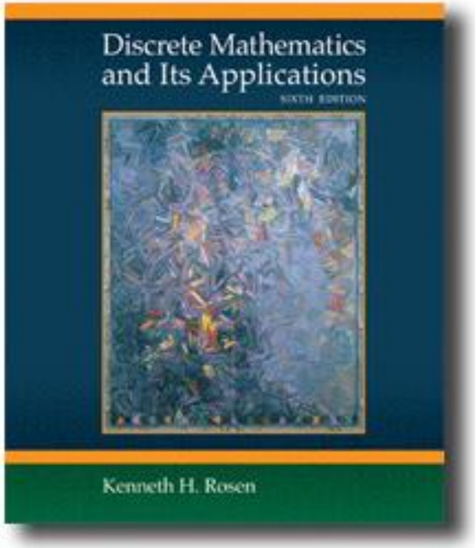


Chapter 2

Basic Structures: Sets, Functions, Sequences, and Sums

Outlines

- 2.1 Sets
- 2.2 Set Operations
- 2.3 Functions
- 2.4 Sequences and Summations

Getting Started

- 2.1 Sets
- 2.2 Set Operations
- 2.3 Functions
- 2.4 Sequences and Summations

2.1 Sets(1/8)

- Definition 1: A ***set*** is an unordered collection of objects
- Definition 2: Objects in a set are called ***elements***, or ***members*** of the set.
  - $a \in A$, $a \notin A$
  - $V = \{a, e, i, o, u\}$
  - $O = \{1, 3, 5, 7, 9\}$
    or $O = \{x \mid x$ is an odd positive integer less than $10\}$
    or $O = \{x \in \mathbf{Z}^+ \mid x$ is odd and $x < 10\}$

2.1 Sets(2/8)

- **N**=$\{0, 1, 2, 3, \ldots\}$, natural numbers
- **Z**=$\{\ldots, -2, -1, 0, 1, 2, \ldots\}$, integers
- $\mathbf{Z}^+$=$\{1, 2, 3, \ldots\}$, positive integers
- **Q**=$\{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, rational numbers
- $\mathbf{Q}^+$=$\{x \in \mathrm{R} \mid x = p/q, \text{ for positive integers } p \text{ and } q\}$
- **R**, real numbers

2.1 Sets(3/8)

- **Definition 3:** Two sets are ***equal*** **if and only if** they have the same elements.
  $A = B$ **iff** $\forall x (x \in A \leftrightarrow x \in B)$
- Venn diagram
  - Universal set U
  - Empty set (null set) $\varnothing$ (or {})

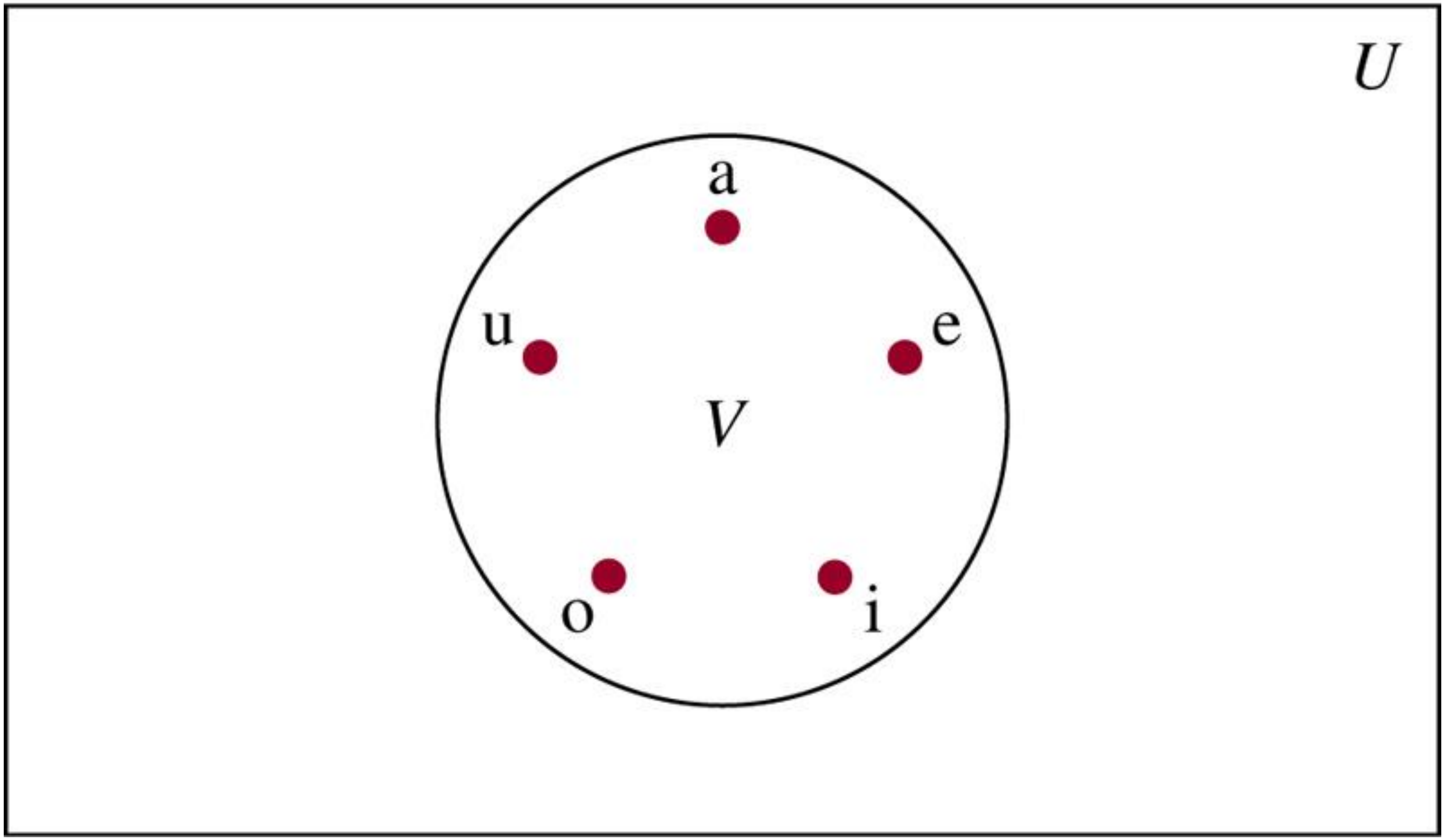


FIGURE 1 Venn Diagram for the Set of Vowels.

Set example

The sets {1, 3, 5} and {3, 5, 1} are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter. Note also that it does not matter if an element of a set is listed more than once, so {1, 3, 3, 3, 5, 5, 5, 5} is the same as the set {1, 3, 5} because they have the same elements.

2.1 Sets(5/8)

- **Definition 4:** The set A is a ***subset*** of B **if and only if** every element of A is also an element of B.
  A $\subseteq$ B iff $\forall x(x \in A \rightarrow x \in B)$
- Theorem 1: For every set S,
  (1) $\emptyset \subseteq$ S and (2) S $\subseteq$ S.
- Proper subset: A $\subset$ B

A is a **proper subset** of B only if

$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$

2.1 Sets(6/8)

- If $A \subseteq B$ and $B \subseteq A$, then $A=B$
- Sets may have other sets as members
  - $A=\{\emptyset, \{a\}, \{b\}, \{a,b\}\}$
    $B=\{x \mid x \text{ is a subset of the set } \{a,b\}\}$
  - $A=B$

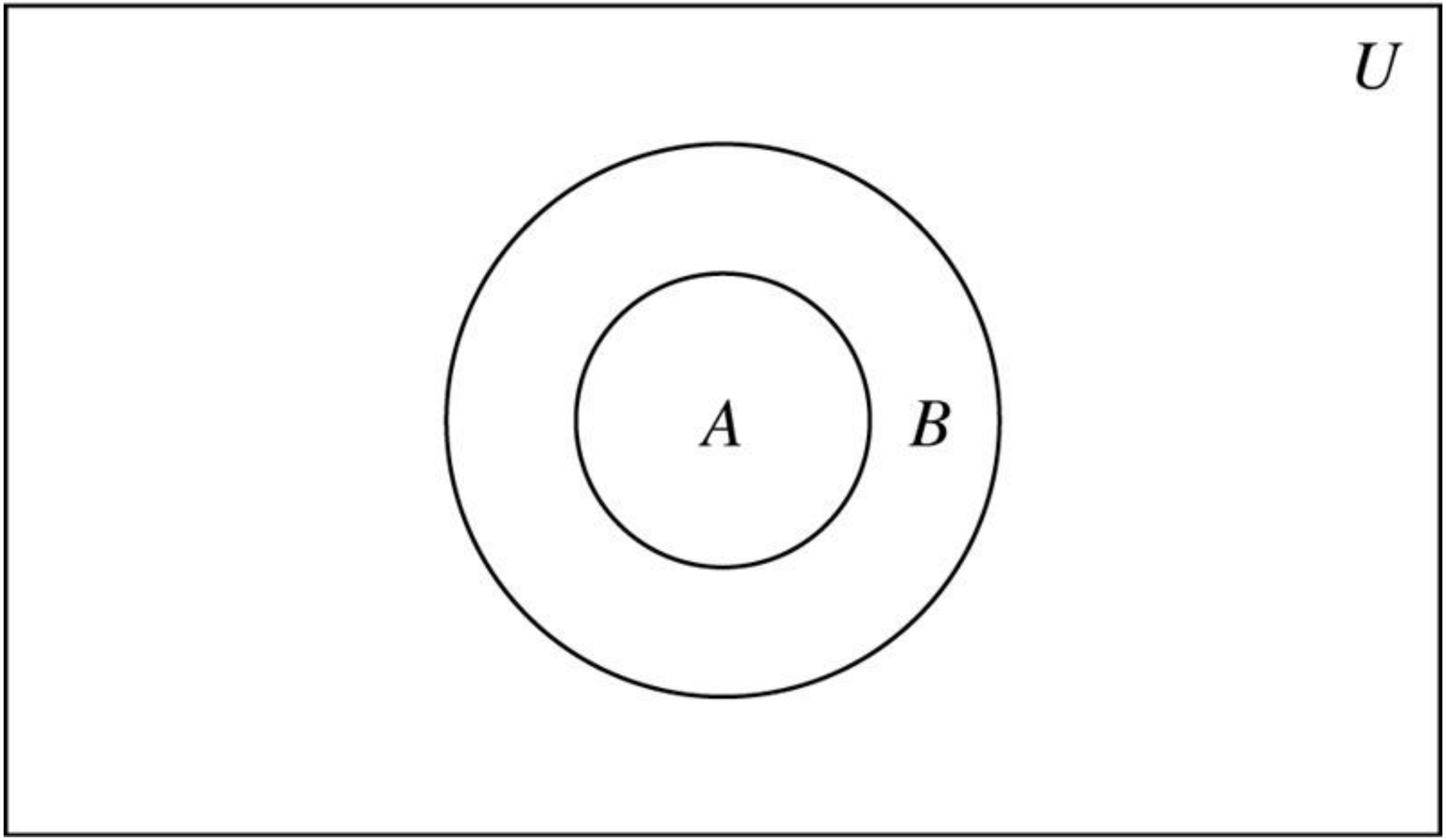


FIGURE 2 Venn Diagram Showing that A Is a Subset of B.

2.1 Sets(8/8)

- **Definition 5:** If there are exactly n distinct members in the set S (n is a nonnegative integer), we say that S is a finite set and that n is the ***cardinality*** of S.
  $|S| = n$
  - $|\varnothing| = 0$
- **Definition 6:** A set is ***infinite*** if it's not finite.
  - $\mathbf{Z}^+$

The Power Set

- **Definition 7:** The ***power set*** of S is the set of all subset of the set S. $P(S)$
  - $P(\{0,1,2\})$

  What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$
  - $P(\emptyset) = \{\emptyset\}$.
  - $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.
- If a set has n elements, then its subset has 2^n elements.

Cartesian Products

- **Definition 8:** ***Ordered n-tuple*** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_i as its ith element for $i=1, 2, \ldots, n$.
- **Definition 9:** ***Cartesian product*** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.
  $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$
  - *E.g.* $A = \{1, 2\}$, $B = \{a, b, c\}$
  - $A \times B$ and $B \times A$ are not equal, unless A=∅ or B=∅ or A=B

- **Definition 10:** *Cartesian product* of A_1, A_2, …, A_n, denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where $a_i \in A_i$ for $i=1,2,\ldots,n$.
  $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i$ for $i=1,2,\ldots,n\}$

- What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Solution: The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A$, $b \in B$, and $c \in C$. Hence,

$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$.

Example

- What are the ordered pairs in the less than or equal to relation, which contains (a, b) if $a \leq b$, on the set $\{0, 1, 2, 3\}$?
  Solution: The ordered pair (a, b) belongs to R if and only if both a and b belong to $\{0, 1, 2, 3\}$ and $a \leq b$. Consequently, the ordered pairs in R are (0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2, 3), and (3, 3).

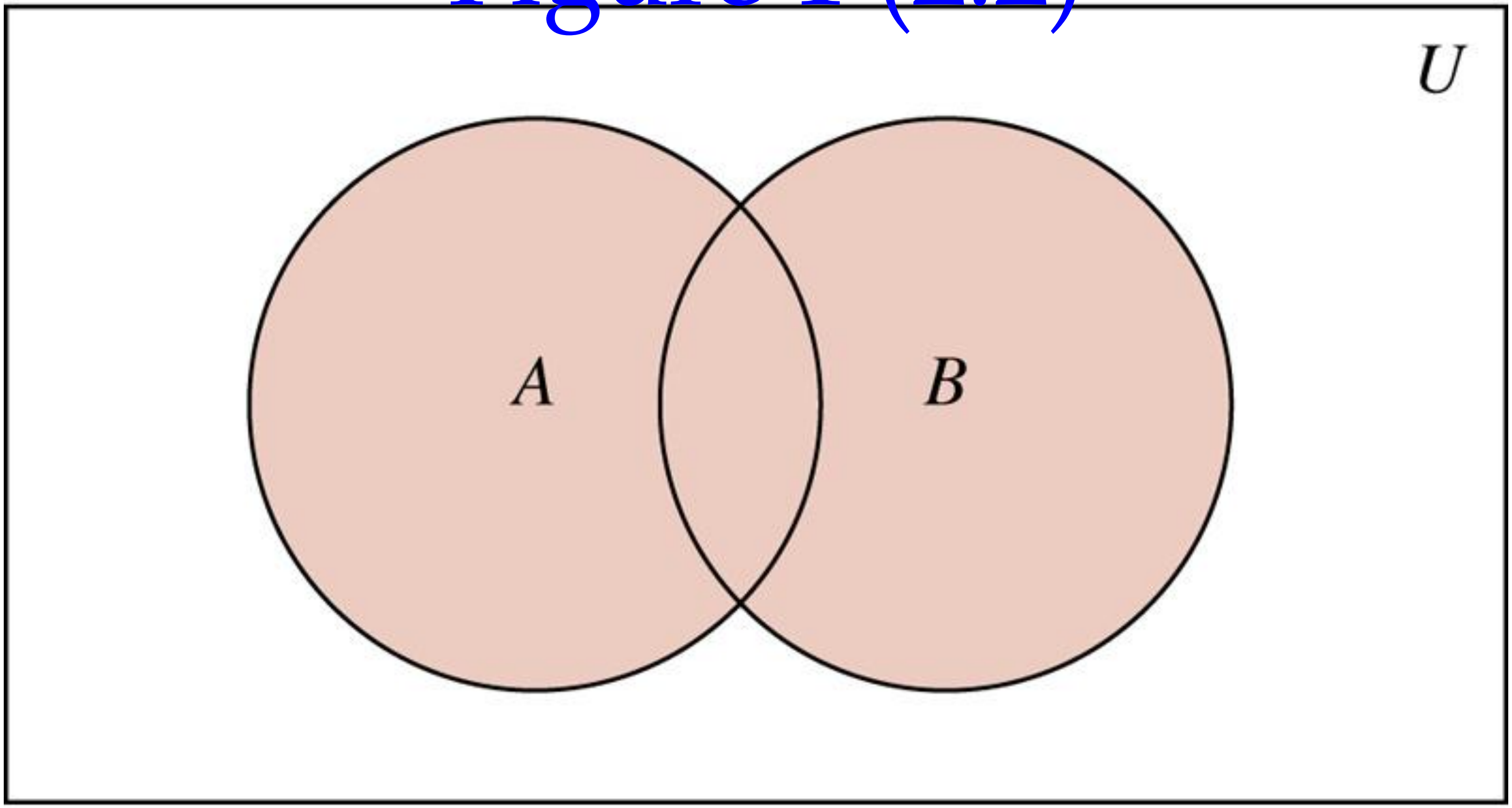
Getting Started

- 2.1 Sets
- 2.2 Set Operations
- 2.3 Functions
- 2.4 Sequences and Summations

2.2 Set Operations

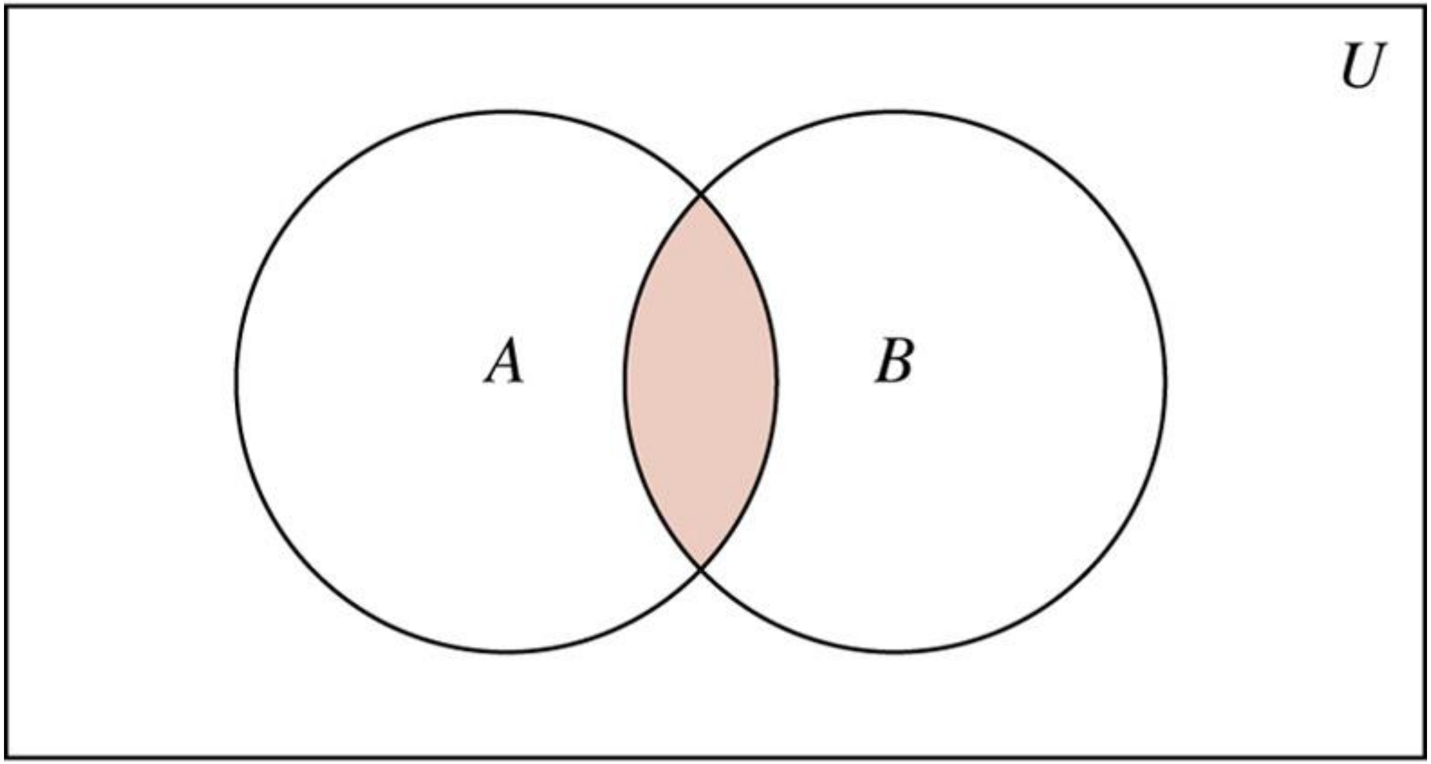
- <u>**Definition 1:**</u> The ***union*** of the sets A and B, denoted by $A \cup B$, is the set containing those elements that are either in A or in B, or in both.
  - $A \cup B = \{x \mid x \in A \lor x \in B\}$
- <u>**Definition 2:**</u> The ***intersection*** of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.
  - $A \cap B = \{x \mid x \in A \land x \in B\}$

Figure 1 (2.2)



$A \cup B$ is shaded.

FIGURE 1 Venn Diagram Representing the Union of *A* and *B*.

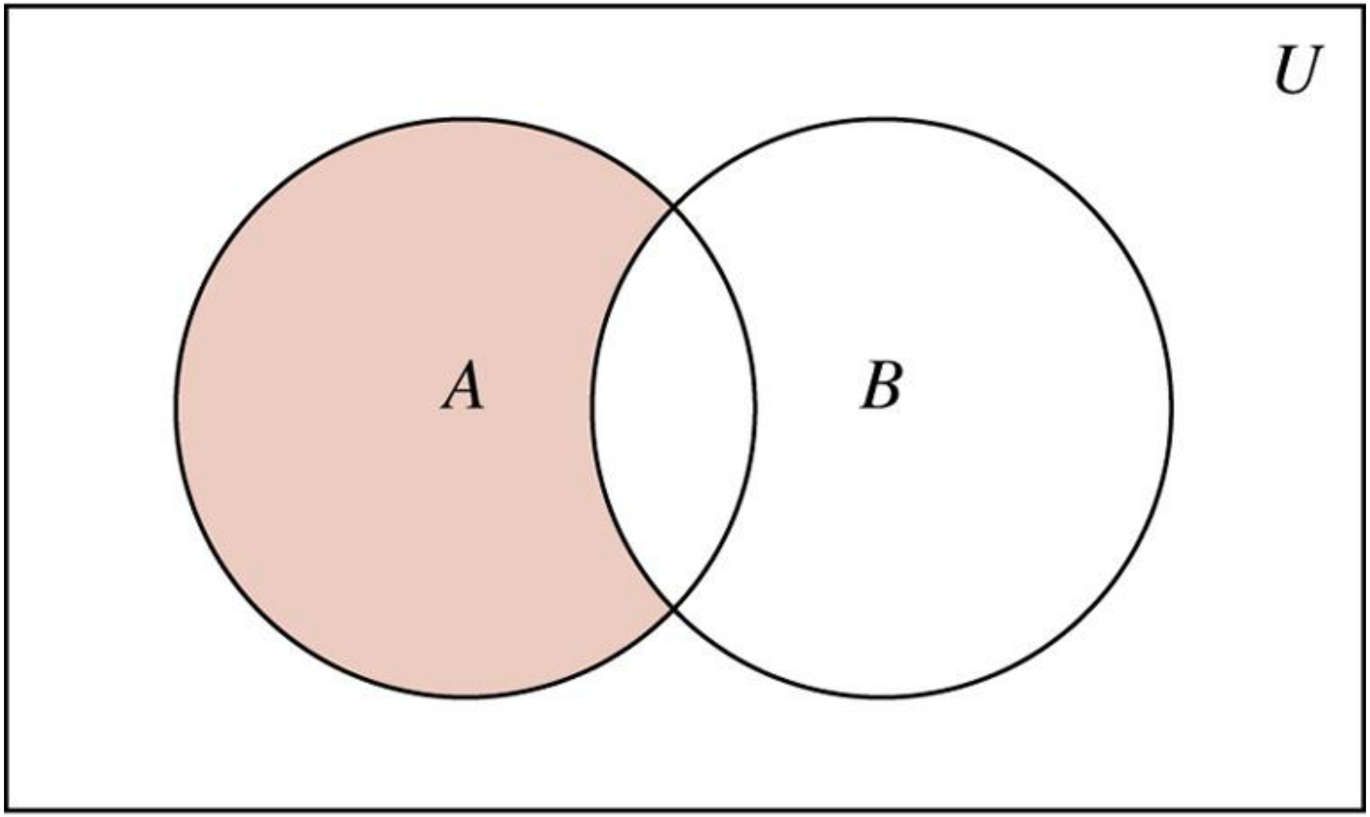


$A \cap B$ is shaded.

FIGURE 2 Venn Diagram Representing the Intersection of *A* and *B*.

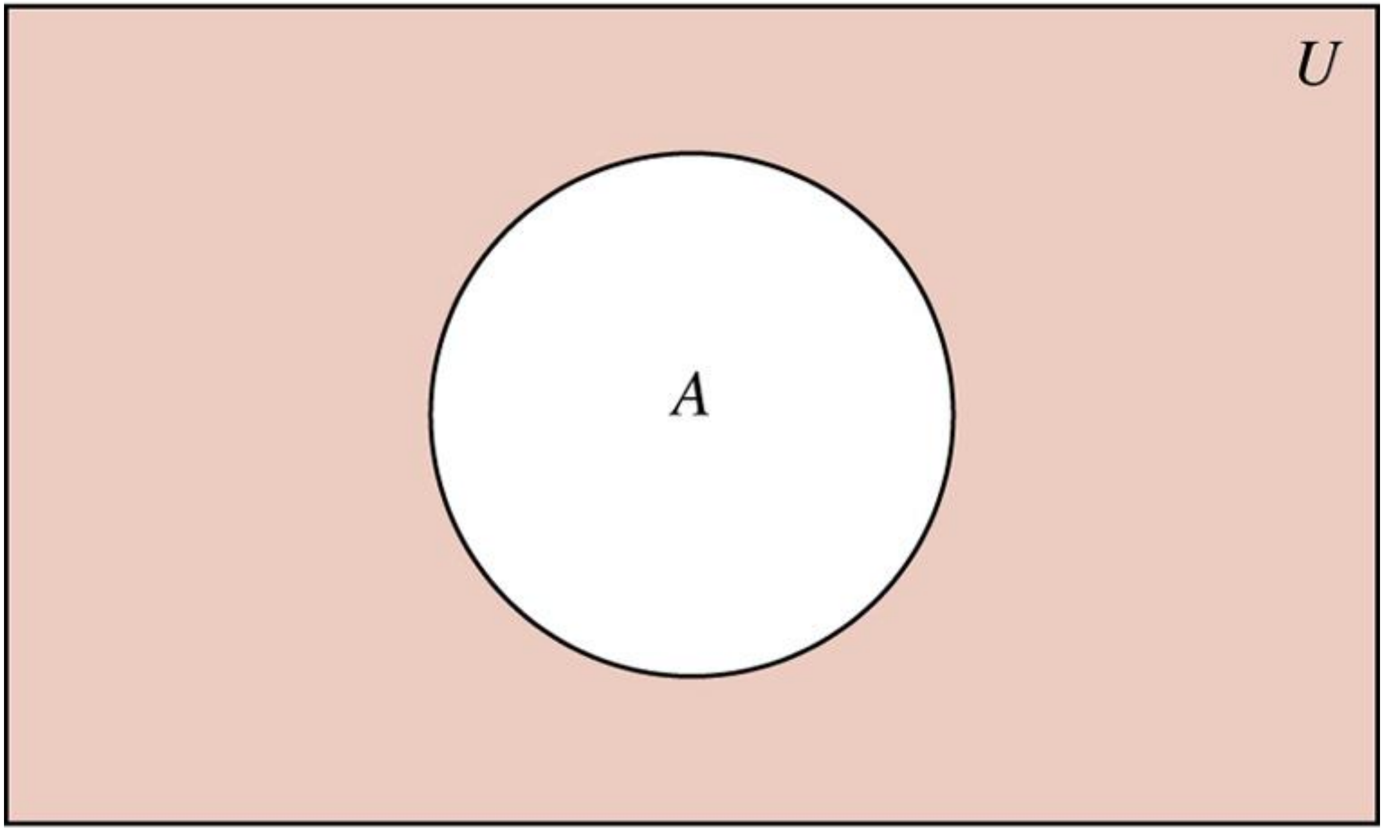
- **Definition 3:** Two sets are ***disjoint*** if their intersection is the empty set.
- $|A \cup B| = |A| + |B| - |A \cap B|$
  - **Principle of inclusion-exclusion**

- **Definition 4:** The ***difference*** of the sets A and B, denoted by A-B, is the set containing those elements that are in A but not in B.
  - Complement of B with respect to A
  - $A-B=\{x \mid x \in A \wedge x \notin B\}$
- **Definition 5:** The ***complement*** of the set A, denoted by $\bar{A}$, is the complement of A with resepect to U.
  - $\bar{A} = \{x \mid x \notin A\}$



$A - B$ is shaded.

FIGURE 3 Venn Diagram for the Difference of A and B.



$\overline{A}$ is shaded.

FIGURE 4 Venn Diagram for the Complement of the Set A.

Set Identities

- To prove set identities
  - Show that each is a subset of the other
  - Using membership tables
  - Using those that we have already proved

TABLE 1 Set Identities.

Identity	*Name*
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws

TABLE 2 (2.2)

TABLE 2 **A Membership Table for the Distributive Property.**

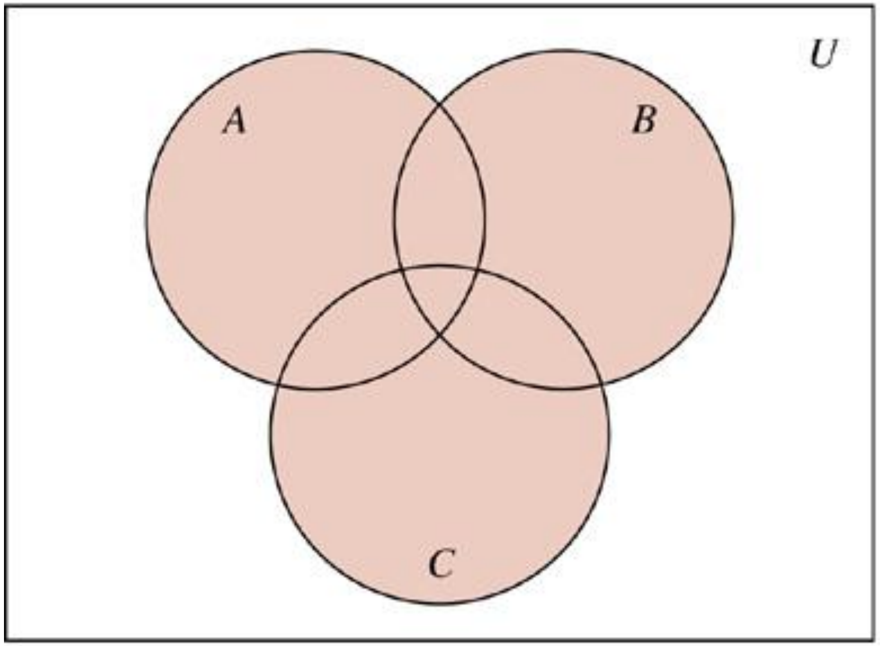
A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

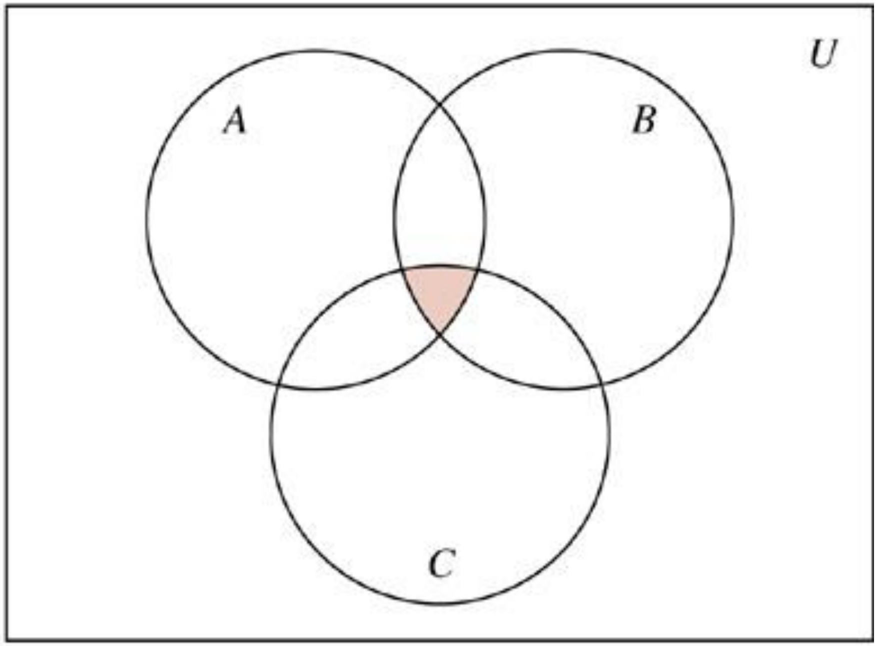
- <u>**Definition 6:**</u> The ***union*** of a collection of sets is the set containing those elements that are members of at least one set in the collection.
  - $A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$
- <u>**Definition 7:**</u> The ***intersection*** of a collection of sets is the set containing those elements that are members of all the sets in the collection.
  - $A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$
- Computer Representation of Sets
  - Using bit strings

- Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?
  Solution: The set $A \cup B \cup C$ contains those elements in at least one of A, B, and C. Hence,
  $$A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}.$$
  The set $A \cap B \cap C$ contains those elements in all three of A, B, and C. Thus,
  $$A \cap B \cap C = \{0\}.$$

FIGURE 5 (2.2)



(a) $A \cup B \cup C$ is shaded.



(b) $A \cap B \cap C$ is shaded.

FIGURE 5 The Union and Intersection of *A*, *B*, and *C*.

Getting Started

- 2.1 Sets
- 2.2 Set Operations
- 2.3 Functions
- 2.4 Sequences and Summations

2.3 Functions

- **Definition 1:** A ***function*** f from A to B is an assignment of exactly one element of B to each element of A. $f: A \to B$
- **Definition 2:** $f: A \to B$.
  - A: ***domain*** of f, B: ***codomain*** of f.
  - $f(a)=b$, a: ***preimage*** of b, b: ***image*** of a.
  - ***Range*** of f: the set of all images of elements of A
  - f: maps A to B

- What are the domain, codomain, and range of the function that assigns grades to students described in the first paragraph of the introduction of this section?

  Solution: Let G be the function that assigns a grade to a student in our discrete mathematics class. Note that $G(\text{Adams}) = A$, for instance. The domain of G is the set {Adams, Chou, Goodfriend, Rodriguez, Stevens}, and the codomain is the set $\{A, B, C, D, F\}$. The range of G is the set $\{A, B, C, F\}$, because each grade except D is assigned to some student.

FIGURE 1

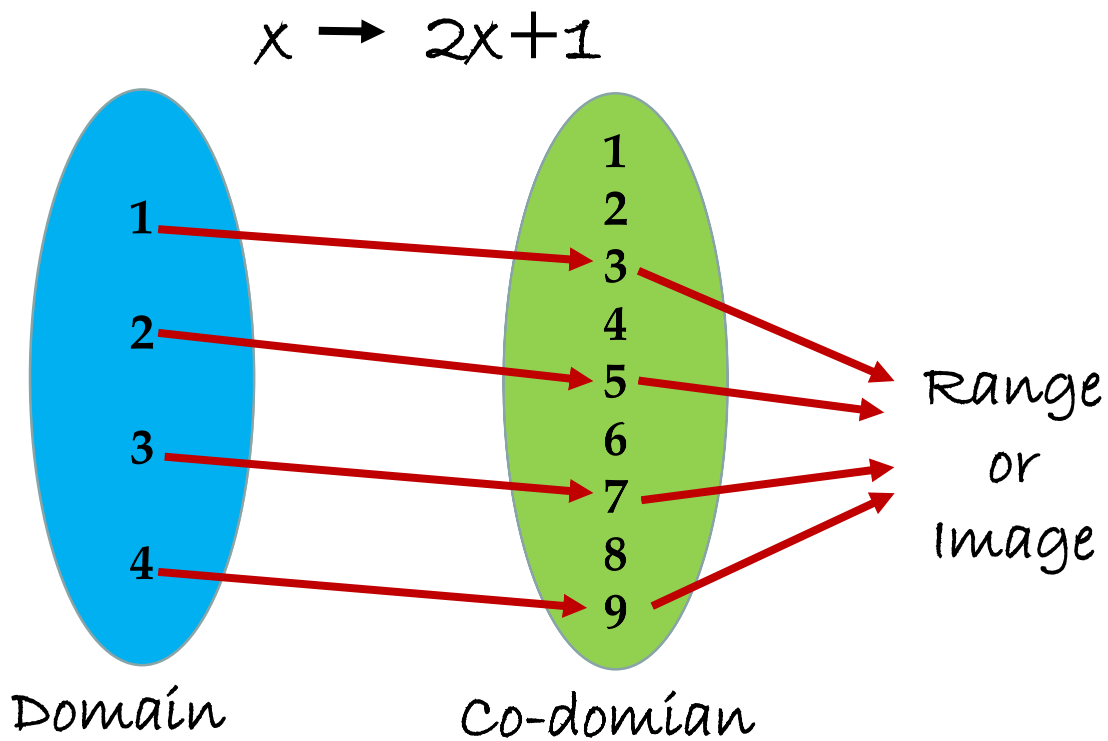


FIGURE 1.1: An example of function with it's components.

FIGURE 2

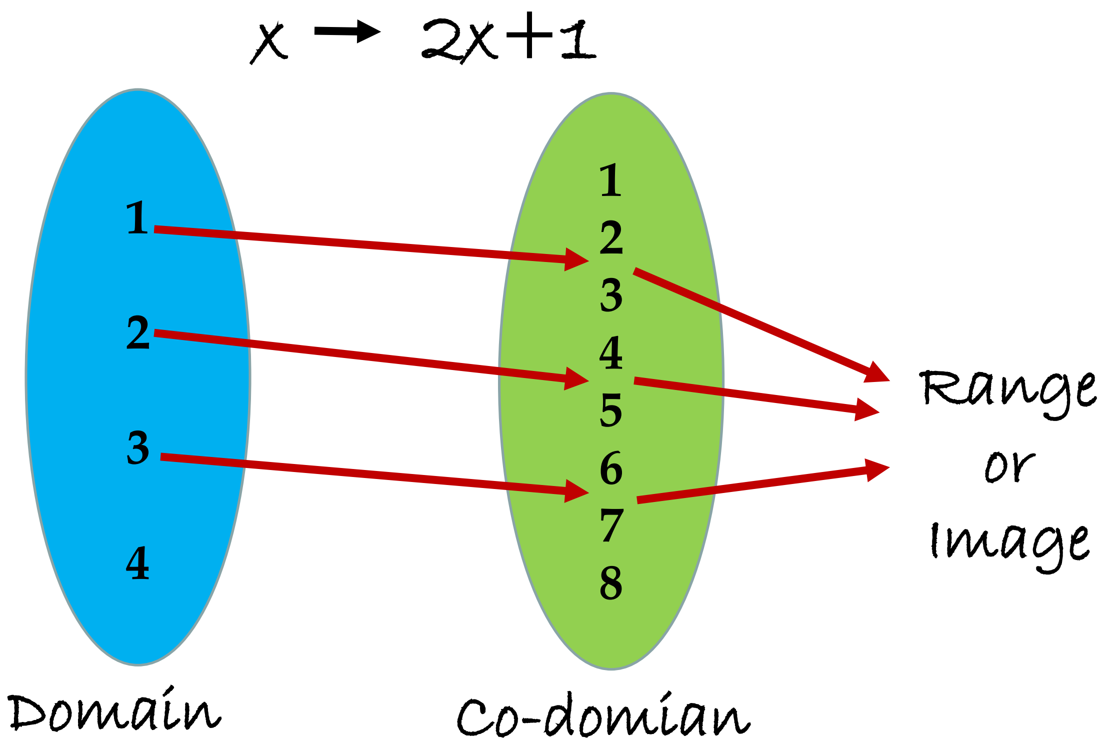


FIGURE 1.2: An example of not being function

FIGURE 3

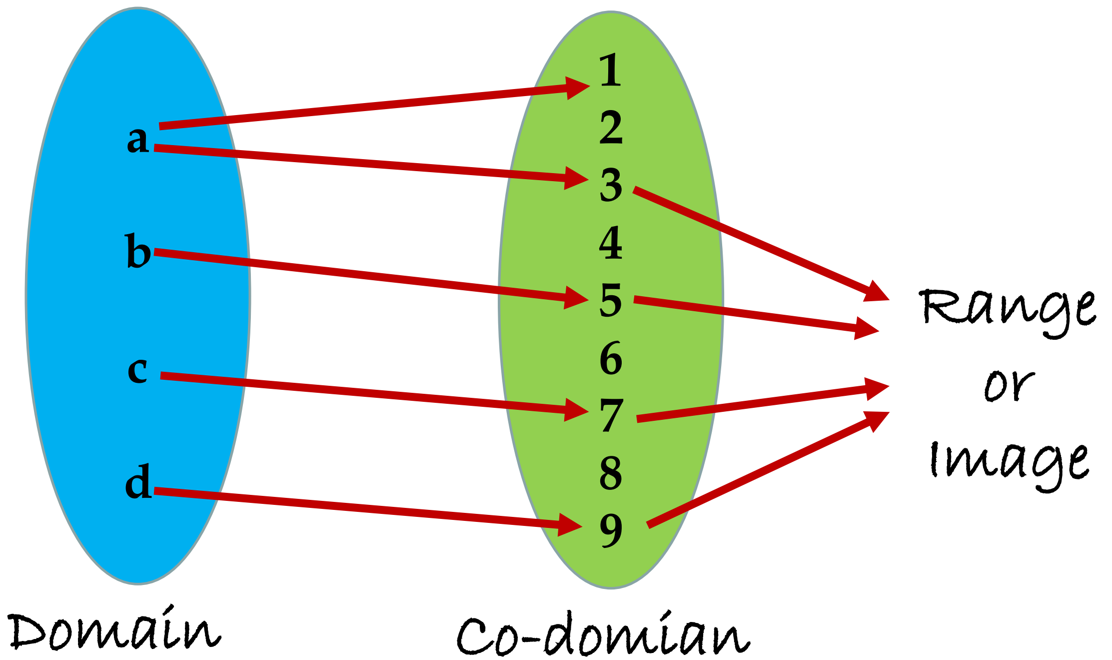


FIGURE 1.3: An example of not being function

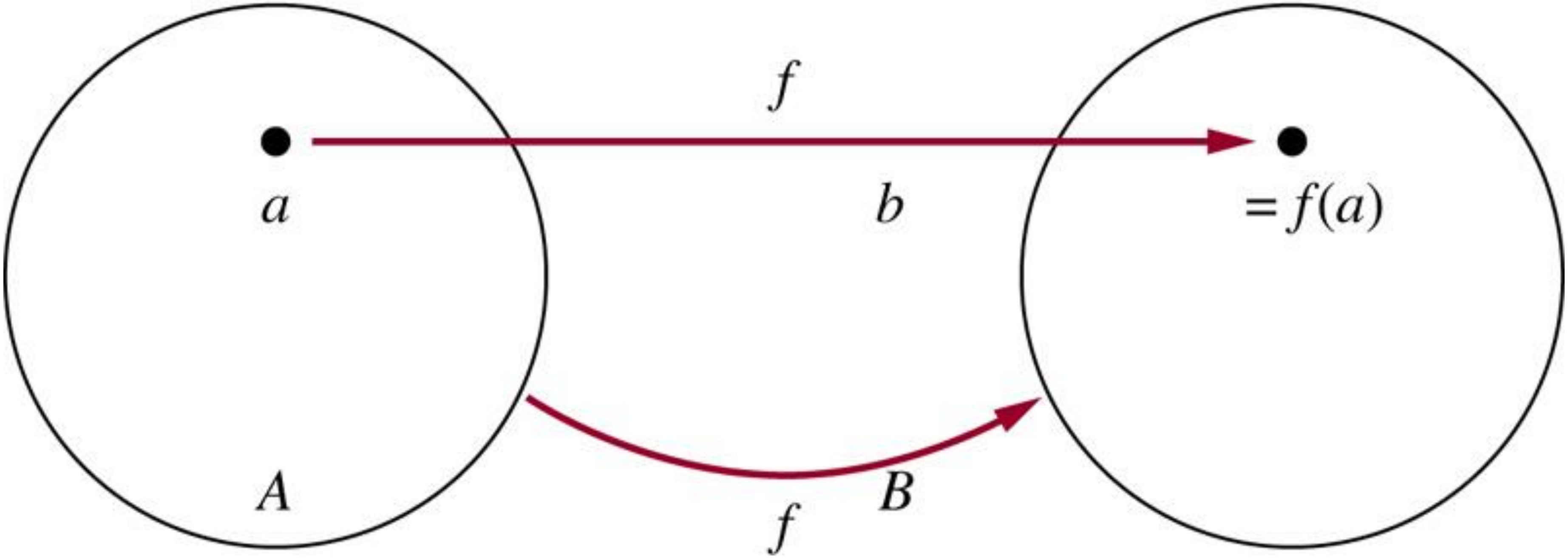


FIGURE 2 The Function f Maps *A* to *B*.

- **Definition 3:** Let f_1 and f_2 be functions from A to **R**. f_1+f_2 and f_1f_2 are also functions from A to **R**:
  - $(f_1+f_2)(x) = f_1(x)+f_2(x)$
  - $(f_1f_2)(x)=f_1(x)f_2(x)$

  Ex: Let $f1$ and $f2$ be functions from **R** to **R** such that $f1(x) = x2$ and $f2(x) = x - x2$. What are the functions $f1 + f2$ and $f1f2$?

  Solution: From the definition of the sum and product of functions, it follows that

  $(f1 + f2)(x) = f1(x) + f2(x) = x2 + (x - x2) = x$

  and

  $(f1f2)(x) = x2(x - x2) = x3 - x4.$

Definition 4: f: $A \to B$, S is a subset of A. The image of S under the function f is the subset of B that consists of the images of the elements of S. We denote the image of S by $f(S)$, so

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}.$$

Ex: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

One-to-One and Onto Functions

- **Definition 5:** A function f is ***one-to-one*** **or** ***injective***, iff $f(a)=f(b)$ implies that $a=b$ for all a and b in the domain of f.
  - $\forall a\, \forall b(\, f(a)=f(b) \rightarrow a=b\,)$ *or*
    $\forall a\, \forall b(\, a \neq b \rightarrow f(a) \neq f(b)\,)$

  a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.
- Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one?

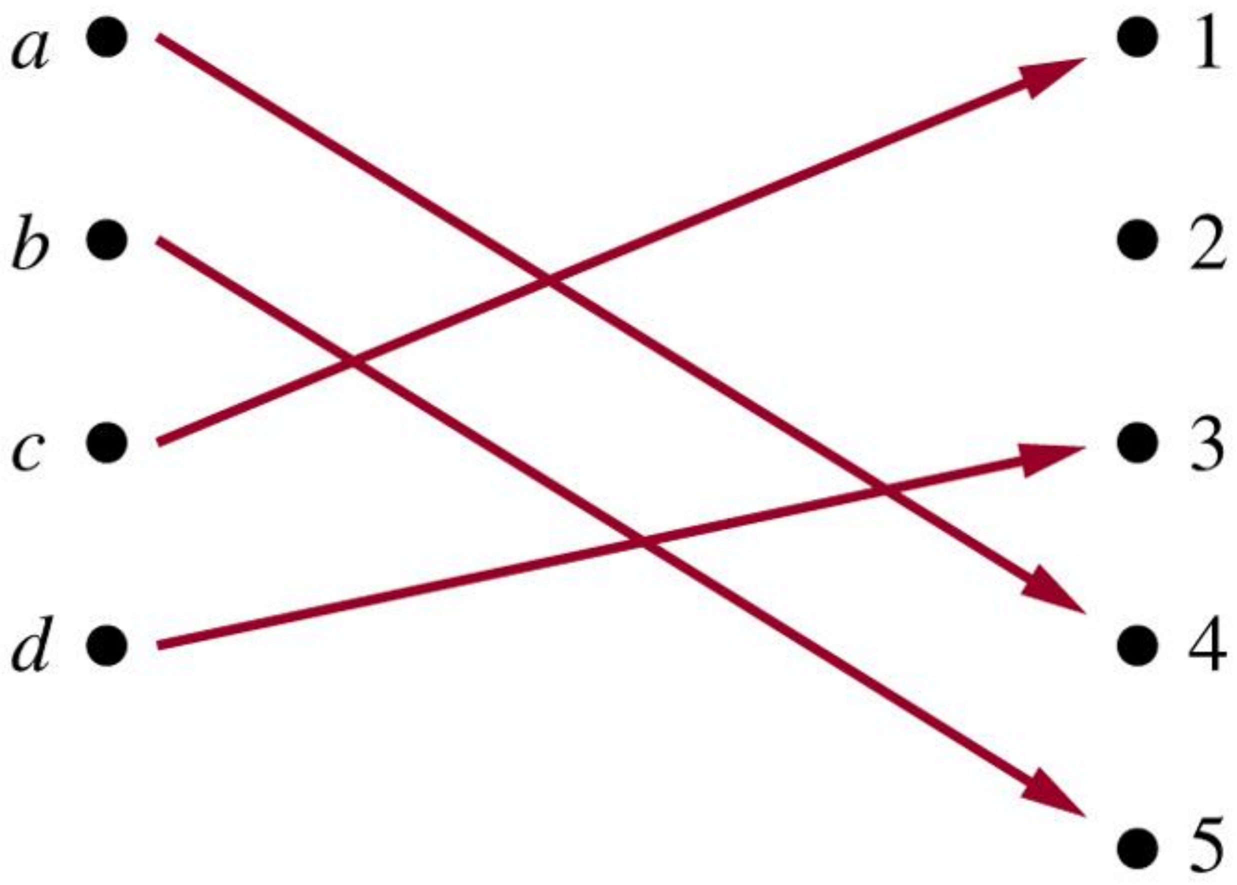


FIGURE 3 A One-to-One Function.

One-to-One and Onto Functions

- **Definition 7:** A function f is ***onto*** **or** ***surjective***, iff for every element $b \in B$ there is an element $a \in A$ with $f(a)=b$.
  - $\forall y \exists x(f(x)=y)$ *or*
    $\forall a \forall b(a \neq b \rightarrow f(a) \neq f(b))$
  - When ***co-domain = range***
- **Definition 8:** A function f is a ***one-to-one correspondence*** **or a** ***bijection*** if it is both one-to-one and onto.
  - Ex: identity function $\iota_A(x)=x$

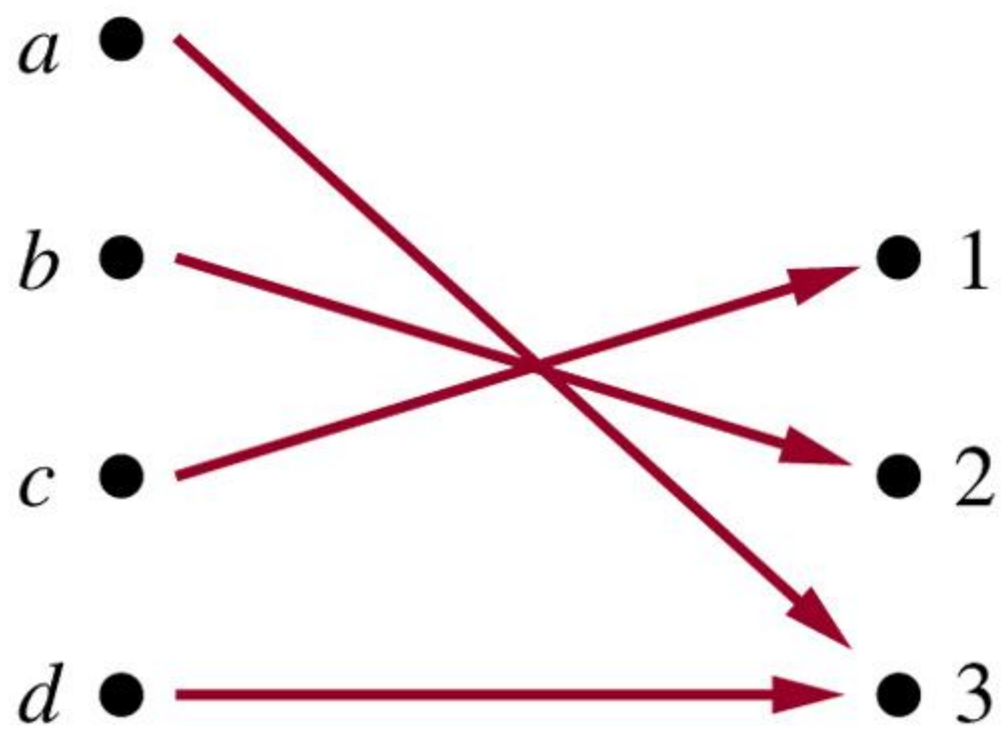


FIGURE 4 An Onto Function.

FIGURE 5 (2.3)

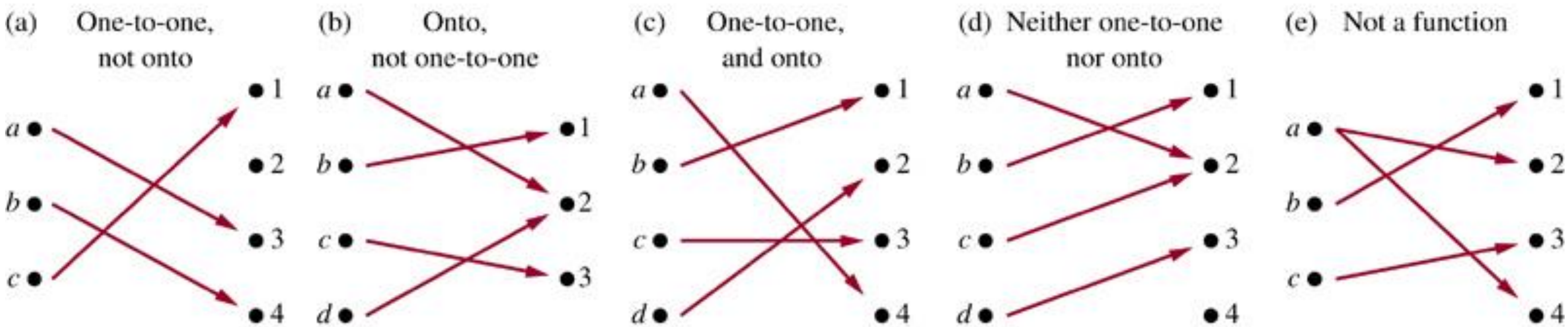


FIGURE 5 Examples of Different Types of Correspondences.

Inverse Functions and Compositions of Functions

- **Definition 9:** Let f be a one-to-one correspondence from A to B. The inverse function of f is the function that assigns to an element b in B the unique element a in A such that f(a)=b.
  - $f^{-1}(b)=a$ when $f(a)=b$

Prove It

If a function is one-to-one correspondence then it's inverse is possible

- Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$.
  Is f invertible, and if it is, what is its inverse?

Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so

$f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$

Let f be the function from R to R with f (x) = x2. Is f invertible? Because f (-2) = f (2) = 4, f is not one-to-one. f is not invertible.

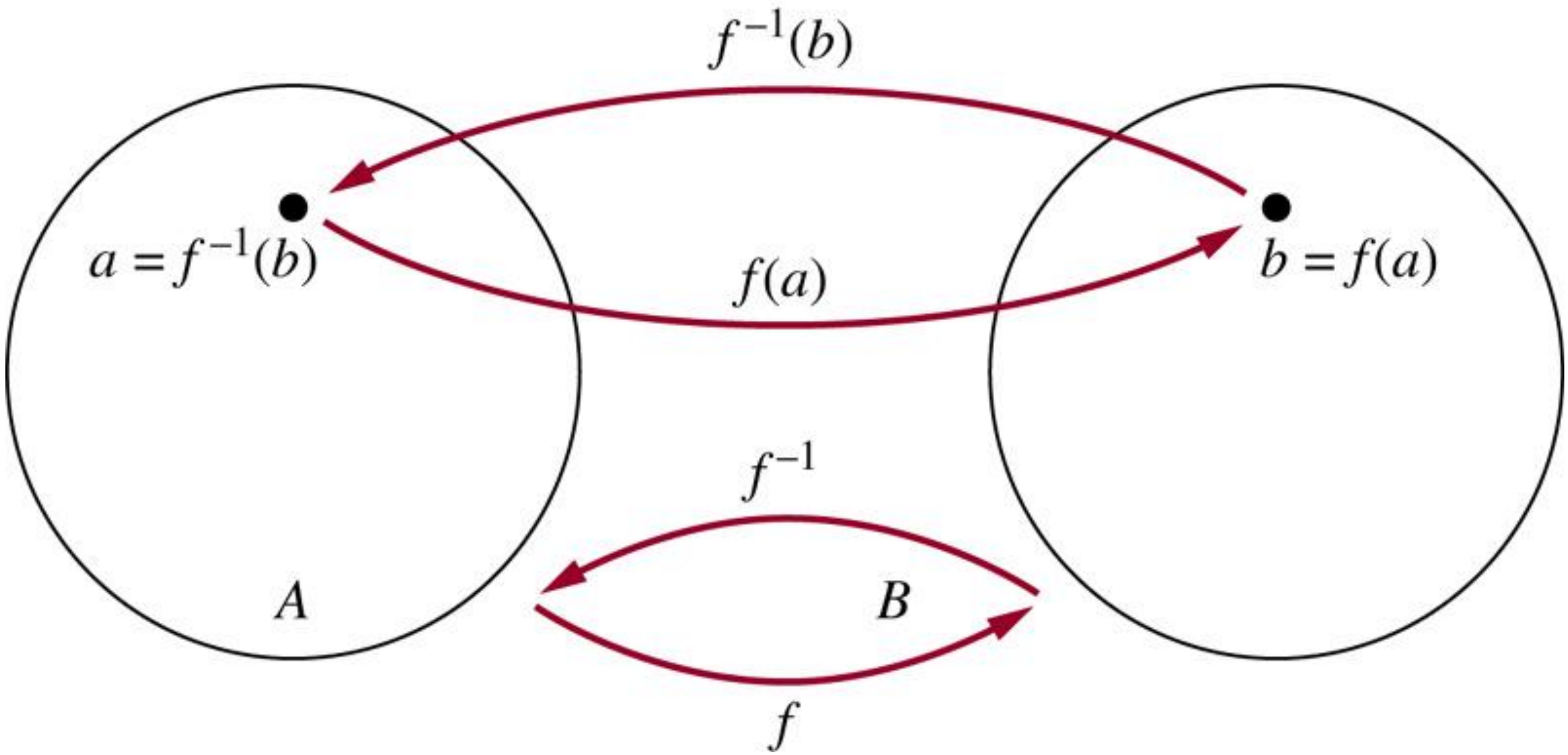


FIGURE 6 The Function f^{-1} Is the Inverse of Function f.

- **Definition 10:** Let g be a function from A to B, and f be a function from B to C. The ***composition*** of functions f and g, denoted by $f \circ g$, is defined by:
  - $f \circ g(a) = f(g(a))$
  - $f \circ g$ and $g \circ f$ are not equal --- ***Prove it***

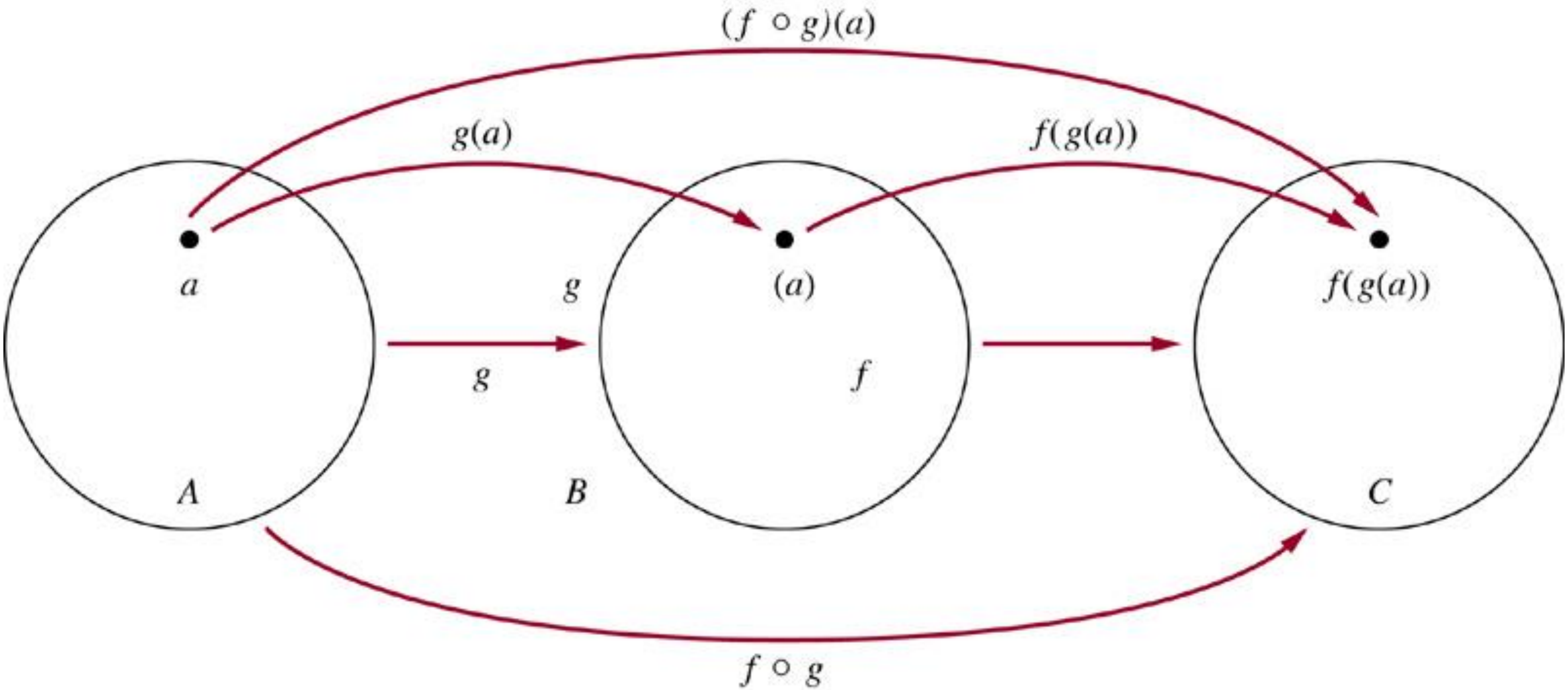


FIGURE 7 The Composition of the Functions f and g.

Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover,

$$(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11.$$

Graphs of Functions

- **Definition 11:** The ***graph*** of function f is the set of ordered pairs $\{(a,b) \mid a \in A \text{ and } f(a)=b\}$

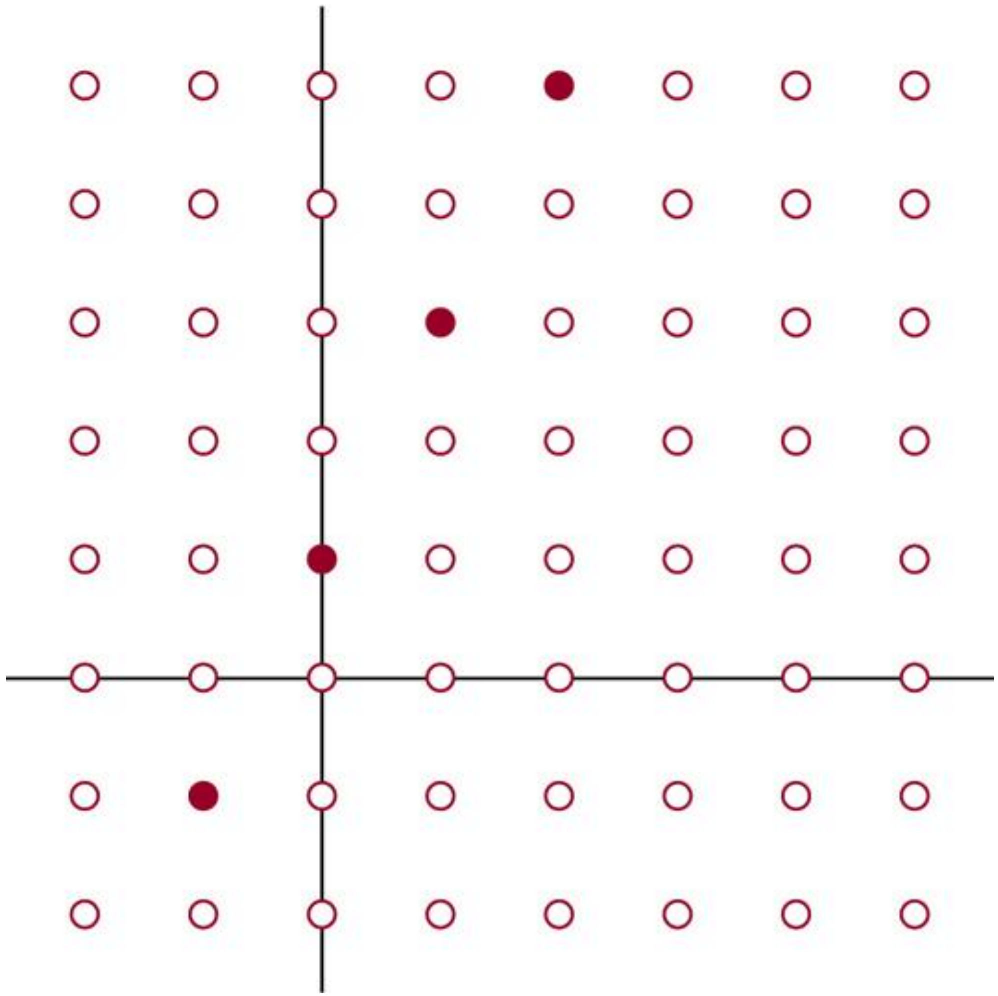


FIGURE 8 The Graph of $f(n) = 2n + 1$ from Z to Z.

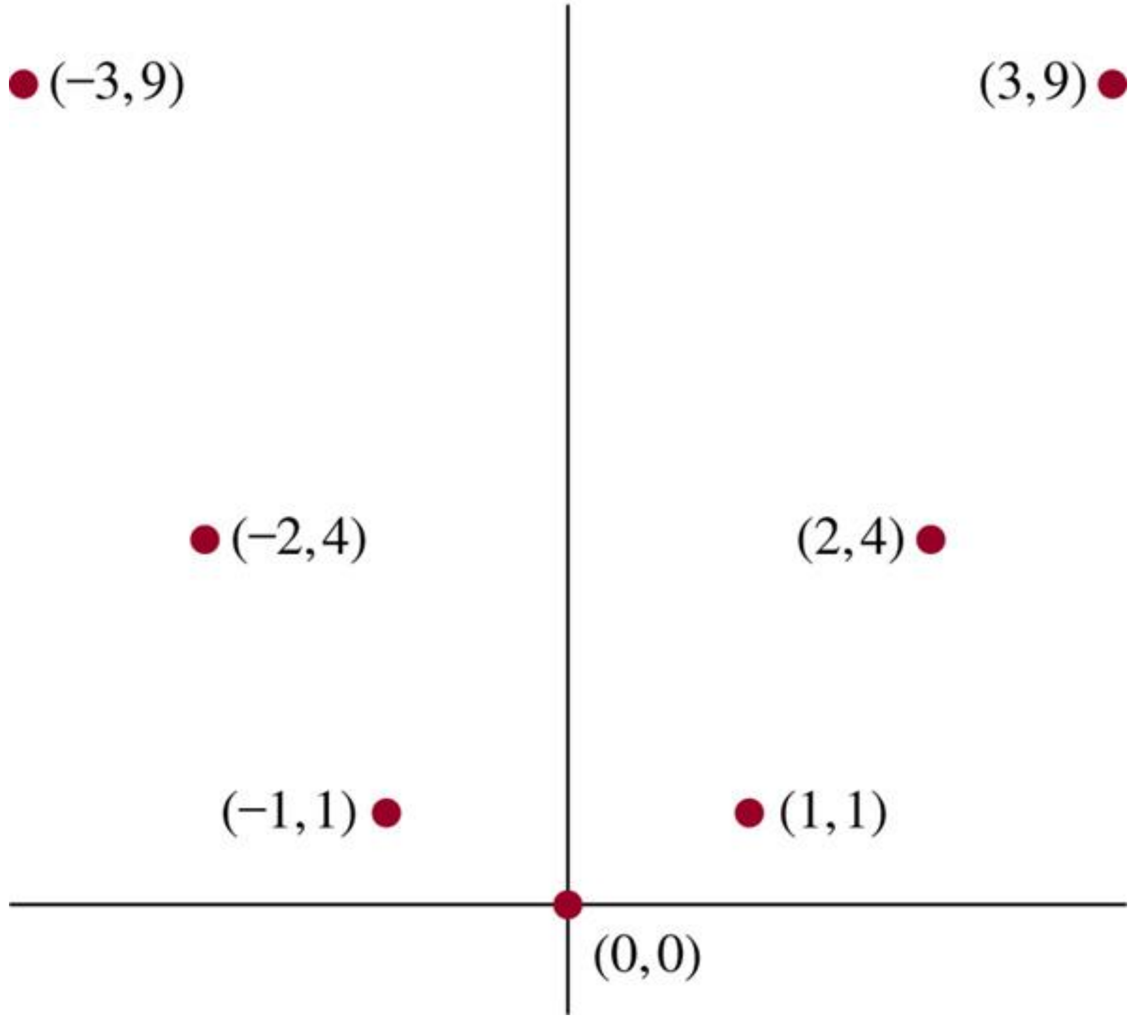


FIGURE 9 The Graph of $f(x) = x^2$ from Z to Z.

Floor and Ceil Functions

- **Definition 12:** The ***floor function*** assigns to x the largest integer that is less than or equal to x ($\lfloor x \rfloor$ or $[x]$)..
- **Definition 13:** The ***ceiling function*** assigns to x the smallest integer that is greater than or equal to x ($\lceil x \rceil$)

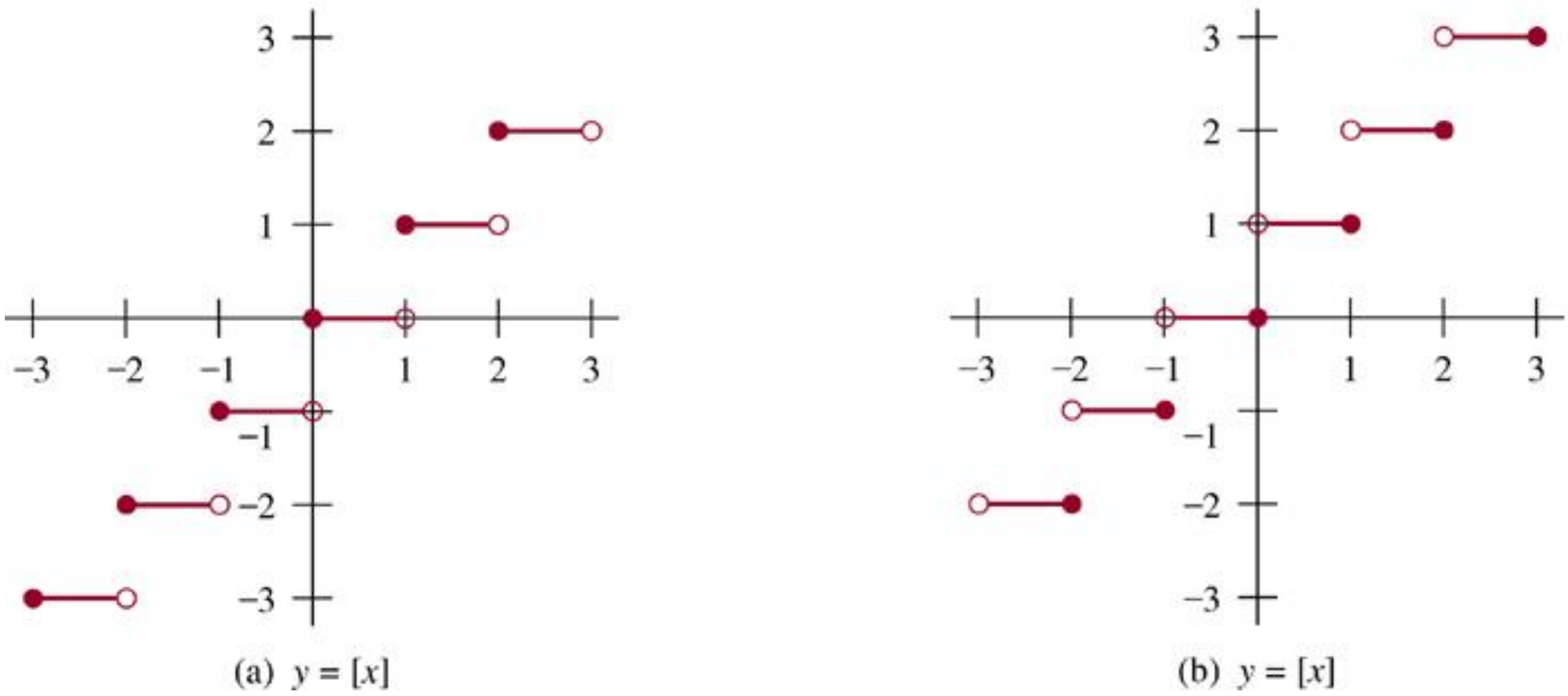


FIGURE 10 Graphs of the (a) Floor and (b) Ceiling Functions.

TABLE 1 (2.3)

TABLE 1 **Useful Properties of the Floor and Ceiling Functions.** (n is an integer, x is a real number)	
(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Proof: 4(a)

- Let $\lfloor x \rfloor = m$ where m is a positive integer.
- By property *1(a)*

$$m \leq x < m + 1$$

- Adding n on both sides

$$m + n \leq x + n < m + n + 1$$

- Using *1(a)* again

$$\lfloor x + n \rfloor = m + n$$

$$\lfloor x + n \rfloor = \lfloor x \rfloor + n$$

- A useful approach for considering statements about the floor function is to let

$$x = n + \varepsilon$$

- *where*

$$n = \lfloor x \rfloor,\ 0 \leq \varepsilon < 1$$

- For ceil

$$x = n - \varepsilon$$

- *where*

$$n = \lceil x \rceil,\ 0 \leq \varepsilon < 1$$

Proof: $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$

- Let $x = n + \varepsilon$ ... ...*(1)*

  where $0 \leq \varepsilon < 1$ and $n = \lfloor x \rfloor$ ......(c) is an integer
- Two cases to consider, depending on whether *ε is less than, or greater than or equal to* $\frac{1}{2}$.

❖ First let $0 \leq \varepsilon < \frac{1}{2}$ ... ...*(2)*

or $0 \leq 2\varepsilon < 1$

❖ From (1) $2x = 2n + 2\varepsilon$ ... ...*(3)*

and $\lfloor 2x \rfloor = 2\,n$

❖ Similarly from (1) $x + \frac{1}{2} = n + \frac{1}{2} + \varepsilon$ and $\lfloor 2x \rfloor = 2\,n$ ... ... *(a)*

❖ From (2) $\frac{1}{2} \leq \varepsilon + \frac{1}{2} < 1$ or $0 < \varepsilon + \frac{1}{2} < 1$

and so $\left\lfloor x + \frac{1}{2} \right\rfloor = n$ ... .*(b)*

❖ From (a) and (b)+(c), $\lfloor 2x \rfloor = 2\,n$ and $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + n = 2n$

So $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$

Proof: $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$

- Let $\frac{1}{2} \leq \varepsilon < 1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(4)$

  or $1 \leq 2\varepsilon < 2$ or $0 \leq 2\varepsilon - 1 < 1$
- From (3) $2x = 2n + 2\varepsilon = (2n + 1) + (2\varepsilon - 1)$

  and $\lfloor 2x \rfloor = 2\,n + 1 \;\ldots .(x)$
- Similarly from (1) $x + \frac{1}{2} = n + \frac{1}{2} + \varepsilon = (n + 1\,) + (\varepsilon - \frac{1}{2}) \;\ldots \ldots (y)$
- From (4) $0 \leq \varepsilon - \frac{1}{2} < \frac{1}{2}$ or $0 < \varepsilon - \frac{1}{2} < 1$

  and so $\left\lfloor x + \frac{1}{2} \right\rfloor = n + 1 \;\ldots \ldots .(b)$
- From (x) and (y)+(c), $\lfloor 2x \rfloor = 2\,n + 1$ and $\lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor = n + n + 1 = 2n + 1$

So $\lfloor 2x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{2} \right\rfloor$