Assignment 1

Geometric Motion Planning

Name

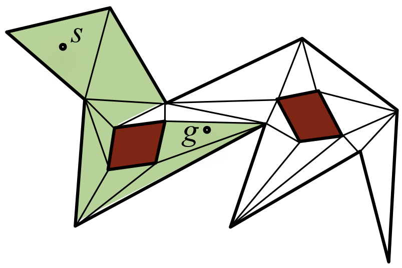
Matriculation Number

Instructions

- Due date: **14 Feb 2023** in class.
- Type up your solutions or write **neatly**.
- Submit your work in hard copy. Make sure to include the cover page.
- The starred questions provide extra-credits.

Problem	**Max Points**	**Points**
1	10	
2	10	
3	10	
4	15	
5	15	
6	15*	
7	25	
8	10	
9	15	
10	10	
11	10 + 5* (proof)	
Total	**130+20***	

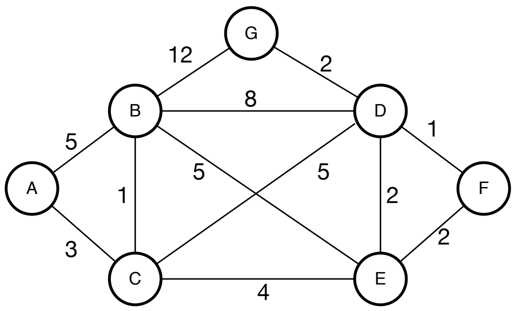
1. For a point robot in a 2-D polygonal environment with obstacles, its free space is decomposed into and represented as a set of triangles. A *channel* is a sequence of adjacent triangles that contains a collision-free path connecting a start position s and a goal position g.



(*a*) Define a search graph G over the triangles so that we can use G to find a channel σ that connects s and g. Describe what the nodes and edges of G represent.

(*b*) Given a channel σ that connects s and g, describe a method to generate a collision-free path from s to g. Draw an example to illustrate your solution.

2. Find the shortest path to node A from every other node in the weighted graph below.



(*a*) Apply the backward dynamic programming algorithm. Let $V^*(s)$ be the shortest-path length from node s to A and $V_i(s)$ be the estimated shortest-path length in the i'th iteration of the dynamic programming algorithm. Show the values for $V_0(s)$, $V_1(s)$, and $V_2(s)$ as well as $V^*(s)$ for all nodes in the graph.

	A	B	C	D	E	F	G
V_0							
V_1							
V_2							
V^*							

(*b*) Apply the Dijkstra's algorithm. The shortest paths form a tree. Draw the shortest-path tree.

3. Let M be a 3×3 orthonormal matrices with determinant $+1$. We have discussed in the class that every such matrix corresponds to a rotation in 3-D space, and vice versa. This implies that M has only 3 *independent* degrees of freedom (DOFs); however, M contains 9 parameters.

(*a*) What are the constraints on the 9 parameters that reduce the number of DOFs of M to 3?

(*b*) M is required to have determinant $+1$. Does this constraint reduce the number of DOFs of M? Why or why not?

4. Give the dimension of the configuration space for the following systems. Briefly justify your answer.

(*a*) An articulated robot in a 2D plane with a fixed base and two revolute joints.

(*b*) Two mobile robots freely translate and rotate in the plane.

(*c*) An aerial manipulator consisting of two manipulators attached to an unmanned aerial vehicle (UAV). Each manipulator has 6 revolute joints.

5. This problem explores the configuration space of lines and that of line segments.
   (*a*) Consider an infinite line ℓ that translates and rotates freely in 3-D space. Give two different parameterizations of the configuration space C for ℓ: one that makes use of angles and one that makes no use of angles.
   (*b*) What is the dimension of C?
   (*c*) Consider a straight-line segment s that translates and rotates freely in 3-D space. What is the dimension of the configuration space for s? Can you use the two parameterizations in part (*a*) for s? What modifications would be needed if any?

6. This problem examines the relationship between distance in the configuration space and distance in the workspace. Specifically, if a robot moves by a certain amount in the configuration space, how much can a point on the robot move in the workspace? Consider a planar robot arm with n sequential links. Each link is a straight-line segment of length L. One endpoint of the link is called the *origin*, and the other is called the *extremity*. The origin of the first link is fixed. The origin of the ith link ($2 \leq i \leq n$) coincides with the extremity of the $(i-1)$th link at a point called a *joint*. A link can rotate freely about the joint.
   (*a*) A configuration q of this robot can be represented by the joint angles $(\theta_1, \theta_2, \ldots, \theta_n)$. The metric d_c in the robot's configuration space is defined as

$$d_c(q, q') = \max_{1 \leq i \leq n} |\theta_i - \theta'_i|$$

   for two configurations q and q'. Suppose that the robot moves from a configuration $q = (\theta_1, \theta_2, \ldots, \theta_n)$ to a configuration $q' = (\theta'_1, \theta'_2, \ldots, \theta'_n)$ along the straight-line segment joining q and q' in the Cartesian space R^n. In other words, the robot moves along the path $(1-\lambda)q + \lambda q'$ for $0 \leq \lambda \leq 1$. Show that no point on the robot traces a path longer than $\alpha d_c(q, q')$ for some positive constant α. Give a bound of α in terms of L, the link length, and n, the number of links.
   (*b*) Let $d_w(q, B)$ denote the minimum distance between the robot placed at a configuration q and a (workspace) obstacle B, *i.e.*, the distance between the closest pair of points on the robot placed at q and B. Using the result from part (*a*), calculate the radius ρ of the neighborhood

$$N(q) = \{q' \mid d_c(q, q') \leq \rho\}$$

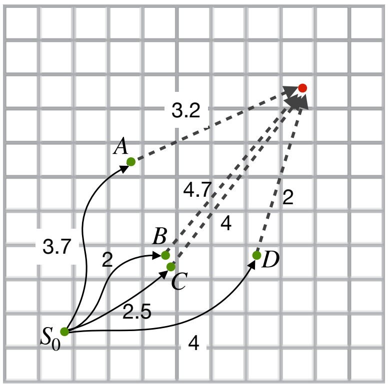
   in which the robot is guaranteed to move freely without colliding with B. Express your answer in terms of α and $d_w(q, B)$.

7. Describe how to sample points uniformly from each set below. For part (*a*) and (*b*), use polar-coordinates parametrization.
   (*a*) A circle with center c and radius r: $S = \{p \in \mathrm{R}^2 \mid \|p - c\| = r\}$.
   (*b*) A disc B_2 with center c and radius r: $B_2 = \{p \in \mathrm{R}^2 \mid \|p - c\| \leq r\}$.
   (*c*) Alternatively, sample the disc B_2 using the rejection method. Sample uniformly from the bounding box of B containing B_2 and reject the samples outside of the disc. Which method would you choose, polar-coordinates parametrization or rejection sampling? Why?
   (*d*) An N-dimensional sphere with center c and radius r: $B_N = \{p \in \mathrm{R}^N \mid \|p - c\| \leq r\}$. If using rejection sampling, discuss the rejection rate and how it scales with the dimension of the space, N.
   (*e*) Can you think of a way to sample B_N without rejection for arbitrary large N?

8. Suppose that the configuration space $\mathcal{C}$ is the unit square $[0,1] \times [0,1]$. The multi-query PRM algorithm, first samples n collision-free configurations and then tries to connect these milestones by calling LINK.
   (*a*) If the algorithm calls LINK for every pair of roadmap nodes, give an asymptotic upper bound on the number of calls to LINK.
   (*b*) Suppose that the algorithms calls LINK only if the Euclidean distance between two milestones is smaller than a threshold t. Give an asymptotic bound on the number of calls to LINK when $t = O(1/\sqrt{n})$. You may assume that the milestones are distributed roughly uniformly in $\mathcal{C}$.

9. Among the motion planning algorithms discussed in the class (dynamic programming, A*, PRM, EST, and RRT), choose the best algorithm for the robot motion planning tasks below. Justify your choice by considering the configuration space dimensions, environment characteristics, computational efficiency, … .
   (*a*) A robot car trying to park itself in an open parking slot.
   (*b*) Because of the COVID-19 pandemic, a cleaning robot must move around and disinfect the busy areas of the hospital every 2 hours.
   (*c*) A self-reconfigurable modular robot consists of many identical modules and can reconfigure its shape to fit a task. See the video for an example.

10. Consider the hybrid A* algorithm described in the class and apply it to plan the motion of an autonomous robot car. Suppose that the A* search starts at the node S, shown in the figure below. By applying four candidate actions, it reaches new nodes: A, B, C, and D. The cost-to-come and the heuristic estimate of the cost-to-go for all the new nodes are shown in the figure.



   (*a*) What is the dimension of the grid used in the hybrid A* search?
   (*b*) What does the priority queue contain at the stage of the hybrid A* search illustrated in the figure? For each item, in the priority queue, specify the node and its associated f-value.

11. A key step in applying the A* algorithm in practice is to design a good heuristic function. Suppose that we want to apply A* to a shortest-path problem and have two heuristic functions $h_1(x)$ and $h_2(x)$, both of which are admissible.
   (*a*) Show that $h(x) = \max\{h_1(x), h_2(x)\}$ is also admissible.
   (*b*) Of the three heuristic functions, $h_1(x), h_2(x)$, and $h(x)$, which one would you use? Why? Give a proof if you can.