A "Not So Obvious" Homeomorphism!

Nripendra Kr Deb

October 2023

Problem:

Let $p : \widetilde{X} \to X$ be a double cover and X is path connected and locally path connected. If the index of $\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $\pi_1(X, x_0)$ is one, then show that $\widetilde{X}$ is homeomorphic to $X \sqcup X$.

Attempted Solution :

Let's first prove some important lemmas that will be used later in solving the main problem.

Lemma 1: Let $p : \widetilde{X} \to X$ be covering map. If the induced map $p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is surjective then loops based at x_0 lift to loops based at $\tilde{x}_0$ under p.

Proof. Consider any loop γ based at x_0 in X. Since p_* is surjective there exists a loop η based at $\tilde{x}_0$ in $\widetilde{X}$ such that $[p \circ \eta] = [\gamma]$. Now consider the lift $\widetilde{\gamma}_{\widetilde{x}_0}$ of γ starting at $\tilde{x}_0$. Note that η is the lift of $p \circ \eta$ starting at $\tilde{x}_0$. Since γ and $p \circ \eta$ are homotopic, hence their lifts starting from the same point must have the same endpoints i.e $\widetilde{\gamma}_{\widetilde{x}_0}(1) = \eta(1) = \tilde{x}_0$. Hence γ lifts to a loop based at $\tilde{x}_0$. □

Remark: Note that lifts of loops based at x_0 starting from $\widetilde{x}_1$ must be loops as well. Indeed if $\widetilde{\gamma}_{\widetilde{x}_1}$ be a path starting a $\widetilde{x}_1$ then $\widetilde{\gamma}_{\widetilde{x}_1}(1) = \widetilde{x}_0$, but then $\overline{\widetilde{\gamma}}_{\widetilde{x}_1}$ is a lift of a loop based at $\widetilde{x}_0$, hence it must be loop.

Lemma 2: Let $p : \widetilde{X} \to X$ be a covering map. If X is locally path connected then so is $\widetilde{X}$. Moreover the path components of $\widetilde{X}$ are open.

Proof. Let $\widetilde{x} \in \widetilde{X}$ and $p(\widetilde{x}) = x$. Let U be any open set containing $\tilde{x}$. If U_x is an evenly covered nbd containing x, then $p(U) \cap U_x$ is an open set containing x. Since X is locally path connected then x has connected nbd V contained in $p(U) \cap U_x$. Then $\widetilde{U} = p^{-1}(V)$ is connected nbd of $\widetilde{x}$ contained in U.
Let C be a path component of $\widetilde{X}$. Let $\widetilde{x} \in C$, since $\widetilde{X}$ is locally path connected so there exists a path connected nbd U of $\widetilde{x}$ contained in $\widetilde{X}$ and hence contained in C as $C \cap U \neq \phi$. Thus C is open. □

We will use the notation $X \sqcup X = (X \times \{0\}) \sqcup (X \times \{1\})$. We fix a point $x_0 \in X$ and let $p^{-1}(x_0) = \{\tilde{x}_0, \tilde{x}_1\}$. Consider the map

$$\Phi : X \sqcup X \to \widetilde{X}, (x, i) \mapsto \widetilde{\gamma}_{\tilde{x}_i}(1)$$

where $\widetilde{\gamma}_{\tilde{x}_i}$ is the unique lift of γ starting at $\tilde{x}_i$ where γ is the path joining x_0 and x with $\gamma(0) = x_0$.

• **Is Φ well-defined?**
Consider any other path η between x_0 and x starting at x_0. Note that $\gamma * \overline{\eta}$ is a loop in X based at x_0. Given $[\pi_1(X, x_0) : p_*(\pi_1(\widetilde{X}, \tilde{x_0}))] = 1$, hence p_* is indeed surjective. So using *lemma 1*, we find that $\gamma * \overline{\eta}$ lifts to a loop in $\widetilde{X}$ based at $\tilde{x}_i$. But note that $\widetilde{\gamma * \overline{\eta}} = \widetilde{\gamma}_{\tilde{x}_i} * \widetilde{\overline{\eta}}_{\widetilde{\gamma}_{\tilde{x}_i}(1)} = \widetilde{\gamma}_{\tilde{x}_i} *$. Let $y \in p^{-1}(x_0)$, then it follows

$\widetilde{\overline{\eta}}_{\widetilde{\gamma}(1)} = \overline{\widetilde{\eta}_y}$, but since $\widetilde{\gamma}_{\widetilde{x}_i} * \overline{\widetilde{\eta}_y}$ is a loop in $\tilde{X}$, so it follows that $y = \widetilde{x}_i$. Hence it follows that $\widetilde{\gamma}_{\widetilde{x}_i}(1) = \widetilde{\eta}_{\widetilde{x}_i}(1)$, which proves the fact that Φ is well-defined.

• **Is Φ one-one?**
Let $\Phi((x,i)) = \phi((y,j))$, hence it follows $\widetilde{\gamma}_{\widetilde{x}_i}(1) = \widetilde{\eta}_{\widetilde{x}_j}(1)$, where γ and η are the paths joining x_0 to x and y respectively. Thus, $p(\widetilde{\gamma}_{\widetilde{x}_i}(1)) = p(\widetilde{\eta}_{\widetilde{x}_j}(1)) \Rightarrow \gamma(1) = \eta(1) \Rightarrow x = y$. Note that $\widetilde{\gamma * \overline{\eta}} = \widetilde{\gamma}_{\widetilde{x}_i} * \widetilde{\overline{\eta}}_{\widetilde{x}_j}$, as $\gamma * \overline{\eta}$ is a loop based at x_0 hence its lift $\widetilde{\gamma}_{\widetilde{x}_i} * \overline{\widetilde{\eta}_{\widetilde{x}_j}}$ must be a loop, thus $\widetilde{\gamma}_{\widetilde{x}_i}(0) = \widetilde{\eta}_{\widetilde{x}_j}(0) \Rightarrow \widetilde{x}_i = \widetilde{x}_j \Rightarrow i = j$. Hence Φ is an injective map.

• **Is Φ surjective?**
Let $\widetilde{y} \in \widetilde{X}$, and $p(\widetilde{y}) = y$ with $p^{-1}(y) = \{\widetilde{y}, \widetilde{y}_1\}$. Since X is path connected, consider the path γ between x_0 and y starting at x_0. Let $\widetilde{\gamma}_{\widetilde{x}_i}$ be the lift of γ starting at $\widetilde{x}_i$. Then $\Phi(y,i) = \widetilde{\gamma}_{\widetilde{x}_i}(1) \in \{\widetilde{y}, \widetilde{y}_1\}$, since Φ is one-one so for some $i \in \{0,1\}$, $\Phi(y,i) = \widetilde{y}$, which proves that Φ is in fact a surjection.

• **Is Φ continuous?**
Consider the map $s_i : X \to \widetilde{X}, x \mapsto \widetilde{\gamma}_{\widetilde{x}_i}(1)$, we want to show that s_i is continuous. Let $\widetilde{V}$ be an open set in $\widetilde{X}$, we need show that $s_i^{-1}(\widetilde{V})$ is open. Let $x \in s_i^{-1}(\widetilde{V})$. Let U_x be an evenly covered nbd of x by p and V_i be the sheet containing $s_i(x)$. Consider $V' = \widetilde{V} \cap V_i$ and $V = p(V')$ which is open since p is an open map. Let V_x be a path-connected open set contained in v containing x. Our claim is $s_i(V_x) \subset \widetilde{V}$. Let $y \in V$, consider γ_1 to be the path joining x_0 and x and γ_2 be a path in V_x joining x and y. Let $\gamma = \gamma_1 * \gamma_2$, note that $\widetilde{x} = \widetilde{\gamma}_{1\tilde{x}_i}(1) \in V'$ and $\widetilde{\gamma_1 * \gamma_2} = \widetilde{\gamma}_{1\tilde{x}_i} * \widetilde{\gamma}_{2\tilde{x}}$. Since $p\mid_{V'} \to V$ is a homeomorphism. And since γ_2 is a path in V so $\widetilde{\gamma}_{2\tilde{x}} = p^{-1} \circ \gamma_2$ by uniqueness of path lifting and it must be a path in V'. Hence $\widetilde{\gamma}_{2\tilde{x}}(1) = s_i(y) \in V' \subset \widetilde{V}$, which gives $s_i(V_x) \subset \widetilde{V}$.
Consider the projection map $q_i : X \times \{i\} \to X, (x,i) \mapsto x$, which is clearly continuous. Note that we can write $\Phi\mid_{X\times\{i\}} = s_i \circ q_i$ which is continuous being the composition of two continuous functions. Now $X \times \{0\}$ and $X \times \{1\}$ are disjoint open sets hence using pasting lemma $\Phi : X \sqcup X \to \widetilde{X}$ is also continuous.

Our next claim is that there is no path joining $\widetilde{x}_0$ and $\widetilde{x}_1$. Indeed if γ is such a path in $\widetilde{X}$ with $\gamma(0) = \widetilde{x}_0$. Then clearly γ is a lift of the path $p \circ \gamma$. But $p(\gamma(0)) = p(\gamma(1)) = x_0$, hence $p \circ \gamma$ is a loop but its lift γ is not, contradiction. By the surjectivity of the map Φ it follows that $\widetilde{X}$ exactly two components C_0 and C_1 containing $\widetilde{x}_0$ and $\widetilde{x}_1$ respectively.
• **Φ is a homeomorphism**
We claim that there is no path joining $\widetilde{x}_0$ and $\widetilde{x}_1$. Indeed if γ is such a path in $\widetilde{X}$ with $\gamma(0) = \widetilde{x}_0$. Then clearly γ is a lift of the path $p \circ \gamma$. But $p(\gamma(0)) = p(\gamma(1)) = x_0$, hence $p \circ \gamma$ is a loop but its lift γ is not, contradiction. By the surjectivity of the map Φ it follows that $\widetilde{X}$ exactly two components C_0 and C_1 containing $\widetilde{x}_0$ and $\widetilde{x}_1$ respectively.
Consider the map $\psi : \widetilde{X} \to X \sqcup X, \widetilde{x} \mapsto (p(\widetilde{x}), i)$ if $\widetilde{x} \in C_i$. Note that $\psi = \Phi^{-1}$. Now our goal is to show that ψ is continuous too. Let $U \times \{i\}$ be an open set in $X \sqcup X$. Then $\psi^{-1}(U \times \{i\}) = \{\widetilde{x} \in \widetilde{X} : \psi(\widetilde{x}) \in U \times \{i\}\} = C_i \cap p^{-1}(U)$ which is open in $\widetilde{X}$. □

Acknowledgement :

• I had a discussion about this problem a few times with *Swapnil*, *Safarul* and *Navprabhat*. Big thanks to them.
• The book *Algebraic Topology* by *Allen Hatcher* helped me a lot while solving this problem.

Thank You