Rational Decision Making

CS4246/CS5446

AI Planning and Decision Making

This lecture will be recorded!

Topics

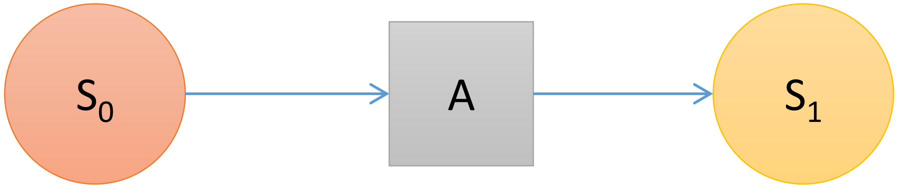
- Nondeterminism
- Decision Making under Uncertainty
- Basis of Utility Theory (15.2)
  - Axioms of Utility Theory
  - Rational preferences lead to utility
- Principle of Maximum Expected Utility (15.1)
- Utility Functions (15.3)

Nondeterminism

Partially observable, nondeterministic, and uncertain environments

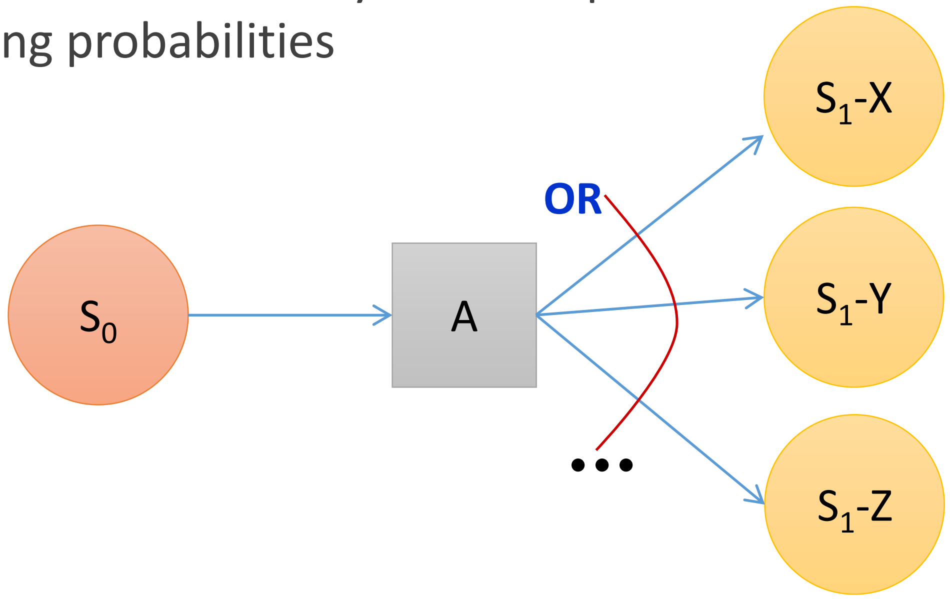
Definition

- Deterministic environment
  - Next state of the environment is completely determined by current state and action executed



Definition

- Non-deterministic environment
  - Actions are characterized by different possible outcomes, without any attaching probabilities



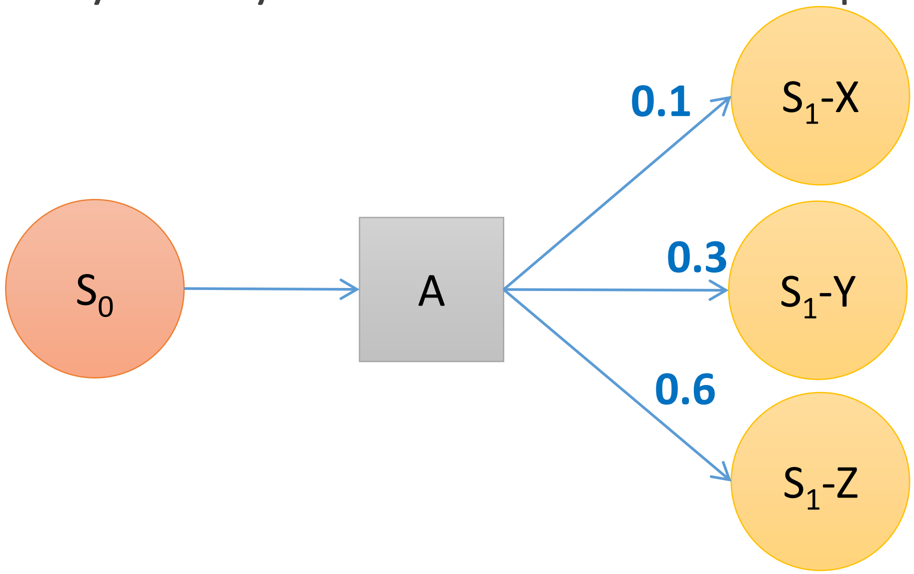
Non-Determinism

- Non-deterministic effects
  - Action may have different possible outcomes
- Controllable (angelic) nondeterminism
  - When an agent itself makes the choices
  - E.g., in hierarchical planning – still deterministic!
- Uncontrollable (demonic) nondeterminism
  - When an adversary or nature makes the choices
  - Planning in nondeterministic or uncertain environments

RN: 11.4.3

Definition

- Stochastic or uncertain environment
  - Outcomes not fully observable or not deterministic
  - Uncertainty usually characterized in terms of probabilities



Some Terminology

- State variables (or fluents)
  - Factored representations of states whose truth values may change over time
- Belief states
  - Sets of physical states the agent might be in
  - Represent agent's current belief about the possible physical states it might be in
  - Entire belief-state space contains every possible set of physical states.
  - If problem P has N fluents, then there are up to 2^N states in the belief space

Example:
What are the belief states for binary fluents:
Have_Cake ∧ Eaten_Cake?

- Percepts
  - Percepts supplied by sensors in acting, but a model of sensors is needed in planning
  - Need to reason about percepts obtained when plan is executed in partially observable environment

Decision Making under Uncertainty

Uncertain states and action effects

Rational Decision Making

- Main ideas:
  - Make decisions based on combining beliefs and desires
  - Choose a strategy for experimentation and action that is logically consistent (cf. right) with:
    - basic judgments about the unknown states or events
    - basic preferences for consequences
- Rationality
  - Assume limited resources
  - Make decisions that will maximize profit/gain (or minimize cost/pain)
  - Rationality as a product (ends) or a process (means)

Types of Decision Theory

- Normative decision theory
  - Describes how ideal, rational agents should behave
- Descriptive decision theory
  - Describes how actual agents (humans) really behave
- Prescriptive decision theory
  - Prescribes guidelines for agents to behave rationally

Decision Theoretic Agents

- Recall: Goal-based agent
  - Has binary distinction between good (goal) and bad (non-goal) states
- Decision-theoretic agent
  - Based on combining probability theory and utility theory
  - Makes rational decisions based on beliefs and desires
  - Operates in contexts with uncertainty and possibly conflicting goals
  - Has continuous measure of outcome quality
- (Normative) Decision Theory
  - Choosing among actions based on desirability of immediate outcomes
  - Assuming episodic, non-deterministic, partially observable environments

Solving Decision Problems

- Decision (Planning) Problem or Model
  - Appropriate abstraction of states, actions, uncertain effects, and goals (wrt costs and values or preferences)
- Decision Algorithm
  - Input: a problem
  - Output: a solution in the form of an optimal action sequence
    - Optimal action at each decision or choice point
- Decision Solution
  - An action sequence or solution from an initial state to the goal state(s)
    - An optimal solution or action sequence; OR
    - An optimal policy that specifies "best" action in each state wrt to costs or values or preferences
  - (Optional) A goal state that satisfies certain properties

Decision Making under Uncertainty

- Decision (Planning) Model:
  - Actions: $a \in A$
  - Uncertain current state: $s \in S$ with probability of reaching: $P(s)$
  - Transition model of uncertain action outcome or effects:
    $P(s'|s, a)$ – probability that action a in state s reaches state s'
  - Outcome of applying action a:
    Result(a) – random variable whose values are outcome states
  - Probability of outcome state s', conditioning on that action a is executed:
    $P(\text{Result}(a) = s') = \sum_s P(s)P(s'|s, a)$
  - Preferences captured by a utility function:
    $U(s)$ – assigns a single number to express the desirability of a state s

Axioms of Utility

Constraints on rational preferences

Modeling Preferences

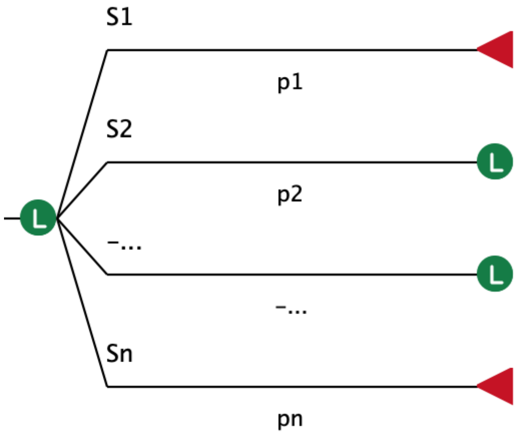
- Preference
  - For any two acts or events A and B, $A \succ B$ (A is preferred to B) or $A \sim B$ (A and B are indifferent)
  - Agent is indifferent between two alternatives only if agent:
    - has considered both alternatives; and
    - is completely willing to trade one for the other
- Utility
  - Classify every item in an individual's preference ordering
    - Applying the conditions of rationality
  - Numeric ranking shows relative importance of an indifference class
    - Relative importance or utility of items in the class
- Utility function or Utility Scale $U(s)$:
  - Assigns a single number to express desirability of a state
  - Numbering items in a preference ordering

Rational Preferences

- Assumptions:
  - $A \succ B$: Agent prefers A to B
  - $A \sim B$: Agent is indifferent between A and B
  - $A \succcurlyeq B$: Agent prefers A over B or is indifferent between them
- Lottery:
  - Action as ticket to lottery L:
    - with set of outcomes S_1,…, S_n
    - with probabilities p_1,…, p_n

  $$L = [p_1, S_1; p_2, S_2, \dots, p_n, S_n]$$

  - Each outcome S_i of a lottery can be either an atomic state or another lottery



Axioms of Utility Theory

- Utility Theory:
  - Understand how preferences between complex lotteries are related to preferences between the underlying states in the lotteries
- 6 axioms of utility
  - Constraints on preferences
  - An agent that violates any axiom will exhibit irrational behavior in some situations

6 Axioms of Utility

- A1: Orderability
- A2: Transitivity
- A3: Continuity
- A4: Substitutability
- A5: Monotonicity
- A6: Decomposibility

A1: Orderability

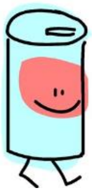
- Definition
  - Given any two lotteries, a rational agent must either:
    - prefer one to the other; OR
    - Rate the two as equally preferable
  - Exactly one of $(A \succ B)$, $(B \succ A)$, or $(A \sim B)$ holds
- Example:
  - A student will prefer
    - taking CS4246 to CS5340
    - taking CS5340 to CS4246
  - Or is indifferent between the two modules

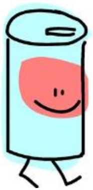
A2: Transitivity

- Definition
  - Given any three lotteries, if an agent prefers A to B and prefers B to C, then the agent must prefer A to C

$$(A \succ B) \wedge (B \succ C) \implies (A \succ C)$$

Coke  $\succ$  Sprite

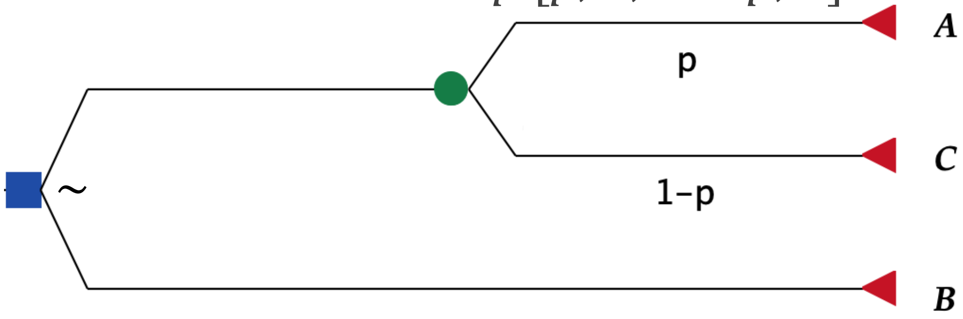
Sprite  $\succ$  Pepsi

Coke  $\succ$  Pepsi

A3: Continuity

- Definition
  - If some lottery B is between A and C in preference, then there is some probability p for which the rational agent will be indifferent between:
    - Betting B for sure; AND
    - The lottery that yields A with probability p and C with probability $1 - p$

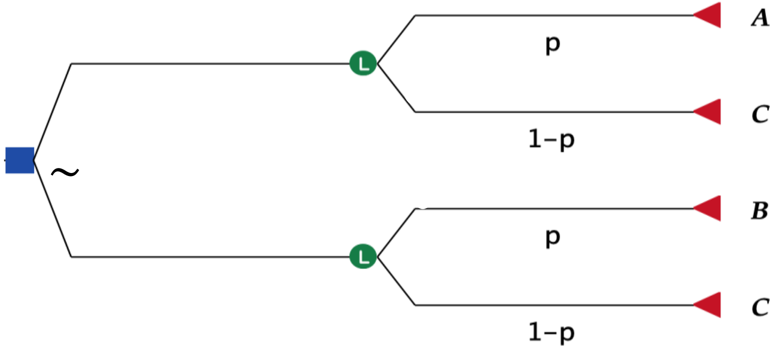
$$A \succ B \succ C \implies \exists p\ [p, A;\ 1 - p, C] \sim B$$



A4: Substitutability

- Definition
  - If an agent is indifferent between two lotteries A and B, then the agent is indifferent between two more complex lotteries that are the same except that B is substituted for A in one of them
  - This holds regardless of probabilities and other outcome(s) in lotteries
  - This also holds if $\succ$ is substituted for $\sim$ in the axiom

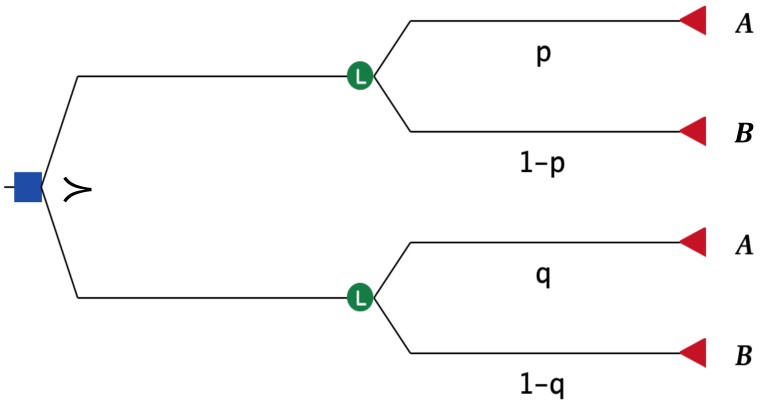
$$A \sim B \implies [p, A;\ 1 - p, C] \sim [p, B;\ 1 - p, C]$$



A5: Monotonicity

- Definition
  - Suppose two lotteries have the same two possible outcomes, A and B.
  - If an agent prefers A to B, then the agent must prefer the lottery that has a higher probability for A (and vice versa)

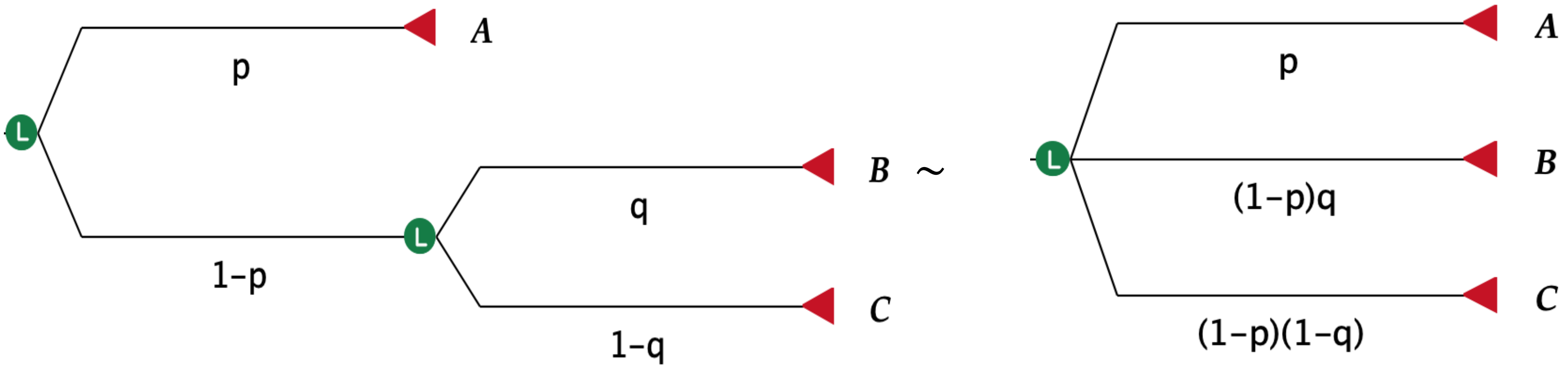
$$A \succ B \implies (p > q \iff [p, A;\ 1 - p, B] \succ [q, A;\ 1 - q, B])$$



A6: Decomposibility

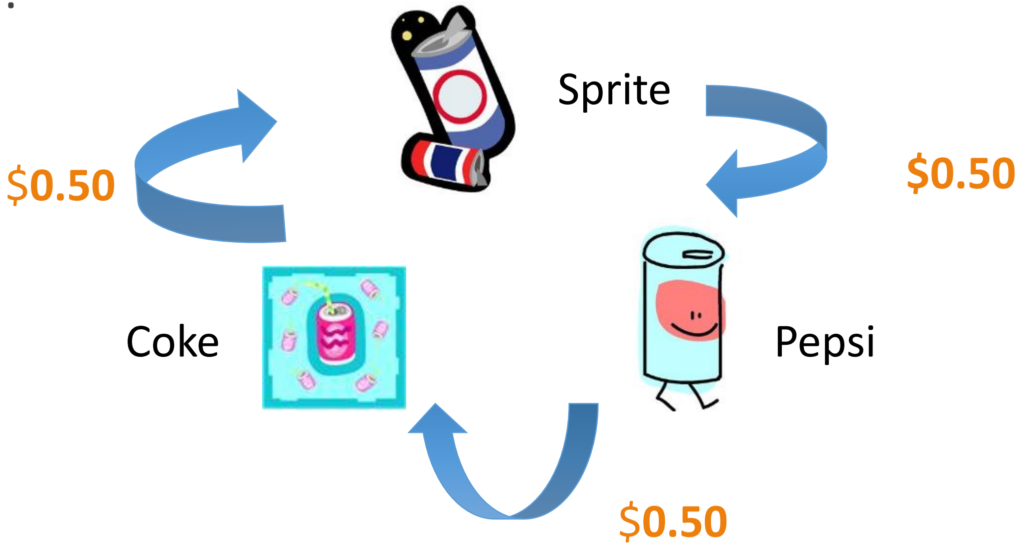
- Definition
  - Compound lotteries can be reduced to simpler ones
  - No fun in gambling rule – two consecutive lotteries can be compressed into a single equivalent lottery

$$[p, A;\ 1 - p, [q, B;\ 1 - q, C]] \sim [p, A;\ (1 - p)q, B;\ (1 - p)(1 - q), C]$$



Exploiting Irrationality

- E.g., Coke≻Sprite≻Pepsi≻Coke
  - How to make money from someone without the above preference structure?



Rational Preferences and Utility

- Note:
  - Axioms of utility are really axioms about preferences
  - Nothing is mentioned about a utility function
  - Need to derive consequences from axioms of utility
- How should a rational agent behave?
  - If an individual can satisfy the conditions of rationality, then, a utility function U can be constructed according to:
    - Von Neumann and Morgenstern (1944) Expected Utility Theorem

Expected Utility Theorem

Von Neumann and Morgenstern (1944)

Existence of Utility Function

- If an agent's preferences obey the axioms of utility; then there exists a function U such that:
  - $U(A) > U(B)$ if and only if A is preferred to B, and
  - $U(A) = U(B)$ if and only if the agent is indifferent between A and B.

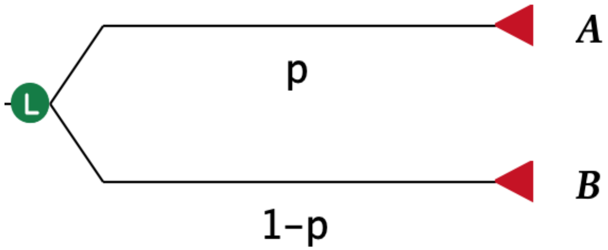
$$U(A) > U(B) \Leftrightarrow A \succ B \qquad \text{and} \qquad U(A) = U(B) \Leftrightarrow A \sim B$$

Expected Utility of A Lottery

- The utility of a lottery is the sum of the probability of each outcome times the utility of that outcome:

$$U([p_i, S_i; \ldots; p_n, S_n]) = \sum_i p_i U(S_i)$$

  - Once the probabilities and utilities of the possible outcome states are specified, the utility of a compound lottery involving those states is completely determined.



$$U(L) = pU(A) + (1 - p)U(B)$$

Computing Expected Utility

- Expected utility
  - Note that outcome of a nondeterministic action is a lottery
  - Expected utility of an action a is the average utility value of the outcomes, weighted by the probability that the outcome occurs
    $EU(a) = \sum_{s'} P(\text{Result}(a) = s')U(s')= \sum_{s'} \sum_{s} P(s)P(s'|s,a)U(s')$
- Non-unique utility functions
  - Agent's behavior doesn't change if U is subjected to an affine transformation:

$$U'(s) = aU(s) + b \text{ with } a > 0$$

Maximum Expected Utility Principle

Basic principle of decision theory

Rational Decision Making

Known utility of each outcome → What is the expected utility of the lottery? → How to choose among the different lotteries?

- An agent can act rationally – consistently with its preferences – only by choosing an action that maximizes expected utility according to:

- Maximum expected utility (MEU) principle
  - A rational agent should choose the action that maximizes its expected utility

$$action = \underset{a}{\operatorname{argmax}}\, EU(a)$$

  - A prescription for intelligence behavior – "do the right thing" – a basis for AI

Exercise

Question:

- Utility of tossing $U(Head) = 5; U(Tail) = 2.$
  - You are asked to choose between 2 coins C_1 and C_2.
  - Probability of getting Head in C_1 is 0.5
  - Probability of getting Head in C_2 is 0.6
  - Which coin should you choose to maximize expected utility?
- Considerations:
  - $EU(choose\ C_1\ ) \ =$
  - $EU(choose\ C_2) \ =$

Rational Decision-Theoretic Agent

- Preference assumption
  - Assumes that agent's preferences satisfy all the axioms or conditions of rationality
- Rationality assumption
  - Assumes that agent always act rationally by choosing the most preferred option
- Ideal agent assumption
  - Agents of the theorem are ideal and hypothetical beings, but can be used as guides for our own problem solving or decision making
- Other assumptions:
  - Existence of a utility function that describes an agent's preference behavior does not necessarily mean that agent is explicitly maximizing that utility function itself
  - An observer can learn about the utility function that represents what the agent is actually trying to achieve through the agent's behavior

MEU as Performance Measures

- Given a set of environments, agents, and relevant perception histories:
  - If an agent acts so as to maximize a utility function that correctly reflects the performance measure, then the agent will achieve the highest possible performance score (averaged over all the possible environments).
- Main ideas:
  - Transition from external performance measure to internal utility function
  - Performance measure:
    - Gives a score for a history—a sequence of states
    - Applied retrospectively after an agent completes a sequence of actions
    - Utility function can be used to guide actions step by step

Computing MEU

- What are the major challenges?
  - Considering many actions a
  - Estimating $P(s)$ over possible states of the world in $P(\text{RESULT}(a) = s')$ requires perception, learning, knowledge representation, and inference
  - Computing $P(\text{RESULT}(a) = s')$ requires a causal model of the world
  - Computing outcome utilities $U(s')$ requires further searching or planning
  - Estimating uncertainty about U
- In summary, decision theory:
  - Provides a basic, general mathematical framework to define the AI problem
  - Cannot solve the AI problem!

Utility Functions

Basic Concepts and Assessment Methods

Encoding Preferences in Utility Functions

- Assess utility or value functions
  - To measure “desirability” of different outcomes and trade-off situations
- Possible scales:
  - Monetary cost
  - Revenue, profit
  - Life expectancy
  - Jobs saved
  - etc.

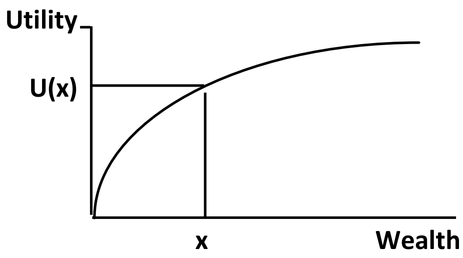
Measuring Utilities

- Provide summary scores that
  - aggregate different aspects of “goodness” measures
  - incorporate attitudes towards risks and “goodness” quantities
- Assessment methods
  - By direct assessment using:
    - probability equivalent
    - certainty equivalent
  - By consensus
  - From published reports
- Multi-attribute utility functions may be used

Utility Functions

- A utility function, U, represents a way to translate dollars or other "desirability" measures, x, into utility units $U(x)$.

Graphical Representation



Mathematical Representation

$U(x) = \log(x)$

$U(x) = 1 - e^{-x/R}$

$U(x) = x^{0.5}$

Tabular Representation

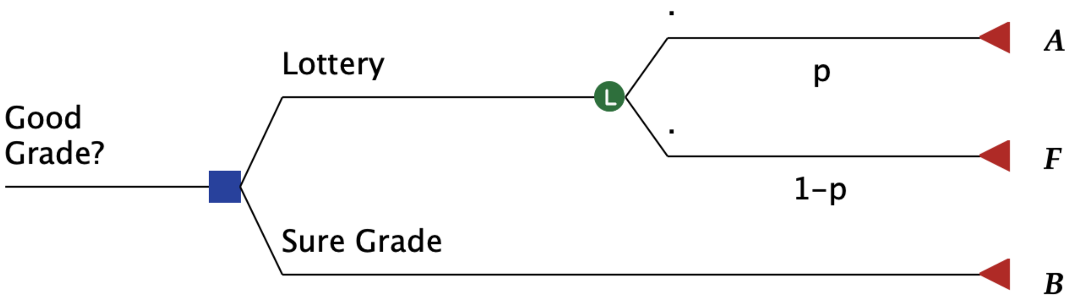
Wealth (x)	Utility Value U(x)
2500	1.50
1500	1.24
1000	0.93
600	0.65
400	0.47
0	0.15

Preference Elicitation: Method 1

- Using probability equivalents
  - Present choices to agents; define the utility function from observed responses
- Method:
  - Fix a scale; fix utilities of 2 outcomes to establish the scale
  - Worst: $u_\perp$; Best: $u_\top$
    - Normalized utilities: $u_\perp = 0; u_\top = 1$
  - Assess utility for prize S by asking the agent to choose between $U(S)$ and the lottery $[p, u_\top; (1-p), u_\perp]$
    - Adjust p until the agent is indifferent between S and lottery

Example: Getting Good Grade

- Demonstration:
  - Grade in CS4246/CS5446: $U(F) = 0; U(A) = 1$; sure grade: B, C, D
  - What value of p would you trade A for B?



  - What about C, D?

Example: Pizza Party

- Demonstration:
  - Alice's preference: $Pasta \succ Pizza \succ Salad$
  - Bob's preference: $Salad \succ Pasta \succ Pizza$
  - Charlie's preference: $Pizza \succ Pasta \succ Salad$
- What is the group's preference?

UNSURE! Compromise and other considerations are needed

- So?
  - Group preference may be non-transitive
  - Studied in social choice theory

Certainty Equivalent

- Certainty Equivalent (CE)
  - Fixed amount of money equivalent to a given situation involving uncertainty
  - Value agent accepts in lieu of the lottery
- Example:
  - You are faced with the following lottery:
    - Win $\$2000$ with probability 0.5
    - Lose $\$20$ with probability 0.5

EMV = $990

  - How much are you willing to sell the lottery for?
  - This amount, say $\$X$, is the least that you would accept for the lottery, then the lottery must be equivalent in your mind to a sure $\$X$

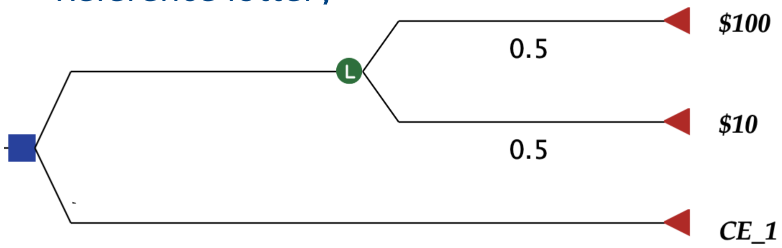
Preference Elicitation: Method 2

- Using certainty equivalents

1. Determine two initial points on curve
2. Arbitrarily assign utility values to the points
3. Create reference lottery to determine CE
4. Find third point on curve by formula:
   - $U(CE_1) = EU(Lottery)$
5. Create additional reference lotteries
6. Proceed as before until enough points are available to plot a curve

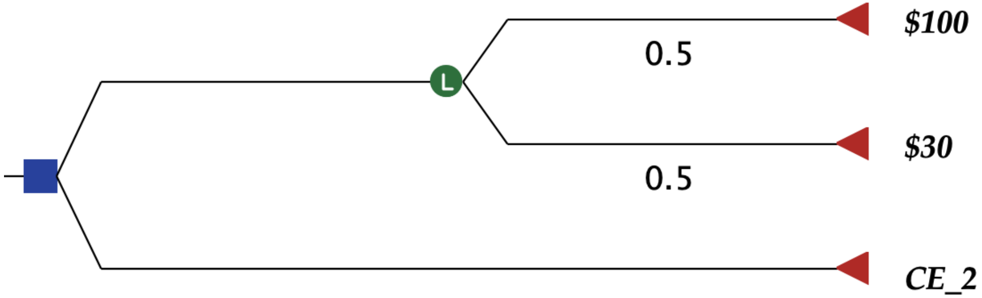
- Example:

1. Assume: worst: \$10, best: \$100
2. Set $U_{\perp}(10) = 0; U_{\top}(100) = 1$
3. Reference lottery



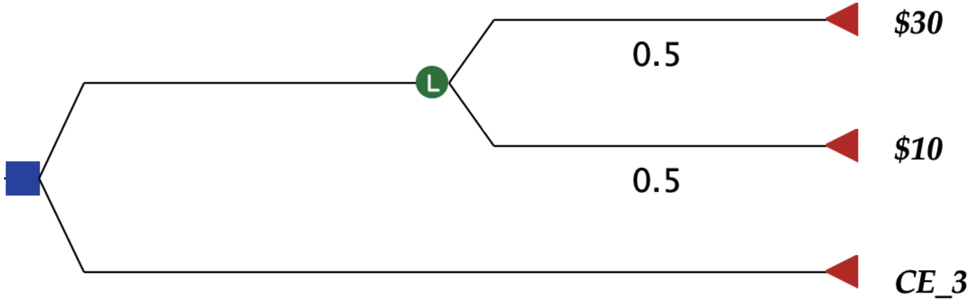
4. Assume: $CE_1 = 30$
5. $U(CE_1) = U(30)$
$= 0.5U(100) + 0.5U(10)$
$= 0.5$

Continue …



CE_2

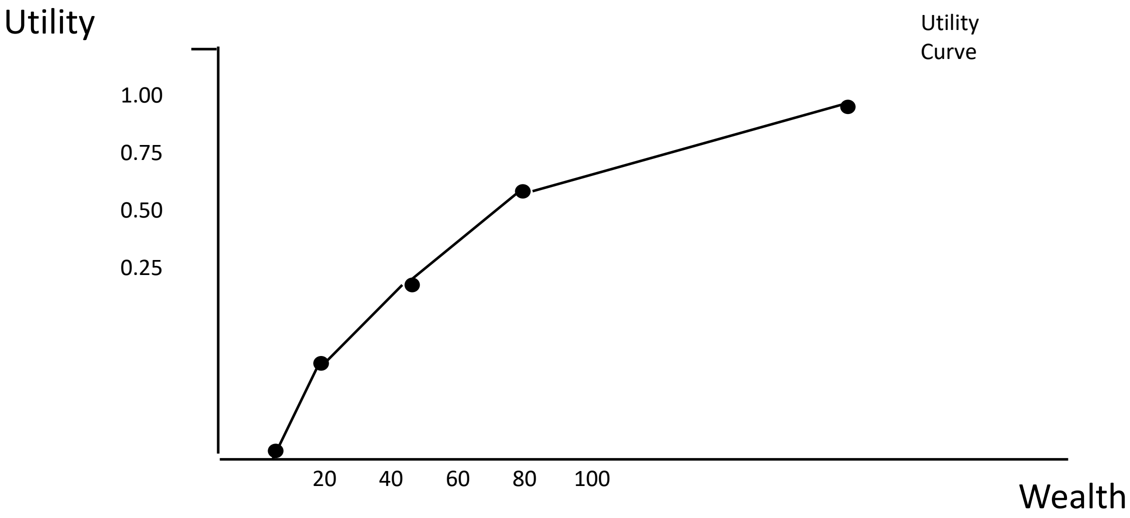
$$U(50) = 0.5U(100) + 0.5U(30) = 0.75$$



CE_3

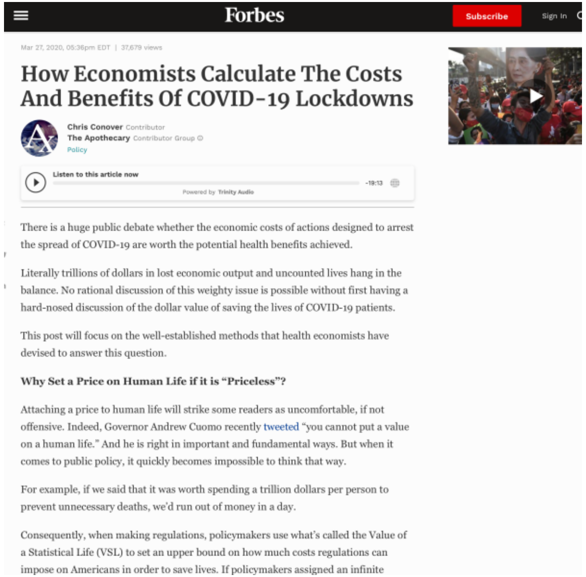
$$U(18) = 0.5U(30) + 0.5U(10) = 0.25$$

Plotting the Utility Function



Example: Value of Statistical Life

- Value of statistical life:
  - How to put a value on human life?
  - Used by agencies of U.S. government, including the Environmental Protection Agency, the Food and Drug Administration, and the Department of Transportation
  - To determine costs and benefits of regulations and interventions
  - How much would it be?
    - Typical value in 2020 (in US) ~ $7.5 million.



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How Economists Calculate The Costs And Benefits Of COVID-19 Lockdowns

Chris Conover Contributor
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There is a huge public debate whether the economic costs of actions designed to arrest the spread of COVID-19 are worth the potential health benefits achieved.

Literally trillions of dollars in lost economic output and uncounted lives hang in the balance. No rational discussion of this weighty issue is possible without first having a hard-nosed discussion of the dollar value of saving the lives of COVID-19 patients.

This post will focus on the well-established methods that health economists have devised to answer this question.

Why Set a Price on Human Life if it is "Priceless"?

Attaching a price to human life will strike some readers as uncomfortable, if not offensive. Indeed, Governor Andrew Cuomo recently tweeted "you cannot put a value on a human life." And he is right in important and fundamental ways. But when it comes to public policy, it quickly becomes impossible to think that way.

For example, if we said that it was worth spending a trillion dollars per person to prevent unnecessary deaths, we'd run out of money in a day.

Consequently, when making regulations, policymakers use what's called the Value of a Statistical Life (VSL) to set an upper bound on how much costs regulations can impose on Americans in order to save lives. If policymakers assigned an infinite

Example: Micromort

value that people place on their own lives

- Micromort:
  - A one in a million chance of death, often used as a unit of risk
  - People appear to be willing to pay about $50 per micromort, e.g., $10,000 for a safer car that halves the risk of death (from 400 to 200 micromort).
  - Only for small risks, most people won't kill themselves for $50 million.
  - See video: https://www.youtube.com/watch?v=G3wolqD-acQ

Example: Quality Adjusted Life Year

- QALY:
  - A scale commonly used in health care literature
  - One year in perfect health is 1 QALY
  - One year bedridden would be less preferred, e.g. 0.5 QALY.
  - Death is 0 QALY
  - See video: https://www.youtube.com/watch?v=3tDXwKVkn68
  - Whose utility values are these?

Utility of Money and Risk Attitudes

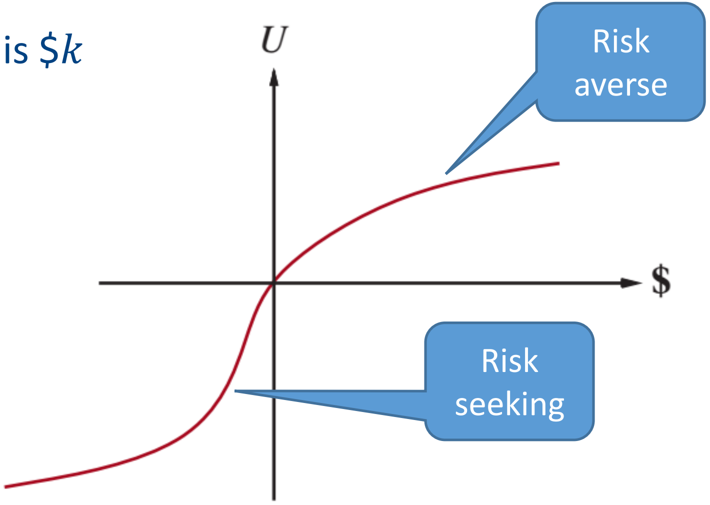
Expected Monetary Value

Are people rational?

- Assumption:
  - Agent prefers more money than less – monotonic preference
- Example:
  - You have won \$1 mil so far
  - Now, flip a coin – Lose all the money (\$0) if Head and gain \$2.5mil if Tail
  - Will you play?
- What is the expected monetary value (EMV) of the lottery?
  - $EMV = [0.5, 0; 0.5, 2.5\ mil] = \$1.25\ mil$
- Is accepting the game the best decision?
  - Yes according to MEU Principle – EMV is higher than \$1 mil won so far!

Utility of Money

- Let S_n denote the state of having $\$n$; current wealth is $\$k$
  - $EU(Accept) = \frac{1}{2}U(S_k) + \frac{1}{2}U(S_{k+2.5mil})$
  - $EU(Decline) = U(S_{k+1mil})$
- To decide:
  - Assign utility values to the outcome states: $S_k, S_{k+2.5mil}, S_{k+1mil}$
  - Will you accept the lottery if: $S_k = 5, S_{k+2.5mil} = 9, S_{k+1mil} = 8$?
  - $EU(Accept) = 0.5(5) + 0.5(9) = 7$
  - $EU(Decline) = 8$
- Note:
  - Utility of money is usually proportional to log of the amount



A typical utility of money function

Source: RN16.2 b

Example: A Choice of Games

- Consider the following two games: EMV

Game 1	Win \$30 with prob 0.5	
	Lose \$1 with prob 0.5	
Game 2	Win \$2000 with prob 0.5	
	Lose \$1900 with prob 0.5	

- Which game would you play? Why?
- What if you are to play a game 10 times?

Limitations of the EMV criterion

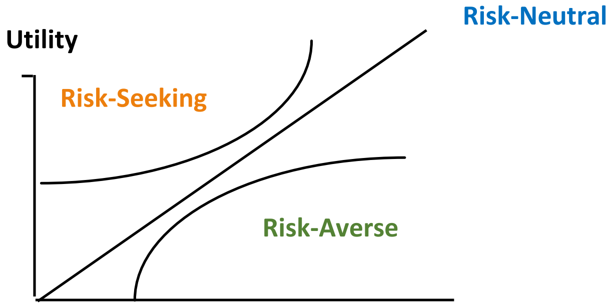
- Using money as decision objective:
  - Intuitive only if objective can be measured in terms of monetary value
  - Considers only the average or expected value; ignores the range of possible values
  - Does not take into account risk attitudes

Risk Attitudes

- A patient is forced to play the following game:
    - Live for another 30 years with probability 0.5
    - Die immediately with probability 0.5
  - Will he choose to live just for 5 years to get out of the game?
- Risk behaviors:
  - If he would trade a lottery for a sure amount that is less than the expected value, he is risk-averse
  - If he would pay an amount more than the expected value to enter the lottery, then he is risk-seeking
  - Otherwise, he is risk-neutral

Risk or Insurance Premium

- Risk attitudes and utility functions

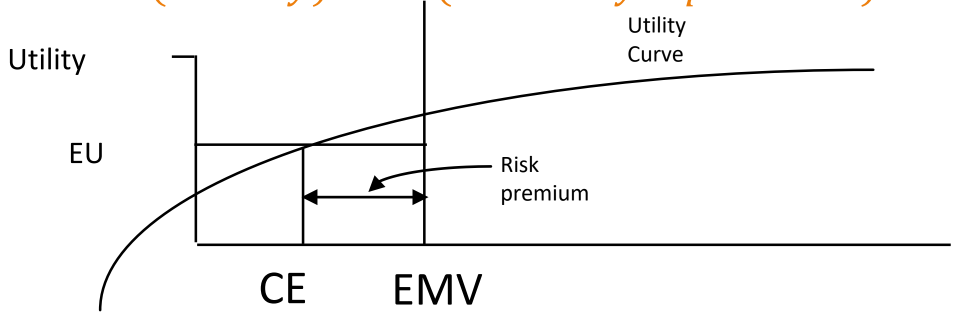


- Risk Premium
  - Risk premium = EMV - Certainty Equivalent
  - Premium paid, in the sense of a lost opportunity, to avoid the risk
- Risk premium and behavior
  - For risk-averse individual, risk premium is positive
  - For risk-seeking individual, risk premium is negative
  - For risk-neutral individual, risk premium is zero

Expected Utility and Certainty Equivalent

- Expected utility of a lottery is equal to the utility of its certainty equivalent:

$$EU(Lottery) \;=\; U(Certainty\ Equivalent)$$



- Note
  - If two alternatives have the same CE, then they must have the same EU; the decision maker would be indifferent to a choice between the two
  - Ranking alternatives by their CEs is the same as ranking them by their expected utilities

Example: Paying to Avoid Risk

- Recall the following gamble:
  - Win $2000 with probability 0.5
  - Lose $20 with probability 0.5
- What is the EMV?
  - EMV = $990
- How much will you be willing to accept to avoid the lottery?
  - Let CE = $300
- What is the risk premium?
  - Risk premium $= \$990 - \$300 = \$690$
- What does this mean?
  - Trading lottery for $300: willing to give up $690 in expected value to avoid risk

Calculating Risk Premium

- Recall:
  - Risk premium = EMV - Certainty Equivalent
  - EU(Lottery) = U(Certainty Equivalent)
- How to calculate risk premium given a lottery and a utility function?

  1) Find EU for the Lottery

  2) Find CE, amount with utility value given by EU

  3) Calculate EMV for the lottery

  4) Risk premium = EMV – CE
- Other methods: Utility functions for non-monetary attributes
  - Utility functions derived for attributes other than money
  - Set up arbitrary scale with extremes (highest and lowest utilities)
  - Use either CE or PE assessment methods

Some Caveats

- Utilities do not add up
  - $U(a+b) \neq U(a) + U(b)$
  - Utility functions are non-linear
  - Calculate net payoffs at end-points of decision tree before transforming to utility values
- Utility differences do not express strengths of preferences
  - Utility provides numerical scale for ordering preferences, not a measure of their strengths
- Utility function
  - A subjective personal statement of an individual's preferences
  - Provides no basis for comparing utilities among individuals
- Certainty equivalent vs expected utility
  - Certainty equivalent is measured in $ (or any basic unit)
  - Expected utility is measured on the utility scale

Other Real-World Challenges

- What about:
  - Multi-attribute utility functions?
  - Unknown preferences?
    - Uncertainties about agent’s own preferences
    - Uncertainties about human agent’s preferences
- Areas of active research and innovation

Homework

- Readings:
  - RN: 15.1 – 15.3.2
  - RN: 15.7 (Unknown Preferences)
- Reviews:
  - Review notes on Probability and Statistics
- References

  - von Neumann, J., O. Morgenstern, and A. Rubinstein, Theory of Games and Economic Behavior (60th Anniversary Commemorative Edition). 1944: Princeton University Press.
  - Clemen, R.T. and T. Reilly, *Making Hard Decisions with DecisionTools*. 2013: Cengage Learning.
  - Abbas, A.E., Foundations of Multiattribute Utility. 2018, Cambridge: Cambridge University Press.

Proof of Expected Utility Theorem

Expected Utility Theorem: Proof

- [Proof Sketch]:
  - An orderable, transitive preference relation satisfies continuity and substitutability if and only if it admits an expected utility representation.
    - Based on Levin, Jonathan. Choice Under Uncertainty. Lecture notes, 2006. Access from: https://web.stanford.edu/~jdlevin/Econ%20202/Uncertainty.pdf

Expected Utility Theorem: Proof

Proof:
We first argue that if U is an expected utility, it satisfies continuity and substitutability.

- Continuity: If $A \succ B \succ C$, then $U(A) > U(B) > U(C)$. Let $p = \frac{U(B)-U(C)}{U(A)-U(C)}$. Then $pU(A) + (1-p)U(c) = U(B)$ showing that this value of p makes $[p, A; 1-p, C] \sim B$.

Continuity:

$A \succ B \succ C \Rightarrow$
$\exists p \; [p, A; 1-p, C] \sim B.$

Expected Utility Theorem: Proof

- Substitutability: If $A \sim B$, then $U(A) = U(B)$. Hence $pU(A) + (1-p)U(C) = pU(B) + (1-p)U(C)$.

Substitutability:

$A \sim B \Rightarrow$
$[p, A; 1-p, C] \sim [p, B; 1-p, C]$.

Now assume that the preference satisfies continuity and substitutability. We will construct an expected utility function U that represents the preference. Let $L_\top$ and $L_\perp$ represent the most and least preferred lotteries. Assume $L_\top \succ L_\perp$ (the result is trivial if $L_\top \sim L_\perp$).

Expected Utility Theorem: Proof

First we show **monotonicity**. If $1 > p > q > 0$, then

$$L_\top \succ [p, L_\top; 1-p, L_\perp] \succ [q, L_\top; 1-q, L_\perp] \succ L_\perp .$$

Aside: argument holds for any L_1, L_2 with $L_1 \succ L_2$.

- First inequality: write $L_\top = [p, L_\top; 1-p, L_\top]$ and the inequality follows from substitutability as $L_\top \succ L_\perp$.
- Second inequality: write LHS as $[(p-q), L_\top; q, L_\top; 1-p, L_\perp]$ and RHS as $[(p-q), L_\perp; q, L_\top; 1-p, L_\perp]$, use substitutability.
- Third inequality: write RHS as $[q, L_\perp; 1-q, L_\perp]$ and use substitutability.

Substitutability : $A \succ B \Rightarrow$
$[p, A; 1-p, C] \succ [p, B; 1-p, C]$.

Monotonicity : $A \succ B \Rightarrow$
$$((p > q) \Leftrightarrow [p, A; 1-p, B] \succ [q, A; 1-q, B]) .$$

Expected Utility Theorem: Proof

For any lottery L, there is a **equivalent lottery** $[p_L, L_\top; (1-p_L), L_\perp]$ with a unique p_L such that $L \sim [p_L, L_\top; (1-p_L), L_\perp]$. Existence follows from continuity while monotonicity implies uniqueness.

Continuity : $A \succ B \succ C \Rightarrow \exists p \; [p, A; 1-p, C] \sim B$.

Monotonicity : $A \succ B \Rightarrow ((p > q) \Leftrightarrow [p, A; 1-p, B] \succ [q, A; 1-q, B])$.

We argue that $U(L) = p_L$ is an expected utility representation of the preference relation.

- First it gives the correct ordering:

$$
\begin{aligned}
& L_1 \succsim L_2 \\
\Leftrightarrow\ & [p_{L_1}, L_\top; (1-p_{L_1}), L_\perp] \succsim [p_{L_2}, L_\top; (1-p_{L_2}), L_\perp] \text{ (equiv lottery)} \\
\Leftrightarrow\ & p_{L_1} \geq p_{L_2} \text{ (monotonicity)} \\
\Leftrightarrow\ & U(L_1) \geq U(L_2) \text{ (defn of utility)}
\end{aligned}
$$

Expected Utility Theorem: Proof

- Also $U(L) = p_L$ is an expected utility, i.e. for any L_1, L_2 and α

$$U([\alpha, L_1; (1-\alpha), L_2]) = \alpha U(L_1) + (1-\alpha)U(L_2).$$

- We know that we can represent L_1 and L_2 by equivalent lotteries

$$L_1 \sim [U(L_1), L_\top; (1 - U(L_1)), L_\perp]$$
$$L_2 \sim [U(L_2), L_\top; (1 - U(L_2)), L_\perp]$$

- Using substitutability ($A \sim B \Rightarrow [p, A; 1-p, C] \sim [p, B; 1-p, C]$)

$$\begin{aligned} &[\alpha, L_1; (1-\alpha), L_2] \\ &\sim [\alpha, [U(L_1), L_\top; (1-U(L_1)), L_\perp]; (1-\alpha), [U(L_2), L_\top; (1-U(L_2)), L_\perp]] \\ &\sim [\alpha U(L_1) + (1-\alpha)U(L_2), L_\top; 1 - \alpha U(L_1) - (1-\alpha)U(L_2), L_\perp] \end{aligned}$$

- Applying definition of utility, we get

$$U([\alpha, L_1; (1-\alpha), L_2]) = \alpha U(L_1) + (1-\alpha)U(L_2).$$

$\square$