Q.2 Show that the function

$$f(z) = \begin{cases} \dfrac{x^2 y^5 (x+iy)}{x^4+y^{10}} & ; \text{ when } z \neq 0 \\ 0 & ; \text{ when } z = 0 \end{cases}$$

is continuous and that the cauchy-Riemann equation are satisfied at the origin, yet $f'(0)$ does n't exist.

1st part

Soln:

Cauchy-Riemann equation:

product function

Given, $f(z) = \dfrac{x^2 y^5 (x+iy)}{x^4+y^{10}}$

will be equal

Real
Imaginary
$y = x$
$\downarrow$
$y^n = x^n$ (4th step)

For this type of math.

$$= \frac{x^3 y^5 + i x^2 y^6}{x^4+y^{10}}$$

$$\Rightarrow f(z) = \frac{x^3 y^5}{x^4+y^{10}} + i \frac{x^2 y^6}{x^4+y^{10}}$$

$$\Rightarrow f(z) = u(x,y) + i\, v(x,y)$$

where, $u(x,y) = \dfrac{x^3 y^5}{x^4+y^{10}}$; $u(0,0) = 0$

$v(x,y) = \dfrac{x^2 y^6}{x^4+y^{10}}$; $v(0,0) = 0$

At origin (0,0),

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y)\ u(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \cdot \left[\frac{(\Delta x)^3 \cdot 0}{(\Delta x)^4 + 0}\right]$$

$$= \lim_{\Delta x \to 0} (0)$$

$$= 0$$

Again, at origin (0,0),

$$\frac{\partial v}{\partial y} = \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \cdot [v(0, \Delta y) - v(0)]$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \cdot \left[\frac{0 \cdot (\Delta y)^6}{0 + (\Delta y)^{10}}\right]$$

$$= \lim_{\Delta y \to 0} (0)$$

$$= 0$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

Again at origin (0,0),

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y}\left[\frac{0 \cdot (\Delta y)^6}{0 + (\Delta y)^{20}}\right]$$

$$= \lim_{\Delta y \to 0} (0)$$

$$= 0$$

Again at origin (0,0),

$$\frac{\partial v}{\partial x} = \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{v(\Delta x, 0) - v(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} (0)$$

$$= 0$$

Here, $\frac{\partial v}{\partial x} = 0$

$$\frac{\partial u}{\partial y} = 0$$

So, we can write

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad [\because 0 = (-1) \times 0]$$

So, C-R equation is satisfied.

$\boxed{2^{nd} \text{ part}}$

From the definition of differentiation we know,

$$\frac{d}{dz} f(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

at origin (0,0),

$$\frac{d}{dz} f(0) = \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z}$$

$$\Rightarrow \frac{d}{dz} f(0) = \lim_{\Delta z \to 0} \frac{1}{\Delta z} [f(\Delta z)]$$

$$\Rightarrow \frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta x)^2 \cdot (\Delta y)^5 (\Delta x + i \Delta y)}{(\Delta x)^4 + (\Delta y)^{10} (\Delta x + i \Delta y)} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{(\Delta x)^2 \cdot (\Delta y)^5}{(\Delta x)^4 + (\Delta y)^{10}}$$

Now along real axis $x \neq 0$, $y = 0$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{(\Delta x)^2 \cdot 0 \, \cancel{(\Delta x + 0)}}{(\Delta x)^4 + 0}$$

$$= 0$$

Now along imaginary axis $y \neq 0$, $x = 0$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} (0) = 0$$

Now along $y = x$

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = \Delta x}} \frac{(\Delta x)^2 \cdot (\Delta x)^5 \, \cancel{(\Delta x + i \Delta x)}}{(\Delta x)^4 + (\Delta x)^{10}}$$

$$\Rightarrow \frac{d}{dz} f(0) = \lim_{\Delta x \to 0} \frac{(\Delta x)^7 \cdot \cancel{\Delta x (1+i)}}{\Delta x^4 \{1 + (\Delta x)^6\}}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x^4 \cancel{(1+i)}}{1 + (\Delta x)^6}$$

$$= \lim_{\Delta x \to 0} (0)$$

$$= 0$$

$\boxed{\text{3nd part}}$

Now along $y^5 = x^2$ we get,

$$\frac{d}{dz} f(0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y^5 = \Delta x^2}} \frac{(\Delta x)^2 \cdot (\Delta x)^2 \cancel{(\Delta x + i\Delta y)}}{(\Delta x)^4 + (\Delta y^5)^2}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^4 \cancel{(\Delta x + i\Delta y)}}{(\Delta x)^4 + (\Delta x)^4}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{0 \cdot (0 + i\Delta y)} (\Delta x)^4}{2(\Delta x)^4}$$

$$= \lim_{\Delta x \to 0} \left(\frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Since, the derivation along real axis, imaginary axis, $y = x$ and along $y^5 = x^2$ is not same. so, $f(0)$ does n't exist.

$\boxed{4^{th} \text{ part}}$

continuity

Given, $f(z) = \dfrac{x^2y^5(x+iy)}{x^4+y^{20}}$

$$\frac{d}{dz} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^5(x+iy)}{x^4+y^{20}}$$

Now,

Along real axis $x\neq 0; y=0$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x\cdot 0\,(x+0)}{x^4+0}$$

$$= 0$$

Along imaginary axis $x=0; y\neq 0$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{0\cdot y^5(0+iy)}{0+y^{20}}$$

$$= 0$$

Now,

Along $y=x$,

$$= \lim_{\substack{x\to 0\\ y=x}} \frac{x^2\cdot x^5(x+ix)}{x^4+x^{20}}$$

$$= \lim_{x\to 0} \frac{x^7\cdot x(1+i)}{x^8\left(\frac{1}{x^4}+x^2\right)}$$

$$= \lim_{x\to 0} \frac{(1+i)}{\left(\frac{1}{x^4}+x^2\right)}$$

$$= \lim_{x\to 0} \frac{1+i}{\infty}$$

$$= \lim_{x\to 0} (0)$$

$$= 0$$

~~so~~ So, this function is continuous.

Complex Integration

Chapter - 3

Cauchy integral formula

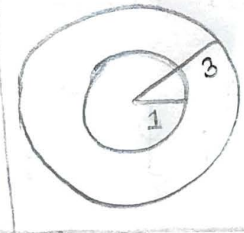
$$\boxed{F(a) = \frac{1}{2\pi i}\oint \frac{F(z)}{z-a}\,dz.}$$

Example

$$F(2) = \frac{1}{2\pi i}\oint \frac{F(z)}{z-2}\,dz$$

$\int$ Line integration

$\oint$ closed integration



0.1 Show that

$$\frac{1}{2\pi i}\oint \frac{e^z}{z-2}\,dz = \begin{cases} e^2; & \text{if } C \text{ in the circle } |z|=3 \\ 0; & \text{if } C \text{ in " " } |z|=1 \end{cases}$$

Soln: We know cauchy integral formula,

$$f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)}\,dz \quad \text{——— (1)}$$

Here, $f(z) = e^z$, $a=2$

Here $a=2$ so ~~that~~ the function $f(z)=e^z$ is analytic ~~at~~ inside the circle $|z|=3$

Since $3>2$ and the function $f(z)=e^z$ is not analytic at $|z|=1$ since $z=2$ outside the circle $|z|=1$; since $1<2$

$z-2=0$

It is in the interval

$z=2$

$|z|=3$

$\sqrt{x^2+y^2}=3$

$x^2+y^2=3^2$

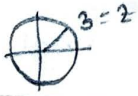
$z-4=0$

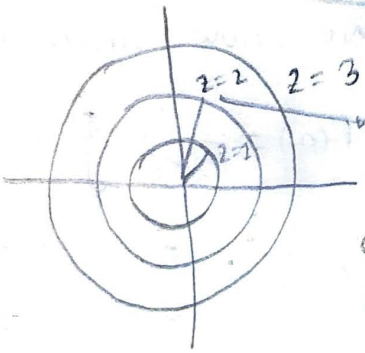
It is outside the interval

$z=4$

$|z|=3$

$x^2+y^2=3^2$





it is valid for $z=3$ (because it is inside the circle $z=3$)

$z = 2$ lie inside the circle $|z| = 3$.

Hence, $\frac{1}{2\pi i} \oint \frac{e^z}{z-2} dz = f(a) = f(z) = f(2) = e^2$

Again, $|z| = 2$ outside the circle $|z| = 1$

so, $F(z) = e^z$ is not analytic for $z = 1$

$\frac{1}{2\pi i} \oint \frac{e^z}{z-2} dz = f(a) = f(z) = 0$.

$\boxed{0.2}$ Show that $\oint \frac{\sin 3z}{z + \frac{\pi}{2}} dz = 2\pi i$, where C is the circle $|z| = 5$.

$\frac{1}{2\pi i}$

Soln: let, $f(z) = (\sin 3z)$

Also let $z + \frac{\pi}{2} = 0$

$|z| = \left|-\frac{\pi}{2}\right| = 1.57 < 5$

Since $1.57 < 5$ so the function $F(z) = \sin 3z$ is analytic inside the circle $|z| = 5$

We know from the cauchy integral formula

$$f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)} dz$$

$$\Rightarrow \frac{1}{2\pi i} \oint \frac{\sin 3z}{z + \pi/2} dz = f(a)$$

$(z = -\pi/2 = a$ replace

Hence it is analytic so $z = a)$

$$\boxed{\begin{array}{l} f(z) = \sin 3z \\ f(a) = \sin 3a \end{array}}$$ replace

$$\Rightarrow \oint \frac{\sin 3z}{z + \pi/2} \, dz = 2\pi i \, f(a)$$

$$\Rightarrow f(z) = \sin 3z = f(a) = \sin 3a$$

$$\Rightarrow f(-\pi/2) = \sin (-3\pi/2) = 1$$

Ans:

Last class (mid)

Show that $\oint \frac{e^{3z}}{z-\pi i} dz = \begin{cases} -2\pi i, & \text{if } C \text{ in the circle } |z-1|=4 \\ 0, & \text{if } C \text{ in the circle } |z-2|=3 \end{cases}$

0.3

$$\boxed{\frac{1}{2\pi i}\oint \frac{f(z)}{z-a}}$$

Soln:

let, $f(z) = e^{3z}$ ——— (1)

inside this circle (analytic)

and let $z-\pi i = 0$

$z = \pi i = (0+i\pi)$

$z = x+iy$

outside this circle (not analytic)

$|z| = |0+i\pi| = \sqrt{(0)^2+(\pi)^2} = \pi = 3.1416$

$|z-1| = 4$

$\Rightarrow |x+iy-1| = 4$

$\Rightarrow |x-1+iy| = 4$

$\Rightarrow \sqrt{(x-1)^2+y^2} = 4$

$\Rightarrow (x-1)^2+y^2 = 4^2$

Here, $3.14164 < 4$ so ~~that~~ the function $f(z)$ in analytic for $|z-1| = 4$

Then from cauchy integral formula.

$$\frac{1}{2\pi i}\oint \frac{f(z)}{z-a} dz = f(a)$$

$$\Rightarrow \frac{1}{2\pi i}\oint \frac{e^{3z}}{z-\pi i} dz = f(\pi i) \quad [\because z=\pi i]$$

$$\Rightarrow \oint \frac{e^{3z}}{z-\pi i}\, dz = 2\pi i \; f(\pi i)$$

$$= 2\pi i \cdot e^{3\pi i} \quad [\because f(z) = e^{3z}]$$

$$= 2\pi i (\cos 3\pi + i \sin 3\pi) \quad [\because e^{i\theta} = \cos\theta + i\sin\theta]$$

$$= 2\pi i (-1+0)$$

$$= -2\pi i$$

Again since $3.1416 > 3$, so the function ~~is~~ ~~not~~ $f(z)$ is not analytic for $|z-2| = 3$

$$\therefore \oint \frac{e^{3z}}{z-\pi i}\, dz = 0$$

(proved)

$\boxed{0.4}$

Show that $\oint \frac{e^{tz}}{z^2+1}\, dz = 2\pi i \sin t$

where C is the curve $|z| = 3$

circle, parabola, Hyperbola

N.B

$$\cos\theta + i\sin\theta = e^{i\theta}$$

$$\cos\theta - i\sin\theta = e^{-i\theta}$$

$$\cos\theta + i\sin\theta + \cos\theta - i\sin\theta = e^{i\theta} + e^{-i\theta}$$

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$$

$$\cos\theta + i\sin\theta - \cos\theta + i\sin\theta = e^{i\theta} - e^{-i\theta}$$

$$\boxed{\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

Soln:

let, $f(z) = e^{tz}$ —— (1)

Also let. $z^2+1 = 0$

$z^2 = -1$

$z = \sqrt{-1} = \pm i$

$z = -i$	$z = +i$
$\lvert z\rvert = 1$	$\lvert z\rvert = 1$

Since $1 < 3$ so the function $f(z)$ is analytic for $|z| = 3$.

Now, From cauchy integral formula,

$$\frac{1}{2\pi i}\oint \frac{f(z)}{z-a} \cdot dz = f(a)$$

$$\Rightarrow \frac{1}{2\pi i}\oint \frac{e^{tz}}{z-i} dz = f(i)$$

$$\Rightarrow \oint \frac{e^{tz}}{z-i} dz = 2\pi i f(i).$$

$$\oint \frac{e^{tz}}{(z-i)} dz = 2\pi i e^{ti} \quad \text{—— (2)}$$

Again $\frac{1}{2\pi i}\oint \frac{e^{tz}}{z-(-i)}\,dz = \cancel{2\pi i}\, f(-i)$

$$\Rightarrow \oint \frac{e^{tz}}{z-(-i)}\,dz = 2\pi i\, f(-i) = 2\pi i\, e^{-ti} \quad \text{—— (3)}$$

In this way we have to prove L.H.S = R.H.S

L.H.S, $\oint \frac{e^{tz}}{z^2-(-1)}\,dz$

$$= \oint \frac{e^{tz}}{z^2-i^2}\,dz \qquad (2)-(3)$$

$$= \oint \frac{e^{tz}}{(z+i)(z-i)}\,dz$$

$$= -\frac{1}{2i}\oint e^{tz}\left(\frac{1}{(z+i)} - \frac{1}{z-i}\right)dz$$

$$\boxed{\begin{aligned} &= -\frac{1}{2i}\cdot\frac{z-i-(z+i)}{(z+i)(z-i)} \\ &= -\frac{1}{2i}\,\frac{2i}{} \end{aligned}}$$

$$= \frac{1}{2i}\oint e^{tz}\left(\frac{1}{(z-i)} - \frac{1}{(z+i)}\right)dz$$

$$= \frac{1}{2i}\left[\oint \frac{e^{tz}}{z-i}\,dz - \oint \frac{e^{tz}}{(z+i)}\,dz\right.$$

$$= \frac{1}{2i}\left(2\pi i \cdot e^{ti} - 2\pi i e^{-ti}\right)$$

$$= \frac{1}{2i}\, 2\pi i\,(e^{ti} - e^{-ti})$$

$$= 2\pi i \left(\frac{e^{ti} - e^{-ti}}{2i}\right)$$

$$= 2\pi i \sin t \quad = R.H.S$$

$$\therefore \oint \frac{e^{tz}}{z^2+1}\,dz = 2\pi i \sin t$$

(Proved)

or,

Another way:

~~if R.H.S co~~

Equation ② − ③

$$\oint \frac{e^{tz}}{z-i}\,dz - \oint \frac{e^{tz}}{z+i}\,dz = 2\pi i e^{ti} - 2\pi i e^{-ti}$$

$$\Rightarrow \oint e^{tz}\left(\frac{1}{z-i} - \frac{1}{z+i}\right) dz = 2\pi i\,(e^{ti} - e^{-ti})$$

$$\Rightarrow \oint e^{tz}\left(\frac{z+i-z+i}{(z-i)(z+i)}\right) dz = 2\pi i\,(e^{ti} - e^{-ti})$$

$$\Rightarrow \oint e^{tz} \frac{2i}{(z^2+1)} dz = 2\pi i (e^{ti} - e^{-ti})$$

$$\Rightarrow \oint \frac{e^{tz}}{z^2+1} dz = 2\pi i \left(\frac{e^{ti} - e^{-ti}}{2i} \right)$$

$$\therefore \oint \frac{e^{tz}}{z^2+1} = 2\pi i \sin t$$

(Proved)

Harmonic

(last topic for mid)

Prove that the function $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic find its harmonic conjugate v and express it $u+iv$ as an analytic function of z.

1st part

Soln:

lep $\boxed{\text{laplace equation}, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}}$

Given, $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\frac{\partial u}{\partial x} = 3y \cdot 2x + 2 \cdot 2x - 0 - 0 = 6xy + 4x$$

Again, $\Rightarrow \frac{\partial^2 u}{\partial x^2} = 6y \cdot 1 + 4 \cdot 1 = 6y + 4$

Again, $\frac{\partial u}{\partial y} = 3 \cdot x^2 \cdot 1 + 0 - 3y^2 - 2 \cdot 2y$

$$\Rightarrow \frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = 0 - 6y - 4$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = -6y - 4$$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$$= 6y + 4 - 6y - 4$$

$$= 0$$

Since u satisfied the Laplace equation so, u is harmonic.

~~Integrating both side with respect to y~~

$$\cancel{v = \int (6xy + 4x)\,dy}$$

$\boxed{\text{2nd part}}$

From, the C-R equation we know,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 6xy + 4x$$

$$\Rightarrow \frac{\partial v}{\partial y} = 6xy + 4x$$

Integrating both side with respect to y,

$$v = \int (6xy + 4x)\, dy$$

$$v = 6x \int y\, dy + 4x \int dy$$

$$= 6x \cdot \frac{y^2}{2} + 4x \cdot y + c$$

$$\therefore u + vi = (3x^2y + 2x^2 - y^3 - 2y) + (3xy^2 + 4xy)i + c$$

A.W-7

Show that $u = 2x - x^3 + 3xy^2$ is harmonic and also find the harmonic conjugate and express it $u + iv$.

$\boxed{1^{st} \text{ part}}$

Soln: Given, $u = 2x - x^3 + 3xy^2$

$$\frac{\partial u}{\partial x} = 2 \cdot 1 - 3x^2 + 3 \cdot 1 \cdot y^2 = 2 - 3x^2 + 3y^2$$

Again, $\frac{\partial^2 u}{\partial x^2} = 0 - 6x + 0 = -6x$

Again, $\frac{\partial u}{\partial y} = 2x - x^3 + 3xy^2$
$= 0 - 0 + 6xy = 6xy$

$$\frac{\partial^2 u}{\partial y^2} = 6xy = 6x$$

Now, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$= -6x + 6x$

$= 0$

Since, u satisfied the laplace equation. So, u is Harmoni

$\boxed{2^{nd} \text{ part}}$

From the C-R equation we know,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2$$

$$\Rightarrow \frac{\partial v}{\partial y} = 2 - 3x^2 + 3y^2$$

Integrating both side with respect to y,

$$v = (2 - 3x^2 + 3y^2)\, dy$$

$$= 2\int dy - 3x^2 \int dy + 3\int y^2 dy$$

$$= 2y - 3x^2 y + 3 \cdot \frac{y^3}{3}$$

$$\therefore v = 2y - 3x^2 y + y^3 + c$$

$$\therefore u + vi = 2x - x^3 + 3xy^2 + (2y + y^3 - 3x^2 y)i + c$$

Ans