School of Computing
National University of Singapore
CS5340: Uncertainty Modeling in AI
Semester 1, AY 2022/23

Exercise 1

Question 1

a)

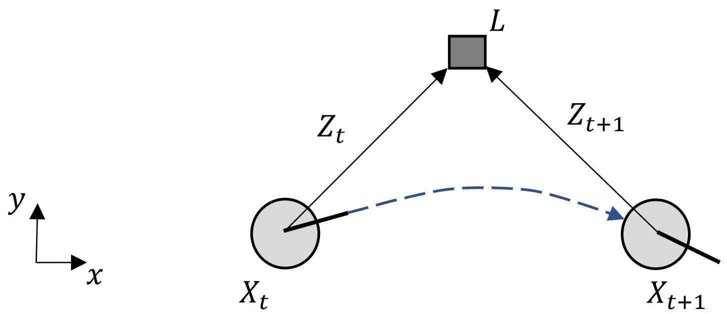


Fig. 1.1

Fig. 1.1 shows a mobile robot that traverses from pose X_t to X_{t+1} over time t to $t+1$. The robot is equipped with an 1-dimensional range sensor that returns the distances Z_t and Z_{t+1} of a landmark structure L in the environment from the poses X_t and X_{t+1} respectively. Let U_t denotes the control command given by the user to move the robot from X_t to X_{t+1}.

(i) Taking $\{U_t, L, X_t, X_{t+1}, Z_t, Z_{t+1}\}$ as random variables, state whether each of these random variables is an observed or latent/hidden random variable. Explain your answers.

(ii) Given the following conditional independencies:

$$L \perp U_t \mid \emptyset, \quad X_t \perp L \mid U_t, \quad X_{t+1} \perp \{L, U_t\} \mid X_t,$$

$$Z_t \perp \{U_t, X_{t+1}\} \mid \{X_t, L\}, \quad Z_{t+1} \perp \{U_t, X_t, Z_t\} \mid \{L, X_{t+1}\}.$$

Write the factorized probability and draw the Bayesian network that represents the joint distribution $p(u_t, l, x_t, x_{t+1}, z_t, z_{t+1})$ assuming the following topological ordering of the random variables:

$$\{U_t, L, X_t, X_{t+1}, Z_t, Z_{t+1}\}.$$

Show all your workings clearly.

(iii) Write the following probability distribution $p(z_t, z_{t+1} \mid l)$ in terms of the factorized probability obtained in (ii). Simplify your answer.

b)

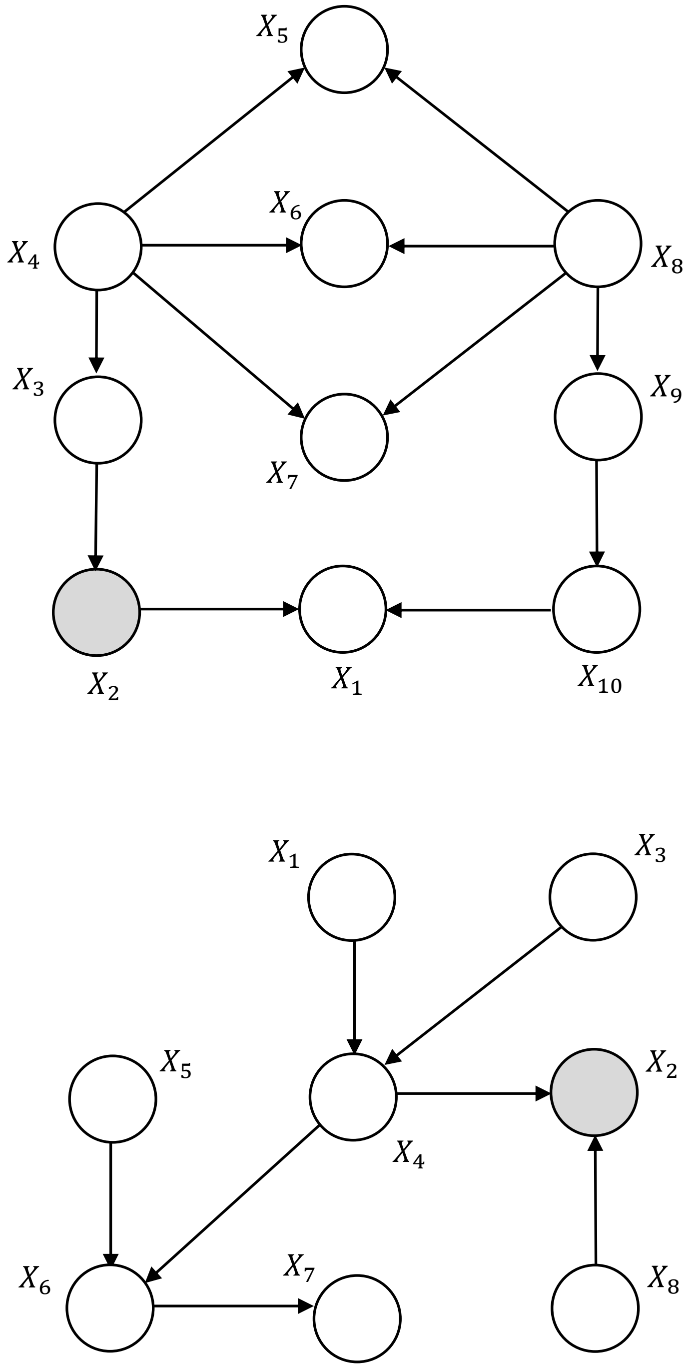


Fig. 1.2

For each of the Bayesian networks shown in Fig. 1.2, determine the largest set of nodes X_B such that $X_1 \perp X_B \mid X_2$. Explain your answers.

Question 2

Consider the graph shown in Fig. 2.1:

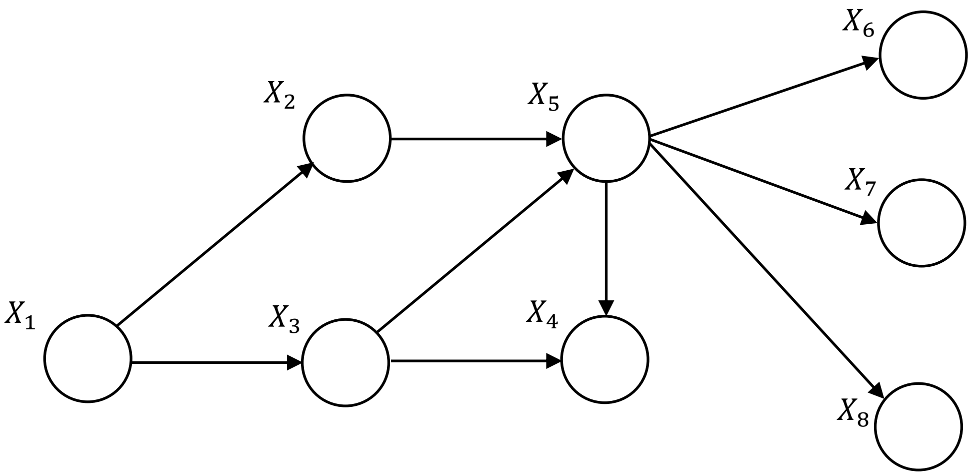


Fig 2.1

a) What is the corresponding moral graph?

b) What is the reconstituted graph from the UNDIRECTEDGRAPHELIMINATE algorithm on the moral graph with the ordering {8,7,6,5,4,3,2,1}?

c) What is the reconstituted graph from the UNDIRECTEDGRAPHELIMINATE algorithm on the moral graph with the ordering {8,5,6,7,4,3,2,1}?

d) Suppose you wish to calculate $p(x_1|x_8)$. Which ordering is preferable? Why?

Question 3

What is the treewidth of the graph below?

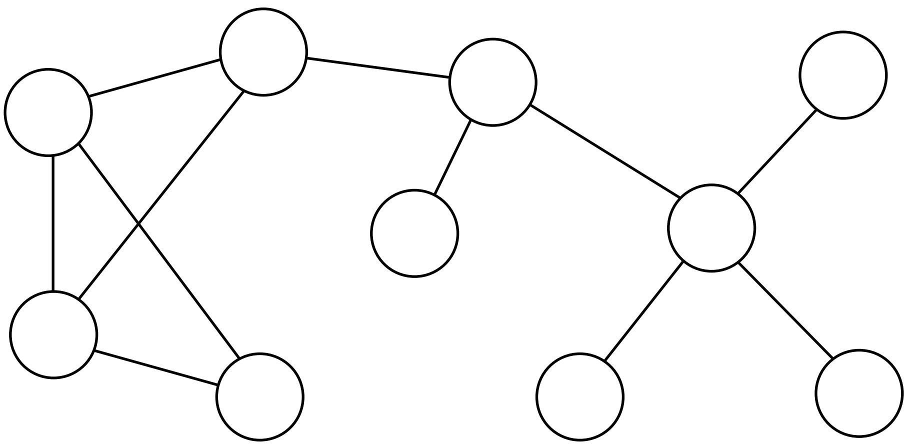


Fig 3.1

Question 4

Consider the following random variables. X_1 and X_2 represent the outcomes of two independent fair coin tosses. X_3 is the indicator function of the event that the outcomes are identical.

a) Specify a directed graphical model that describes the joint probability distribution (i.e. specify the graph and the conditional distributions).

b) Specify an undirected graphical model that describes the joint probability distribution (i.e. give the graph and specify the clique potentials).

c) In both cases, list all conditional independencies that are implied by the graph.

d) In both cases, list any additional conditional dependencies that are displayed by this probability distribution but are not implied by the graph.

Question 5

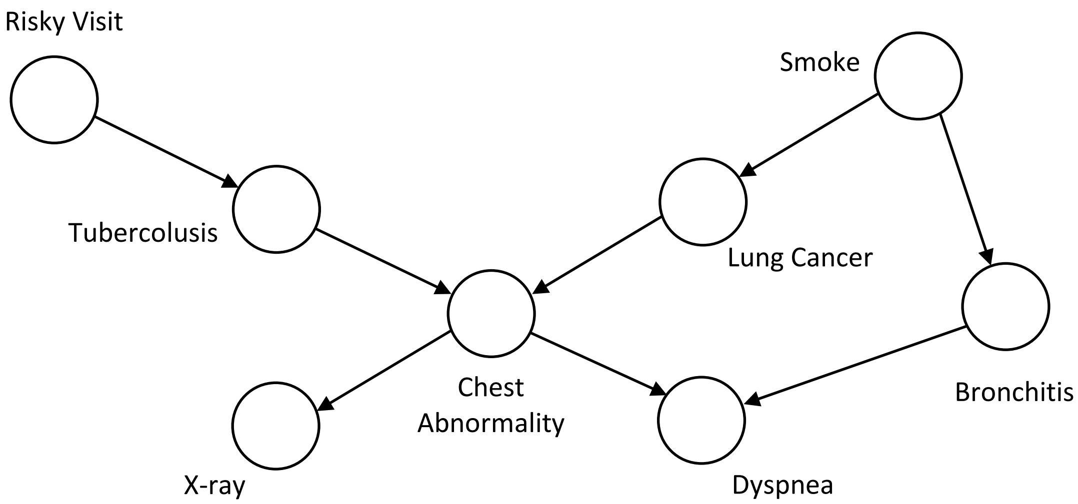


Fig. 5.1

The graphical model shown above describes some relationships among variables associated with chest abnormality. Answer the following questions based on the graphical model.

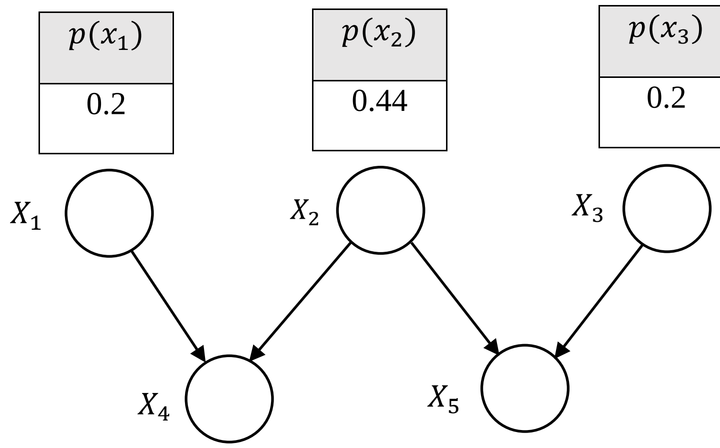
a) True or False. Justify your choice. $Smoke \perp Dyspnea \mid Bronchitis$.
b) True or False. Justify your choice. $Bronchitis \perp X\text{-}ray \mid Cancer$.
c) True or False. Justify your choice. $Smoke \perp Risky\ Visit \mid Dyspnea$.
d) True of False. Justify your choice. $X\text{-}ray \perp Smoke \mid \{Cancer, Bronchitis\}$.

Question 6

Evaluate (give the distribution tables) the following probabilities:

$$p(x_1\,|x_5),\quad p(x_2\,|x_4),\quad p(x_3\,|x_2),\quad p(x_4\,|x_3),\quad p(x_5)$$

for the Bayesian network shown in Fig. 6.1, where each random variable takes a binary state, i.e. $x_i \in \{T, F\}$. Show all your workings clearly.



X_1	X_2	$p(x_4\|x_1, x_2)$
T	T	0.35
T	F	0.6
F	T	0.01
F	F	0.95

X_2	X_3	$p(x_5\|x_2, x_3)$
T	T	0.35
T	F	0.6
F	T	0.01
F	F	0.95

Fig. 6.1

Question 7

Give the junction tree of the Bayesian network shown in Fig. 7.1 using the following elimination order: $\{X_7, X_6, X_5, X_4, X_3, X_2, X_1\}$. Show all your workings clearly.

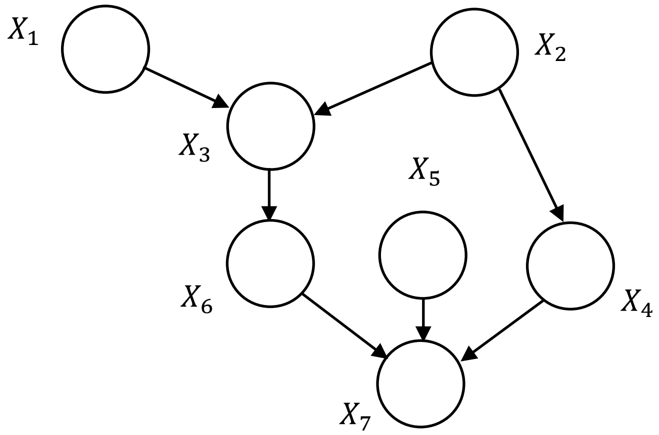


Fig. 7.1

Question 8

Figure 8.1 shows a Bayesian network with five random variables X_1, X_2, X_3, X_4, X_5 , where $x_i \in \{0,1\}$ for $i = 1, 2, 4$, and $x_i \in \{0,1,2\}$ for $i = 3, 5$.

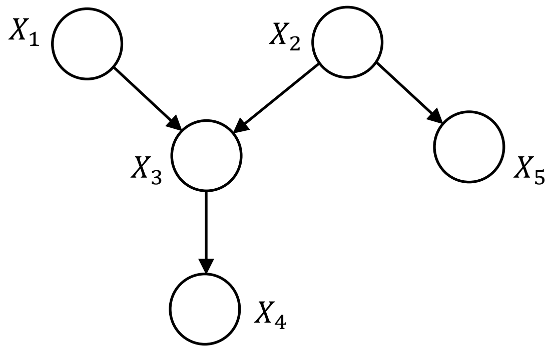


Figure 8.1

(a) Write down all the conditional independences given by the Bayesian network.

(b) Write down the factorized expression of the joint probability given by the Bayesian network.

(c) Convert the Bayesian network into a factor graph. Draw the factor graph and write down the expression of each factor clearly in your answer.

(d) Table 8.1 gives the probability tables of the Bayesian network, find the conditional probability $p(x_1|x_3 = 1, x_2)$. Show all your workings clearly.

X_1	X_2	X_3	$p(x_3\|x_1, x_2)$
0	0	0	0.3
0	0	1	0.4
0	1	0	0.9
0	1	1	0.08
1	0	0	0.05
1	0	1	0.25
1	1	0	0.5
1	1	1	0.3

X_1	$p(x_1)$
0	0.6

X_2	$p(x_2)$
0	0.7

X_3	X_4	$p(x_4\|x_3)$
0	0	0.1
1	0	0.4
2	0	0.99

Table 8.1

Question 9

Figure 9.1 shows a graphical model with six binary-state latent random variables $Z = \{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6\}$, $z_i \in \{0,1\}$, and six binary-state observed random variables $X = \{X_1, X_2, X_3, X_4, X_5, X_6\}$, $x_i \in \{0,1\}$. Table 9.1 gives the pairwise potentials $\phi(z_i, z_j)$, $\forall\, ij \in \mathcal{E}_Z$ and conditional probability $p(x_i|z_i)$ for $i = 1, \ldots 6$, where $\mathcal{E}_Z$ denotes all the edges between the latent random variables in the graphical model. Find the configuration of Z that maximizes the joint probability $p(X, Z)$.

(**Hint**: convert the graphical model into a factor graph, where the respective pairwise potential and conditional probability are represented as a single factor.)

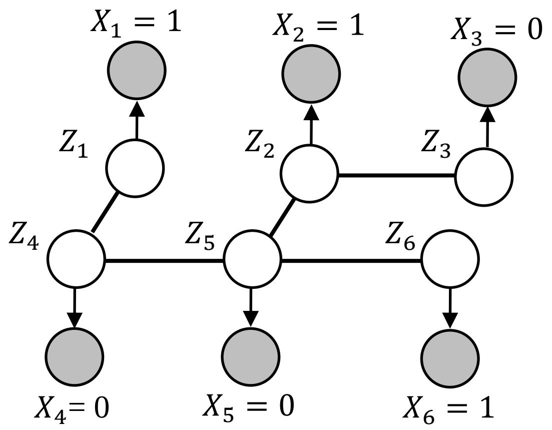


Figure 9.1

Z_i	Z_j	$\phi(z_i, z_j)$
0	0	0
0	1	2
1	0	2
1	1	0

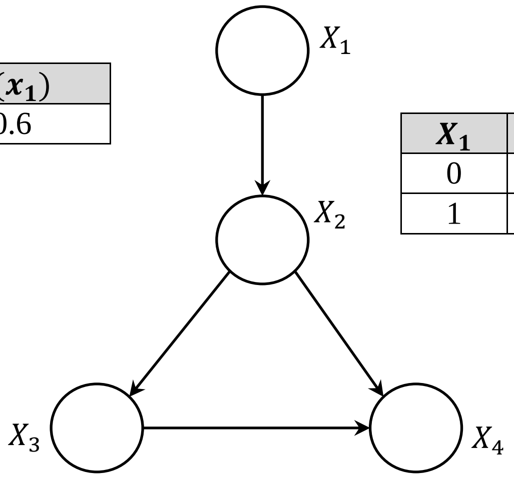
X_i	Z_i	$p(x_i \| z_i)$
0	0	0.9
0	1	0.05
1	0	0.1
1	1	0.95

Table 9.1

Question 10

Figure 10.1 shows a Bayesian Network with four random variables X_1, X_2, X_3 and X_4, where $x_i \in \{0,1\}$. The respective prior and conditional probability distribution tables are also given.

X_1	$p(x_1)$
0	0.6



X_1	X_2	$p(x_2 \mid x_1)$
0	0	0.6
1	0	0.3

X_2	X_3	$p(x_3 \mid x_2)$
0	0	0.8
1	0	0.2

X_2	X_3	X_4	$p(x_4 \| x_2, x_3)$
0	0	0	0.1
0	1	0	0.8
1	0	0	0.5
1	1	0	0.6

Figure 10.1

Find the following marginal probabilities:

a. $p(x_2)$
b. $p(x_3)$
c. $p(x_4)$
d. $p(x_3, x_4)$

Question 11

a) Assume that the day of the week that females are born on, X, is independent of the day of the week, Y, on which males are born. Assume, however, that the old rhyme is true, and that personality is dependent on the day of the week you're born on. If A represents the female personality type and B the male personality type, then $(A \top X)$ and $(B \top Y)$, but $(A \perp B)$. Whether or not a male and a female are married, M, depends strongly on their personality types, $(M \top A, B)$, but is independent of X and Y if we know A and B. Draw a graphical model that can represent this setting. What can we say about the (graphical) dependency between the days of the week that John and Jane are born on, given that they are not married?

Show all your workings clearly.

Note: $(A \top X)$ is the shorthand for A is dependent on X.

b) Prove that the following relations are true or false:

i. $(X \perp Y)$ & $(Y \perp Z) \Longrightarrow (X \perp Z)$?

ii. $(X \perp Y \mid Z) \Longrightarrow (X \perp Y, W \mid Z)$?

iii. Given that $(A \perp C \mid D, B)$ & $(A \perp B \mid D) \Longrightarrow (A \perp B, C \mid D)$, does $(A \perp B \mid D, C)$ & $(A \perp C \mid D, B) \Longrightarrow (A \perp B, C \mid D)$?

iv. $(X, Y, Z \perp A, B, C \mid D, E, F) \Longrightarrow (X \perp A, B \mid D, E, F)$ & $(X, Y \perp A \mid D, E, F)$ ..., i.e. $(any\ subset\ of\ \{X, Y, Z\} \perp any\ subset\ of\ \{A, B, C\} \mid D, E, F)$?

Show all your workings clearly.

Question 12

Figure 12.1 shows a directed graphical model with five random variables X_1, X_2, X_3, X_4, X_5, where $X_i \in \{0,1\}$. The respective conditional probabilities are shown in Table 12.1, where a, b, c and d are unknown values. Given that the minimal probability 0.00216 occurs at the configuration of $X_1 = 1, X_2 = 0, X_3 = 1, X_4 = 1, X_5 = 0$, and the maximal probability 0.10976 occurs at the configuration of $X_1 = 0, X_2 = 0, X_3 = 0, X_4 = 1, X_5 = 1$, find the **numerical values** of the probability distribution $p(X_2, X_3, X_5)$. **Show all your workings clearly**.

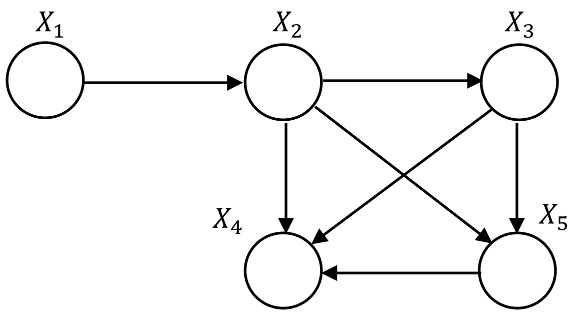


Figure 12.1.

X_2	X_3	X_5	$p(X_5 \mid X_2, X_3)$
0	0	0	0.3
0	0	1	0.7
0	1	0	0.1
0	1	1	0.9
1	0	0	0.5
1	0	1	0.5
1	1	0	0.8
1	1	1	0.2

X_2	X_3	$p(X_3 \mid X_2)$
0	0	0.7
0	1	0.3
1	0	0.6
1	1	0.4

X_1	X_2	$p(X_2 \mid X_1)$
0	0	0.8
0	1	0.2
1	0	c
1	1	d

X_4	X_2	X_3	X_5	$p(X_4 \mid X_2, X_3, X_5)$
0	0	0	0	0.2
0	0	0	1	0.3
0	0	1	0	0.6
0	0	1	1	0.9
0	1	0	0	0.2
0	1	0	1	0.3
0	1	1	0	0.2
0	1	1	1	0.6
1	0	0	0	0.8
1	0	0	1	0.7
1	0	1	0	0.4
1	0	1	1	0.1
1	1	0	0	0.8
1	1	0	1	0.7
1	1	1	0	0.8
1	1	1	1	0.4

X_1	$p(X_1)$
0	a
1	b

Table 12.1.

Question 13

Prove the following conditional independences are true:

a) $(X \perp Y, W \mid Z) \Longrightarrow (X \perp Y \mid Z)$. This is also known as **Decomposition**.

b) $(X \perp Y, W \mid Z) \Longrightarrow (X \perp Y \mid Z, W)$. This is also known as **Weak Union**.

c) $(X \perp Y \mid Z)$ and $(X \perp W \mid Z, Y) \Longrightarrow (X \perp W, Y \mid Z)$. This is also known as **Contraction**.

Question 14

Figure 4.1 shows a three-node Bayesian network with random variables $X_i \in \mathbb{R}$. Furthermore, $p(X_1 \mid \mu_1, \sigma_1^2) = \mathcal{N}(X_1 \mid \mu_1, \sigma_1^2)$, $p(X_2 \mid \mu_2, \sigma_2^2) = \mathcal{N}(X_2 \mid \mu_2, \sigma_2^2)$, and $p(X_3 \mid X_1, X_2, w_0, w_1, w_2, \sigma_3^2) = \mathcal{N}(X_3 \mid w_0 + w_1 X_1 + w_2 X_2, \sigma_3^2)$, where $\mathcal{N}(X \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-0.5 \frac{(X-\mu)^2}{\sigma^2}\right\}$ is the Gaussian distribution parameterized by the mean μ and variance σ^2. We have the linear-Gaussian distribution when the mean μ is parameterized by $w_0, \ldots w_M$ as a weighted sum of the parent nodes $\{X_{\pi,1} \ldots X_{\pi,M}\}$ of X, i.e., $\mu = w_0 + \sum_m w_m X_{\pi,m}$. Given 10 observations of the three-node Bayesian network in Table 4.1, find the probability density at $p(X_1 = 1.5, X_2 = 0.5, X_3 = 1.0)$. **Show all your workings clearly.**

Useful equation: $\frac{d}{dx} \ln(f(x)) = \frac{1}{f(x)} \frac{df(x)}{dx}$.

(25 mark)

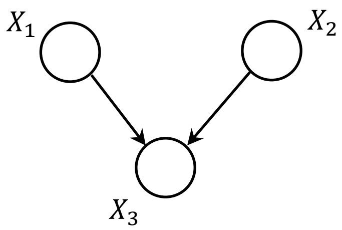


Figure. 4.1

Observation #	X_1	X_2	X_3
1	1.86	-0.03	-0.31
2	1.68	1.09	1.48
3	2.41	-0.45	2.18
4	2.40	-0.33	2.98
5	0.87	0.99	3.08
6	1.96	2.83	2.24
7	1.78	0.07	-0.89
8	2.81	1.44	1.18
9	3.42	0.72	0.88
10	3.44	2.34	6.15

Table. 4.1

--End--