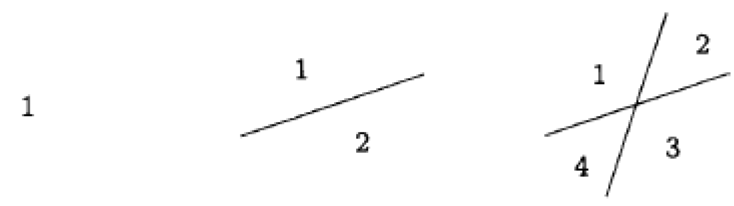
Lines in the plane:

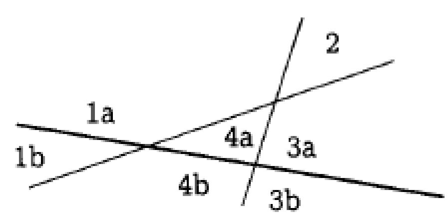
The second sample problem has a more geometric flavor. How many regions can a person obtain by making n straight cuts? Or, more academically: What is the maximum number of regions defined by n lines (not parallel lines, must be intersecting lines) in the plane? This problem was first solved in 1826, by the Swiss mathematician Jacob Steiner. Suppose, L(n) defines the maximum number of regions where n is the number of lines.

Again we start by looking at small cases, remembering to begin with the smallest of all. The plane with no lines has one region; with one line it has two regions; and with two lines it has four regions:



$L_0 = 1$ $L_1 = 2$ $L_2 = 4$

Sure, we think, $L = 2^n$; of course! Adding a new line simply doubles the number of regions. Unfortunately this is WRONG. With n=3, we get the following regions L(n)=7:



So, now the appropriate generalization for the recurrence relation of this problem can be explained as follows. The number of regions L(n) for n lines will firstly, depend on the number of regions for old lines (n-1) that is L(n-1). Secondly, now how many regions will increase from previous case for the new number of lines (n)? This will depend on the intersection points. Clearly, the nth line (new line) increases the number of regions if and only if it splits the old regions and it splits the old regions if and only if it hits the old lines (n-1) in n-1 different points or places. Only then the number of regions increase atleast 1 from the old regions. So, for this case the increment of regions for the new line (n) would depend on (n-1+1=n) n. So, now we have,

L(n)>=L(n-1)+n

So, the recurrence relation for the lines of plane problem would be defined as follows –

$L(0)=1$

$L(n)=L(n-1)+n$ for $n>0$

Now, the solution for this recurrence relation is solved as follows using iterative method –

$L(n)=L(n-1)+n$

$=L(n-2)+(n-1)+n$

$= L(n-3)+(n-2)+(n-1)+n$

$=L(0)+1+2+3$ [Putting $n=3$]

.

.

$= L(0)+1+2+3+\ldots\ldots+n$

$= 1+1+2+3+\ldots\ldots+n$ [$L(0)=1$]

$= 1+(1+2+3+\ldots\ldots+n)$

$= 1+[n(n+1)/2]$ [$1+2+3+\ldots..+n=n(n+1)/2$]

The solution for the recursive relation of lines of plane is $1+[n(n+1)/2]$