Computer Graphics - HW 2

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Question 1:

a) What is the implicit representation of the 2D plane that is tangent to the sphere S at the point (4,3,4)

$S: (x-1)^2 + (y-2)^2 + z^2 - 26 = 0$

$\Rightarrow$ the center of the sphere is $(1,2,0)$

let $\vec{n}$ be the vector from the center to the point $p_0 = (4,3,4)$

$\Rightarrow \vec{n} = (4,3,4) - (1,2,0) = (4-1, 3-2, 4-0) = (3,1,4)$

since the plane tangent to the sphere $\vec{n}$ is vertical to it.

let $p = (x,y,z)$

$\Rightarrow \vec{n}(p - p_0) = 0$

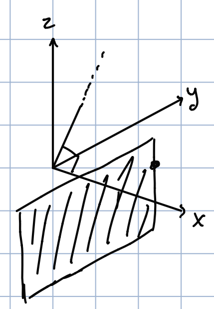
$\Rightarrow (3,1,4)(x-4, y-3, z-4) = 0$

$3(x-4) + 1(y-3) + 4(z-4) = 0$

$3x - 12 + y - 3 + 4z - 16 = 0$

$3x + y + 4z - 31 = 0$

b) What is the implicit representation of the 2D plane that passes through the point (6,1,0) and is parallel to a line that lies on the yz-plane? How many possible solutions exists?



- a plane that parallel to a line lies on the yz-plane means the plane also parallel to the plane yz $\Rightarrow$ the equation is $x = a$. However we know the plane passes through the point $(6,1,0) \Rightarrow x = 6$ is the implicit representation of the plane.

Only one solution exists since any other $x = a \neq 6$ won't passes through (6,1,0)

c) Let $f_1(x,y,z)$ and $f_2(x,y,z)$ be implicit representation of two spheres. Also assume that the spheres intersect on more than two points. Explain in detail what does the following surface represent:

$S = \{(x,y,z) \mid f_1(x,y,z)=0 \text{ and } f_2(x,y,z)>0\} \cup \{(x,y,z) \mid f_2(x,y,z)=0 \text{ and } f_1(x,y,z)>0\}$

In other words, S is the set of points (x,y,z) that satisfy either of the following

a: $f_1(x,y,z)=0$ and $f_2(x,y,z)>0$

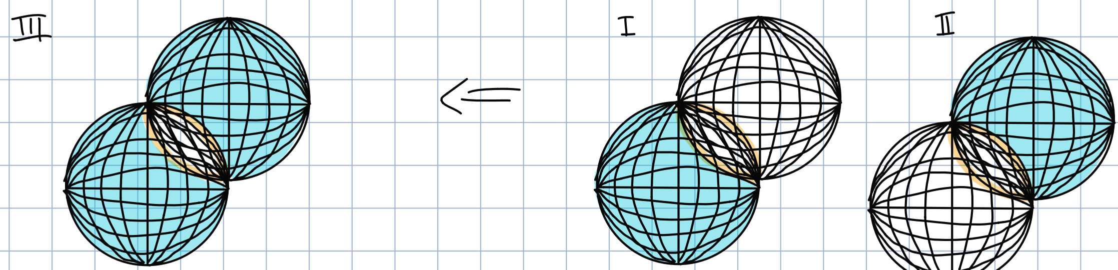
b: $f_2(x,y,z)=0$ and $f_1(x,y,z)>0$

Is this surface closed? If not, how would you change S so that the surface is closed?

→ first define open set: U is open if for any point $p \in U$ there exist $r>0$ such that $B(p,r) \subseteq U$ when B is an open ball

closed sets: U is closed set if it complement set is open

S is a set of points that lies on the light blue surfaces:



S contains points on the casing of f_1 sphere such that not on f_2 casing sphere (I) or points on the casing of f_2 sphere such that not on f_1 casing like (II). So the union we get any points from (III).

claim: S is open surface.

we will show that for any point $p=(x,y,z) \in S$ there exist an open ball $B((x,y,z),r)$ that also in S.

Let $p=(x_p,y_p,z_p) \in S$ and without limiting generality assume $p \in \{f_1(x,y,z)=0 \wedge f_2(x,y,z)>0\}$

we mark f_2 as follows: $f_2(x,y,z) = ax^2+by^2+cz^2-d$.

for $p=(x_p,y_p,z_p) \rightarrow ax_p^2+by_p^2+cz_p^2-d>0 \rightarrow$ lets mark the value we get as $r \rightarrow axp^2 + byp^2 + czp^2 - d = r > r/2 > 0$.

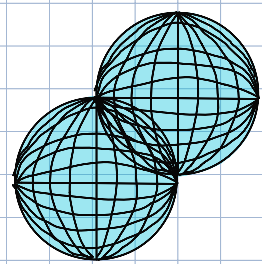
So if we take an open ball with radius of $r/2$ on the casing of f_1, we still get an open surface of p which in S.

The same claim hold for $p \in \{f_2(x,y,z)=0 \wedge f_1(x,y,z)>0\}$.

→ If we want that S will be a closed surface we need to change it definition that satisfy either of the following:

a: $f_1(x,y,z)=0$ and $f_2(x,y,z) \geq 0$

b: $f_2(x,y,z)=0$ and $f_1(x,y,z) \geq 0$



In that way the surface will be including points of the intersaction: $f_1(x,y,z)=0 \wedge f_2(x,y,z)=0$ and for those points we don't have an open ball that still in S.

d) The point $p=(a,7,b)$ lies on the plane $2x+y+z+1=0$. Determine all possible values of value of a and b. What does the set of all possible points $p=(a,7,b)$ represent?

$\Rightarrow 2a+7+b+1=0 \rightarrow$ a line on the plane

$\Rightarrow 2a+b+8=0 \rightarrow b=-2a-8$

$\Rightarrow$ all the points such that $(a,7,-2a-8)$ where a can be any real numbel

e) A line ℓ lies on the plane: $x+y+z+1=0$, the line direction is $(a,1,0)$ for some value $a \in \mathbb{R}$:

1) Can you determine the line equation from the given details above? How many possible line exists?

→ let $p=(x_p, y_p, z_p)$ be a point on the plane and mark the direction $(a,1,0) = \vec{v}$. so $\ell(t) = p + t \cdot \vec{v}$. The normat to the plane $n=(1,1,1)$ is also the normal to $\ell \Rightarrow n \cdot \vec{v} = 0 \Rightarrow 1 \cdot a + 1 \cdot 1 + 1 \cdot 0 = 0 \Rightarrow a = -1$

$\Rightarrow \ell(t) = p + t \cdot (-1,1,0) \Rightarrow$ there are infinity possiblities for p - a point on the plane.

2) If the line also passes through the point $(1,-1,-1)$ can you determine the line equation? How many lines exists?

→ we know the line direction vector $(-1,1,0)$ and a point the line passes through $\Rightarrow p = (1,-1,-1) \Rightarrow \ell: (1,-1,-1) + t(-1,1,0)$

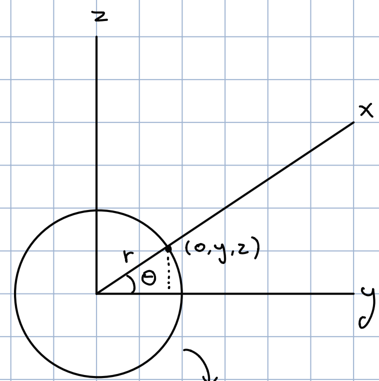
only one line ℓ.

f) A 3D sphere is centered at the origin, and a 2D plane π is given by the implicit equation $\pi: ax + by + cz + d = 0$

The intersection of the sphere and the plane π forms the circle given by the parametric representation : $x=0$, $y = \frac{3}{\sqrt{2}} \cdot \sin\theta$, $z = \frac{3}{\sqrt{2}} \cdot \cos\theta$ where $\theta \in [0°, 360°]$.

Calculate the radius of the sphere and the plane's equation π.

$\rightarrow$ Since the center of the sphere is the origin, the radius start at the origin and end at the case of the sphere. From the circle parametric representation, $x=0$ and we know the circle lies on the plane π



$0^2 + y^2 + z^2 = r^2$

$\Rightarrow$ π is the yz-plane, and the radius is the size of the vector that start at the origin $(0,0,0)$ and end at the point $\left(0, \frac{3}{\sqrt{2}}\sin\theta, \frac{3}{\sqrt{2}}\cos\theta\right)$

so let choose $\theta = 90°$ and get :

$$r = \|(x,y,z)\| = \left\|\left(0, \tfrac{3}{\sqrt{2}}\sin\theta, \tfrac{3}{\sqrt{2}}\cos\theta\right)\right\| = \sqrt{0^2 + \left(\tfrac{3}{\sqrt{2}}\sin 90°\right)^2 + \left(\tfrac{3}{\sqrt{2}}\cos 90°\right)^2}$$

$$r = \sqrt{\left(\tfrac{3}{\sqrt{2}}\right)^2} = \frac{3}{\sqrt{2}}$$

g) Find the intersection of the 2D line and a plane below using only their parametric representation: $l: (3,2,0) + (-1,1,5)\cdot t$

$\pi: (0,-1,4) + (1,1,0)t + (5,1,-2)s$

$\Rightarrow l(t_1): (3-t_1,\ 2+t_1,\ 5t_1)$

$\pi(t_2,s): (t_2+5s,\ -1+t_2+s,\ 4-2s)$

we want to find t_1, t_2, s such that $l(t_1) = \pi(t_2, s)$

$3-t_1 = t_2+5s \rightarrow t_2 = 3-t_1-5s$

$t_2 = 3-1-5\cdot\left(-\frac{1}{2}\right) = 2+\frac{5}{2} = 4.5$

$2+t_1 = -1+t_2+s \rightarrow 2+t_1 = -1+3-t_1-4s \rightarrow 2t_1 = -4s \rightarrow t_1 = -2s$

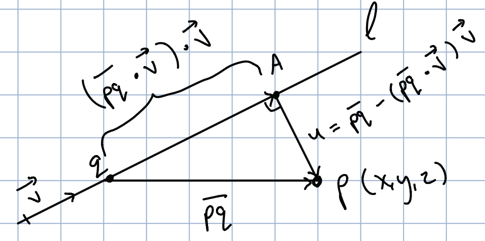
$t_1 = -2\cdot\left(-\frac{1}{2}\right) = 1$

$5t_1 = 4-2s \rightarrow 5\cdot(-2s) = 4-2s \rightarrow -10s = 4-2s \rightarrow -8s = 4 \rightarrow s = -\frac{1}{2}$

$\Rightarrow l(t_1 = 1): (3-1,\ 2+1,\ 0+5) = (2, 3, 5)$

Question 2

a) Let $l(t) = q + vt$ be the parametric representation of a line, where v is a unit norm vector and q is a point on the line. Find the projection of a point $p = (x, y, z)$ on the line $l(t)$



→ mark $q = (x_q, y_q, z_q)$

$\Rightarrow \overline{pq} = (x, y, z) - (x_q, y_q, z_q)$

$\overline{pq} = (x - x_q, y - y_q, z - z_q)$

→ let mark A the point on the line that most colse to p and the vector $\vec{u}$ is $p - A$ $\Rightarrow$ $A = p - \vec{u}$ and A is the projection.

$$\Rightarrow \vec{u} = \overline{pq} - (\overline{pq} \circ \vec{v}) \circ \vec{v} = (x - x_q, y - y_q, z - z_q) - ((x - x_q, y - y_q, z - z_q) \circ \vec{v}) \circ \vec{v}$$

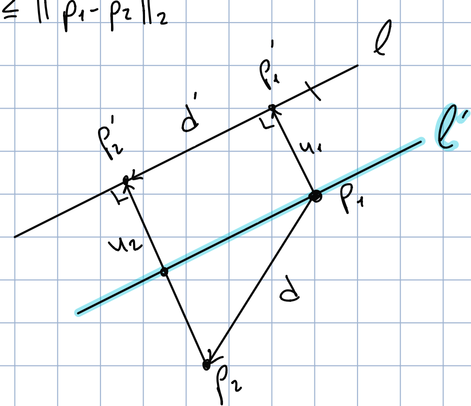
$$= (x - x_q, y - y_q, z - z_q) - (\underbrace{(x - x_q)v_x + (y - y_q)v_y + (z - z_q)v_z}_{\text{mark this as } \alpha}) \cdot (v_x, v_y, v_z)$$

$$= (x - x_q - \alpha v_x,\ y - y_q - \alpha v_y,\ z - z_q - \alpha v_z)$$

$$\Rightarrow A = p - \vec{u} = (x, y, z) - (x - x_q - \alpha v_x,\ y - y_q - \alpha v_y,\ z - z_q - \alpha v_z)$$

$$A = (x_q + \alpha v_x,\ y_q + \alpha v_y,\ z_q + \alpha v_z)$$

b) Let p_1 and p_2 be two points and let p_1' and p_2' be their projection on the line l respectively. Prove that $\|p_1' - p_2'\|_2 \leq \|p_1 - p_2\|_2$



→ p_1' and p_2' are the projection points

$\Rightarrow$ the vectors $u_1 = \overline{p_1' p_1}$ and $u_2 = \overline{p_2' p_2}$ are ortogonals to the line l

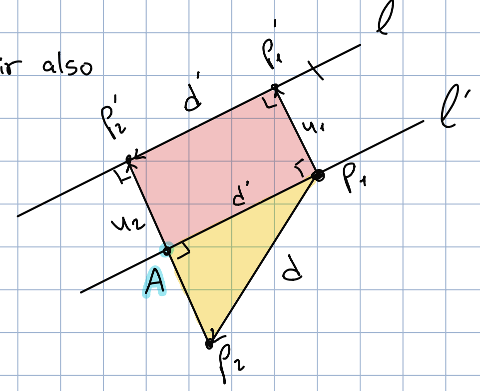
let mark the distance between p_1' to p_2' as d'

we can write l as follows: $l = p_1' + t \cdot (p_2' - p_1')$ for the vector $(p_2' - p_1')$

let l' be a parallel line that passes through p_1, $l' = p_1 + t(p_2' - p_1')$

l and l' have the same direction $(p_2' - p_1')$.

Since $u_2 = (p_2' - p_1)$ and $u_1 = (p_1' - p_1)$ is ortogonal to ℓ their also ortogonal to ℓ'. Therefore the point A is the projection of p_2 on line ℓ'.



p_2', p_1', p_1, A create a rectangle since there are 4 $90°$ angles $\Rightarrow$ the distane between $|p_1' - p_2'| = d'$ is equal to the distance $|p_1 - A| = d'$.

So we get a $90°$ traingle p_1, A, p_2

from pythagoras $\Rightarrow \|p_1 - A\|^2 + \|p_2 - A\|^2 = \|p_1 - p_2\|^2$

$\Rightarrow \|p_1' - p_2'\|^2 + \|p_2 - A\|^2 = \|p_1 - p_2\|^2$

since $\|p_2 - A\|^2$ is a distance that $\geq 0 \Rightarrow \|p_1' - p_2'\|^2 \leq \|p_1 - p_2\|^2$

we can take out the root $\Rightarrow \|p_1' - p_2'\| \leq \|p_1 - p_2\|$

c) Can the argument in section (b) be extended to 2D planes?

Meaning, let p_1 and p_2 be two points and let p_1' and p_2' be their projections on the 2D plane π. Prove or Disprove that $\|p_1' - p_2'\| \leq \|p_1 - p_2\|$ for any two points p_1, p_2. You may assume the plane π is given by the unit normal n and a point q.

→ Yes the argument can be extended to 2D planes and we can prove with similar way as we did in (b).

Define the plane $\pi : p_1' + t \cdot \vec{v} + s \cdot \vec{u}$ for $\vec{v}$ and $\vec{u}$ some two vectors on the plane. We can define a parallel plane of π that passes through $p_1 \rightarrow \pi' : p_1 + t\vec{v} + s\vec{u}$ (we can use the same $\vec{v}$ and $\vec{u}$ because the planes π and π' are parallel.

let mark the ortogonals vector $(p_1' - p_1) = n_1$ and $(p_2' - p_2) = n_2$

Fotermore we know that p_2 has a projection point on π' also

with the same ortogonal vector n_2 and a scalar $c \to c \cdot n_2$

let mark this projection as A.

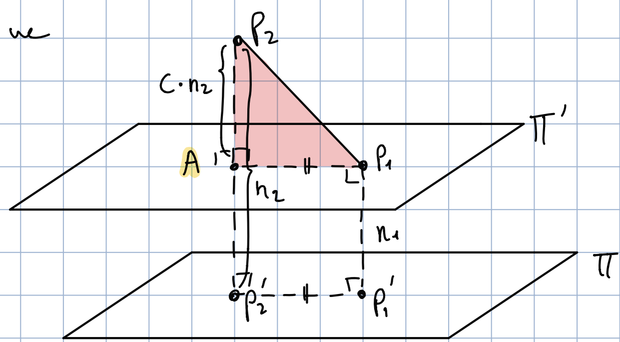
In the same way as in section (b) we

get a retangale A, p_1, p_1', p_2'

therefore the distance $\| p_1' - p_2' \|$

is equal to $\| A - p_1 \|$

and we get a $90°$ triangle of A, p_2, p_1



and according to pythagoras:

$$\| A - p_1 \|^2 + \| A - p_2 \|^2 = \| p_1 - p_2 \|^2 \Rightarrow \| p_1' - p_2' \|^2 + \| A - p_2 \|^2 = \| p_1 - p_2 \|^2$$

$$\Rightarrow \| A - p_2 \| \geq 0 \Rightarrow \| p_1' - p_2' \|^2 \leq \| p_1 - p_2 \|^2 \Longrightarrow \| p_1' - p_2' \| \leq \| p_1 - p_2 \|$$

root out

Question 3

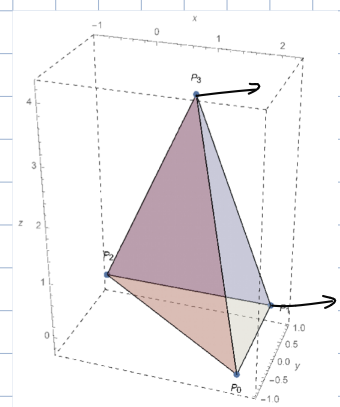
A tetrahedral is given by the points:

$$P_0 = (2,-1,0),\quad P_1 = (2,1,0),\quad P_2 = (-1,1,0),\quad P_3 = (1,0,4)$$

a) Determine the unit normal (facing outside) of the triangles faces of the tetrahedral.

→ let mark the 4 faces as f_1, f_2, f_3 and f_4

f_1 contains the vectors $v_1 = p_2 - p_0$ and $u_1 = p_3 - p_0$
f_2 contains the vectors $v_2 = p_2 - p_1$ and $u_2 = p_3 - p_1$
f_3 contains the vectors $v_3 = p_3 - p_0$ and $u_3 = p_3 - p_1$
f_4 contains the vectors $v_4 = p_2 - p_0$ and $u_4 = p_2 - p_1$



$v_1 = (-1,1,0) - (2,-1,0) = (-3,2,0)$ $\quad u_1 = (1,0,4) - (2,-1,0) = (-1,1,4)$

$$n_1 = v_1 \times u_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 4 - 0\cdot 1 \\ 0\cdot(-1) - (-3)\cdot 4 \\ -3\cdot 1 - 2\cdot(-1) \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \\ -1 \end{pmatrix}$$

$$\hat{n}_1 = \frac{1}{\sqrt{8^2 + 12^2 + (-1)^2}} \cdot (8,12,-1) = \frac{1}{\sqrt{209}}(8,12,-1)\underbrace{\left(\cdot(-1)\right)}_{\text{for facing outside}} = \boxed{\frac{1}{\sqrt{209}}(-8,-12,1)}$$

$v_2 = (-1,1,0) - (2,1,0) = (-3,0,0)$ $\quad u_2 = (1,0,4) - (2,1,0) = (-1,-1,4)$

$$n_2 = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0\cdot 4 - 0\cdot(-1) \\ 0\cdot(-1) - (-3)\cdot 4 \\ -3\cdot(-1) - 0\cdot(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \\ 3 \end{pmatrix}$$

$$\hat{n}_2 = \frac{1}{\sqrt{0^2 + 12^2 + 3^2}}(0,12,3) = \boxed{\frac{1}{\sqrt{153}}(0,12,3)}$$

$v_3 = (1,0,4) - (2,-1,0) = u_1 = (-1,1,4)$ $\quad u_3 = p_3 - p_1 = u_2 = (-1,-1,4)$

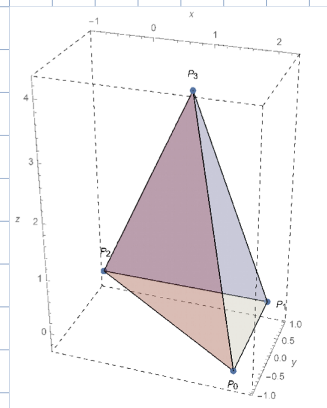
$$n_3 = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 - (-4) \\ -4 - (-4) \\ 1 - (-1) \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 2 \end{pmatrix} \qquad \hat{n}_3 = \frac{1}{\sqrt{8^2 + 0^2 + 2^2}} \cdot (8,0,2) = \boxed{\frac{1}{\sqrt{68}}(8,0,2)}$$

$v_4 = v_1 = (-3,2,0)$ $\quad u_4 = v_2 = (-3,0,0)$

$$n_4 = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0-0 \\ 0-(-6) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} \qquad \hat{n}_4 = \frac{1}{\sqrt{0^2+0^2+6^2}}(0,0,6) = \frac{1}{6}(0,0,6)\underbrace{\left(\cdot(-1)\right)}_{\text{for facing outside}} = \boxed{\frac{1}{6}(0,0,-6)}$$

b) Calculate the implicit equation of te planes that contain the tetrahedral's faces

→ let mark the 4 faces as f_1, f_2, f_3 and f_4



f_1 contains the vectors $v_1 = p_2 - p_0$ and $u_1 = p_3 - p_0$

f_2 contains the vectors $v_2 = p_2 - p_1$ and $u_2 = p_3 - p_1$

f_3 contains the vectors $v_3 = p_3 - p_0$ and $u_3 = p_3 - p_1$

f_4 contains the vectors $v_4 = p_2 - p_0$ and $u_4 = p_2 - p_1$

$v_1 = (-1,1,0) - (2,-1,0) = (-3,2,0) \qquad u_1 = (1,0,4) - (2,-1,0) = (-1,1,4)$

$$n_1 = v_1 \times u_1 = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 4 - 0\cdot 1 \\ 0\cdot(-1) - (-3)\cdot 4 \\ -3\cdot 1 - 2\cdot(-1) \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \\ -1 \end{pmatrix}$$

$\Rightarrow f_1: 8x + 12y - z + d = 0 \rightarrow f_1(p_0) = 8\cdot 2 + 12\cdot(-1) - 0 + d = 0$

$16 - 12 = -d \rightarrow d = -4$

$f_1: 8x + 12y - z - 4 = 0 \mid \cdot(-1) \rightarrow f_1: -8x - 12y + z + 4 = 0$

$v_2 = (-1,1,0) - (2,1,0) = (-3,0,0) \qquad u_2 = (1,0,4) - (2,1,0) = (-1,-1,4)$

$$n_2 = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0\cdot 4 - 0\cdot(-1) \\ 0\cdot(-1) - (-3)\cdot 4 \\ -3\cdot(-1) - 0\cdot(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 12 \\ 3 \end{pmatrix}$$

$\Rightarrow f_2: 0\cdot x + 12y + 3z + d = 0 \rightarrow f_2(p_1) = 0 + 12\cdot 1 + 0 + d = 0 \rightarrow d = -12$

$f_2: 0 + 12y + 3z - 12 = 0$

$v_3 = (1,0,4) - (2,-1,0) = u_1 = (-1,1,4) \qquad u_3 = p_3 - p_1 = u_2 = (-1,-1,4)$

$$n_3 = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 - (-4) \\ -4 - (-4) \\ 1 - (-1) \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 2 \end{pmatrix}$$

$\Rightarrow f_3: 8\cdot x + 0\cdot y + 2z + d = 0 \qquad f_3(p_1) = 8\cdot 2 + 0 + 0 + d = 0$

$d = -16$

$f_3: 8x + 0 + 2z - 16 = 0$

$v_4 = v_1 = (-3,2,0) \qquad u_4 = v_2 = (-3,0,0)$

$$n_4 = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ 0 - 0 \\ 0 - (-6) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

$\Rightarrow f_4: 0\cdot x + 0\cdot y + 6z + d = 0 \qquad f_4(p_2) = 0 + 0 + 6\cdot 0 + d = 0$

$d = 0$

$f_4: 0 + 0 + 6z + 0 = 0 \mid (-1)$

$f_4: -6z = 0$

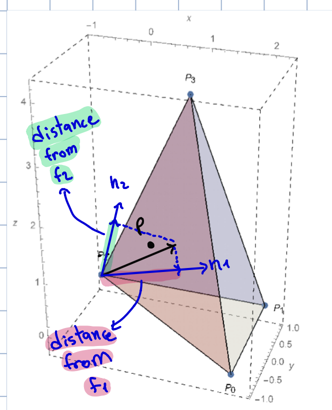
c) Explain in-detail how to determine if a point $p=(x,y,z)$ is inside the tetrahedral.

→ To determine if a point $p=(x,y,z)$ is inside the tetrahedral we need to put p's values in each of the equations we got in section (b). If at all the equations we recived a negative value it means that the point p is in the tetrahedral.

If at one or more equation we get a positive value it is not in the tetrahedral.

d) A point $p=(x,y,z)$ lies inside the tetrahedral. Determine the distance of the point $p=(x,y,z)$ from the tetrahedral surface.

→ We need to find the projection point of p on each of the faces of the tetrahedral. Then the size of the vector between p and each projection will be the distance from each face respectively.



for f_1 we will take p_2 that on this face and define the vector $(p-p_2)=(x+1,y-1,z)$. Now we project this vector on $n_1 \rightarrow \left((x+1,y-1,z)\cdot \hat{n}_1\right) n_1$

$$= \frac{-8(x+1)-12(y-1)+2}{\sqrt{209}}(-8,-12,1) \rightarrow \left| \frac{-8(x+1)-12(y-1)+2}{\sqrt{209}}(-8,-12,1) \right|$$

and this is the distance we need to take off p to get to the projection of p on f_1 ⇒ this is exactly the distance from f_1.

for f_2 we also can take $p_2 \rightarrow (x+1,y-1,z)$ and we need to project this vector on $n_2 \rightarrow \left((x+1,y-1,z)\cdot \hat{n}_2\right) n_2 = \frac{12(y-1)+3z}{\sqrt{153}}(0,12,3)$

$$\rightarrow \left| \frac{12(y-1)+3z}{\sqrt{153}}(0,12,3) \right|$$ as the same way this the distance from f_2.

for f_3 we can take the point p_1 which lies on this face. Define the vector $(p-p_1) = (x-2, y-1, z)$ and the projection of this vector on $n_3 \to ((x-2, y-1, z) \cdot \hat{n}_3)\, n_3 = \frac{8(x-2)+2z}{\sqrt{68}}(8,0,2) \to \left|\frac{8(x-2)+2z}{\sqrt{68}}(8,0,2)\right|$

and this the distance from f_3.

for f_4 we also can take p_1 so we get the vetor $(x-2, y-1, z)$ and need to project it on $n_4 \to \frac{-6z}{6}(0,0,-6) \to \left|\frac{-6z}{6}(0,0,-6)\right|$

and this the distance from f_4.