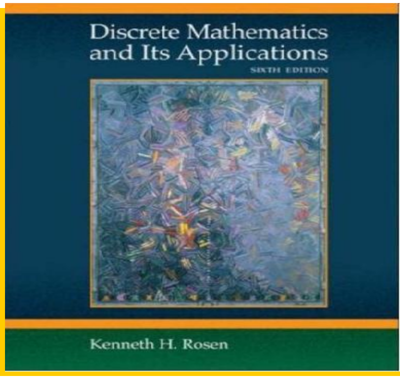
Chapter 8: Relations



- Relations(8.1)
- n-any Relations & their Applications (8.2)
- Representing Relations (8.3)
- Equivalence Relations (8.5)

Relations (8.1)

- Introduction

  – Relationship between a program and its variables

  – Integers that are congruent modulo k

  – Pairs of cities linked by airline flights in a network

Relations (8.1) (cont.)

- Relations & their properties

  – Definition 1

Let A and B be sets. A binary relation from A to B is a subset of A * B.

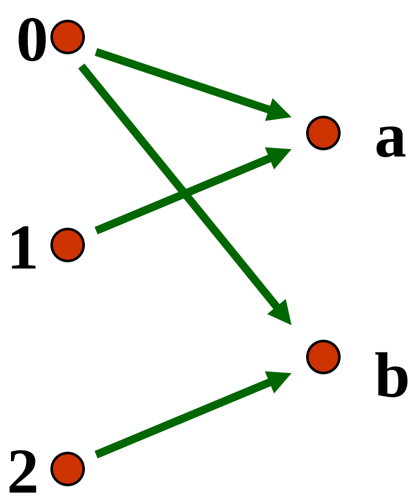
In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

Relations (8.1) (cont.)

– Notation:

$$aRb \Leftrightarrow (a, b) \in R$$

$$a \not R b \Leftrightarrow (a, b) \notin R$$



R	a	b
0	X	X
1	X	
2		X

Relations (8.1) (cont.)

– Example:
A = set of all cities
B = set of the 50 states in the USA
Define the relation R by specifying that (a, b) belongs to R if city a is in state b.

$$\left.\begin{array}{l} (\textit{Boulder}, \textit{Colorado}) \\ (\textit{Bangor}, \textit{Maine}) \\ (\textit{Ann Arbor}, \textit{Michigan}) \\ (\textit{Cupertino}, \textit{California}) \\ \textit{Red Bank}, \textit{New Jersey}) \end{array}\right\} \textit{are in R.}$$

Relations (8.1) (cont.)

Functions as relations

- The graph of a function f is the set of ordered pairs (a, b) such that b = f(a)

- The graph of f is a subset of A * B $\Rightarrow$ it is a relation from A to B

- Conversely, if R is a relation from A to B such that every element in A is the first element of exactly one ordered pair of R, then a function can be defined with R as its graph

Relations (8.1) (cont.)

- Relations on a set
  - Definition 2

A relation on the set A is a relation from A to A.

  - Example: A = set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}

Solution: Since (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b

R = {(1,1), (1,2), (1.3), (1.4), (2,2), (2,4), (3,3), (4,4)}

Relations (8.1) (cont.)

Properties of Relations

- Definition 3

A relation R on a set A is called reflexive if (a, a) $\in$ R for every element a $\in$ A.

Relations (8.1) (cont.)

– Example (a): Consider the following relations on {1, 2, 3, 4}

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Which of these relations are reflexive?

Relations (8.1) (cont.)

Solution:

R_3 and R_5: reflexive $\Leftarrow$ both contain all pairs of the form (a, a): (1,1), (2,2), (3,3) & (4,4).

R_1, R_2, R_4 and R_6: not reflexive $\Leftarrow$ not contain all of these ordered pairs. (3,3) is not in any of these relations.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (8.1) (cont.)

– Definition 4:

A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.
A relation R on a set A such that $(a, b) \in R$ and $(b, a) \in R$ only if $a = b$, for all $a, b \in A$, is called antisymmetric.

Relations (8.1) (cont.)

– Example: Which of the relations from example (a) are symmetric and which are antisymmetric?

Solution:

❖ R_2 & R_3: symmetric $\Leftarrow$ each case (b, a) belongs to the relation whenever (a, b) does.
For R_2: only thing to check that both (1,2) & (2,1) belong to the relation
For R_3: it is necessary to check that both (1,2) & (2,1) belong to the relation.
None of the other relations is symmetric: find a pair (a, b) so that it is in the relation but (b, a) is not.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (8.1) (cont.)

Solution (cont.):

- R_4, R_5 and R_6: antisymmetric $\Leftarrow$ for each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a, b) and (b, a) belong to the relation.
  None of the other relations is antisymmetric.: find a pair (a, b) with $a \neq b$ so that (a, b) and (b, a) are both in the relation.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (8.1) (cont.)

– Definition 5:

A relation R on a set A is called transitive if whenever (a, b) $\in$ R and (b,c) $\in$ R, then (a, c) $\in$ R, for all a, b, c $\in$ R.

Relations (8.1) (cont.)

– Example: Which of the relations in example (a) are transitive?

- R_4 , R_5 & R_6 : transitive $\Leftarrow$ verify that if (a, b) and (b, c) belong to this relation then (a, c) belongs also to the relation
  R_4 transitive since (3,2) and (2,1), (4,2) and (2,1), (4,3) and (3,1), and (4,3) and (3,2) are the only such sets of pairs, and (3,1) , (4,1) and (4,2) belong to R_4.
  Same reasoning for R_5 and R_6.
- R_1 : not transitive $\Leftarrow$ (3,4) and (4,1) belong to R_1, but (3,1) does not.
- R_2 : not transitive $\Leftarrow$ (2,1) and (1,2) belong to R_2, but (2,2) does not.
- R_3 : not transitive $\Leftarrow$ (4,1) and (1,2) belong to R_3, but (4,2) does not.

R_1 = {(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)}
R_2 = {(1,1), (1,2), (2,1)}
R_3 = {(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (3,4), (4,1), (4,4)}
R_4 = {(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)}
R_5 = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)}
R_6 = {(3,4)}

Relations (8.1) (cont.)

- Combining relations

  – Example:

    Let A = {1, 2, 3} and B = {1, 2, 3, 4, }. The relations R_1 = {(1,1), (2,2), (3,3)} and R_2 = {(1,1), (1,2), (1,3), (1,4)} can be combined to obtain:

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\}$$

$$R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

Relations (8.1) (cont.)

– Definition 6:

Let R be a relation from a set A to a set B and S a relation from B to a set C.
The composite of R and S is the relation consisting of ordered pairs (a, c), where a $\in$ A, c $\in$ C, and for which there exists an element b $\in$ B such that (a, b) $\in$ R and (b, c) $\in$ S. We denote the composite of R and S by S $\circ$ R.

Relations (8.1) (cont.)

– Example: What is the composite of the relations R and S where R is the relation from {1,2,3} to {1,2,3,4} with R = {(1,1), (1,4), (2,3), (3,1), (3,4)} and S is the relation from {1,2,3,4} to {0,1,2} with S = {(1,0), (2,0), (3,1), (3,2), (4,1)}?

Solution: S ∘ R is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered in R agrees with the first element of the ordered pair in S.
For example, the ordered pairs (2,3) in R and (3,1) in S produce the ordered pair (2,1) in S ∘ R. Computing all the ordered pairs in the composite, we find

S ∘ R = ((1,0), (1,1), (2,1), (2,2), (3,0), (3,1)}

N-ary Relations & their Applications (8.2)

- Relationship among elements of more than 2 sets often arise: n-ary relations
- Airline, flight number, starting point, destination, departure time, arrival time

N-ary Relations & their Applications (8.2) (cont.)

N-ary relations

– Definition 1:

Let A_1, A_2, …, A_n be sets. An n-ary relation on these sets is a subset of $A_1 * A_2 * \ldots * A_n$ where A_i are the domains of the relation, and n is called its degree.

– Example: Let R be the relation on N * N * N consisting of triples (a, b, c) where a, b, and c are integers with a<b<c. Then $(1,2,3) \in R$, but $(2,4,3) \notin R$. The degree of this relation is 3. Its domains are equal to the set of integers.

N-ary Relations & their Applications (8.2) (cont.)

Databases & Relations

– Relational database model has been developed for information processing

– A database consists of records, which are n-tuples made up of fields

– The fields contains information such as:

- Name
- Student #
- Major
- Grade point average of the student

N-ary Relations & their Applications (8.2) (cont.)

– The relational database model represents a database of records or n-ary relation

– The relation is R(Student-Name, Id-number, Major, GPA)

N-ary Relations & their Applications (8.2) (cont.)

– Example of records

(Smith, 3214, Mathematics, 3.9)

(Stevens, 1412, Computer Science, 4.0)
(Rao, 6633, Physics, 3.5)
(Adams, 1320, Biology, 3.0)
(Lee, 1030, Computer Science, 3.7)

N-ary Relations & their Applications (8.2) (cont.)

TABLE A: Students

Students Names	ID #	Major	GPA
Smith	3214	Mathematics	3.9
Stevens	1412	Computer Science	4.0
Rao	6633	Physics	3.5
Adams	1320	Biology	3.0
Lee	1030	Computer Science	3.7

N-ary Relations & their Applications (8.2) (cont.)

Operations on n-ary relations

– There are varieties of operations that are applied on n-ary relations in order to create new relations that answer eventual queries of a database

– Definition 2:

Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the selection operator s_C maps n-ary relation R to the n-ary relation of all n-tuples from R that satisfy the condition C.

N-ary Relations & their Applications (8.2) (cont.)

– Example:

if s_C = "Major = "computer science" $\wedge$ GPA > 3.5" then the result of this selection consists of the 2 four-tuples:

(Stevens, 1412, Computer Science, 4.0)

(Lee, 1030, Computer Science, 3.7)

N-ary Relations & their Applications (8.2) (cont.)

– Definition 3:

The projection $P_{i_1, i_2, \dots, i_m}$ maps the n-tuple $(a_1, a_2, \dots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \dots, a_{i_m})$ where $m \leq n$.

In other words, the projection $P_{i_1, i_2, \dots, i_m}$ deletes n – m of the components of n-tuple, leaving the i_1th, i_2th, …, and i_mth components.

N-ary Relations & their Applications (8.2) (cont.)

– Example: What relation results when the projection $P_{1,4}$ is applied to the relation in Table A?

Solution: When the projection $P_{1,4}$ is used, the second and third columns of the table are deleted, and pairs representing student names and GPA are obtained. Table B displays the results of this projection.

TABLE B: GPAs

Students Names	GPA
Smith	3.9
Stevens	4.0
Rao	3.5
Adams	3.0
Lee	3.7

N-ary Relations & their Applications (8.2) (cont.)

– Definition 4:

Let R be a relation of degree m and S a relation of degree n. The join $J_p(R,S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

N-ary Relations & their Applications (8.2) (cont.)

– Example: What relation results when the operator J_2 is used to combine the relation displayed in tables C and D?

TABLE C: Teaching Assignments

Professor	Dpt	Course #
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE D: Class Schedule

Dpt	Course #	Room	Time
Computer Science	518	N521	2:00 PM
Mathematics	575	N502	3:00 PM
Mathematics	611	N521	4:00 PM
Physics	544	B505	4:00 PM
Psychology	501	A100	3:00 PM
Psychology	617	A110	11:00 AM
Zoology	335	A100	9:00 AM
Zoology	412	A100	8:00 AM

N-ary Relations & their Applications (8.2) (cont.)

Solution: The join J_2 produces the relation shown in Table E

Table E: Teaching Schedule

Professor	Dpt	Course #	Room	Time
Cruz	Zoology	335	A100	9:00 AM
Cruz	Zoology	412	A100	8:00 AM
Farber	Psychology	501	A100	3:00 PM
Farber	Psychology	617	A110	11:00 AM
Grammer	Physics	544	B505	4:00 PM
Rosen	Computer Science	518	N521	2:00 PM
Rosen	Mathematics	575	N502	3:00 PM

N-ary Relations & their Applications (8.2) (cont.)

– Example: We will illustrate how SQL (Structured Query Language) is used to express queries by showing how SQL can be employed to make a query about airline flights using Table F. The SQL statements

```
SELECT departure_time
FROM Flights
WHERE destination = 'Detroit'
```

are used to find the projection P_5 (on the departure_time attribute) of the selection of 5-tuples in the flights database that satisfy the condition: destination = 'Detroit'. The output would be a list containing the times of flights that have Detroit as their destination, namely, 08:10, 08:47, and 9:44.

N-ary Relations & their Applications (8.2) (cont.)

Table F: Flights

Airline	Flight #	Gate	Destination	Departure time
Nadir	122	34	Detroit	08:10
Acme	221	22	Denver	08:17
Acme	122	33	Anchorage	08:22
Acme	323	34	Honolulu	08:30
Nadir	199	13	Detroit	08”47
Acme	222	22	Denver	09:10
Nadir	322	34	Detroit	09:44

Representing Relations (8.3)

- First way is to list the ordered pairs
- Second way is through matrices
- Third way is through direct graphs

Representing Relations (8.3)

- Representing relations through matrices
- $$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

  – Example: Suppose that the relation R on a set is represented by the matrix: $$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

  Is R reflexive, symmetric, and/or antisymmetric?

  Solution: Since all the diagonal elements of this matrix are equal to 1, R is reflexive. Moreover, since M_R is symmetric $\Rightarrow$ R is symmetric. R is not antisymmetric.

Representing Relations (8.3)

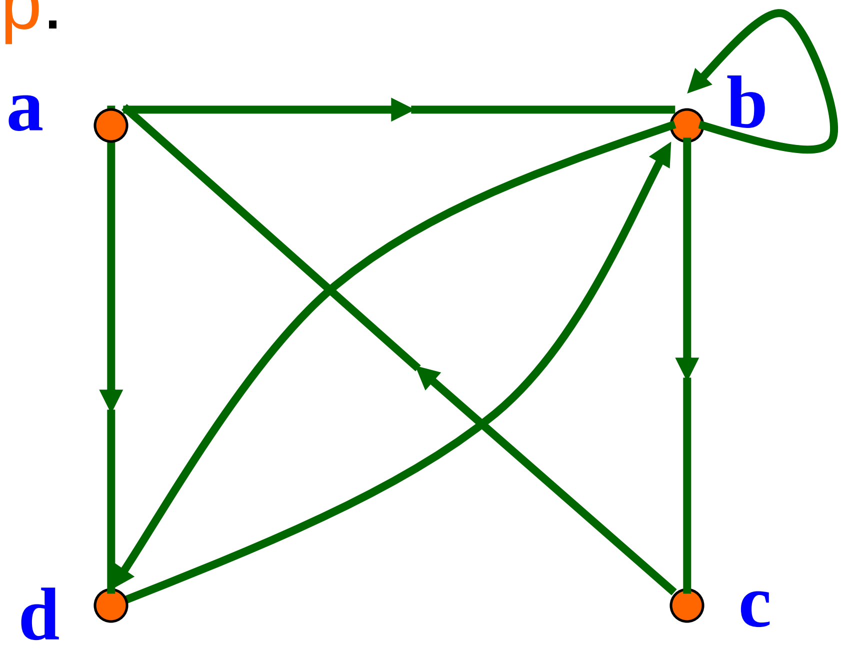
- Representing relations using diagraphs

  – Definition 1:

A directed graph, or diagraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs). The vertex a is called the initial vertex of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

Representing Relations (8.3)

– Example: The directed graph with vertices a, b, c and d , and edges (a,b), (a,d), (b,b), (b,d), (c,a) and (d,b). The edge (b,b) is called a loop.



Equivalence Relations (8.5)

- Students registration time with respect to the first letter of their names
- R contains (x,y) $\Leftrightarrow$ x and y are students with last names beginning with letters in the same block
- 3 blocks are considered: A-F, G-O, P-Z
- R is reflexive, symmetric & transitive
- The set of student is therefore divided in 3 classes depending on the first letter of their names

Equivalence Relations (8.5)

- Definition 1

> A relation on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

- Examples
:
  - Suppose that R is the relation on the set of strings of English letters such that *aRb* if and only if l(a) = l(b), where l(x) is the length of the string x. Is R an equivalence relation?
    Solution: R is reflexive, symmetric and transitive $\Rightarrow$ R is an equivalence relation
  - A divides b; is it an equivalence relation?

Equivalence Relations (8.5)

Equivalence classes

– Definition 2:

> Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the equivalence class of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we will delete the subscript R and write [a] for this equivalence class.

Equivalence Relations (8.5)

– Example: What are the equivalences classes of 0 and 1 for congruence modulo 4?

Solution:
The equivalence class of 0 contains all the integers a such that a ≡ 0 (mod 4). Hence, the equivalence class of 0 for this relation is

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$$

The equivalence class of 1 contains all the integers a such that a ≡ 1 (mod 4). The integers in this class are those that have a remainder of 1 when divided by 4. Hence, the equivalence class of 1 for this relation is

$$[1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$

Equivalence Relations (8.5)

Equivalence classes & partitions

– Theorem 1:

Let R be an equivalence relation on a set A. These statements are equivalent:

i. $a\ R\ b$
ii. $[a] = [b]$
iii. $[a] \cap [b] \neq \emptyset$

Equivalence Relations (8.5)

– Theorem 2:

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.

Equivalence Relations (8.5)

– Example: List the ordered pairs in the equivalence relation R produced by the partition A_1 = [1,2,3}, A_2 = {4,5} and A_3 = {6} of S = {1,2,3,4,5,6}

Solution: The subsets in the partition are the equivalences classes of R. The pair (a,b) $\in$ R if and only if a and b are in the same subset of the partition.

The pairs (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2) and (3,3) $\in$ R $\Leftarrow$ A_1 = [1,2,3} is an equivalence class. The pairs (4,4), (4,5), (5,4) and (5,5) $\in$ R $\Leftarrow$ A_2 = {4,5} is an equivalence class.
The pair (6,6) $\in$ R $\Leftarrow$ {6} is an equivalence class.

No pairs other than those listed belongs to R.