NUS National University of Singapore | School of Computing

CS5340
Uncertainty Modelling in AI

Lecture 1:
Introduction to Probabilistic Reasonings

Assoc. Prof. Lee Gim Hee

AY 2022/23

Semester 1

Course Schedule

Week	Date	Topic	Remarks
1	10 Aug	Introduction to probabilistic reasoning	**Assignment 0:** Python Numpy Tutorial (Ungraded)
2	17 Aug	Bayesian networks (Directed graphical models)	
3	24 Aug	Markov random Fields (Undirected graphical models)	
4	31 Aug	Variable elimination and belief propagation	**Assignment 1:** Belief propagation and maximal probability (15%)
5	07 Sep	Factor graph and the junction tree algorithm	
6	14 Sep	Parameter learning with complete data	**Assignment 1:** Due **Assignment 2:** Junction tree and parameter learning (15%)
-	21 Sep	Recess week	**No lecture**
7	28 Sep	Mixture models and the EM algorithm	**Assignment 2:** Due
8	05 Oct	Hidden Markov Models (HMM)	**Assignment 3:** Hidden Markov model (15%)
9	12 Oct	Monte Carlo inference (Sampling)	
*	15 Oct	Variational inference	Makeup Lecture (Venue TBD) Time: 9.30am – 12.30pm (Saturday)
10	19 Oct	Variational Auto-Encoder and Mixture Density Networks	**Assignment 3:** Due **Assignment 4:** MCMC Sampling (15%)
11	26 Oct	No Lecture	I will be traveling
12	02 Nov	Graph-cut and alpha expansion	**Assignment 4:** Due
13	09 Nov	-	

Acknowledgements

- A lot of slides and content of this lecture are adopted from:

1. Simon Prince, “Computer Vision: Models, Learning, and Inference”, Chapter 1 and 2.
2. Daphne Koller and Nir Friedman, "Probabilistic graphical models”, Chapter 2.
3. Christopher Bishop, “Pattern Recognition and Machine Learning”, Chapter 2.

Learning Outcomes

Students should be able to:

1. Describe uncertain quantities with random variables and joint probabilities.

2. Explain the basic rules of probability – sum, product, Bayes', independence and expectation rules.

3. Use the common probabilities distributions – Bernoulli, categoricial, univariate and multivariate normal distributions.

4. Explain the use of conjugate distributions.

Probability Space

- A probability space (Ω, E, P) models a process consisting of outcomes that occur randomly.

- Consists of three parts:

  1. Outcome or sample space Ω
  2. Event space E
  3. Probability distribution $P: E \rightarrow \mathbb{R}$

Outcome/Sample and Event Spaces

- Outcome/sample space is an agreed upon set of possible outcomes, denoted by Ω.

- Event space $E \subseteq 2^{\Omega}$ is a subset of the power set of Ω, it is the set of measurable events to which we assign probabilities.

Outcome and Event Spaces

- Event space must satisfy three basis properties:

1. It **must contain** the empty event ϕ, and the trivial event Ω.

2. It is closed under countable unions, i.e. if $\alpha_i \in E \ \forall \ i = 1, 2, \ldots$, then so is $\bigcup_{i=1}^{\infty} \alpha_i$.

3. It is closed under complements, i.e. if $\alpha \in E$, then so is $\Omega - \alpha$.

Outcome/Sample and Event Spaces

Example 1:

Let's consider a 6-faced dice. The outcome/sample space is given by $\Omega = \{1, 2, 3, 4, 5, 6\}$.

A possible event space is $E = \{\{1,3,5\}, \{2,4,6\}, \emptyset, \{1,2,3,4,5,6\}\}$, i.e. event of a throw is even or odd.

Check:

1. E contains the empty $\emptyset$ and trivial $\{1,2,3,4,5,6\}$ sets.
2. Let $\alpha_1 = \{1,3,5\}$ and $\alpha_2 = \{2,4,6\}$, i.e., $\alpha_1, \alpha_2 \in E$, then $\alpha_1 \cup \alpha_2 = \{1,2,3,4,5,6\} \in E$ because E is closed under countable unions.
3. Let $\alpha = \{2,4,6\} \in E$, then $\{1,2,3,4,5,6\} - \{2,4,6\} = \{1,3,5\} = \{\Omega - \alpha\} \in E$ because E is closed under complement.

Remark: Check for yourself that 2 and 3 are always true $\forall\ \alpha \in E$!

Outcome/Sample and Event Spaces

Example 2:

Let's consider measuring the lifetime of a lightbulb. The outcome/sample space is given by $\Omega = [0, \infty)$.

A possible event space is $E = \{[0, 90), [90, \infty), \emptyset, [0, \infty)\}$, i.e. event of lightbulb lifespan ≥ 90.

Check:

1. E contains the empty $\emptyset$ and trivial $[0, \infty)$sets.
2. Let $\alpha_1 = \emptyset$ and $\alpha_2 = [0, \infty)$, i.e., $\alpha_1, \alpha_2 \in E$, then $\alpha_1 \cup \alpha_2 = [0, \infty) \in E$ because E is closed under countable unions.
3. Let $\alpha = [0, 90) \in E$, then $[0, \infty) - [0, 90) = [90, \infty) = \{\Omega - \alpha\} \in E$ because E is closed under complement.

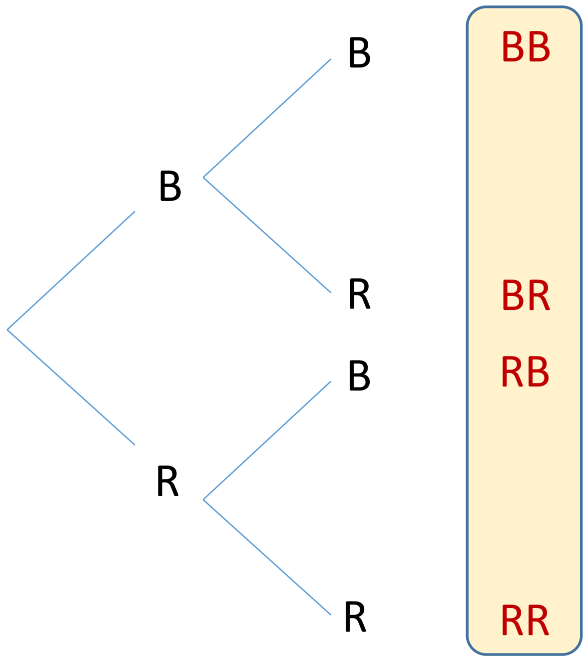
Remark: Check for yourself that 2 and 3 are always true $\forall\, \alpha \in E$!

Outcome/Sample and Event Spaces

Example 3:

Picking 2 marbles, one at a time, from a bag that contains many blue and red marbles. Find the sample space?

Tree Diagram



List

{BB, BR, RB, RR}

Table

	B	R
B	BB	BR
R	RB	RR

Probability Distributions

- A probability distribution P over (Ω, E) is a mapping from events in E to real values ($P: E \rightarrow \mathbb{R}$) that satisfies the following conditions, i.e. axioms of probability:

1. Non-negativity, i.e. $P(\alpha) \geq 0, \ \forall \alpha \in E$.
2. Probability of all outcomes sums to 1, i.e. $P(\Omega) = 1$.
3. Mutually disjoint events: If $\alpha, \beta \in E$ and $\alpha \cap \beta = \emptyset$, then $P(\alpha \cup \beta) = P(\alpha) + P(\beta)$.

Random Variables

- A random variable, denoted as X (upper case), is the formal machinery for discussing attributes and their values in different outcomes.

- More formally: given a probability space (Ω, E, P), a random variable is a function $X: \Omega \rightarrow S$ that maps a set of possible outcomes Ω to a measurable space S.

- Typically, S is the set of real numbers, i.e. $S \in \mathbb{R}$.

Random Variables

- The set of values that a random variable X can take is denoted as $Val(X)$.
- A lower case letter, e.g. x, is used to refer to a generic value of a random variable X, a.k.a. realization of the random variable.

  Example: We write $P(X = x)$ for all $x \in Val(X)$.

- $P(x)$ is often used as a shorthand notation for $P(X = x)$.
- We use the notation x^i to represent a specific value of X.

Random Variables

- The value of a random variable $Val(X)$ can be:
  - ➢ Discrete, i.e. takes values from a predefined set, or
  - ➢ Continuous, i.e. takes values that are real numbers.

Examples:

Random variables with discrete values

- Rolling a six-faced die: $Val(X) = \{1,2, \dots, 6\}$
- Weather conditions: $Val(X) = \{\text{"rain"}, \text{"cloud"}, \text{"snow"}, \text{"sun"}, \text{"wind"}\}$
- Number of people on the next train: $Val(X) = \mathbb{Z}_{\geq 0}$

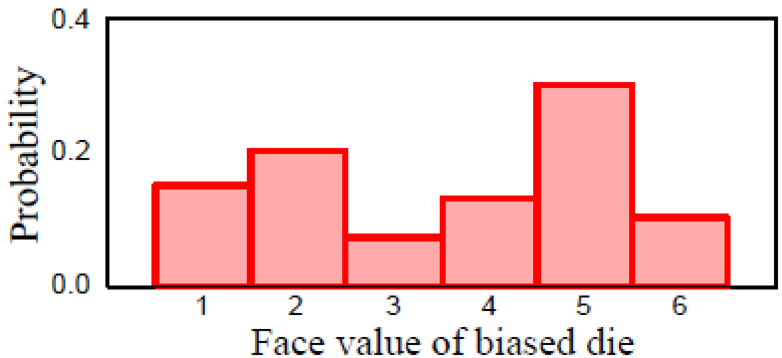
Continuous random variables

- Time taken to finish an exam: $Val(X) = [1,2]$ hours
- Height of a tree: $Val(X) = \mathbb{R}_{>0}$
- Ambient Temperature: $Val(X) = \mathbb{R}$

Probability Distributions: Discrete Vs Continuous

- Discrete: Probability mass function, $P(x)$

$P(x)$



$$Val(X) = \{1,2,3,4,5,6\}$$

$$\sum_{i=1}^{K} P\left(X = x^i\right) = 1$$

$$0 \leq P\left(X = x^i\right) \leq \ 1,$$

$$\forall\, i = 1, \ldots K,$$

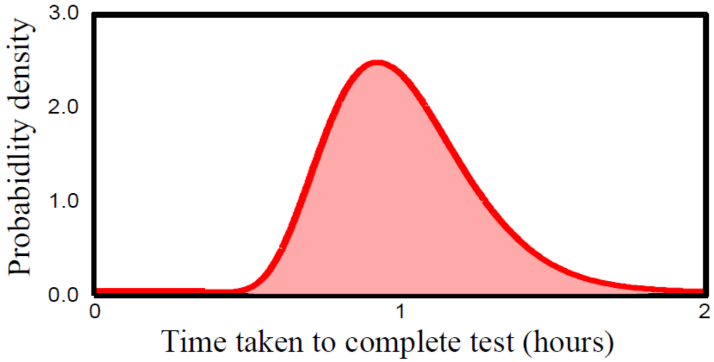
where $K = |Val(X)|$

Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

Probability Distributions: Discrete Vs Continuous

- Continuous: Probability density function is a function (denoted by a lower case p) $p(x)\colon \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$.

$p(x)$



$$Val(X) = [0,2]$$

$$\int_{Val(X)} p(x)dx = 1\,;$$

$$p\left(X = x^i\right) \geq 0, \quad \forall \ x^i \in Val(X)$$

Images Source: “Computer Vision: Models, Learning, and Inference”, Simon Prince

Probability Distributions: Discrete Vs Continuous

- Continuous: Probability density function is a function (denoted by a lower case p) $p(x): \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$.

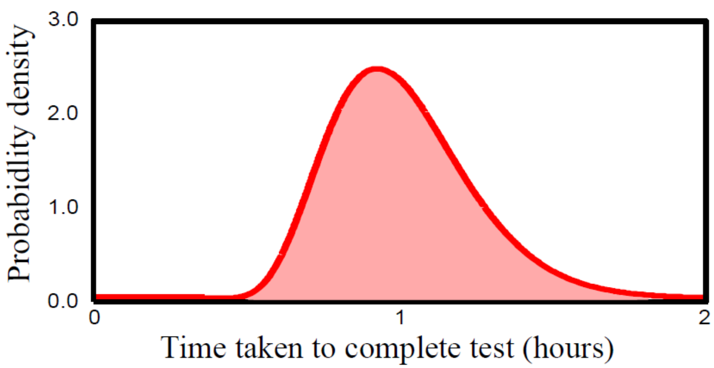
$P(X)$ is the cumulative function of X:

$$P(X \leq a) = \int_{-\infty}^{a} p(x)\, dx$$

$$P(a \leq X \leq b) = \int_{a}^{b} p(x)\, dx$$

$$P\left(X = x^i\right) = \int_{x^i}^{x^i} p(x)\, dx = 0, \quad \forall\ x^i \in Val(X)$$

$p(x)$



$Val(X) = [0,2]$

Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

Probability Distributions: Discrete Vs Continuous

In this course, we abuse the notation by denoting both the probability mass function and probability density function as the lower case $p(x)$!

We silently note the property differences in $P(x)$ when X is discrete or continuous.

Probabilistic Reasoning

Probabilistic Modeling:

- The central paradigm of probabilistic reasoning is to:

    1. Identify all relevant variables $X_1, \ldots X_N$ of the environment, and

    2. make a probabilistic model $p(X_1, \ldots X_N)$ of their interactions.

Probabilistic Reasoning

Probabilistic Inference:

- Reasoning (inference) is then performed by:

1. Introducing evidence that sets variables in known state, and

2. subsequently computing probabilities of interest, conditioned on this evidence.

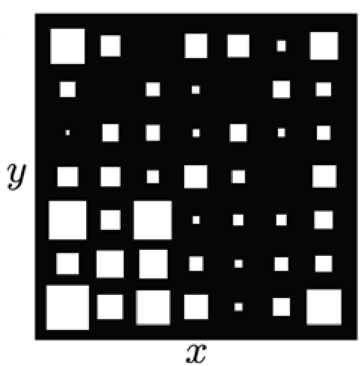
Probabilistic Reasoning

- To this end, we require the definitions of joint probability, marginalization, conditional probability, Bayes' rule, and independence.
- In this lecture, we look at the use of these definitions for probabilistic modeling and inference of a small number of variables.
- In the subsequent lectures, we will look at the use of these definitions with **graphical models** for large number of variables.

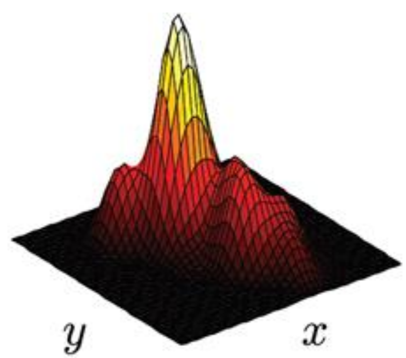
Probability: Joint Probability

- Consider all combination of events of two random variables X and Y.
- Some combinations of outcomes are more likely than others.

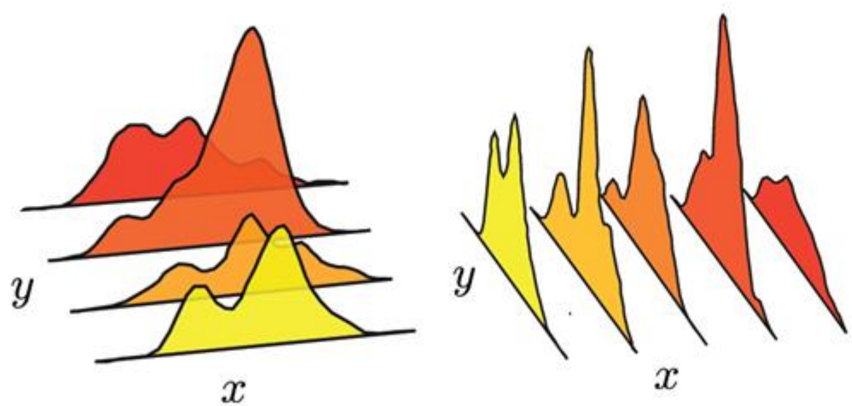
Discrete



Continuous



Discrete-Continuous



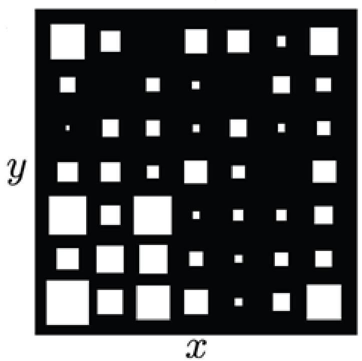
Images Source: “Computer Vision: Models, Learning, and Inference”, Simon Prince

Probability: Joint Probability

- This is captured in the joint probability distribution $p(x, y)$.
- Read as "probability of X and Y".
- Can be more than two random variables, i.e. $p(a, b, c, \dots)$.

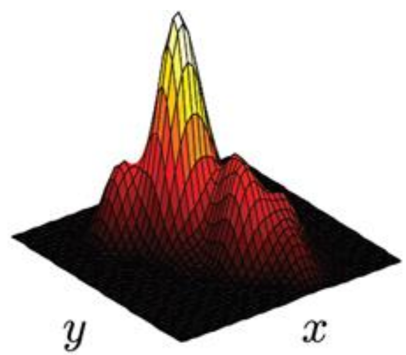
Discrete

$p(x, y)$



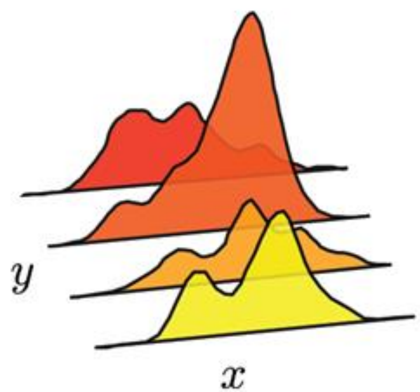
Continuous

$p(x, y)$

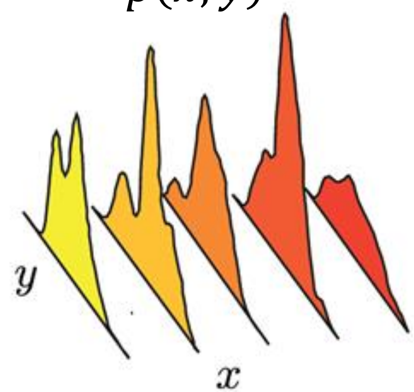


Discrete-Continuous

$p(x, y)$



$p(x, y)$



Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

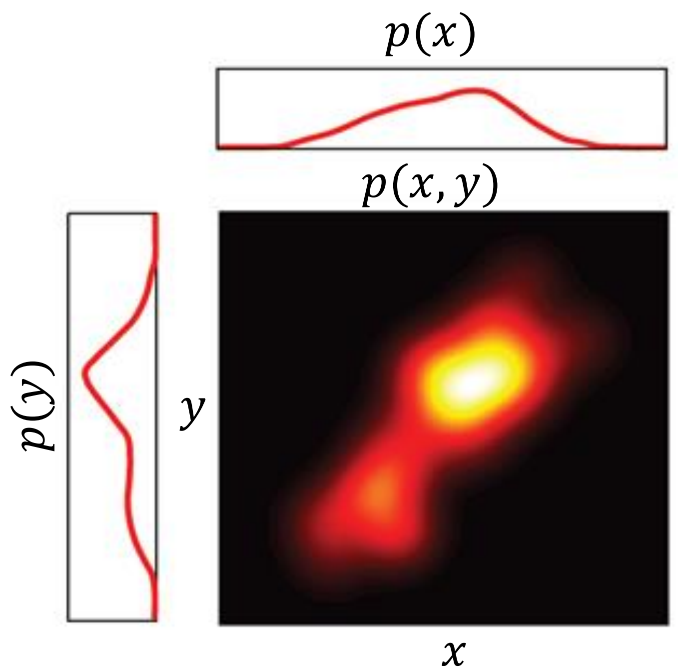
Probability: Marginalization

- Recover probability distribution of any variable in a joint distribution by integrating (or summing) over all other variables.
- Also known as the "sum rule" of probability.

Continuous:

$$p(x) = \int p(x, y)\, dy$$

$$p(y) = \int p(x, y)\, dx$$



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Probability: Marginalization

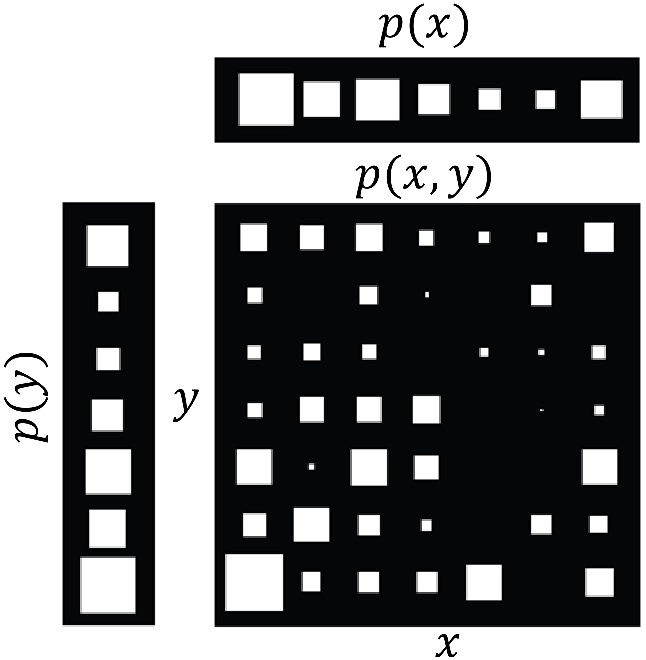
- Recover probability distribution of any variable in a joint distribution by integrating (or summing) over all other variables.

- Also known as the "sum rule" of probability.

Discrete:

$$p(x) = \sum_y p(x, y)$$

$$p(y) = \sum_x p(x, y)$$



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Probability: Marginalization

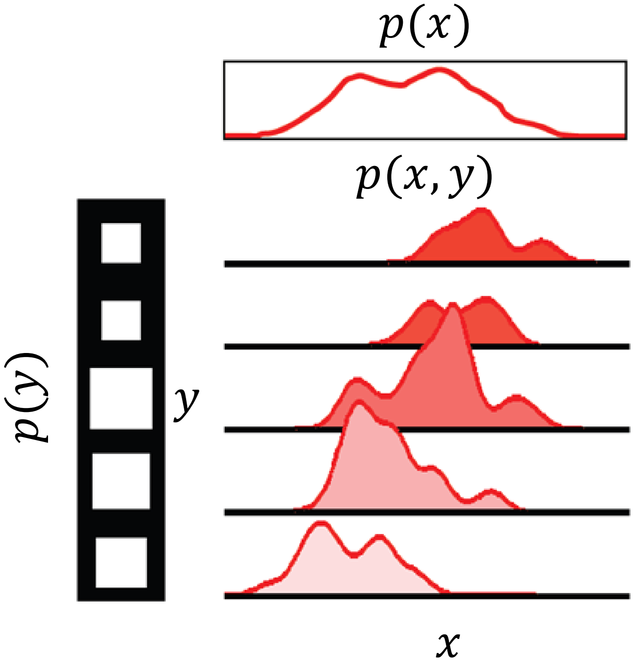
- Recover probability distribution of any variable in a joint distribution by integrating (or summing) over all other variables.

- Also known as the "sum rule" of probability.

Discrete-continuous:

$$p(x) = \sum_{y} p(x, y)$$

$$p(y) = \int p(x, y)\, dx$$



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Probability: Marginalization

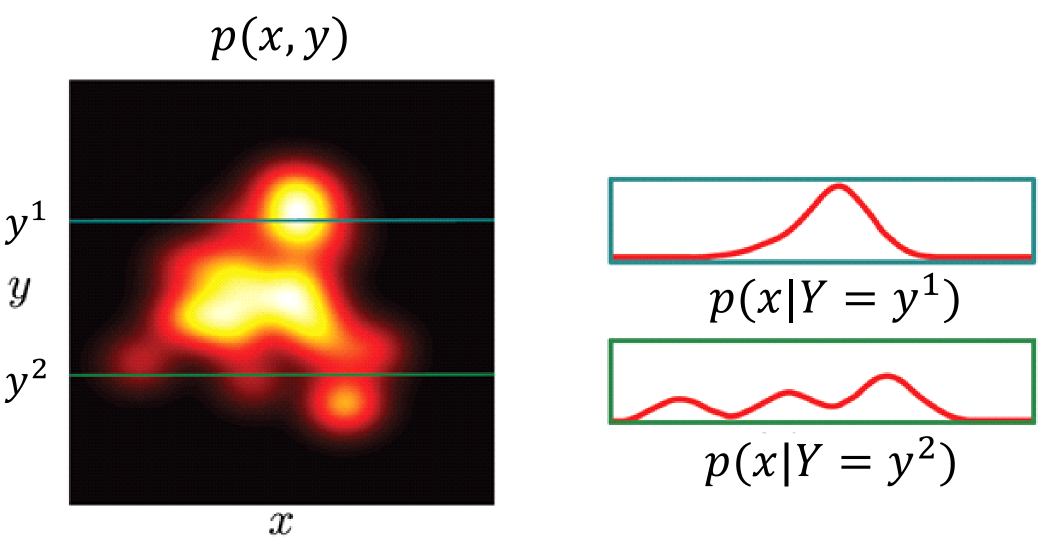
- Works in higher dimensions too!

Example:

$$p(x, y) = \sum_{w} \int p(w, x, y, z)\, dz$$

Probability: Conditional Probability

- $p(x|Y = y^*)$: "probability of X given $Y = y^*$".
- Also known as "chain rule" or "product rule" of probability.
- Relative propensity of the random variable X to take different outcomes given that the random variable Y is fixed to value y^*.



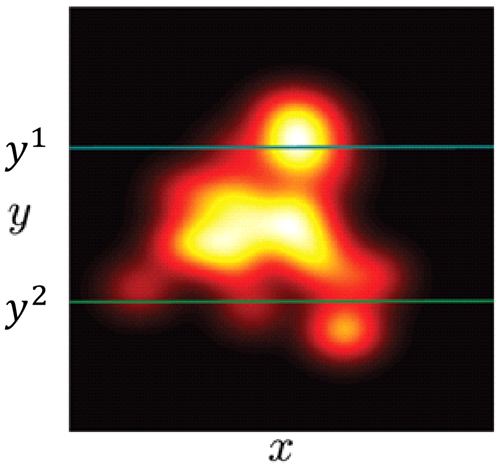
Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

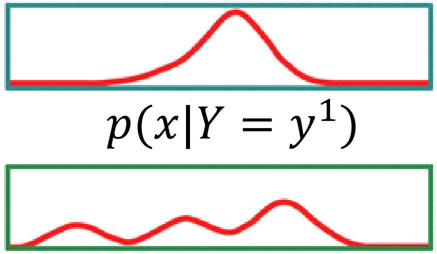
Probability: Conditional Probability

- Conditional probability can be extracted from joint probability.
- Extract appropriate slice and normalize (so that the area is 1):

$$P(x|Y = y^*) = \frac{p(x, Y = y^*)}{\int p(x, Y = y^*)dx} = \frac{p(x, Y = y^*)}{p(Y = y^*)}$$

$p(x, y)$





$p(x|Y = y^1)$

$p(x|Y = y^2)$

Images Source: “Computer Vision: Models, Learning, and Inference”, Simon Prince

Probability: Conditional Probability

$$P(x|Y=y^*) = \frac{p(x, Y=y^*)}{\int p(x, Y=y^*)dx} = \frac{p(x, Y=y^*)}{p(Y=y^*)}$$

- Usually written in compact form:

$$p(x|y) = \frac{p(x,y)}{p(y)}$$

- Which can be re-arranged to give:

$$p(x,y) = p(x|y)p(y)$$
$$p(x,y) = \underbrace{p(y|x)p(x)}_{\text{Hence, the name “product rule”!}}$$

Probability: Conditional Probability

$$p(x, y) = p(x|y)p(y)$$

- Works for higher dimensions too!

Example:

$$\begin{aligned} p(w, x, y, z) &= p(w, x, y|z)p(z) \\ &= p(w, x|y, z)p(y|z)p(z) \\ &= p(w|x, y, z)p(x|y, z)p(y|z)p(z) \end{aligned}$$

Probability: Bayes' Rule

- Recall:

$$p(x,y) = p(x|y)p(y)$$
$$p(x,y) = p(y|x)p(x)$$

Thomas Bayes
1701–1761

- Eliminating $p(x,y)$, we get:

$$p(y|x)p(x) = p(x|y)p(y)$$

- Rearranging:

$$p(y|x) = \frac{p(x|y)p(y)}{p(x)} = \frac{p(x|y)p(y)}{\int p(x,y)dy} = \frac{p(x|y)p(y)}{\int p(x|y)p(y)dy}$$

Image source: "Pattern Recognition and Machine Learning", Christopher Bishop

Probability: Bayes' Rule

Terminology:

Likelihood – propensity for observing a certain value of X given a certain value of Y

Prior – what we know about Y before seeing X

$$p(y|x) = \frac{p(x|y)p(y)}{\int p(x|y)p(y)dy}$$

Posterior – what we know about Y after observing X

Evidence –a constant to ensure that the left hand side is a valid distribution

Probability: Example

Let random variables B and F represent the box color and type of fruit respectively, where $Val(B) = \{r, b\}$ and $Val(F) = \{a, o\}$.

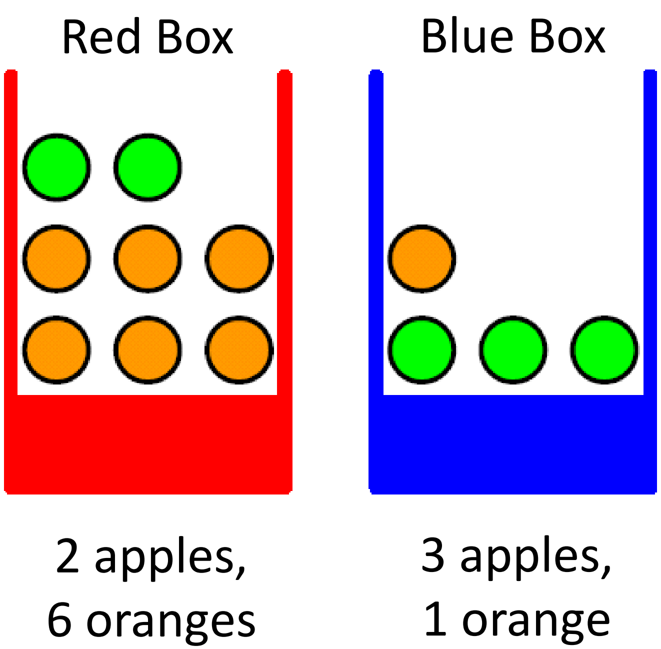
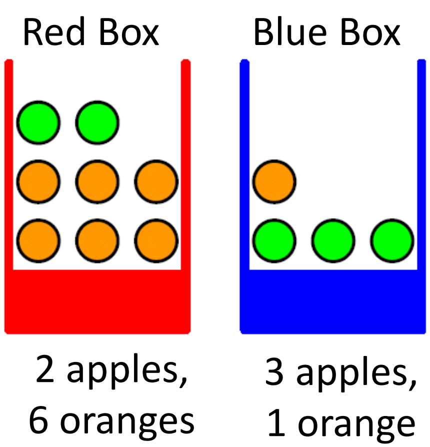


Image source: “Pattern Recognition and Machine Learning”, Christopher Bishop

Probability: Example

Given:



- Probabilities of selecting either the red or the blue boxes,

$$p(B = r) = 0.4$$
$$p(B = b) = 0.6$$

- Conditional probabilities for the type of fruit, given the selected box,

$$p(F = a|B = r) = 0.25$$
$$p(F = o|B = r) = 0.75$$
$$p(F = a|B = b) = 0.75$$
$$p(F = o|B = b) = 0.25$$

Image source: “Pattern Recognition and Machine Learning”, Christopher Bishop

Probability: Example

Find:

a) The overall probability of choosing an apple.

b) Identify the color of the box if we observed that an orange has been selected.

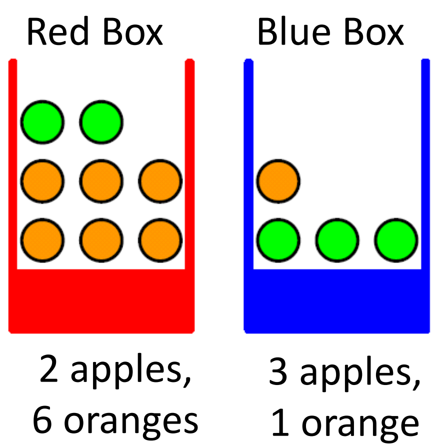
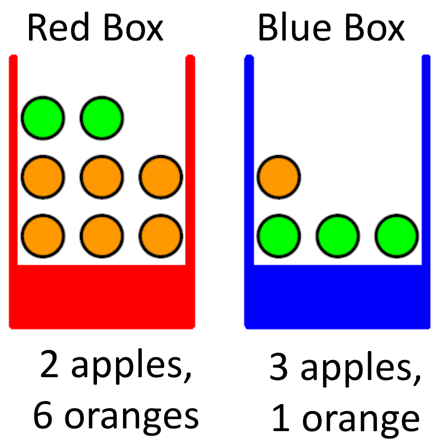


Image source: “Pattern Recognition and Machine Learning”, Christopher Bishop

Probability: Example

Solution:

a) The overall probability of choosing an apple.



Using the sum and product rules of probability:

$$p(F = a) = \sum_B p(F = a|B)p(B)$$

$$= p(F = a|B = r)p(B = r) + p(F = a|B = b)p(B = b)$$

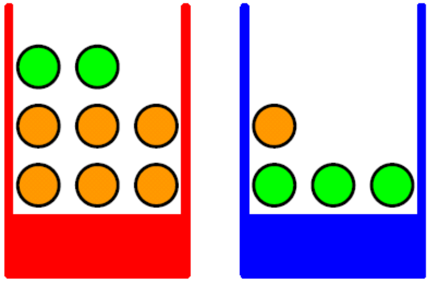
$$= (0.25)(0.4) + (0.75)(0.6) = 0.55$$

Image source: "Pattern Recognition and Machine Learning", Christopher Bishop

Probability: Example

Solution:

b) Identify the color of the box if we observed that an orange has been selected.



Using Bayes' theorem:

$$p(B=r|F=o) = \frac{p(F=o|B=r)p(B=r)}{p(F=o)}$$

$$= \frac{p(F=o|B=r)p(B=r)}{1-p(F=a)} = \frac{(0.75)(0.4)}{1-0.55}$$

$$= 0.667$$

$$p(B=b|F=o) = 1 - p(B=r|F=o) = 1 - 0.667 = 0.333$$

The orange is more likely to be selected from the red box!

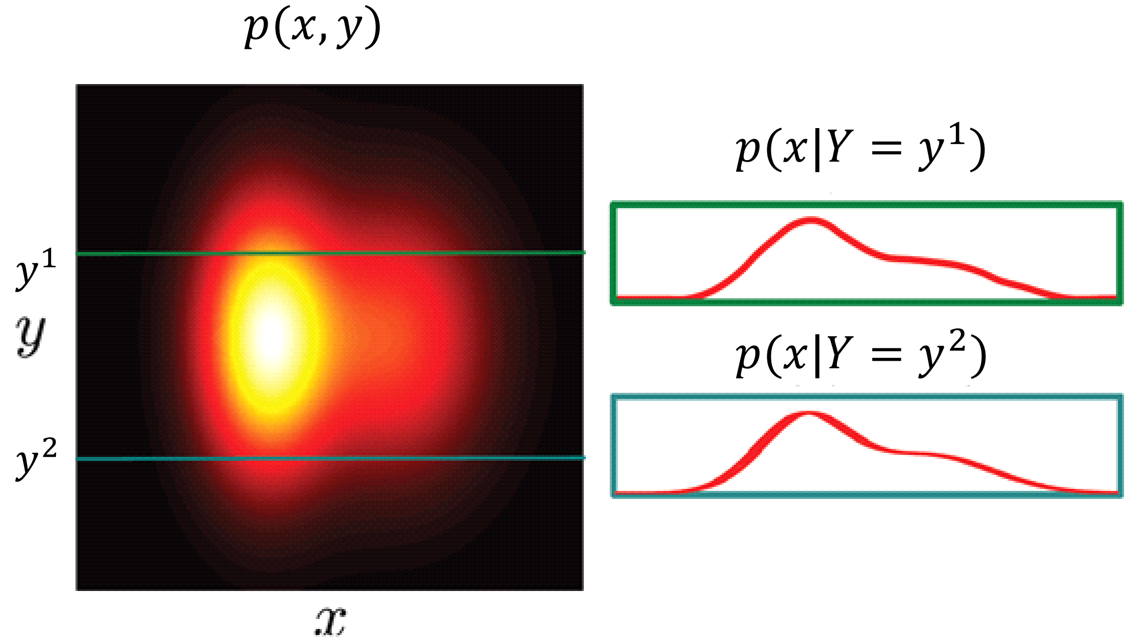
Image source: "Pattern Recognition and Machine Learning", Christopher Bishop

Probability: Independence

- The independence of X and Y means that every conditional distribution is the same.
- The value of Y tells us nothing about X and vice-versa.

$$p(x|y) = p(x)$$

$$p(y|x) = p(y)$$



Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

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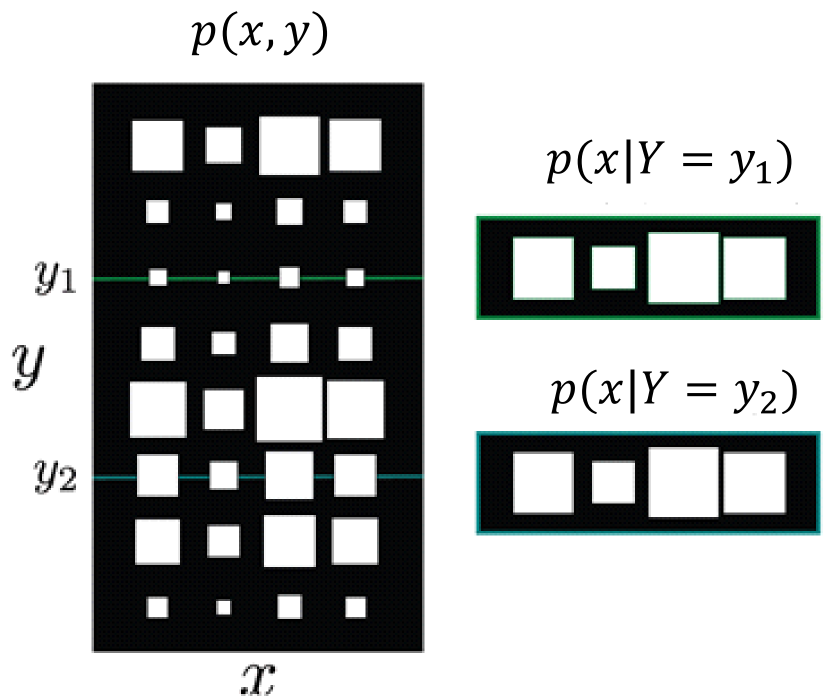

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Probability: Independence

- When variables are independent, the joint factorizes into a product of the marginals:

$$p(x, y) = p(x|y)p(y)$$
$$= p(x)p(y)$$

Probability: Expectation

- The expected or average value of some function $f[x]$ taking into account the distribution of X.

Definition:

$$E\big[f[x]\big] = \sum_{x} f[x]p(x)$$

$$E\big[f[x]\big] = \int f[x]p(x)dx$$

Probability: Rules of Expectation

- **Rule 1**: Expected value of a constant is the constant.

$$E[\kappa] = \kappa$$

- **Rule 2**: Expected value of constant times function is constant times expected value of function.

$$E\left[\kappa f[x]\right] = \kappa E[f[x]]$$

Probability: Rules of Expectation

- **Rule 3**: Expectation of sum of functions is sum of expectation of functions.

$$E\big[f[x] + g[x]\big] = E\big[f[x]\big] + E[g[x]]$$

- **Rule 4**: Expectation of product of functions in variables X and Y is product of expectations of functions if X and Y are independent.

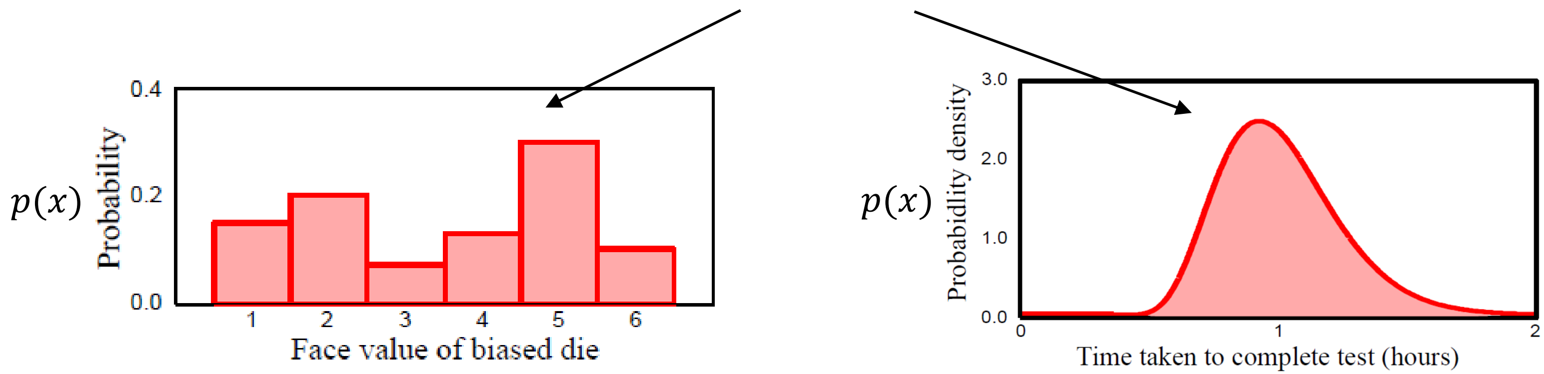
$$E\big[f[x]g[y]\big] = E\big[f[x]\big]E[g[y]],$$

if X and Y are independent

Probability Distributions

- We have seen the definitions of random variables, probability, and rules for manipulating probabilities.
- One question that remains unanswered is: "How do we assign the values of $p(x)$?"

How to get $p(x)$?



$val(X) = \{1,2,3,4,5,6\}$ $val(x) = [0,2]$

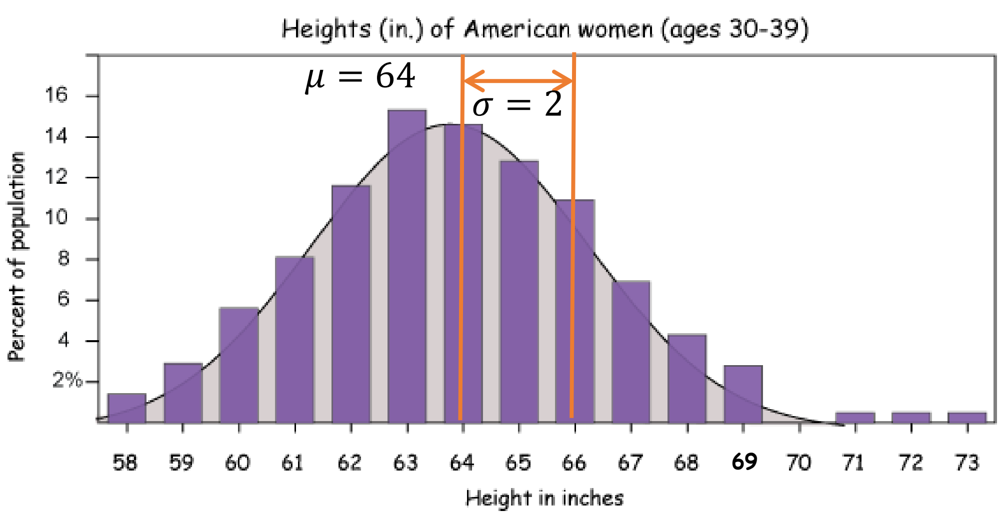
Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

Probability Distributions

Q: "How do we assign the probability values?"

A: Use probability distributions defined over some parameters learned from data!

Example:



Fitting a Normal distribution to the heights of a population:

$$p(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{(x-\mu)^2}{2\sigma^2}$$

Parameters: mean $\mu = 64$, variance $\sigma^2 = 4$ are learned from data.

Image source: http://www.drcruzan.com/ProbStat_Distributions.html

Common Probability Distributions

- The choice of distribution depends on the type/domain of data to be modeled.

Data Type	Domain	Distribution
univariate, discrete, binary	$x \in \{0, 1\}$	Bernoulli
univariate, discrete, multi-valued	$x \in \{1, 2, \ldots, K\}$	categorical
univariate, continuous, unbounded	$x \in \mathbb{R}$	univariate normal
univariate, continuous, bounded	$x \in [0, 1]$	beta
multivariate, continuous, unbounded	$\mathbf{x} \in \mathbb{R}^K$	multivariate normal
multivariate, continuous, bounded, sums to one	$\mathbf{x} = [x_1, x_2, \ldots, x_K]^T$ $x_k \in [0, 1], \sum_{k=1}^{K} x_k = 1$	Dirichlet
bivariate, continuous, x_1 unbounded, x_2 bounded below	$\mathbf{x} = [x_1, x_2]$ $x_1 \in \mathbb{R}$ $x_2 \in \mathbb{R}^+$	normal-scaled inverse gamma
multivariate vector $\mathbf{x}$ and matrix $\mathbf{X}$, $\mathbf{x}$ unbounded, $\mathbf{X}$ square, positive definite	$\mathbf{x} \in \mathbb{R}^K$ $\mathbf{X} \in \mathbb{R}^{K \times K}$ $\mathbf{z}^T \mathbf{X} \mathbf{z} > 0 \quad \forall\, \mathbf{z} \in \mathbb{R}^K$	normal inverse Wishart

Bernoulli Distribution

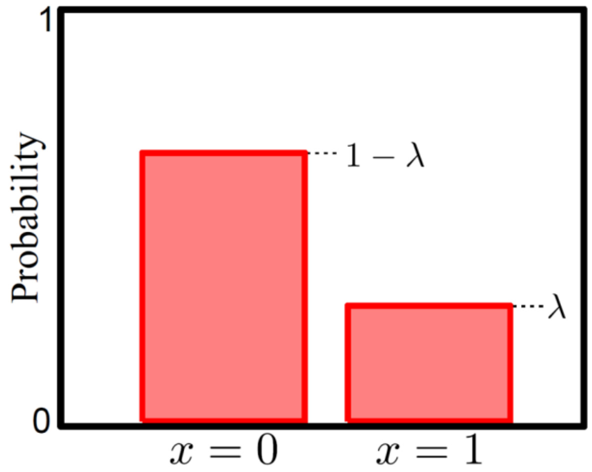
- Single binary random variable X, i.e. $x \in \{0,1\}$
- A single parameter $\lambda \in [0,1]$.

$$\begin{aligned} p(X=0 \mid \lambda) &= 1-\lambda \\ p(X=1 \mid \lambda) &= \lambda \end{aligned}$$

Or

$$p(x) = \lambda^x (1-\lambda)^{1-x},$$

$$p(x) = \mathrm{Bern}_x[\lambda]$$



Example:
X is the outcome of flipping a coin, $X = 1$ represents 'heads', and $X = 0$ represents 'tails'.

Images source: "Pattern Recognition and Machine Learning", Christopher Bishop
"Computer Vision: Models, Learning, and Inference", Simon Prince

Categorical Distribution

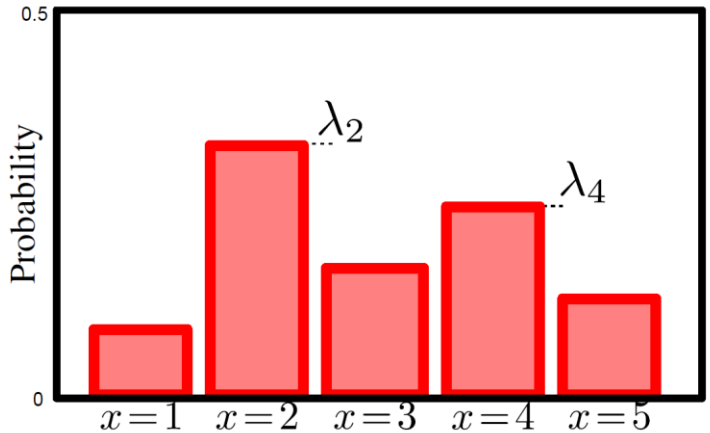
- Discrete variables $\boldsymbol{X}$ that take on 1-of-*K* possible mutually exclusive states, e.g. a *K*-faced die.
- $\boldsymbol{x}$ is represented by a *K*-dimensional vector $\mathbf{e}_k$ in which one of the elements $x_k = 1$, and $\sum_{k=1}^{K} x_k = 1$.
- e.g. $K = 5$, and $\boldsymbol{x} = \mathbf{e}_3 = [0,0,1,0,0]^\top$.
- *K* parameters $\lambda = [\lambda_1, \dots, \lambda_K]^\top$, where $\lambda \geq 0$, $\sum_k \lambda_k = 1$.

$$p(\boldsymbol{X} = \mathbf{e}_k \mid \lambda) = \lambda_k$$

Or

$$p(\boldsymbol{x}) = \prod_{k=1}^{K} \lambda_k^{x_k} = \lambda_k,$$

$$p(\boldsymbol{x}) = \mathrm{Cat}_x[\lambda]$$



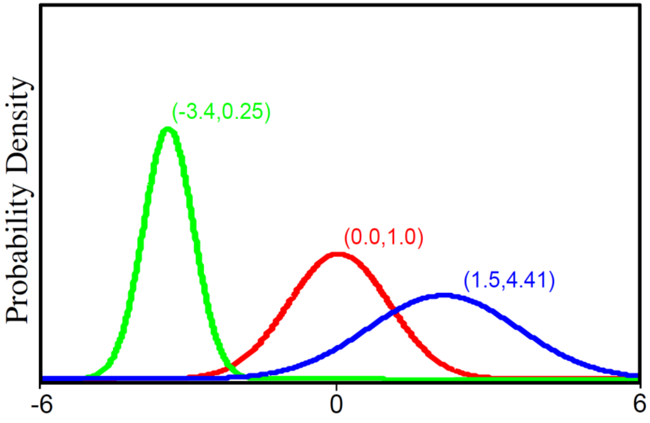
Images Source: “Computer Vision: Models, Learning, and Inference”, Simon Prince

Univariate Normal Distribution

- Also known as the Gaussian distribution.
- Univariate normal distribution describes single continuous variable X, i.e. $x \in \mathbb{R}$.
- Two parameters $\mu \in \mathbb{R}$ (mean) and $\sigma^2 > 0$ (variance).

Carl Friedrich Gauss
1777–1855

$$p(X = a \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{(a-\mu)^2}{2\sigma^2}, \quad a \in \mathbb{R}$$



Or

$$\boxed{\begin{aligned} p(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp - \frac{(x-\mu)^2}{2\sigma^2} \\ p(x) &= \text{Norm}_x[\mu, \sigma^2] \end{aligned}}$$

Image sources: “Pattern Recognition and Machine Learning”, Christopher Bishop
“Computer Vision: Models, Learning, and Inference”, Simon Prince

Multivariate Normal Distribution

- Multivariate normal distribution describes a *D*-dimensional continuous variable $\boldsymbol{X}$, i.e. $\boldsymbol{x} \in \mathbb{R}^D$.
- *D*-dimensional mean $\boldsymbol{\mu} \in \mathbb{R}^D$, and $D \times D$ symmetrical positive definite covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}_+^{D \times D}$.

$$p(\boldsymbol{X} = \boldsymbol{a} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\{ -0.5(\boldsymbol{a} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{a} - \boldsymbol{\mu}) \}, \quad \boldsymbol{a} \in \mathbb{R}^D$$

Or

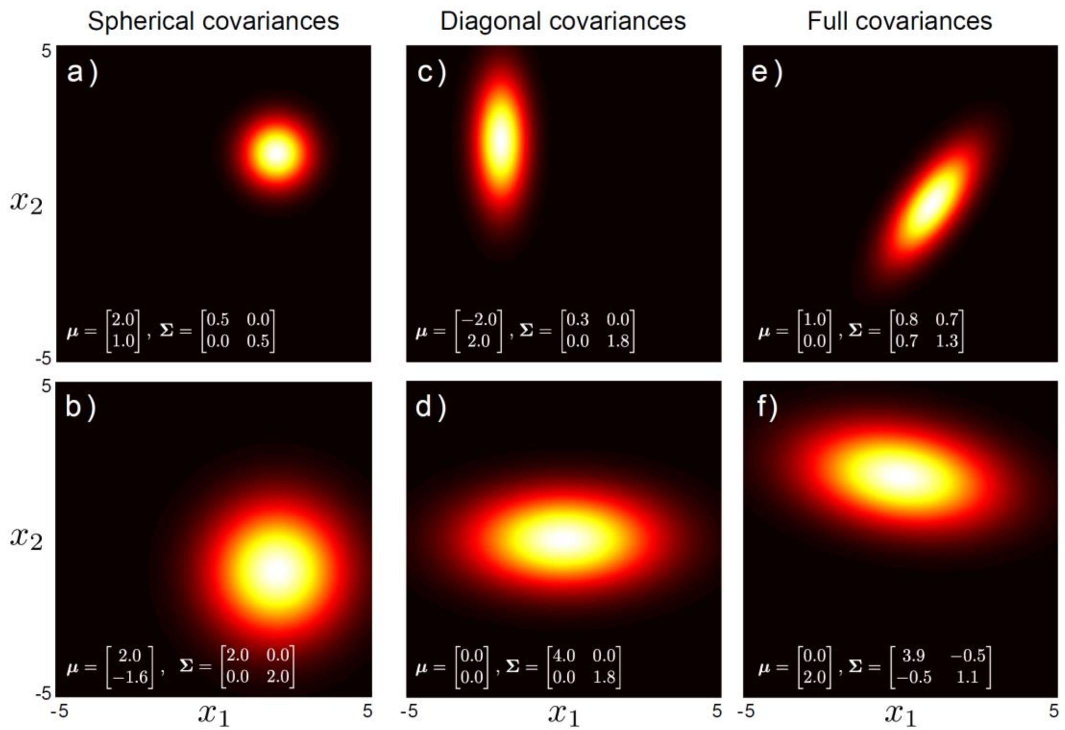
$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\{ -0.5(\boldsymbol{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}) \}$$

$$p(\boldsymbol{x}) = \mathrm{Norm}_{\boldsymbol{x}}[\boldsymbol{\mu}, \boldsymbol{\Sigma}]$$

Types of Covariance

- Covariance matrix has three forms: spherical, diagonal and full.

$$\boldsymbol{\Sigma}_{spher} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \quad \boldsymbol{\Sigma}_{diag} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \quad \boldsymbol{\Sigma}_{full} = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{bmatrix}$$



Images Source: “Computer Vision: Models, Learning, and Inference”, Simon Prince

Conjugate Distributions

- Conjugate distributions model the parameters of the probability distributions.
- Product of a probability distribution and its conjugate has the same form as the conjugate times a constant.
- Parameters of conjugate distributions are known as hyperparameters because they control the parameter distributions.

Distribution	**Domain**	**Parameters modeled by**
Bernoulli	$x \in \{0, 1\}$	beta
categorical	$x \in \{1, 2, \ldots, K\}$	Dirichlet
univariate normal	$x \in \mathbb{R}$	normal inverse gamma
multivariate normal	$\mathbf{x} \in \mathbb{R}^k$	normal inverse Wishart

Importance of Conjugate Distributions

1. Learning the parameters θ of a probability distribution:

Recall the Bayes' Rule:

1. Choose prior that is conjugate to likelihood

$$p(\theta|x) = \frac{p(x|\theta)p(\theta)}{\int p(x|\theta)p(\theta)d\theta}$$

2. Implies that posterior must have same form as conjugate prior distribution, i.e. closed-form.

3. Posterior must be a distribution which implies that evidence must equal constant κ from conjugate relation.

Importance of Conjugate Distributions

2. Marginalizing over parameters:

$$p(x^*|\boldsymbol{x}) = \int p(x^*|\theta\ )p(\theta|\boldsymbol{x})d\theta$$

ii. Integral becomes easy --the product becomes a constant times a distribution.

i. Chosen as conjugate to other term.

Integral of constant times probability distribution
= constant times integral of <u>probability distribution</u>
= constant x <u>1</u> = constant

Importance of Conjugate Distributions

Proof:

$$p(x^*|x) = \frac{p(x^*, x)}{p(x)}$$ (Conditional probability)

$$= \frac{\int p(x^*, x, \theta) d\theta}{p(x)}$$ (Marginal probability)

$$= \frac{\int p(x^*, \theta | x) \cancel{p(x)} d\theta}{\cancel{p(x)}}$$ (Conditional probability)

$$= \int p(x^* | x, \theta) p(\theta | x) d\theta$$ (Conditional probability)

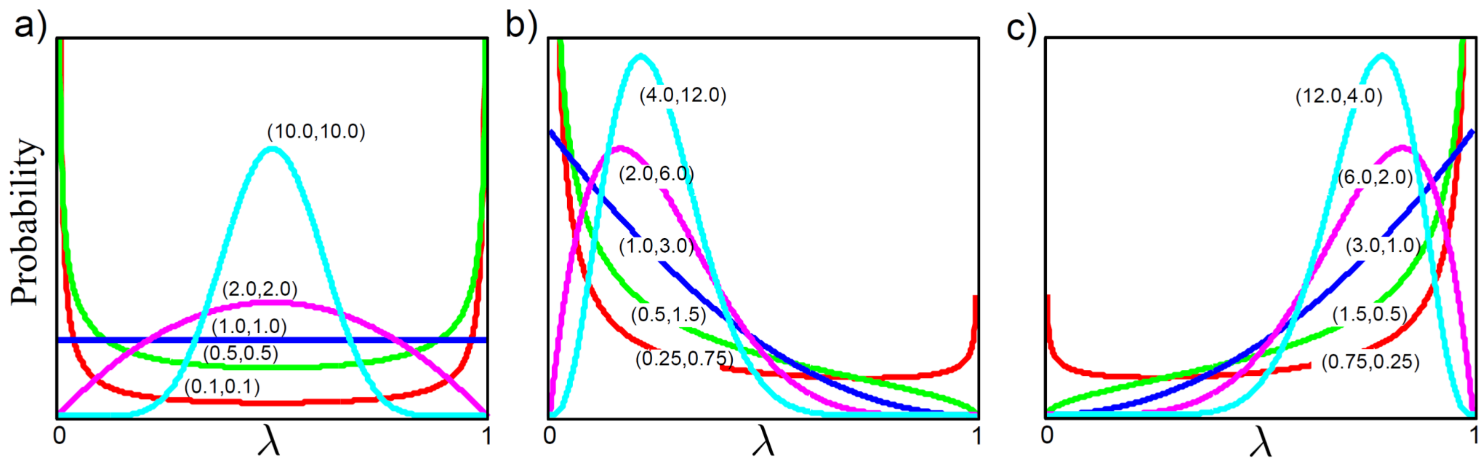
$$= \int p(x^* | \theta) p(\theta | x) d\theta$$ (Conditional Independence)

Conjugate Distribution: Beta Distribution

- Conjugate distribution of Bernoulli distribution.
- Defined over parameter of the Bernoulli distribution $\lambda \in [0,1]$.

$$p(\lambda) = \frac{\Gamma[\alpha+\beta]}{\Gamma[\alpha]\Gamma[\beta]} \lambda^{\alpha-1}(1-\lambda)^{\beta-1}$$

$$p(\lambda) = \text{Beta}_{\lambda}[\alpha, \beta]$$



Images Source: “Computer Vision: Models, Learning, and Inference”, Simon Prince

Conjugate Distribution: Beta Distribution

$$p(\lambda) = \frac{\Gamma[\alpha+\beta]}{\Gamma[\alpha]\Gamma[\beta]}\lambda^{\alpha-1}(1-\lambda)^{\beta-1}$$

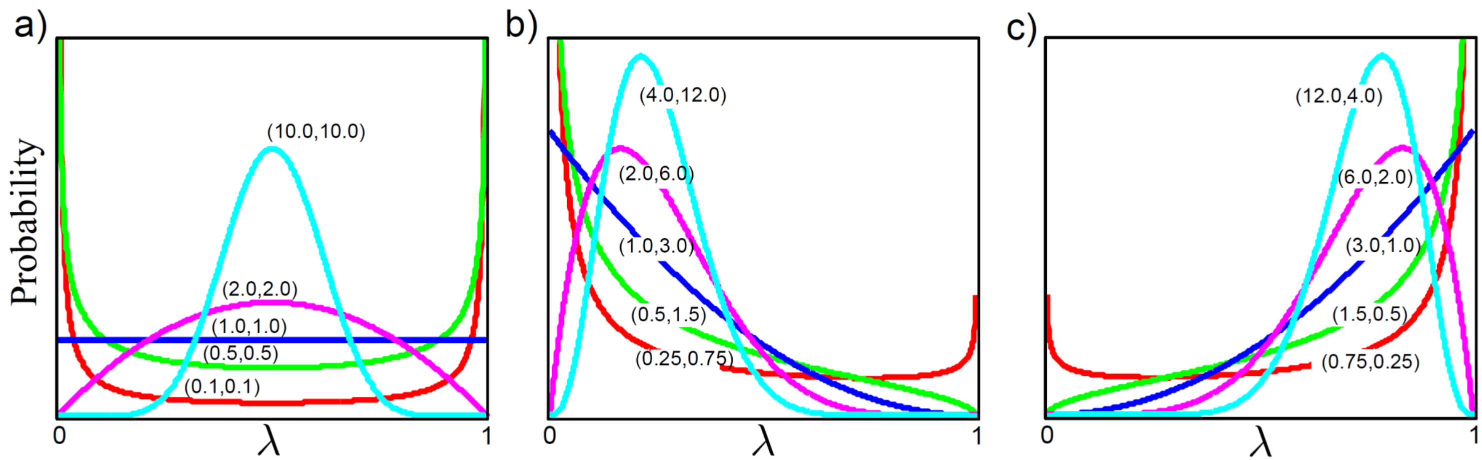
$$p(\lambda) = \text{Beta}_\lambda[\alpha,\beta]$$

Gamma Function:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt, \qquad z \in \mathbb{C}$$

$$\Gamma(n) = (n-1)!\,, \qquad n \in \mathbb{R}_{>0}$$

- Two hyperparameters $\alpha, \beta > 0$.



Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

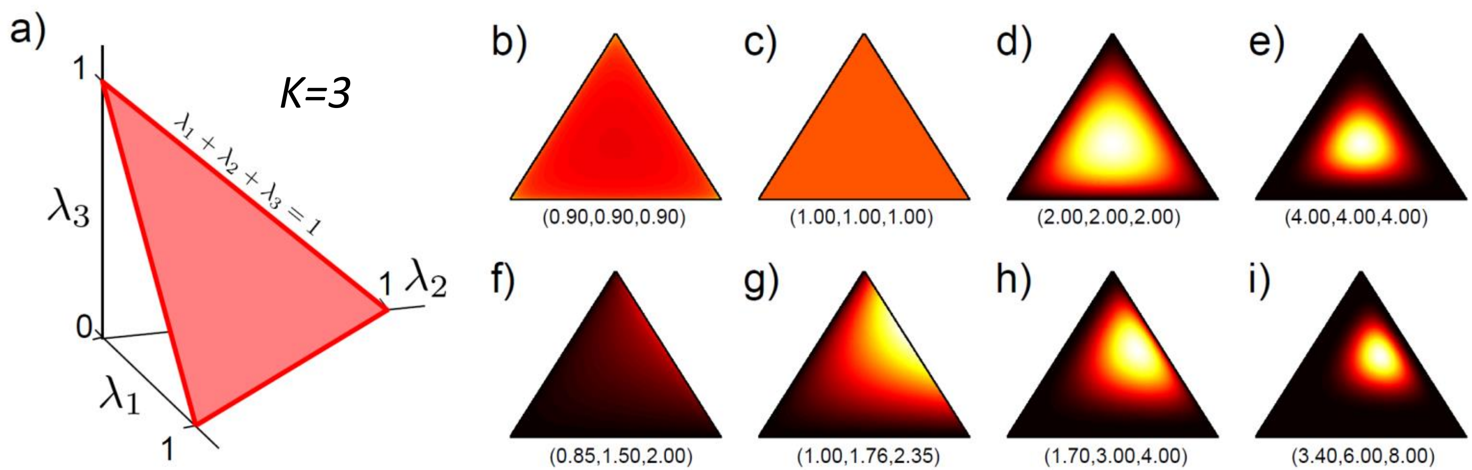
Conjugate Distribution: Dirichlet Distribution

- Conjugate distribution of categorical distribution.
- Defined over K parameters of Categorical distribution, $\lambda_k \in [0,1]$, where $\sum_k \lambda_k = 1$.

$$p(\lambda_1, \dots, \lambda_K) = \frac{\Gamma[\sum_{k=1}^{K} \alpha_k]}{\prod_{k=1}^{K} \Gamma[\alpha_k]} \prod_{k=1}^{K} \lambda_k^{\alpha_k - 1},$$

$$p(\lambda_1, \dots, \lambda_K) = \mathrm{Dir}_{\lambda_{1\dots K}}[\alpha_1, \dots \alpha_K]$$

Peter Gustav Lejeune Dirichlet (1805-1859)



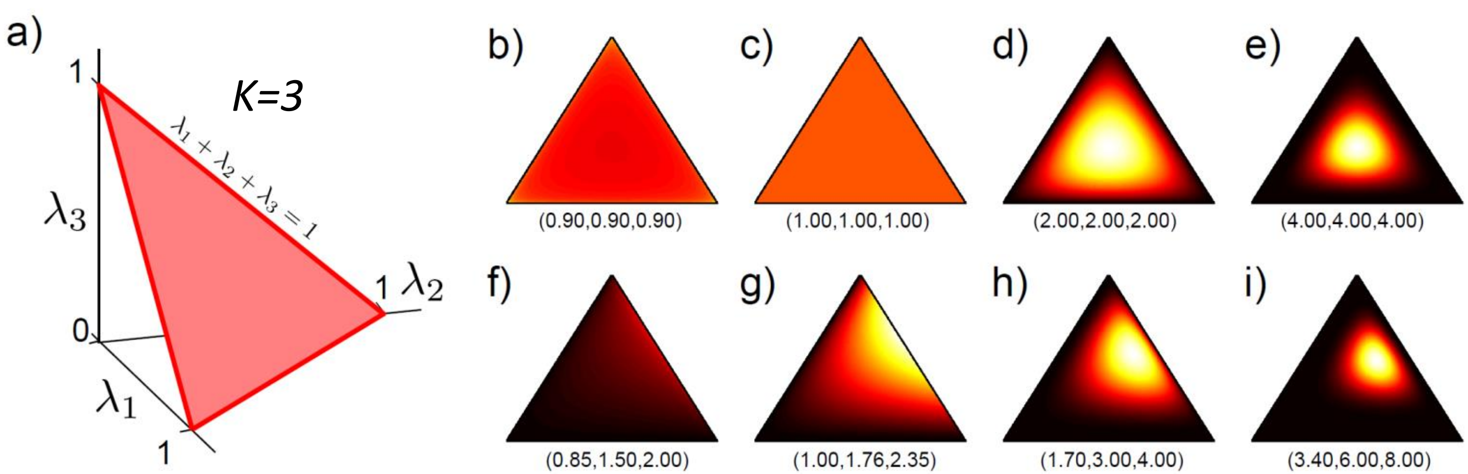
Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince
http://www.amt.edu.au/biogdirichlet.html

Conjugate Distribution: Dirichlet Distribution

$$p(\lambda_1, \dots, \lambda_K) = \frac{\Gamma[\sum_{k=1}^{K} \alpha_k]}{\prod_{k=1}^{K} \Gamma[\alpha_k]} \prod_{k=1}^{K} \lambda_k^{\alpha_k - 1},$$

$$p(\lambda_1, \dots, \lambda_K) = \mathrm{Dir}_{\lambda_{1\dots K}}[\alpha_1, \dots \alpha_K]$$

- *K* hyperparameters $\alpha_k > 0$.



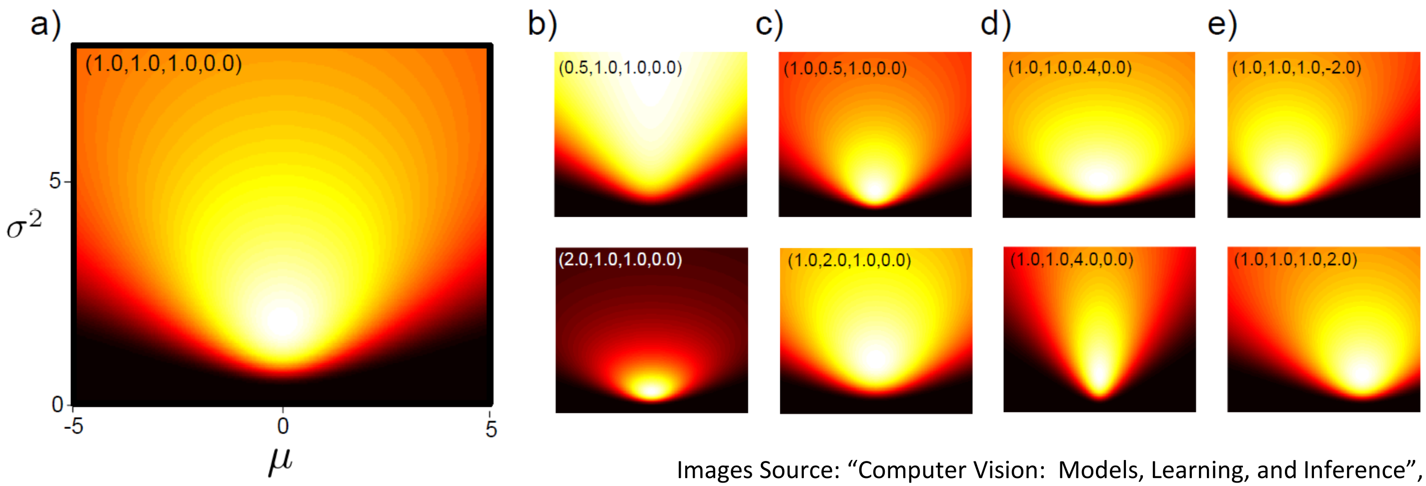
Images Source: "Computer Vision: Models, Learning, and Inference", Simon Prince

Conjugate Distribution: Normal Inverse Gamma Distribution

- Conjugate distribution of univariate normal distribution.
- Defined on parameters $\mu, \sigma^2 > 0$ of univariate normal distribution.

$$p(\mu, \sigma^2) = \frac{\sqrt{\gamma}}{\sigma\sqrt{2\pi}} \frac{\beta^\alpha}{\Gamma[\alpha]} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} \exp\left[-\frac{2\beta + \gamma(\delta - \mu)^2}{2\sigma^2}\right]$$

$$p(\mu, \sigma^2) = \mathrm{NormInvGam}_{\mu,\sigma^2}[\alpha, \beta, \gamma, \delta]$$



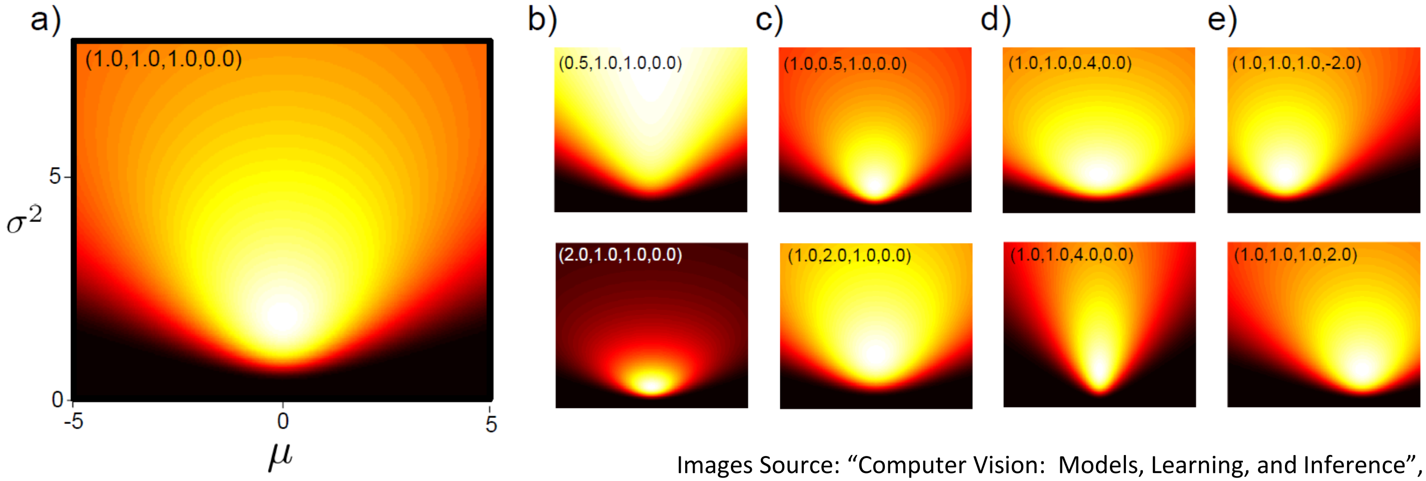
Images Source: “Computer Vision: Models, Learning, and Inference”, Simon Prince

Conjugate Distribution: Normal Inverse Gamma Distribution

$$p(\mu,\sigma^2) = \frac{\sqrt{\gamma}}{\sigma\sqrt{2\pi}} \frac{\beta^\alpha}{\Gamma[\alpha]} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} \exp\left[-\frac{2\beta+\gamma(\delta-\mu)^2}{2\sigma^2}\right]$$

$$p(\mu,\sigma^2) = \text{NormInvGam}_{\mu,\sigma^2}[\alpha,\beta,\gamma,\delta]$$

- Four hyperparameters $\alpha, \beta, \gamma > 0$ and $\delta \in \mathbb{R}$.



Images Source: “Computer Vision: Models, Learning, and Inference”, Simon Prince

Conjugate Distribution: Normal Inverse Wishart

John Wishart (1898-1956)

- Conjugate distribution of multivariate normal distribution.
- Defined on parameters $\boldsymbol{\mu}, \boldsymbol{\Sigma}$ of multivariate normal distribution.

$$p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{\gamma^{D/2}|\boldsymbol{\Psi}|^{\alpha/2}\exp[-0.5\left(\mathrm{Tr}\left[\boldsymbol{\Psi}\boldsymbol{\Sigma}^{-1}\right] + \gamma(\boldsymbol{\mu}-\boldsymbol{\delta})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\boldsymbol{\delta})\right)]}{2^{\alpha D/2}(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{(\alpha+D+2)/2}\Gamma_D[\alpha/2]}$$

$$p(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \mathrm{NorIWis}_{\boldsymbol{\mu},\boldsymbol{\Sigma}}[\alpha, \boldsymbol{\Psi}, \gamma, \boldsymbol{\delta}]$$

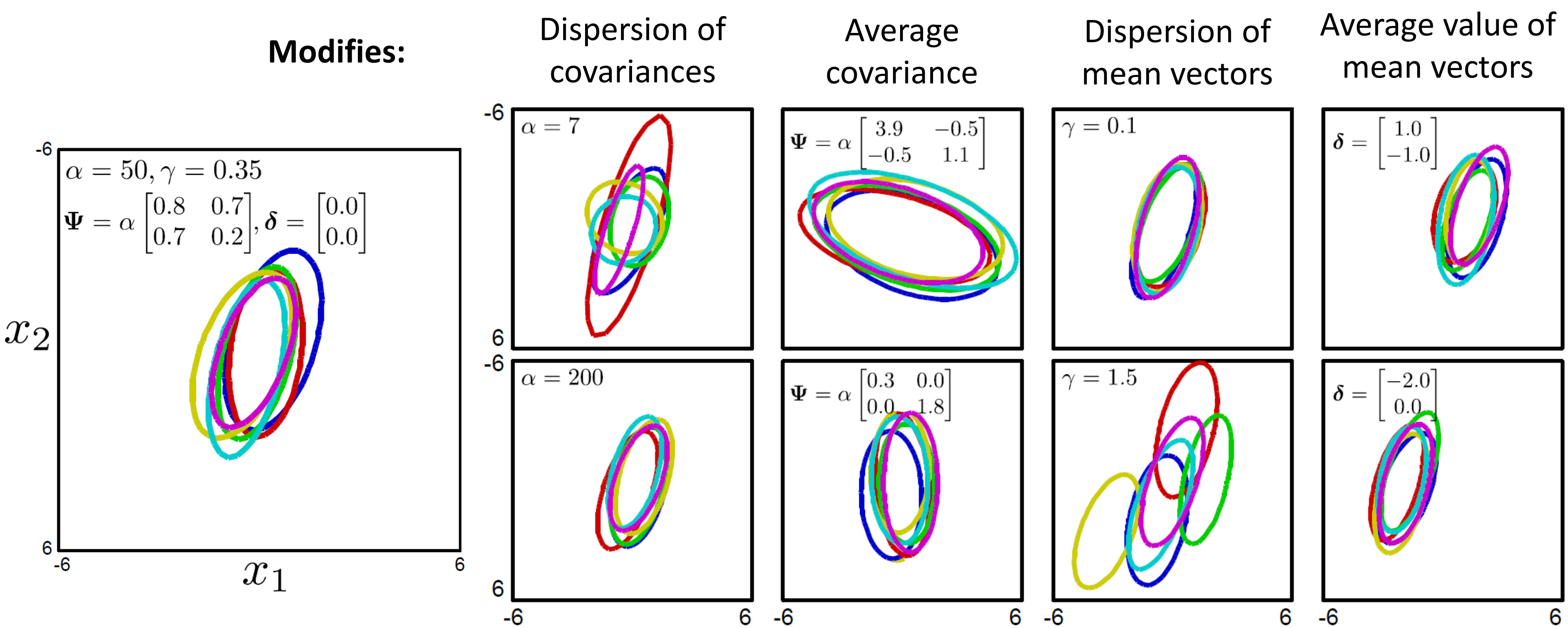
- Four hyperparameters: a positive scalar α, a positive definite matrix $\boldsymbol{\Psi} \in \mathbb{R}_+^{D\times D}$, a positive scalar γ, and a vector $\boldsymbol{\delta} \in \mathbb{R}^D$.

Multivariate gamma function:

$$\Gamma_D[a] = \pi^{a(a-1)/4} \prod_{j=1}^{a} \Gamma[a + (1-j)/2]$$

Conjugate Distribution: Normal Inverse Wishart

- Samples from Normal Inverse Wishart:



Images Source: “Computer Vision: Models, Learning, and Inference”, Simon Prince

Example 1: Conjugate Distribution

Find: The posterior distribution of the parameter (μ, σ) from a univariate Gaussian distribution.

Solution: Using Bayes' rule:

$$p(\theta|x) = \frac{\prod_{i=1}^{N} p(x[i] \mid \theta)p(\theta)}{p(x)} = \frac{\prod_{i=1}^{N} p(x[i] \mid \theta)p(\theta)}{\int \prod_{i=1}^{N} p(x[i] \mid \theta)p(\theta)\, d\theta}$$

where:

$$\prod_{i=1}^{N} p(x[i] \mid \theta)p(\theta) = \prod_{i=1}^{N} \mathrm{Norm}_{x[i]}[\mu, \sigma^2]\, \mathrm{NormInvGam}_{\mu,\sigma^2}[\alpha, \beta, \gamma, \delta]$$

Example 1: Conjugate Distribution

We have

$$\prod_{i=1}^{N} p(x[i]|\theta)p(\theta) = \prod_{i=1}^{N} \mathrm{Norm}_{x[i]}[\mu,\sigma^2]\,\mathrm{NormInvGam}_{\mu,\sigma^2}[\alpha,\beta,\gamma,\delta]$$

Rearranging:

$$\prod_{i=1}^{N} p(x[i]|\theta)p(\theta) = \underbrace{\kappa[\alpha,\beta,\gamma,\delta,x]}_{\text{Constant}}\mathrm{NormInvGam}_{\mu,\sigma^2}[\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}]$$

where

$$\tilde{\alpha} = \alpha + \frac{N}{2}, \qquad \tilde{\delta} = \frac{(\gamma\delta + \sum_i x[i])}{\gamma + N},$$

$$\tilde{\gamma} = \gamma + N, \qquad \tilde{\beta} = \frac{\sum_i x[i]^2}{2} + \beta + \frac{\gamma\delta^2}{2} - \frac{(\gamma\delta + \sum_i x[i])^2}{2(\gamma + N)}.$$

Example 1: Conjugate Distribution

Putting into the Bayes' rule, we get:

$$p(\theta|x) = \frac{\prod_{i=1}^{N} p(x[i]|\theta)p(\theta)}{p(x)} = \frac{\prod_{i=1}^{N} p(x[i]|\theta)p(\theta)}{\int \prod_{i=1}^{N} p(x[i]|\theta)p(\theta)\, d\theta}$$

$$p(\theta|x) = \frac{\cancel{\kappa[\alpha,\beta,\gamma,\delta,x]}\text{NormInvGam}_{\mu,\sigma^2}[\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}]}{\cancel{\kappa[\alpha,\beta,\gamma,\delta,x]}\underbrace{\int\int \text{NormInvGam}_{\mu,\sigma^2}[\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}]d\mu d\sigma^2}_{=1}}$$

$$\boxed{p(\theta|x) = \text{NormInvGam}_{\mu,\sigma^2}[\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}]}$$

Same form as conjugate prior,
i.e., Normalized Inverse Gamma!

Example 2: Conjugate Distribution

Find: $p(x^*|\boldsymbol{x})$ for a univariate Gaussian.

Solution:

$$p(x^*|x) = \int\int p(x^*|\mu,\sigma^2)p(\mu,\sigma^2|x)d\mu d\sigma^2$$

$$= \int\int \mathrm{Norm}_{x^*}[\mu,\sigma^2]\mathrm{NormInvGam}_{\mu,\sigma^2}[\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta}]d\mu d\sigma^2$$

$$= \kappa[\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta},x^*]\underbrace{\int\int \mathrm{NormInvGam}_{\mu,\sigma^2}[\breve{\alpha},\breve{\beta},\breve{\gamma},\breve{\delta}]d\mu d\sigma^2}_{=1}$$

$$= \kappa[\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta},x^*]$$

This is a constant (function of $\{x^*, x[1], \dots, x[N]\}$)!

Example 2: Conjugate Distribution

This is a constant (function of $\{x^*, x[1], \ldots, x[N]\}$)!

$$p(x^*|x) = \kappa[\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, x^*] = \frac{1}{\sqrt{2\pi}} \frac{\tilde{\beta}\tilde{\alpha}\sqrt{\tilde{\gamma}}}{\breve{\beta}\breve{\alpha}\sqrt{\breve{\gamma}}} \frac{\Gamma[\breve{\alpha}]}{\Gamma[\tilde{\alpha}]}$$

where

$$\breve{\alpha} = \tilde{\alpha} + 1/2, \qquad \breve{\gamma} = \tilde{\gamma} + 1$$

$$\breve{\beta} = \frac{x^{*2}}{2} + \tilde{\beta} + \frac{\tilde{\gamma}\tilde{\delta}^2}{2} - \frac{(\tilde{\gamma}\tilde{\delta} + x^*)^2}{2(\tilde{\gamma} + 1)}.$$

Summary

You have learned how to:

1. Describe uncertain quantities with random variables and joint probabilities.

2. Explain the basic rules of probability – sum, product, Bayes', independence and expectation rules.

3. Use the common probabilities distributions – Bernoulli, categoricial, univariate and multivariate normal distributions.

4. Explain the use of conjugate distributions.