine Equations (3)

Example: $5x + 3y = 2$ \rightarrow many solutions

$$\gcd(5, 3) = 1$$

As $1 = 2 \cdot 5 + 3 \cdot 3$, then one solution is:

$$x = 2 \cdot 2 = 4$$

$$y = -3 \cdot 2 = -6$$

As $1 = (-10) \cdot 5 + 17 \cdot 3$, then another solution is:

$$x = -10 \cdot 2 = -20$$

$$y = 17 \cdot 2 = 34$$

Example: $15x + 9y = 8$ \rightarrow no solutions

Whereas, $\gcd(15, 9) = 3$, then the equation can be expressed as

$$3 \cdot (5x + 3y) = 8.$$

The left-hand side of the equation is divisible by 3, but the right-hand side is not, therefore the equality cannot be valid for any integer x and y .



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More about Linear Diophantine Equations (1)

- **General solution** of a Diophantine equation $ax + by = c$ is

$$\begin{cases} x &= x_0 + \frac{kb}{\gcd(a,b)} \\ y &= y_0 - \frac{ka}{\gcd(a,b)} \end{cases}$$

where x_0 and y_0 are particular solutions and k is an integer.

- Particular solutions can be found by means of Euclidean algorithm:

$$\begin{cases} x_0 &= \frac{cs}{\gcd(a,b)} \\ y_0 &= \frac{ct}{\gcd(a,b)} \end{cases}$$

- This equation has a solution (where x and y are integers) if and only if $\gcd(a,b) \mid c$
- The general solution above provides **all** integer solutions of the equation (see proof in http://en.wikipedia.org/wiki/Diophantine_equation)



More about Linear Diophantine Equations (2)

Example: $5x + 3y = 2$

We have found, that $\gcd(5, 3) = 1$ and its particular solutions are $x_0 = 4$ and $y_0 = -6$.

Thus, for any $k \in \mathbb{Z}$:

$$\begin{cases} x &= 4 + 3k \\ y &= -6 - 5k \end{cases}$$

Solutions of the equation for $k = \dots, -3, -2, -1, 0, 1, 2, 3, \dots$ are infinite sequences of numbers:

$$\begin{array}{cccccccccc} x &= & \dots, & -5, & -2, & 1, & 4, & 7, & 10, & 13, & \dots \\ y &= & \dots, & 9, & 4, & -1, & -6, & -11, & -16, & -21, & \dots \end{array}$$

Among others, if $k = -8$, then we get the solution $x = -20$ ja $y = 34$.



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Prime and composite numbers

Every integer greater than 1 is either **prime** or **composite**, but not both:

- A positive integer p is called **prime** if it has just two divisors, namely 1 and p . By convention, 1 is not prime

Prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, ...

- An integer that has three or more divisors is called **composite**

Composite numbers: 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, ...



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Another application of EA

The Fundamental Theorem of Arithmetic

Every positive integer n can be written uniquely as a product of primes:

$$n = p_1 \dots p_m = \prod_{k=1}^m p_k, \quad p_1 \leq \dots \leq p_m$$

Proof. Suppose we have two factorizations into primes

$$n = p_1 \dots p_m = q_1 \dots q_k, \quad p_1 \leq \dots \leq p_m \text{ and } q_1 \leq \dots \leq q_k$$

Assume that $p_1 < q_1$. Since p_1 and q_1 are primes, $\gcd(p_1, q_1) = 1$. That means that EA defines integers s and t that $sp_1 + tq_1 = 1$. Therefore

$$sp_1 q_2 \dots q_k + tq_1 q_2 \dots q_k = q_2 \dots q_k$$

Now p_1 divides both terms on the left, thus $q_2 \dots q_k / p_1$ is integer that contradicts with $p_1 < q_1$. This means that $p_1 = q_1$.

Similarly, using induction we can prove that $p_2 = q_2$, $p_3 = q_3$, etc

Q.E.D.



Canonical form of integers

- Every positive integer n can be represented uniquely as a product

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} = \prod_p p^{n_p}, \quad \text{where each } n_p \geq 0$$

For example:

$$600 = 2^3 \cdot 3^1 \cdot 5^2 \cdot 7^0 \cdot 11^0 \cdots$$

$$35 = 2^0 \cdot 3^0 \cdot 5^1 \cdot 7^1 \cdot 11^0 \cdots$$

$$5\,251\,400 = 2^3 \cdot 3^0 \cdot 5^2 \cdot 7^1 \cdot 11^2 \cdot 13^0 \cdots 29^0 \cdot 31^1 \cdot 37^0 \cdots$$



Prime-exponent representation of integers

- Canonical form of an integer $n = \prod_p p^{n_p}$ provides a sequence of powers $\langle n_1, n_2, \dots \rangle$ as another representation.

For example:

$$600 = \langle 3, 1, 2, 0, 0, 0, \dots \rangle$$

$$35 = \langle 0, 0, 1, 1, 0, 0, 0, \dots \rangle$$

$$5 \ 251 \ 400 = \langle 3, 0, 2, 1, 2, 0, 0, 0, 0, 0, 1, 0, 0, \dots \rangle$$



Prime-exponent representation and arithmetic operations

Multiplication

Let

$$m = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} = \prod_p p^{m_p}$$

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} = \prod_p p^{n_p}$$

Then

$$mn = p_1^{m_1+n_1} p_2^{m_2+n_2} \cdots p_k^{m_k+n_k} = \prod_p p^{m_p+n_p}$$

Using prime-exponent representation:

$$mn = \langle m_1 + n_1, m_2 + n_2, m_3 + n_3, \dots \rangle$$

For example

$$\begin{aligned} 600 \cdot 35 &= \langle 3, 1, 2, 0, 0, 0, \dots \rangle \cdot \langle 0, 0, 1, 1, 0, 0, 0, \dots \rangle \\ &= \langle 3+0, 1+0, 2+1, 0+1, 0+0, 0+0, \dots \rangle \\ &= \langle 3, 1, 3, 1, 0, 0, \dots \rangle = 21\,000 \end{aligned}$$



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Some other operations

The greatest common divisor and the least common multiple (*lcm*)

$$\gcd(m, n) = \langle \min(m_1, n_1), \min(m_2, n_2), \min(m_3, n_3), \dots \rangle$$

$$\text{lcm}(m, n) = \langle \max(m_1, n_1), \max(m_2, n_2), \max(m_3, n_3), \dots \rangle$$

Example

$$120 = 2^3 \cdot 3^1 \cdot 5^1 = \langle 3, 1, 1, 0, 0, \dots \rangle$$

$$36 = 2^2 \cdot 3^2 = \langle 2, 2, 0, 0, \dots \rangle$$

$$\gcd(120, 36) = 2^{\min(3,2)} \cdot 3^{\min(1,2)} \cdot 5^{\min(1,0)} = 2^2 \cdot 3^1 = \langle 2, 1, 0, 0, \dots \rangle = 12$$

$$\text{lcm}(120, 36) = 2^{\max(3,2)} \cdot 3^{\max(1,2)} \cdot 5^{\max(1,0)} = 2^3 \cdot 3^2 \cdot 5^1 = \langle 3, 2, 1, 0, 0, \dots \rangle = 360$$



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$$120 = 2^3 \cdot 3^1 \cdot 5^1 = \langle 3, 1, 1, 0, 0, \dots \rangle$$

$$36 = 2^2 \cdot 3^2 = \langle 2, 2, 0, 0, \dots \rangle$$

$$\gcd(120, 36) = 2^{\min(3,2)} \cdot 3^{\min(1,2)} \cdot 5^{\min(1,0)} = 2^2 \cdot 3^1 = \langle 2, 1, 0, 0, \dots \rangle = 12$$

$$\text{lcm}(120, 36) = 2^{\max(3,2)} \cdot 3^{\max(1,2)} \cdot 5^{\max(1,0)} = 2^3 \cdot 3^2 \cdot 5^1 = \langle 3, 2, 1, 0, 0, \dots \rangle = 360$$



Properties of the GCD

Homogeneity

$$\gcd(na, nb) = n \cdot \gcd(a, b) \text{ for every positive integer } n.$$

Proof.

Let $a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $b = p_1^{\beta_1} \cdots p_k^{\beta_k}$, and $\gcd(a, b) = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$, where $\gamma_i = \min(\alpha_i, \beta_i)$. If $n = p_1^{n_1} \cdots p_k^{n_k}$, then

$$\begin{aligned} \gcd(na, nb) &= p_1^{\min(\alpha_1 + n_1, \beta_1 + n_1)} \cdots p_k^{\min(\alpha_k + n_k, \beta_k + n_k)} = \\ &= p_1^{\min(\alpha_1, \beta_1)} p_1^{n_1} \cdots p_k^{\min(\alpha_k, \beta_k)} p_k^{n_k} = \\ &= p_1^{n_1} \cdots p_k^{n_k} p_1^{\gamma_1} \cdots p_k^{\gamma_k} = n \cdot \gcd(a, b) \end{aligned}$$

Q.E.D.



Properties of the GCD

GCD and LCM

$\gcd(a, b) \cdot \text{lcm}(a, b) = ab$ for every two positive integers a and b

Proof.

$$\begin{aligned}\gcd(a, b) \cdot \text{lcm}(a, b) &= p_1^{\min(\alpha_1, \beta_1)} \dots p_k^{\min(\alpha_k, \beta_k)} \cdot p_1^{\max(\alpha_1, \beta_1)} \dots p_k^{\max(\alpha_k, \beta_k)} = \\ &= p_1^{\min(\alpha_1, \beta_1) + \max(\alpha_1, \beta_1)} \dots p_k^{\min(\alpha_k, \beta_k) + \max(\alpha_k, \beta_k)} = \\ &= p_1^{\alpha_1 + \beta_1} \dots p_k^{\alpha_k + \beta_k} = ab\end{aligned}$$

Q.E.D.



Relatively prime numbers

Definition

Two integers a and b are said to be **relatively prime** (or **co-prime**) if the only positive integer that evenly divides both of them is 1.

Notations used:

- $\gcd(a, b) = 1$
- $a \perp b$

For example

$16 \perp 25$ and $99 \perp 100$

Some simple properties:

- Dividing a and b by their greatest common divisor yields relatively primes:

$$\gcd\left(\frac{a}{\gcd(a, b)}, \frac{b}{\gcd(a, b)}\right) = 1$$

- Any two positive integers a and b can be represented as $a = a'd$ and $b = b'd$, where $d = \gcd(a, b)$ and $a' \perp b'$



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Properties of relatively prime numbers

Theorem

If $a \perp b$, then $\gcd(ac, b) = \gcd(c, b)$ for every positive integer c .

Proof.

Assuming canonic representation of $a = \prod_p p^{\alpha_p}$, $b = \prod_p p^{\beta_p}$ and $c = \prod_p p^{\gamma_p}$, one can conclude that for any prime p :

- The premise $a \perp b$ implies that $p^{\min(\alpha_p, \beta_p)} = 1$, it is that either $\alpha_p = 0$ or $\beta_p = 0$.
- If $\alpha_p = 0$, then $p^{\min(\alpha_p + \gamma_p, \beta_p)} = p^{\min(\gamma_p, \beta_p)}$.
- If $\beta_p = 0$, then $p^{\min(\alpha_p + \gamma_p, \beta_p)} = p^{\min(\alpha_p + \gamma_p, 0)} = 1 = p^{\min(\gamma_p, 0)} = p^{\min(\gamma_p, \beta_p)}$.

Hence, the set of common divisors of ac and b is equal to the set of common divisors of c and b .

Q.E.D.



Divisibility

Observation

Let

$$a = \prod_p p^{\alpha_p}$$

and

$$b = \prod_p p^{\beta_p}.$$

Then $a|b$ iff $\alpha_p \leq \beta_p$ for every prime p .



Consequences from the theorems above

- 1 If $a \perp c$ and $b \perp c$, then $ab \perp c$
- 2 If $a|bc$ and $a \perp b$, then $a|c$
- 3 If $a|c$, $b|c$ and $a \perp b$, then $ab|c$

Example: compute $\gcd(560, 315)$

$$\begin{aligned}\gcd(560, 315) &= \gcd(5 \cdot 112, 5 \cdot 63) = \\ &= 5 \cdot \gcd(112, 63) = \\ &= 5 \cdot \gcd(2^4 \cdot 7, 63) = \\ &= 5 \cdot \gcd(7, 63) \\ &= 5 \cdot 7 = 35\end{aligned}$$



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The number of divisors

- Canonic form of a positive integer permits to compute the number of its factors without factorization:

- If

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

then any divisor of n can be constructed by multiplying $0, 1, \dots, n_1$ times the prime divisor p_1 , then $0, 1, \dots, n_2$ times the prime divisor p_2 etc.

- Then the number of divisors of n should be

$$(n_1 + 1)(n_2 + 1) \cdots (n_k + 1).$$

Example

Integer 694 575 has $694\,575 = 3^4 \cdot 5^2 \cdot 7^3$ on $(4 + 1)(2 + 1)(3 + 1) = 60$ factors.



Next subsection

1 Prime and Composite Numbers

- Divisibility

2 Greatest Common Divisor

- Definition
- The Euclidean algorithm

3 Primes

- The Fundamental Theorem of Arithmetic
- Distribution of prime numbers



Number of primes

Euclid's theorem

There are infinitely many prime numbers.

Proof. Let's assume that there is finite number of primes:

$$p_1, p_2, p_3, \dots, p_k.$$

Consider

$$n = p_1 p_2 p_3 \cdots p_k + 1.$$

Like any other natural number, n is divisible at least by 1 and itself, i.e. it can be prime. Dividing n by p_1, p_2, p_3, \dots or p_k yields the remainder 1. So, n should be prime that differs from any of numbers $p_1, p_2, p_3, \dots, p_k$, that leads to a contradiction with the assumption that the set of primes is finite.

Q.E.D.



Number of primes (another proof)

Theorem

There are infinitely many prime numbers.

Proof. For any natural number n , there exists a prime number greater than n :

Let p be the smallest divisor of $n! + 1$ that is greater than 1. Then

- p is a prime number, as otherwise it wouldn't be the smallest divisor.
- $p > n$, as otherwise $p|n!$ and $p|n! + 1$ and $p|(n! + 1) - n! = p|1$.

Q.E.D.



Number of primes: A proof by Paul Erdős

Theorem

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty$$



Number of primes: A proof by Paul Erdős

Theorem

$$\sum_{p \text{ prime}} \frac{1}{p} = \infty$$

By contradiction, assume $\sum_{p \text{ prime}} \frac{1}{p} < \infty$.

- Call a prime p *large* if $\sum_{q \text{ prime} \geq p} \frac{1}{q} < \frac{1}{2}$. Let N be the number of *small* primes.

- For $m \geq 1$ let $U_m = \{1 \leq n \leq m \mid n \text{ only has small prime factors}\}$.

For $n \in U_m$ it is $n = d \cdot k^2$ where q is a product of distinct small primes. Then

$$|U_m| \leq 2^N \cdot \sqrt{m}$$

- For $m \geq 1$ and p prime let $D_{m,p} = \{1 \leq n \leq m \mid p \nmid n\}$. Then

$$|\{1, \dots, m\} \setminus U_m| \leq \sum_{p \text{ large prime}} |D_{m,p}| < \frac{m}{2}$$

- Then $\frac{m}{2} \leq |U_m| \leq 2^N \sqrt{m}$: this is false for m large enough.



Primes are distributed “very irregularly”

- Since all primes except 2 are odd, the difference between two primes must be at least two, except 2 and 3.
- Two primes whose difference is two are called **twin primes**. For example (17, 19) or (3557 and 3559). There is **no proof** of the hypothesis that there are infinitely many twin primes.

Theorem

For every positive integer k , there exist k consecutive composite integers.

Proof. Let $n = k + 1$ and consider the numbers $n! + 2, n! + 3, \dots, n! + n$. All these numbers are composite because of $i | n! + i$ for every $i = 2, 3, \dots, n$.

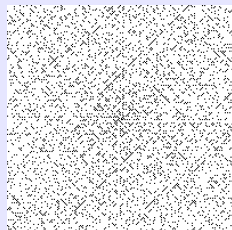
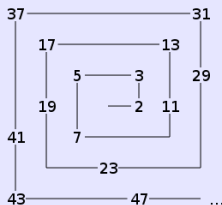
Q.E.D.



Distribution diagrams for primes



37-36-35-34-33-32-31
 38 17-16-15-14-13 30
 39 18 5-4-3 12 29
 40 19 6 1-2 11 28
 41 20 7-8-9-10 27
 42 21-22-23-24-25-26
 43-44-45-46-47-48-49...



The prime counting function $\pi(n)$

- Definition:

$\pi(n)$ = number of primes in the set $\{1, 2, \dots, n\}$

- The first values:

$\pi(1) = 0$	$\pi(2) = 1$	$\pi(3) = 2$	$\pi(4) = 2$
$\pi(5) = 3$	$\pi(6) = 3$	$\pi(7) = 4$	$\pi(8) = 4$



The Prime Number Theorem

Theorem

The quotient of division of $\pi(n)$ by $n/\ln n$ will be arbitrarily close to 1 as n gets large. It is also denoted as

$$\pi(n) \sim \frac{n}{\ln n}$$

- Studying prime tables C. F. Gauss come up with the formula in ~ 1791 .
- J. Hadamard and C. de la Vallée Poussin proved the theorem independently from each other in 1896.



The Prime Number Theorem (2)

Example: How many primes are with 200 digits?

- The total number of positive integers with 200 digits:

$$10^{200} - 10^{199} = 9 \cdot 10^{199}$$

- Approximate number of primes with 200 digits

$$\pi(10^{200}) - \pi(10^{199}) \approx \frac{10^{200}}{200 \ln 10} - \frac{10^{199}}{199 \ln 10} \approx 1,95 \cdot 10^{197}$$

- Percentage of primes

$$\frac{1,95 \cdot 10^{197}}{9 \cdot 10^{199}} \approx \frac{1}{460} = 0.22\%$$



Warmup: Extending $\pi(x)$ to positive reals

Problem

Let $\pi(x)$ be the number of primes which are not larger than $x \in \mathbb{R}$.
Prove or disprove: $\pi(x) - \pi(x-1) = [x \text{ is prime}]$.



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Problem

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Prove or disprove: $\pi(x) - \pi(x-1) = [x \text{ is prime}]$.

Solution

The formula is true if x is integer: but x is real . . .

But clearly $\pi(x) = \pi(\lfloor x \rfloor)$: then

$$\begin{aligned}\pi(x) - \pi(x-1) &= \pi(\lfloor x \rfloor) - \pi(\lfloor x-1 \rfloor) \\ &= \pi(\lfloor x \rfloor) - \pi(\lfloor x \rfloor - 1) \\ &= [\lfloor x \rfloor \text{ is prime}] ,\end{aligned}$$

which is true.

