# Prime-Capturing Polynomial!!?

**Nripendra Kumar Deb**

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**Problem :** Show that there exists no non-constant multivariate polynomial $f$ with integer coefficients such that for all naturals $x_1, \ldots, x_n$ we have $f(x_1, \ldots, x_n)$ is a prime.

**Solution:** It is enough to prove that there is no such uni-variate polynomial. To the contrary lets assume $f$ is such a uni-variate polynomial. We will be using the fact that

$$a \equiv b \pmod{m} \implies f(a) \equiv f(b) \pmod{m}$$

We choose some $n_0 \in \mathbb{N}$ such that [1]$|f(n_0)| > 1$ and let $p = f(n_0)$, which must be a prime by our hypothesis. Then we have

$$f(n_0 + pt) \equiv f(n_0) \pmod{p} \quad t \in \mathbb{Z}$$

But since $p$ divides $f(n_0)$ hence $p$ must divide $f(n_0 + pt)$. But since $f(n_0 + pt)$ is a prime so we have $|f(n_0 + pt)| = p$ for all $t \in \mathbb{Z}$, which contradicts the fact that $|f(x)| \to \infty$ as $x \to \infty$. $\square$

**HW 2 (ii):** Show that there exists no non-constant multivariate polynomial $f$ with rational coefficients such that for all naturals $x_1, \ldots, x_n$ we have $f(x_1, \ldots, x_n)$ is a prime.

**Solution:** It is enough to prove that there is no such uni-variate polynomial. To the contrary lets assume $f$ is such a uni-variate polynomial. Note that we can write

$$f(x) = \frac{g(x)}{d}, \qquad d \in \mathbb{Z} \text{ and } g(x) \in \mathbb{Z}[x]$$

Let $f(1) = p$ which is a prime by our hypothesis. Then $g(1) = pd$ and we have

$$g(1 + pdt) \equiv g(1) \pmod{pd} \quad t \in \mathbb{Z}$$

Hence $pd$ divides $g(1 + pdt)$ and thus $p$ divides $f(1 + pdt) = \frac{g(1+pdt)}{d}$. But $f(1 + pdt)$ is a prime so $|f(1 + pdt)| = p$ for all $t \in \mathbb{Z}$. It contradicts the fact that $f$ is a non-constant polynomial. $\square$

[1] From our hypothesis $n_0 = 1$ also works