CS5562: Trustworthy Machine Learning

Part II Lecture 4: Differentially Private Learning

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[a]Acknowledgment. The wonderful teaching assistants: Hongyan Chang, Martin Strobel, Jiashu Tao, Yao Tong, Jiayuan Ye

Contents

Differential Privacy

Differential Privacy

- Consider $x = \langle x_1, x_2, \cdots, x_i, \cdots x_n \rangle$
- Consider a neighboring dataset $x' = \langle x_1, x_2, \cdots, \cancel{x_i}, \cdots x_n \rangle$
- Definition: ϵ-DP

$$\forall y, x, x' : \qquad \ln\left(\frac{\Pr[Y = y | X = x]}{\Pr[Y = y | X = x']}\right) \leq \epsilon$$

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(b) Small ε

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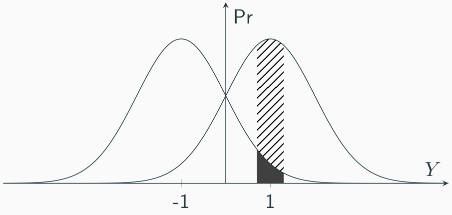

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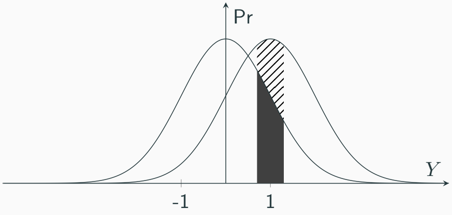

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An Approximate Notion of Differential Privacy

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- Consider a neighboring dataset $x' = \langle x_1, x_2, \cdots, \cancel{x_i}, \cdots x_n \rangle$
- Definition: (ϵ, δ)-DP

$$\forall x, x' : \qquad \Pr\left[\overbrace{\ln(\frac{\Pr[Y = y | X = x]}{\Pr[Y = y | X = x']}) > \epsilon}^{\text{violating } \epsilon\text{-DP}} \right] < \delta$$

where the randomness of probability is over output y drawn from the output distribution $Pr[Y | X = x]$

- The chance that we have unbounded privacy loss is very small (δ)

An Approximate Notion of Differential Privacy

$$\Pr[Y = y|X = x] \leq e^{\epsilon} \Pr[Y = y|X = x'] + \delta$$

Differentially Private Mechanisms

Example: Counting Queries

- Assume there is a sensitive dataset, and the analyst is interested in counting how many records in the dataset match a given predicate (the query)
- How much can a small modification in the dataset change the output?
- Definition: **Sensitivity** of a function $f : (x_1, \cdots, x_n) \mapsto (y_1, \cdots, y_k)$ with respect to a norm $\|\cdot\|$ is $\Delta f = \max_{\text{neighboring datasets } x, x'} \|f(x) - f(x')\|$
- Sensitivity of the counting function is 1
- How to randomize true counts to satisfy differential privacy?

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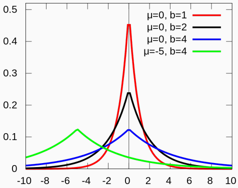
Laplace Mechanism



- Laplace distribution (centered at 0, with scale b):

$$Lap(z; b) = \frac{1}{2b} e^{\frac{-|z|}{b}}$$

- Laplace mechanism: $M(x, f, \epsilon) = f(x) + noise$, where coordinates of $noise \overset{\text{i.i.d}}{\sim} Lap(\Delta f / \epsilon)$
- $\Delta f = \max_{x,x'} \|f(x) - f(x')\|_1$, where x, x' are neighboring datasets

Laplace Mechanism is Differentially Private

- We prove for one-dimensional case, i.e. $f(x)$ is real number.

$$
\begin{aligned}
\frac{\Pr[M(x,f,\epsilon)=y]}{\Pr[M(x',f,\epsilon)=y]} &= \frac{e^{\frac{-|f(x)-y|}{\Delta f/\epsilon}}}{e^{\frac{-|f(x')-y|}{\Delta f/\epsilon}}} \\
&= e^{\frac{\epsilon}{\Delta f}(|f(x')-y|-|f(x)-y|)} \\
&\le e^{\frac{\epsilon}{\Delta f}(|f(x')-f(x)|)} \qquad \text{triangle inequality} \\
&\le e^{\epsilon} \qquad \text{sensitivity}
\end{aligned}
$$

Source: [Dwork and Roth, 2014]

Gaussian Mechanism

- Gaussian distribution (centered at 0, with standard deviation σ):

$$z \sim N(0, \sigma^2), \quad p(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(\frac{-z^2}{2\sigma^2})$$

- Gaussian mechanism:

$M(x, f, \epsilon, \delta) = f(x) + noise,$ where coordinates of $noise \overset{\text{i.i.d}}{\sim} N\left(0, \sigma^2\right)$

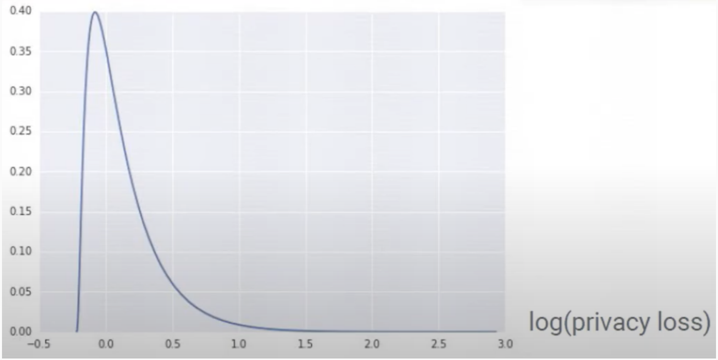
$$\text{for } \sigma = \frac{\Delta f}{\epsilon}\sqrt{2 \log \frac{5}{4\delta}}, \text{for } \epsilon \in (0, 1)$$

- $\Delta f = \max_{x,x'} \|f(x) - f(x')\|_2$, where x, x' are neighboring datasets

Source: [Dwork and Roth, 2014]

Privacy loss random variable has a long tail

- **privacy loss random variable** $L = \frac{\Pr[Y=y|X=x]}{\Pr[Y=y|X=x']}$, $y \sim M(x', f, \epsilon, \delta)$



Source: [Abadi et al., 2016]

How to bound the tail: moment method and Markov inequality

- We need tail bound $\Pr[L \geq e^{\epsilon}] < \delta$ for the random variable $L = \frac{\Pr[Y=y|X=x]}{\Pr[Y=y|X=x']}$, $y \sim M(x', f, \epsilon, \delta)$
- The λ-**th moment** $(\lambda \geq 0)$ of the random variable L : $\mathbb{E}\left[L^{\lambda}\right]$
  Example: the first order moment of random variable L is its mean
- The **Markov inequality** for non-negative random variable L:

$$\Pr[L \geq e^{\epsilon}] \leq \frac{\mathbb{E}[L^{\lambda}]}{e^{\lambda\epsilon}}.$$

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Source: [Abadi et al., 2016]

Gaussian Mechanism is (ϵ, δ)-differentially private

- Without loss of generality, let $f(x') = 0$ and $f(x) = f(x') - \Delta_f$.

$$L = \frac{\Pr[Y = y|X = x]}{\Pr[Y = y|X = x']} = \frac{e^{-\frac{(f(x)-y)^2}{2\sigma^2}}}{e^{-\frac{(f(x')-y)^2}{2\sigma^2}}}$$

$$= e^{-\frac{\Delta_f^2 + 2y\Delta_f}{2\sigma^2}} \text{ for } \sigma = \frac{\Delta f}{\epsilon}\sqrt{2\log\frac{5}{4\delta}}$$

- Compute moments $\mathbb{E}[L^\lambda]$ for $\lambda \geq 0$ and use Markov inequality

$$\Pr[L \geq e^\epsilon] \leq e^{-\lambda\epsilon}\mathbb{E}[L^\lambda] = e^{-\lambda\epsilon}\int e^{-\frac{\lambda\Delta_f^2 + \lambda\cdot 2y\Delta f}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{y^2}{2\sigma^2}} dy$$

$$= e^{-\lambda\epsilon - \frac{\lambda\Delta_f^2}{2\sigma^2} + \frac{\lambda^2\Delta_f^2}{2\sigma^2}} = e^{-\lambda\epsilon - \frac{\lambda\epsilon^2}{4\log 5/(4\delta)} + \frac{\lambda^2\epsilon^2}{4\log 5/(4\delta)}}$$

$$< \delta \quad \text{(by setting } \lambda = \frac{2\log(1/\delta)}{\epsilon}\text{)}$$

Source: [Abadi et al., 2016]

Comparison: Laplace mechanism and Gaussian mechanism

- Consider real-valued function f with $\Delta_f = 1$
- To ensure ε-DP for $\varepsilon = 1$, we need Laplace noise $noise_L \sim Lap(1)$.
- To ensure (ε, δ)-DP for $\varepsilon = 1$, we need Gaussian noise $noise_G \sim N(0, \sigma^2)$ with $\sigma = \sqrt{2 \log \frac{5}{4\delta}}$.

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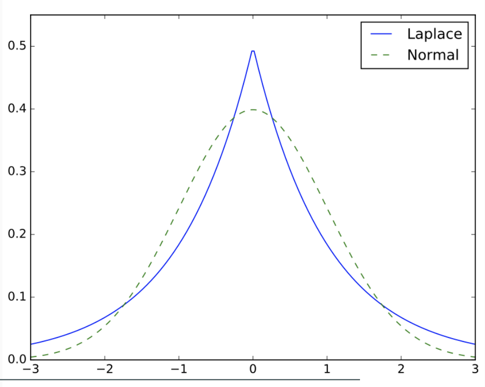
Comparison: Laplace mechanism and Gaussian mechanism

- Laplace noise $noise_L$ satisfies $p(|noise_L| = z) \propto e^{-|z|}$
- Gaussian noise $noise_G$ satisfies $p(|noise_G| = z) \propto e^{-\frac{z^2}{4 \log \frac{5}{4\delta}}}$
- As $z \to \infty$, we have $p(|noise_L| = z) \gg p(|noise_G| = z)$



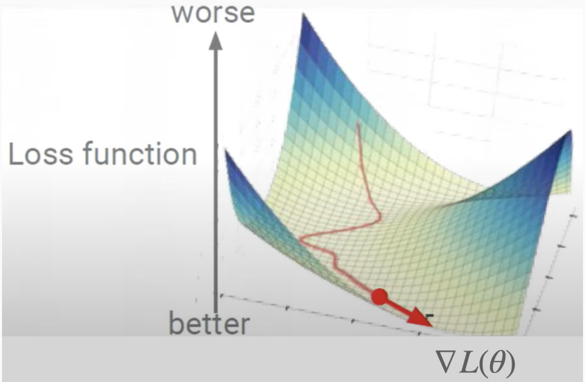
$\Rightarrow$ Laplace noise has a longer tail, thus tends to give larger error

Source: Blog by John D. Cook

Differentially Private SGD

Training a machine learning model with SGD

- How does SGD work? In iteration t of the algorithm, we
  - Choose a mini-batch B_t of the training data
  - Compute the average gradient $g = \frac{1}{|B_t|} \sum_{z \in B_t} \nabla L(\theta, z)$
  - Take a step (with stepsize η_t) in the opposite direction of the average gradient: $\theta \leftarrow \theta - \eta_t g$



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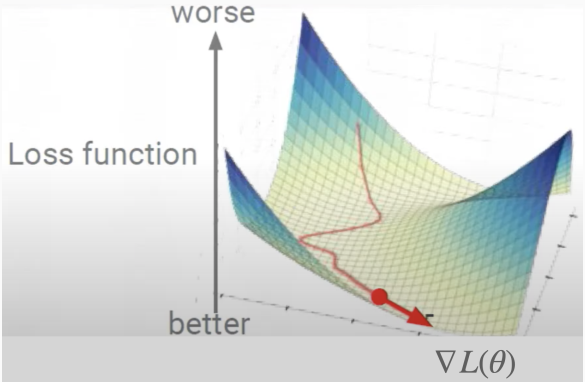

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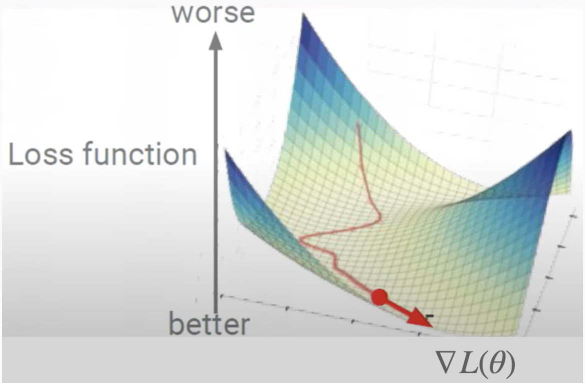

How can we design a differentially private SGD algorithm?

- What is the data dependent computation?
- What is its sensitivity?
- Which DP mechanism should we use?

Training a machine learning model with DP-SGD

- What is the data dependent computation? gradient $\nabla L(\theta, z)$
- What is its sensitivity? unbounded
  - Can we bound the sensitivity?
  - Use norm-bounding: Normalize the gradient vector to a given L2-norm C
  - This is an extremely bad way of bounding sensitivity, because it sets the sensitivity to the range of the function (but we don't know how to do better than this)
- Which DP mechanism should we use? Gaussian mechanism, as compared with the Laplace mechanism, we impose less error

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Training a machine learning model with DP-SGD

Input: Examples $\{x_1, \ldots, x_N\}$, loss function $\mathcal{L}(\theta) = \frac{1}{N}\sum_i \mathcal{L}(\theta, x_i)$. Parameters: learning rate η_t, noise scale σ, group size L, gradient norm bound C.

Initialize θ_0 randomly

for $t \in [T]$ **do**

Take a random sample L_t with sampling probability L/N

Compute gradient

For each $i \in L_t$, compute $\mathbf{g}_t(x_i) \leftarrow \nabla_{\theta_t}\mathcal{L}(\theta_t, x_i)$

Clip gradient

$\bar{\mathbf{g}}_t(x_i) \leftarrow \mathbf{g}_t(x_i) / \max\left(1, \frac{\|\mathbf{g}_t(x_i)\|_2}{C}\right)$

Add noise

$\tilde{\mathbf{g}}_t \leftarrow \frac{1}{L}\left(\sum_i \bar{\mathbf{g}}_t(x_i) + \mathcal{N}(0, \sigma^2 C^2 \mathbf{I})\right)$

Descent

$\theta_{t+1} \leftarrow \theta_t - \eta_t \tilde{\mathbf{g}}_t$

Output θ_T and compute the overall privacy cost (ε, δ) using a privacy accounting method.

DP-SGD Algorithm [Abadi et al., 2016]

How private is the DP-SGD algorithm?

- For running a single iteration:
  - We use one (ε, δ)-differentially private Gaussian Mechanism to compute noisy gradient
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Amplification by Sub-sampling

- Denote $M(x, f, \epsilon, \delta)$ as a (ϵ, δ)-differentially private Gaussian mechanism
- Let $\text{Poi}(x, q)$ be the Poisson sub-sampling mechanism on dataset x that includes each record x_i independently with probability q
- Sub-sampled Gaussian mechanism

$$M_q(x, f, \epsilon, \delta) = f \circ \text{Poi}(x, q) + noise$$

where coordinates of $noise \overset{\text{i.i.d}}{\sim} N(0, \sigma^2)$ with $\sigma = \frac{\Delta f}{\epsilon}\sqrt{2 \log \frac{5}{4\delta}}$

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Amplification by Sub-sampling for moments

- Denote the privacy loss random variable for a Gaussian mechanism as $L = \frac{\Pr[M(x,f,\epsilon,\delta)=\theta]}{\Pr[M(x',f,\epsilon,\delta)=\theta]}, \theta \sim M(x', f, \epsilon, \delta)$
- Denote the privacy loss random variable for a sub-sampled Gaussian mechanism $L_q = \frac{\Pr[M_q(x,f,\epsilon,\delta)=\theta]}{\Pr[M_q(x',f,\epsilon,\delta)=\theta]}, \theta \sim M_q(x', f, \epsilon, \delta)$ where q is the sub-sampling probability
- Then, we can prove $\ln \mathbb{E}[L_q^\lambda] \le \frac{q^2}{1-q} \ln \mathbb{E}[L^\lambda] + O(q^3\lambda^3/\sigma^3)$

Source: [Abadi et al., 2016, Lemma 3, Theorem 1]

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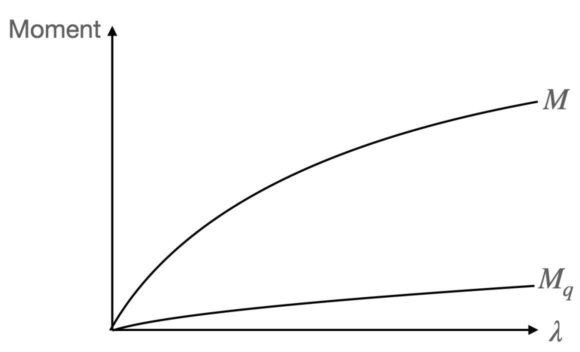
Amplification by Sub-sampling



- For moments, we prove that $\ln \mathbb{E}[L_q^\lambda] \leq \frac{q^2}{1-q} \ln \mathbb{E}[L^\lambda] + O(q^3\lambda^3/\sigma^3)$
- Therefore, by applying Markov inequality, we can prove that $M_q(x, f, \epsilon, \delta)$ is approximately $(q\epsilon, \delta)$-DP

Source: [Abadi et al., 2016, Lemma 3, Theorem 1]

How private is the DP-SGD algorithm?

- For running a single iteration:
  - We use one approximately $(q\varepsilon, \delta)$-differentially private sub-sampled Gaussian Mechanism to compute noisy gradient, where $q = \frac{L}{N}$
- For running multiple iterations:
  - How to compose the privacy bound for each iteration to obtain a privacy bound for the whole algorithm?

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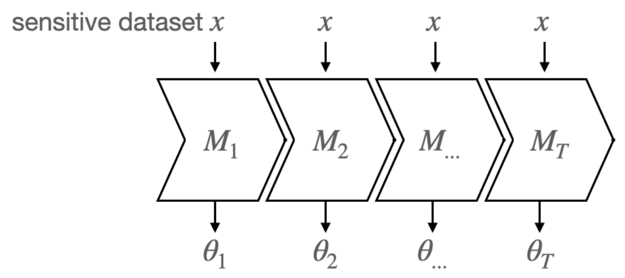
How to compose privacy bound for multiple iterations?

- Assume, we run an iterative algorithm that uses a DP mechanism M_i on a sensitive dataset, in the i-th iteration of computation.
- Let the total number of iterations be T
- Let the randomness used by the T DP mechanisms be independent
- Outputting $M_i(x)$ is (ϵ, δ)-differentially private $\Rightarrow$ How private is outputting the **composition** of $M_1(x), \cdots, M_T(x)$?



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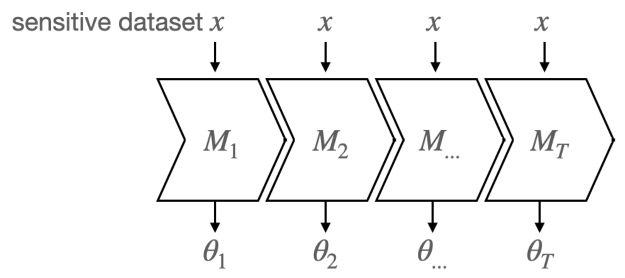

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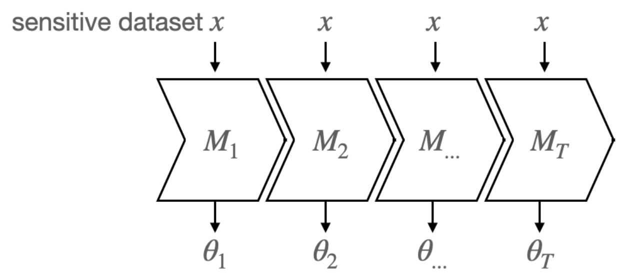

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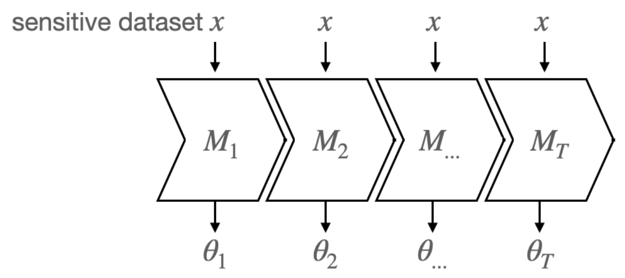

Moments for composition of DP mechanisms

- Let the privacy loss random variable for i-th mechanism be $L_i = \frac{\Pr[M_i(x,f,\epsilon,\delta)=y]}{\Pr[M_i(x',f,\epsilon,\delta)=y]}, y \sim M_i(x', f, \epsilon, \delta)$
- Then the composition of T sequential DP mechanisms is $M_{com}(x, f, \epsilon, \delta) : x \mapsto (M_1(x), \cdots, M_T(x))$.
- We need to analyze the moment of privacy loss random variable for composed mechanism

$$L_{com} = \frac{\Pr[M_{com}(x, f, \epsilon, \delta) = (\theta_1, \cdots, \theta_T)]}{\Pr[M_q(x', f, \epsilon, \delta) = (\theta_1, \cdots, \theta_T)]}$$

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Moments for composition of DP mechanisms

- By the independence between the randomness used in T DP mechanisms $M_1, \cdots, M_T$, we have

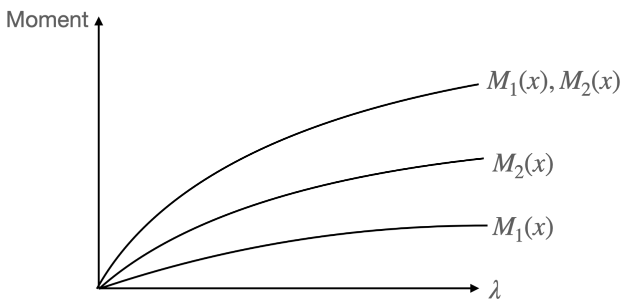
$$\begin{aligned} L_{compose} &= \frac{\Pr[M_{compose}(x) = (\theta_1, \cdots, \theta_T)]}{\Pr[M_{compose}(x') = (\theta_1, \cdots, \theta_T)]} \\ &= \frac{\Pr[M_1(x) = \theta_1] \cdots \Pr[M_T(x) = \theta_T]}{\Pr[M_1(x') = \theta_1] \cdots \Pr[M_T(x') = \theta_T]} \\ &= L_1 \cdots L_T \end{aligned}$$

  where $L_1, \cdots, L_T$ are the independent privacy loss random variables for mechanisms $M_1, \cdots, M_T$ respectively.
- Therefore, we have $\mathbb{E}[L_{com}^{\lambda}] \leq \prod_{i=1}^{T} \mathbb{E}[L_i^{\lambda}]$

Source: [Abadi et al., 2016]

Moment accountant for composition of DP mechanisms

- By Markov inequality, the overall computation over T steps is approximately $(\epsilon\sqrt{T}, \delta)$-DP [Abadi et al., 2016, Theorem 1]
- Example: if each step is $(0.005, 10^{-5})$-DP, and after 1000 steps, the algorithm will be approximately $(0.15, 10^{-5})$-DP



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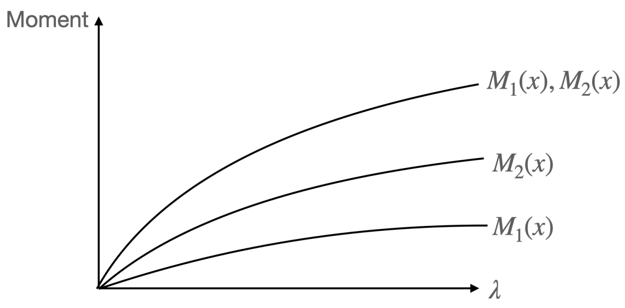

Advanced Topic: can we prove privacy bound for DP-SGD that is smaller than moment accountant?

- Observe that the composability of moment accountant implicitly assumes that the outputs after all k step $(y_1, \cdots, y_k)$ are released
- In reality, only the final output y_k is released, while all the preceding outputs $y_1, \cdots, y_{k-1}$ are hidden
- Under this more realistic hidden-state assumption, the privacy bound may converge, if one of the following condition holds
    - The loss function is strongly convex and smooth on unconstrained space $\mathbb{R}^d$ [Chourasia et al., 2021, Ye and Shokri, 2022, Altschuler and Talwar, 2022]
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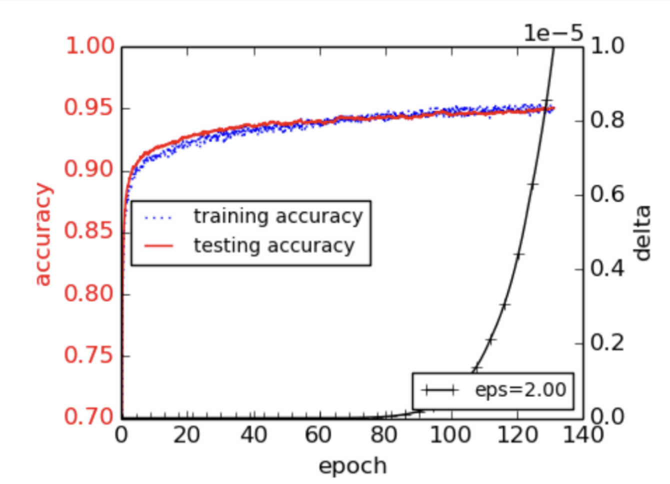
Training a machine learning model with DP-SGD



Training a NN on MNIST dataset using DP-SGD Algorithm [Abadi et al., 2016]

Same NN has 98.30% accuracy in ≈ 100 epochs, when trained non-privately

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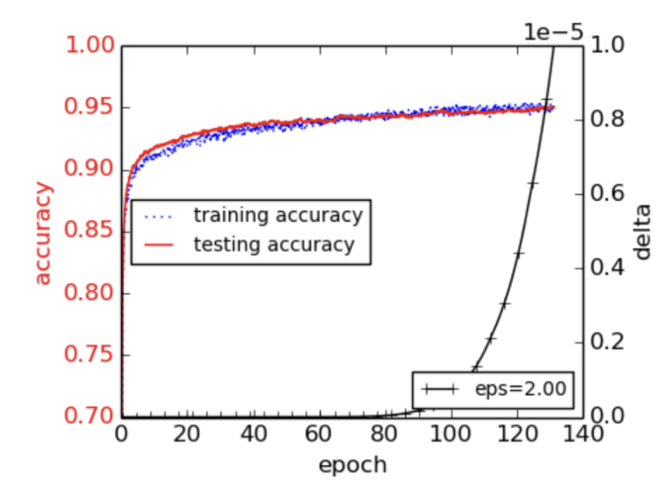

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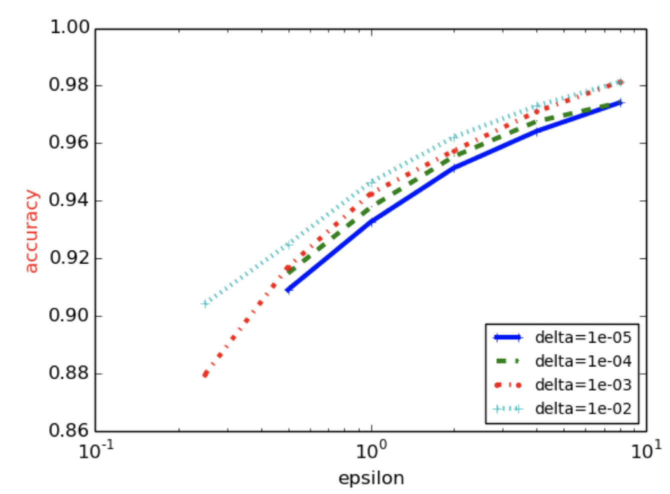
Trade-off between privacy and accuracy of DP-SGD



Best accuracy of training a NN on MNIST dataset using DP-SGD, when constrained within different differential privacy budget (ϵ, δ)

Advanced topic: How to improve the trade-off between privacy and accuracy for learning algorithm?

- Start from better features for training, rather than training from scratch
  - Features of pretrained models (that did not access private dataset) [Abadi et al., 2016]
  - Handcraft features using prior insights [Tramer and Boneh, 2020]
- Use optimization algorithm that converges with fewer epochs (s.t. consumed privacy budget ϵ is also smaller), rather than vanilla SGD
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  - Project noisy gradient to a lower-dimensional space, and then add smaller amount of noise [Yu et al., 2021, Zhou et al., 2021]

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