Counting

Lecturer , CSE

Motivation

- Combinatorics is the study of collections of objects. Specifically, ***counting*** objects, ***arrangement***, ***derangement***, etc. along with their mathematical properties
- Counting objects is important in order to analyze algorithms and compute discrete probabilities
- Originally, combinatorics was motivated by gambling: counting configurations is essential to elementary probability
  - A simple example: How many arrangements are there of a deck of 52 cards?
- In addition, combinatorics can be used as a proof technique
  - A ***combinatorial proof*** is a proof method that uses counting arguments to prove a statement

Product Rule

- If two events are ***not*** **mutually exclusive (that is we do them separately)**, then we apply the product rule
- **Theorem**: Product Rule

  Suppose a procedure can be accomplished with two disjoint subtasks. If there are

  - n_1 ways of doing the first task and
  - n_2 ways of doing the second task,

  then there are $n_1.n_2$ ways of doing the overall procedure

Product Rule

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution: The procedure of assigning offices to these two employees consists of assigning an office to Sanchez, which can be done in 12 ways, then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways. By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

Product Rule

Counting Functions How many functions are there from a set with m elements to a set with n elements?

Solution: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdot \cdots \cdot n = n^m$ functions from a set with m elements to one with n elements. For example, there are $5^3 = 125$ different functions from a set with three elements to a set with five elements.

Product Rule

Counting One-to-One Functions How many one-to-one functions are there from a set with m elements to one with n elements?

Solution: First note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.

Now let $m \leq n$. Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$. There are n ways to choose the value of the function at a_1. Because the function is one-to-one, the value of the function at a_2 can be picked in $n - 1$ ways (because the value used for a_1 cannot be used again). In general, the value of the function at a_k can be chosen in $n - k + 1$ ways. By the product rule, there are $n(n - 1)(n - 2) \cdots (n - m + 1)$ one-to-one functions from a set with m elements to one with n elements.

For example, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions from a set with three elements to a set with five elements. ◀

Sum Rule (1)

- If two events ***are*** **mutually exclusive, that is, they cannot be done at the same time**, then we must apply the sum rule
- **Theorem**: Sum Rule. If
  - an event e_1 can be done in n_1 ways,
  - an event e_2 can be done in n_2 ways, and
  - e_1 and e_2 are mutually exclusive

  then the number of ways of both events occurring is $n_1 + n_2$

Sum Rule (1)

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Solution: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

Sum Rule (2)

- There is a natural generalization to any <u>sequence</u> of m tasks; namely the number of ways m mutually events can occur

$$n_1 + n_2 + \ldots + n_{m-1} + n_m$$

- We can give another formulation in terms of sets. Let $A_1, A_2, \ldots, A_m$ be <u>pairwise disjoint sets.</u> Then

$$\mathbf{|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| \cup |A_2| \cup \ldots \cup |A_m|}$$

(In fact, this is a special case of the general Principal of Inclusion-Exclusion (PIE))

More Complex Counting Problems

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution: Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, $P = P_6 + P_7 + P_8$. We will now find P_6, P_7, and P_8. Finding P_6 directly is difficult. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6. Hence,

$$P_6 = 36^6 - 26^6 = 2{,}176{,}782{,}336 - 308{,}915{,}776 = 1{,}867{,}866{,}560.$$

Similarly, we have

$$P_7 = 36^7 - 26^7 = 78{,}364{,}164{,}096 - 8{,}031{,}810{,}176 = 70{,}332{,}353{,}920$$

and

$$\begin{aligned} P_8 &= 36^8 - 26^8 = 2{,}821{,}109{,}907{,}456 - 208{,}827{,}064{,}576 \\ &= 2{,}612{,}282{,}842{,}880. \end{aligned}$$

Consequently,

$$P = P_6 + P_7 + P_8 = 2{,}684{,}483{,}063{,}360.$$

The Subtraction Rule (Inclusion–Exclusion for Two Sets)

- Suppose that a task can be done in one of two ways, but some of the ways to do it are common to both ways.
- In this situation, we cannot use the sum rule to count the number of ways to do the task.
- If we add the number of ways to do the tasks in these two ways, **we get an overcount of the total number of ways** to do it, because the ways to do the task that are common to the two ways are counted twice.

The Subtraction Rule (Inclusion–Exclusion for Two Sets)

- To correctly count the number of ways to do the two tasks, we must subtract the number of ways that are counted twice.
- This leads us to an important counting rule ***THE SUBTRACTION RULE*** or ***principle of inclusion–exclusion***

The Subtraction Rule

THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

The Subtraction Rule

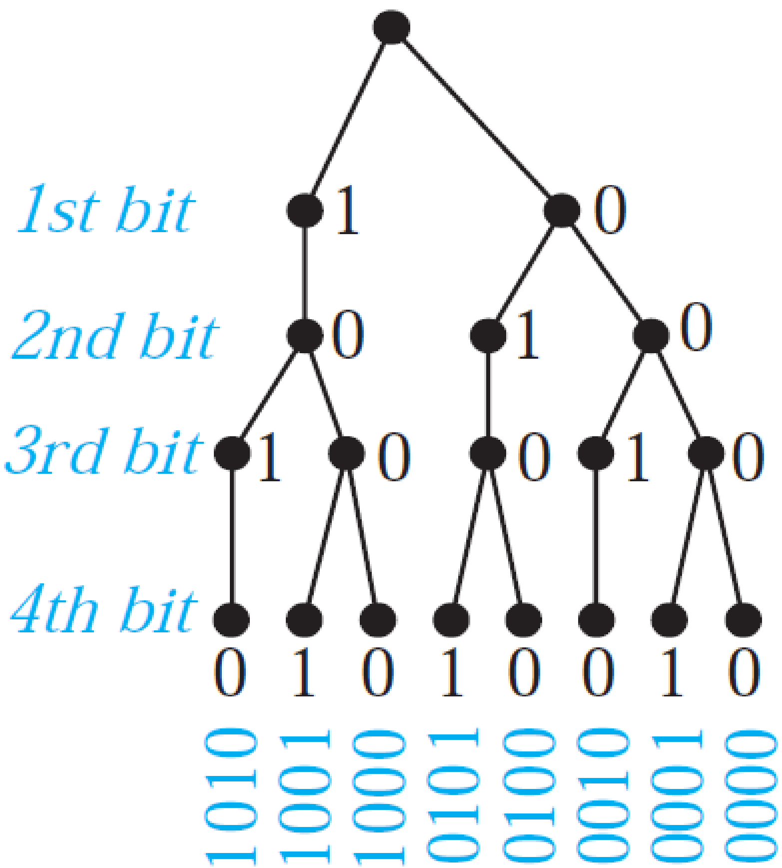
- How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
- We can construct a bit string of length eight that begins with a 1 in $2^7 = 128$ ways
- Similarly, we can construct a bit string of length eight ending with the two bits 00, in $2^6 = 64$ ways.
- Some of the ways to construct a bit string of length eight starting with a 1 are the same as the ways to construct a bit string of length eight that ends with the two bits 00 ,in $2^5 = 32$ ways.
- So, the solution is $= 128 + 64 – 32 = 160$

Solving Counting Problems Using Tree Diagrams

- A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the endpoints of other branches.
- To use trees in counting, we use a **branch to represent each possible choice**.
- We **represent the possible outcomes by the leaves**, which are the endpoints of branches not having other branches starting at them.

Solving Counting Problems Using Tree Diagrams

- How many bit strings of length four do not have two consecutive 1s?
- ***Answer : 8***



Pigeonhole Principle

- This principal is a fundamental tool of elementary discrete mathematics.
- It is also known as the ***Dirichlet Drawer Principle or Dirichlet Box Pinciple***

Pigeonhole Principle

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it.

Pigeonhole Principle

THE PIGEONHOLE PRINCIPLE If k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Proof : Pigeonhole Principle

- We prove the pigeonhole principle using a proof by contraposition.
- Suppose that ***none*** of the ***k boxes*** contains ***more than one object***.
- Then the ***total number of objects*** would be ***at most k***.
- This is a ***contradiction***, because there are ***at least k + 1 objects***.

Pigeonhole Principle

EXAMPLE 1 Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.

EXAMPLE 2 In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.

Pigeonhole Principle

- Prove that "A function f from a set with $k + 1$ or more elements to a set with k elements is not one-to-one" using ***Pigeonhole Principle***.

Generalized Pigeonhole Principle

THE GENERALIZED PIGEONHOLE PRINCIPLE If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

EXAMPLE 5 Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

Some Elegant Applications of the Pigeonhole Principle

During a month with 30 days, a baseball team plays at least one game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Solution: Let a_j be the number of games played on or before the jth day of the month. Then $a_1, a_2, \ldots, a_{30}$ is an increasing sequence of distinct positive integers, with $1 \leq a_j \leq 45$. Moreover, $a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ is also an increasing sequence of distinct positive integers, with $15 \leq a_j + 14 \leq 59$.

The 60 positive integers $a_1, a_2, \ldots, a_{30}, a_1 + 14, a_2 + 14, \ldots, a_{30} + 14$ are all less than or equal to 59. Hence, by the pigeonhole principle two of these integers are equal. Because the integers a_j, $j = 1, 2, \ldots, 30$ are all distinct and the integers $a_j + 14$, $j = 1, 2, \ldots, 30$ are all distinct, there must be indices i and j with $a_i = a_j + 14$. This means that exactly 14 games were played from day $j + 1$ to day i. ◀

Some Elegant Applications of the Pigeonhole Principle

Show that among any $n+1$ positive integers not exceeding $2n$ there must be an integer that divides one of the other integers.

Every sequence of n^2+1 distinct real numbers contains a subsequence of length $n+1$ that is either strictly increasing or strictly decreasing.

Permutations and Combinations

- Many counting problems can be solved by finding **the number of ways to arrange a specified number of distinct elements of a set of a particular size**, where **<u>the order of these elements matters</u>**. ***Permutations***
- Many other counting problems can be solved by **finding the number of ways to select a particular number of elements from a set of a particular size**, **<u>where the order of the elements selected does not matter</u>**. ***Combinations***

Permutations

- In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

- By the product rule, there are **$5 \cdot 4 \cdot 3 = 60$** ways to **select three students from a group of five students**.
- By the product rule, there are **$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$** ways to **arrange all five students**

Permutations

- A **permutation** of a set of distinct objects is an ordered arrangement of these objects.
- An ordered arrangement of r elements of a set is called an ***r*-permutation**.
- The number of ***r*-permutations of a set with *n* elements** is denoted by ***P(n, r)***.

Permutations

Let $S = \{a, b, c\}$. The 2-permutations of S are the ordered arrangements $a, b; a, c; b, a; b, c; c, a;$ and c, b. Consequently, there are six 2-permutations of this set with three elements. There are always six 2-permutations of a set with three elements. There are three ways to choose the first element of the arrangement. There are two ways to choose the second element of the arrangement, because it must be different from the first element. Hence, by the product rule, we see that $P(3, 2) = 3 \cdot 2 = 6$. the first element. By the product rule, it follows that $P(3, 2) = 3 \cdot 2 = 6$.

Permutations

THEOREM 1 If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Permutations

COROLLARY 1 If n and r are integers with $0 \le r \le n$, then $P(n, r) = \dfrac{n!}{(n-r)!}$.

Combinations

THEOREM 2 The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n,r) = \frac{n!}{r!\,(n-r)!}.$$

$$P(n,r) = C(n,r) \cdot P(r,r).$$

Combinations

COROLLARY 2 Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Combinatorial Proof

A *combinatorial proof* of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called *double counting proofs* and *bijective proofs*, respectively.

Binomial Coefficients and Identities

- A **binomial** expression is simply the sum of two terms, such as *x* + *y*. (The terms can be products of constants and variables, but that does not concern us here.)

THEOREM 1 THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Binomial Coefficients and Identities

EXAMPLE 3 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Solution: From the binomial theorem it follows that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.$$

THEOREM 1 **THE BINOMIAL THEOREM** Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Binomial Coefficients and Identities

THEOREM 2 **PASCAL'S IDENTITY** Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Permutations with Repetition

THEOREM 1 The number of r-permutations of a set of n objects with repetition allowed is n^r.

Combinations with Repetition

THEOREM 2 There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

Combinations with Repetition

EXAMPLE 5 How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers?

Solution: To count the number of solutions, we note that a solution corresponds to a way of selecting 11 items from a set with three elements so that x_1 items of type one, x_2 items of type two, and x_3 items of type three are chosen. Hence, the number of solutions is equal to the number of 11-combinations with repetition allowed from a set with three elements. From Theorem 2 it follows that there are

$$C(3 + 11 - 1, 11) = C(13, 11) = C(13, 2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$$

Generalized Permutations and Combinations

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

The End