CS103
Discrete Mathematics

Propositional Logic - say a bit…

This week we're using propositional logic as a foundation for formal proofs.

Propositional logic is also the key to writing good code…you can't do any kind of conditional (if) statement without understanding the condition you're testing.

Propositional Logic - 2 more defn…

A *tautology* is a proposition that's always TRUE.

A *contradiction* is a proposition that's always FALSE.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Propositional Logic - an unfamous ≡

`if NOT (blue AND NOT red) OR red then…`

$\neg(p \wedge \neg q) \vee q \equiv \neg p \vee q$

$\neg(p \wedge \neg q) \vee q$	$\equiv$	$(\neg p \vee \neg\neg q) \vee q$	DeMorgan's
	$\equiv$	$(\neg p \vee q) \vee q$	Double negation
	$\equiv$	$\neg p \vee (q \vee q)$	Associativity
	$\equiv$	$\neg p \vee q$	Idempotent

Propositional Logic - one last proof

- Show that $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.
- We use $\equiv$ to show that $[p \wedge (p \rightarrow q)] \rightarrow q \equiv T$.

$[p \wedge (p \rightarrow q)] \rightarrow q$

$\equiv [p \wedge (\neg p \vee q)] \rightarrow q$	substitution for $\rightarrow$
$\equiv [(p \wedge \neg p) \vee (p \wedge q)] \rightarrow q$	distributive
$\equiv [\, F \vee (p \wedge q)] \rightarrow q$	uniqueness
$\equiv (p \wedge q) \rightarrow q$	identity
$\equiv \neg(p \wedge q) \vee q$	substitution for $\rightarrow$
$\equiv (\neg p \vee \neg q) \vee q$	DeMorgan's
$\equiv \neg p \vee (\neg q \vee q)$	associative
$\equiv \neg p \vee T$	excluded middle
$\equiv T$	domination

Predicate Logic - everybody loves somebody

Proposition, YES or NO?

$3 + 2 = 5$ YES

$X + 2 = 5$ NO

$X + 2 = 5$ for any choice of X in $\{1, 2, 3\}$ YES

$X + 2 = 5$ for some X in $\{1, 2, 3\}$ YES

Predicate Logic - everybody loves somebody

Alicia eats pizza at least once a week.
Garrett eats pizza at least once a week.
Allison eats pizza at least once a week.
Gregg eats pizza at least once a week.
Ryan eats pizza at least once a week.
Meera eats pizza at least once a week.
Ariel eats pizza at least once a week.

$\vdots$

Predicates

Alicia eats pizza at least once a week.

⋮

Define:

EP(*x*) = "*x* eats pizza at least once a week."

Universe of Discourse - *x* is a student in cse1207

A *predicate*, or propositional function, is a function that takes some variable(s) as arguments and returns True or False.

Note that *EP*(*x*) is not a proposition, *EP*(*Ariel*) is.

Predicates

Suppose $Q(x,y) =$ "$x > y$"

Proposition, YES or NO?

$Q(x,y)$ NO

$Q(3,4)$ YES

$Q(x,9)$ NO

Predicate, YES or NO?

$Q(x,y)$ YES

$Q(3,4)$ NO

$Q(x,9)$ YES

Predicates - the universal quantifier

Another way of changing a predicate into a proposition.

Suppose $P(x)$ is a predicate on some universe of discourse.

Ex. $B(x)$ = "x is carrying a backpack," x is set of cse1207 students.

The universal quantifier of $P(x)$ is the **proposition**:

"$P(x)$ is true for all x in the universe of discourse."

We write it $\forall x\, P(x)$, and say "for all x, $P(x)$"

$\forall x\, P(x)$ is TRUE if $P(x)$ is true for every single x.

$\forall x\, P(x)$ is FALSE if there is an x for which $P(x)$ is false.

$\forall x\, B(x)$?

Predicates - the existential quantifier

Another way of changing a predicate into a proposition.

Suppose $P(x)$ is a predicate on some universe of discourse.
Ex. $C(x)$ = "x has a candy bar," x is set of cs173 students.

The existential quantifier of $P(x)$ is the **proposition**:
"$P(x)$ is true for some x in the universe of discourse."

We write it $\exists x\, P(x)$, and say "for some x, $P(x)$"

$\exists x\, P(x)$ is TRUE if there is an x for which $P(x)$ is true.
$\exists x\, P(x)$ is FALSE if $P(x)$ is false for every single x.

$\exists x\, C(x)$?

Predicates - the existential quantifier

TABLE 1 Quantifiers.		
Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

- A domain must always be specified when a statement $\exists x P(x)$ is used. Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x P(x)$ has no meaning.

Predicates - more examples

L(x) = "x is a lion."
F(x) = "x is fierce."
C(x) = "x drinks coffee."

Universe of discourse is all creatures.

All lions are fierce.

$\forall x\,(L(x) \rightarrow F(x))$

Some lions don't drink

$\exists x\,(L(x) \wedge \neg C(x))$

Some fierce creatures don't dr

$\exists x\,(F(x) \wedge \neg C(x))$

Predicates - more examples

$L(x)$ = "x is a lion."
$F(x)$ = "x is fierce."
$C(x)$ = "x drinks coffee."

Universe of discourse is all creatures.

- Notice that the second statement cannot be written as $\exists x(P(x) \rightarrow \neg R(x))$. The reason is that $P(x) \rightarrow \neg R(x)$ is true whenever x is not a lion, so that $\exists x(P(x) \rightarrow \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee.

Predicates - more examples

B(x) = "x is a hummingbird."
L(x) = "x is a large bird."
H(x) = "x lives on honey."
R(x) = "x is richly colored."

Universe of discourse is all creatures.

All hummingbirds are richly colore

$\forall x\, (B(x) \rightarrow R(x))$

No large birds live on honey.

$\neg\exists x\, (L(x) \wedge H(x))$

Birds that do not live on hone

$\forall x\, (\neg H(x) \rightarrow \neg R(x))$

Predicates - quantifier negation

Not all large birds live on honey. $\neg\forall x\,(L(x) \rightarrow H(x))$

$\forall x\, P(x)$ means "$P(x)$ is true for every x."

What about $\neg\forall x\, P(x)$?

Not ["$P(x)$ is true for every x."]

"There is an x for which $P(x)$ is not true."

$\exists x\, \neg P(x)$

So, $\neg\forall x\, P(x)$ is the same as $\exists x\, \neg P(x)$.

$\exists x\, \neg(L(x) \rightarrow H(x))$

Negations of Quantified Statements

Everyone likes football.

Every mother loves her child.

What is the negation of this statement?

Not everyone likes football = There exists someone who doesn’t like football.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

(generalized) DeMorgan’s Law

Say the domain has only three values.

$$\equiv \neg (P(1) \wedge P(2) \wedge P(3))$$

$$\equiv \neg (P(1) \wedge P(2)) \vee \neg P(3))$$

$$\equiv \neg P(1) \vee \neg P(2) \vee \neg P(3) \quad \equiv \exists x \neg P(x)$$

The same idea can be used to prove it for any number of variables.

Predicates - quantifier negation

No large birds live on honey.

$\neg\exists x\,(L(x) \wedge H(x))$

$\exists x\, P(x)$ means "$P(x)$ is true for some x."

What about $\neg\exists x\, P(x)$?

Not ["$P(x)$ is true for some x."]

"$P(x)$ is not true for all x."

$\forall x\, \neg P(x)$

So, $\neg\exists x\, P(x)$ is the same as $\forall x\, \neg P(x)$.

$\forall x\, \neg(L(x) \wedge H(x))$

Negations of Quantified Statements

There is a plant that can fly.

What is the negation of this statement?

Not exists a plant that can fly = every plant cannot fly.

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

(generalized) DeMorgan's Law

Say the domain has only three values.

$$\begin{aligned} \neg \exists x P(x) &\equiv \neg(P(1) \vee P(2) \vee P(3)) \\ &\equiv \neg(P(1) \vee P(2)) \wedge \neg P(3)) \\ &\equiv \neg P(1) \wedge \neg P(2) \wedge \neg P(3) \\ &\equiv \forall x \neg P(x) \end{aligned}$$

The same idea can be used to prove it for any number of variables.

Predicates - quantifier negation

So, $\neg\forall x\, P(x)$ is the same as $\exists x\, \neg P(x)$.

So, $\neg\exists x\, P(x)$ is the same as $\forall x\, \neg P(x)$.

General rule: to negate a quantifier, move negation to the right, changing quantifiers as you go.

Predicates - quantifier negation

No large birds live on honey.

$\neg\exists x\,(L(x) \wedge H(x)) \equiv \forall x\,\neg(L(x) \wedge H(x))$	Negation rule
$\equiv \forall x\,(\neg L(x) \vee \neg H(x))$	DeMorgan's
$\equiv \forall x\,(L(x) \rightarrow \neg H(x))$	Subst for $\rightarrow$

What's wrong with this proof?

Predicates - quantifier negation

Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.