Number Theory

ITT9131 Konkreetne Matemaatika

Chapter Four

- Divisibility
- Primes
- Prime examples
- Factorial Factors
- Relative primality
- ‘MOD’: the Congruence Relation
- Independent Residues
- Additional Applications
- Phi and Mu

Contents

1 Modular arithmetic

2 Primality test
- Fermat' theorem
- Fermat' test
- Rabin-Miller test

3 Phi and Mu

Next section

1 Modular arithmetic

2 Primality test
- Fermat' theorem
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3 Phi and Mu

Congruences

Definition

Integer a is congruent to integer b modulo $m > 0$, if a and b give the same remainder when divided by m. Notation $a \equiv b \pmod{m}$.

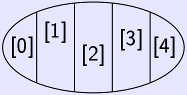
Alternative definition: $a \equiv b \pmod{m}$ iff $m|(b-a)$. Congruence is

a *equivalence relation:*

Reflectivity: $a \equiv a \pmod{m}$

Symmetry: $a \equiv b \pmod{m} \quad \Rightarrow \quad b \equiv a \pmod{m}$

Transitivity: $a \equiv b \pmod{m}$ ja $b \equiv c \pmod{m} \quad \Rightarrow \quad a \equiv c \pmod{m}$



Properties of the congruence relation

- If $a \equiv b \pmod{m}$ and $d|m$, then $a \equiv b \pmod{d}$
- If $a \equiv b \pmod{m_1}, a \equiv b \pmod{m_2}, \ldots, a \equiv b \pmod{m_k}$, then $a \equiv b \pmod{lcm(m_1, m_2, \ldots, m_k)}$
- If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$
- If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $ac \equiv bd \pmod{m}$
- If $a \equiv b \pmod{m}$, then $ak \equiv bk \pmod{m}$ for any integer k
- If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a - c \equiv b - d \pmod{m}$
- If $a \equiv b \pmod{m}$, then $a + um \equiv b + vm \pmod{m}$ for every integers u and v
- If $ka \equiv kb \pmod{m}$ and $gcd(k, m) = 1$, then $a \equiv b \pmod{m}$
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Warmup: An impossible Josephus problem

The problem

Ten people are sitting in circle, and every mth person is executed.
Prove that, for every $k \geqslant 1$, the first, second, and third person executed *cannot* be 10, k, and $k+1$, in this order.

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Solution

- If 10 is the first to be executed, then $10|m$.
- If k is the second to be executed, then $m \equiv k \pmod 9$.
- If $k+1$ is the third to be executed, then $m \equiv 1 \pmod 8$, because $k+1$ is the first one after k.

But if $10|m$, then m is even, and if $m \equiv 1 \pmod 8$, then m is odd: it cannot be both at the same time.

Application of congruence relation

Example 1: Find the remainder of the division of $a = 1395^4 \cdot 675^3 + 12 \cdot 17 \cdot 22$ by 7.

As $1395 \equiv 2 \pmod 7$, $675 \equiv 3 \pmod 7$, $12 \equiv 5 \pmod 7$, $17 \equiv 3 \pmod 7$ and $22 \equiv 1 \pmod 7$, then

$$a \equiv 2^4 \cdot 3^3 + 5 \cdot 3 \cdot 1 \pmod 7$$

As $2^4 = 16 \equiv 2 \pmod 7$, $3^3 = 27 \equiv 6 \pmod 7$, and $5 \cdot 3 \cdot 1 = 15 \equiv 1 \pmod 7$ it follows

$$a \equiv 2 \cdot 6 + 1 = 13 \equiv 6 \pmod 7$$

Application of congruence relation

Example 2: Find the remainder of the division of $a = 53 \cdot 47 \cdot 51 \cdot 43$ by 56.

A. As $53 \cdot 47 = 2491 \equiv 27 \pmod{56}$ and $51 \cdot 43 = 2193 \equiv 9 \pmod{56}$, then

$$a \equiv 27 \cdot 9 = 243 \equiv 19 \pmod{56}$$

B. As $53 \equiv -3 \pmod{56}$, $47 \equiv -9 \pmod{56}$, $51 \equiv -5 \pmod{56}$ and $43 \equiv -13 \pmod{56}$, then

$$a \equiv (-3) \cdot (-9) \cdot (-5) \cdot (-13) = 1755 \equiv 19 \pmod{56}$$

Application of congruence relation

Example 3: Find a remainder of dividing 45^{69} by 89

Make use of so called *method of squares*:

$$\begin{aligned}
45 &\equiv 45 \pmod{89}\\
45^2 &= 2025 \equiv 67 \pmod{89}\\
45^4 &= (45^2)^2 \equiv 67^2 = 4489 \equiv 39 \pmod{89}\\
45^8 &= (45^4)^2 \equiv 39^2 = 1521 \equiv 8 \pmod{89}\\
45^{16} &= (45^8)^2 \equiv 8^2 = 64 \equiv 64 \pmod{89}\\
45^{32} &= (45^{16})^2 \equiv 64^2 = 4096 \equiv 2 \pmod{89}\\
45^{64} &= (45^{32})^2 \equiv 2^2 = 4 \equiv 4 \pmod{89}
\end{aligned}$$

As $69 = 64 + 4 + 1$, then

$$45^{69} = 45^{64} \cdot 45^4 \cdot 45^1 \equiv 4 \cdot 39 \cdot 45 \equiv 7020 \equiv 78 \pmod{89}$$

Application of congruence relation

Let $n = a_k \cdot 10^k + a_{k-1} \cdot 10^{k-1} + \ldots + a_1 \cdot 10 + a_0$, where $a_i \in \{0, 1, \ldots, 9\}$ are digits of its decimal representation.

Theorem: An integer n is divisible by 11 iff the difference of the sums of the odd numbered digits and the even numbered digits is divisible by 11 :

$$11 | (a_0 + a_2 + \ldots) - (a_1 + a_3 + \ldots)$$

Proof.

Note, that $10 \equiv -1 \pmod{11}$. Then $10^i \equiv (-1)^i \pmod{11}$ for any i. Hence,

$$\begin{aligned} n &\equiv a_k(-1)^k + a_{k-1}(-1)^{k-1} + \ldots - a_1 + a_0 = \\ &= (a_0 + a_2 + \ldots) - (a_1 + a_3 + \ldots) \pmod{11} \qquad \text{Q.E.D.} \end{aligned}$$

Example 4: 34425730438 is divisible by 11

Indeed, due to the following expression is divisible by 11:

$$(8 + 4 + 3 + 5 + 4 + 3) - (3 + 0 + 7 + 2 + 4) = 27 - 16 = 11$$

Strange numbers: “arithmetic of days of the week”

Addition:

$\oplus$	Su	Mo	Tu	We	Th	Fr	Sa
Su	Su	Mo	Tu	We	Th	Fr	Sa
Mo	Mo	Tu	We	Th	Fr	Sa	Su
Tu	Tu	We	Th	Fr	Sa	Su	Mo
We	We	Th	Fr	Sa	Su	Mo	Tu
Th	Th	Fr	Sa	Su	Mo	Tu	We
Fr	Fr	Sa	Su	Mo	Tu	We	Th
Sa	Sa	Su	Mo	Tu	We	Th	Fr

Multiplication:

$\odot$	Su	Mo	Tu	We	Th	Fr	Sa
Su	Su	Su	Su	Su	Su	Su	Su
Mo	Su	Mo	Tu	We	Th	Fr	Sa
Tu	Su	Tu	Th	Sa	Mo	We	Fr
We	Su	We	Sa	Tu	Fr	Mo	Th
Th	Su	Th	Mo	Fr	Tu	Sa	We
Fr	Su	Fr	We	Mo	Sa	Th	Tu
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Th	Th	Fr	Sa	Su	Mo	Tu	We
Fr	Fr	Sa	Su	Mo	Tu	We	Th
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We	Su	We	Sa	Tu	Fr	Mo	Th
Th	Su	Th	Mo	Fr	Tu	Sa	We
Fr	Su	Fr	We	Mo	Sa	Th	Tu
Sa	Su	Sa	Fr	Th	We	Tu	Mo

Commutativity:

$$Tu + Fr = Fr + Tu \qquad\qquad Tu \cdot Fr = Fr \cdot Tu$$

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Su	Su	Mo	Tu	We	Th	Fr	Sa
Mo	Mo	Tu	We	Th	Fr	Sa	Su
Tu	Tu	We	Th	Fr	Sa	Su	Mo
We	We	Th	Fr	Sa	Su	Mo	Tu
Th	Th	Fr	Sa	Su	Mo	Tu	We
Fr	Fr	Sa	Su	Mo	Tu	We	Th
Sa	Sa	Su	Mo	Tu	We	Th	Fr

Multiplication:

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Su	Su	Su	Su	Su	Su	Su	Su
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Th	Su	Th	Mo	Fr	Tu	Sa	We
Fr	Su	Fr	We	Mo	Sa	Th	Tu
Sa	Su	Sa	Fr	Th	We	Tu	Mo

Associativity:

$$(Mo + We) + Fr = Mo + (We + Fr)(Mo \cdot We) \cdot Fr = Mo \cdot (We \cdot Fr)$$

Strange numbers: “arithmetic of days of the week”

Addition:

⊕	Su	Mo	Tu	We	Th	Fr	Sa
Su	Su	Mo	Tu	We	Th	Fr	Sa
Mo	Mo	Tu	We	Th	Fr	Sa	Su
Tu	Tu	We	Th	Fr	Sa	Su	Mo
We	We	Th	Fr	Sa	Su	Mo	Tu
Th	Th	Fr	Sa	Su	Mo	Tu	We
Fr	Fr	Sa	Su	Mo	Tu	We	Th
Sa	Sa	Su	Mo	Tu	We	Th	Fr

Multiplication:

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Su	Su	Su	Su	Su	Su	Su	Su
Mo	Su	Mo	Tu	We	Th	Fr	Sa
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Th	Su	Th	Mo	Fr	Tu	Sa	We
Fr	Su	Fr	We	Mo	Sa	Th	Tu
Sa	Su	Sa	Fr	Th	We	Tu	Mo

Subtraction is inverse operation of addition:

$$Th - We = (Mo + We) - We = Mo$$

Strange numbers: “arithmetic of days of the week”

Addition:

$\oplus$	Su	Mo	Tu	We	Th	Fr	Sa
Su	Su	Mo	Tu	We	Th	Fr	Sa
Mo	Mo	Tu	We	Th	Fr	Sa	Su
Tu	Tu	We	Th	Fr	Sa	Su	Mo
We	We	Th	Fr	Sa	Su	Mo	Tu
Th	Th	Fr	Sa	Su	Mo	Tu	We
Fr	Fr	Sa	Su	Mo	Tu	We	Th
Sa	Sa	Su	Mo	Tu	We	Th	Fr

Multiplication:

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Su	Su	Su	Su	Su	Su	Su	Su
Mo	Su	Mo	Tu	We	Th	Fr	Sa
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We	Su	We	Sa	Tu	Fr	Mo	Th
Th	Su	Th	Mo	Fr	Tu	Sa	We
Fr	Su	Fr	We	Mo	Sa	Th	Tu
Sa	Su	Sa	Fr	Th	We	Tu	Mo

Su is zero element:

$$We + Su = We \qquad\qquad We \cdot Su = Su$$

Strange numbers: “arithmetic of days of the week”

Addition:

⊕	Su	Mo	Tu	We	Th	Fr	Sa
Su	Su	Mo	Tu	We	Th	Fr	Sa
Mo	Mo	Tu	We	Th	Fr	Sa	Su
Tu	Tu	We	Th	Fr	Sa	Su	Mo
We	We	Th	Fr	Sa	Su	Mo	Tu
Th	Th	Fr	Sa	Su	Mo	Tu	We
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Multiplication:

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Su	Su	Su	Su	Su	Su	Su	Su
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Th	Su	Th	Mo	Fr	Tu	Sa	We
Fr	Su	Fr	We	Mo	Sa	Th	Tu
Sa	Su	Sa	Fr	Th	We	Tu	Mo

Mo is unit:

$$We \cdot Mo = We$$

Arithmetic modulo m

- Numbers are denoted by $\overline{0}, \overline{1}, \ldots, \overline{m-1}$, where $\overline{a}$ represents the class of all integers that dividing by m give remainder a.
- Operations are defined as follows

$$\begin{aligned} \overline{a}+\overline{b}=\overline{c} \quad &\text{iff} \quad a+b\equiv c \pmod{m} \\ \overline{a}\cdot\overline{b}=\overline{c} \quad &\text{iff} \quad a\cdot b\equiv c \pmod{m} \end{aligned}$$

Examples

- “arithmetic of days of the week", modulus 7
- Boolean algebra, modulus 2

Division in modular arithmetic

- Dividing $\overline{a}$ by $\overline{b}$ means to find a quotient x, such that $\overline{b} \cdot x = \overline{a}$, s.o. $\overline{a}/\overline{b} = x$

In "arithmetic of days of the week":

- $Mo/Tu = Th$ ja $Tu/Mo = Tu$.
- We cannot divide by Su, exceptionally Su/Su could be any day.
- A quotient is well defined for $\overline{a}/\overline{b}$ for every $\overline{b} \neq \overline{0}$, if the modulus is a prime number.

⊙	Su	Mo	Tu	We	Th	Fr	Sa
Su	Su	Su	Su	Su	Su	Su	Su
Mo	Su	Mo	Tu	We	Th	Fr	Sa
Tu	Su	Tu	Th	Sa	Mo	We	Fr
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Th	Su	Th	Mo	Fr	Tu	Sa	We
Fr	Su	Fr	We	Mo	Sa	Th	Tu
Sa	Su	Sa	Fr	Th	We	Tu	Mo

Division modulo prime p

Theorem

If m is a prime number and $x < m$, then the numbers

$$\overline{x}\cdot\overline{0}, \overline{x}\cdot\overline{1}, \ldots, \overline{x}\cdot\overline{m-1}$$

are pairwise different.

Proof. Assume contrary, that the remainders of dividing $x \cdot i$ and $x \cdot j$, where $i < j$, by m are equal. Then $m|(j-i)x$, that is impossible as $j - i < m$ and $gcd(m,x) = 1$. Hence, $\overline{x}\cdot\overline{i} \neq \overline{x}\cdot\overline{j}$ Q.E.D.

Corollary

If m is prime number, then the quotient of the division $\overline{x} = \overline{a}/\overline{b}$ modulo m is well defined for every $b \neq 0$.

If the modulus is not prime ...

The quotient is not well defined, for example:

$$\overline{1} = \overline{2}/\overline{2} = \overline{3}$$

$\odot$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{0}$	$\overline{2}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

Computing of $\overline{x} = \overline{a}/\overline{b}$ modulo p (where p is a prime number)

In two steps:

1. Compute $\overline{y} = \overline{1}/\overline{b}$
2. Compute $\overline{x} = \overline{y} \cdot \overline{a}$

How to compute $\overline{y} = \overline{1}/\overline{b}$ i.e. find such a $\overline{y}$, that $\overline{b} \cdot \overline{y} = \overline{1}$

Algorithm:

```
1 Using Euclidean algorithm, compute $gcd(p, b) = \ldots = 1$
2 Find the coefficients $s$ and $t$, such that $ps + bt = 1$
3 if $t \geqslant p$ then $t := t$ mod $p$ fi
4 return(t)                                   % Property: $\overline{t} = \overline{1}/\overline{b}$
```

Division modulo p

Example: compute $\overline{53}/\overline{2}$ modulo 234 527

- At first, we find $\overline{1}/\overline{2}$. For that we compute GCD of the divisor and modulus:
$$gcd(234527, 2) = gcd(2, 1) = 1$$
- The remainder can be expressed by modulus ad divisor as follows:
$$1 = 2(-117263) + 234527 \text{ or}$$
$$-117263 \cdot 2 \equiv 117264 \pmod{234527}$$

  Thus, $\overline{1}/\overline{2} = \overline{117264}$
- Due to $x = 53 \cdot 117264 \equiv 117290 \pmod{234527}$, the result is $\overline{x} = \overline{53} \cdot \overline{117264} = \overline{117290}$.

Linear equations

Solve the equation $\overline{7}\overline{x}+\overline{3}=\overline{0}$ modulo 47

Solution can be written as $\overline{x}=-\overline{3}/\overline{7}$

- Compute GCD using Euclidean algorithm

$$gcd(47,7)=gcd(7,5)=gcd(5,2)=gcd(2,1)=1,$$

  that yields the relations

$$1=5-2\cdot 2 \qquad 2=7-5 \qquad 5=47-6\cdot 7$$

- Find coefficients of 47 and 7:

$$\begin{aligned}1&=5-2\cdot 2=\\&=(47-6\cdot 7)-2\cdot(7-5)=\\&=47-8\cdot 7+2\cdot 5=\\&=47-8\cdot 7+2\cdot(47-6\cdot 7)=\\&=3\cdot 47-20\cdot 7\end{aligned}$$

Continues on the next slide ...

Linear equations (2)

Solve the equation $\overline{7}\overline{x}+\overline{3}=\overline{0}$ modulo 47

- The previous expansion of the $gcd(47,7)$ shows that $-20\cdot 7\equiv 1 \pmod{47}$ i.e. $27\cdot 7\equiv 1 \pmod{47}$
  Hence, $\overline{1}/\overline{7}=\overline{-20}=\overline{27}$
- The solution is $\overline{x}=\overline{-3}\cdot\overline{27}=\overline{13}$

The latter equality follows from the congruence relation $44\equiv -3 \pmod{47}$, therefore $x=44\cdot 27=1188\equiv 13 \pmod{47}$

Solving a system of equations using elimination method

Example

Assuming modulus 127, find integers x and y such that:

$$\begin{cases} \overline{12}\,\overline{x}+\overline{31}\,\overline{y}=\overline{2} \\ \overline{2}\,\overline{x}+\overline{89}\,\overline{y}=\overline{23} \end{cases}$$

Accordingly to the elimination method, multiply the second equation by $-\overline{6}$ and sum pu the equations, we get

$$\overline{y}=\frac{\overline{2}-\overline{6}\cdot\overline{23}}{\overline{31}-\overline{6}\cdot\overline{89}}$$

Due to $6\cdot 23=138\equiv 11 \pmod{127}$ and $6\cdot 89=534\equiv 26 \pmod{127}$, the latter equality can be transformed as follows:

$$\overline{y}=\frac{\overline{2}-\overline{11}}{\overline{31}-\overline{26}}=\frac{-\overline{9}}{\overline{5}}$$

Substituting $\overline{y}$ into the second equation, express $\overline{x}$ and transform it further considering that $5\cdot 23=115\equiv -12 \pmod{127}$ and $9\cdot 89=801\equiv 39 \pmod{127}$:

$$\overline{x}=\frac{\overline{23}-\overline{89}\,\overline{y}}{\overline{2}}=\frac{\overline{23}\cdot\overline{5}-\overline{899}}{\overline{10}}=\frac{\overline{-12}+\overline{39}}{\overline{10}}=\frac{\overline{27}}{\overline{10}}$$

Solving a system of equations using elimination method (2)

Continuation of the last example ...

Computing:

$$\begin{cases} \overline{x} = \overline{27}/\overline{10} \\ \overline{y} = -\overline{9}/\overline{5} \end{cases}$$

if the modulus is 127.

Apply the Euclidean algorithm:

$$gcd(127,5) = gcd(5,2) = gcd(2,1) = 1$$
$$gcd(127,10) = gcd(10,7) = gcd(7,3) = gcd(3,1) = 1$$

That gives the equalities:

$$1 = 5 - 2\cdot 2 = 5 - 2(127 - 25\cdot 5) = (-2)127 + 51\cdot 5$$
$$1 = 7 - 2\cdot 3 = 127 - 12\cdot 10 - 2(10 - 127 + 12\cdot 10) = 3\cdot 127 - 38\cdot 10$$

Hence, division by $\overline{5}$ is equivalent to multiplication by $\overline{51}$ and division by $\overline{10}$ to multiplication to $-\overline{38}$. Then the solution of the system is

$$\begin{cases} \overline{x} = \overline{27}/\overline{10} = -\overline{27}\cdot\overline{38} = -\overline{1026} = \overline{117} \\ \overline{y} = -\overline{9}/\overline{5} = -\overline{9}\cdot\overline{51} = -\overline{459} = \overline{49} \end{cases}$$

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- Fermat’ test
- Rabin-Miller test

3 Phi and Mu

For determining whether a number n is prime.

There are alternatives:

- Try all numbers $2, \ldots, n-1$. If n is not dividisble by none of them, then it is prime.
- Same as above, only try numbers $2, \ldots, \sqrt{n}$.
- Probabilistic algorithms with polynomial complexity (the Fermat' test, the Miller-Rabin test, etc.).
- Deterministic primality-proving algorithm by Agrawal–Kayal–Saxena (2002).

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Fermat's "Little" Theorem

Theorem

If p is prime and a is an integer not divisible by p, then

$$p | a^{p-1} - 1$$

Lemma

If p is prime and $0 < k < p$, then $p | \binom{p}{k}$

Proof. This follows from the equality

$$\binom{p}{k} = \frac{p^{\underline{k}}}{k!} = \frac{p(p-1)\cdots(p-k+1)}{k(k-1)\cdots 1}$$

Pierre de Fermat

(1601–1665)

Another formulation of the theorem

Fermat's "little" theorem

If p is prime, and a is an integer, then $p|a^p - a$.

Proof.

- If a is not divisible by p, then $p|a^{p-1}-1$ iff $p|(a^{p-1}-1)a$
- The assertion is trivally true if $a=0$. To prove it for $a>0$ by induction, set $a=b+1$. Hence,

$$\begin{aligned} a^p - a &= (b+1)^p - (b+1) = \\ &= \binom{p}{0} b^p + \binom{p}{1} b^{p-1} + \cdots + \binom{p}{p-1} b + \binom{p}{p} - b - 1 = \\ &= (b^p - b) + \binom{p}{1} b^{p-1} + \cdots + \binom{p}{p-1} b \end{aligned}$$

Here the expression $(b^p - b)$ is divisible by p by the induction hypothesis, while other terms are divisible by p by the Lemma. Q.E.D.

Application of the Fermat' theorem

Example: Find a remainder of division the integer 3^{4565} by 13.

Fermat' theorem gives $3^{12} \equiv 1 \pmod{13}$. Let's divide 4565 by 12 and compute the remainder: $4565 = 380 \cdot 12 + 5$. Then

$$3^{4565} = (3^{12})^{380} 3^5 \equiv 1^{380} 3^5 = 81 \cdot 3 \equiv 3 \cdot 3 = 9 \pmod{13}$$

Application of the Fermat' theorem (2)

Prove that $n^{18}+n^{17}-n^2-n$ is divisible by 51 for any positive integer n.

Let's factorize

$$\begin{aligned}
A &= n^{18}+n^{17}-n^2-n= \\
&= n(n^{17}-n)+n^{17}-n= \\
&= (n+1)(n^{17}-n)= && \text{\% From Fermat' theorem } \Rightarrow 17|A \\
&= (n+1)n(n^{16}-1)= \\
&= (n+1)n(n^8-1)(n^8+1)= \\
&= (n+1)n(n^4-1)(n^4+1)(n^8+1)= \\
&= (n+1)n(n^2-1)(n^2+1)(n^4+1)(n^8+1)= \\
&= \underbrace{(n+1)n(n-1)}_{\text{divisible by 3}}(n+1)(n^2+1)(n^4+1)(n^8+1)
\end{aligned}$$

Hence, A is divisible by $17\cdot 3=51$.

Pseudoprimes

A pseudoprime is a probable prime (an integer that shares a property common to all prime numbers) that is not actually prime.

- The assertion of the Fermat' theorem is valid also for some composite numbers.
- For instance, if $p = 341 = 11 \cdot 31$ and $a = 2$, then dividing

  $$2^{340} = (2^{10})^{34} = 1024^{34}$$

  by 341 yields the remainder 1, because of dividing 1024 gives the remainder 1.
- Integer 341 is a Fermat' pseudoprime to base 2.
- However, 341 the assertion of Fermat' theorem is not satisfied for the base 3. Dividing 3^{340} by 341 results in the remainder 56.

Carmichael numbers

Definition

An integer n that is a Fermat pseudoprime for every base a that are coprime to n is called a Carmichael number.

Example: let $p = 561 = 3 \cdot 11 \cdot 17$ and $gcd(a, p) = 1$.

$$
\begin{aligned}
a^{560} &= (a^2)^{280} \text{ gives the remainder 1, if divded by 3} \\
a^{560} &= (a^{10})^{56} \text{ gives the remainder 1, if divded by 11} \\
a^{560} &= (a^{16})^{35} \text{ gives the remainder 1, if divded by 17}
\end{aligned}
$$

Thus $a^{560} - 1$ is divisible by 3, by 11 and by 17.

- See `http://oeis.org/search?q=Carmichael`, sequence nr A002997

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Fermat' test

Fermat' theorem: If p is prime and integer a is such that $1 \leqslant a < p$, then

$$a^{p-1} \equiv 1 \pmod{p}.$$

To test, whether n is prime or composite number:

- Check validity of $a^{n-1} \equiv 1 \pmod{n}$ for every $a = 2, 3, \ldots, n-1$.
- If the condtion is not satisfiable for one or more value of a, then n is composite, otherwise prime.

Example: is 221 prime?

$$\begin{aligned} 2^{220} &= \left(2^{11}\right)^{20} \equiv 59^{20} = \left(59^{4}\right)^{5} \equiv 152^{5} = \\ &= 152 \cdot \left(152^{2}\right)^{2} \equiv 152 \cdot 120^{2} \equiv 152 \cdot 35 = 5320 \equiv 16 \pmod{221} \end{aligned}$$

Hence, 221 is a composite number. Indeed, $221 = 13 \cdot 17$

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Hence, 221 is a composite number. Indeed, $221 = 13 \cdot 17$

Problems of the Fermat' test

- Computing of LARGE powers $\rightsquigarrow$ *method of squares*
- Computing with LARGE numbers$\rightsquigarrow$ *modular arithmetic*
- *n* is a pseudoprime $\rightsquigarrow$ choose *a randomly* and repeat
- *n* is a Carmichael number $\rightsquigarrow$ use better methods, for example *Rabin-Miller test*

Modified Fermat' test

Input: n – a value to test for primality
k – the number of times to test for primality
Output: "n is composite" or "n is probably prime"

```
for $i := 0$ step 1 to $k$
do
    pick $a$ randomly, such that $1 < a < n$
    if $a^{n-1} \not\equiv 1 \pmod{n}$ return("$n$ is composite"); exit
od
return("$n$ is probably prime")
```

Example, $n = 221$, randomly picked values for a are 38 ja 26

$a^{n-1} = 38^{220} \equiv 1 \pmod{221}$ ⇝ 38 is pseudoprime

$a^{n-1} = 26^{220} \equiv 169 \not\equiv 1 \pmod{221}$ ⇝ 221is composite number

Does not work, if n is a Carmichael number: $561, 1105, 1729, 2465, 2821, 6601, 8911, \ldots$

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od
return("$n$ is probably prime")
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Example, $n = 221$, randomly picked values for a are 38 ja 26

$$a^{n-1} = 38^{220} \equiv 1 \pmod{221} \qquad \rightsquigarrow \text{38 is pseudoprime}$$

$$a^{n-1} = 26^{220} \equiv 169 \not\equiv 1 \pmod{221} \qquad \rightsquigarrow \text{221is composite number}$$

Does not work, if n is a Carmichael number: $561, 1105, 1729, 2465, 2821, 6601, 8911, \ldots$

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An idea, how to battle against Carmichael numbers

- Let n be an odd positive integer to be tested against primality
- Randomly pick an integer a from the interval $0 \leqslant a \leqslant n-1$.
- Consider the expression $a^n - a = a(a^{n-1} - 1)$ and until possible, transform it applying the identity $x^2 - 1 = (x-1)(x+1)$
- If the expression $a^n - a$ is not divisible by n, then all its divisors are also not divisible by n.
- If at least one factor is divisible by n, then n is probably prime. To increase this probability, it is need to repeat with another randomly chosen value of a.

Example: $n = 221$

- Let's factorize:

$$\begin{aligned} a^{221} - a = a(a^{220} - 1) &= \\ &= a(a^{110} - 1)(a^{110} + 1) = \\ &= a(a^{55} - 1)(a^{55} + 1)(a^{110} + 1) \end{aligned}$$

- If $a = 174$, then $174^{110} = (174^2)^{55} \equiv (220)^{55} = 220 \cdot (220^2)^{27} \equiv 220 \cdot 1^{27} \equiv 220 \equiv -1 \pmod{221}$. Thus 221 is either prime or pseudoprime to the base 174.
- If $a = 137$, then $221 \nmid a, 221 \nmid (a^{55} - 1), 221 \nmid (a^{55} + 1), 221 \nmid (a^{110} + 1)$. Consequently, 221 is a composite number

Rabin-Miller test

Input: $n > 3$ – a value to test for primality
k – the number of times to test for primality
Output: "n is composite" or "n is probably prime"

```
Factorize $n - 1 = 2^s \cdot d$, where $d$ is an odd number
for $i := 0$ step 1 to $k$
{
    Randomly pick value for $a \in \{2, 3, \dots, n-1\}$;
    $x := a^d$ mod $n$;
    if $x = 1$ or $x = n - 1$ then { continue; }
    for $r := 1$ step 1 to $s - 1$
    {
        $x := x^2$ mod $n$
        if $x = 1$ then { return("$n$ is composite"); exit; }
        if $x = n - 1$ then { break; }
    }
    return("$n$ is composite"); exit;
}
return("$n$ is probably prime");
```

Complexity of the algorithm is $\mathcal{O}(k \log_2^3 n)$

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3. Phi and Mu

Euler's totient function ϕ

Euler's totient function

Euler's totient function ϕ is defined for $m \geqslant 2$ as

$$\phi(m) = |\{n \in \{0, \ldots, m-1\} \mid gcd(m,n) = 1\}|$$

n	2	3	4	5	6	7	8	9	10	11	12	13
$\phi(n)$	1	2	2	4	2	6	4	6	4	10	4	12

Computing Euler's function

Theorem

1. If $p \geqslant 2$ is prime and $k \geqslant 1$, then $\phi(p^k) = p^{k-1} \cdot (p-1)$.
2. If $m, n \geqslant 1$ are relatively prime, then $\phi(m \cdot n) = \phi(m) \cdot \phi(n)$.

Proof

1. Exactly every pth number n, starting with 0, has $gcd(p^k, n) \geqslant p > 1$.
   Then $\phi(p^k) = p^k - p^k/p = p^{k-1} \cdot (p-1)$.
2. If $m \perp n$, then for every $k \geqslant 1$ it is $k \perp mn$ if and only if both $m \perp k$ and $n \perp k$.

Multiplicative functions

Definition

$f : \mathbb{N}^+ \to \mathbb{N}^+$ is *multiplicative* if it satisfies the following condition:
For every $m, n \geqslant 1$, if $m \perp n$, then $f(m \cdot n) = f(m) \cdot f(n)$

Theorem

If $g(m) = \sum_{d|m} f(d)$ is multiplicative, then so is f.

- $g(1) = g(1) \cdot g(1) = f(1)$ must be either 0 or 1.
- If $m = m_1 m_2$ with $m_1 \perp m_2$, then by induction

$$\begin{aligned}
g(m_1 m_2) &= \sum_{d_1 d_2 | m_1 m_2} f(d_1 d_2) \\
&= \left(\sum_{d_1 | m_1} f(d_1) \right) \left(\sum_{d_2 | m_1} f(d_2) \right) - f(m_1) f(m_2) + f(m_1 m_2) \\
&\quad \text{with } d_1 \perp d_2 \\
&= g(m_1) g(m_2) - f(m_1) f(m_2) + f(m_1 m_2) :
\end{aligned}$$

whence $f(m_1 m_2) = f(m_1) f(m_2)$ as $g(m_1 m_2) = g(m_1) g(m_2)$.

$\sum_{d|m} \phi(d) = m$: Example

The fractions

$$\frac{0}{12}, \frac{1}{12}, \frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{5}{12}, \frac{6}{12}, \frac{7}{12}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}, \frac{11}{12}$$

are simplified into:

$$\frac{0}{1}, \frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{2}{3}, \frac{3}{4}, \frac{15}{6}, \frac{11}{12}.$$

The divisors of 12 are 1, 2, 3, 4, 6, and 12. Of these:

- The denominator 1 appears $\phi(1) = 1$ time: $0/1$.
- The denominator 2 appears $\phi(2) = 1$ time: $1/2$.
- The denominator 3 appears $\phi(3) = 2$ times: $1/3$, $2/3$.
- The denominator 4 appears $\phi(4) = 2$ times: $1/4$, $3/4$.
- The denominator 6 appears $\phi(6) = 2$ times: $1/6$, $5/6$.
- The denominator 12 appears $\phi(12) = 4$ times: $1/12$, $5/12$, $7/12$, $11/12$.

We have thus found: $\phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(6) + \phi(12) = 12$.

$\sum_{d|m}\phi(d) = m$: Proof

Call a fraction a/b basic if $0 \leqslant a < b$.
After simplifying any of the m basic fractions with denominator m, the denominator d of the resulting fraction must be a divisor of m.

Lemma

In the simplification of the m basic fractions with denominator m, for every divisor d of m, the denominator d appears exactly $\phi(d)$ times.

It follows immediately that $\sum_{d|m}\phi(d) = m$.

Proof

- After simplification, the fraction k/d only appears if $gcd(k,d) = 1$: for every d there are at most $\phi(d)$ such k.
- But each such k appears in the fraction kh/n, where $h \cdot d = n$.

Euler's theorem

Statement

If m and n are positive integers and $n \perp m$, then $n^{\phi(m)} \equiv 1 \pmod{m}$.

Note: Fermat's little theorem is a special case of Euler's theorem for $m = p$ prime.

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Statement

If m and n are positive integers and $n \perp m$, then $n^{\phi(m)} \equiv 1 \pmod{m}$.

Note: Fermat's little theorem is a special case of Euler's theorem for $m = p$ prime.

Proof with $m \geqslant 2$ (cf. Exercise 4.32)

Let $U_m = \{0 \leqslant a < m \mid a \perp m\} = \{a_1, \ldots, a_{\phi(m)}\}$ in increasing order.

- The function $f(a) = na \pmod{m}$ is a permutation of U_m:
  If $f(a_i) = f(a_j)$, then $m | n(a_i - a_j)$, which is only possible if $a_i = a_j$.
- Consequently,

$$n^{\phi(m)} \prod_{i=1}^{\phi(m)} a_i \equiv \prod_{i=1}^{\phi(m)} a_i \pmod{m}$$

- But by construction, $\prod_{i=1}^{\phi(m)} a_i \perp m$: we can thus simplify and obtain the thesis.

Möbius function μ

Mobius function

Mobius' function μ is defined for $m \geqslant 1$ by the formula

$$\sum_{d|m} \mu(d) = [m = 1]$$

m	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mu(m)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0	-1

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$\mu(m)$	1	-1	-1	0	-1	1	-1	0	0	1	-1	0	-1

As $[m = 1]$ is clearly multiplicative, so is μ!

Computing the Möbius function

Theorem

For every $m \geqslant 1$,

$$\mu(m) = \begin{cases} (-1)^k & \text{if } m = p_1 p_2 \cdots p_k \text{ distinct primes}\,, \\ 0 & \text{if } p^2 | m \text{ for some prime } p\,. \end{cases}$$

Indeed, let p be prime. Then, as $\mu(1) = 1$:

- $\mu(1) + \mu(p) = 0$, hence $\mu(p) = -1$.
  The first formula then follows by multiplicativity.
- $\mu(1) + \mu(p) + \mu(p^2) = 0$, hence $\mu(p^2) = 0$.
  The second formula then follows, again by multiplicativity.

Möbius inversion formula

Theorem

Let $f, g : \mathbb{Z}^+ \to \mathbb{Z}^+$. The following are equivalent:

1. For every $m \geqslant 1$, $g(m) = \sum_{d|m} f(d)$.
2. For every $m \geqslant 1$, $f(m) = \sum_{d|m} \mu(d) g\left(\frac{m}{d}\right)$.

Corollary

For every $m \geqslant 1$,

$$\phi(m) = \sum_{d|m} \mu(d) \cdot \frac{m}{d} \ :$$

because we know that $\sum_{d|m} \phi(d) = m$.

Proof of Möbius inversion formula

Suppose $g(m) = \sum_{d|m} f(d)$ for every $m \geqslant 1$. Then for every $m \geqslant 1$:

$$\begin{aligned}
\sum_{d|m} \mu(d) g\left(\frac{m}{d}\right) &= \sum_{d|m} \mu\left(\frac{m}{d}\right) g(d) \\
&= \sum_{d|m} \mu\left(\frac{m}{d}\right) \sum_{k|d} f(k) \\
&= \sum_{k|m} \left(\sum_{d|(m/k)} \mu\left(\frac{m}{kd}\right) \right) f(k) \\
&= \sum_{k|m} \left(\sum_{d|(m/k)} \mu(d) \right) f(k) \\
&= \sum_{k|m} \left[\frac{m}{k} = 1 \right] f(k) \\
&= f(m).
\end{aligned}$$

The converse implication is proved similarly.