NUS National University of Singapore | Computing

Partially Observable Markov Decision Process

CS4246/CS5446

AI Planning and Decision Making

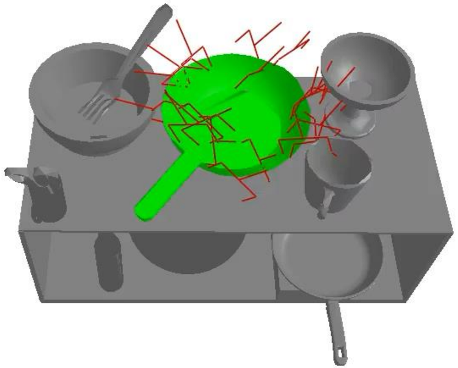
Please join:
pollev.com/anarayan

This lecture will be recorded!

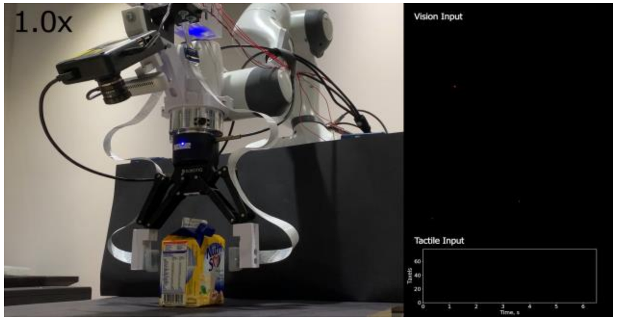
About me

Name: *Tasbolat Taunyazov*, P.h.D student
website: www.tasbolat.com

Control



Manipulation



Perception

Topics

- Partially Observable Markov Decision Process (16.4)
  - Belief states and definitions
- Solution algorithms (16.5)
  - Value iteration
  - Online methods (16.5.2 and 3rd ed. 17.4.3)

`

Solving Sequential Decision Problems

- Decision (Planning) Problem or Model
  - Appropriate abstraction of states, actions, uncertain effects, goals (wrt costs and values or preferences), and time horizon + observations (through sensing)
- Decision Algorithm
  - Input: a problem
  - Output: a solution as an optimal action sequence or policy over time horizon
- Decision Solution
  - An action sequence or solution from an initial state to the goal state(s)
    - An optional solution or action sequence; OR
    - An optimal policy that specifies "best" action in each state wrt to costs or values or preferences
  - (Optional) A goal state that satisfies certain properties

Recall: Decision Making under Uncertainty

- Decision Model:
  - Actions: $a \in A$
  - Uncertain current state: $s \in S$ with probability of reaching: $P(s)$
  - Transition model of uncertain action outcome or effects:
    $P(s'|s,a)$ – probability that action a in state s reaches state s'
  - Outcome of applying action a:
    Result(a) – random variable whose values are outcome states
  - Probability of outcome state s', conditioning on that action a is executed:
    $P(\text{Result}(a) = s') = \sum_s P(s)P(s'|s,a)$
  - Preferences captured by a utility function:
    $U(s)$ – assigns a single number to express the desirability of a state s

Sequential Decision Problems

- What are sequential decision problems?
  - An agent’s utility depends on a sequence of decisions
  - Incorporate utilities, uncertainty, and sensing
  - Search and planning problems are special cases
  - Decision (Planning) Models:
    - Markov decision process (MDP)
    - Partially observable Markov decision process (POMDP)
    - Reinforcement learning: sequential decision making + learning

Quiz

Quiz answer

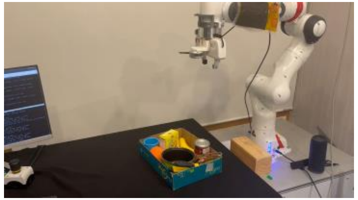
Recap: Markov Decision Process (MDP)

- Formally:
  - An MDP $M \triangleq (S, A, T, R, \gamma)$ consists of:
  - A set S of states
  - A set A of actions
  - A transition function $T: S \times A \times S \rightarrow [0,1]$ that satisfies Markov Property such that:

$$\forall s \in S, \forall a \in A: \sum_{\{s' \in S\}} T(s, a, s') = \sum_{\{s' \in S\}} P(s' | s, a) = 1$$

  - A reward function $R: S \rightarrow \mathbb{R}$ or $R: S \times A \times S \rightarrow \mathbb{R}$
  - A discount factor $0 < \gamma < 1$
  - Solution is a **policy** – a function to recommend an action in each state: $\pi: S \rightarrow A$
    - Solution involves careful balancing of risk and reward

Is MDP enough?



Robotics: grasping an item in a clutter
evidence by pixels

Navigation: driving from A to B
evidence by road signs/buildings/etc

Medicine: detecting a diagnosis of a patient
evidence by symptoms

Partially Observable Markov Decision Process (POMDP)

- An POMDP $M \triangleq (S, A, E, T, O, R)$ consists of
- A set S of states
- A set A of actions
- A set E of evidences or observations or percepts
- A transition function $T: S \times A \times S \rightarrow [0,1]$ such that:

$$\forall s \in S, \forall a \in A: \sum_{s\prime \in S} T(s, a, s') = \sum_{s\prime \in S} P(s'|s, a) = 1$$

- An observation function O: $S \times E \rightarrow [0, 1]$ such that:

$$\forall s \in S, \forall \mathrm{e} \in \mathrm{E}: \sum_{\mathrm{e} \in \mathrm{E}} O(s, e) = \sum_{e \in E} P(e|s) = 1$$

- A reward function $R: S \rightarrow \Re$ or $R: S \times A \times S \rightarrow \Re$
- Solution: What is a policy in POMDP?

Why Study POMDPs?

- Uncertainty in action outcomes
  - MDP: fully observable environment
  - Agent knows exactly which state it is in
- Uncertainty in observations
  - POMDP: partially observable environment
  - Agent does not know exactly which state it is in – cannot observe the state directly
  - Some noisy observations projected from a state
- Real-world challenges
  - Model sequential decision problem for an uncertain, dynamic, partially observable environment
  - If the state is not directly observable, how does an agent reason about its decisions?

Observation Function and Sensor Model

- Observation function:
  - $O(s, e) = P(e|s)$ is the probability of observing (or perceiving evidence) e from state s
  - Define $O(s, e)$ for all $s \in S$ and $e \in E$
  - Assumption of Markov property
- Sensor model
  - The observation function defines the sensor model
  - An observation is also called a measurement or test

Quiz

Quiz answer

Belief State as Probability Distribution

- Belief State:
  - Actual state of the system is unknown, but we can track the probability distribution or belief state over the possible states
  - An action a changes the belief state b, not just the physical state s
  - $b(s)$ denotes probability assigned to actual state s by belief state b
  - If $b(s)$ is the current belief, the agent executes action a and receives evidence e', then the updated belief is given by filtering:

$$b'(s') = \alpha \underbrace{\boxed{P(e'|s')}}_{\text{Sensor model}} \sum_s \underbrace{\boxed{P(s'|s,a)}}_{\text{Transition model}} b(s)$$

where α is the normalizing constant that makes the belief state sum to 1

- Filtering Function:

$$b' = \text{FORWARD}(b, a, e')$$

Intuition on Belief State Update

- Recall that Bayes Theorem:

$$P(A|B) = \alpha\, P(B|A)P(A)$$

  where α is normalizing constant.

- Then:

$$\begin{aligned} b'(s') &= P(s'|e', a) && \text{(by definition)} \\ &= \alpha\, P(e'|s')P(s', a) && \text{(Bayes Theorem)} \\ &= \alpha\, P(e'|s') \textstyle\sum_s P(s'|s, a)b(s). && \text{(marginalization over } s\text{)} \end{aligned}$$

Exercise

$$b'(s') = \alpha P(e'|s') \sum_{s} P(s'|s,a)b(s)$$

- Consider a problem with two states s_1 and s_2:
  - Current belief $b(s_1) = 0.6$ and $b(s_2) = 0.4$.
- For action a:
  - Let the transition probabilities be: $P(s_1|s_1,a) = 0.2$ and $P(s_2|s_1,a) = 0.8$; $P(s_2|s_2,a) = 0.3$ and $P(s_1|s_2,a) = 0.7$
  - Let the observation probabilities be: $P(o_2|s_1) = 0.7$ and $P(o_2|s_2) = 0.2$
- Assume that evidence $e' = o_2$ is received. What is b'?

$$\begin{aligned} b'(s_1) &= \alpha P(o_2|s_1)[P(s_1|s_1,a)b(s_1) + P(s_1|s_2,a)b(s_2)] \\ &= \alpha 0.7[0.2 \times 0.6 + 0.7 \times 0.4] = \quad ? \end{aligned}$$

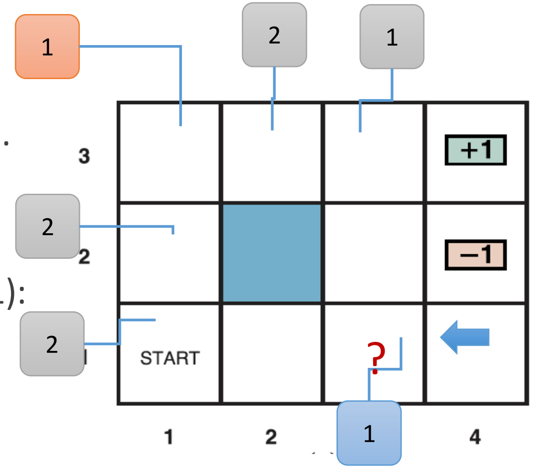
$$\begin{aligned} b'(s_2) &= \alpha P(o_2|s_2)[P(s_2|s_1,a)b(s_1) + P(s_2|s_2,a)b(s_2)] \\ &= \alpha 0.2[0.8 \times 0.6 + 0.3 \times 0.4] = \quad ? \end{aligned}$$

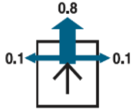
Decision Making in POMDPs

- Main ideas:
  - Optimal action depends only on agent's current belief
  - Optimal policy can be described by a mapping $\pi^*(b)$ from belief to action
  - Optimal policy does not depend on the actual state agent is in!
  - Include value of information as a component in decision making
- Decision cycle of a POMDP agent:
  - Given current belief b, execute action $a = \pi^*(b)$
  - Receive percept e'
  - Set belief to $b' = \text{FORWARD}(b, a, e')$ and repeat

Example: Navigation in Grid World

- POMDP :
  - States S, actions A, transition T , reward R, and observation model O.
- Assume:
  - An observation function measures no. of adjacent walls in a state s
  - For all non-terminal states except those in column 3 (where value = 1):
  - $O(s,2) = 0.9, O(s,*) = 0.1$ (where * indicates a wrong value)
- Belief state:
  - $b = \langle 1/9, 1/9/, 1/9, 1/9, 1/9, 1/9, 1/9, 1/9, 1/9, 0, 0 \rangle$
  - $b((1,1)) = 1/9, b((1,2)) = 1/9, \ldots , b((3,4)) = 0$
- Suppose agent moves Left and sensor reports 1 adjacent wall
  - It is quite likely that agent is now in $(3,1)$
- What is the exact probability values of the new belief state?





Source: RN Chapter 16

Calculating New Belief State

- What is the probability that an agent in belief state b reaches belief state b' after executing action a?
  - With current belief $b(s)$, agent executes action a and perceives e', updated belief is:

$$b'(s') = \alpha P(e'|s') \sum_s P(s'|s, a) b(s) = \text{FORWARD}(b, a, e')$$

    where α is normalizing constant for belief state to sum to 1
  - New belief b' is a conditional probability over actual state given sequence of percepts and actions so far
  - If action and subsequent percept were known, deterministic update to the belief using $b' = \text{FORWARD}(b, a, e')$
  - But subsequent percept is not yet known, so agent might arrive in one of several possible belief states b', depending on percept received

Calculating Probability of Percept

- Probability of percept
  - Probability of perceiving e', given that a was performed starting in belief state b, is given by summing over all the actual states s' that the agent might reach:

$$P(e'|a,b) = \sum_{s'} P(e'|a,s',b)P(s'|a,b)$$

$$= \sum_{s'} P(e'|s')P(s'|a,b)$$

$$= \sum_{s'} \underbrace{\boxed{P(e'|s')}}_{\text{Sensor model}} \sum_{s} \underbrace{\boxed{P(s'|s,a)}}_{\text{Transition model}} b(s)$$

Calculating Transition Model and Reward Model

- Probability of reaching b' from b, given action a:
  - Transition model for belief state space:

$$P(b'|a,b) = \sum_{e'} P(b'|e',a,b)P(e'|a,b)$$
$$= \sum_{e'} P(b'|e',a,b) \sum_{s'} \underbrace{\boxed{P(e'|s')}}_{\text{Sensor model}} \sum_s \underbrace{\boxed{P(s'|s,a)}}_{\text{Transition model}} b(s) \,.$$

where $P(b'|e',a,b)$ is 1 if $b' = \text{FORWARD}(b,a,e')$ and 0 otherwise

- Reward function of belief state space:
  - Expected reward if the agent does a in belief state b:

$$\rho(b,a) = \sum_s b(s) \sum_{s'} P(s'|s,a)\,R(s,a,s')$$

Reducing POMDP into an MDP

- Belief space MDP:
  - Transition model $P(b'|b, a)$ and reward model $\rho(b, \mathrm{a})$ define an observable MDP on the space of belief states!
  - Solving a POMDP on a physical state space – solving a continuous, usually high-dimensional MDP on the corresponding belief-state space
  - An optimal policy for this MDP, $\pi^*(b)$ is also an optimal policy for the original POMDP
- Remember:
  - The belief state is always observable to the agent, by definition

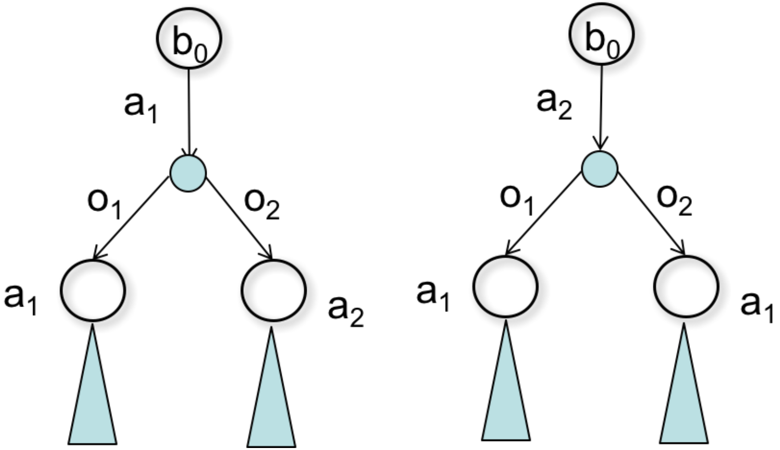
Value Iteration

Value Iteration for POMPDs

- Policy and conditional plan
  - Consider an optimal policy π^* and its application in a specific belief state b
  - Policy generates an action, then for each subsequent observation or percept, belief state is updated and new action is generated, and so on
  - For b, the policy is equivalent to a conditional plan
- Main issue:
  - How does the expected utility of executing a fixed conditional plan varies with the initial belief state?

Conditional Plan

- A policy at a belief b_0 is a conditional plan

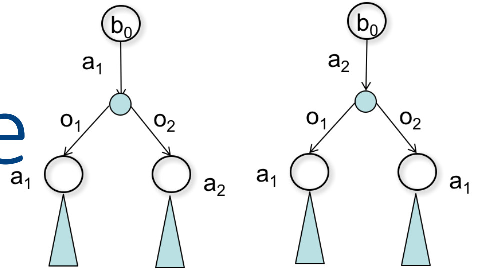


- Multiple conditional plans are possible

Utility Function for Belief State

- Note:
  - A belief state b is a probability distribution
  - Each utility value in a POMDP is a function of an entire probability distribution
- Problems:
  - Probability distributions are continuous
  - Huge complexity of belief spaces
- Solution:
  - For finite state, action, and observation spaces and planning horizon:
    utility functions represented by piecewise linear functions over belief space

Utility Function of Belief State



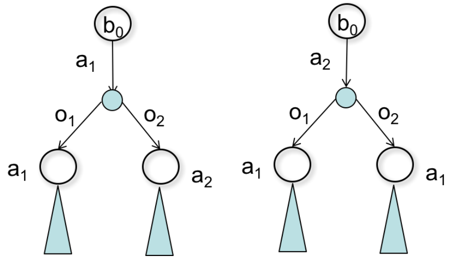
- Let the utility of executing a fixed conditional plan p starting in physical state s be $\alpha_p(s)$ – alpha vector

- Expected utility of executing p in belief state b is:

  $U_p(b) = \sum_s b(s)\alpha_p(s)$ or $b \cdot \alpha_p$ (inner product of vectors b, α_p)

  - A linear function of b, corresponding to a hyperplane in belief space

Utility Function of Belief State



- At any particular belief state b, optimal policy is to choose the conditional plan with highest utility:

$$U^{\pi^*}(b) = U(b) = \max_p b \cdot \alpha_p$$

- Utility or value function $U(b)$ on belief states, being the maximum of a collection of hyperplanes, will be piecewise linear and convex
- For finite depth, there are only a finite number of conditional plans
  - With $|A|$ actions, $|E|$ observations, there are $|A|^{O(|E|^{d-1})}$ distinct depth-d plans

Example: A Two-State World

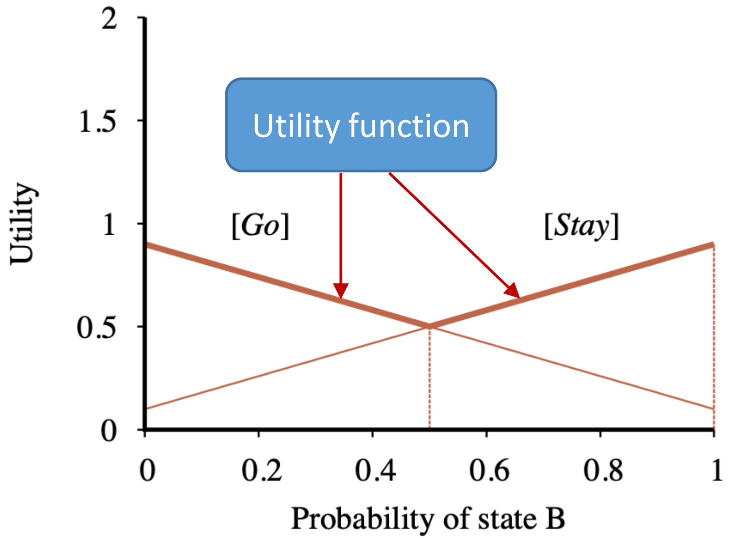
- Given:
  - Two states: $A\,,B$ (or $0, 1$ in RN 3e) [Belief space is 1-dimensional]
  - Rewards: $R(.,.,A) = 0\,,\ R(.,.,B) = 1$ [Any transition ending in A is 0, …]
  - Two actions:
    - Stay – Stays put with probability = 0.9
    - Go – Switches to the other state with probability = 0.9
  - Discount factor: $\gamma = 1$
  - Sensor: Reports correct state with probability = 0.6
- Obviously:
  - Agent should: Stay when it thinks it is in state B and Go when it thinks it is in state A
- What are the values for 1-step plans (α-vectors)?

Example: Solving a POMDP, $d = 1$

- Consider one-step plans: [Stay] and [Go]
  - Each receives reward for one transition as follows:

$$\alpha_{[Stay]}(A) = 0.9\ R(A, Stay, A) + 0.1\ R(A, Stay, B) = 0.1$$
$$\alpha_{[Stay]}(B) = 0.1\ R(B, Stay, A) + 0.9\ R(B, Stay, B) = 0.9$$
$$\alpha_{[Go]}(A) \quad = 0.1\ R(A, Go, A) + 0.9\ R(A, Go, B) \qquad = 0.9$$
$$\alpha_{[Go]}(B) \quad = 0.9\ R(B, Go, A) + 0.1\ R(B, Go, B). \qquad = 0.1$$

- Hyperplanes for $b \cdot \alpha_{[Stay]}$ and $b \cdot \alpha_{[Go]}$
  - Utility or value function: max of the two linear functions
- What is the one-step optimal policy?
  - Optimal policy is to Stay if $b(B) > 0.5$ and Go otherwise



Utility of Belief-State $b(B)$: $d = 1$

Source: RN Figure 17.15 (a)

Example: Solving a POMDP, $d = 2$

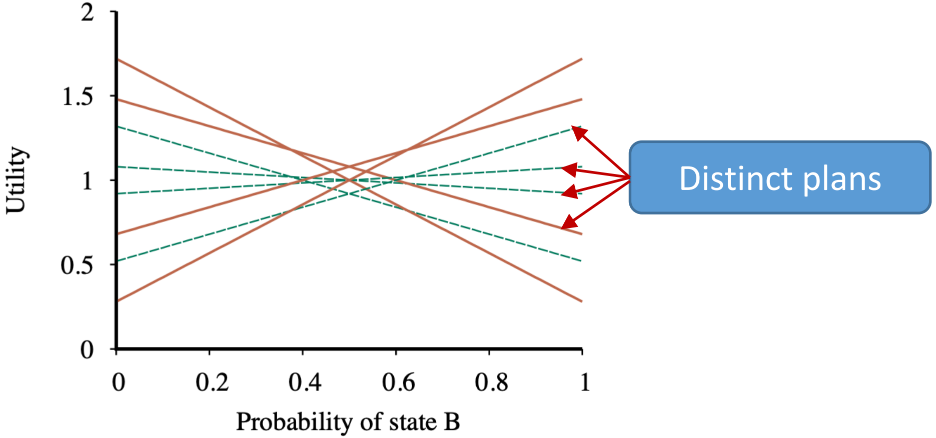
- Deriving a solution:
  - Obtain utilities $\alpha_p(s)$ for all the conditional plans p of depth 1 in each physical state s
  - Compute utilities for conditional plans of depth 2 by considering:
    - each possible first action
    - each possible subsequent percept, and
    - each way of choosing a depth-1 plan to execute for each percept:

      [Stay; **if** Percept = A **then** Stay **else** Stay]

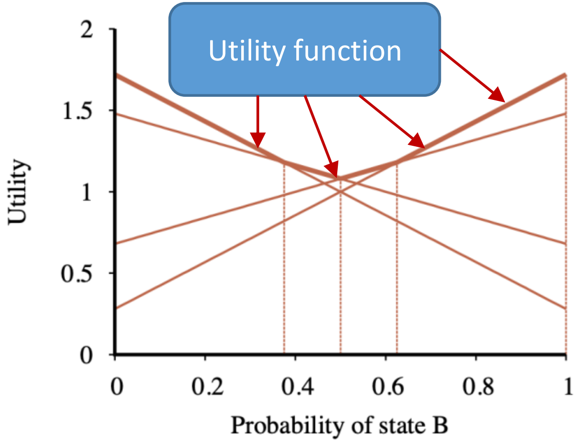
      [Stay; **if** Percept = A **then** Stay **else** Go]

      [Go; **if** Percept = A **then** Stay **else** Stay] …
  - There are 8 distinct depth-2 plans in all
    - 4 suboptimal and dominated plans – dominated plans are never optimal
    - 4 undominated plans, each optimal in a specific region
  - The regions partition the belief-state space

Example: Utility of Belief-State $b(B)$: $d = 2$



Utilities of 8 distinct 2-step plans

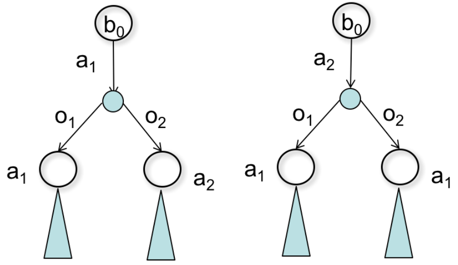


Utilities of 4 undominated 2-step plans

- Continuous belief space is divided into regions; the regions partition the belief-state space
  - Each region corresponds to a conditional plan that is optimal for that region
- $U(b)$ is piecewise linear and convex

Source: RN Figure 16.15 (b) and (c)

Value Iteration in POMDP



- In general:
  - Let p be a depth-d conditional plan with initial action a followed by depth $(d-1)$ subplans $p \cdot e'$ for percept e'

Transition model

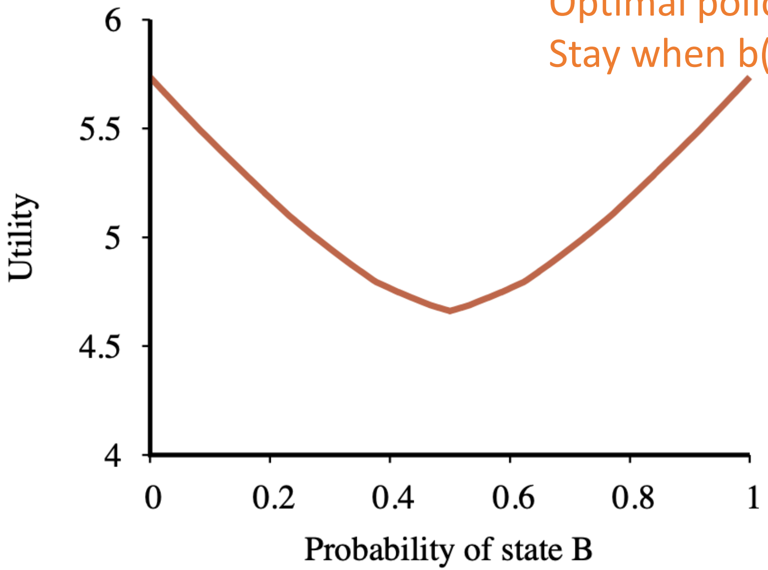
Observation model

$$\alpha_p(s) = \sum_{s'} P(s'|s,a) \left[R(s,a,s') + \gamma \sum_{e'} P(e'|s') \alpha_{p \cdot e'}(s') \right]$$

- Given utility function:
  - Executable policy extracted by:
    - looking at which hyperplane is optimal at any given belief state b
    - executing the first action of the corresponding plan
- POMDP Value Iteration (VI):
  - maintains a collection of undominated plans $\{p\}$ with their utility hyperplanes $\{alpha\ vectors\ \alpha_p\}$
  - Cf MDP VI computes one utility number, $U(s)$ for each state s

Example: Utility of Belief-State $b(B)$: $d = 8$

Optimal policy:
Stay when b(B) > 0.5 and Go otherwise



Utility function for optimal 8-step plans

Source: RN Figure 17.15 (d)

POMDP Value Iteration Algorithm

function POMDP-VALUE-ITERATION(*pomdp*, ϵ) **returns** a utility function
  inputs: *pomdp*, a POMDP with states S, actions $A(s)$, transition model $P(s' \mid s, a)$,
      sensor model $P(e \mid s)$, rewards $R(s, a, s')$, discount γ
    ϵ, the maximum error allowed in the utility of any state
  local variables: U, U', sets of plans p with associated utility vectors α_p

  $U' \leftarrow$ a set containing all one-step plans $[a]$, with $\alpha_{[a]}(s) = \sum_{s'} P(s' \mid s, a)\, R(s, a, s')$
  repeat
    $U \leftarrow U'$
    $U' \leftarrow$ the set of all plans consisting of an action and, for each possible next percept,
      a plan in U with utility vectors computed according to Equation (16.18)
    $U' \leftarrow$ REMOVE-DOMINATED-PLANS(U')
  until MAX-DIFFERENCE$(U, U') \leq \epsilon(1-\gamma)/\gamma$
  return U

Source: RN Figure 16.16

Notes on Value Iteration

- Algorithm's complexity depends primarily on how many plans get generated.
  - Given $|A|$ actions and $|E|$ observations, there are $|A|^{O(|E|^{d-1})}$ distinct depth-d plans.
  - $|A|n^{|E|}$ conditional plans generated at level $d+1$ before elimination, where n is the no. of conditional plans at level d
  - Elimination of dominated plans to reduce doubly exponential growth
  - P-SPACE hard – very inefficient in practice
  - Approximate methods necessary
- Intermediate belief states have lower value than state A and state B
  - In the intermediate states the agent lacks information needed to choose a good action.
  - Information has value and optimal policies in POMDPs often include information-gathering actions.

Online Methods

Approximate solutions

Scaling POMDP solvers

- POMDP solvers need to solve two problems
  - Belief tracking/filtering
    - Given the history observed, what is the current belief?
  - Planning
    - Given the current belief, what is the optimal action to take?
- To scale up, need to scale up for both problems

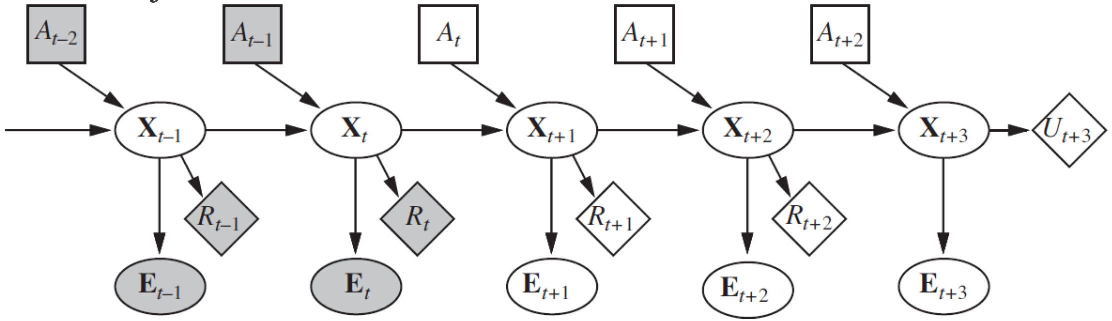
Online Agent for POMDP

- Main ideas:
  - Represent transition model and sensor model by a dynamic decision network (DDN)
  - Belief tracking – Inference in DDN – Exact inference is computationally intractable
    - No known polynomial time algorithm, as number of state variables grow
  - Approximate solution: Deploy a filtering algorithm (exact or particle filtering) to incorporate each new percept and action and to update belief state representation
  - Projecting forward possible action sequences and choosing the best one
- Note:
  - A DDN can actually be used as inputs for any POMDP solution algorithm

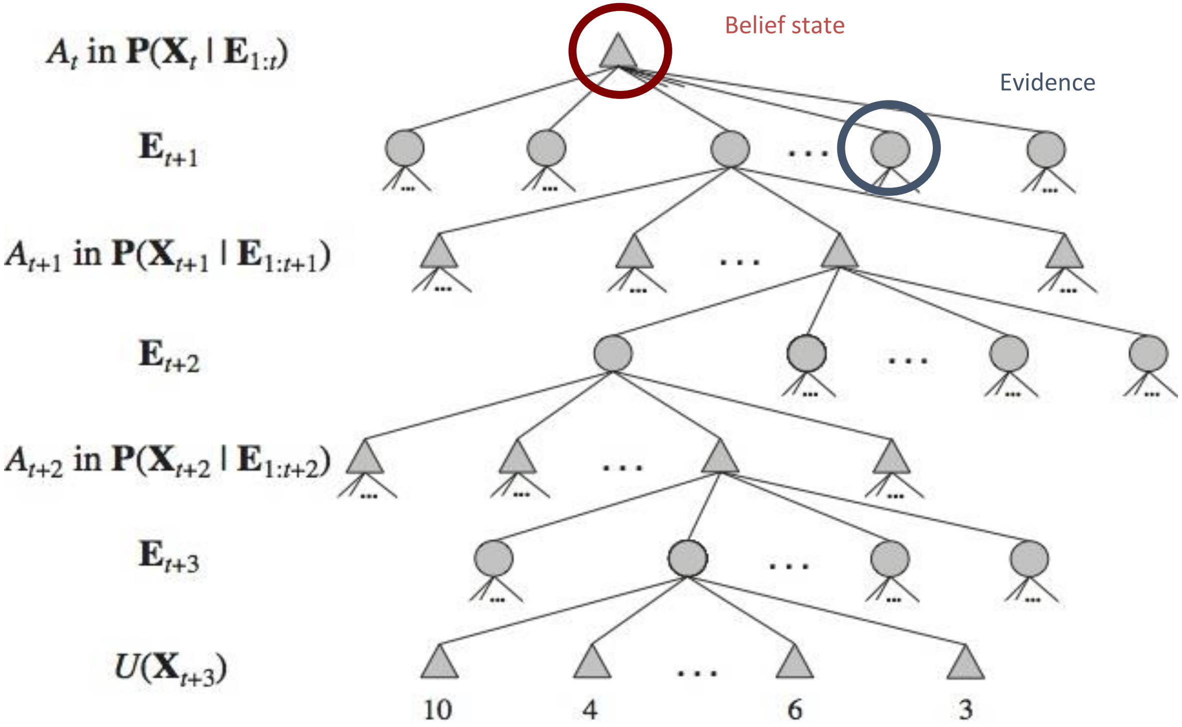
Dynamic Decision Network (DDN)

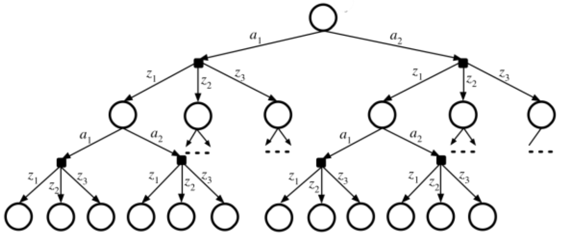
- Execution of POMDP over time can be represented as a DDN
  - Transition and sensor models represented by a Dynamic Bayesian Network (DBN)
  - Add decision and utility nodes to get DDN
- In DDN:
  - State S_t becomes set of variables $\mathbf{X}_t$
  - Evidence/observation variables are $\mathbf{E}_t$
  - Action at time t is A_t
  - Transition: $P(\mathbf{X}_{t+1}|\mathbf{X}_t, A_t)$
  - Sensor model: $P(\mathbf{E}_t|\mathbf{X}_t)$

Note:
- Variables with known values are shaded
- Current time is t and agent must decide what to do



Look-Ahead Solution of POMDP





Alternate form

Time complexity: $O\left(|A|^d|E|^d\right)$

Source: RN 3e Figure 16.11

Notes on Online Solution

- Main ideas:
  - Project action sequences forward from current belief state
  - Determine utility estimates at projected depth based on future rewards
  - Extract decision by backing up utilities from the leaves
    - Averaging chance nodes and maximizing/minimizing decision nodes
  - Non-leaf states may have rewards
  - Decision nodes correspond to belief states rather than actual states
- Complexity
  - Time complexity of exhaustive search to depth d is $O\left(|A|^d|E|^d\right)$ where $|A|$ is the no. of available actions and $|E|$ is the number of possible percepts

POMCP[1]

Partially Observable Monte Carlo Planning

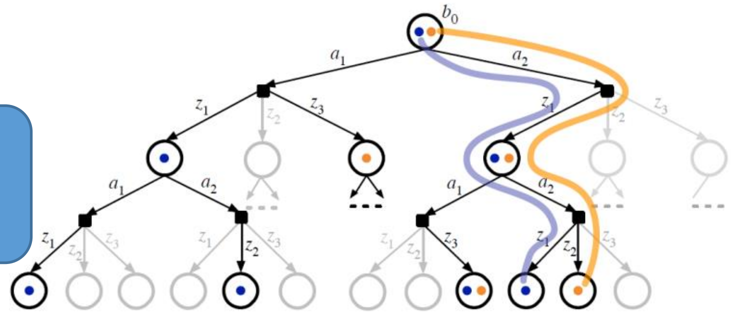
MCTS:
- Select
- Expand
- Simulate
- Backup

- To run UCT on POMDP, we need to represent beliefs in the nodes
  - Inference to construct beliefs is intractable in general
  - For MDP, states are easy to generate
  - Beliefs are difficult to generate, so POMCP uses action-observation history
  - History is equivalent to belief assuming same initial belief; |A|^d k^d
- Instead of propagating beliefs forward in a trial
  - POMCP samples a state at the root from the initial belief.
  - Run the simulation using the state to generate action-observation history
  - Only need to sample from $P(s'|s,a)$ then $P(e'|s')$ to generate observation e' for the action-observation history
  - Avoid constructing beliefs

Silver, D. and J. Veness, *Monte-Carlo planning in large POMDPs*, in *Advances in Neural Information Processing Systems 23: 24th Annual Conference on Neural Information Processing Systems 2010.* . 2010: Vancouver, British Columbia, Canada. p. 2164-2172

DESPOT[2]

Determinized Sparse Partially Observable Tree



Source: Ye et al 2017

- Construct search tree differently! (make tree smaller)
  - Construct k different search trees
  - Each search tree, sample a state at the root to initialize
  - At every action node, try all actions on the (single) state at that node
  - At every observation node, sample a single observation from the (single) state at node
- Size of the tree:
  - Each of the k trees have size A^d. Combine together to get tree of size $O(|A|^d k)$
  - Exponentially smaller than $|A|^d|E|^d$
- Can show that searching sampled tree is sufficient if a small good policy exists
  - Use heuristics to do anytime search of this tree

[2]Ye, N., et al., *DESPOT: online POMDP planning with regularization.* J. Artif. Int. Res., 2017. **58**(1): p. 231–266.

POMDP Applications

- Autonomous driving golf cart in UTown
  - https://youtu.be/y_9VMD_sQhw
  - DESPOT is used there
  - Works in real-time
- Robotics
- Airborne Collision Avoidance System X
  - ACAS X
  - POMDPs with neural networks for function approximation
- Education
- Dialog system/Chatbot
  - Assistive agent for dementia patients

ARTICLE

Automated handwashing assistance for persons with dementia using video and a partially observable Markov decision process

Authors: Jesse Hoey, Pascal Poupart, Axel von Bertoldi, Tammy Craig, Craig Boutilier, Alex Mihailidis Authors Info & Claims

Computer Vision and Image Understanding, Volume 114, Issue 5 • May, 2010 • pp 503–519 • https://doi.org/10.1016/j.cviu.2009.06.008

Summary

- POMDPs
  - Compute optimal policy under partially observability.
  - For finite horizon problems, resulting optimal utility (value) functions are continuous, piecewise linear, and convex.
  - In each iteration, no. of linear constraints grows exponentially.
  - Approximate solvers can scale to to large size problems; solvers mainly remain in research settings
    - For example, the SARSOP solver from NUS does the Bellman update only on a small subset of beliefs and use heuristics to explore the belief space
    - https://www.comp.nus.edu.sg/~leews/publications/rss08.pdf
  - Online search tends to scale better.

Homework

- Readings
  - RN 16.4, 16.5
  - *RN 14.2.1, 14.5.3 (Filtering and Particle Filtering – Optional)*
- References

  - Kaelbling, L.P., M.L. Littman, and A.R. Cassandra, *Planning and acting in partially observable stochastic domains.* Artif. Intell., 1998. **101**(1–2): p. 99–134.
  - Silver, D. and J. Veness, *Monte-Carlo planning in large POMDPs*, in *Advances in Neural Information Processing Systems 23: 24th Annual Conference on Neural Information Processing Systems 2010.* . 2010: Vancouver, British Columbia, Canada. p. 2164-2172.
  - Ye, N., et al., *DESPOT: online POMDP planning with regularization.* J. Artif. Int. Res., 2017. **58**(1): p. 231–266.