NUS National University of Singapore | School of Computing

CS5340
Uncertainty Modeling in AI

Lecture 2:
Bayesian Networks
(Directed Graphical Models)

Assoc. Prof. Lee Gim Hee

AY 2022/23

Semester 1

Course Schedule

Week	Date	Topic	Remarks
1	10 Aug	Introduction to probabilistic reasoning	**Assignment 0:** Python Numpy Tutorial (Ungraded)
2	17 Aug	Bayesian networks (Directed graphical models)	
3	24 Aug	Markov random Fields (Undirected graphical models)	
4	31 Aug	Variable elimination and belief propagation	**Assignment 1:** Belief propagation and maximal probability (15%)
5	07 Sep	Factor graph and the junction tree algorithm	
6	14 Sep	Parameter learning with complete data	**Assignment 1:** Due **Assignment 2:** Junction tree and parameter learning (15%)
-	21 Sep	Recess week	**No lecture**
7	28 Sep	Mixture models and the EM algorithm	**Assignment 2:** Due
8	05 Oct	Hidden Markov Models (HMM)	**Assignment 3:** Hidden Markov model (15%)
9	12 Oct	Monte Carlo inference (Sampling)	
*	15 Oct	Variational inference	Makeup Lecture (Venue TBD) Time: 9.30am – 12.30pm (Saturday)
10	19 Oct	Variational Auto-Encoder and Mixture Density Networks	**Assignment 3:** Due **Assignment 4:** MCMC Sampling (15%)
11	26 Oct	No Lecture	I will be traveling
12	02 Nov	Graph-cut and alpha expansion	**Assignment 4:** Due
13	09 Nov	-	

Acknowledgements

- A lot of slides and content of this lecture are adopted from:

1. “An introduction to probabilistic graphical models”, Michael I. Jordan, 2002 http://people.eecs.berkeley.edu/~jordan/prelims/chapter2.pdf (Section 2.1)
2. "Pattern recognition and machine learning", Christopher Bishop (Chapter 8, Section 8.1 and 8.2).
3. "Machine learning - a probabilistic approach", Kevin Murphy (Chapter 10)
4. "Probabilistic graphical models", Koller and Friedman (Chapter 3)

Learning Outcomes

- Students should be able to:

    1. Explain the concepts of conditional independence.
    2. Use the Bayesian network to represent conditional independence in joint distributions.
    3. Describe d-separation using the three canonical 3-node graph.
    4. Deduce all conditional independence in a Bayesian network using the Bayes ball algorithm.
    5. Explain the concepts of Markov Blanket.

Why the Need for Graphical Models?

It difficult to work with joint probabilities $p(x_1, \dots, x_N \,|\theta)$ with fully correlated random variables.

Why?

Why the Need for Graphical Models?

- Let's illustrate this with *N* discrete random variables $X_1, \dots, X_N$, where $x_i \in \{1, \dots, K\}$.

- We need $O(K^N)$ parameters θ to represent the joint distribution $p(x_1, \dots, x_N | \theta)$.

- Inference becomes intractable when N is large, and a huge amount of data is needed to learn all parameters.

Why the Need for Graphical Models?

Easy solution?

- Fully independent assumption on all random variables reduces the number of parameters to $O(NK)$.

$$p(x_1, \ldots, x_N \mid \theta) = \prod_{i=1}^{N} p(x_i \mid \theta_i)$$

- Inference becomes tractable products of $p(x_i \mid \theta_i)$, and smaller amount of data is needed to learn all parameters.

Why the Need for Graphical Models?

- Fully independent assumption:

$$p(x_1, \dots, x_N \,|\theta) = \prod_{i=1}^{N} p(x_i|\theta_i)$$

- Is it always correct to assume that all random variables are fully independent?

Examples:

Probability of the next letter in a word:

" probabili ▯ "

Is ▯ independent of the letters that are already known in the word?

Why the Need for Graphical Models?

- Fully independent assumption :

$$p(x_1, \ldots, x_N \,|\theta) = \prod_{i=1}^{N} p(x_i|\theta_i)$$

- Is it always correct to assume that all random variables are fully independent?

Examples:

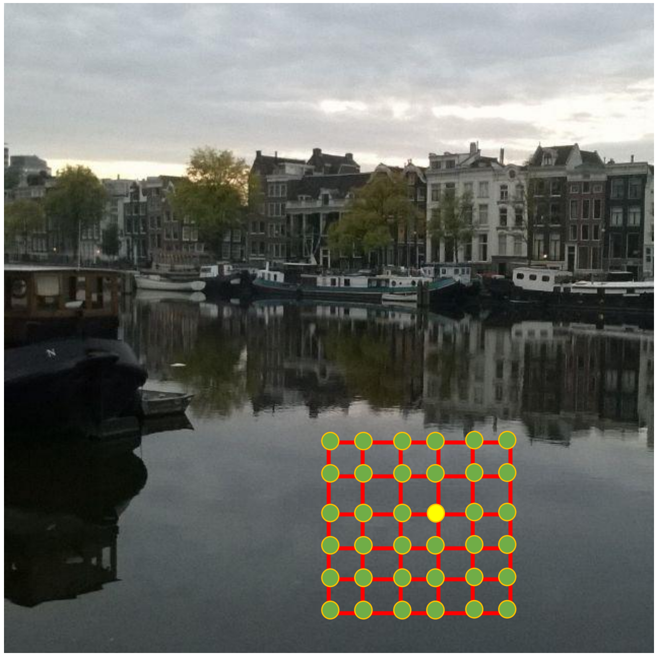
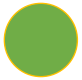
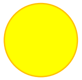


Photo Source:
G.H. Lee "Amsterdam"


: pixel is labeled as "water"


: is this pixel more likely to be "water" or "sky"?

Why the Need for Graphical Models?

Random variables are often NOT fully independent, how can we:

- Compactly represent the joint distribution $p(x_1 , \ldots , x_N|\theta)$ of multiple correlated variables?
- Use the joint distribution to *infer* one set of variables given another in a reasonable amount of computation time?
- *Learn* the parameters of the joint distribution with a reasonable amount of data?

Use Graphical Models!!!

Conditional Independence

- We have seen that:
  - Fully independent assumption is insufficient to model real-world random variables which are unlikely to be fully independent.
  - Fully correlated joint distributions can become intractable.
- A good compromise is by assuming an intermediate degree of dependency among the random variables.
- This is conditional independence.

Conditional Independence

- More formally, two random variables X_A and X_C are conditionally independent given X_B if:

$$p(x_A, x_C | x_B) = p(x_A | x_B) p(x_C | x_B)$$

- Or alternatively:

$$p(x_A | x_B, x_C) = p(x_A | x_B), \qquad \forall\, X_B : p(x_B) > 0$$

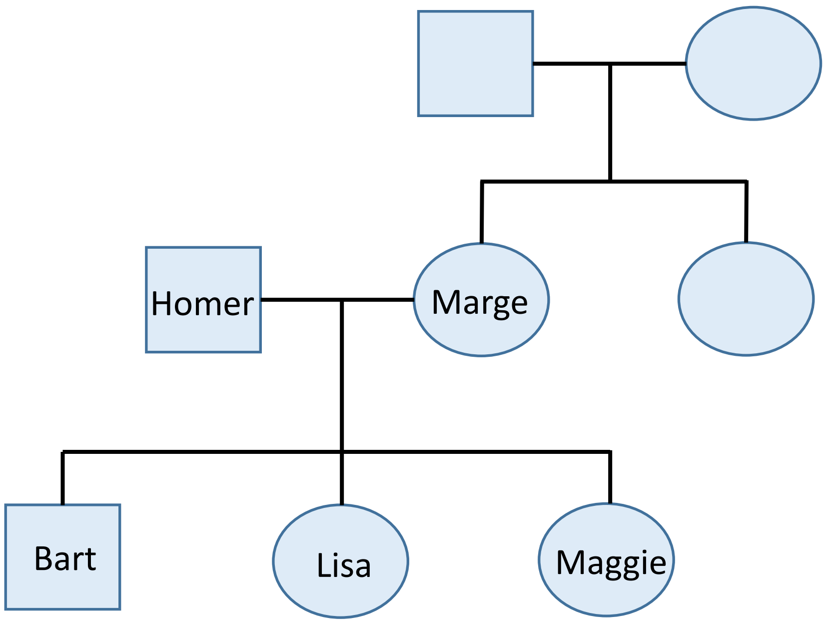
- That is, learning the values of X_C does not change prediction of X_A once we know the value of X_B.

- Written as $X_A \perp X_C \mid X_B$.

Conditional Independence

Example: Family Trees (Pedigree)

A node represents an individual's genotype.



Conditional Independence:

$$X_{Bart} \perp (X_{nonDesc} \setminus X_{Parent}) \mid X_{Parent}$$

Non-descendants

Any random variable is locally dependent on only its parent nodes, also known as the Markov Assumption.

Bayesian Networks: Definitions

- We use a directed acyclic graph (DAG) to represent conditional independence.
- A DAG is a pair $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set nodes and $\mathcal{E}$ a set of oriented edges.

Example of a Directed Graphical Model (DGM), i.e. Bayesian Network:

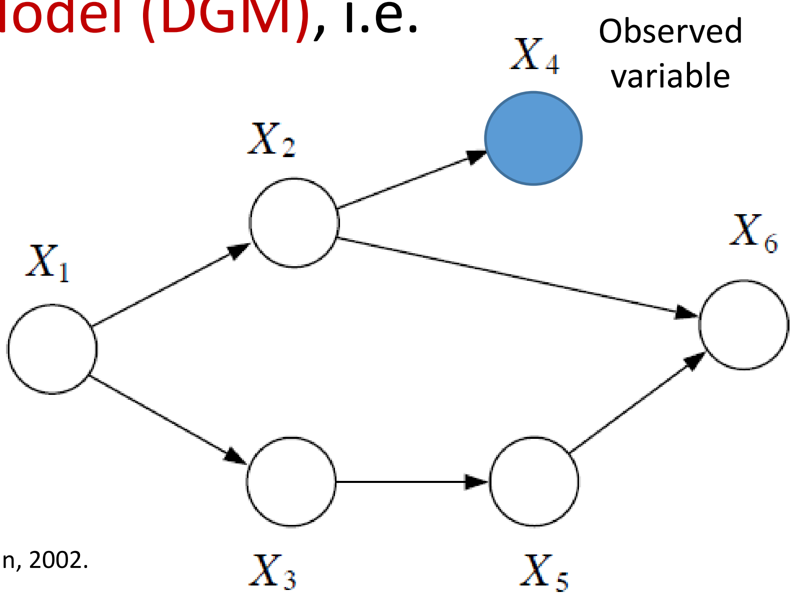


Image modified from: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Bayesian Networks: Definitions

- $\mathcal{G}$ does not contain any cycles.
- Shaded node refers to observed variable.
- Unshaded node refers to latent/hidden (i.e. unobserved) variables.

Example of a Directed Graphical Model (DGM), i.e. Bayesian Network:

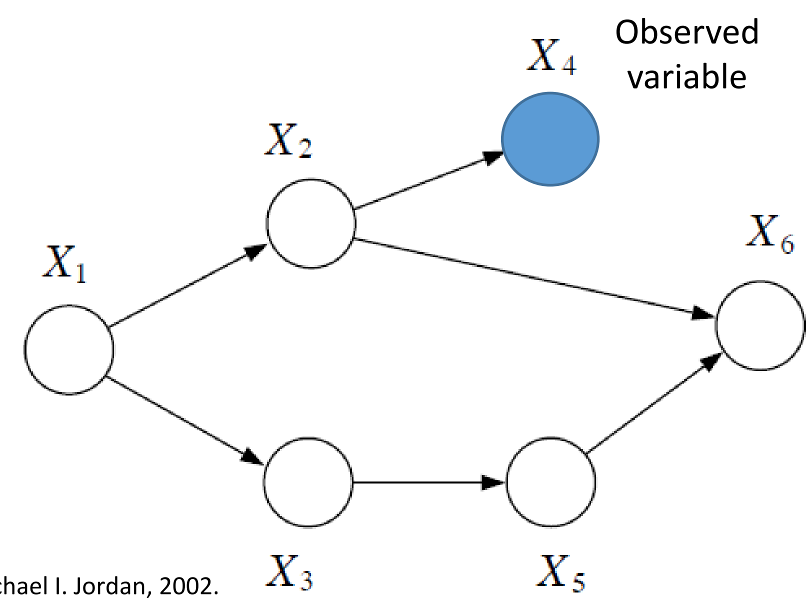
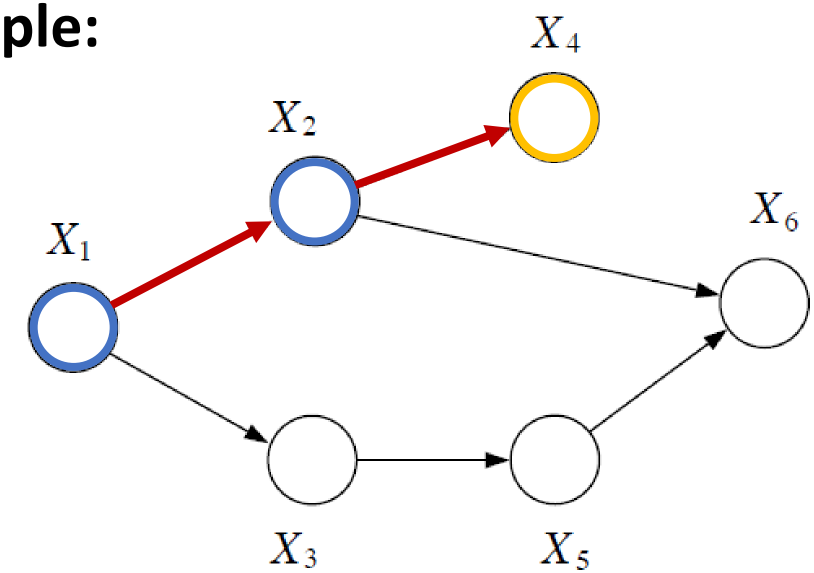


Image modified from: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Bayesian Networks: Definitions

- Ancestors are the parents, grand-parents, etc of a node.
- The ancestors of t is the set of node s that connect to t via a trail: $\mathrm{anc}(t) \triangleq \{s: s \rightsquigarrow t\}$.

Example:



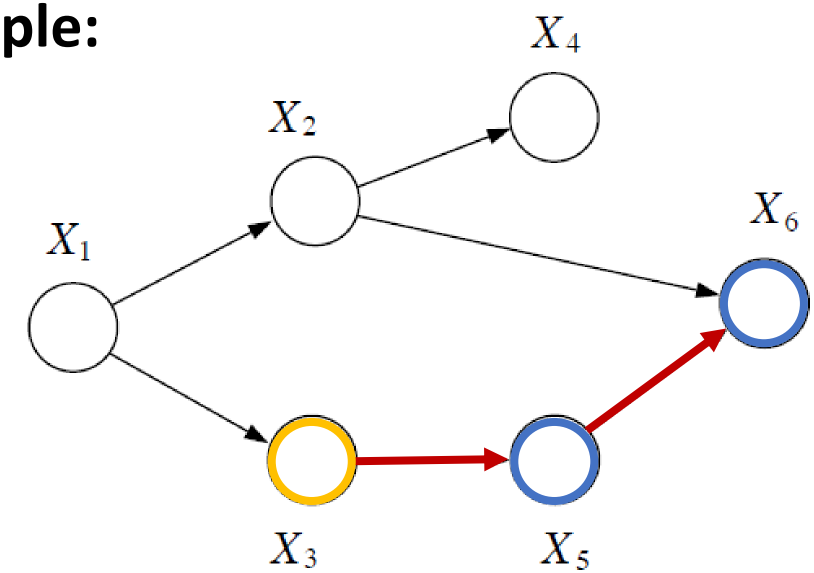
$$\mathrm{anc}(X_4) = \{X_1, X_2\}$$

Image modified from: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Bayesian Networks: Definitions

- Descendants are the children, grand-children, etc of a node.
- The descendants of s is the set of nodes that can be reached via trials from s: $\text{desc}(s) \triangleq \{t: s \rightsquigarrow t\}$.

Example:



$$\text{desc}(X_3) = \{X_5, X_6\}$$

Image modified from: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Bayesian Networks: Definitions

- There is an associated random variable X_i for each $i \in \mathcal{V}$.
- Each node $i \in \mathcal{V}$ has a set of parent nodes π_i, which can be the empty set.
- Let X_{π_i} represent all the random variables that are parents to the random variable X_i.

Example:

$$X_{\pi_1} = \emptyset, \quad X_{\pi_2} = X_1$$
$$X_{\pi_3} = X_1, \quad X_{\pi_4} = X_2$$
$$X_{\pi_5} = X_3, \quad X_{\pi_6} = \{X_2, X_5\}$$

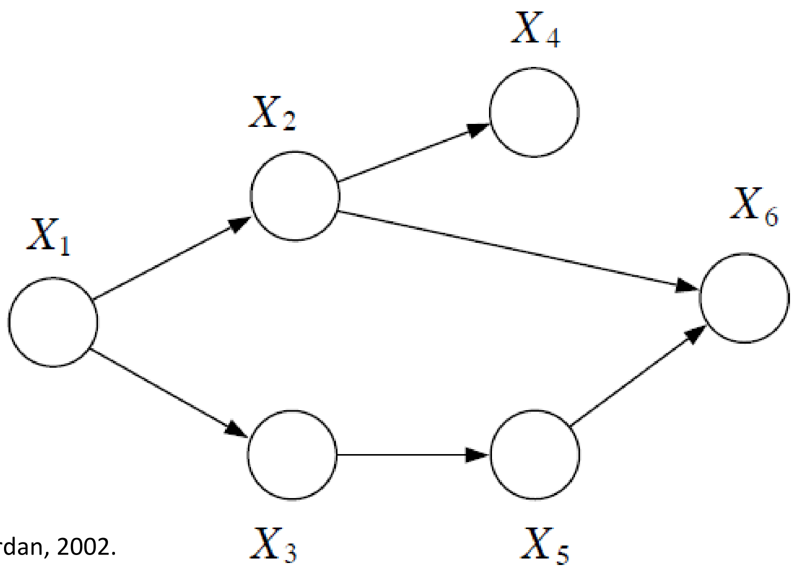


Image source: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Markov Assumption

- Markov assumption: Each random variable X_i is independent of its non-descendants $X_{\text{nonDesc}(X_i)}$ given its parents X_{π_i}.
- The following set of basic conditional independence statements can be associated to the DGM:

$$\{X_i \perp (X_{\text{nonDesc}(X_i)} \setminus X_{\pi_i}) \mid X_{\pi_i}\}$$

Example:

Non-descendants of X_2 are outlined blue.

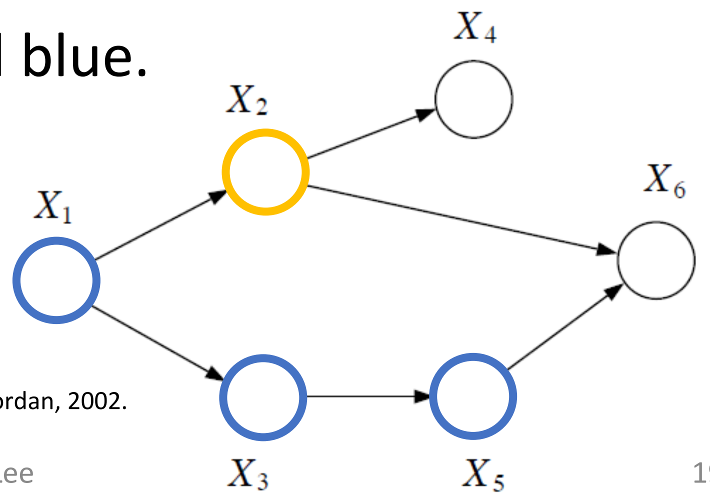


Image modified from: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Markov Assumption

Example:

We have the following set of basic conditional independence from the given Bayesian network.

$X_1 \perp \emptyset \mid \emptyset,$

$X_2 \perp \{X_3, X_5,\} \mid X_1,$

$X_3 \perp \{X_2, X_4\} \mid X_1,$

$X_4 \perp \{X_1, X_3, X_5, X_6\} \mid X_2,$

$X_5 \perp \{X_1, X_2, X_4\} \mid X_3,$

$X_6 \perp \{X_1, X_3, X_4\} \mid \{X_2, X_5\}$

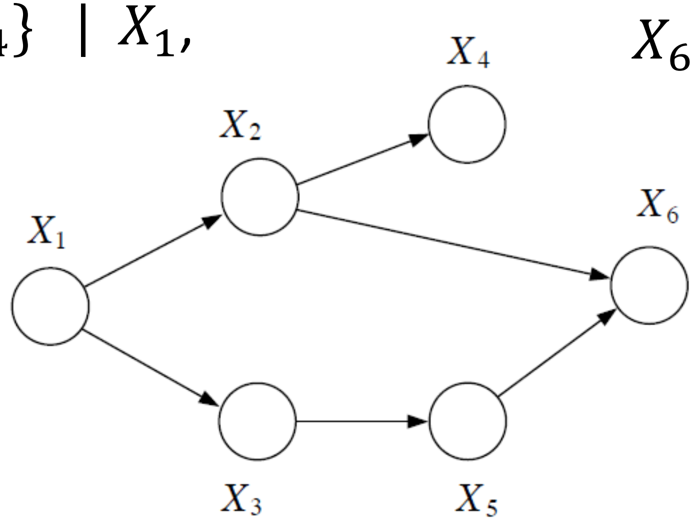


Image source: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Markov Assumption

- The basic conditional independence statements in the DGM give rise to a set of conditional probabilities:

$$\{X_i \perp (X_{\text{nonDesc}(x_i)} \setminus X_{\pi_i}) \mid X_{\pi_i}\}$$

$$\Downarrow$$

$$p(x_i | x_{\pi_i}), \qquad i = 1, \dots, N$$

- $p(x_i | x_{\pi_i})$ is defined locally according to the parent-child relationship specified by the DGM.

Bayesian Networks: Joint Probability

- Locality of the parent-child relationship is used to construct economical representations of the joint distribution.

- The parent-child represents conditional independence:

$$p(x_i|x_{\pi_i})$$

- Joint probability can be read off the graph as the product of all local conditional independence:

$$p(x_1, \dots, x_N) = \prod_{i=1}^{N} p(x_i|x_{\pi_i})$$

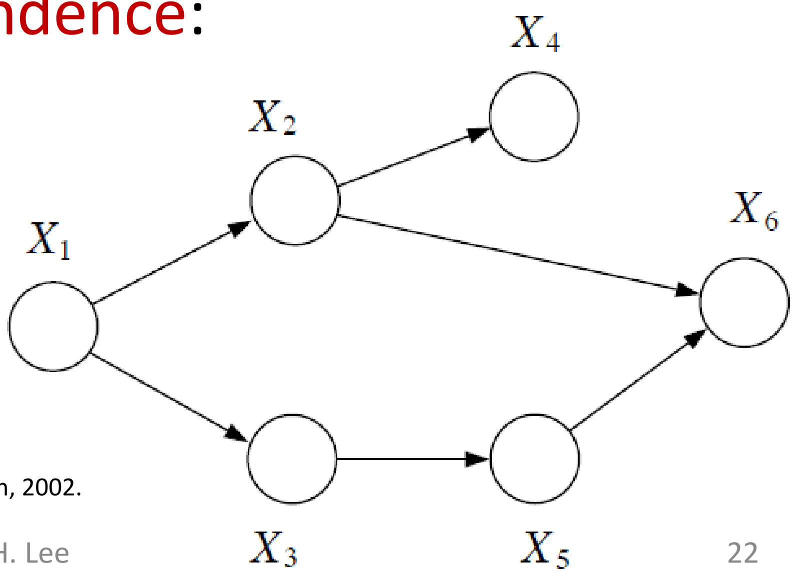


Image source: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Bayesian Networks: Joint Probability

Proof Sketch:

$$p(x_1, \dots, x_N) = p(x_1) \prod_{i=2}^{N} p(x_i | x_{x_1, \dots, x_{i-1}})$$ (chain rule)

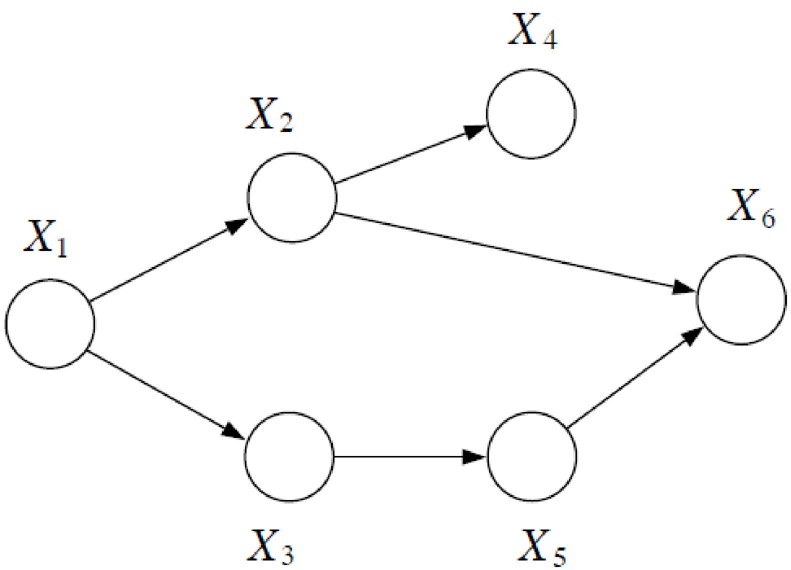
$$\Big\downarrow \quad \{X_i \perp (X_{\mathrm{nonDesc}(x_i)} \setminus X_{\pi_i}) \mid X_{\pi_i}\}$$

$$p(x_1, \dots, x_N) = \prod_{i=1}^{N} p(x_i | x_{\pi_i})$$

(Assuming topological ordering of DGM)

Bayesian Networks: Joint Probability

Example:



$$\boxed{p(x_1, \dots, x_N) = \prod_{i=1}^{N} p\left(x_i \middle| x_{\pi_i}\right)}$$

$$p(x_1, \dots, x_6) = p(\mathrm{x}_1)\mathrm{p}(\mathrm{x}_2|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_3|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_4|\mathrm{x}_2)\mathrm{p}(\mathrm{x}_5|\mathrm{x}_3)\mathrm{p}(\mathrm{x}_6|\mathrm{x}_2, \mathrm{x}_5)$$

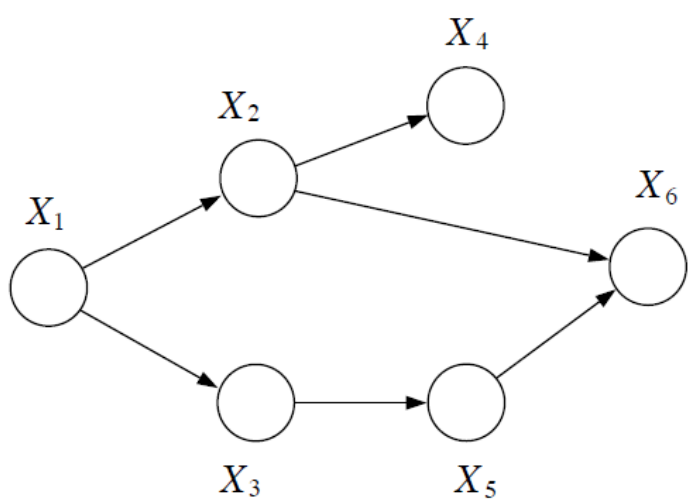
Image source: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Bayesian Networks: Joint Probability

Example:

Let's verify that the basic sets of conditional independence are indeed represented in the joint probability:

$$p(x_1, \dots, x_6) = p(x_1)\mathrm{p}(x_2|x_1)\mathrm{p}(x_3|x_1)\mathrm{p}(x_4|x_2)\mathrm{p}(x_5|x_3)\mathrm{p}(x_6|x_2, x_5)$$



$X_1 \perp \emptyset \mid \emptyset,$

$X_2 \perp \{X_3, X_5,\} \mid X_1,$

$X_3 \perp \{X_2, X_4\} \mid X_1,$

$X_4 \perp \{X_1, X_3, X_5, X_6\} \mid X_2,$

$X_5 \perp \{X_1, X_2, X_4\} \mid X_3,$

$X_6 \perp \{X_1, X_3, X_4\} \mid \{X_2, X_5\}$

Bayesian Networks: Joint Probability

Example:

Let's verify that X_1 and X_3 are independent of X_4 given X_2, i.e. we want to prove the following:

$$p(x_4 \,|\, x_1, x_2, x_3) = p(x_4 \,|\, x_2)$$

First, we compute the marginal probability of $\{X_1, X_2, X_3, X_4\}$:

$$\begin{aligned}
p(x_1, x_2, x_3, x_4) &= \sum_{x_5} \sum_{x_6} p(x_1, x_2, x_3, x_4, x_5, x_6) \\
&= \sum_{x_5} \sum_{x_6} p(x_1) p(x_2 \,|\, x_1) p(x_3 \,|\, x_1) p(x_4 \,|\, x_2) p(x_5 \,|\, x_3) p(x_6 \,|\, x_2, x_5) \\
&= p(x_1) p(x_2 \,|\, x_1) p(x_3 \,|\, x_1) p(x_4 \,|\, x_2) \sum_{x_5} p(x_5 \,|\, x_3) \sum_{x_6} p(x_6 \,|\, x_2, x_5) \\
&= p(x_1) p(x_2 \,|\, x_1) p(x_3 \,|\, x_1) p(x_4 \,|\, x_2),
\end{aligned}$$

Bayesian Networks: Joint Probability

Example:

Let's verify that X_1 and X_3 are independent of X_4 given X_2.

Next, we compute the marginal probability of $\{X_1, X_2, X_3\}$:

$$\begin{aligned} p(x_1, x_2, x_3) &= \sum_{x_4} p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1)p(x_4 \mid x_2) \\ &= p(x_1)p(x_2 \mid x_1)p(x_3 \mid x_1). \end{aligned}$$

Dividing the two marginal yields the desired conditional:

$$p(x_4 \mid x_1, x_2, x_3) = p(x_4 \mid x_2),$$

Which demonstrates the conditional independence relationship $X_4 \perp \{X_1, X_3\} \mid X_2$.

Bayesian Networks: Parameter Reduction

- Let m_i denote the number of parents of node X_i, and each node takes on K values.

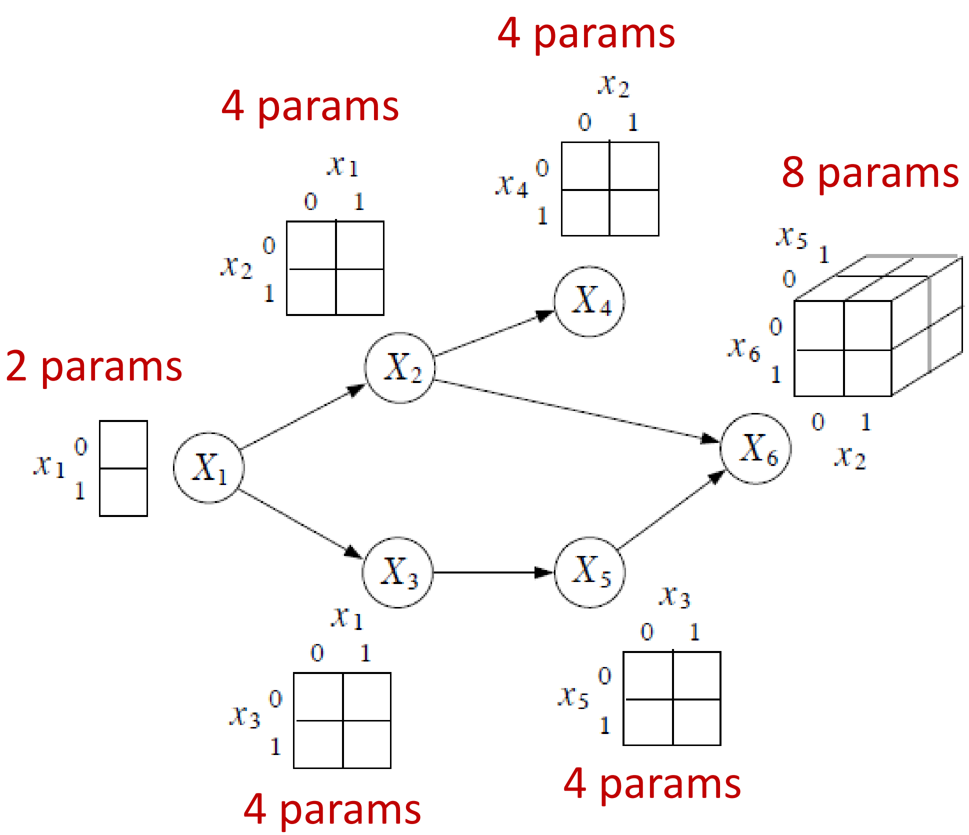
- The conditional probability associated with X_i can be represented with a table of size K^{m_i+1}.

- Results in huge reduction of parameters needed to represent the joint probability, i.e. from $O(K^N)$ to $O(K^{m+1}), m \ll N$.

Bayesian Networks: Parameter Reduction

Example:

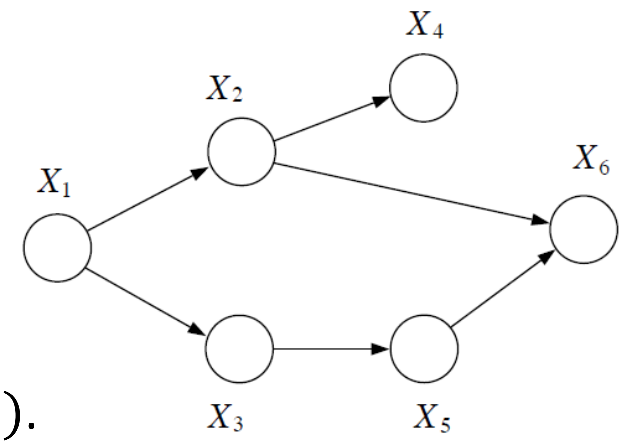
Binary state random variable $x_i \in \{0,1\}$.



- Total parameters = 26.
- Total parameters needed for fully dependent joint probability = $2^6 = 64$.
- More significant difference with higher number of nodes.

Additional Conditional Independence?

It turns out $X_1 \perp X_6 \mid \{X_2, X_3\}$ is also a conditional independence, but not directly observed from the parent-child relation.



Proof:

We want to show that: $p(x_1|x_2, x_3, x_6) = p(x_1|x_2, x_3)$.

$$p(x_1, x_2, x_3, x_6) = \sum_{x_4} \sum_{x_5} p(\mathrm{x}_1)\mathrm{p}(\mathrm{x}_2|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_3|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_4|\mathrm{x}_2)\mathrm{p}(\mathrm{x}_5|\mathrm{x}_3)\mathrm{p}(\mathrm{x}_6|\mathrm{x}_2, \mathrm{x}_5)$$

$$= p(\mathrm{x}_1)\mathrm{p}(\mathrm{x}_2|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_3|\mathrm{x}_1) \overbrace{\sum_{x_4} \mathrm{p}(\mathrm{x}_4|\mathrm{x}_2)}^{1} \sum_{x_5} \mathrm{p}(\mathrm{x}_5|\mathrm{x}_3)\mathrm{p}(\mathrm{x}_6|\mathrm{x}_2, \mathrm{x}_5)$$

$$= p(\mathrm{x}_1)\mathrm{p}(\mathrm{x}_2|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_3|\mathrm{x}_1) \sum_{x_5} \mathrm{p}(\mathrm{x}_5|\mathrm{x}_3)\mathrm{p}(\mathrm{x}_6|\mathrm{x}_2, \mathrm{x}_5)$$

$$p(x_2, x_3, x_6) = \sum_{x_1} \sum_{x_4} \sum_{x_5} p(\mathrm{x}_1)\mathrm{p}(\mathrm{x}_2|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_3|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_4|\mathrm{x}_2)\mathrm{p}(\mathrm{x}_5|\mathrm{x}_3)\mathrm{p}(\mathrm{x}_6|\mathrm{x}_2, \mathrm{x}_5)$$

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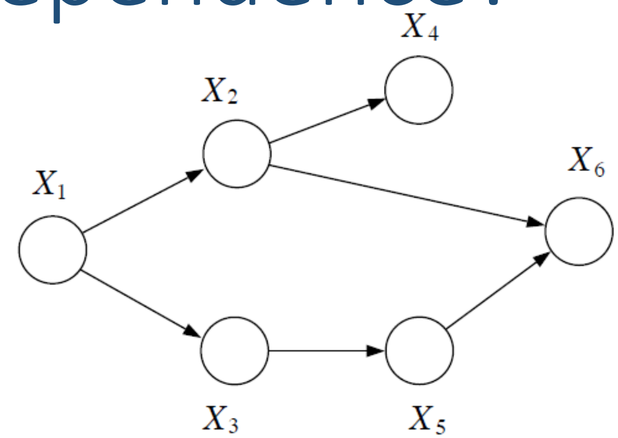


Proof:

$$p(x_1, x_2, x_3, x_6) = p(\mathrm{x}_1)\mathrm{p}(\mathrm{x}_2|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_3|\mathrm{x}_1) \sum_{x_5} \mathrm{p}(\mathrm{x}_5|\mathrm{x}_3)\mathrm{p}(\mathrm{x}_6|\mathrm{x}_2, \mathrm{x}_5)$$

$$p(x_2, x_3, x_6) = \sum_{x_1} p(\mathrm{x}_1)\mathrm{p}(\mathrm{x}_2|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_3|\mathrm{x}_1) \sum_{x_5} \mathrm{p}(\mathrm{x}_5|\mathrm{x}_3)\mathrm{p}(\mathrm{x}_6|\mathrm{x}_2, \mathrm{x}_5)$$

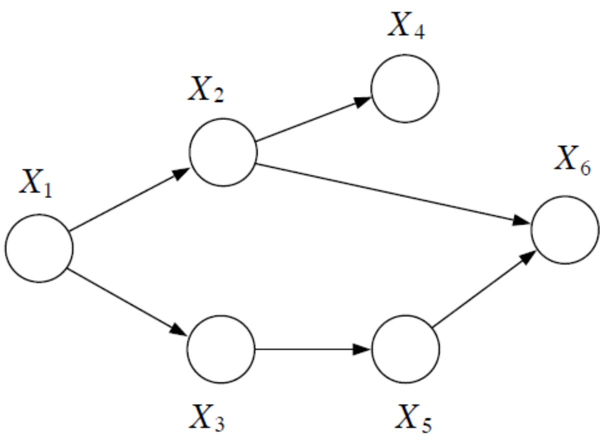
$$p(x_1|x_2, x_3, x_6) = \frac{p(x_1, x_2, x_3, x_6)}{p(x_2, x_3, x_6)}$$

$$= \frac{p(\mathrm{x}_1)\mathrm{p}(\mathrm{x}_2|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_3|\mathrm{x}_1) \cancel{\sum_{x_5} \mathrm{p}(\mathrm{x}_5|\mathrm{x}_3)\mathrm{p}(\mathrm{x}_6|\mathrm{x}_2, \mathrm{x}_5)}}{\sum_{x_1} p(\mathrm{x}_1)\mathrm{p}(\mathrm{x}_2|\mathrm{x}_1)\mathrm{p}(\mathrm{x}_3|\mathrm{x}_1) \cancel{\sum_{x_5} \mathrm{p}(\mathrm{x}_5|\mathrm{x}_3)\mathrm{p}(\mathrm{x}_6|\mathrm{x}_2, \mathrm{x}_5)}}$$

$$= \frac{p(\mathrm{x}_1, x_2, x_3)}{\sum_{x_1} p(\mathrm{x}_1, x_2, x_3)} = \frac{p(\mathrm{x}_1, x_2, x_3)}{p(x_2, x_3)} = p(x_1|x_2, x_3)$$

Additional Conditional Independence?

It turns out $X_1 \perp X_6 \mid \{X_2, X_3\}$ is also a conditional independence, but not directly observed from the parent-child relation.



Proof:

$$p(x_1|x_2, x_3, x_6) = p(x_1|x_2, x_3)$$

Question: Can we write all other conditional independencies by just inspecting the DGM without going through the complicated mathematics?

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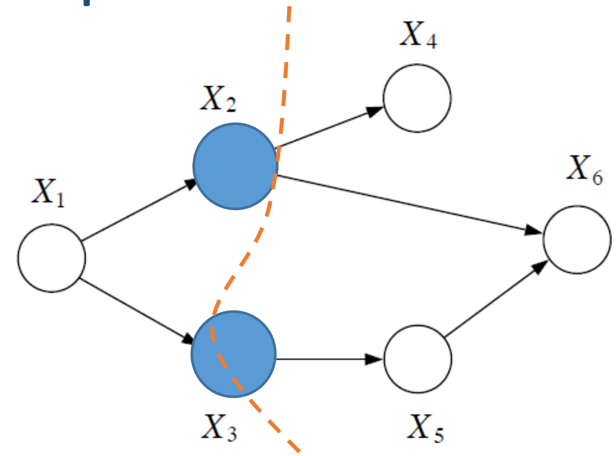


Proof:

$$p(x_1|x_2, x_3, x_6) = p(x_1|x_2, x_3)$$

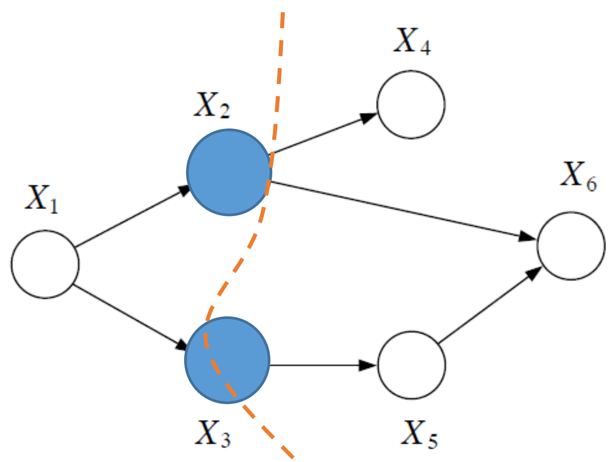
The nodes $\{X_2, X_3\}$ "block" X_1 from X_6.

Question: Can we write all other conditional independencies by just inspecting the DGM without going through the complicated mathematics?

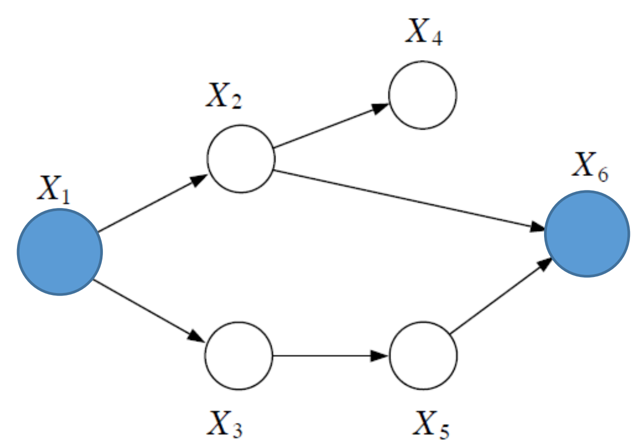
Answer: Yes, observe that the nodes $\{X_2, X_3\}$ "block" all paths from X_1 to X_6. This suggests the notion of graph separation for inferring conditional independence.

Additional Conditional Independence?

- We must be careful in making the notion of "blocking".
- For example, X_2 is NOT independent of X_3 given X_1 and X_6 as would be suggested by a naïve interpretation of "blocking".
- Precise definition of "blocking" must be done through the "three canonical 3-node graphs", and "d-separation".



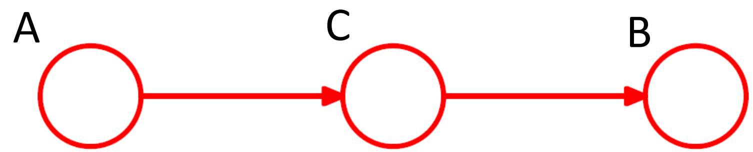
The nodes $\{X_2, X_3\}$ **"block"** X_1 from X_6.



X_2 is **NOT independent** of X_3 given $\{X_1, X_6\}$.

Three Canonical 3-Node Graphs

1.



Joint distribution corresponding to this graph:

$$p(a, b, c) = p(a)p(c|a)p(b|c)$$

If none of the variables are observed, we can see that *A* and *B* are NOT independent by marginalizing over *C*:

$$p(a, b) = p(a) \boxed{\sum_c p(c|a)p(b|c)} = p(a)p(b|a)$$

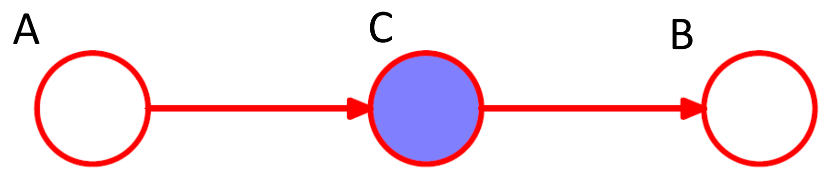
$$\sum_c p(c|a)p(b|c) = \sum_c \frac{p(a)p(c|a)p(b|c)}{p(a)} = \sum_c \frac{p(a,b,c)}{p(a)} = \frac{p(a,b)}{p(a)} = p(b|a)$$

which in general does not factorize into $p(a)p(b)$, and so

$$A \not\perp B \mid \emptyset$$

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs



If we condition on node *C*, using Bayes' theorem together with the joint distribution, we get:

$$\begin{aligned} p(a,b|c) &= \frac{p(a,b,c)}{p(c)} \\ &= \frac{\boxed{p(a)p(c|a)}p(b|c)}{p(c)} \\ &= p(a|c)p(b|c) \end{aligned}$$

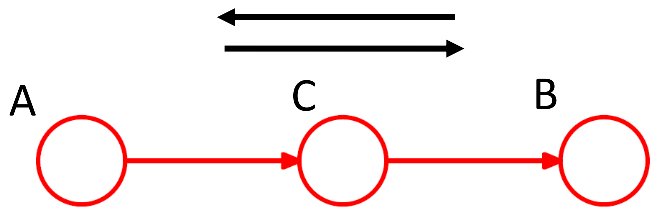
(Bayes rule)
$p(a)p(c \mid a) = p(a \mid c)p(c)$

which shows the conditional independence property:

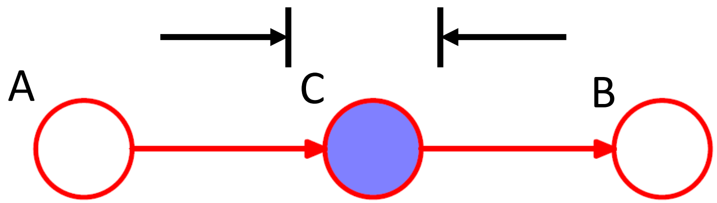
$$A \perp B \mid C$$

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs



$$A \not\perp B \mid \emptyset$$

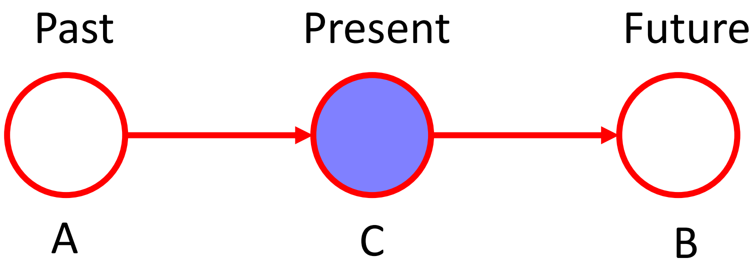


$$A \perp B \mid C$$

- The node *C* is said to be *head-to-tail* with respect to the path from node *A* to node *B*.
- Such a path connects nodes *A* and *B* and renders them dependent.
- The observation of *C* 'blocks' the path from *A* to *B* and so we obtain the conditional independence property.

Three Canonical 3-Node Graphs

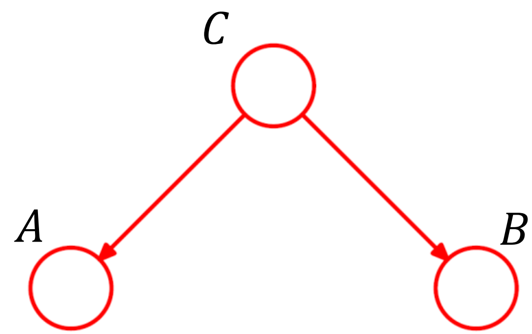
Intuitive interpretation:



- The conditional independence $A \perp B \mid C$ translates into the statement: "the past is independent of the future given the present".
- This is an example of a simple classical Markov Chain.

Three Canonical 3-Node Graphs

2.



Joint distribution corresponding to this graph:

$$p(a, b, c) = p(a|c)p(b|c)p(c)$$

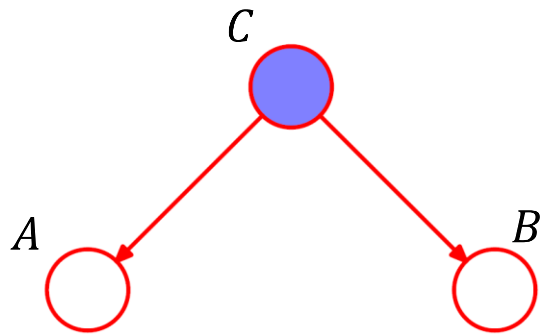
If none of the variables are observed, we can see that *A* and *B* are NOT independent by marginalizing both sides over *C*:

$$p(a, b) = \sum_c p(a|c)p(b|c)p(c)$$

which in general does not factorize into $p(a)p(b)$, and so

$$A \not\perp B \mid \emptyset$$

Three Canonical 3-Node Graphs



If we condition on node C, we can easily write down the conditional distribution of A and B, given C, in the form:

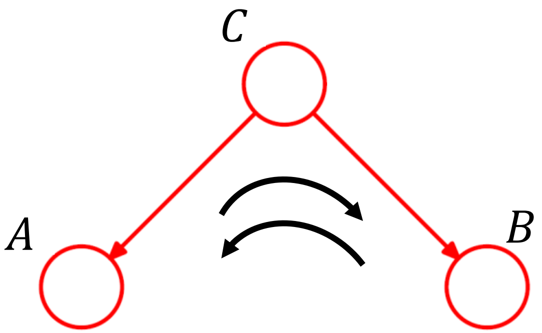
$$\begin{aligned} p(a, b|c) &= \frac{p(a, b, c)}{p(c)} \\ &= p(a|c)p(b|c) \end{aligned}$$

$$p(a, b, c) = p(a \mid c)p(b \mid c)p(c)$$

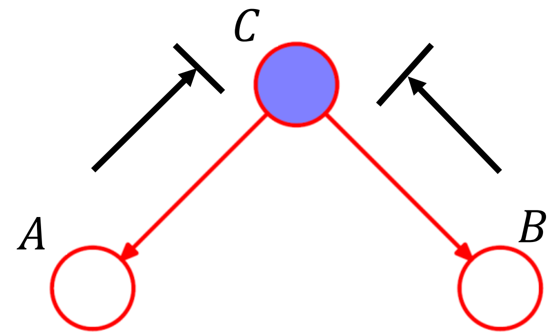
which shows the conditional independence property:

$$A \perp B \mid C$$

Three Canonical 3-Node Graphs



$A \not\perp B \mid \emptyset$

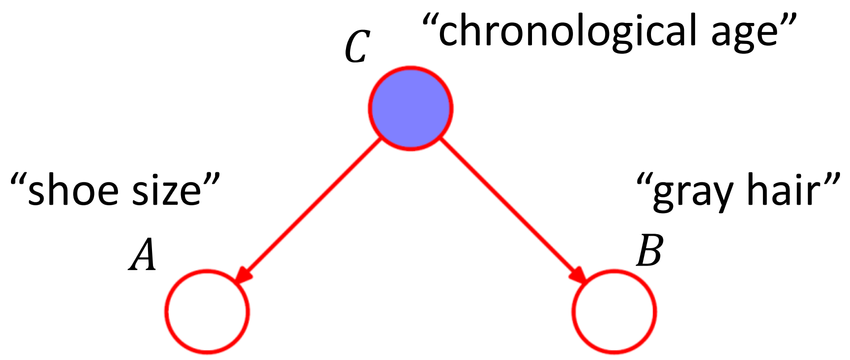


$A \perp B \mid C$

- The node C is said to be *tail-to-tail* with respect to the path from node A to B.
- Such a path connects nodes A and B and renders them dependent.
- The observation of C 'blocks' the path from A to B, and we obtain the conditional independence property.

Three Canonical 3-Node Graphs

Intuitive interpretation:

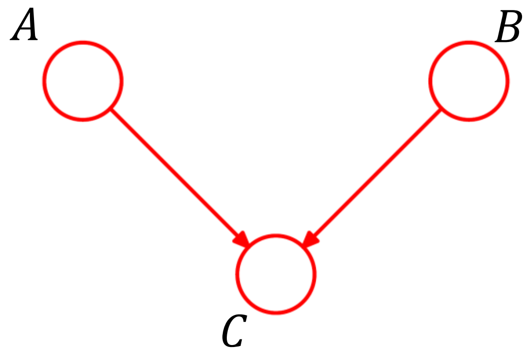


- Given the age of a person, there is no further relationship between the size of his feet and the amount of gray hair he has.
- We say that the variable C "explains" all the observed dependence between A and B.

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs

3.



Joint distribution corresponding to this graph:

$$p(a, b, c) = p(a)p(b)p(c|a, b)$$

If none of the variables are observed, marginalizing both sides over *c* we obtain:

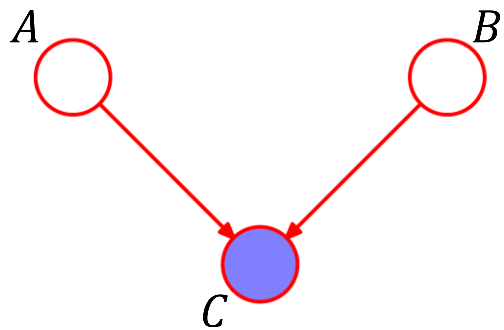
$$p(a, b) = \sum_c p(a)p(b)p(c \mid a, b)$$
$$= p(a)p(b)$$

A and *B* are independent with no variables observed, in contrast to the two cases:

$$A \perp B \mid \emptyset$$

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs



If we condition on node C, the conditional distribution of A and B is given by:

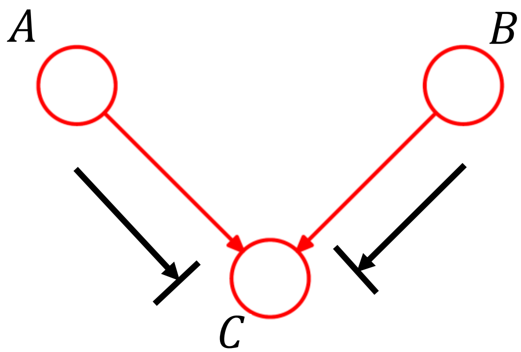
$$
\begin{aligned}
p(a, b|c) &= \frac{p(a, b, c)}{p(c)} \\
&= \frac{p(a)p(b)p(c|a, b)}{p(c)}
\end{aligned}
$$

which in general does not factorize into the product $p(a)p(b)$, and so

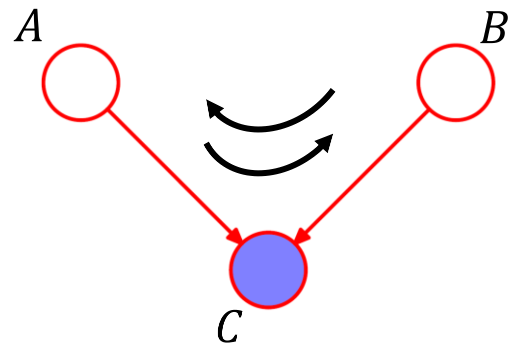
$$A \not\perp B \mid C$$

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs



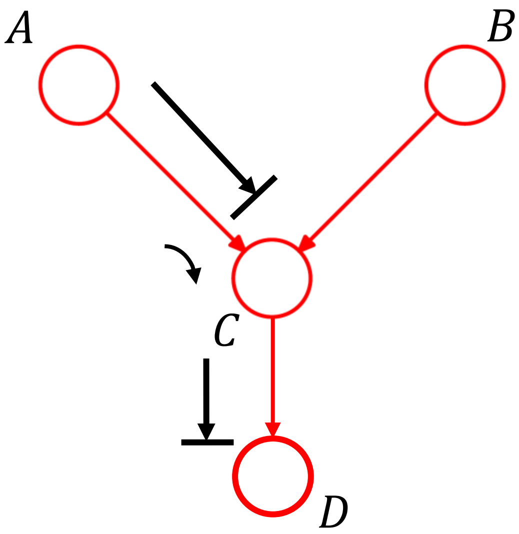
$$A \perp B \mid \emptyset$$



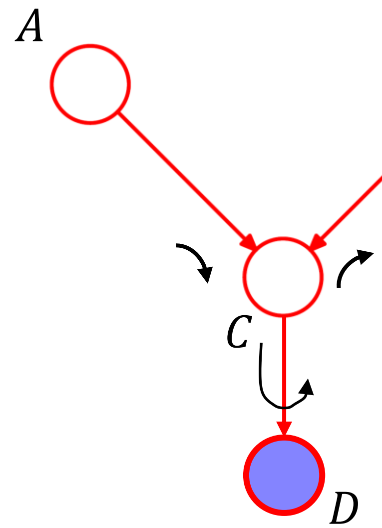
$$A \not\perp B \mid C$$

- Node C is *head-to-head* with respect to the path from A to B, also known as the "v-structure".
- When node C is unobserved, it "blocks" the path, and the variables A and B are independent.
- However, conditioning on C "unblocks" the path and renders A and B dependent.

Three Canonical 3-Node Graphs



$$A \perp B \mid \emptyset$$

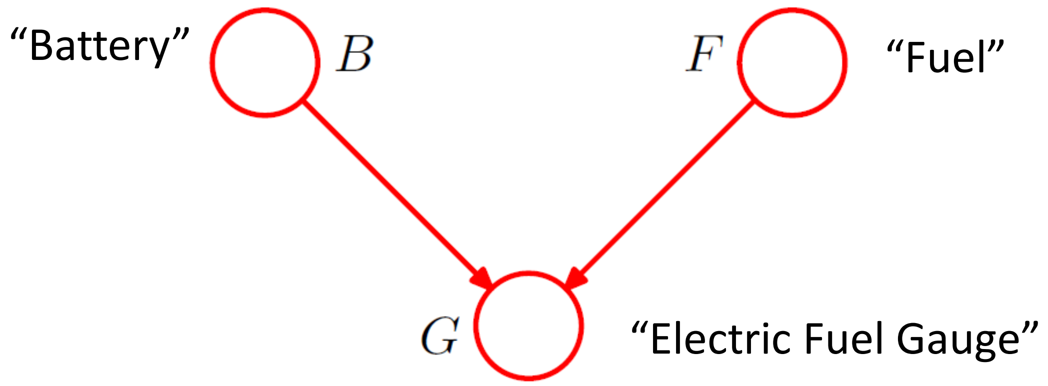


$$A \not\perp B \mid D$$

- The observation of any descendent node of *C* “unblocks” the path from *A* to *B*.

Three Canonical 3-Node Graphs

Intuitive interpretation:



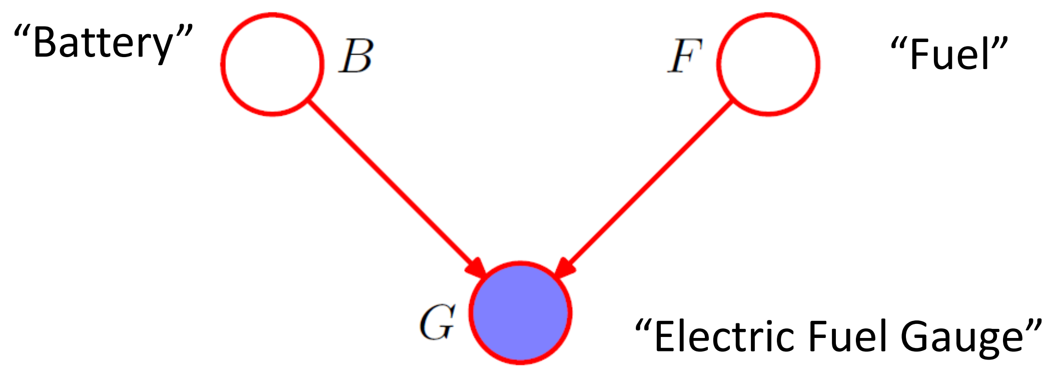
- No relation between the battery and fuel status if fuel gauge is not read.

- This implies conditional independence of "battery" and "fuel" when "gauge" is not observed.

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs

Intuitive interpretation:

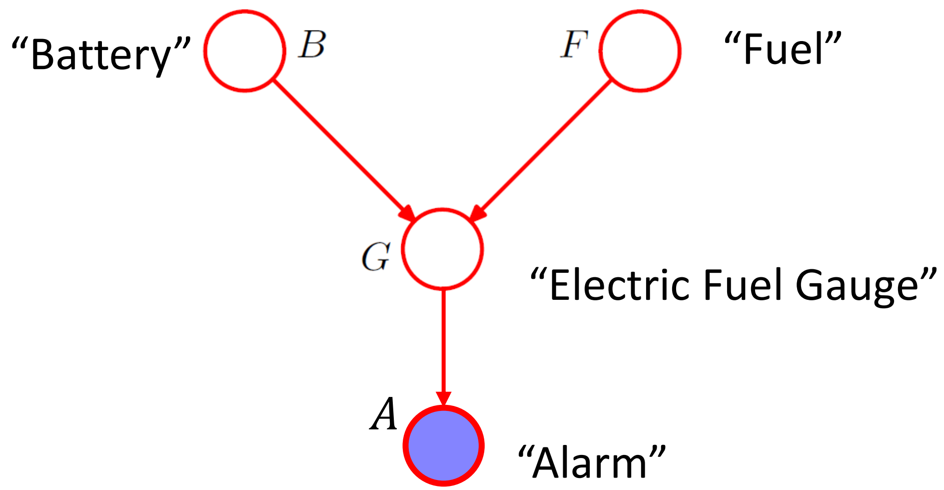


- Suppose the fuel gauge shows “empty”, knowing that the battery is flat lowers our belief that the fuel tank is empty.
- Battery and fuel status are now no longer independent.
- This is known as the “explaining-away” effect.

Image Source: “Pattern Recognition and Machine Learning”, Christopher Bishop

Three Canonical 3-Node Graphs

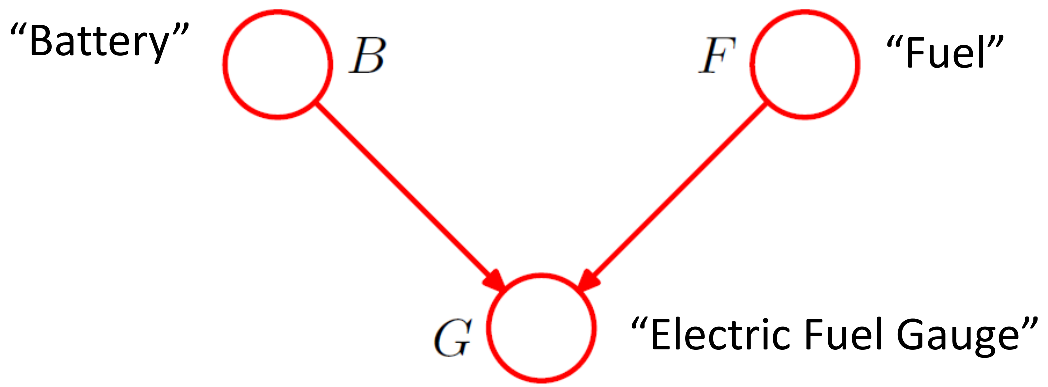
Intuitive interpretation:



- Alarm goes off when fuel gauge is empty.
- Suppose alarm goes off, we know that the fuel gauge shows "empty".
- Knowing that the is battery is flat lowers our belief that the fuel tank is empty, i.e. battery and fuel status are now no longer independent.

Three Canonical 3-Node Graphs

Numerical Example:

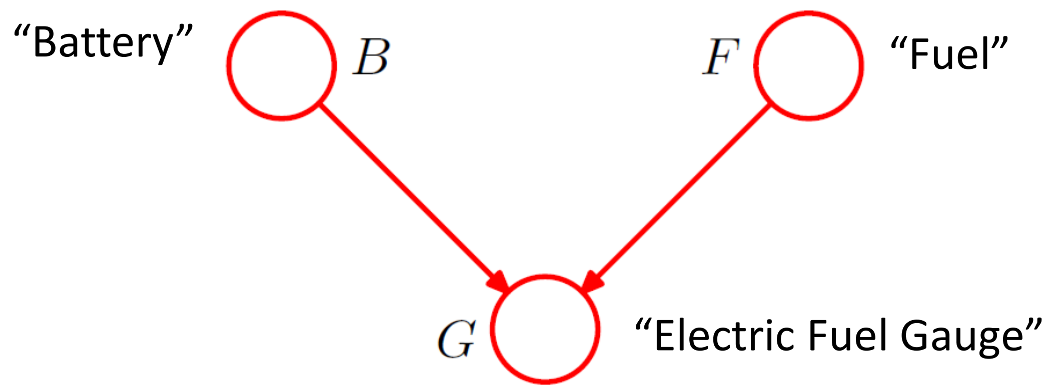


- B : battery state that is either charged (B = 1) or flat (B = 0).
- F : fuel tank state that is either full of fuel (F = 1) or empty (F = 0).
- G : electric fuel gauge state which indicates either full (G = 1) or empty (G = 0).

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs

Numerical Example:



Given:

$$
\begin{aligned}
p(B=1) &= 0.9 \\
p(F=1) &= 0.9
\end{aligned}
$$

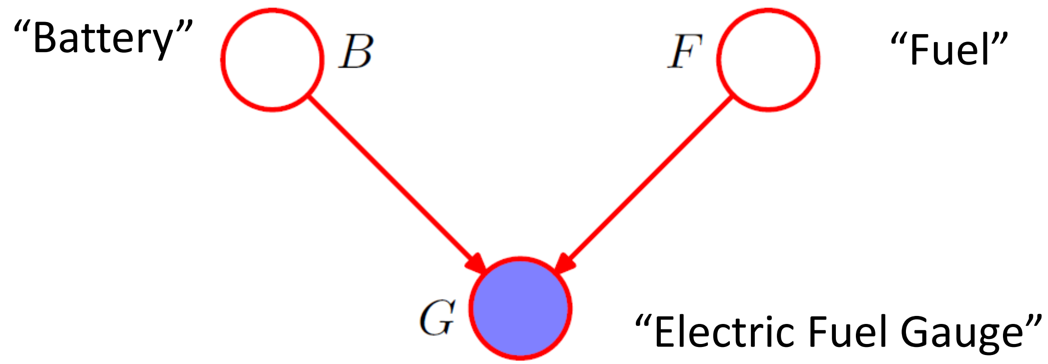
$$
\begin{aligned}
p(G=1|B=1,F=1) &= 0.8 \\
p(G=1|B=1,F=0) &= 0.2 \\
p(G=1|B=0,F=1) &= 0.2 \\
p(G=1|B=0,F=0) &= 0.1
\end{aligned}
$$

Before we observe any data, the prior probability of the fuel tank being empty is $p(F = 0) = 0.1$.

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs

Numerical Example:



Suppose that we observe the fuel gauge and it reads empty, i.e., $G = 0$, we have:

$$p(F=0|G=0) = \frac{p(G=0|F=0)p(F=0)}{p(G=0)} \simeq 0.257$$

where

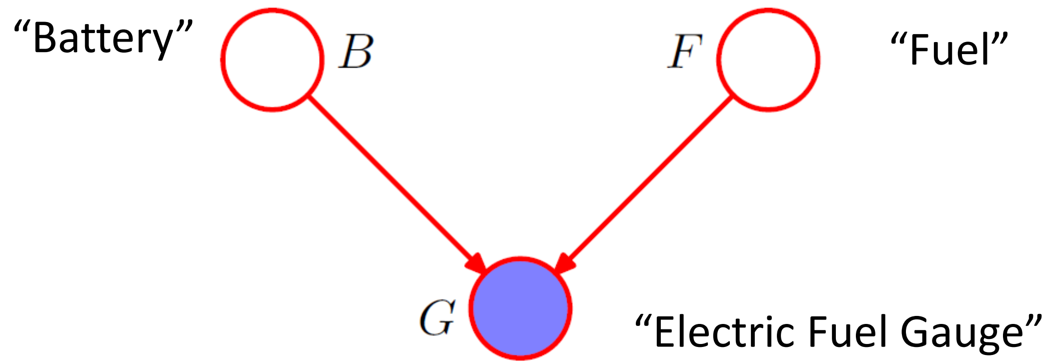
$$p(G=0) = \sum_{b\in\{0,1\}} \sum_{f\in\{0,1\}} p(G=0|B,F)p(B)p(F) = 0.315$$

$$p(G=0|F=0) = \sum_{b\in\{0,1\}} p(G=0|B,F=0)p(B) = 0.81$$

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs

Numerical Example:



Hence,

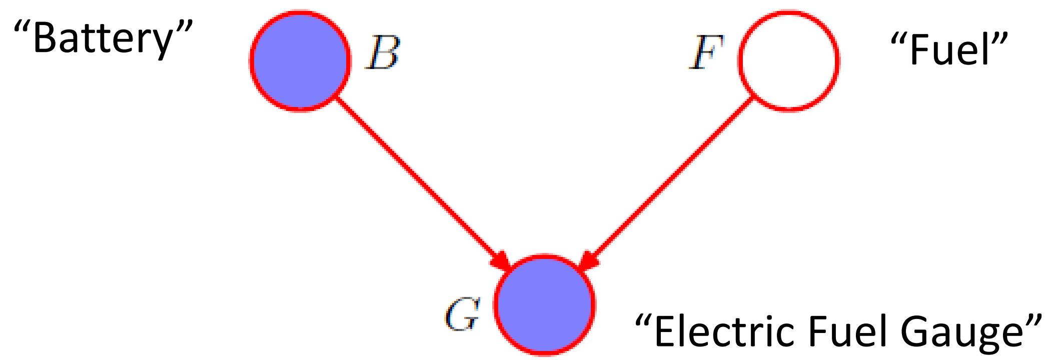
$$\overset{0.257}{p(F=0|G=0)} > \overset{0.1}{p(F=0)}$$

Observing that the gauge reads empty makes it more likely that the tank is indeed empty, as we would intuitively expect.

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs

Numerical Example:

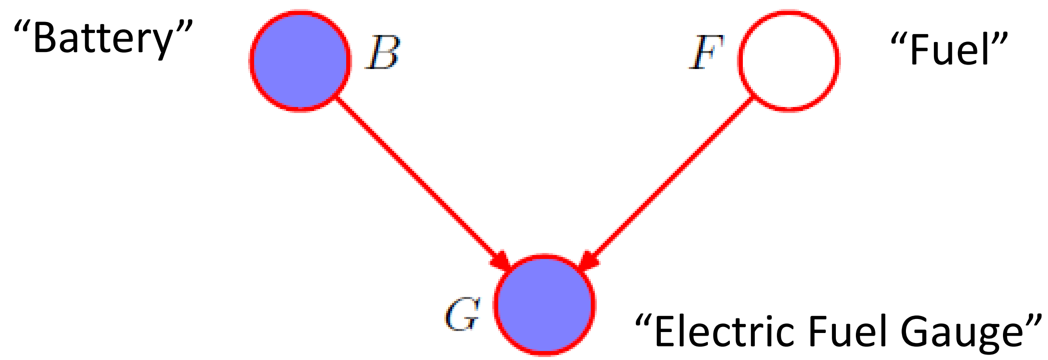


- If we also check the state of the battery and find that it is flat, i.e., $B = 0$.
- We have now observed the states of both fuel gauge and battery.

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs

Numerical Example:



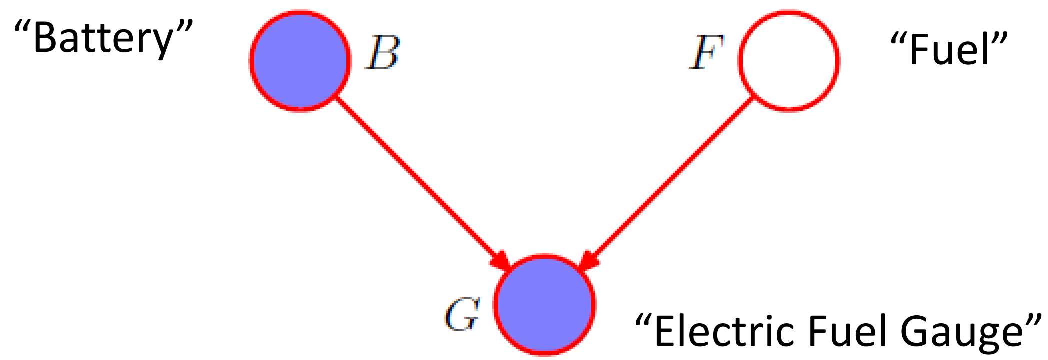
- Posterior probability that fuel tank is empty given observations of both fuel gauge and battery state is:

$$p(F=0|G=0,B=0) = \frac{p(G=0|B=0,F=0)p(F=0)}{\sum_{F\in\{0,1\}} p(G=0|B=0,F)p(F)} \simeq 0.111$$

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Three Canonical 3-Node Graphs

Numerical Example:



0.257 0.111

$$p(F = 0|G = 0) > p(F = 0|G = 0, B = 0)$$

- Finding out that battery is flat *explains away* observation that the fuel gauge reads empty!

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Graph Separation

- We have seen earlier that $A \perp B \mid C$ if all paths from nodes in set A are "blocked" from nodes in set B when all nodes from set C are observed.

- *A* is said to be d-separated from *B* by *C*, and the joint distribution over all of the variables in the graph will satisfy $A \perp B \mid C$.

Graph Separation

- From the three canonical 3-node graphs, any path is said to be "blocked" / d-separated if it includes a node such that either:

  a) The arrows on the path meet either head-to-tail or tail-to-tail at the node, and the node is in the set C, or

  b) The arrows meet head-to-head at the node, and neither the node, nor any of its descendants, is in the set C.

Bayes Ball Algorithm

- This is a "reachability" algorithm:

1. Shade the nodes in set C.
2. Place a ball at each of the nodes in set A.
3. Let the balls bounce around the graph according to the d-separation rules:

IF none of the balls reach B THEN

$$A \perp B \mid C,$$

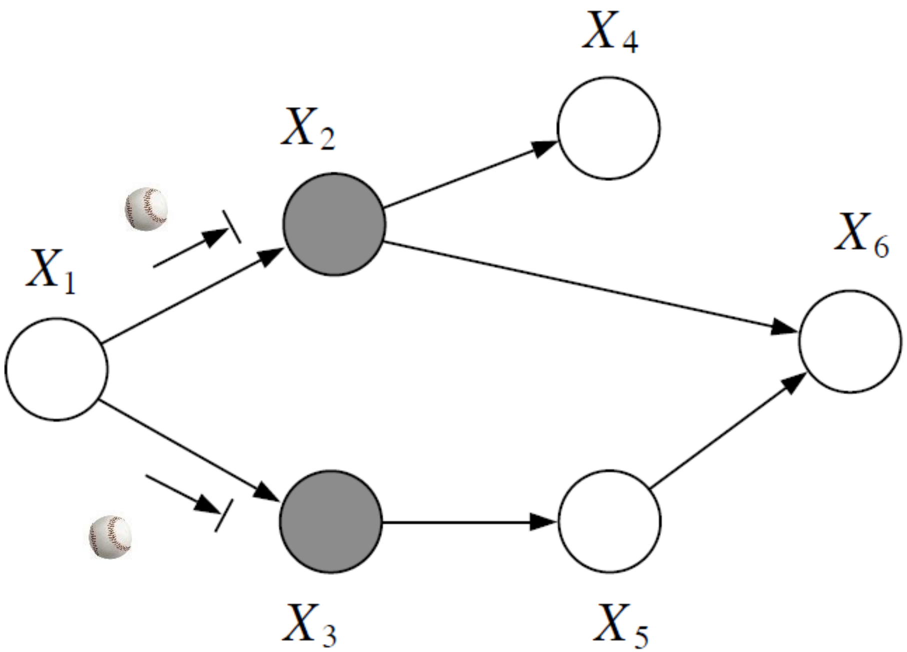
ELSE

$$A \not\perp B \mid C,$$

- Can be implemented as a breadth-first search.

Bayes Ball Algorithm

Example 1:

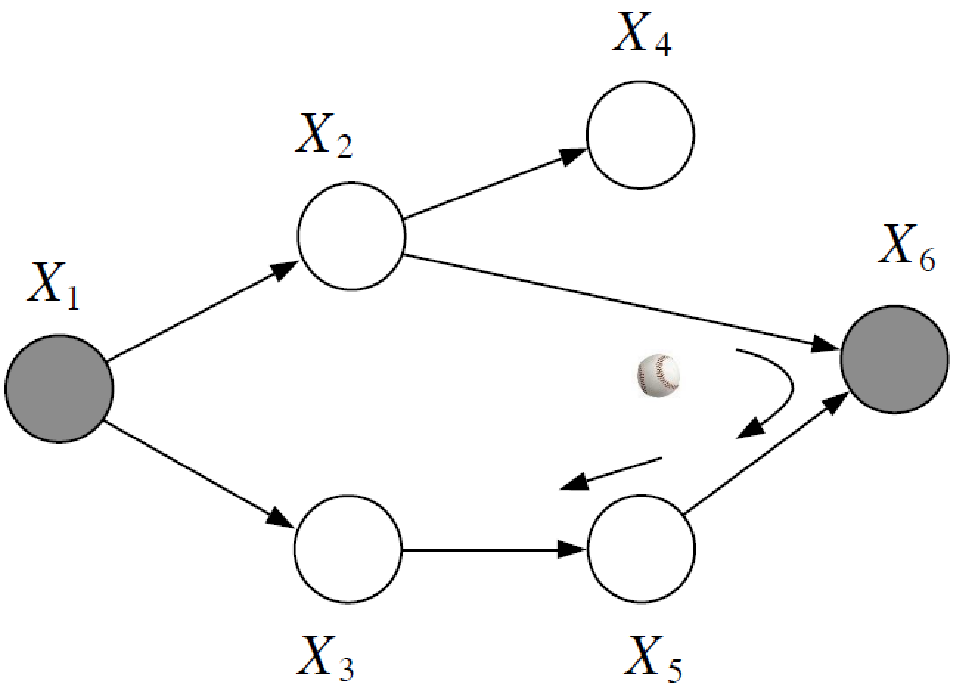


- A ball cannot pass through X_2 to X_6 nor through X_3 i.e. $X_1 \perp X_6 \mid \{X_2, X_3\}$.

Image modified from: “An introduction to probabilistic graphical models”, Michael I. Jordan, 2002.

Bayes Ball Algorithm

Example 2:



- A ball can pass through X_2 to X_6 through X_5 and X_3 i.e. $X_2 \not\perp X_3 \mid \{X_1, X_6\}$.

Image modified from: "An introduction to probabilistic graphical models", Michael I. Jordan, 2002.

Bayes Ball Algorithm

Example 3: **Naïve Bayes Classifier**

- Classify e-mails as spam ($Y = 1$) or not spam ($Y = 0$).
- Let $1{:}N$ index the words in our vocabulary (e.g., English)
- $$X_n = \begin{cases} 0 & \text{word } n \text{ appears in an e−mail} \\ 1 & \text{otherwise} \end{cases}$$
- E-mails are drawn according to some distribution $p(x_1, \ldots, x_N, Y)$

Bayes Ball Algorithm

Example 3: **Naïve Bayes Classifier**

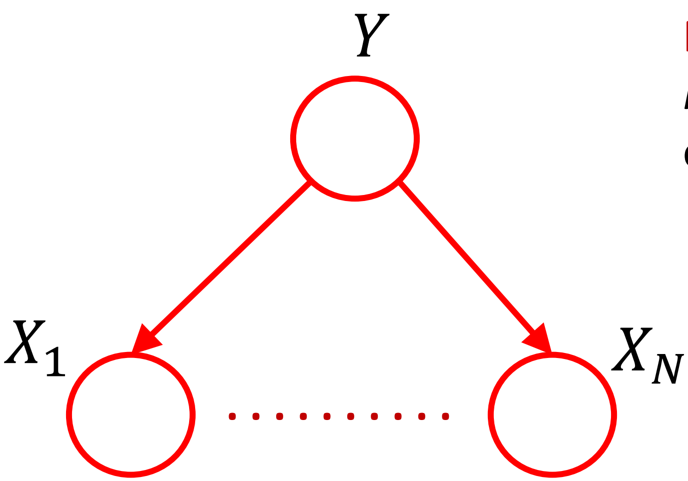
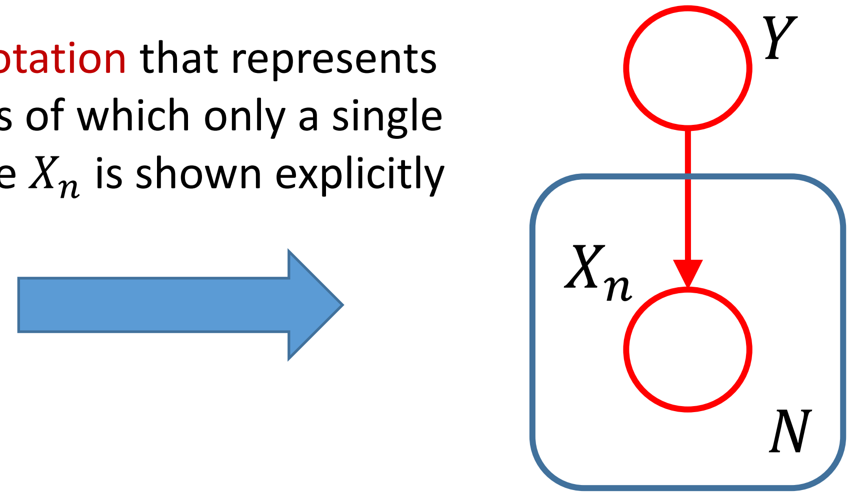


Plate notation that represents N nodes of which only a single example X_n is shown explicitly

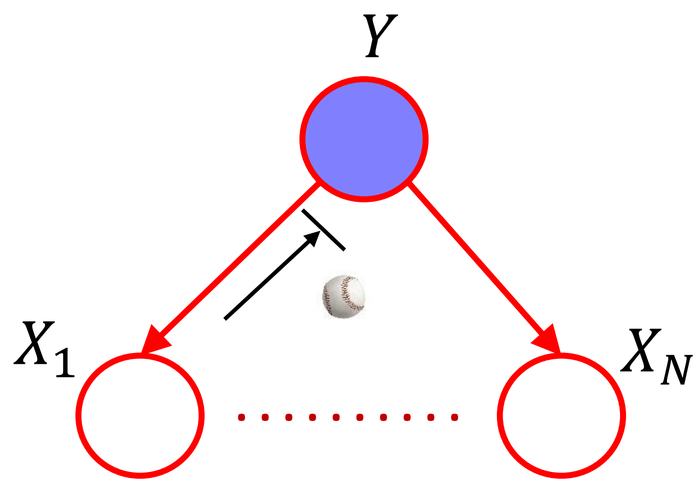


Joint distribution:

$$p(x_1, \ldots x_N, Y) = p(x_1, \ldots x_N | Y) p(Y)$$

Bayes Ball Algorithm

Example 3: **Naïve Bayes Classifier**

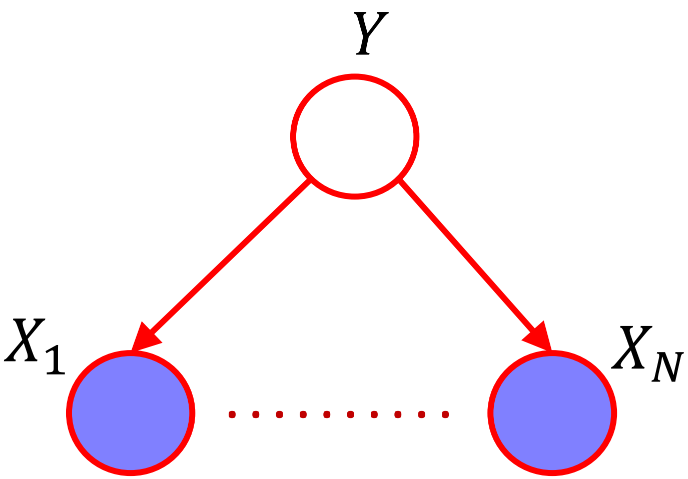


Joint distribution:

$$p(x_1, \dots x_N, Y) = p(x_1, \dots x_N \mid Y)\, p(Y)$$
$$= p(x_1|Y) \dots p(x_N|Y) p(Y)$$
$$= \prod_{n=1}^{N} p(x_n|Y) p(Y)$$

Bayes Ball Algorithm

Example 3: **Naïve Bayes Classifier**



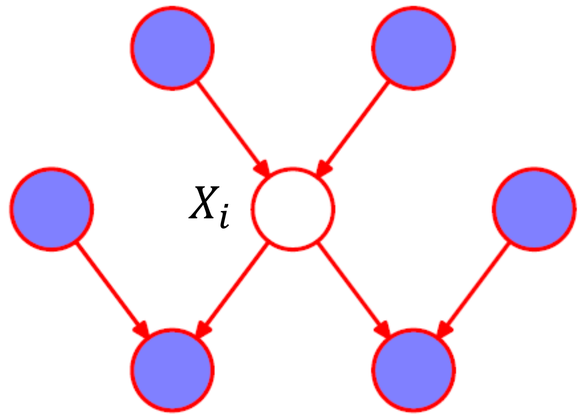
Joint distribution:

$$p(x_1, \dots x_N, Y) = \prod_{n=1}^{N} p(x_n | Y) p(Y)$$

Predict with Bayes rule:

$$p(Y = 1 \mid x_1, \dots, x_N) = \frac{p(y = 1) \prod_{n=1}^{N} p(x_n \mid Y = 1)}{\sum_{y=\{0,1\}} p(Y = y) \prod_{n=1}^{N} p(x_n \mid Y = y)}$$

Markov Blanket



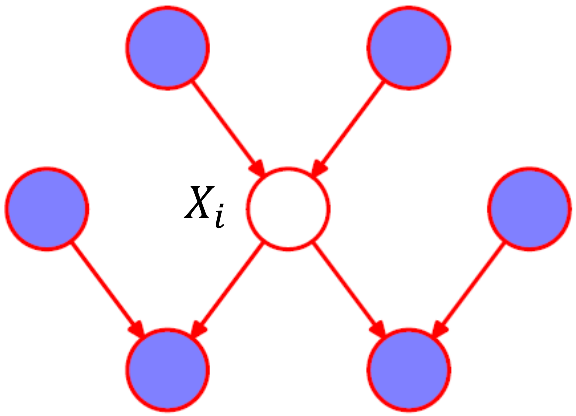
- The Markov blanket of a node X_i comprises the set of parents, children and co-parents of the node.

- Conditional distribution of X_i, conditioned on all the remaining variables in the graph, is dependent only on the variables in the Markov blanket.

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Markov Blanket

Proof:



$$p(x_i|x_{\{j\neq i\}}) = \frac{p(x_1,\dots,x_D)}{\int p(x_1,\dots,x_D)dx_i} = \frac{\prod_k p(x_k|x_{\pi_k})}{\int \prod_k p(x_k|x_{\pi_k})\, dx_i}$$

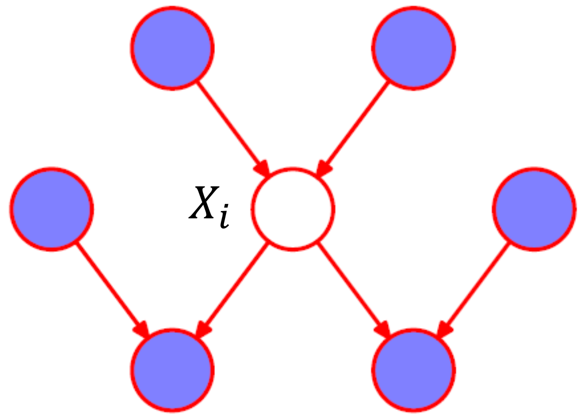
$$= \frac{\cancel{\prod_{l\neq\{m,i\}} p(x_l|x_{\pi_l})} \prod_m p(x_m|x_{\pi_m} : x_i \in x_{\pi_m})\, p(x_i|x_{\pi_i})}{\cancel{\prod_{l\neq\{m,i\}} p(x_l|x_{\pi_l})} \int \prod_m p(x_m|x_{\pi_m} : x_i \in x_{\pi_m})\, p(x_i|x_{\pi_i}) dx_i}$$

$$= \frac{\prod_m p(x_m|x_{\pi_m} : x_i \in x_{\pi_m})\, p(x_i|x_{\pi_i})}{\int \prod_m p(x_m|x_{\pi_m} : x_i \in x_{\pi_m})\, p(x_i|x_{\pi_i}) dx_i}$$

Image Source: “Pattern Recognition and Machine Learning”, Christopher Bishop

Markov Blanket

Proof:

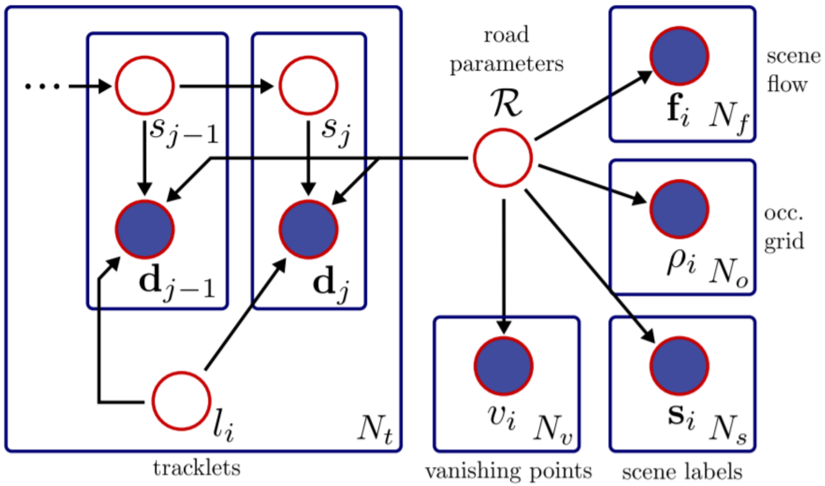


$$p(x_i|x_{\{j\neq i\}}) = \frac{\prod_m p\left(x_m \middle| x_{\pi_m} : x_i \in x_{\pi_m}\right) p(x_i|x_{\pi_i})}{\int \prod_m \underbrace{p\left(x_m \middle| x_{\pi_m} : x_i \in x_{\pi_m}\right)}_{\text{children and co-parents of } X_i} \underbrace{p(x_i|x_{\pi_i})}_{\text{parents of } X_i} dx_i}$$

Image Source: "Pattern Recognition and Machine Learning", Christopher Bishop

Examples of Bayesian Network

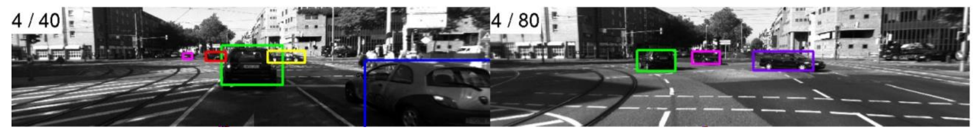
Urban Scene Understanding:



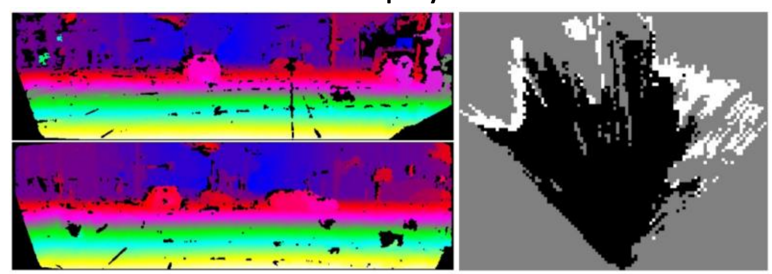
$$
\begin{aligned}
p(\mathcal{E}, \mathcal{R} | \Theta) \;=\; & \underbrace{p(\mathcal{R}|\Theta)}_{\text{Prior}} \; \underbrace{\left[\prod_{i=1}^{N_t} p(\mathbf{t}_i | \mathcal{R}, \Theta) \right]}_{\text{Vehicle Tracklets}} \\
& \times \underbrace{\prod_{i=1}^{N_v} p(v_i | \mathcal{R}, \Theta)}_{\text{Vanishing Points}} \; \underbrace{\prod_{i=1}^{N_s} p(\mathbf{s}_i | \mathcal{R}, \Theta)}_{\text{Scene Labels}} \\
& \times \underbrace{\prod_{i=1}^{N_f} p(\mathbf{f}_i | \mathcal{R}, \Theta)}_{\text{Scene Flow}} \; \underbrace{\prod_{i=1}^{N_o} p(\rho_i | \mathcal{R}, \Theta)}_{\text{Occupancy Grid}}
\end{aligned}
$$

Vanishing Points

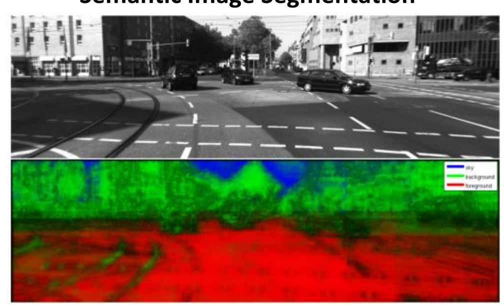
Tracklets



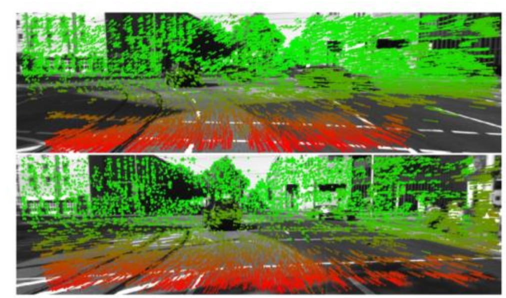
Occupany Grid



Semantic Image Segmentation



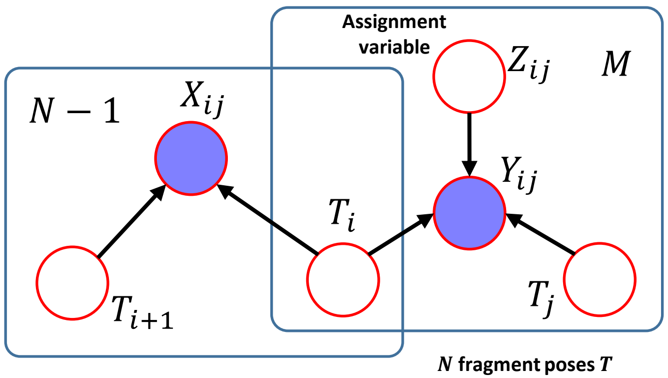
Scene Flow

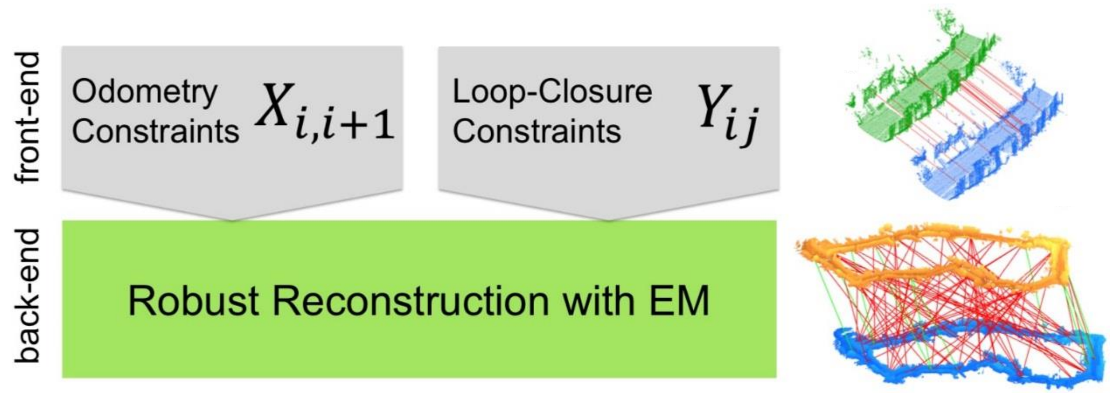


Andreas Geiger, PhD Thesis 2013

Examples of Bayesian Network

Robust 3D Point Cloud Reconstruction:





Expectation step:

$$Q_{EM} = \sum_z p\left(Z \mid Y, T^{old}\right) \ln p(X, Y, Z \mid T)$$

Maximization step:

$$\underset{T}{\operatorname{argmax}}\, Q_{EM}$$

Ziquan Lan, Zi Jian Yew, Gim Hee Lee, "Robust Point Cloud Based Reconstruction of Large-Scale Outdoor Scenes", CVPR 2019

Summary

- You have learned how to:

1. Explain the concepts of conditional independence.
2. Use the Bayesian network to represent conditional independence in joint distributions.
3. Describe d-separation using the three canonical 3-node graph.
4. Deduce all conditional independence in a Bayesian network using the Bayes ball algorithm.
5. Explain the concepts of Markov Blanket.