# An Analysis of Fornberg's Numerical Conformal Mapping Method

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ABSTRACT

In this thesis, we first look at conformal mappings, their geometric characteristics with connections to holomorphic functions, and their relation to the Riemann Mapping Theorem. We then proceed to analyze Fornberg's numerical conformal mapping method for mapping the unit disk to a smooth simply connected region through the use of Taylor series. We then examine a modification done by Baddredine et al., and finally end by implementing the code given by Baddredine and proceed to plot computed conformal maps for different regions.

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# Introduction

Conformal mapping, a powerful tool in mathematics, physics, and engineering, has been a subject of study for centuries. Its ability to preserve angles and orientation while transforming complex domains into simpler ones has found wide-ranging applications across various fields, from fluid dynamics to electrical engineering.

The concept of conformal mapping traces back to the early developments in complex analysis during the 18th and 19th centuries. Mathematicians like Leonhard Euler, Carl Friedrich Gauss, and Bernhard Riemann laid the foundational principles of conformal mapping, unraveling its profound implications in understanding complex functions and their geometries [7].

While the theoretical framework of conformal mapping has been well-established, in the late 1950s its practical implementation through the use of numerical methods gained prominence with the advent of computers. In recent decades, advancements in computational mathematics and the availability of more powerful computing resources have opened new avenues for exploring numerical conformal mapping [7].

In the next chapters, we will define conformal mappings, their geometric properties, and state the infamous Riemann mapping theorem given by Bernhard Riemann. We will also analyze Bengt Fornberg's numerical conformal mapping method for the unit disk to a smooth simply connected region, look at a modern-day modification and implementation of said method by Badreddine et al., and finally visualize the numerical conformal mapping method.

# 2

# The Geometry of Conformal Mapping

In this chapter, we introduce the concept of conformal mappings and explore their geometric characteristics. A conformal map is defined as a mapping that maintains both angles and orientation at every point within its domain. Below are two examples provided to get the idea of what a conformal map is.

**Example 2.0.1.** Let $\Omega := \{z \in \mathbb{C} \mid Re(z) > 0\}$ and define a mapping $f : \Omega \mapsto \mathbb{C}$ as $f(z) = e^{i\,\frac{\pi}{4}} z$ with $z = |z|e^{i\,arg(z)}$. Then,

$$|f(z)| = |e^{i\,\frac{\pi}{4}} z| = |e^{i\,\frac{\pi}{4}}||z| = \left|cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right| |z| = \left|\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right| |z| = |z|$$

and

$$\begin{aligned} arg(f(z)) &= arg(e^{i\,\frac{\pi}{4}} z) = arg\left(|z|e^{i\,\frac{\pi}{4}} e^{iarg(z)}\right) = arg\left(|z|e^{i(\frac{\pi}{4}+arg(z))}\right) \\ &= arg(|z|) + arg\left(e^{i(\frac{\pi}{4}+arg(z))}\right) = \frac{\pi}{4} + arg(z). \end{aligned}$$

Thus, we see that the map $f$ is a rigid rotation about the origin by $\frac{\pi}{4}$ and conserves angles as well as orientation. Therefore, $f$ is a conformal map.

**Example 2.0.2.** Let $\Omega := \{z \in \mathbb{C} \mid Re(z) > 0\}$, and define a mapping $f : \Omega \mapsto \mathbb{C}$ as the complex conjugate function, $f(z) = \bar{z}$.

Then, $f$ is the reflection across the real axis. Thus, $f$ is not conformal as it preserves angles but reverses the orientation.

**Definition 2.0.1** (Angle Preserving)**.** A $2 \times 2$ matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ preserves angles if and only if there exists some $p > 0$ such that $A^T A = pI$ where $I$ is the identity matrix. This is because A preserves angles (as a linear transformation on $\mathbb{R}^2$) if it preserves dot products up to a scalar p (because $cos(\theta) = \frac{u \cdot v}{\sqrt{u \cdot u}\sqrt{v \cdot v}}$). In other words, preserving angles means that if two vectors $u$ and $v$ form an angle $\theta$, then the angle between $Au$ and $Av$ is also $\theta$.

*Proof.* Let $u, v \in \mathbb{R}^2$ and $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Then,

$$\begin{aligned} A^T A &= pI \\ (A^T A)u^T v &= pu^T v \\ u^T A^T A v &= pu^T v \\ (Au)^T A v &= pu^T v \\ Au \cdot Av &= pu \cdot v. \end{aligned}$$

This means that if an angle $\theta'$ is the angle between $Au$ and $Av$ then

$$\begin{aligned} cos(\theta') = \frac{Au \cdot Av}{\sqrt{Au \cdot Au}\sqrt{Av \cdot Av}} &= \frac{pu \cdot v}{\sqrt{u \cdot u}\sqrt{pv \cdot v}} = \frac{pu \cdot v}{\sqrt{p}||u||\sqrt{p}||v||} \\ &= \frac{u \cdot v}{||u|| * ||v||} = cos(\theta). \end{aligned}$$

Hence, $\theta' = \theta$ and A is angle preserving.

∎

Let $f : \Omega \mapsto \mathbb{C}$ be defined as $f = u + iv$, where $u : \Omega \mapsto \mathbb{R}$ and $v : \Omega \mapsto \mathbb{R}$. The mapping $f : \Omega \mapsto \mathbb{C}$ induces a real-valued mapping $f_{\mathbb{R}} : \Omega \mapsto \mathbb{R}^2$ given by $f_{\mathbb{R}}(x, y) = (u(x, y), v(x, y))$. So, $f_{\mathbb{R}}$ creates a mapping from the domain $\Omega$ to points in the 2D plane by considering the real and imaginary parts of the complex function $f$.

**Definition 2.0.2** (Orientation Preserving)**.** A mapping $f$ preserves orientation at $(x_0, y_0)$ if and only if the Jacobian matrix, $\mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0)$, has a positive determinant.

**Example 2.0.3.** Let $f(z) = e^{i\,\frac{\pi}{4}}z$. Then,

$$f(z) = e^{i\,\frac{\pi}{4}}z = \frac{\sqrt{2}}{2}(1+i)(x+iy) = \left(\frac{\sqrt{2}(x-y)}{2}\right) + i\left(\frac{\sqrt{2}(x+y)}{2}\right) = u + iv.$$

and $f_{\mathbb{R}}(x, y) = \left(\frac{\sqrt{2}(x-y)}{2}, \frac{\sqrt{2}(x+y)}{2}\right)$. Taking the determinant of the corresponding Jacobian matrix yields,

$$\det\left(\mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0)\right) = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = 1 > 0.$$

Hence we see that $f$ preserves orientation at each point confirming our result from Example 2.0.1.

**Example 2.0.4.** Let $f(z) = \bar{z} = x - iy$. Then, $f_{\mathbb{R}}(x, y) = (x, -y)$ and we see that,

$$\det\left(\mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0)\right) = \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1 < 0.$$

**Example 2.0.5.** Let $f(z) = z^2 + \bar{z}$. Then,

$$f(z) = z^2 + \bar{z} = (x + iy)^2 + (x - iy) = x^2 + 2ixy - y^2 + x - iy$$
$$= (x^2 - y^2 + x) + i(2xy - y) = u(x, y) + iv(x, y).$$

and $f_{\mathbb{R}}(x, y) = ((x^2 - y^2 + x), (2xy - y))$. The determinant of the corresponding Jacobian matrix gives,

$$\det\left(\mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0)\right) = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \det\begin{bmatrix} 2x+1 & -2y \\ 2y & 2x-1 \end{bmatrix} = 4x^2 + 4y^2 - 1.$$

Here we see that $\det\left(\mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0)\right) > 0$ if and only if $x^2 + y^2 > \left(\frac{1}{2}\right)^2$. So $f$ preserves orientation for all values outside of the circle centered at 0 with radius $\frac{1}{2}$ and does not preserve orientation for any values on or inside the circle.

**Definition 2.0.3.** Let $\Omega$ be an open subset of $\mathbb{R}^2$. A map $f : \Omega \mapsto \mathbb{R}^2$ is conformal if and only if $f$ is differentiable on $\Omega$ and $\mathbf{J}_f(x_0, y_0)$ preserves angle and orientation at each $(x_0, y_0) \in \Omega$.

**Theorem 1.** Let $\Omega \subset \mathbb{C}$. Then $f : \Omega \mapsto \mathbb{C}$ is holomorphic with $f'(z_0) \neq 0$ for all $z_0 \in \Omega$ if and only if $f_{\mathbb{R}} : \Omega \mapsto \mathbb{R}^2$ is conformal.

*Proof.* Recall that if $f = u + iv$ then $f_{\mathbb{R}}(x, y) = (u(x, y), v(x, y))$ and

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z},$$

where $\Delta z$ approaches zero from any direction. Since $f$ is holomorphic, approaching from any direction gives the same value for the limit. Therefore, we can approach from the x direction and let $\Delta z = \Delta x + 0i$. Then,

$$\begin{aligned}
\frac{df}{dz} &= \lim_{\Delta x \to 0} \frac{f((x+\Delta x)+iy) - f(x+iy)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u((x+\Delta x),y) + iv((x+\Delta x),y) - (u(x,y)+iv(x,y))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{u((x+\Delta x),y) - u(x,y) + i\left(v((x+\Delta x),y) - v(x,y)\right)}{\Delta x} \\
&= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}
\end{aligned}$$

Using the Cauchy-Riemann equations, $A = \mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix}$.

Therefore,

$$\begin{aligned}
A^T A = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ -\frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix} &= \begin{bmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 & 0 \\ 0 & \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \end{bmatrix} \\
&= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

which implies that A preserves angles at each point in $\Omega$ where $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 > 0$. Also,

$$\det\left(\mathbf{J}_{f_{\mathbb{R}}}(x_0, y_0)\right) = \det \begin{bmatrix} \frac{\partial u}{\partial x} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{bmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

implies that A preserves orientation when $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 > 0$, but $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\left(\frac{\partial u}{\partial x}\right) + i\left(\frac{\partial v}{\partial x}\right)\right|^2 = \left|\frac{df}{dz}\right|^2 > 0$ since $\frac{df}{dz} \neq 0$ on $\Omega$. The backwards direction of the proof is omitted.

■

With this theorem, we establish the connection between the geometric point of view of conformal maps and the analytic point of view of holomorphic functions with nonzero complex derivative. Therefore, we can conclude that a biholomorphism, a holomorphic function with a holomorphic inverse, refers to the same concept of a bijective conformal mapping.

**Theorem 2** (Riemann Mapping Theorem)**.** Let $\Omega \subset \mathbb{C}$ be a non-empty simply connected set that is not all of $\mathbb{C}$. Then for any $z_0 \in \Omega$, there exists a unique biholomorphism $F : \Omega \mapsto \mathbb{D}$, where $\mathbb{D}$ represents the unit disk centered at the origin, such that

$$F(z_0) = 0, \text{ and } F'(z_0) > 0.$$

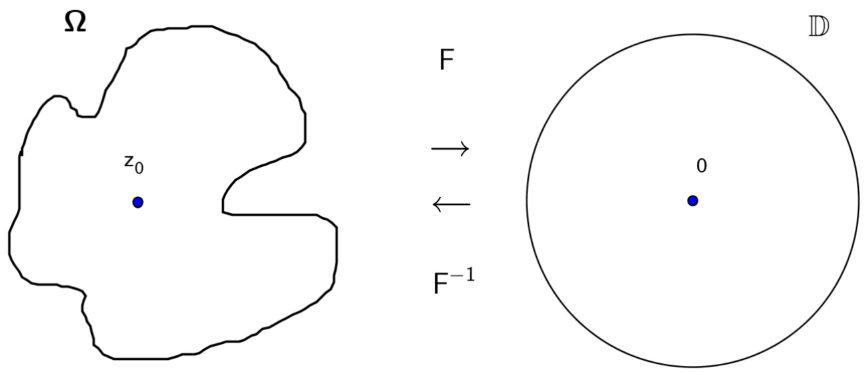



Figure 2.1: Visualization of the Riemann Mapping Theorem

**Example 2.0.6.** Let $\mathbb{H} := \{z \in \mathbb{C} \mid Im(z) > 0\}$, i.e. the upper half plane. Then $\mathbb{H}$ can be mapped to the unit disk $\mathbb{D}$ via a Möbius transformation. Let $f$ be the Möbius transformation that maps $0, 1, \infty$ to $1, i, -1$ respectively. Then the line through $0, 1, \infty$ (the real axis) must be mapped to the circle through $1, i, -1$ (the unit circle).

Furthermore, the domain to the left of the real axis (when we orientate the real axis from 0 to $\infty$) is mapped to the domain to the left of the unit circle, with orientation given by the ordering of the points on the circle (counter clockwise). The restriction of the Möbius transformation $f$ to $\mathbb{H}$ thus maps $\mathbb{H}$ onto $\mathbb{D}$. We can find the formula for $f$ as follows:

1. $f$ has the form $f(z) = \dfrac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Since $f(\infty) \neq \infty$, we have $c \neq 0$, and as the expression $cz + d$ determines the behavior of $f(z)$ we can therefore assume $c = 1$. Thus $f(z) = \dfrac{az+b}{z+d}$.

2. If $f(\infty) = -1$, then for $z = \infty$ we have $f(\infty) = \dfrac{a \cdot \infty + b}{\infty + d} = -1$. As $z \to \infty$, the dominant term in $\dfrac{a \cdot \infty + b}{\infty + d}$ is $az$. Since $f(\infty)$ must be finite, $az$ must tend to a finite value and in order for $f(\infty) = -1$, $a$ must be $-1$. Then $f(z) = \dfrac{-z + b}{z + d}$.
3. Since $f(0) = 1$, we have $\dfrac{b}{d} = 1$, so $b = d$. Then $f(z) = \dfrac{-z + b}{z + b}$.
4. Since $f(1) = i$ we have $\dfrac{-1 + b}{1 + b} = i$. Solving for $b$ yields

$$\frac{-1+b}{1+b} = i \iff -1 + b = i(1+b) \iff b(1-i) = (i+1)$$
$$\iff b = \frac{i+1}{1-i} = \frac{(i+1)(1+i)}{(1-i)(1+i)} = \frac{i-1+1+i}{1+1} = i$$

Therefore, we find $f(z) = \dfrac{-z + i}{z + i}$ conformally maps the upper half plane $\mathbb{H}$ onto the unit disk $\mathbb{D}$.

**Example 2.0.7.** Let $Q := \{z \in \mathbb{C} \mid Im(z) > 0 \land Re(z) > 0\}$, i.e. the first quadrant. Since the map $f$ from Example 1.0.6 maps 0 to 1, $i$ to 0, and $\infty$ to $-1$, it maps the line through $0, i, \infty$ (the imaginary axis) to the line through $1, 0, -1$ (the real axis). Hence, the restriction of $f$ to $Q$ conformally maps $Q$ onto the upper half of the unit disk denoted as $\mathbb{D}^+$.

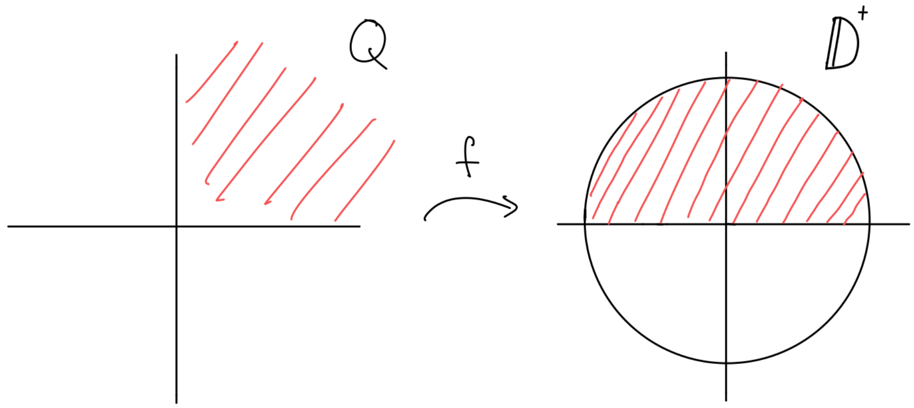



Figure 2.2: Visualization of $f$

**Example 2.0.8.** Let $g(z) = z^2$, then $g$ is injective and analytic in $Q$. Therefore, $g$ maps $Q$ conformally onto its image, namely the upper half plane $\mathbb{H}$.

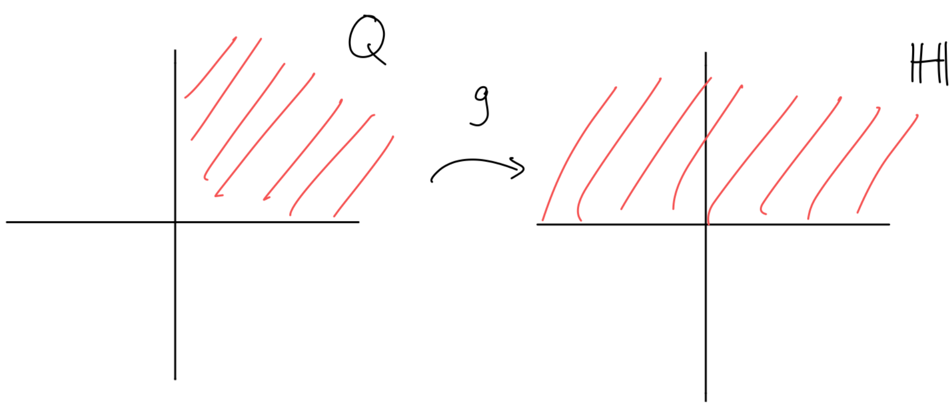



Figure 2.3: Visualization of $g$

**Example 2.0.9.** Combining Examples 1.0.6, 1.0.7, and 1.0.8 leads us to the construction of a Riemann map from $\mathbb{D}^+$ to $\mathbb{D}$ as $h = f \circ g \circ f^{-1}$.

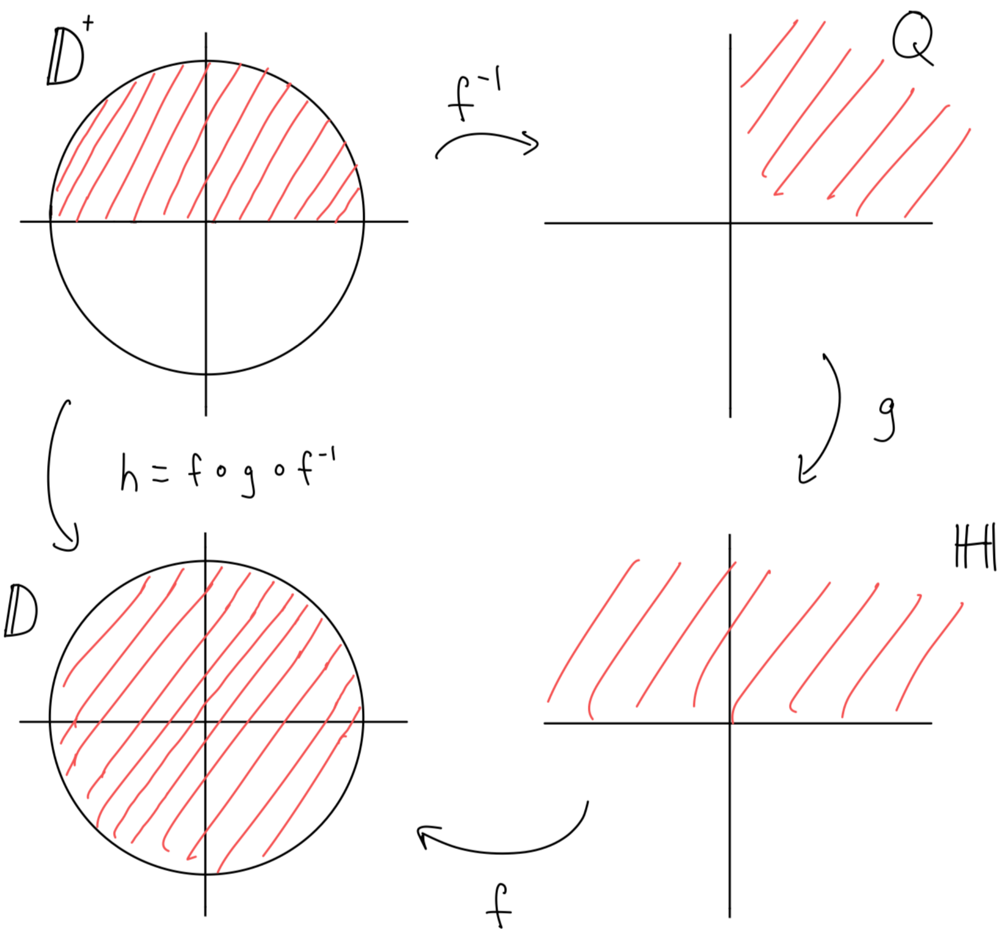



Figure 2.4: Construction of the Riemann Map $h$

# 3

# Fornberg's Method For The Disk

Fornberg [5] proposed a solution for calculating the inverse of the unique mapping defined in the Riemann Mapping Theorem (Theorem 2). The method computes the conformal map $f^{-1}$ from the interior of the unit disk to the interior of a smooth simply connected curve by finding the leading Taylor coefficients of

$f^{-1}$. Most methods for computing conformal maps solve the problem of mapping to or from the unit circle by first establishing a mapping of the boundaries then computing the complete mapping function. However, Fornberg's method simultaneously computes the boundary correspondence and the Taylor coefficients (mapping function) through the use of a quadratically convergent outer iteration and a super-linearly convergent inner iteration, which leads to an efficient time complexity of $\mathcal{O}(N \log N)$.

### 3.0.1 Fornberg's Original Method

The method for finding the conformal mapping starts by introducing $N$ complex points $\zeta_i$ ordered monotonically along the boundary curve $J$ of the set $\Omega$. We want to be able to find a way to move the points along $J$ to a position where we can then use the unique mapping (currently unknown) so that the points on $J$ correspond to the $N^{th}$ roots of unity $z_i$ on the unit circle. For each guess of the positions of the points $\zeta_i$, an analytic function

$$\zeta(z) = \sum_{\nu = -(N/2)+1}^{N/2} d_\nu z^\nu \tag{3.1}$$

is introduced [5]. We obtain the coefficients $d_\nu$ through the use of a complex discrete Fourier transform applied to all $\zeta_i$. This function is only one particular case that maps the points from the unit circle onto the points in the boundary. Clearly, we can see a problem though as this function does not satisfy the constraint given by the Riemann Mapping Theorem that $\zeta(0) = 0$. To solve this issue, Fornberg introduces a quadratically convergent strategy for the boundary curve $J$ in which the points $\zeta_i$ are moved along the boundary to the point where $\zeta(z)$ will lose its singularities inside the unit circle and therefore satisfy the constraint $\zeta(0) = 0$. The coefficients $d_\nu$ will be 0 for $\nu = 0, -1, -2, \ldots, -N/2 + 1$ (removing singularities), and will then approximate the Taylor coefficients of the mapping function for $\nu = 1, 2, \ldots, N/2$. Every step of this process creates a linear problem which is then solved using the super-linearly convergent inner iteration which utilizes the conjugate gradient method [5].

### 3.0.2 Fornberg's outer iteration

Since we assumed the boundary curve $J$ is smooth (differentiable at every point), simply connected (no holes and can be deformed to a point without leaving the region), and enclosing the origin in $\mathbb{C}$, we can represent the true mapping function $\xi(z)$ by a convergent Taylor series

$$\xi(z) = \sum_{\nu=1}^{\infty} c_\nu z^\nu, \quad |z| \leq 1. \tag{3.2}$$

Here, $\xi(z)$ denotes the true mapping function while $\zeta(z)$ denotes the approximation of $\xi(z)$. We still require the constraint that $\xi(0) = 0$ implying that $\zeta(0) = 0$ should also hold, in order to define a unique mapping. Often times the condition that $\partial\xi/\partial z > 0$ is used to ensure the uniqueness of the map (which would also ensure $\xi(z)$ preserves orientation by Definition 1.0.2). Fornberg on the other hand takes a different approach. He requires $\xi(1)$ to lie at a specific point on $J$. Then, for values of $z$ on the unit circle, i.e., $z(\theta) = e^{2\pi i\theta}$, $0 \leq \theta \leq 1$, (3.2) becomes

$$\xi(z(\theta)) = \sum_{\nu=1}^{\infty} c_\nu e^{2\pi i \nu \theta}. \tag{3.3}$$

Fornberg coins this function as the "boundary correspondence function", which is a uniquely determined periodic function of $\theta$ [5]. Taking (3.3) with the $\theta$ values $\theta_k = k/N,\ k = 0, 1, \ldots, N-1$, and assuming $N$ is even gives

$$\xi_k = \xi(z(\theta_k)) = \sum_{\nu=-(N/2)+1}^{N/2} g_\nu e^{2\pi i \nu \frac{k}{N}} \tag{3.4}$$

where

$$g_\nu = \sum_{j=0}^{\infty} c_{\nu+jN}, \tag{3.5}$$

therefore defining $c_\nu = 0$ for $\nu \leq 0$, which will be later used to remove singularities inside the unit circle and satisfy the constraint $\xi(0) = 0$. Fornberg defines the error in accepting $g_\nu$ as an approximation for $c_\nu$ as

$$g_\nu = \sum_{j=0}^{\infty} c_{\nu+jN} \Longleftrightarrow g_\nu - c_\nu = \sum_{j=1}^{\infty} c_{\nu+jN}. \tag{3.6}$$

By choosing $N$ to be large enough, the error for $\nu = 1, 2, \ldots, N/2$ is within our wanted tolerance [5]. The discrete Fourier transform given as

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i \frac{k}{N} n}, \tag{3.7}$$

and its inverse,

$$x_n = \sum_{k=0}^{N-1} X_k \cdot e^{2\pi i \frac{k}{N} n} \tag{3.8}$$

can be used to transform (3.4) as (3.7) can be evaluated using indices $[-N/2+1, N/2]$ if $N$ is even. Thus, we see (3.4) become

$$\xi_k = \sum_{\nu=-(N/2)+1}^{N/2} g_\nu e^{2\pi i \nu \frac{k}{N}} \Longleftrightarrow g_\nu = \frac{1}{N} \sum_{k=0}^{N-1} \xi_k e^{-2\pi i \nu \frac{k}{N}} \tag{3.9}$$

for $\nu = -N/2+1, \ldots, N/2$. With the known positions of the $N$ points $\xi_k$, we therefore obtain all the coefficients of $g_\nu$ by applying a single FFT to the DFT in (3.9). With $N$ large enough, the values of $g_\nu$ for $\nu = -(N/2)+1, \ldots, 0$ become arbitrarily small (removing the singularities inside the unit circle), and the values for $g_\nu$ for $\nu = 1, \ldots, (N/2)$ become arbitrarily close to the values of the true mapping's coefficients $c_\nu$ in (3.2). We therefore consider the approximate equivalent of (3.9)

$$d_\nu = \frac{1}{N} \sum_{k=0}^{N-1} \zeta_k e^{-2\pi i \nu \frac{k}{N}} \tag{3.10}$$

for $\nu = -N/2+1, \ldots, N/2$. Here, the points $\zeta_k$ represent guesses for $\xi_k$ along the boundary curve where the conformal map will be determined and are ordered monotonically along $J$. We want to adjust the positions of these points $\zeta_k$ on $J$ such that certain coefficients $d_\nu$ in (3.10) of the mapping function $\zeta(z)$ become 0 for $\nu = -N/2+1, \ldots, 0$. This adjustment ensures the removal of singularities inside the unit circle and satisfies the constraint $\zeta(0) = 0$. With $N$ free real parameters we want to make $N/2$ of the complex numbers ($d_\nu$ for $\nu = -N/2 + 1, \ldots, 0$) zero, suggesting that there should be an equal number of equations and unknowns. However, we have not yet satisfied the constraint that we must have a fixed point on $J$ to obtain our wanted unique solution. After linearization, the system of equations derived from this process results in $N$ equations (one for each $d_\nu$) with $N$ unknowns (the positions of the points $\zeta_k$), but it's noted that due to the constraint of fixing the position of one point, the system's rank is only $N-1$, therefore accounting for the constraint (within truncation errors) [5].

Fornberg uses a two-step process to move the points $\zeta_k$. Given the tangential directions $b_k$ (with $|b_k| = 1$) at the points $\zeta_k$ on $J$, we move these points in the tangential directions by distances of $t_k$ in a specific way so as $d_0, d_{-1}, \ldots, d_{-N/2+1} = 0$ [5]. This yields

$$0 = \frac{1}{N} \sum_{k=0}^{N-1} (\zeta_k + t_k b_k) e^{-2\pi i \nu \frac{k}{N}} \tag{3.11}$$

for $\nu = -N/2 + 1, \ldots, 0$. Afterwards, the points $\zeta_k + t_k b_k$ are moved back to the curve $J$ to make sure the points remain on the boundary curve after the adjustment. Since $J$ is smooth, the distance from the curve is $\mathcal{O}(t_i^2)$. The quadratic dependence of the movement on the distances $t_i$ ensures that the adjustments are gradual and do not introduce significant deviations.

Subtracting the $\nu^{th}$ equation in (3.11) from the $\nu^{th}$ equation in (3.10) gives

$N/2$ complex linear equations for the $N$ real unknowns $t_k$ as

$$\begin{aligned} d_\nu &= \frac{1}{N}\sum_{k=0}^{N-1}\zeta_k e^{-2\pi i\nu\frac{k}{N}} - \left(\frac{1}{N}\sum_{k=0}^{N-1}(\zeta_k + t_k b_k)e^{-2\pi i\nu\frac{k}{N}}\right) \\ &= \frac{1}{N}\sum_{k=0}^{N-1}\zeta_k e^{-2\pi i\nu\frac{k}{N}} - \left(\zeta_k \cdot e^{-2\pi i\nu\frac{k}{N}} + t_k b_k \cdot e^{-2\pi i\nu\frac{k}{N}}\right) \\ &= -\frac{1}{N}\sum_{k=0}^{N-1} t_k b_k e^{-2\pi i\nu\frac{k}{N}} \end{aligned} \tag{3.12a}$$

for $\nu = -N/2+1, \ldots, 0$ [5]. (3.12a) can be equivalently represented using matrix notation as

$$\begin{bmatrix} d_0 \\ d_{-1} \\ d_{-2} \\ \vdots \\ d_{-N/2+1} \end{bmatrix} = -\frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & 1 & \ldots & 1 \\ 1 & e^{2\pi i/N} & e^{4\pi i/N} & e^{6\pi i/N} & \ldots & e^{2(N-1)\pi i/N} \\ 1 & e^{4\pi i/N} & e^{8\pi i/N} & e^{12\pi i/N} & \ldots & e^{2(2N-2)\pi i/N} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & e^{2(N/2-1)\pi i/N} & e^{2(N-2)\pi i/N} & e^{2(3N/2-3)\pi i/N} & \ldots & e^{2((N/2-1)(N-1))\pi i/N} \end{bmatrix} \cdot \begin{bmatrix} t_0 b_0 \\ t_1 b_1 \\ t_2 b_2 \\ \vdots \\ t_{N-1} b_{N-1} \end{bmatrix}, \tag{3.12b}$$

with $(\cdot)$ indicating matrix multiplication. Fornberg then goes on to reformulate the linear system to give a very fast solution (in $\mathcal{O}(N \log N)$ time) using the special structure of the coefficient matrix in (3.12b), and he also discusses how the position of one point, e.g. $\zeta_0$, should be arbitrary in (3.12). These details were omitted here but can be found in [5].

### 3.0.3 Badreddine et al. Modification of Fornberg's Method

We seek a function $f$ that maps points from the unit disk $\mathbb{D}$ to points inside the region $\Omega$ which is bounded by the curve $J$, where the boundary $J$ is parameterized by $S$, i.e. $J : j(S),\ 0 \leq S \leq L,\ j(0) = j(L)$. Similar to Fornberg's method, we use a normalization imposed on $f$ as $f(0) = a \in \Omega$ and $f(1) = j(0)$ [3]. This means that the conformal map $f$ should map the origin of the unit disk to a point $a \in \Omega$, and the point on the unit circle corresponding to 1 to the starting point $j(0)$ on the boundary $J$. Finding $f$ is equivalent to finding the boundary correspondence function $S = S(\theta)$ such that $f(e^{i\theta}) = j(S(\theta))$ like in Fornberg's method [3]. If $f(e^{i\theta})$ has the Fourier series expansion

$$f(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta},$$

then f extends analytically into $\mathbb{D}$ if and only if

$$a_{-k} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{ik\theta} dt = 0$$

for $k = 1, 2, \ldots$ [3]. This integral representation is similar to Fornberg's method, as we must remove any singularities inside the unit disk in order for $f$ to be smoothly extended across the entire disk. By ensuring $a_{-k} = 0$ for $k = 1, 2, \ldots,$ we can remove any such singularities inside the unit disk.

Let $S^{(k)}(\theta)$ be an approximation to $S(\theta)$ at the $k^{th}$ step. Then we can iteratively refine an approximation to $S(\theta)$ at each $k^{th}$ step with a Newton-like update. At the $k^{th}$ Newton step, we aim to find a $2\pi$ periodic correction $U^{(k)}(\theta)$ such that the boundary values of the conformal mapping $f(e^{i\theta})$ match those of the boundary curve $J$ at $e^{i\theta}$, i.e.

$$f(e^{i\theta}) = j(S^{(k)}(\theta) + U^{(k)}(\theta)).$$

However, directly finding the desired correction $U^{(k)}(\theta)$ is difficult. Badreddine et al. overcome this difficulty by linearizing about $S^{(k)}(\theta)$ [3]. This linearization involves approximating the conformal map $f(e^{i\theta})$ as

$$f(e^{i\theta}) \approx j(S^{(k)}(\theta)) + j'(S^{(k)}(\theta))U^{(k)}(\theta).$$

Essentially, we are taking the point on the boundary curve $J$ corresponding to the current approximation $S^{(k)}(\theta)$, i.e. $j(S^{(k)}(\theta))$, and adding that to the tangential direction $(j'(S^{(k)}(\theta)))$ multiplied by the correction function $(U^{(k)}(\theta))$ to adjust the boundary correspondence function. The condition $a_{-k} = 0$ for $k = 1, 2, \ldots$ linearizes the problem of finding the correction function $U^{(k)}(\theta)$ as it leads to a system of linear equations, where each equation corresponds to setting the coefficient $a_{-k}$ to zero for a particular value of $k$. Each equation in this system involves the unknown correction function $U^{(k)}(\theta)$, hence forming the linear problem. Badreddine et al. discretize this system with $N$-point trigonometric interpolation which results in a symmetric positive definite system,

$$AU = b,$$

where $A$ is the discretization of the identity matrix plus a low rank operator [3]. $A$ being the discretization of the identity matrix ensures that the original boundary correspondence function $S^{(k)}(\theta)$ remains unchanged. In addition, the positive definiteness of $A$ ensures that the eigenvalues of $A$ are all positive, which guarantees that the system $AU = b$ will have a unique solution, which is another constraint imposed on the conformal map $f(e^{i\theta})$. This system is solved in the same efficient manner as Fornberg's system, using the conjugate gradient method and the fast Fourier transform to produce the previously mentioned time complexity of $\mathcal{O}(N \log N)$. The Newton update is then given as

$$S^{(k+1)}(\theta) = S^{(k)}(\theta) + U^{(k)}(\theta),$$

and near quadratic convergence is generally observed for a sufficiently close initial guess. Badreddine et al. state that an initial guess of $S^{(0)}(\theta) = L\theta/2\pi$ will work for domains that are nearly circular [3]. Once we know the boundary correspondence function, we can compute the Taylor series, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ for the map. Using Cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - a}\, dz,$$

we can get the Taylor series for $f(z)$ with $|z| < |\zeta| = 1$, $\zeta = e^{i\theta}$, $d\zeta = ie^{i\theta}\, d\theta$ as,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{j(S(\theta))}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i} \oint_{|\zeta|=1} j(S(\theta)) \cdot \frac{1}{\zeta - z}\, d\zeta. \end{aligned} \tag{3.13}$$

Before completing the integral, we will focus on expanding $1/(\zeta - z)$. We can expand this expression using a geometric series as follows,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} \cdot \frac{1}{1 - \frac{z}{\zeta}} = \frac{1}{\zeta} \cdot \sum_{n\,=\,0}^{\infty} \left(\frac{z}{\zeta}\right)^n \tag{3.14}$$

Substituting (3.14) back into (3.13) gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{|\zeta|=1} j(S(\theta)) \left( \sum_{k=0}^{\infty} \left(\frac{z}{\zeta}\right)^k \right) \frac{d\zeta}{\zeta} \\ &= \sum_{k\,=\,0}^{\infty} \frac{1}{2\pi i} \oint_{|\zeta|=1} j(S(\theta)) \left(\frac{z}{\zeta}\right)^k \frac{d\zeta}{\zeta} \\ &= \sum_{k\,=\,0}^{\infty} \frac{1}{2\pi i} \oint_{|\zeta|=1} j(S(\theta)) \left(\frac{z}{e^{i\theta}}\right)^k \frac{ie^{i\theta}\, d\theta}{e^{i\theta}} \\ &= \sum_{k\,=\,0}^{\infty} \left( \frac{1}{2\pi} \int_0^{2\pi} j(S(\theta)) \left(\frac{z}{e^{i\theta}}\right)^k d\theta \right) \\ &= \sum_{k\,=\,0}^{\infty} \left( \frac{1}{2\pi} \int_0^{2\pi} j(S(\theta)) e^{-ik\theta}\, d\theta \right) z^k = \sum_{k\,=\,0}^{\infty} a_k z^k. \end{aligned}$$

Hence, the Taylor coefficients of $f(z)$ are the Fourier coefficients of the $2\pi$ periodic function, $j(S(\theta))$, i.e.

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} j(S(\theta)) e^{-ik\theta}\, d\theta,$$

and we need only know the computed boundary correspondence function $S(\theta)$ to be able to obtain a conformal map from the unit disk $\mathbb{D}$ to the region $\Omega$ bounded by $J$.

# 4

# Testing Fornberg's Method

In this chapter, we test Fornberg's method on various different smooth simply connected regions. A plot of the wanted boundary curve is shown beside the computed boundary, and the parameterized points $j(S(\theta))$ are shown next to the corresponding real and imaginary parts of the computed Fourier boundary coefficients $a_k$. 256 points are used with 100 max iterations. The detailed MATLAB implementation of Fornberg's method can be found in [2]. The code was ran in GNU Octave version 8.4.0 with all plots and a few minor tweaks to the original method made by myself.

### 4.0.1 Boundary Curve: Unit Disk

For the first example, we will compute the conformal map from the unit disk to itself using Fornberg's method.

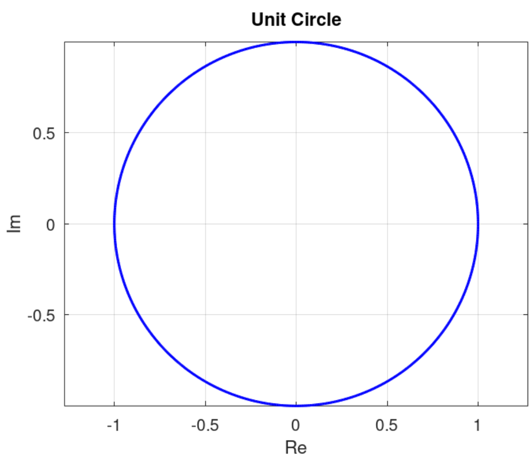



Figure 4.1: Boundary Curve to be Computed

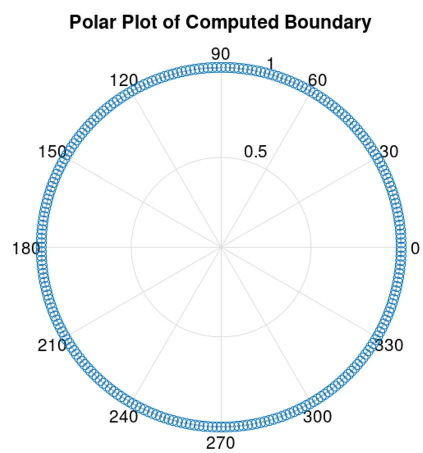



Figure 4.2: Computed Boundary w/ Fornberg's Method

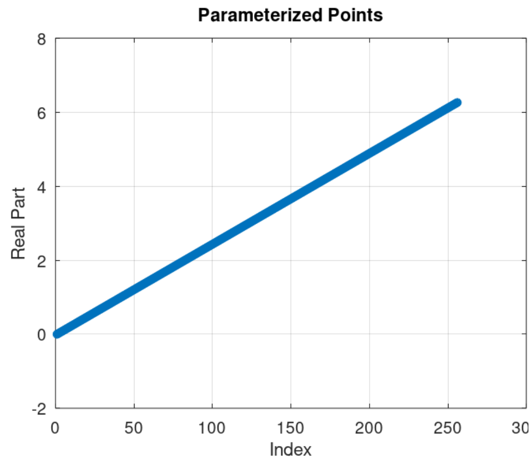



Figure 4.3: Computed Parameterized Points of Boundary Curve

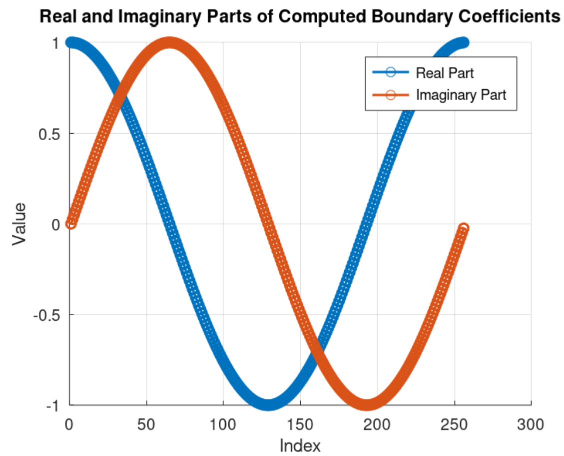
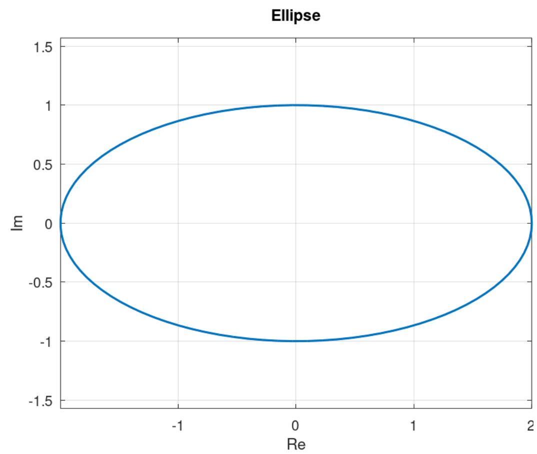



Figure 4.4: Computed Coefficients of Boundary Curve

### 4.0.2 Boundary Curve: Ellipse

For the next example, we will compute the conformal map from the unit disk to an ellipse.



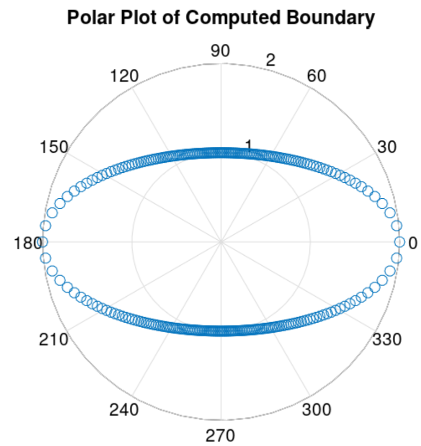
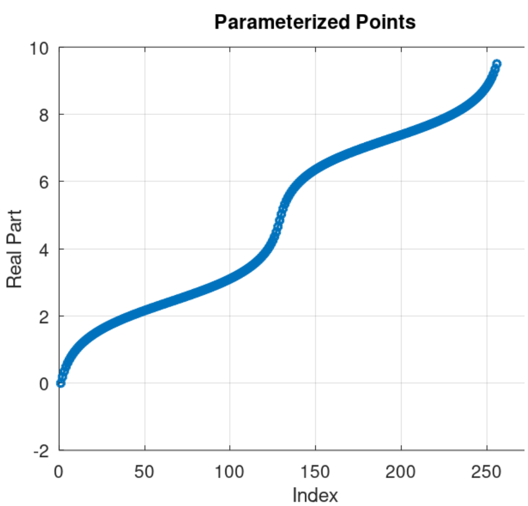




Figure 4.5: Boundary Curve to be Computed

Figure 4.6: Computed Boundary w/ Fornberg's Method



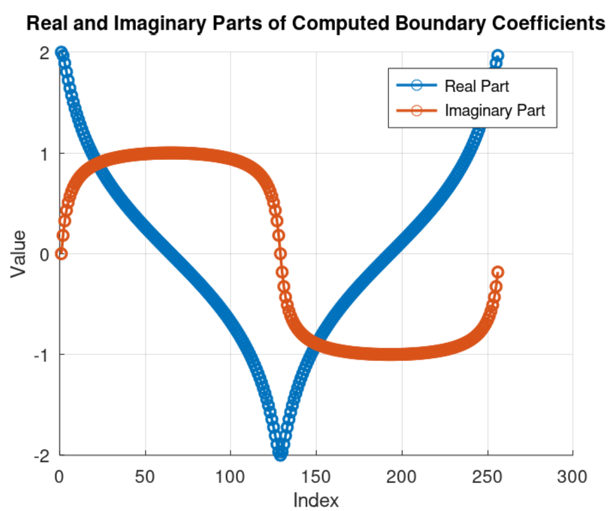




Figure 4.7: Computed Parameterized Points of Boundary Curve

Figure 4.8: Computed Coefficients of Boundary Curve

### 4.0.3 Boundary Curve: Inverted Ellipse

Continuing on, we will compute the conformal map from the unit disk to an inverted ellipse.

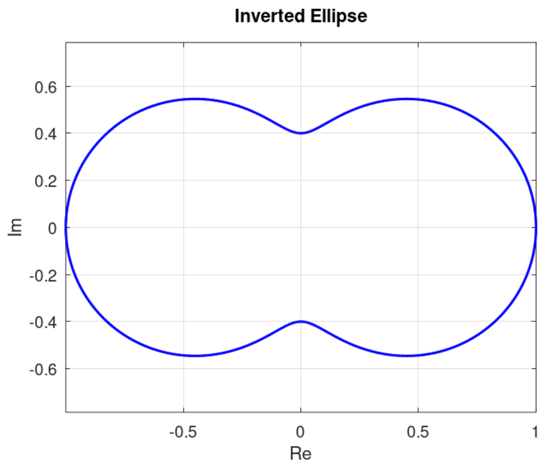



Figure 4.9: Boundary Curve to be Computed

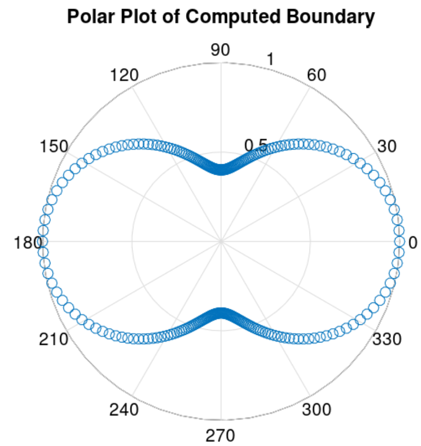



Figure 4.10: Computed Boundary w/ Fornberg's Method

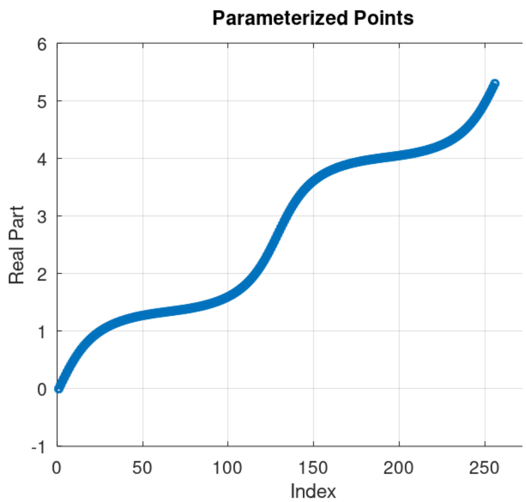



Figure 4.11: Computed Parameterized Points of Boundary Curve

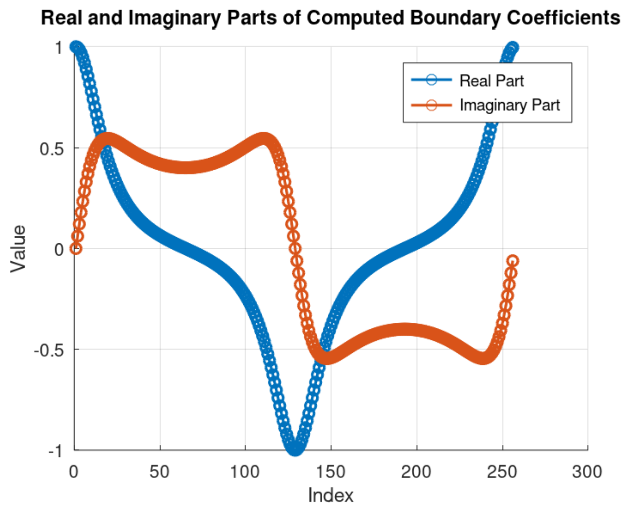



Figure 4.12: Computed Coefficients of Boundary Curve

### 4.0.4 Boundary Curve: Cassini Ovals

Lastly, we will compute the conformal map from the unit disk to a Cassini Oval.

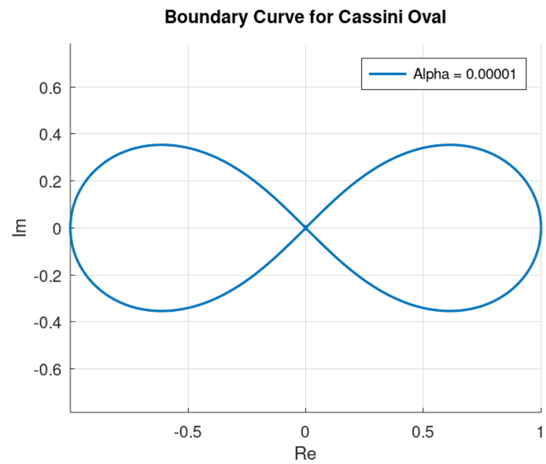



Figure 4.13: Boundary Curve to be Computed

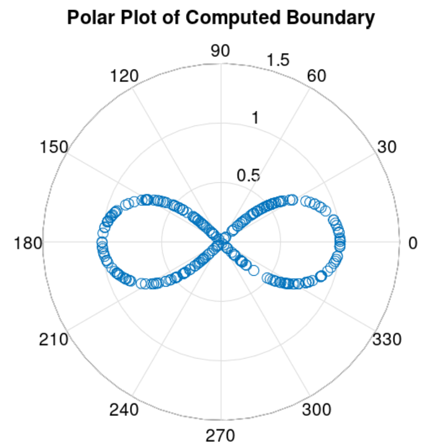



Figure 4.14: Computed Boundary w/ Fornberg's Method

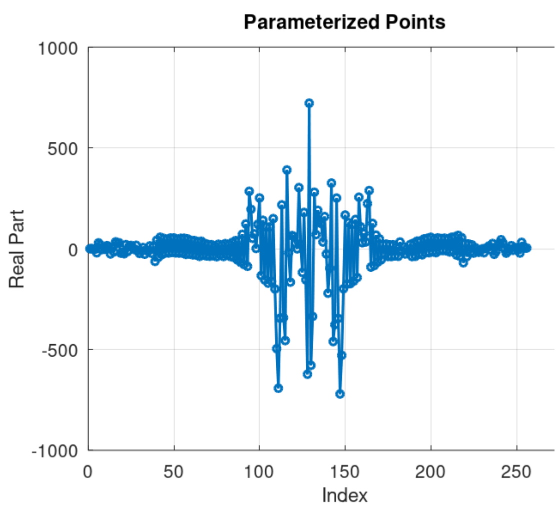



Figure 4.15: Computed Parameterized Points of Boundary Curve

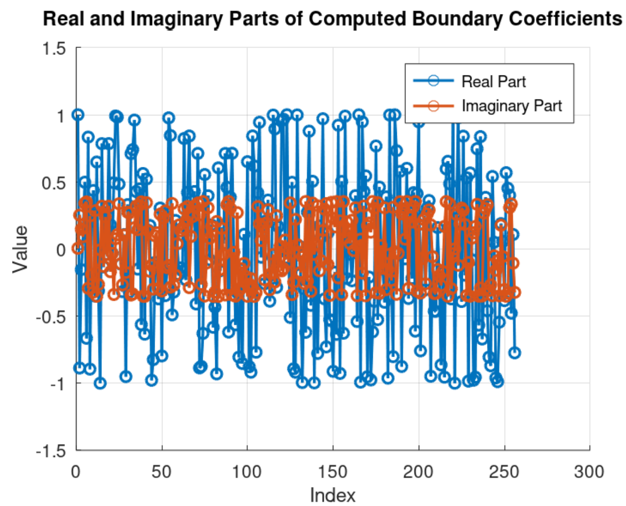



Figure 4.16: Computed Coefficients of Boundary Curve

# 5

# Conclusion

In conclusion, the exploration of conformal mappings and their properties led to an elegant relationship between holomorphic functions and the geometries of conformal maps. We also saw how the Riemann mapping theorem plays a crucial role in both conformal mappings and, more specifically, Fornberg's method for mapping the unit disk to a smooth simply connected domain. We further analyzed Fornberg's method, which computes the leading Taylor coefficients of $F^{-1}$ specified in the Riemann mapping theorem, looked at a modification done by Badreddine et al., and implemented Badreddine et al.'s code to create plots of the computed maps for various regions.

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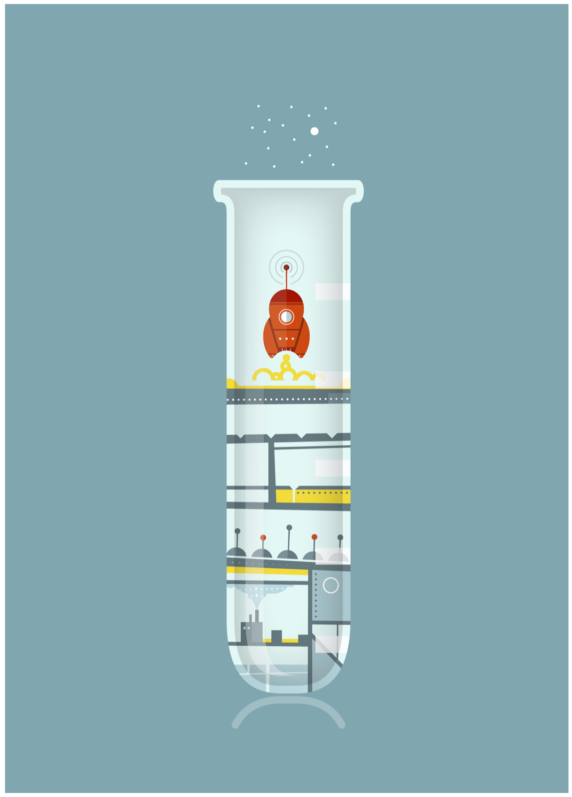

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