An Analysis of Fornberg's Numerical Conformal Mapping Method

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What is a Conformal Map?

In the simplest sense, a conformal mapping, also called conformal transformation, or biholomorphic map, is a transformation that preserves angles and orientation.

The stereographic projection, used in ancient times, and Mercator's projection, developed in the sixteenth century, were early examples of conformal mappings.

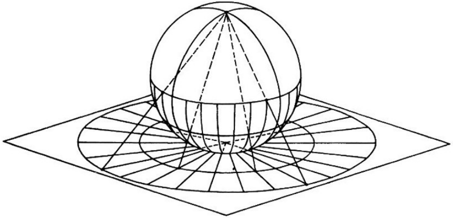

Figure: Stereographic Projection
(from https://ncatlab.org/nlab/show/stereographic+projection)

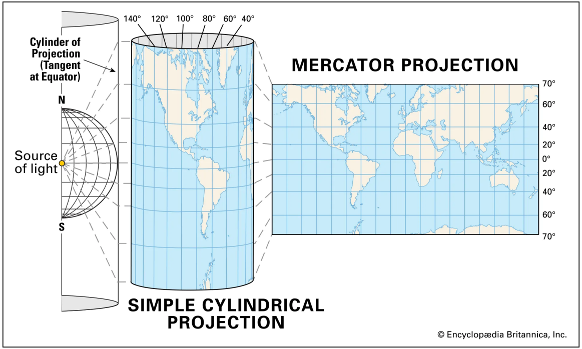

Figure: Mercator Projection
(from https://www.britannica.com/science/Mercator-projection)

Conformal Mappings in Math, Physics, and Engineering

Conformal mappings are invaluable for solving problems in math, physics, and engineering that can be expressed in terms of functions of a complex variable(s) yet exhibit inconvenient geometries.

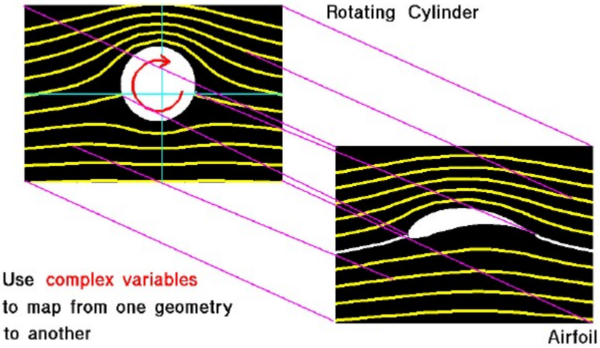

Figure: Joukowski Conformal Mapping of an Airfoil
(from https://www.grc.nasa.gov/www/k-12/VirtualAero/BottleRocket/airplane/map.html)

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In this section, we mathematically define the concept of conformal mappings and explore their geometric characteristics. We also state the infamous Riemann Mapping Theorem given by Bernhard Riemann in his 1851 PhD thesis and illustrate an example of constructing a Riemann map.

Figure: Bernhard Riemann
(from https://mathshistory.st-andrews.ac.uk/Biographies/Riemann/)

Conformal Mapping Example 1

Example

Let $\Omega := \{z \in \mathbb{C} \mid Re(z) > 0\}$ *and define a mapping* $f : \Omega \mapsto \mathbb{C}$ *as* $f(z) = e^{i\frac{\pi}{4}}z$ *with* $z = |z|e^{i\,arg(z)}$. *Then,*

$$|f(z)| = |e^{i\frac{\pi}{4}}z| = |e^{i\frac{\pi}{4}}||z| = \left|cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right||z| = \left|\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right||z| = |z|$$

and

$$\begin{aligned} arg(f(z)) &= arg(e^{i\frac{\pi}{4}}z) = arg\left(|z|e^{i\frac{\pi}{4}}e^{iarg(z)}\right) = arg\left(|z|e^{i(\frac{\pi}{4}+arg(z))}\right) \\ &= arg(|z|) + arg\left(e^{i(\frac{\pi}{4}+arg(z))}\right) = \frac{\pi}{4} + arg(z). \end{aligned}$$

Conformal Mapping Example 2

Example

Let $\Omega := \{z \in \mathbb{C} \mid Re(z) > 0\}$, *and define a mapping* $f : \Omega \mapsto \mathbb{C}$ *as the complex conjugate function,* $f(z) = \bar{z}$.

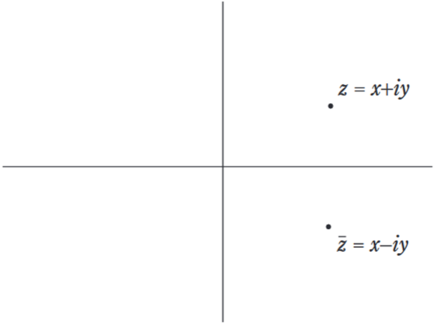


Figure: Complex Conjugate Function

Angle Preserving

Definition

A 2×2 *matrix* $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ *preserves angles if and only if there exists some* $p > 0$ *such that* $A^T A = pI$ *where* I *is the identity matrix. This is because* A *preserves angles (as a linear transformation on* $\mathbb{R}^2$*) if it preserves dot products up to a scalar* p *(because* $cos(\theta) = \frac{u \cdot v}{\sqrt{u \cdot u}\sqrt{v \cdot v}}$*).*

Angle Preserving

Proof.

Let $u, v \in \mathbb{R}^2$ and $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then,

$$\begin{aligned} A^T A &= pI \\ (A^T A) u^T v &= p u^T v \\ (Au)^T Av &= p u^T v \\ Au \cdot Av &= pu \cdot v. \end{aligned}$$

This means that if an angle θ' is the angle between Au and Av then

$$cos(\theta') = \frac{Au \cdot Av}{\sqrt{Au \cdot Au}\sqrt{Av \cdot Av}} = \frac{pu \cdot v}{\sqrt{u \cdot u}\sqrt{pv \cdot v}} = \frac{pu \cdot v}{\sqrt{p}||u||\sqrt{p}||v||}$$
$$= \frac{u \cdot v}{||u|| * ||v||} = cos(\theta). \quad \blacksquare$$

Orientation Preserving

Definition

Let $f : \Omega \mapsto \mathbb{C}$ *be defined as* $f = u + iv$*, where* $u : \Omega \mapsto \mathbb{R}$ *and* $v : \Omega \mapsto \mathbb{R}$*. The mapping* $f : \Omega \mapsto \mathbb{C}$ *induces a real-valued mapping* $f_{\mathbb{R}} : \Omega \mapsto \mathbb{R}^2$ *given by* $f_{\mathbb{R}}(x, y) = (u(x, y), v(x, y))$*. This map preserves orientation at* (x_0, y_0) *if and only if the Jacobian matrix,* $\mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0)$*, has a positive determinant.*

Conformal Mapping Example 1 Revisited

Example

Let $f(z) = e^{i\frac{\pi}{4}}z$. *Then,*

$$f(z) = e^{i\frac{\pi}{4}}z = \frac{\sqrt{2}}{2}(1+i)(x+iy) = \left(\frac{\sqrt{2}(x-y)}{2}\right) + i\left(\frac{\sqrt{2}(x+y)}{2}\right) = u + iv.$$

and $f_{\mathbb{R}}(x,y) = \left(\frac{\sqrt{2}(x-y)}{2}, \frac{\sqrt{2}(x+y)}{2}\right)$. *Taking the determinant of the corresponding Jacobian matrix yields,*

$$\det\left(\mathbf{J}_{f_{\mathbb{R}}}(x_0,y_0)\right) = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = 1 > 0.$$

Conformal Mapping Example 2 Revisited

Example

Let $f(z) = \bar{z} = x - iy$. *Then,* $f_{\mathbb{R}}(x, y) = (x, -y)$ *and we see that,*

$$\det\left(\mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0)\right) = \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1 < 0.$$

Conformality

Definition

Let Ω be an open subset of $\mathbb{R}^2$. A map $f : \Omega \mapsto \mathbb{R}^2$ is conformal if and only if f is differentiable on Ω and $\mathbf{J}_f(x_0, y_0)$ preserves angle and orientation at each $(x_0, y_0) \in \Omega$.

Holomorphic Function and its Relation to Conformality

Theorem

Let $\Omega \subset \mathbb{C}$. Then $f : \Omega \mapsto \mathbb{C}$ is holomorphic with $f'(z_0) \neq 0$ for all $z_0 \in \Omega$ if and only if $f_{\mathbb{R}} : \Omega \mapsto \mathbb{R}^2$ is conformal.

Proof.

Recall that if $f = u + iv$ then $f_{\mathbb{R}}(x, y) = (u(x, y), v(x, y))$ and

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

where Δz approaches zero from any direction. Since f is holomorphic, approaching from any direction gives the same value for the limit. Therefore, we can approach from the x direction and let $\Delta z = \Delta x + 0i$.

Proof.

Then,

$$\begin{aligned}\frac{df}{dz} &= \lim_{\Delta x \to 0} \frac{f((x+\Delta x)+iy) - f(x+iy)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u((x+\Delta x),y) + iv((x+\Delta x),y) - (u(x,y)+iv(x,y))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u((x+\Delta x),y) - u(x,y) + i\left(v((x+\Delta x),y) - v(x,y)\right)}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\end{aligned}$$

Using the Cauchy-Riemann equations, $A = \mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix}$.

Proof.

Therefore,

$$A^T A = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ -\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 & 0 \\ 0 & \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \end{bmatrix}$$
$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which implies that A preserves angles at each point in Ω where $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 > 0$.

Proof.

Also,

$$\det\left(\mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0)\right) = \det\begin{bmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

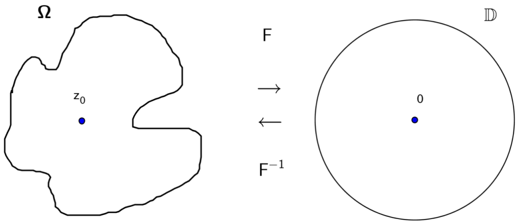
implies that A preserves orientation when $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 > 0$, but $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\left(\frac{\partial u}{\partial x}\right) + i\left(\frac{\partial v}{\partial x}\right)\right|^2 = \left|\frac{df}{dz}\right|^2 > 0$ since $\frac{df}{dz} \neq 0$ on Ω. ■

Riemann Mapping Theorem

Theorem

Let $\Omega \subset \mathbb{C}$ be a non-empty simply connected set that is not all of $\mathbb{C}$. Then for any $z_0 \in \Omega$, there exists a unique biholomorphism $F : \Omega \mapsto \mathbb{D}$, where $\mathbb{D}$ represents the unit disk centered at the origin, such that

$$F(z_0) = 0, \text{ and } F'(z_0) > 0.$$



Riemann Mapping Theorem Example

Example

Let $\mathbb{H} := \{z \in \mathbb{C} \mid Im(z) > 0\}$*, i.e. the upper half plane. Then* $\mathbb{H}$ *can be mapped to the unit disk* $\mathbb{D}$ *via a Möbius transformation. Let* f *be the Möbius transformation that maps* $0, 1, \infty$ *to* $1, i, -1$ *respectively. Then the line through* $0, 1, \infty$ *(the real axis) must be mapped to the circle through* $1, i, -1$ *(the unit circle).*

Riemann Mapping Theorem Example Cont.

Example

The restriction of the Möbius transformation f to $\mathbb{H}$ thus maps $\mathbb{H}$ onto $\mathbb{D}$. We can find the formula for f as follows:

1) f has the form $f(z) = \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Since $f(\infty) \neq \infty$, we have $c \neq 0$, and as the expression $cz + d$ determines the behavior of $f(z)$ we can therefore assume $c = 1$. Thus $f(z) = \dfrac{az+b}{z+d}$.

2) If $f(\infty) = -1$, then for $z = \infty$ we have $f(\infty) = \dfrac{a \cdot \infty + b}{\infty + d} = -1$. As $z \to \infty$, the dominant term in $\dfrac{a \cdot \infty + b}{\infty + d}$ is az. Since $f(\infty)$ must be finite, az must tend to a finite value and in order for $f(\infty) = -1$, a must be -1. Then $f(z) = \dfrac{-z+b}{z+d}$.

Riemann Mapping Theorem Example Cont.

Example

3) Since $f(0) = 1$, *we have* $\dfrac{b}{d} = 1$, *so* $b = d$. *Then* $f(z) = \dfrac{-z+b}{z+b}$.

4) Since $f(1) = i$ *we have* $\dfrac{-1+b}{1+b} = i$. *Solving for* b *yields*

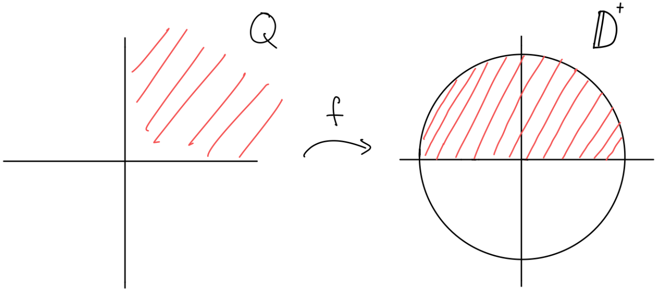
$$\frac{-1+b}{1+b} = i \iff -1+b = i(1+b) \iff b(1-i) = (i+1)$$
$$\iff b = \frac{i+1}{1-i} = \frac{(i+1)(1+i)}{(1-i)(1+i)} = \frac{i-1+1+i}{1+1} = i$$

Therefore, we find $f(z) = \dfrac{-z+i}{z+i}$ *conformally maps the upper half plane* $\mathbb{H}$ *onto the unit disk* $\mathbb{D}$.

Riemann Mapping Theorem Example Cont.

Example

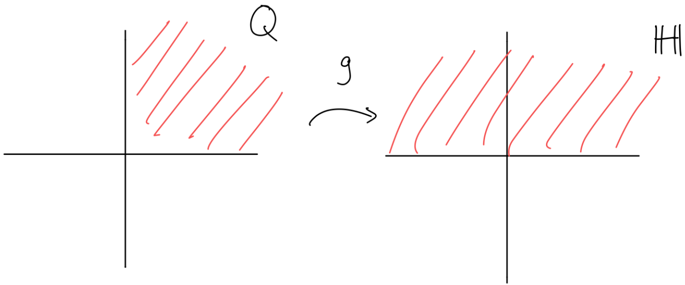
Let $Q := \{z \in \mathbb{C} \mid Im(z) > 0 \wedge Re(z) > 0\}$, *i.e. the first quadrant. Since the map* f *from the previous slide maps 0 to 1,* i *to 0, and* ∞ *to* -1, *it maps the line through* $0, i, \infty$ *(the imaginary axis) to the line through* $1, 0, -1$ *(the real axis).*



Riemann Mapping Theorem Example Cont.

Example

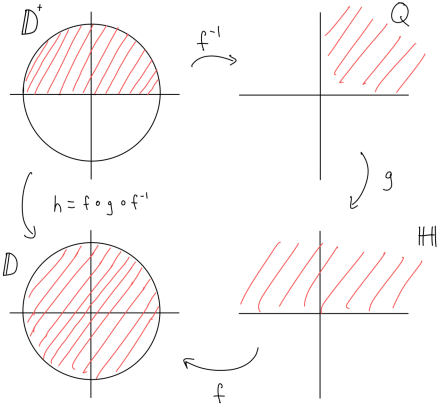
Let $g(z) = z^2$*, then* g *is injective and analytic in* Q*. Therefore,* g *maps* Q *conformally onto its image, namely the upper half plane* $\mathbb{H}$*.*



Riemann Mapping Theorem Example Cont.

Example

Combining the three examples from the previous slides leads us to the construction of a Riemann map from $\mathbb{D}^+$ *to* $\mathbb{D}$ *as* $h = f \circ g \circ f^{-1}$



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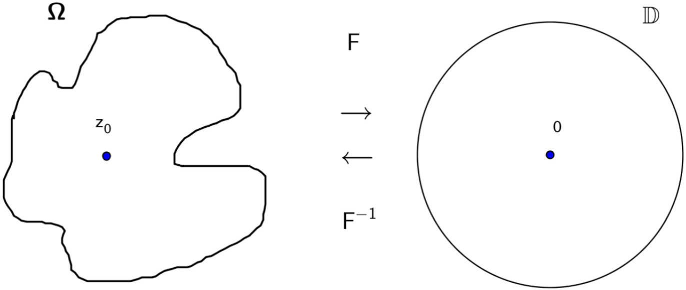
Fornberg's Method

Fornberg proposed a solution for calculating the inverse of the unique mapping defined in the Riemann Mapping Theorem. The method computes the conformal map F^{-1} from the interior of the unit disk to the interior of a smooth simply connected curve by finding the leading Taylor coefficients of F^{-1}.



Fornberg's Method Cont.

The method for finding the conformal mapping starts by introducing N complex points ζ_i ordered monotonically along the boundary curve J of the set Ω. We want to be able to find a way to move the points along J to a position where we can then use the unique mapping so that the points on J correspond to the N^{th} roots of unity z_i on the unit circle.

$$\zeta(z) = \sum_{\nu = -(N/2)+1}^{N/2} d_\nu z^\nu \tag{1}$$

However, we have a problem as this function does not satisfy the constraint given by the Riemann Mapping Theorem that $\zeta(0) = 0$.

Fornberg's Method Cont.

Assuming the boundary curve J is smooth (differentiable at every point), simply connected (no holes and can be deformed to a point without leaving the region), and enclosing the origin in $\mathbb{C}$, we can represent the true mapping function $\xi(z)$ by a convergent Taylor series

$$\xi(z) = \sum_{\nu = 1}^{\infty} c_\nu z^\nu, \quad |z| \leq 1. \tag{2}$$

Fornberg's Method Cont.

Then, for values of z on the unit circle, i.e., $z(\theta) = e^{2\pi i\theta}$, $0 \leq \theta \leq 1$, the Taylor series (2) in the previous slide becomes

$$\xi(z(\theta)) = \sum_{\nu=1}^{\infty} c_\nu e^{2\pi i\nu\theta}. \tag{3}$$

Fornberg coins this function as the "boundary correspondence function", which is a uniquely determined periodic function of θ.

Fornberg's Method Cont.

Taking the boundary correspondence function (3) with the θ values $\theta_k = k/N,\ k = 0, 1, \ldots, N-1$, and assuming N is even gives

$$\xi_k = \xi(z(\theta_k)) = \sum_{\nu = -(N/2)+1}^{N/2} g_\nu e^{2\pi i \nu \frac{k}{N}} \tag{4}$$

where

$$g_\nu = \sum_{j=0}^{\infty} c_{\nu + jN}, \tag{5}$$

therefore defining $c_\nu = 0$ for $\nu \leq 0$, which will be later used to remove singularities inside the unit circle and satisfy the constraint $\xi(0) = 0$.

Fornberg's Method Cont.

The discrete Fourier transform given as

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i \frac{k}{N} n}, \tag{6}$$

and its inverse,

$$x_n = \sum_{k=0}^{N-1} X_k \cdot e^{2\pi i \frac{k}{N} n} \tag{7}$$

can be used to transform (4) as

$$\xi_k = \sum_{\nu=-(N/2)+1}^{N/2} g_\nu e^{2\pi i \nu \frac{k}{N}} \iff g_\nu = \frac{1}{N} \sum_{k=0}^{N-1} \xi_k e^{-2\pi i \nu \frac{k}{N}}, \quad \nu = -N/2+1, \ldots, N/2. \tag{8}$$

Fornberg's Method Cont.

We therefore consider the approximate equivalent of (8)

$$d_\nu = \frac{1}{N} \sum_{k=0}^{N-1} \zeta_k e^{-2\pi i \nu \frac{k}{N}}, \quad \nu = -N/2+1, \ldots, N/2. \tag{9}$$

Here, the points ζ_k represent guesses for ξ_k along the boundary curve where the conformal map will be determined and are ordered monotonically along J.

Fornberg's Method Cont.

Fornberg uses a two-step process to move the points ζ_k. Given the tangential directions b_k (with $|b_k| = 1$) at the points ζ_k on J, we move these points in the tangential directions by distances of t_k in a specific way so as $d_0, d_{-1}, \ldots, d_{-N/2+1} = 0$. This yields

$$0 = \frac{1}{N} \sum_{k=0}^{N-1} (\zeta_k + t_k b_k) e^{-2\pi i \nu \frac{k}{N}}, \qquad \nu = -N/2 + 1, \ldots, 0. \tag{10}$$

Fornberg's Method Cont.

Subtracting the ν^{th} equation in (10) from the ν^{th} equation in (9) gives $N/2$ complex linear equations for the N real unknowns t_k as

$$
\begin{aligned}
d_\nu &= \frac{1}{N}\sum_{k=0}^{N-1} \zeta_k e^{-2\pi i \nu \frac{k}{N}} - \left(\frac{1}{N}\sum_{k=0}^{N-1} (\zeta_k + t_k b_k) e^{-2\pi i \nu \frac{k}{N}}\right) \\
&= \frac{1}{N}\sum_{k=0}^{N-1} \zeta_k e^{-2\pi i \nu \frac{k}{N}} - \left(\zeta_k \cdot e^{-2\pi i \nu \frac{k}{N}} + t_k b_k \cdot e^{-2\pi i \nu \frac{k}{N}}\right) \\
&= -\frac{1}{N}\sum_{k=0}^{N-1} t_k b_k e^{-2\pi i \nu \frac{k}{N}}, \quad \nu = -N/2+1, \ldots, 0. \qquad (11a)
\end{aligned}
$$

Fornberg's Method Cont.

The previous slide's series (11a) can be equivalently represented using matrix notation as

$$\begin{bmatrix} d_0 \\ d_{-1} \\ d_{-2} \\ \vdots \\ d_{-N/2+1} \end{bmatrix} = -\frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & e^{2\pi i/N} & e^{4\pi i/N} & e^{6\pi i/N} & \dots & e^{2(N-1)\pi i/N} \\ 1 & e^{4\pi i/N} & e^{8\pi i/N} & e^{12\pi i/N} & \dots & e^{2(2N-2)\pi i/N} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & e^{2(N/2-1)\pi i/N} & e^{2(N-2)\pi i/N} & e^{2(3N/2-3)\pi i/N} & \dots & e^{2((N/2-1)(N-1))\pi i/N} \end{bmatrix} \cdot \begin{bmatrix} t_0 b_0 \\ t_1 b_1 \\ t_2 b_2 \\ \vdots \\ t_{N-1} b_{N-1} \end{bmatrix}. \tag{11b}$$

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Bardeddine et al.'s Modification

We still seek a function f that maps points from the unit disk $\mathbb{D}$ to points inside the region Ω which is bounded by the curve J, where the boundary J is parameterized by S, i.e. $J : j(S),\ 0 \leq S \leq L,\ j(0) = j(L)$. Similar to Fornberg's method, we use a normalization imposed on f as $f(0) = a \in \Omega$ and $f(1) = j(0)$.

Bardeddine et al.'s Modification Cont.

Finding f is equivalent to finding the boundary correspondence function $S = S(\theta)$ such that $f(e^{i\theta}) = j(S(\theta))$ like in Fornberg's method. If $f(e^{i\theta})$ has the Fourier series expansion

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta},$$

then f extends analytically into $\mathbb{D}$ if and only if

$$a_{-k} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{ik\theta}dt = 0, \quad k = 1, 2, \ldots.$$

Bardeddine et al.'s Modification Cont.

Let $S^{(k)}(\theta)$ be an approximation to $S(\theta)$ at the k^{th} step. Then we can iteratively refine an approximation to $S(\theta)$ at each k^{th} step with a Newton-like update. At the k^{th} Newton step, we aim to find a 2π periodic correction $U^{(k)}(\theta)$ such that the boundary values of the conformal mapping $f(e^{i\theta})$ match those of the boundary curve J at $e^{i\theta}$, i.e.

$$f(e^{i\theta}) = j(S^{(k)}(\theta) + U^{(k)}(\theta)).$$

However, directly finding the desired correction $U^{(k)}(\theta)$ is difficult. Badreddine et al. overcome this difficulty by linearizing about $S^{(k)}(\theta)$. This linearization involves approximating the conformal map $f(e^{i\theta})$ as

$$f(e^{i\theta}) \approx j(S^{(k)}(\theta)) + j'(S^{(k)}(\theta))U^{(k)}(\theta).$$

Bardeddine et al.'s Modification Cont.

Badreddine et al. discretize this system with N-point trigonometric interpolation which results in a symmetric positive definite system,

$$AU = b,$$

where A is the discretization of the identity matrix plus a low rank operator.

Bardeddine et al.'s Modification Cont.

We use the conjugate gradient method and the fast Fourier transform to produce the previously mentioned time complexity of $\mathcal{O}(N \log N)$. The Newton update is then given as

$$S^{(k+1)}(\theta) = S^{(k)}(\theta) + U^{(k)}(\theta),$$

and near quadratic convergence is generally observed for a sufficiently close initial guess.

Bardeddine et al.'s Modification Cont.

Once we know the boundary correspondence function, we can compute the Taylor series, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for the map. Using Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-a}\, dz,$$

we can get the Taylor series for $f(z)$ with $|z| < |\zeta| = 1$, $\zeta = e^{i\theta}$, $d\zeta = ie^{i\theta}\, d\theta$ as,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{j(S(\theta))}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \oint_{|\zeta|=1} j(S(\theta)) \cdot \frac{1}{\zeta - z}\, d\zeta. \end{aligned} \tag{12}$$

Bardeddine et al.'s Modification Cont.

Before completing the integral, we will focus on expanding $1/(\zeta - z)$. We can expand this expression using a geometric series as follows,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \cdot \frac{1}{1 - \frac{z}{\zeta}} = \frac{1}{\zeta} \cdot \sum_{n=0}^{\infty} \left(\frac{z}{\zeta}\right)^n \tag{13}$$

Bardeddine et al.'s Modification Cont.

Substituting (13) back into (12) gives

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_{|\zeta|=1} j(S(\theta)) \left(\sum_{k=0}^{\infty} \left(\frac{z}{\zeta} \right)^k \right) \frac{d\zeta}{\zeta} \\
&= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \oint_{|\zeta|=1} j(S(\theta)) \left(\frac{z}{\zeta} \right)^k \frac{d\zeta}{\zeta} \\
&= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \oint_{|\zeta|=1} j(S(\theta)) \left(\frac{z}{e^{i\theta}} \right)^k \frac{ie^{i\theta}\, d\theta}{e^{i\theta}} \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} j(S(\theta)) \left(\frac{z}{e^{i\theta}} \right)^k d\theta \right) \\
&= \sum_{k=0}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} j(S(\theta)) e^{-ik\theta}\, d\theta \right) z^k = \sum_{k=0}^{\infty} a_k z^k.
\end{aligned}
$$

Bardeddine et al.'s Modification Cont.

Therefore, the Taylor coefficients of $f(z)$ are the Fourier coefficients of the 2π periodic function, $j(S(\theta))$, i.e.

$$a_k := \frac{1}{2\pi}\int_0^{2\pi} j(S(\theta))e^{-ik\theta}\,d\theta,$$

and we need only know the computed boundary correspondence function $S(\theta)$ to be able to obtain a conformal map from the unit disk $\mathbb{D}$ to the region Ω bounded by J.

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Boundary Curve: Unit Disk

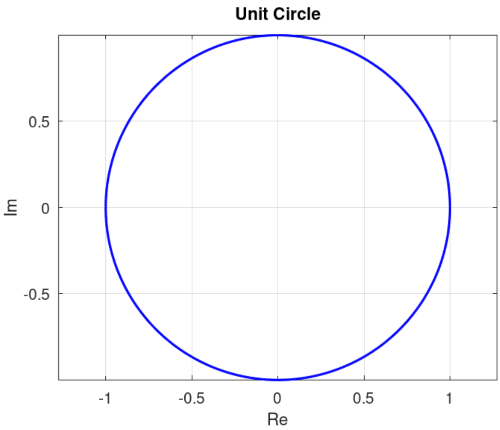

Figure: Boundary Curve to be Computed

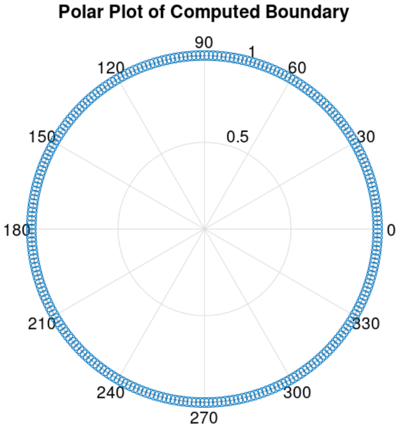

Figure: Computed Boundary w/ Fornberg's Method

Boundary Curve: Unit Disk

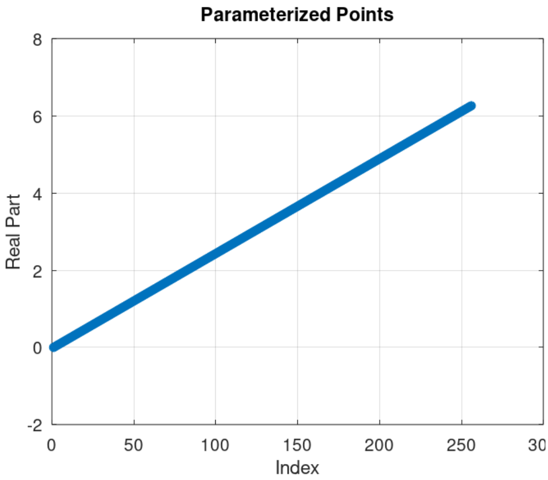

Figure: Computed Parameterized Points of Boundary Curve

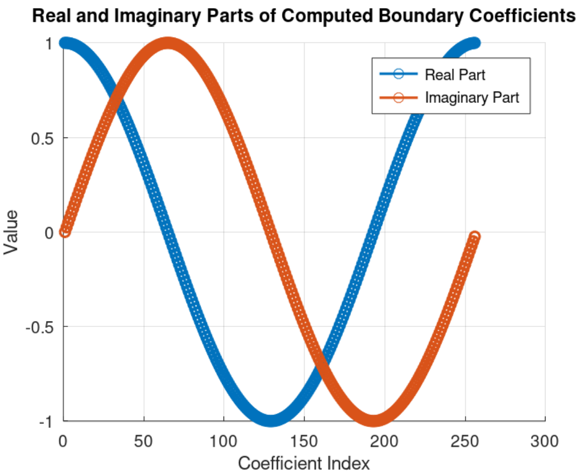

Figure: Computed Coefficients of Boundary Curve

Boundary Curve: Ellipse

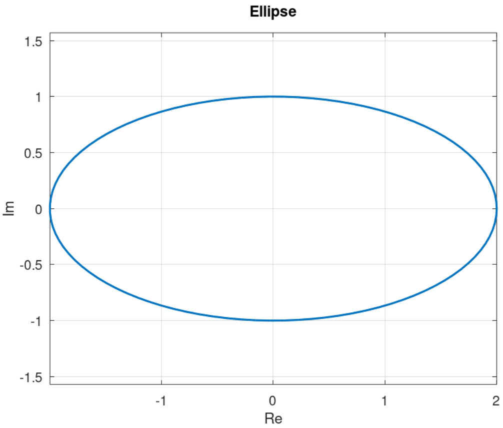

Figure: Boundary Curve to be Computed

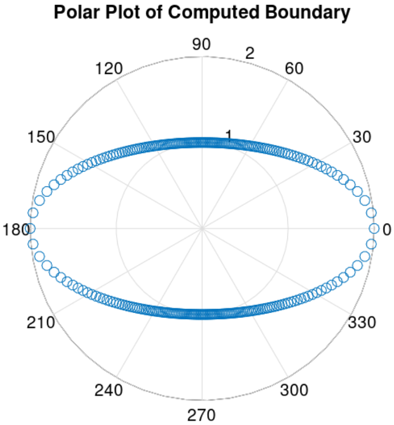

Figure: Computed Boundary w/ Fornberg's Method

Boundary Curve: Ellipse

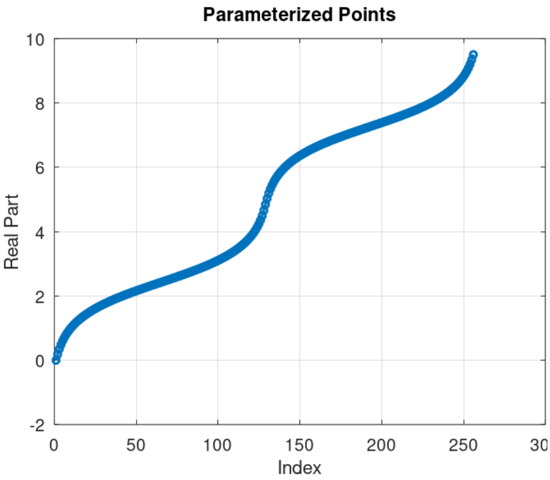

Figure: Computed Parameterized Points of Boundary Curve

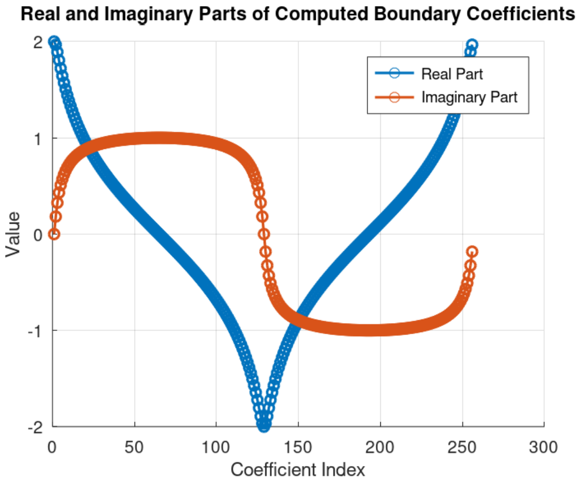

Figure: Computed Coefficients of Boundary Curve

Boundary Curve: Inverted Ellipse

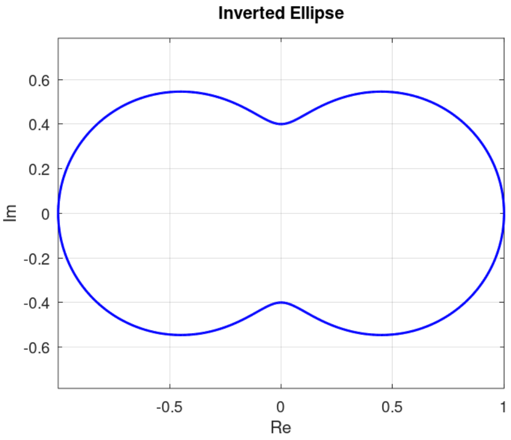

Figure: Boundary Curve to be Computed

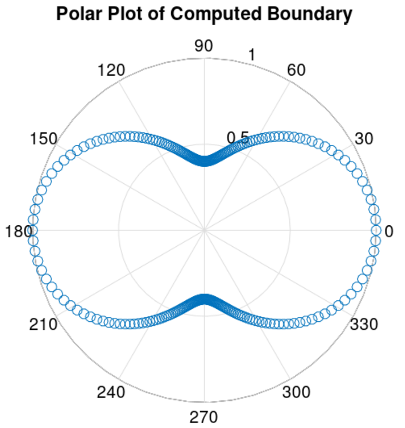

Figure: Computed Boundary w/ Fornberg's Method

Boundary Curve: Inverted Ellipse

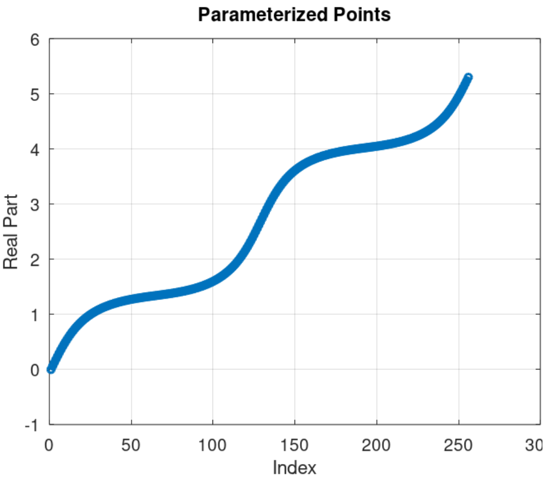

Figure: Computed Parameterized Points of Boundary Curve

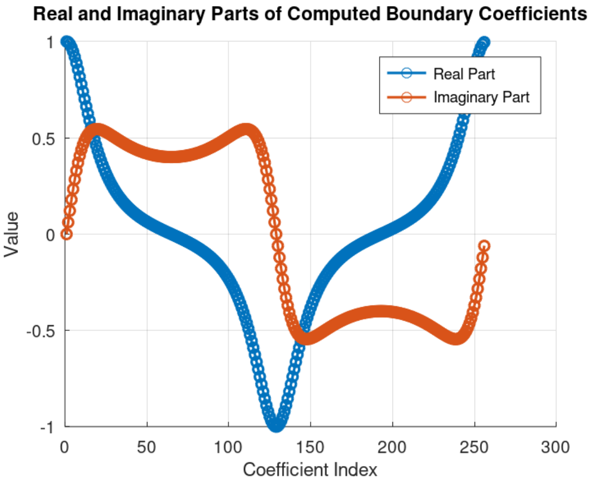

Figure: Computed Coefficients of Boundary Curve

Boundary Curve: Cassini Oval

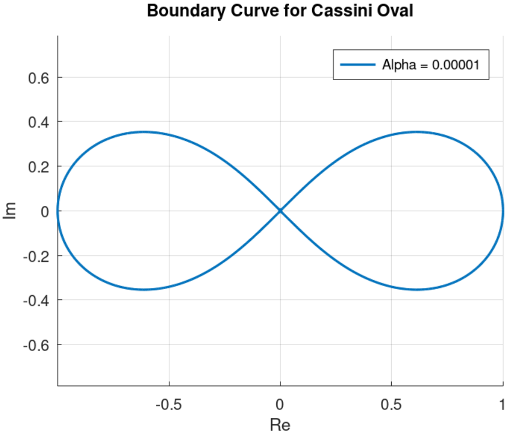

Figure: Boundary Curve to be Computed

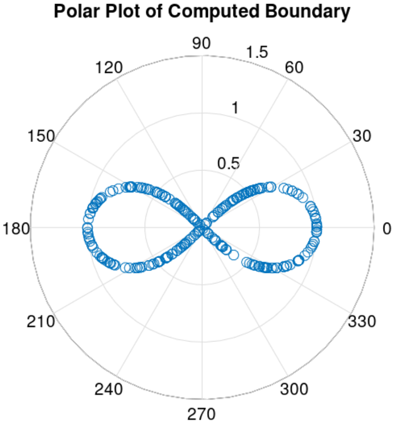

Figure: Computed Boundary w/ Fornberg's Method

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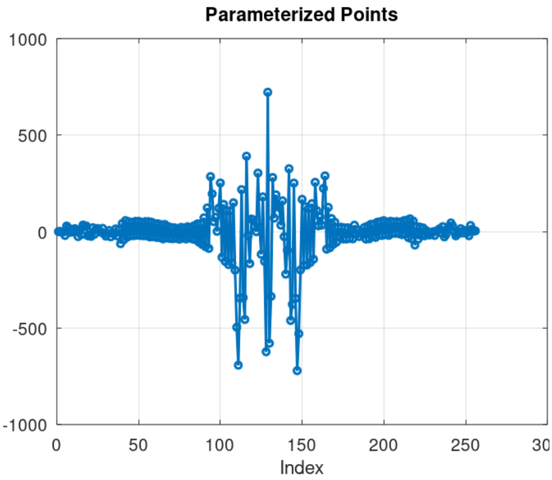

Figure: Computed Parameterized Points of Boundary Curve

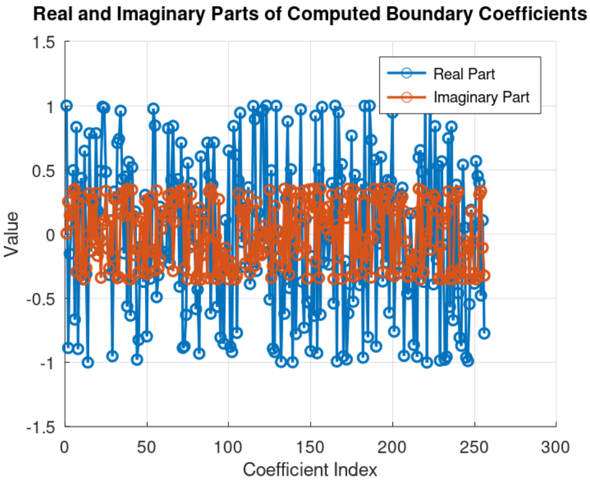

Figure: Computed Coefficients of Boundary Curve

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