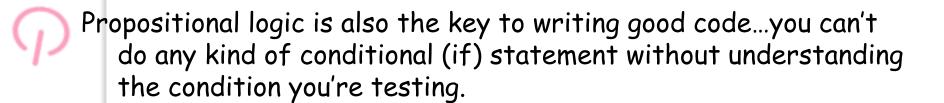


CS103 Discrete Mathematics

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Propositional Logic - say a bit...

This week we're using propositional logic as a foundation for formal proofs.





Propositional Logic - 2 more defn...

A tautology is a proposition that's always TRUE.

A contradiction is a proposition that's always FALSE.



р	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
Т	F	Т	F
F	Т	Т	F



Propositional Logic - an unfamous ≡

if NOT (blue AND NOT red) OR red then ...

$$\neg (p \land \neg q) \lor q \equiv \neg p \lor q$$



$$\neg (p \land \neg q) \lor q \equiv (\neg p \lor \neg \neg q) \lor q$$
 DeMorgan's

$$\equiv (\neg p \lor q) \lor q$$

Double negation

$$\equiv \neg p \lor (q \lor q)$$

Associativity

$$= \neg p \lor q$$

Idempotent





Propositional Logic - one last proof

- Show that $[p \land (p \rightarrow q)] \rightarrow q$ is a tautology.
- We use \equiv to show that $[p \land (p \rightarrow q)] \rightarrow q \equiv T$.

$$[p \land (p \to q)] \to q$$

substitution for \rightarrow

distributive

uniqueness

identity

substitution for \rightarrow

DeMorgan's

associative

excluded middle

domination



Predicate Logic - everybody loves somebody

Proposition, YES or NO?

$$3 + 2 = 5$$
 YES

$$X + 2 = 5$$

$$X + 2 = 5$$
 for any choice of X in $\{1, 2, 3\}$

$$X + 2 = 5$$
 for some X in $\{1, 2, 3\}$

YES

YES





Predicate Logic - everybody loves somebody



Alicia eats pizza at least once a week.

Garrett eats pizza at least once a week.

Allison eats pizza at least once a week.

Gregg eats pizza at least once a week.

Ryan eats pizza at least once a week.

Meera eats pizza at least once a week.

Ariel eats pizza at least once a week.

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Predicates

Alicia eats pizza at least once a week.

•

Define:

EP(x) = "x eats pizza at least once a week." Universe of Discourse - x is a student in cse1207

A predicate, or propositional function, is a function that takes some variable(s) as arguments and returns True or False.

Note that EP(x) is not a proposition, EP(Ariel) is.





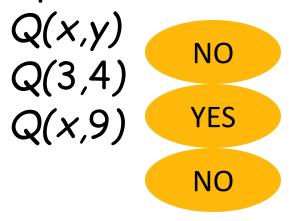


Predicates

Suppose
$$Q(x,y) = "x > y"$$



Proposition, YES or NO?



Predicate, YES or NO?

$$Q(x,y)$$
 YES
 $Q(3,4)$ NO
 $Q(x,9)$ YES





Predicates - the universal quantifier

Another way of changing a predicate into a proposition.

Suppose P(x) is a predicate on some universe of discourse. Ex. B(x) = x is carrying a backpack, x is set of cse1207 students.

The universal quantifier of P(x) is the **proposition**: "P(x) is true for all x in the universe of discourse."

We write it $\forall x P(x)$, and say "for all x, P(x)"

 $\forall x P(x)$ is TRUE if P(x) is true for every single x. $\forall x P(x)$ is FALSE if there is an x for which P(x) is false.







Predicates - the existential quantifier

Another way of changing a predicate into a proposition.

Suppose P(x) is a predicate on some universe of discourse. Ex. C(x) = x has a candy bar, x is set of cs173 students.

The existential quantifier of P(x) is the **proposition**: "P(x) is true for some x in the universe of discourse."

We write it $\exists x P(x)$, and say "for some x, P(x)"

 $\exists x P(x)$ is TRUE if there is an x for which P(x) is true.

 $\exists x P(x) \text{ is } FALSE \text{ if } P(x) \text{ is } false \text{ for every single } x.$







Predicates - the existential quantifier

TABLE 1 Quantifiers.				
Statement	When True?	When False?		
$\forall x P(x)$	P(x) is true for every x .	There is an x for which $P(x)$ is false.		
$\exists x P(x)$	There is an x for which $P(x)$ is true.	P(x) is false for every x .		



• A domain must always be specified when a statement $\exists x P(x)$ is used. Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x P(x)$ has no meaning.





Predicates - more examples

$$L(x) = "x is a lion."$$

Universe of discourse is all creatures.

$$F(x) = "x is fierce."$$

$$C(x) = x drinks coffee.$$



All lions are fierce.

Some lions don't drink $\forall x (L(x) \rightarrow F(x))$

Some fierce creatures don't dr. $\exists x (L(x) \land \neg C(x))$



 $\exists x (F(x) \land \neg C(x))$



Predicates - more examples

$$L(x) = x is a lion.$$

F(x) = x is fierce.

Universe of discourse is all creatures.

C(x) = x drinks coffee.



• Notice that the second statement cannot be written as $\exists x(P(x) \rightarrow \neg R(x))$. The reason is that $P(x) \rightarrow \neg R(x)$ is true whenever x is not a lion, so that $\exists x(P(x) \rightarrow \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee.





Predicates - more examples



$$L(x) = "x is a large bird."$$

$$H(x) = x$$
 lives on honey.

R(x) = x is richly colored."

Universe of discourse is all creatures.

All humming birds are richly color $\forall x (B(x) \rightarrow R(x))$

No large birds live on honey.

Birds that do not live on hone, $\neg \exists x (L(x) \land H(x))$



$$\forall x (\neg H(x) \rightarrow \neg R(x))$$



Not all large birds live on honey. $\neg \forall x (L(x) \rightarrow H(x))$

$$\neg \forall x (L(x) \rightarrow H(x))$$



 $\forall x P(x)$ means "P(x) is true for every x." What about $\neg \forall x P(x)$?

Not ["P(x) is true for every x."]

"There is an x for which P(x) is not true."

$$\exists x \neg P(x)$$

So, $\neg \forall x P(x)$ is the same as $\exists x \neg P(x)$.



$$\exists x \neg (L(x) \rightarrow H(x))$$

Negations of Quantified Statements

Everyone likes football.

Every mother loves her child.

What is the negation of this statement?

Not everyone likes football = There exists someone who doesn't like football.

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

(generalized) DeMorgan's Law

Say the domain has only three values.

The same idea can be used to prove it for any number of variables.



No large birds live on honey.

$$\neg \exists x (L(x) \land H(x))$$



 $\exists x \ P(x) \ \text{means "}P(x) \ \text{is true for some } x."$ What about $\neg \exists x \ P(x)$?

Not ["P(x) is true for some x."]

"P(x) is not true for all x." $\forall x \ \neg P(x)$

So, $\neg \exists x P(x)$ is the same as $\forall x \neg P(x)$.



 $\forall x \neg (L(x) \wedge H(x))$

Negations of Quantified Statements

There is a plant that can fly.

What is the negation of this statement?

Not exists a plant that can fly = every plant cannot fly.

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

(generalized) DeMorgan's Law

Say the domain has only three values.

$$\neg \exists x P(x) \equiv \neg (P(1) \lor P(2) \lor P(3))$$

$$\equiv \neg (P(1) \lor P(2)) \land \neg P(3))$$

$$\equiv \neg P(1) \land \neg P(2) \land \neg P(3)$$

$$\equiv \forall x \neg P(x)$$

The same idea can be used to prove it for any number of yariables.





So,
$$\neg \forall x P(x)$$
 is the same as $\exists x \neg P(x)$.

So, $\neg \exists x P(x)$ is the same as $\forall x \neg P(x)$.

General rule: to negate a quantifier, move negation to the right, changing quantifiers as you go.





No large birds live on honey.



$$\neg\exists x \ (L(x) \land H(x)) \equiv \forall x \neg (L(x) \land H(x)) \qquad \text{Negation}$$

$$\text{rule}$$

$$\equiv \forall x \ (\neg L(x) \lor \neg H(x)) \qquad \text{DeMorgan's}$$

$$\equiv \forall x \ (L(x) \to \neg H(x)) \qquad \text{Subst for } \to$$

What's wrong with this proof?





Show that $\neg \forall x (P(x) \rightarrow Q(x))$ and $\exists x (P(x) \land \neg Q(x))$ are logically equivalent.



TABLE 2 De Morgan's Laws for Quantifiers.					
Negation	Equivalent Statement	When Is Negation True?	When False?		
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x , $P(x)$ is false.	There is an x for which $P(x)$ is true.		
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	P(x) is true for every x .		

