National University of Singapore
School of Computing

Semester 1, AY2023-24

CS4246/CS5446 AI Planning and Decision Making

Tutorial Week 11: Game Theory

Guidelines

You may discuss the content of the questions with your classmates. But everyone should work on and be ready to present ALL the solutions.

Problem 1: Battle of the Sexes

Husband and wife would like to go on a date-night out and there are only two venues for entertainment that night: a Ballet and a K-pop Concert. The wife wants to see the Ballet while the husband wants to see the Concert. But both of them prefer being together than being alone. Out of love for each other, they do not explicitly tell each other their own preferences (Bad idea!). The payoff matrix is shown below where the husband is the row player and the wife is the column player. Please work out the Nash equilibria for them.

	$Ballet$	$Concert$
$Ballet$	$1, 2$	$0, 0$
$Concert$	$0, 0$	$2, 1$

a) Explain why are the strategy profiles $\langle Ballet, Ballet \rangle$ and $\langle Concert, Concert \rangle$ Nash equilibria.

b) Find a Nash equilibrium where both players play mixed strategies.

c) Compute the expected utility of all three equilibria for the husband. Do the same for the wife.

d) Compute the utility of both players going to their preferred activity, and the expected utility of for both players when they both select each activity randomly with equal probability.

Solution:

a) In the profile, $\langle Ballet, Ballet \rangle$, if the husband chooses to switch, his utility will drop from 1 to 0, while if the wife chooses to switch, her utility will drop from 2 to 0. In the profile, $\langle Concert, Concert \rangle$, if the husband chooses to switch, his utility will drop from 2 to 0, while if the wife chooses to switch, her utility will drop from 1 to 0.

b) Assume that the husband plays $Ballet$ with probability p. To select a value p where the expected utility of the wife is the same regardless of the choice of the wife, we have

$$p \times 2 + (1 - p) \times 0 = p \times 0 + (1 - p) \times 1.$$

Solving we get $p = 1/3$. Now assume that the wife plays $Ballet$ with probability q. To select a value q where the expected utility of the husband is the same regardless of the choice of the husband, we have

$$q \times 1 + (1 - q) \times 0 = q \times 0 + (1 - q) \times 2.$$

Solving we get $q = 2/3$.

c) For the equilibrium $\langle Ballet, Ballet\rangle$, the husband's utility is 1 while the wife's utility is 2, For the equilibrium $\langle Concert, Ballet\rangle$ the husband's utility is 2 while the wife's utility is 1. For the mixed strategy Nash equilibrium, the husband's expected utility is $1/3 \times 2/3 \times 1 + 1/3 \times 2/3 \times 0 + 2/3 \times 1/3 \times 0 + 2/3 \times 1/3 \times 2 = 2/3$ and the wife's expected utility is $1/3 \times 2/3 \times 2 + 1/3 \times 2/3 \times 0 + 2/3 \times 1/3 \times 0 + 2/3 \times 1/3 \times 1 = 2/3$.

d) If both the husband and wife does their preferred activity, both their utilities are 0. If they both randomly choose an activity with equal probability, the expected utility for the husband (and by symmetry for the wife) is $1/2 \times 1/2 \times 1 + 1/2 \times 1/2 \times 0 + 1/2 \times 1/2 \times 0 + 1/2 \times 1/2 \times 2 = 3/4$.

Problem 2: Pure Maxmin and Minmax Strategies

Consider the following 3×3 two-person, zero-sum matrix game:

	t_1	t_2	t_3
s_1	1	6	0
s_2	2	0	3
s_3	3	2	4

In this question, we assume that only pure strategies are considered.

a) Find the maxmin strategy and value for the row player.

b) Find the minmax strategy for the column player against the row player and the minmax value for the row player.

c) Prove that, in general (i.e., not just for the game above), $\max_s \min_t u(s, t) \leq \min_t \max_s u(s, t)$.

Solution:

a) First, for each $s_i, i = 1, 2, 3$, place a star on the payoff that is minimum over t_1, t_2, t_3 as shown below:

	t_1	t_2	t_3
s_1	1	6	0^*
s_2	2	0^*	3
s_3	3	2^*	4

Then, among those payoffs indicated with stars, select the largest payoff, which is 2. This payoff of 2 is the maxmin value and the maxmin strategy for the row player is s_3.

b) First, for each $t_j, j = 1, 2, 3$, place a star on the payoff that is maximum over s_1, s_2, s_3 as shown below:

	t_1	t_2	t_3
s_1	1	6^*	0
s_2	2	0	3
s_3	3^*	2	4^*

Then, among those payoffs indicated with stars, select the smallest payoff, which is 3. This payoff of 3 is the minmax value and the minmax strategy for the column player is t_1.

c) Notice from questions 1 and 2 that the maxmin value is not more than the minmax value. In general,

$$
\begin{aligned}
& \forall s\ \forall t \min_{t'}\ u(s,t') \leq u(s,t) \\
\Rightarrow\quad & \forall t\ \max_{s'} \min_{t} u(s',t') \leq \max_{s} u(s,t) \\
\Rightarrow\quad & \max_{s'} \min_{t'} u(s',t') \leq \min_{t} \max_{s} u(s,t)
\end{aligned}
$$