CSE 103: Discrete Mathematics
Chapter 1.1.-1.3: Propositional Logic

Outline

Propositions

A proposition is a declarative sentence that is either true or false but not both. Examples of propositions:

- The Moon is made of green cheese.
- Trenton is the capital of New Jersey.
- Toronto is the capital of Canada.
- $1 + 0 = 1$
- $0 + 0 = 2$

Examples that are not propositions.

- Sit down!
- What time is it?
- $x + 1 = 2$
- $x + y = z$

Propositional Logic

Constructing Propositions

- Propositional Variables: $p, q, r, s, \ldots$
- The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.
- Compound Propositions; constructed from logical connectives and other propositions
- Negation $\neg$
- Conjunction $\wedge$
- Disjunction $\vee$
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Propositional Logic - negation

Suppose p is a proposition.
The *negation* of p is written $\neg p$ and has meaning:

"It is not the case that p."

- Ex. CS1207 is NOT Belal's favorite class.

Truth table for negation:

p	$\neg p$
T	F
F	T

Notice that $\neg p$ is a propositio n!

Disjunction

The disjunction of propositions p and q is denoted by $p \vee q$ and has this truth table:

Disjunction corresponds to English "or."
$p \vee q$ is when p or q (or both) are true.

- Ex. Michael is brave OR nuts.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conjunction

The disjunction of propositions p and q is denoted by $p \wedge q$ and has this truth table:

Conjunction corresponds to English “and.”
$p \wedge q$ is true exactly when p and q are both true.

- Ex. Amy is curious AND clever.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Implication

- If p and q are propositions, then $p \rightarrow q$ is a conditional statement or implication which is read as "if p, then q" and has this truth table: 2 meaning
  1.If p then q that means if p is true then q must be true, the p→q will be true.
  2. p implies q means if p true then q will be true.

p	***q***	***p* →*q***
T	T	T
T	F	F
F	T	T
F	F	T

- In $p \rightarrow q$, p is the hypothesis (antecedent or premise) and q is the conclusion (or consequence).

  Implication can be expressed by $p \rightarrow q \equiv \neg p \vee q$

Understanding Implication

- In $p \rightarrow q$ there does not need to be any connection between the antecedent or the consequent. The meaning depends only on the truth values of p and q.
- One way to view the logical conditional is to think of an obligation or contract. "If I am elected, then I will lower taxes." Let p = I am elected & q = I will lower taxes
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Different Ways of Expressing $p \rightarrow q$

if p, then q
if p, q
q unless $\neg p$
q if p
p is sufficient for q
q is necessary for p
a sufficient condition for q is p

p implies q
p only if q
q when p
q whenever p
q follows from p
a necessary condition for p is q

Converse, Contrapositive, and Inverse

- $q \to p$ is the **converse** of $p \to q$
- $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
- $\neg p \to \neg q$ is the **inverse** of $p \to q$

Example: Find the converse, inverse, and contrapositive of "It is raining is a sufficient condition for my not going to town."

Solution:

converse: If I do not go to town, then it is raining.

inverse: If it is not raining, then I will go to town.

contrapositive: If I go to town, then it is not raining.

How do the converse, contrapositive, and inverse relate to $p \to q$?

$p \to q \equiv \neg p \vee q$ so $\neg(p \to q) \equiv p \vee \neg q$ so $p \to q \equiv \neg q \to \neg p$

- **converse** $\equiv$ **contrapositive** ?
- **converse** $\equiv$ **inverse** ?
- **contrapositive** $\equiv$ **inverse** ?

Biconditional

If p and q are propositions, then the biconditional proposition $p \leftrightarrow q$ has this truth table. “If I am elected, then I will lower taxes.”

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \leftrightarrow q$ also reads as

- p if and only if q
- p **iff** q.
- p is necessary and sufficient for q
- if p then q, and conversely
- p implies q, and vice-versa

Precedence of Logical Operators

- $\neg$
- $\wedge$
- $\vee$
- $\rightarrow$
- $\leftrightarrow$

Thus $p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$.
If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Satisfiability, Tautology, Contradiction

A proposition is

- satisfiable, if its truth table contains **true** at least once. Example: $p \wedge q$.
- a tautology, if it is always true. Example: $p \vee \neg p$.
- a contradiction, if it always false. Example: $p \wedge \neg p$.
- a contingency, if it is neither a tautology nor a contradiction. Example: p.

Logical Equivalence

[redacted]

Two compound propositions p and q are logically equivalent if the columns in a truth table giving their truth values agree.
This is written as $p \equiv q$.

It is easy to show:

Fact

$p \equiv q$ if and only if $p \leftrightarrow q$ is a tautology.

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Truth table proving De Morgan's second law.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Important Logical Equivalences

Domination laws:	$p \vee \mathbf{T} \equiv \mathbf{T}$, $p \wedge \mathbf{F} \equiv \mathbf{F}$
Identity laws:	$p \wedge \mathbf{T} \equiv p$, $p \vee \mathbf{F} \equiv p$
Idempotent laws:	$p \wedge p \equiv p$, $p \vee p \equiv p$
Double negation law:	$\neg(\neg p) \equiv p$
Negation laws:	$p \vee \neg p \equiv \mathbf{T}$, $p \wedge \neg p \equiv \mathbf{F}$

The first of the Negation laws is also called “law of excluded middle”. Latin: “tertium non datur”.

Commutative laws:	$p \wedge q \equiv q \wedge p$, $p \vee q \equiv q \vee p$
Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Absorption laws:	$p \vee (p \wedge q) \equiv p$, $p \wedge (p \vee q) \equiv p$

More Logical Equivalences

TABLE 7 **Logical Equivalences Involving Conditional Statements.**

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

TABLE 8 **Logical Equivalences Involving Biconditional Statements.**

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

A Proof in Propositional Logic

To prove: $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$\neg(p \vee (\neg p \wedge q))$	$\equiv$	$\neg p \wedge \neg(\neg p \wedge q)$	by De Morgan's 2nd law
	$\equiv$	$\neg p \wedge (\neg(\neg p) \vee \neg q)$	by De Morgan's first law
	$\equiv$	$\neg p \wedge (p \vee \neg q)$	by the double negation law
	$\equiv$	$(\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the 2nd distributive law
	$\equiv$	$\mathbf{F} \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv \mathbf{F}$
	$\equiv$	$(\neg p \wedge \neg q) \vee \mathbf{F}$	by commutativity of disj.
	$\equiv$	$\neg p \wedge \neg q$	by the identity law for **F**

Reference:

Discrete Mathematics and Its Applications
- Rosen