Fourier Series

Leibnitz Successive differentiation :

$$(uv)_n = u_n v + n u_{n-1} v_1 + \frac{n(n-1)}{2!} u_{n-2} v_2 + \frac{n(n-1)(n-2)}{3!} u_{n-3} v_3 + \cdots + u v_n$$

Leibnitz successive Integration :

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$u' \to$ differentiation

$v_1 \to$ Integration

LIATE

- L → (log, ln) logarithm
- I → Inverse (x^{-1}) sin
- A → Algebric ($x^3, 2x^5$)
- T → trigonomentry ($\sin x, \cos x$)
- E → Exponential (e^x, a^x)

(trigono.) T E (exponential)

this two together then it will not work / invalid

Ex: $\int x^4 \cos x\,dx$

$$= x^4 \cdot (\sin x) - 4x^3 \cdot (-\cos x) + 12x^2 \cdot (-\sin x) - 24x(+\cos x) + 24 \cdot (\sin x) - 0$$

$$\begin{cases} \sin nx\,dx = -\dfrac{\cos nx}{n} \\ \sin n\pi = 0 \\ \sin 2n\pi = 0 \\ \cos \pi = (-1)^n \\ \cos 2n\pi = (-1)^{2n} \end{cases}$$

Some Important formulas

$\sin n\pi = 0$; $n \in I_+$ (Positive integer)

$\cos n\pi = (-1)^n$; $n \in$ positive integer (I_+)

$\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + c$

$\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}[a\sin bx - b\cos bx]}{a^2+b^2} + c$

$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0$; if $\boxed{m \neq n}$

$\int_{-\pi}^{\pi} \sin mx \cdot \sin nx \cdot dx = \pi$; if $\boxed{m = n}$

$\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = 0$; if $\boxed{m \neq n}$

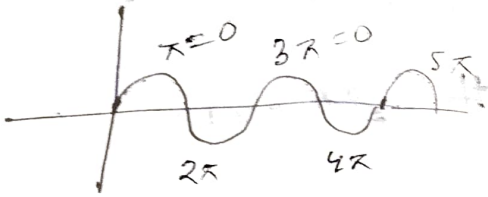
$\int_{-\pi}^{\pi} \cos mx \cdot \cos nx \, dx = \pi$; if $\boxed{m = n}$

$\int_{-\pi}^{\pi} \sin mx \cdot \cos nx \, dx = 0$ | $\boxed{m \neq n}$ (odd) | $\boxed{m = n}$

$\int_{-\pi}^{\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = 0$

$$\# \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \quad \Big| \; m \neq n$$

$$= \int_{-\pi}^{\pi} \sin 3x \sin 2x \, dx$$



$$= \frac{1}{2} \int_{-\pi}^{\pi} 2 \sin 3x \sin 2x \, dx \qquad \sin n\pi = 0$$

$$= \frac{1}{2} \left[\int_{-\pi}^{\pi} \{ \cos(3x-2x) - \cos(3x+2x) \} dx \right]$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \cos x \, dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos 5x \, dx$$

$$= \frac{1}{2} \left[\sin x \right]_{-\pi}^{\pi} - \frac{1}{2} \left[\frac{\sin 5x}{5} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} \left[\left[\sin \pi - \sin(-\pi) \right] \right] - \frac{1}{10} \left[\sin 5\pi - \sin(-5\pi) \right]$$

$$= 0$$

$m = n$

$$\int_{-\pi}^{\pi} \sin 3x \cdot \sin 3x \, dx$$

$$\boxed{\begin{aligned} 1 - \cos 2\theta &= 2\sin^2\theta \\ 1 + \cos 2\theta &= 2\cos^2\theta \end{aligned}}$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\sin^2 3x \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2 \cdot 3x) \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 6x \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 6x \, dx$$

$$= \frac{1}{2}\left[x\right]_{-\pi}^{\pi} - \frac{1}{2}\left[\frac{\sin 6x}{6}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\{\pi - (-\pi)\} - 0$$

$$= \frac{2\pi}{2}$$

$$= \pi$$

Fourier Series:

A Function $f(x)$ defined in finite interval and expression of sine and cosine function, then the expression is called fourier series.

(Periodic)

There are two types of ~~fourier~~ fourier series:

(1) Full range fourier series with period $2L$ or range $-L$ to L or 0 to $2L$.

Here, L = "+" values

(2) Half range fourier series with period 0 to L.

$L - (-L) = 2L$

Full range fourier series:

Suppose $f(x)$ be any periodic function defined in the intervals $-L$ to L, with period $2L$ then the fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$n = 1, 2, 3, \ldots$

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{L} + a_2 \cos\frac{2\pi x}{L} + a_3 \cos\frac{3\pi x}{L} + \cdots + b_1 \sin\frac{\pi x}{L} + b_2 \sin\frac{2\pi x}{L} + b_3 \sin\frac{3\pi x}{L} + \cdots$$

function defined in the intervals $-\pi$ to π with period $2L$

if it will define in question

$2L = \pi - (-\pi) = 2\pi$

$\therefore L = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad [\because L = \pi]$$

~~where~~ a_n ~~and~~ ~~bn~~

where, a_0, a_n and b_n are called fourier coefficent.

$$a_0 = \frac{1}{2} \int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin nx\, dx$$

if we consider a_0 ~~rather then~~ then it will be $\frac{1}{2L}$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Q: From fourier series determine the coefficient. W

Process of determining the coefficent a_0, a_n, b_n.

→ Let us first determine a_0

From the Fourier series we know

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

Integrating both side of (1) with respect to x within limit $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\int_{-\pi}^{\pi} a_n \cos nx\, dx + \int_{-\pi}^{\pi} b_n \sin nx\, dx \right] \longrightarrow (2)$$

Now, $\frac{a_0}{2} \int_{-\pi}^{\pi} dx$

$$= \frac{a_0}{2} [x]_{-\pi}^{\pi}$$

$$= \frac{a_0}{2} [\pi - (-\pi)]$$

$$= \frac{a_0}{2} \cdot 2\pi$$

$$= a_0 \pi$$

Now, $a_n \int_{-\pi}^{\pi} \cos nx \, dx = a_n \left[\frac{\sin nx}{n} \right]_{-\pi}^{\pi}$

$= a_n \cdot 0$

$= 0$

Again, $b_n \int_{-\pi}^{\pi} \sin nx \, dx = 0$ [$\because$ odd function]

From ② No equation,

$$\int_{-\pi}^{\pi} F(x) \, dx = a_0 \pi + (0+0)$$

$$\therefore \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \, dx$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \Big|_{-\pi \text{ to } \pi}$$

Process of determining the coefficient a_0, a_n and b_n

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Determine a_n:

From the definition of fourier series we know that,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

Multiplying both side by $\cos mx$ [m is fixed value] and then integrate both side with respect to x within $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x) \cos mx\, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \cos mx\, dx + \sum_{n=1}^{\infty} \left[\int_{-\pi}^{\pi} a_n \cos mx \cos mx\, dx + \int_{-\pi}^{\pi} b_n \sin nx \cos mx\, dx \right]$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mx\,dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos nx \cos mx\,dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin nx \cos mx\,dx \quad \text{--- (2)}$$

Now, $\int_{-\pi}^{\pi}\cos mx\,dx = 0$

and $\int_{-\pi}^{\pi}\cos nx\cos mx\,dx = \begin{cases} 0 ; & m \neq n \\ \pi ; & m = n. \end{cases}$

and $\int_{-\pi}^{\pi}\sin nx\cos mx\,dx = 0$

Put this value in equation (2)

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\cdot 0 + a_m \pi + 0$$

$a_m = a_n$ (same thing) as when $m = n$ then ans. π

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$\cos nx \approx \cos mx$

$$\boxed{a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$$

Determining b_n:

from the definition of fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

Multiplying both side by $\sin mx$ [m is any fixed value] and then integrate both side with respect to x within $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x) \cdot \sin mx \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \sin mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx$$

$$+ \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx \longrightarrow (2)$$

Now; $\frac{a_0}{2} \int_{-\pi}^{\pi} \sin mx \, dx = 0$

and $\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0$

and $\int_{-\pi}^{\pi} \sin nx \sin mx = \begin{cases} 0 ; & m \neq n \\ \pi ; & m = n \end{cases}$

Put the above value in equation (2)

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \frac{a_0}{2} \cdot 0 + \sum_{n=1}^{\infty} a_n \cdot 0 + b_m \cdot \pi$$

$$\therefore b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx$$

(same)

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx \, dx$$

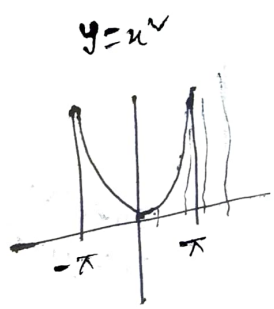
Question:

From fourier series determine the coefficients.

" " " a_0, a_n and b_n.

Problem - 1

Find the fourier series of $f(x) = x^2$ from $-\pi$ to π.



Soln:

Given, $f(x) = x^2$

From the definition of fourier series

we know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\Rightarrow x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad \text{--- (1)}$$

	-1	x	y	z	3	4
	-x	y	1	4	9	16

$a_1 \cos x + a_2 \cos 2x$

$a_3 \cos 3x + \cdots$

$\frac{a_0}{2} + \cdots + b_1 \sin x + b_2 \sin 2x$

$+ b_3 \sin 3x + \cdots$

$\frac{a_0}{2}$

where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi}\left[\pi^3 - (-\pi)^3\right]$$

$$= \frac{1}{3\pi}\cdot 2\pi^3$$

$$\therefore \boxed{a_0 = \frac{2}{3}\pi^2}$$

Again, ~~and~~ $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cdot\cos nx\,dx$$

Successive integration

$$= \frac{1}{\pi}\left[x^2\cdot\frac{\sin nx}{n} - 2x\cdot\frac{1}{n}\left(\frac{-\cos nx}{n}\right) + 2\cdot\frac{1}{n^2}\left(\frac{-\sin nx}{n}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cdot x^2\sin nx + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left\{\pi^2 \sin n\pi - (-\pi)^2 \sin n(-\pi)\right\} + \frac{2}{n^2}\left\{\pi \cos n\pi - (-\pi)\cos n(-\pi)\right\} - \frac{2}{n^3}\left\{\sin n\pi - \sin n(-\pi)\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\left(\pi^2 \sin n\pi + \pi^2 \sin n\pi\right) + \frac{2}{n^2}\left(\pi \cos n\pi + \pi \cos n\pi\right) - \frac{2}{n^3}\left(\sin n\pi + \sin n\pi\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cdot 0 + \frac{2}{n^2}\cdot 2\pi(-1)^n - \frac{2}{n^3}\cdot 0\right]$$

$$a_n = \frac{1}{\pi}\cdot\frac{4\pi(-1)^n}{n^2}$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^n}$$

Again, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin x\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx = 0$$

Put the value of a_0, a_n and b_n in (1)

$$x^2 = \frac{1}{2}\cdot\frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n \cos nx + 0\cdot \sin nx\right)$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$\therefore x^2 = \frac{\pi^2}{3} + \frac{4\cdot(-1)^2}{1^2}\cos x + \frac{4(-1)^2}{2^2}\cos 2x + \frac{4(-1)^3}{3^2}\cos 3x + \cdots$$

Ans:

14.012.19

Lec-08

Find the fourier series of the function $f(x) = x + x^2$.

in the interval $-\pi$ to π or $-\pi < x < \pi$

or,

Find the fourier series of sines and cosine of multiples of x which will represent $x + x^2$ in the interval $-\pi$ to π.

or,

Represents a signal $f(x) = x + x^2$ by the combination of basic sine and cosine in the intervals $-\pi$ to π or $-\pi < x < \pi$

<u>Soln:</u> Given, $f(x) = x + x^2$

From fourier series ~~in the intervals~~ we know,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

$$x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx \longrightarrow (2)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx \longrightarrow (3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \longrightarrow (4)$$

From equation (2)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} \quad \left[\because \int_{-\pi}^{\pi} x\,dx = \text{odd} = 0\right]$$

$$= \frac{1}{3\pi}\left[\pi^3 - (-\pi^3)\right]$$

$$= \frac{1}{3\pi}\cdot(\pi^3 + \pi^3)$$

$$= \frac{1}{3\pi} \times 2\pi^3$$

for exm 2/3 mathn

$$a_0 = \frac{2\pi^2}{3}$$

From (3)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} \underset{u}{x^2}\,\underset{v}{\cos nx}\,dx$$

$$= 0 + \frac{1}{\pi}\left[x^2\cdot\frac{\sin nx}{n} - 2x\cdot\frac{1}{n}\left(-\frac{\cos nx}{n}\right) + 2\cdot 1\cdot\frac{1}{n^2}\left(-\frac{\sin nx}{n}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cdot x^2 \sin nx + \frac{2}{n^2} x\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\{\pi^2 \sin n\pi - (-\pi)^2 \sin n(-\pi)\} + \frac{2}{n^2}\{\pi\cos n\pi - (-\pi)\cos n(-\pi)\} - \frac{2}{n^3}\{\sin n(\pi) - \sin n(-\pi)\}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}(\pi^2 \sin n\pi + \pi^2 \sin n\pi) + \frac{2}{n^2}(\pi\cos n\pi + \pi\cos n\pi) - \frac{2}{n^3}(\sin n\pi + \sin n\pi)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cdot 0 + \frac{2}{n^2}\{\pi(-1)^n\} - \frac{2}{n^3}\cdot 0\right.$$

$$= \frac{1}{\pi}\cdot\frac{2}{n^2}\cdot 2\cdot(-1)^n$$

$$\therefore a_n = \frac{4}{n^2}(-1)^n$$

A.W ($b_n = 2$)

From equation (4)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

not connect

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\, dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx$$

$$= \cancel{\frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\, dx +}$$

$$= 0 + \frac{1}{\pi}\left[x^2 \cdot \left(\frac{-\cos nx}{n}\right) - 2x \cdot \frac{1}{n} \cdot \left(\frac{-\sin nx}{n}\right) + 2 \cdot 1 \cdot \frac{1}{n^2} \cdot \left(\frac{\cos nx}{n}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\{-x^2 \cos nx\} + \frac{2}{n^2} x \sin nx + \frac{2}{n^3} \cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\{-\pi^2 \cos n\pi\} + \frac{2}{n^2} \cdot \pi \sin n\pi + \frac{2}{n^3} \cos n\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\{(-\pi^2) \cdot (-1)^n\} + 0 + \frac{2}{n^3} \cdot (-1)^n\right]$$

$$= -\frac{\pi}{n} \cdot (-1)^n + \frac{2}{n^3}(-1)^n$$

$$= (-1)^n \left(-\frac{\pi}{n} + \frac{2}{n^3}\right)$$

connect

From equ (4)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx + \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x \frac{(-\cos nx)}{n} - 1 \cdot \frac{1}{n}\left(\frac{-\sin nx}{n}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n} x \cos nx + \frac{1}{n^2} \sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\{\pi \cos n\pi - (-\pi)\cos n(-\pi)\} + \frac{1}{n^2}\{\sin n\pi - \sin n(-\pi)\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n} \pi \cos n\pi + \pi \cos n\pi + \frac{1}{n^2} \sin n\pi + \sin n\pi\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\{\pi(-1)^n + \pi(-1)^n\} + \frac{1}{n^2} \cdot 0\right]$$

$$= \frac{1}{\pi} \cdot \left(-\frac{1}{n}\right) 2\pi(-1)^n = -\frac{2}{n}(-1)^n$$

Put values in (1) no equ →

$$x + x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left[\frac{4}{n^2}(-1)^n \cos nx + \left\{-\frac{2}{n}(-1)^n\right\} \sin nx\right]$$

Ans:

17.012.19

Lec-09

$\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

$\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$

$\int u v \, dx$ → this formula will not work

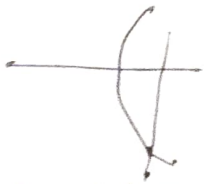
LIATE $\int e^{2x} \sin 3x \, dx$ → cyclic integration

<u>Example: 1</u>

$$\int_{-\pi}^{\pi} e^{2x} \sin 3x \, dx$$

$$= \left[\frac{e^{2x}[2\sin 3x - 3\cos 3x]}{2^2+3^2} \right]_{-\pi}^{\pi}$$

Obtain the fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.



$-L$ to L

$0 < x < 4\pi$ $\quad 2\pi$.

$2\pi - 0 = 2\pi$

$2L = 2\pi$

$\therefore L = \pi$

to determine limit

$0 < x < 4\pi$

$2L = 4\pi$

$\therefore L = 2\pi$ ✓

$2L = \pi - 0 = \pi$

$\therefore L = \frac{\pi}{2}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

For limit $-\pi < x < \pi$ & $0 < x < 2\pi$ only

$2L = 2\pi$
$\therefore L = \pi$

Soln: let, $f(x) = e^{-x}$

From the fourier series we know that

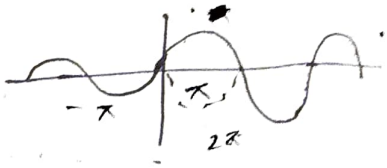
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\Rightarrow e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

Here, $$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx \longrightarrow (2)$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx \longrightarrow (3)$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx \longrightarrow (4)$$



Now from (2)

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} e^{-x}\, dx$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{-1} \right]_0^{2\pi}$$

$$= -\frac{1}{\pi} \left[e^{-x} \right]_0^{2\pi} = -\frac{1}{\pi} (e^{-2\pi} - e^{0})$$

$$\therefore a_0 = -\frac{1}{\pi} (e^{-2\pi} - 1) = \frac{1}{\pi} (1 - e^{-2\pi})$$

From equ (3)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-1\cdot x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}(-1\cdot\cos nx + n\sin nx)}{(-1)^2+n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-x}\,n\sin nx - e^{-x}\cos nx\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\cdot\left[n(e^{-2\pi}\cdot\sin n\cdot 2\pi - e^{0}\sin n\cdot 0) - (e^{-2\pi}\cos n\cdot 2\pi - e^{0}\cos n\cdot 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[n(e^{-2\pi}\cdot\sin 2n\pi - e^{0}\cdot\sin 0) - (e^{-2\pi}\cdot\cos 2\cdot n\pi - 1)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left\{-e^{-2\pi}(2\cos^2 n\pi - 1) - 1\right\}$$

$$= \frac{1}{\pi(1+n^2)}\left[-e^{-2\pi}\cdot 2\cdot(\cos n\pi)^2 - 2\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[-2e^{-2\pi}\cdot(-1)^{2n} - 2\right]$$

$$\therefore a_n = \frac{-2}{\pi(1+n^2)}\left[e^{-2\pi}(-1)^n + 1\right]$$

$$\therefore a_n = \frac{-2}{\pi(1+n^2)}\left(e^{-2\pi}+1\right)$$

$1+\cos 2A = 2\cos^2 A$

$\therefore \cos 2A = 2\cos^2 A - 1$

$\cos n\pi = (-1)^n$

$\Downarrow$

$\cos 2n\pi = (-1)^{2n}$

Again, $b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-1x}\sin n\cdot x\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-1x}(-1\sin nx - n\cos nx)}{(-1)^2+n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[-e^{-x}\cdot\sin nx - ne^{-x}\cos nx\right]_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[-(e^{-2\pi}\sin n\cdot 2\pi - e^{0}\sin n\cdot 0) - n(e^{-2\pi}\cos n\cdot 2\pi - e^{0}\cos n\cdot 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\cdot\left[-n\{e^{-2\pi}(-1)^{2n} - 1\}\right]$$

$$\therefore b_n = \frac{1}{\pi(1+n^2)}\left[-n\{e^{-2\pi}\underline{(-1)^{2n}} - 1\}\right]$$

Hence,
$(-1)^{2n} = 1$

Hence, $n = 1, 2, 3 \ldots$

$2 \times n =$ even

so, $(-1)^{2n} =$ even

(from book)

$$b_n = \frac{1}{\pi(1+n^2)}\left[-n\{e^{-2\pi} - 1\}\right]$$

Put the value of a_0, a_n and b_n in equ (1)

$$e^{-x} = \frac{1/\pi(1-e^{-2\pi})}{2} + \sum_{n=1}^{\infty}\left[\underbrace{\left\{\frac{-3}{\pi(1+n^2)}\left[e^{-2\pi}(-1)^n + 1\right]\right\}}_{a_n}\cos nx + \underbrace{\frac{1}{\pi(1+n^2)}\left(-n(e^{-2\pi} - 1)\right)}_{b_n}\sin nx\right]$$

Ans:

$\boxed{\text{A.W}}$

① Find the fourier series for $f(x) = e^x$ in the intervals $-\pi$ to π.

Soln: let, $f(x) = e^x$

From the fourier series we know that,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

Here, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \longrightarrow (2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \longrightarrow (3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx \longrightarrow (4)$$

From equ (2) $\rightarrow$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\, dx$$

$$= \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$\therefore a_0 = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^x(1 \cdot \cos nx + n \sin nx)}{(1)^2 + n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)} \left[\{ e^x (\cos nx + n \sin nx) \} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)} \left[\{ e^{\pi} (\cos n\pi - e^{-\pi} \cos n(-\pi) \} - \{ e^{\pi} n \sin n \cdot \pi - e^{-\pi} n \sin n(-\pi) \} \right]$$

$$= \frac{1}{\pi(1+n^2)} \left[\{ e^{\pi} (-1)^n - e^{-\pi} (-1)^n \} - 0 \right]$$

$$\therefore a_n = \frac{1}{\pi(1+n^2)} \left[e^{\pi} - e^{-\pi} \right]$$

From equ (4)

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx = \frac{1}{\pi} \left[\frac{e^x (1 \cdot \sin nx - n \cos nx)}{(1)^2 + n^2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)} \left[\{ e^x (\sin nx - n \cos nx) \} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(1+n^2)} \left[\{ e^{\pi} \sin n\pi - e^{-\pi} \sin n(-\pi) \} - \{ e^{\pi} \cdot n \cos n\pi - e^{-\pi} \cdot n \cos n(-\pi) \} \right]$$

$$= \frac{1}{\pi(1+n^2)} \left[0 - \{ e^{\pi} \cdot n(-1)^n - e^{-\pi} \cdot n(-1)^n \} \right]$$

$$\therefore b_n = \frac{1}{\pi(1+n^2)} \left[n e^{\pi} - n e^{-\pi} \right]$$

Put the value of a_0, a_n and b_n in equation (1) $\rightarrow$

$$e^x = \frac{\frac{1}{\pi}(e^{\pi} - e^{-\pi})}{2} + \sum_{n=1}^{\infty} \left[\left\{ \frac{1}{\pi(1+n^2)} (e^{\pi} - e^{-\pi}) \right\} \cos nx + \left\{ \frac{1}{\pi(1+n^2)} (n e^{\pi} - n e^{-\pi}) \right\} \sin nx \right]$$

Ans.

② Find the fourier series for $f(x) = x\cos x$ in the intervals $-\pi$ to π.

<u>Soln:</u> let, $f(x) = x\cos x$

From fourier series we know that,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

Here, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \longrightarrow (2)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx \longrightarrow (3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx \longrightarrow (4)$$

From equ ②

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x\, dx$$

$$= \frac{1}{\pi}\left[x\cdot(\sin x) - 1\cdot(-\cos x)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[x\sin x + \cos x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\{\pi \sin\pi - \pi\sin(-\pi)\} - \{\cos\pi - \cos(-\pi)\}\right]$$

$$= \frac{1}{\pi}\left[(0-0) - (1-1)\right]$$

$$= 0$$

$\therefore a_0 = 0$

From equ (3)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x \cdot \cos nx \, dx$$

$$= \frac{1}{\pi}\cdot\frac{x}{2}\int_{-\pi}^{\pi} 2\cos x \cdot \cos nx \, dx$$

$$= \frac{1}{\pi}\frac{x}{2}\left[\int_{-\pi}^{\pi}\{\cos(x-nx) + \cos(x+nx)\}\, dx\right]$$

$$= \frac{x}{2\pi}\int_{-\pi}^{\pi}\cos(x-nx)\, dx + \frac{x}{2\pi}\int_{-\pi}^{\pi}\cos(x+nx)\, dx$$

$$= \frac{1}{2\pi}\left[\{\pi\cos(\pi-n\pi) - (-\pi)\cdot\cos(-\pi-n(-\pi)\}\right] +$$

$$\frac{1}{2\pi}\left[\pi\cos(\pi+n\pi) - (-\pi)\cos\{-\pi+n(-\pi)\}\right]$$

$$= \frac{1}{2\pi}\left[\pi\cos\pi - \pi\cos n\pi + \pi\cos\pi + \pi\cos n\pi\right] +$$

$$\frac{1}{2\pi}\left[\pi\cos\pi + \pi\cos n\pi + \pi\cos\pi + \pi\cos n\pi\right]$$

$$= \frac{1}{2\pi}\left[-1 - \pi(-1)^n + (-1) + \pi\cdot(-1)^n + (-1) + \pi(-1)^n + (-1)\right.$$

$$\left. + \pi(-1)^n\right] \cancel{(-1) + \pi(-1)^n]}$$

$$= \frac{1}{2\pi}\left[-1 - \pi(-1)^n - 1 + \pi(-1)^n - 1 + \pi(-1)^n - 1 + \pi(-1)^n\right]$$

$$= \frac{1}{2\pi}\left[2\pi(-1)^n - 4\right]$$

$$= \frac{2[\pi(-1)^n - 2]}{2\pi}$$

$$= \frac{\pi(-1)^n - 2}{\pi} = \frac{\pi\{(-1)^n - \frac{1}{\pi}2\}}{\pi} = (-1)^n - \frac{2}{\pi}$$

$$\therefore a_n = (-1)^n - \frac{2}{\pi}$$

From equation (4)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos x \sin nx \, dx$$

$\therefore b_n = 0$

Put the value of a_0, a_n and b_n in equ (1)

$$x\cos x = 0 + \sum_{n=1}^{\infty}\left[\left\{(-1)^n - \frac{2}{\pi}\right\}\cos nx + 0\right]$$

$$\therefore x\cos x = \sum_{n=1}^{\infty}\left[\left\{(-1)^n - \frac{2}{\pi}\right\}\cos nx\right]$$ Ans:

Lec-10

21.12.19
Sat-day

(5*) for exm (must 1 Question common)

Obtain the fourier series of the function

N.B: Piece wise Function

$$f(x) = \begin{cases} 0 ; & -\pi \le x \le 0 \\ 1 ; & 0 \le x \le \pi \end{cases}$$ lowest to highest range

$2L = 2\pi$
$\therefore L = \pi$

Soln: Given function $f(x) = \begin{cases} 0 ; & -\pi \le x \le 0 \\ 1 ; & 0 \le x \le \pi \end{cases}$

Here the period $2L = \{\pi - (-\pi)\}$

$2L = 2\pi$

$\therefore L = \pi$

From the definition of fourier series we know,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

Here, $L = \pi$

So, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right)$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$

In exam start from here (S.C) ✓

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \longrightarrow (2)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \longrightarrow (3)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \longrightarrow (4)$$

From equation (2)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} 0 \cdot dx + \frac{1}{\pi}\int_{0}^{\pi} 1 \cdot dx$$

$$= \frac{1}{\pi}\left[x\right]_0^{\pi} = \cancel{\frac{1}{\pi}[\pi]}$$

$$= \frac{1}{\pi}\left[\pi - 0\right]$$

$$= \frac{1}{\pi}\cdot\pi$$

$$= 1$$

From equation ③

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0.\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1.\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = \frac{1}{n\pi}\left[\sin n\pi - \sin 0\right]$$

$$= 0$$

$\therefore a_n = 0$

From equation ④

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{-1}{n\pi}\left[\cos n\pi - \cos n.0\right]$$

$$= \frac{-1}{n\pi}\left\{(-1)^n - 1\right\}$$

$$\therefore b_n = -\frac{1}{n\pi}\left\{(-1)^n - 1\right\}$$

Put the value of a_0, a_n & b_n in equ ① →

$$F(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[0.\cos nx + \left[-\frac{1}{n\pi}\left\{(-1)^n - 1\right\}\right]\sin nx\right]$$

$$\therefore F(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[-\frac{1}{n\pi}\left\{(-1)^n - 1\right\}\sin nx\right]$$

Ans:

Find the fourier series expansion of the function

$$F(x) = \begin{cases} 0 \; ; & -\pi < x \leq 0 \\ x \; ; & 0 < x < \pi \end{cases}$$

and prove that when $x=0$ then $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Part-1

Soln: We know,

From fourier series, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$

Here, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \longrightarrow (2)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \longrightarrow (3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \longrightarrow (4)$$

From equation (2) $\longrightarrow$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0\right]$$

$$= \frac{\pi}{2}$$

$\therefore a_0 = \frac{\pi}{2}$

From equation (3)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= 0 + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$\cancel{= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}}$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\frac{1}{n}\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi} = \frac{1}{\pi}\left[0 - 0 + \frac{1}{n^2}\cos n\pi - \cos 0\right]$$

$$\therefore a_n = \frac{1}{\pi}\left[\frac{1}{n^2}\{(-1)^n - 1\}\right]$$

From equation (4)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0 + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\frac{1}{n}\left(\frac{-\sin nx}{n}\right) + 0\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{1}{n}\pi\cos n\pi + \frac{1}{n}\cdot 0\cdot\cos n\cdot 0\right\} + \left\{\frac{1}{n^2}\sin n\cdot\pi - \frac{1}{n^2}\sin n\cdot 0\right\}\right]$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\{\pi(-1)^n\}\right] + 0$$

$$\cancel{= \frac{1}{\pi}\left[-\frac{1}{n}\{\pi(-1)^n\}\right]}$$

$$\therefore b_n = \frac{1}{\pi}\left[-\frac{1}{n}\{\pi(-1)^n\}\right] = -\frac{1}{n}(-1)^n$$

Put the value of a_0, a_n & b_n in equation (1) $\rightarrow$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\left[\frac{1}{\pi}\left\{\frac{1}{n^2}(-1)^n - 1\right\} \cos nx + \frac{1}{\pi}\left[-\frac{1}{n}\{(-1)^n\}\right] \sin nx \right]\right.$$

$$\text{or, } f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi}\left[(-1)^n - 1\right] \cos nx + \sum_{n=1}^{\infty} -\frac{1}{n}(-1)^n \sin nx$$

Part-2

For, $x = 0$

$$f(x) = \frac{\pi}{4} + \frac{1}{(1)^2\pi}\{(-1)^1 - 1\} \cos 1.x + \frac{1}{2^2\pi}((-1)^2 - 1) \cos 2x$$

$$+ \frac{1}{3^2\pi}((-1)^3 - 1) \cos 3x + \frac{1}{4^2\pi}((-1)^4 - 1) \cos 4x +$$

$$\frac{1}{5^2\pi}((-1)^5 - 1) \cos 5x + \dots + \left(-\frac{1}{1}\right)(-1)^1 \sin 1.x +$$

$$\left(-\frac{1}{2}\right)(-1)^2 \sin 2x + \left(-\frac{1}{3}\right)(-1)^3 \sin 3x + \dots + \dots$$

Now, put $x = 0$,

$$0 = \frac{\pi}{4} + \frac{1}{\pi}(-2) + \frac{1}{2^2\pi} \cdot 0 + \frac{1}{3^2\pi} \cdot (-2) + \frac{1}{4^2\pi} \cdot 0 + \frac{1}{5^2\pi} \cdot (-2)$$

$$+ \dots \dots + 0 + 0 + 0 + 0 + \dots$$

$$\Rightarrow 0 = \frac{\pi}{4} + \boxed{\frac{1}{\pi}(-2)} + \frac{1}{3^2\pi}(-2) + \frac{1}{5^2\pi}(-2) + \dots\dots$$

$$\Rightarrow 0 = \frac{\pi}{4} - 2 \cdot \frac{1}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\dots\right)$$

$$\Rightarrow \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}.$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

(Proved)

Here, $n = 1$, $\frac{1}{(2.1-1)^2} = \frac{1}{(2-1)^2} = \frac{1}{1^2}$

N.B.: $\frac{1}{(2.3-1)^2} = \frac{1}{(6-1)^2} = \frac{1}{5^2}$

Expand in a series of sines and cosines of multiplications of x the function given by

$$f(x) = \begin{cases} (x-\pi); & \text{when } -\pi < x < 0 \\ (\pi - x); & \text{when } 0 < x < \pi \end{cases}$$

range: $-\pi$ to π

value: $0, \pi, -\pi$

20 marks

this type of math (must)

also prove that when $x = \pm\pi$ and 0 then

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Part-1

Soln: Given Function $f(x) = \begin{cases} (x-\pi); & \text{when } -\pi < x < 0 \\ (\pi - x); & \text{when } 0 < x < \pi \end{cases}$

Here, $2L = \pi - (-\pi)$

$2L = 2\pi$

$\therefore L = \pi$

From the definition of fourier series we know that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \longrightarrow (1)$$

Here, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \longrightarrow (2)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx \longrightarrow (3)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx \longrightarrow (4)$$

From equation (2)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,\cancel{\cos}\cdot dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (x-\pi)\,dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\,dx - \frac{1}{\pi}\int_{-\pi}^{0} \pi\,dx + \frac{1}{\pi}\int_{0}^{\pi} \pi\,dx - \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} - \frac{1}{\pi}\cdot\pi\,[x]_{-\pi}^{0} + \frac{1}{\pi}\cdot\pi\,[x]_{0}^{\pi} - \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\,[x^2]_{-\pi}^{0} - [x]_{-\pi}^{0} + [x]_{0}^{\pi} - \frac{1}{2\pi}\,[x^2]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\,[0^2 - (-\pi)^2] - [0 - (-\pi)] + (\pi - 0) - \frac{1}{2\pi}(\pi^2 - 0^2)$$

$$= \frac{1}{2\pi}(-\pi)^2 - \pi + \pi - \frac{1}{2\pi}\cdot\pi^2$$

$$= -\frac{1}{2\pi}\cdot\pi^2 - \frac{1}{2\pi}\cdot\pi^2$$

$$= -2\cdot\frac{1}{2\pi}\cdot\pi^2$$

$$a_0 = -\pi$$

From equation (3)

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (x-\pi)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi-x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx - \frac{1}{\pi}\int_{-\pi}^{0} \pi\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \pi\cos nx\,dx$$

$$- \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - 1\cdot\frac{1}{n}\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} - \left[\frac{\sin nx}{n}\right]_{-\pi}^{0}$$

$$+ \left[\frac{\sin nx}{n}\right]_{0}^{\pi} - \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\frac{1}{n}\cdot\left(\frac{-\cos nx}{n}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}(x\sin nx) + \frac{1}{n^2}(\cos nx)\right]_{-\pi}^{0} - \frac{1}{n}\left[\sin nx\right]_{-\pi}^{0}$$

$$+ \frac{1}{n}\left[\sin nx\right]_{0}^{\pi} - \frac{1}{\pi}\left[\frac{1}{n}(x\sin nx) + \frac{1}{n^2}(\cos nx)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\{0\cdot\sin 0 - (-\pi)\sin n(-\pi)\} + \frac{1}{n^2}\{\cos 0 - \cos n(-\pi)\}\right]$$

$$- \frac{1}{n}\{\sin 0 - \sin n(-\pi)\} + \frac{1}{n}(\sin n\pi - \sin 0) - \frac{1}{\pi}$$

$$\left[\frac{1}{n}\{\pi\sin n\pi - 0\sin 0\} + \frac{1}{n^2}\{\cos n\pi - \cos 0\}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cdot\{0 - \pi\sin n\pi\} + \frac{1}{n^2}\{1-(-1)^n\}\right] - \frac{1}{n}(0+0) + \frac{1}{n}\cdot(0-0)$$

$$- \frac{1}{\pi}\left[\frac{1}{n}\cdot(0) + \frac{1}{n^2}\{(-1)^n - 1\}\right]$$

$$= \frac{1}{\pi}\cdot\frac{1}{n^2}\{1-(-1)^n\} - \frac{1}{\pi}\cdot\frac{1}{n^2}\{(-1)^n - 1\}$$

$$= \frac{1}{\pi n^2}\{1-(-1)^n\} + \frac{1}{n\pi^2}\{1-(-1)^n\}$$

$$= \frac{2}{\pi n^2}\{1-(-1)^n\}$$

$$\therefore a_n = \frac{2}{\pi n^2}\{1-(-1)^n\}$$

From equation (4)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (x-\pi)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx - \frac{1}{\pi}\int_{-\pi}^{0} \pi\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} \pi\sin nx\,dx$$

$$- \frac{1}{\pi}\int_{0}^{\pi} x\,\cancel{\cos}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\frac{1}{n}\left(\frac{-\sin nx}{n}\right)\right]_{-\pi}^{0} - \left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} +$$

$$\left[\frac{-\cos nx}{n}\right]_{0}^{\pi} - \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\frac{1}{n}\left(\frac{-\sin nx}{n}\right)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}(x\cos nx) + \frac{1}{n^2}(\sin nx)\right]_{-\pi}^{0} + \frac{1}{n}[\cos nx]_{-\pi}^{0}$$

$$-\frac{1}{n}[\cos nx]_{0}^{\pi} - \frac{1}{\pi}\left[-\frac{1}{n}(x\cos nx) + \frac{1}{n^2}(\sin nx)\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\{0.\cos 0 - (-\pi)\cos n(-\pi)\} + \frac{1}{n^2}\{\sin 0 - \sin(-\pi)\}\right]$$

$$+ \frac{1}{n}\{\cos 0 - \cos n(-\pi)\} - \frac{1}{\pi}\left[-\frac{1}{n}(\pi\cos n\pi - 0\cos 0) + \frac{1}{n^2}(\sin n\pi - \sin 0)\right]$$

$$-\frac{1}{n}(\cos n\pi - \cos 0)$$

$$= \frac{1}{\pi}\left[-\frac{1}{n}(0 + \pi\cos n\pi) + \frac{1}{n^2}\cdot 0\right] + \frac{1}{n}\{1-(-1)^n\}$$

$$-\frac{1}{n}\{(-1)^n - 1\} - \frac{1}{\pi}\left[-\frac{1}{n}(\pi\cos n\pi - 0) + \frac{1}{n^2}\cdot 0\right]$$

$$= \underline{-\frac{1}{\pi}\cdot\frac{1}{n}\cdot\pi\cos n\pi} + \frac{1}{n}\{1-(-1)^n\} + \frac{1}{n}\{1-(-1)^n\} - \underline{+\frac{1}{\pi\cdot n}\cdot\pi\cos n\pi}$$

$$= \frac{2}{n}\{1-(-1)^n\}$$

$$\therefore b_n = \frac{2}{n}\{1-(-1)^n\}$$

Put the value of ~~an~~ a_0, a_n and b_n in equ (1)

$$f(x) = -\frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi n^2}\{1-(-1)^n\}\cos nx + \frac{2}{n}\{1-(-1)^n\}\sin nx\right]$$

$$\therefore f(x) = -\frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\{1-(-1)^n\}\cos nx + \sum_{n=1}^{\infty}\frac{2}{n}\{1-(-1)^n\}\sin nx$$

Part - 2

For, $x = 0$

$$f(x) = \frac{\pi}{4} + -\frac{\pi}{2} + \frac{2}{\pi(1)^2}\{1-(-1)^1\} \cancel{\cos nx} \cdot \cos 1.x +$$

$$\cancel{+ \frac{2}{1}\{1-(-1)^1\} \sin 1x}$$

$$f(x) = -\frac{\pi}{2} + \frac{2}{\pi(1)^2}\{1-(-1)^1\} \cos 1.x + \frac{2}{\pi(2)^2}\{1-(-1)^2\}$$

$$\cos 2x + \frac{2}{\pi(3)^2}\{1-(-1)^3\} \cos 3x + \frac{2}{\pi(4)^2}$$

$$\{1-(-1)^4\} \cos 4x + \frac{2}{\pi(5)^2}\{1-(-1)^5\} \cos 5x$$

$$+ \dots\dots + \frac{2}{1}\{1-(-1)^n\} \sin 1x + \frac{2}{2}\{1-(-1)^2\} \sin 2x$$

$$+ \frac{2}{3}\{1-(-1)^3\} \sin 3x + \dots + \dots$$

Now, put $x = 0$

$$0 = -\frac{\pi}{2} + \left(\frac{2}{\pi} \times 2\right) \cancel{\cos x} + \frac{2}{4\pi} \times 0 + \left(\frac{2}{9\pi} \times 2\right) +$$

$$\frac{2}{16\pi} \times 0 + \frac{2}{25\pi} \times 2 + \dots\dots + 0 + 0 + 0 + 0 + \dots$$

$[\because \sin 0^\circ = 0]$

$[\because \cos 0^\circ = 1]$

$$0 = -\frac{\pi}{2} + \frac{4}{\pi} + \frac{4}{9\pi} + \frac{4}{25\pi} + \dots\dots$$

$$\Rightarrow \frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right)$$

$$\Rightarrow \therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots$$

For, $x = \pi$

$$\pi = -\frac{\pi}{2} - \frac{4}{\pi} - \frac{4}{9\pi} - \frac{4}{25\pi} - \cdots\cdots$$

$$\Rightarrow \pi - \frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\right)$$

$$\Rightarrow \frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\right)$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$$

For, $x = -\pi$

$$-\pi = -\frac{\pi}{2} - \frac{4}{\pi} - \frac{4}{9\pi} - \frac{4}{25\pi} + \cdots\cdots$$

$$\Rightarrow -\pi - \frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\right)$$

$$\Rightarrow \frac{\pi}{2} = \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\right)$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$$

(Proved)