6. Inner Product Spaces

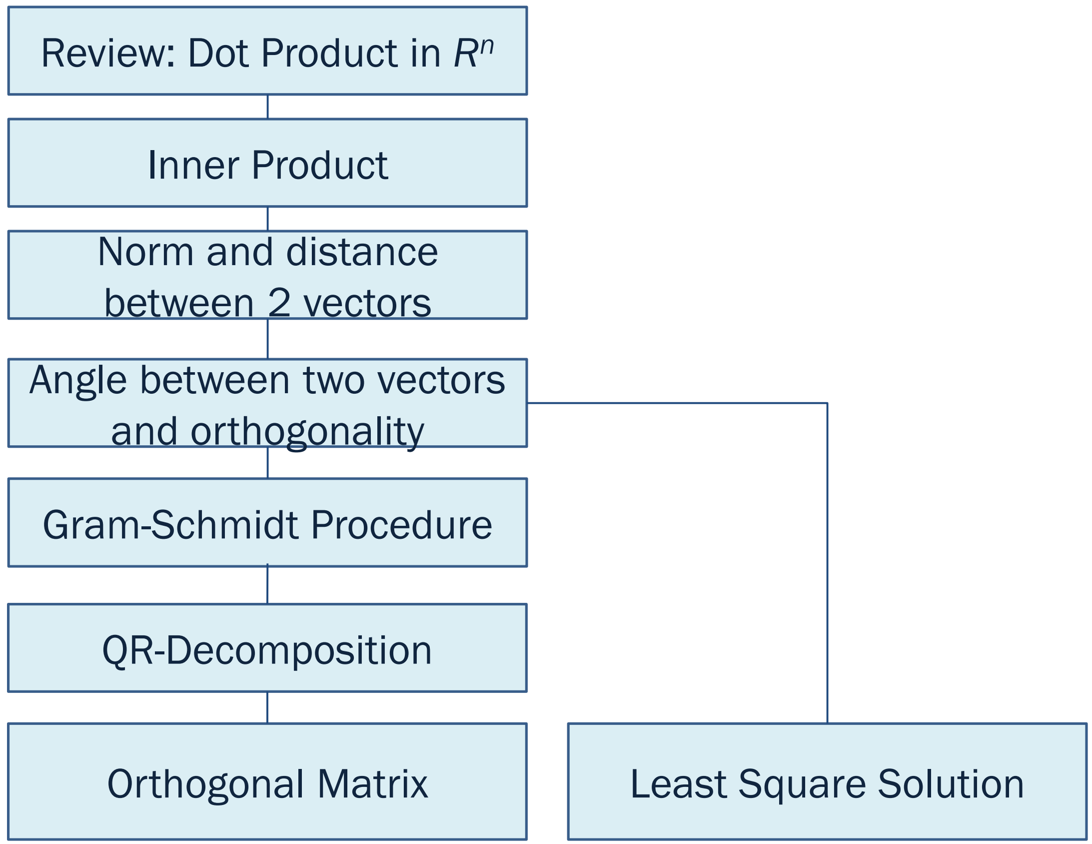
MODUL E-LEARNING ALJABAR LINIER
FAKULTAS ILMU KOMPUTER, UNIVERSITAS INDONESIA

Learning Objectives

After completing this module, students should be able to :

1. Explain the definition of inner product spaces;
2. Find the norm of a vector and the angle between two vectors;
3. Explain the relation between null space, row space and column space;
4. Apply the Gram-Schmidt procedure on a basis of a finite-dimension vector space;
5. Find the QR-decomposition of a matrix where its columns are linearly independent;
6. Find the least square solution from the normal system of an SLE.

Scope


Review: Dot Product in R^n
Inner Product
Norm and distance between 2 vectors
Angle between two vectors and orthogonality
Gram-Schmidt Procedure
QR-Decomposition
Orthogonal Matrix
Least Square Solution

Pre-test Module

Pre-test

- Answer the following questions:

1. Do you think that vectors are a mathematical element that can be drawn as a line segment that has a direction?
2. Is matrix considered not to be a vector because you cannot determine its magnitude and direction?

How is your *pre-test*?

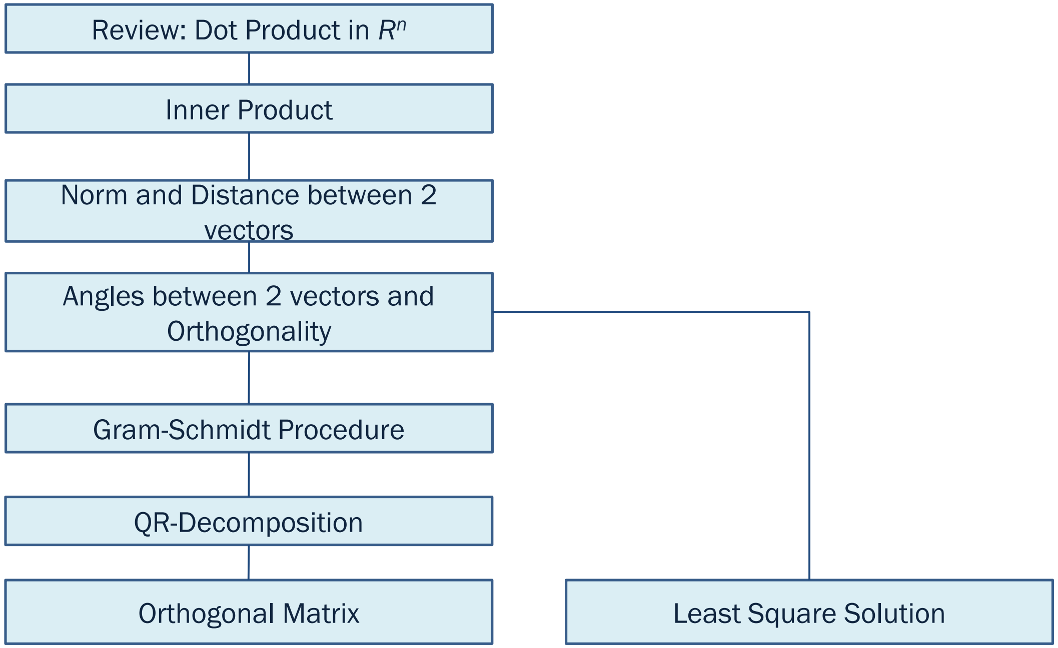
Compare your answers.

If both of your answers are: YES

it is possible that you are still not familiar with these new concepts. It is difficult to change the knowledge we know since ages ago. Try to review again on general vector spaces, You will understand that a matrix can be considered as a vector and it is not always the case that we can draw vectors as line segments that has direction (arrow).

If both of your answers are: NO

CONGRATULATIONS, vector is an element of a vector space. Vector can be any object such as matrices, continuous functions and even real numbers; these kind of vector cannot be represented as a line segment with direction.



Review: Dot Product in R^n

Dot Product in R^n

- Is **a.b** = **b.a**?
- Write two different formulas of : **a.b**
- Can you give a sample to validate the formula **a.b** = **b.a** ?
- Note:

  A sample helps to bring an abstract concept to be more concrete.

  A sample may help to think more generally, but it is not enough to validate a general concept

Dot Product in R^n

- Is $(k\mathbf{a}).\mathbf{b} = k(\mathbf{b}.\mathbf{a})$?

- ❑ Use some examples/samples to guess if it is true or false
- ❑ Then, prove that the property is generally applicable.

Dot Product in R^n

- Is **a**.(**b** + **c**) = **a**.**b** + **a**.**c** ?

- ❑ Use some examples/samples to guess if it is true or false
- ❑ Then, prove that the property is generally applicable.

Dot Product in R^n

- Find **a.a**
- In which cases does **a.a** = 0?

- ❑ Use some examples/samples to guess if it is true or false
- ❑ Then, prove that the property is generally applicable.

4 Properties of the Dot Product

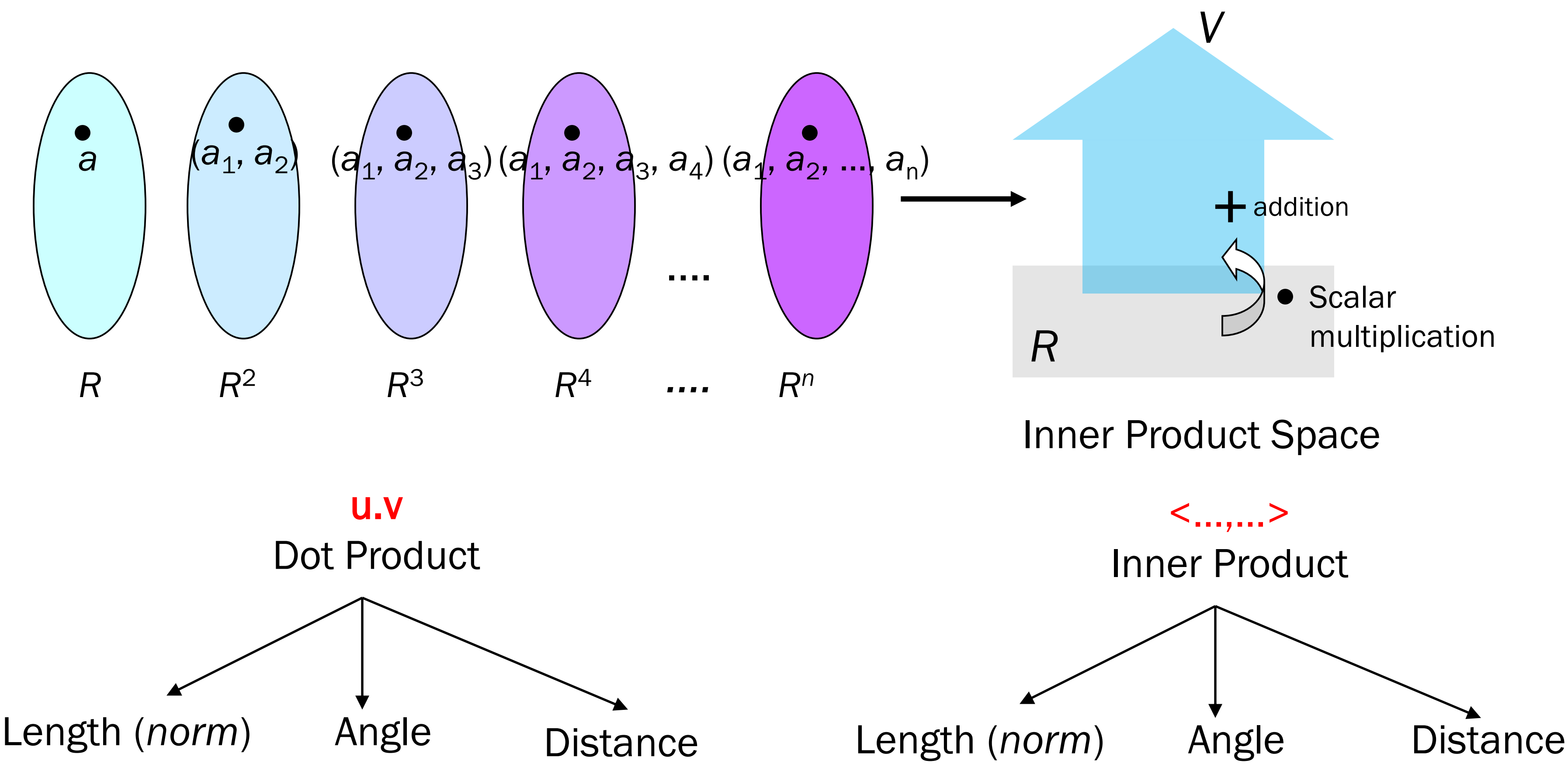
- Symmetry
- Homogeneity
- Additive
- Positive

Norm of a Vector, Angle between 2 Vectors

- Find the norm of vector **a** in R^n

- Find the angle between **a** and **b** in R^n

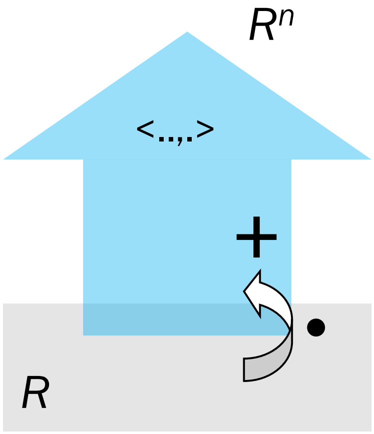
Euclidean Vector Space



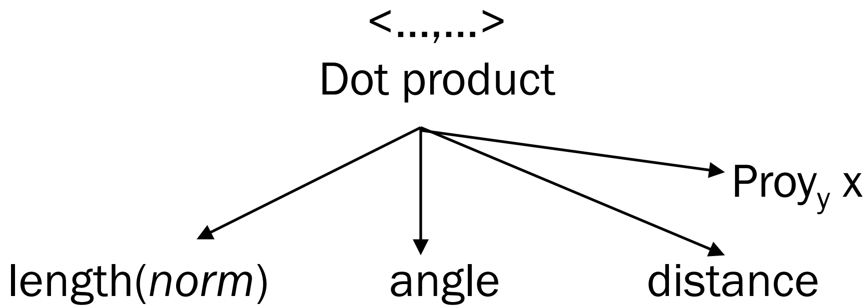
Properties of the Dot Product

The dot product in R^n satisfies 4 properties, i.e.

1. **u.v = v.u** (symmetry)
2. (k **u**).**v** = k (**u.v**) (homogeneity)
3. **u.**(**v** + **w**) = **u.v** + **u.w** (additive)
4. **v.v** = 0, **v.v** = 0 if and only if **v** = 0 (positive)



Euclid vector space



Next, we will define the inner product as a generalization of the dot product.

Review: Dot Product in R^n

Inner Product

Norm and Distance between 2 vectors

Angles between 2 vectors and Orthogonality

Gram-Schmidt Procedure

QR-Decomposition

Orthogonal Matrix

Least Square Solution

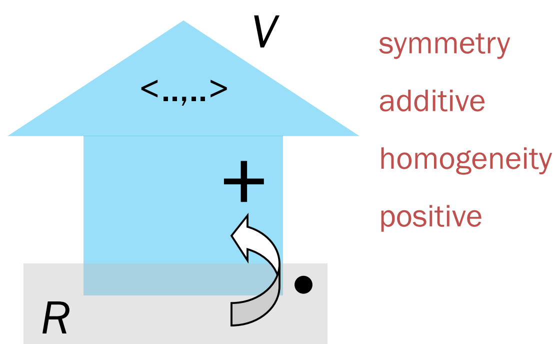
6.1 Inner Product

Inner Product

Definition 6.1.: *Inner Product*

An inner product of a vector space *V* over real numbers is a function that associates a real number <**u**,**v**> with every pair of vectors **u** and **v** in *V* in such a way that the following axioms are satisfied for all vectors **u**, **v**, and **w** in *V* and all scalars *k* :

1. <**u**, **v**> = <**v**, **u**> (symmetry axiom)
2. <**u** + **v**, **w**> = <**u**, **w**> + <**v**, **w**> (additive axiom)
3. <*k* **u**, **v**> = *k* <**u**, **v**> (homogeneity axiom)
4. <**v**, **v**> $\geq$ 0 where <**v**, **v**> = 0 if and only if **v** = **0** (positivity axiom)

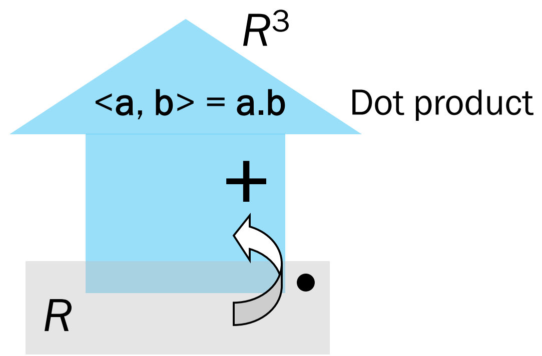


A vector space over real numbers with an inner product is called an **inner product space**

Euclid Inner Product

Let **u** and **v** be any two vectors in R^3, and it is defined that:

<**u**, **v**> = **u**.**v** (dot product)



Inner product space (on R)

As it is previously explained, the 4 following properties are satisfied:

1. <**u**, **v**> = <**v**, **u**>

$$\langle \mathbf{u},\mathbf{v}\rangle = u_1v_1 + u_2v_2 + u_3v_3$$
$$= v_1u_1 + v_2u_2 + v_3u_3$$
$$= \mathbf{v}.\mathbf{u}$$

It is proved that <**u**, **v**> = <**v**, **u**>

2. <(**u** + **v**), **w**> = <**u**, **w**> + <**v**, **w**>

$$\langle (\mathbf{u}+\mathbf{v}),\mathbf{w}\rangle = (u_1+v_1, u_2+v_2, u_3+v)_3.(w_1, w_2, w_3)$$
$$= (u_1+v_1)w_1 + (u_2+v_2)w_2 + (u_3+v_3)w_3$$
$$= (u_1w_1 + v_1w_1) + (u_2w_2 + v_2w_2) + (u_3w_3 + v_3w_3)$$
$$= \langle \mathbf{u},\mathbf{w}\rangle + \langle \mathbf{v},\mathbf{w}\rangle$$

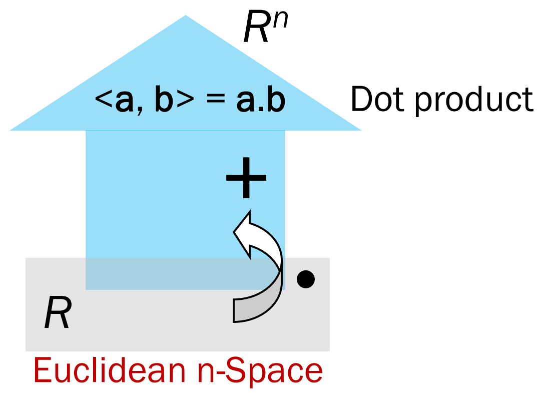
It is proved that <(**u** + **v**), **w**> = <**u**, **w**> + <**v**, **w**>

Proving properties 3 and 4 are for your exercise/homework. R^3 is an inner product space (Euclidean n-space)

Euclidean Inner Product

The dot product in R^n satisfies the 4 properties:

1. $<\mathbf{u}, \mathbf{v}> = <\mathbf{v}, \mathbf{u}>$
2. $<(\mathbf{u} + \mathbf{v}), \mathbf{w}> = <\mathbf{u}, \mathbf{w}> + <\mathbf{v}, \mathbf{w}>$
3. $<k\mathbf{u}, \mathbf{v}> = k <\mathbf{u}, \mathbf{v}>$
4. $<\mathbf{v}, \mathbf{v}> \geq 0$ where $<\mathbf{v}, \mathbf{v}> = 0$ if and only if $\mathbf{v} = \mathbf{0}$

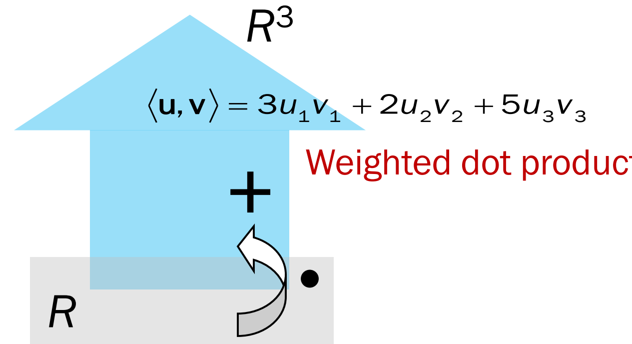


The dot product is an inner product of R^n, and it is called as the Euclidean inner product. R^n with a Euclidean inner product is called the Euclidean n-space.

Weighted Inner Product

If **u** and **v** are any two vectors in R^3, it is defined that:

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2 + 5u_3v_3$$



Inner product space (over R)

Try to find whether these 4 axioms are satisfied or not.

1. <**u**, **v**> = <**v**, **u**>
2. <(**u** + **v**), **w**> = <**u**, **w**> + <**v**, **w**>
3. <k **u**, **v**> = k <**u**, **v**>
4. <**v**, **v**> $\geq$ 0 where <**v**, **v**> = 0 if and only if **v** = **0**

Proof : Positive-weighted Inner Product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 2u_2v_2 + 5u_3v_3$$

$$\begin{aligned} 1. \langle \mathbf{u}, \mathbf{v} \rangle &= 3u_1v_1 + 2u_2v_2 + 5u_3v_3 \\ &= 3v_1u_1 + 2v_2u_2 + 5v_3u_3 \\ &= \langle \mathbf{v}, \mathbf{u} \rangle \end{aligned}$$

$$\begin{aligned} 2. \langle (\mathbf{u}, \mathbf{v}), \mathbf{w} \rangle &= \langle ((u_1 + v_1), (u_2 + v_2), (u_3 + v_3)), (w_1, w_2, w_3) \rangle \\ &= 3(u_1 + v_1)w_1 + 2(u_2 + v_2)w_2 + 5(u_3 + v_3)w_3 \\ &= (3u_1w_1 + 2u_2w_2 + 5u_3w_3) + (3v_1w_1 + 2v_2w_2 + 5v_3w_3) \\ &= \langle (\mathbf{u}, \mathbf{w}), (\mathbf{v}, \mathbf{w}) \rangle \end{aligned}$$

$$\begin{aligned} 3. \langle k\mathbf{u}, \mathbf{v} \rangle &= \langle k(u_1, u_2, u_3), (v_1, v_2, v_3) \rangle \\ &= \langle (ku_1, ku_2, ku_3), (v_1, v_2, v_3) \rangle \\ &= (3ku_1v_1 + 2ku_2v_2 + 5ku_3v_3) \\ &= k(3u_1v_1 + 2u_2v_2 + 5u_3v_3) \\ &= k\langle \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

Conclusion: a positive-weighted inner product function is an inner product in R^3, it is called as a weighted Euclidean inner product

4. <**v, v**> $\geq$ 0 where <**v, v**> = 0 if and only if **v = 0**
   (try to prove it as your exercise)

Exercise 1: negative weights

1. Let **u** and **v** be any two vectors in R^3, and it is defined that:

$$f(\mathbf{u},\mathbf{v}) = 3u_1v_1 + 2u_2v_2 - 5u_3v_3$$

Show that *f* is not an inner product.

Answer:

Let $\mathbf{v} = (1, 1, 1)$

$f(\mathbf{v}, \mathbf{v}) = 3.1.1.+ 2.1.1 -5.1.1 = 0$ where $\mathbf{v} \neq 0$.

The positive axiom is NOT satisfied, so *f* is not an inner product. If there is a negative weight, then that function is not an inner product.

2. Give an example of a function that maps $f(R^4 x R^4) \rightarrow R$ which is also a weighted inner product.

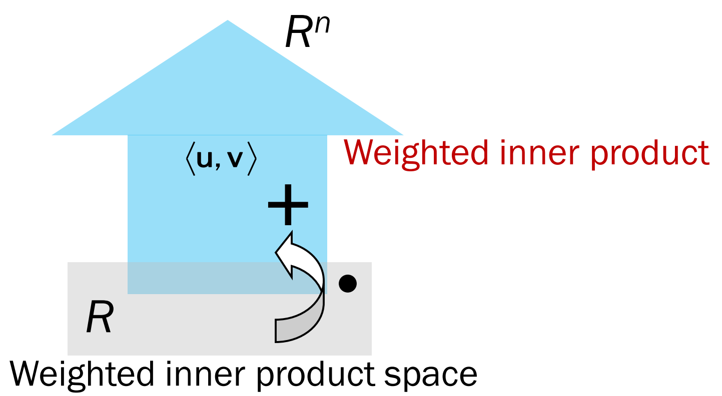
3. . Give an example of a function that maps $f(R^4 x R^4) \rightarrow R$ which is NOT an inner product.

Weighted Inner Products in R^n

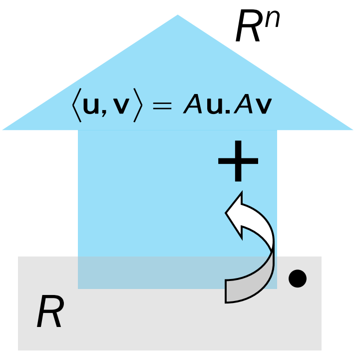
Definition 6.2.: *Weighted inner product* in R^n

If $\mathbf{u} = (u_1, u_2, u_3,\ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, v_3,\ldots, v_n)$ are vectors in R^n, then the weighted inner product with weights $w_1, w_2, w_3,\ldots, w_n$ where each weights is a real <u>positive</u> number, is defined as:

$$\langle \mathbf{u}, \mathbf{v} \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots + w_n u_n v_n$$



Inner Product on R^n generated by Matrices



A is a nxn matrix that is invertible. The inner product is defined as $<\mathbf{u}, \mathbf{v}> = A\mathbf{u}.A\mathbf{v}$

4 properties are satisfied, i.e.:

1. $A\mathbf{u}. A\mathbf{v} = A\mathbf{v}.A\mathbf{u}$
2. $A\ (\mathbf{u} + \mathbf{v}). A\mathbf{w} = A\mathbf{u}.A\mathbf{w} + A\mathbf{v}.A\mathbf{w}$
3. $A\ (k\ \mathbf{u}.A\mathbf{v}) = k\ (A\mathbf{u}.A\mathbf{v})$
4. $A\mathbf{v}.A\mathbf{v} \geq 0$ where $A\mathbf{v}.A\mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$

Note:

$A\mathbf{u}$ and $A\mathbf{v}$ are vectors in R^n. Such that, $A\mathbf{u}. A\mathbf{v}$ is a scalar. It is easy to show that the 4 axioms are satisfied by this inner product.

That vector space R^n is said to be an inner product space R^n that is generated by matrix A.

Exercise 2

1. For the inner product that is generated by matrix A, if A is the identity matrix, what kind of inner product will you get?

2. If all weights (in the weighted Euclidean inner product) are 1, what kind of inner product will you get?

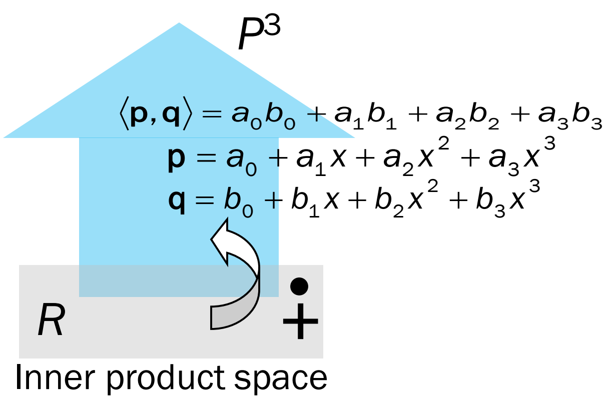
Inner Product on P^3 (1)

Any two vectors in P^3

$$\mathbf{p} = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$\mathbf{q} = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

The inner product is defined as:

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$



The following properties are satisfied:

1. <**u**, **v**> = <**v**, **u**>
2. <(**u** + **v**), **w**> = <**u**, **w**> + <**v**, **w**>
3. <k **u**, **v**> = k <**u**, **v**>
4. <**v**, **v**> $\geq$ 0 where <**v**, **v**> = 0 if and only if **v** = **0**

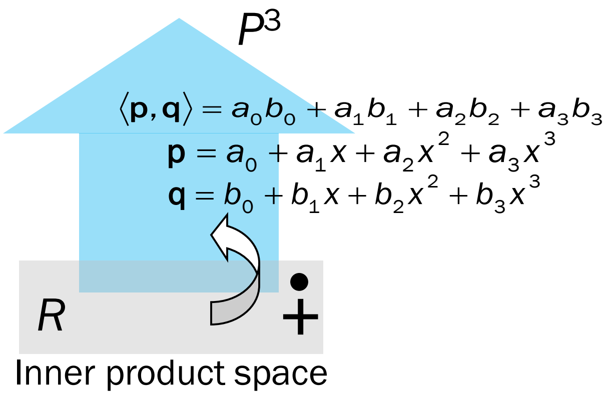
Conclusion:

Function <**p**, **q**> is an inner product on P^3

Inner Product on P^3 (2)

Any two vectors in P^3

$$\mathbf{p} = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$\mathbf{q} = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$



The inner product space is defined as:

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + 2a_1 b_1 + 7a_2 b_2 + 3a_3 b_3$$

The following four properties are satisfied:

1. <**u**, **v**> = <**v**, **u**>
2. <(**u** + **v**), **w**> = <**u**, **w**> + <**v**, **w**>
3. <*k* **u**, **v**> = *k* <**u**, **v**>
4. <**v**, **v**> $\geq$ 0 where <**v**, **v**> = 0 if and only if **v** = **0**

Conclusion:

The function <**p**, **q**> is an inner product on P^3

Not an inner product in P^3

Any two vectors in P^3

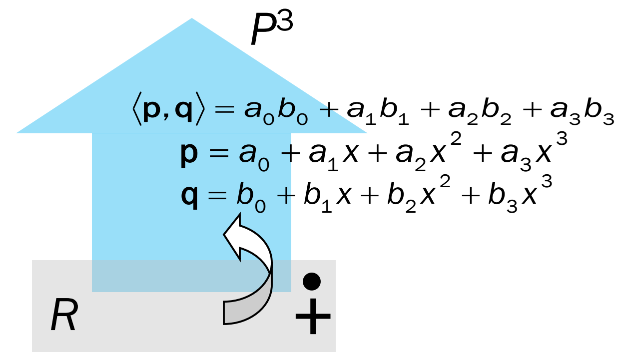
$$\mathbf{p} = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$\mathbf{q} = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

The inner product is defined as:

$$\langle \mathbf{p}, \mathbf{q} \rangle = 2a_0 b_0 + 2a_1 b_1 - 7a_2 b_2 + 3a_3 b_3$$

Is <**p**, **q**> an inner product on P^3?

[Hint: try for $\mathbf{p} = 1 + x + x^2 + x^3$ show that the 4th axiom is not satisfied]



Inner product space

Inner product on P^3 (3)

Any two vectors in P^3

$$\mathbf{p} = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$\mathbf{q} = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$

The inner product is defined as:

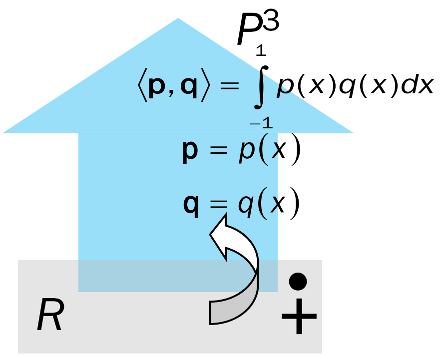
$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x) q(x) dx$$

The following four properties are satisfied:

1. <**u**, **v**> = <**v**, **u**>
2. <(**u** + **v**), **w**> = <**u**, **w**> + <**v**, **w**>
3. <k **u**, **v>** = k <**u**, **v**>
4. <**v**, **v**> $\geq$ 0 where <**v**, **v**> = 0 if and only if **v** = **0**

Conclusion:

<**p**, **q**> is an inner product on P^3



Inner product space

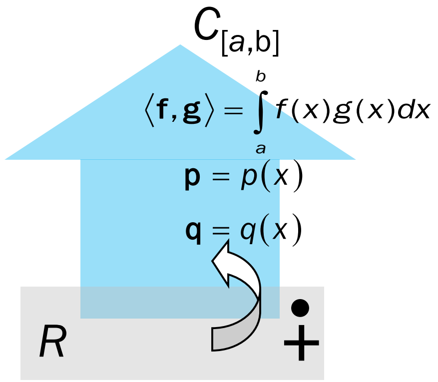
The definite integral: inner product on $C_{[a,b]}$

$C_{[a.b]}$ is a vector space that consists of continuous functions on the interval

$[a, b]$, $b > a$.

f and **g** are two vectors in $C_{[a,\,b]}$, where $\mathbf{f} = f(x)$, $\mathbf{g} = g(x)$

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_a^b f(x)g(x)dx$$



Inner product space

Inner product in M_{nxn}

M_{nxn} is a vector space that consists of *nxn* matrices

If $\mathbf{u} = A$, $\mathbf{v} = B$ are matrices in M_{nxn}, the inner product is defined as

$$<\mathbf{u}, \mathbf{v}> = \text{trace}(A^T B).$$

The function above is an inner product because the 4 axioms are satisfied.

1. $<\mathbf{u}, \mathbf{v}> = <\mathbf{v}, \mathbf{u}>$
2. $<\mathbf{u} + \mathbf{v}, \mathbf{w}> = <\mathbf{u}, \mathbf{w}> + <\mathbf{v}, \mathbf{w}>$
3. $<k\,\mathbf{u}, \mathbf{v}> = k<\mathbf{u}, \mathbf{v}>$
4. $<\mathbf{v}, \mathbf{v}> \geq 0$ where $<\mathbf{v}, \mathbf{v}> = 0$ if and only if $\mathbf{v} = \mathbf{0}$

So, M_{nxn} with the inner product above is an inner product space.

- Review: Dot Product in R^n
- Inner Product
- Norm and Distance between 2 vectors
- Angles between 2 vectors and Orthogonality
  - Gram-Schmidt Procedure
  - QR-Decomposition
  - Orthogonal Matrix
  - Least Square Solution

6.2 Norm and Distance between 2 Vectors

Norm (length) and Distance

Definition 6.3: Length and distance in an inner product space

Given an inner product space *V*, the *norm* (length) of a vector **v**, is defined as follows $\|\mathbf{v}\| = \sqrt{\langle v, v \rangle}$

The distance between **u** and **v** is $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

Example 1:

(a) **u** = (2, 3, 1) in the Euclidean space R^3

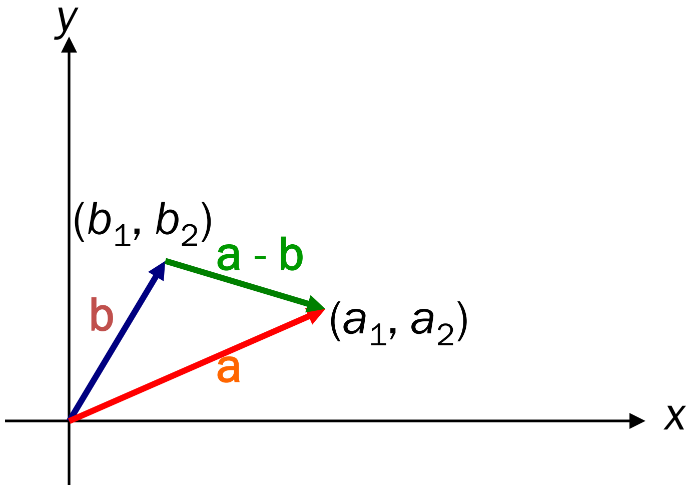
$$\|\mathbf{u}\| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

(b) **u** = (2, 3, 1) in R^3 where it has a weighted inner product, with weights 2, 1, 1 consecutively.

$$\|\mathbf{u}\| = \sqrt{2.2^2 + 1.3^2 + 1.1^2} = \sqrt{18}$$

Distance between two vectors in the Euclidean vector space

Distance between two vectors is the norm of the vector that is the difference between the two vectors



Distance between **a** and **b** is

$$\|\mathbf{a}-\mathbf{b}\| = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2}$$

Example 2: norm of matrix

M_{nxn} is a vector space that consists of *nxn* matrices. If **u** = *A*, **v** = *B* are matrices in M_{nxn}, the inner product is defined as <**u, v**>= trace(A^TB)

Let there be a matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

The norm of *A* is obtained as follows.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \mathbf{u}$$

$$\langle \mathbf{u}, \mathbf{u} \rangle = a_{11} a_{11} + a_{12} a_{12} + a_{21} a_{21} + a_{22} a_{22}$$

$$\|\mathbf{u}\| = \langle \mathbf{u}, \mathbf{u} \rangle^{1/2} = \sqrt{a_{11}{}^2 + a_{12}{}^2 + a_{21}{}^2 + a_{22}{}^2}$$

$$= \sqrt{1^2 + 2^2 + 2^2 + 0^2}$$

$$= 3$$

Example 3: norm of a polynomial

p = $p(x) = x^2 + 1$ in P^3 with the inner product defined as the definite integral from -1 to 1. Then, the norm of **p** is

$$\|\mathbf{p}\| = \sqrt{\int_{-1}^{1} (x^2+1)^2 dx} = \sqrt{\left[\tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 + x\right]_{-1}^{1}} = \ldots.$$

Example 4: distance between two polynomials

Based on the definition of inner product on P^3:

$$\mathbf{p} = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$
$$\mathbf{q} = b_0 + b_1 x + b_2 x^2 + b_3 x^3$$
$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$

Distance between two vectors:

$$\mathbf{p} = 2 + x + x^3$$

$$\mathbf{q} = 1 + 3x + x^2$$

$$\mathbf{p} - \mathbf{q} = 1 - 2x - 1x^2 + x^3$$

$$\|\mathbf{p} - \mathbf{q}\| = \sqrt{\langle \mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} \rangle}$$

$$= \sqrt{1 + 4 + 1 + 1} = \sqrt{7}$$

Properties of the Norm

Theorem 6.1: Properties of the norm

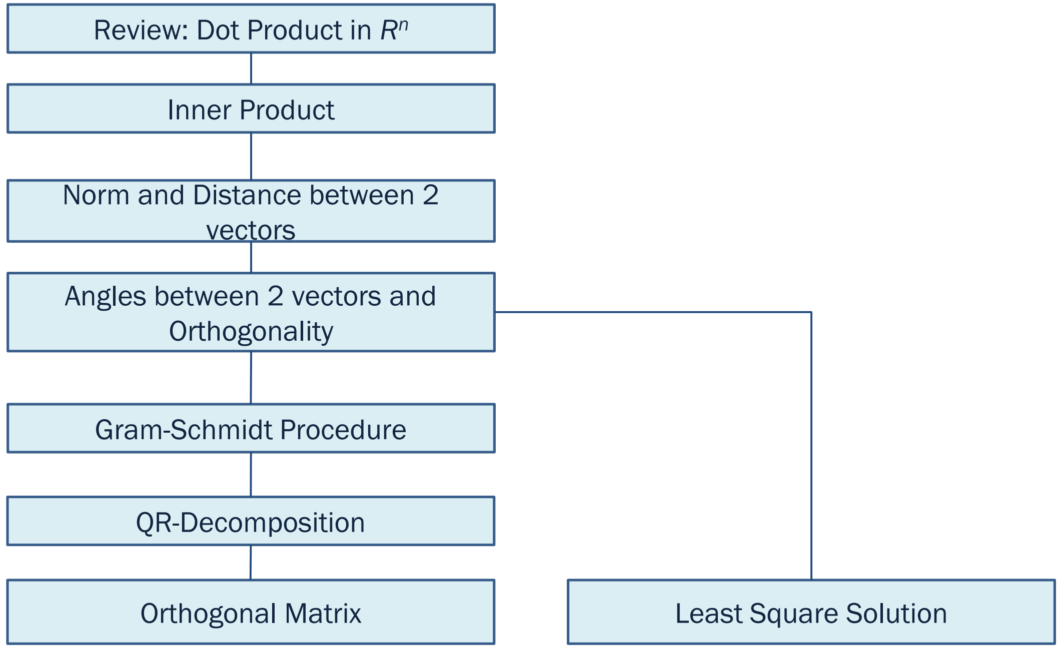
If **u** and **v** are vectors in the inner product space *V*, if *k* is a scalar, then

(1) $\|\mathbf{u}\| \geq 0$

(2) $\|\mathbf{u}\| = 0$, if and only if $\mathbf{u} = \mathbf{0}$

(3) $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$

(4) $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$



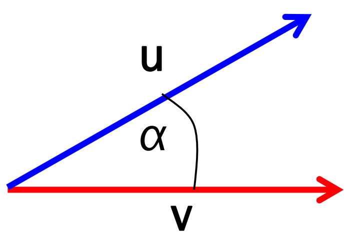
6.3 Angle between 2 vectors and orthogonality

The angle between two vectors in the Euclidean n-space

If **u** and **v** are nonzero vectors in R^2, then:

$$\mathbf{u}.\mathbf{v} = u_1 v_1 + u_2 v_2 = \|\mathbf{u}\|\|\mathbf{v}\| \cos \alpha$$

where α is the angle between **u** and **v**



$$\cos \alpha = \frac{\mathbf{u}.\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

The angle between two vectors in the general vector space

Definition 6.4.: Angle between 2 vectors in an inner product space *V*

If **u** and **v** are nonzero vectors in an inner product space, then:

$$\cos\alpha = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

As it is in the Euclidean vector space, $-1 \leq \cos\alpha \leq 1$.

This is guaranteed by the following theorem:

Theorem 6.2.: Cauchy- Schwartz Inequality

If **u** and **v** are two vectors in an inner product space over real numbers, then

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

Example 3: the cosinus of the angle between two vectors

Let **u =** (0, 1, 0), **v** = (2, 1, 2), where the inner product is defined as follows:

$$<\mathbf{u}, \mathbf{v}> = u_1v_1+u_2v_2+u_3v_3$$

Determine the cosinus angle between **u** and **v.**

Answer:

$$\cos\alpha = \frac{\mathbf{u.v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{0.2+1.1+2.0}{1.3} = \frac{1}{3}$$

Example 4: cosinus angle between two vectors

Let **u =** (0, 1, 0), **v** = (2, 1, 2) where the weighted inner product on R^n is defined as follows:

$$\langle \mathbf{u}, \mathbf{v} \rangle = 1u_1v_1 + 2u_2v_2 + 3u_3v_3$$

Determine the cosinus angle between **u** and **v** in the Euclidean vector space R^3

Answer: $$\cos \alpha = \frac{\mathbf{u}.\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1.0.2 + 2.1.1 + 3.2.0}{1.3} = \frac{2}{3}$$

Example 5: cosinus of the angle between two vectors

An inner product is defined in M^{nxn}: **<u, v>** = trace(A^TB)

Let there be two vectors: $A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

The cosinus angle between those two vectors is

$$\cos\alpha = \frac{\mathbf{u}.\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{2.2 + 3.1 + 1.1 + 1.3}{\left(\sqrt{15}\right)^2} = \frac{11}{15}$$

Note:

$$\|\mathbf{u}\| = \sqrt{2.2 + 3.3 + 1.1 + 1.1} = \sqrt{15}$$

$$\|\mathbf{v}\| = \sqrt{2.2 + 1.1 + 1.1 + 3.3} = \sqrt{15}$$

Orthogonal Vectors

*D*efinition 6.5.: orthogonal vectors

Two vectors **a, b** in an inner product space *V* are called orthogonal if and only if <**a, b**> = 0

Example 6:

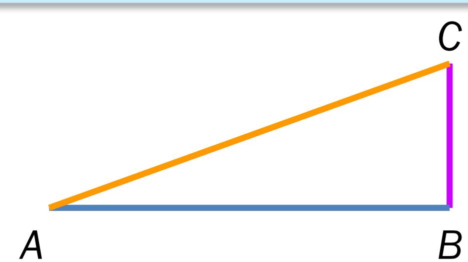
Let **p** = *p*(x) = *x* dan **q** =*q*(x) = x^2 in P^3 where <**p.q**> is defined as:

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$$

Such that, $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} xx^2 dx = \frac{1}{4}x^4 \Big]_{-1}^{1} = 0$

Hence, **p** and **q** are orthogonal

Pythagorean Theorem



$AC^2 = AB^2 + BC^2$

The Pythagorean Theorem also applies to inner product spaces.

Let **u** and **v** be orthogonal vectors, <**u, v**> = 0

$$\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle$$ (*norm* definition)

$$= \langle \mathbf{u}+\mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u}+\mathbf{v}, \mathbf{v} \rangle$$ (inner product property)

$$= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle$$ (inner product property)

$$= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$$ (<**u, v**> = 0 because they are orthogonal)

$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Theorem 6.3: Pythagorean Theorem

If **u** and **v** are orthogonal vectors in an inner product space, then:

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Example 7: Applying the Pythagorean Theorem

Let there be two orthogonal vectors in R^4 :

$$\mathbf{u} = (0,1,0,0)$$
$$\mathbf{v} = (1,0,1,2)$$

then,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \langle \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v} \rangle = \langle (1,1,1,2),(1,1,1,2) \rangle = 7$$

$$\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = 0+1+0+0 = 1$$

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 1+0+1+4 = 6$$

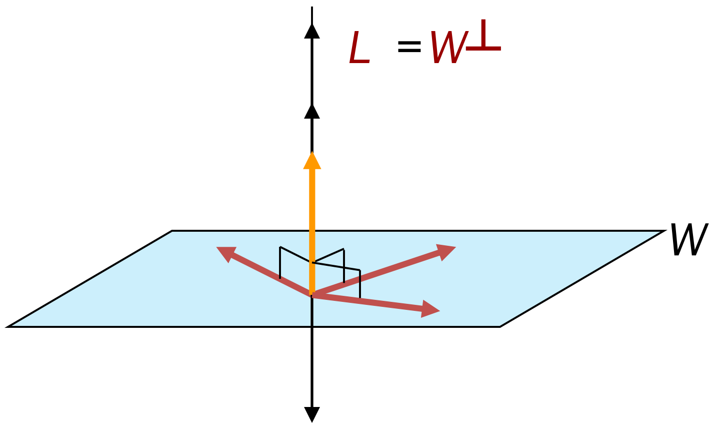
It is shown that: $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = 7$

Orthogonal Complement

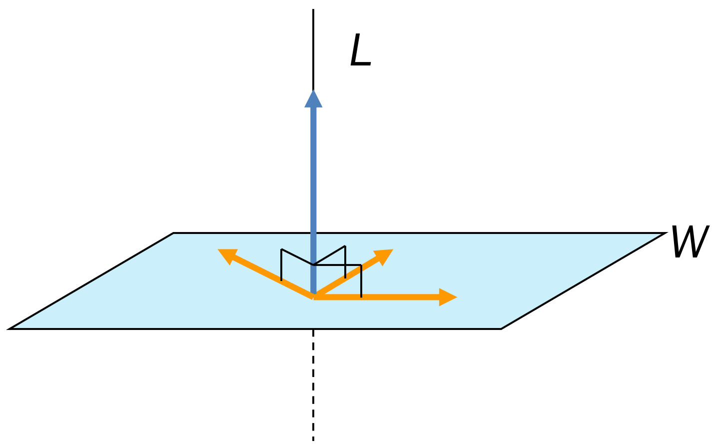
Definition 6.6.:

Let W be a subspace of the inner product space V.
Vector **v** is orthogonal to W if **v** is orthogonal to every other vector in W.

The set of all vectors that are orthogonal to W is called the orthogonal complement of W, and it is denoted by $W^{\perp}$ (read: W perp)



Orthogonal Complement (cont)



Line L and the plane W are subspaces of R^3. Every vector on L is orthogonal to every vector on the *xy*-plane. If two subspaces are orthogonal complement to each other (e.g. L is orthogonal complement of W and vice versa), then $L = W^{\perp}$

In geometry, the symbol $\perp$ means "orthogonal".
$W^{\perp}$ is read as "W perp", from the word "perpendicular".

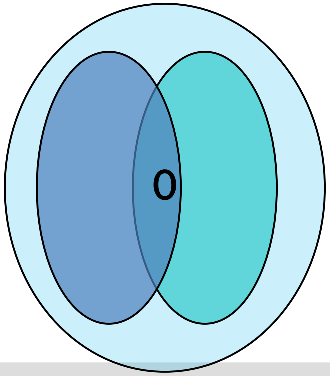
The Basic Property of Orthogonal Complement

If W is a subspace of the finite dimensional vector space V, then:

1. $W^{\perp}$ is a subspace of V
2. The vector which is a member of both W and $W^{\perp}$ is only the zero vector **0**.
3. Orthogonal complement of $W^{\perp}$ is W

To show that $W^{\perp}$ is a subspace, we need to show that $W^{\perp}$ is not empty, closed under addition, and closed under scalar multiplication.

If **a** is a vector in W and $W^{\perp}$, then $\langle \mathbf{a}, \mathbf{a} \rangle = 0$, is satisfied if and only if $\mathbf{a} = \mathbf{0}$ (positivity axiom of an inner product space)



Example 5

1. If V is an inner product space, determine the orthogonal complement of $\{\mathbf{0}\}$.

   Answer: V.

2. If V is an inner product space, and W is a subspace of V.

   If $W = W^{\perp}$. What can you conclude based on that info?

   Answer: $V = \{\mathbf{0}\}$

3. W is the $y = -x$ line on the coordinate system of R^2. Determine $W^{\perp}$.

   Answer: the $y = x$ line.

4. If W is the yz-plane on the coordinate system of R^3. Determine $W^{\perp}$.

   Answer: the x-axis

Null(A) and Row (A) are Orthogonal Complement to each other

Proof:

1. Let there be any vector **v** that is orthogonal to every vector in Row(A), show that $A\mathbf{v} = \mathbf{0}$, where **v** is an element of Null(A)
2. If $A\mathbf{v} = \mathbf{0}$, show that **v** is orthogonal to every vector in Row(A).

v is orthogonal to every vector in Row(A), then **v** is orthogonal to the row space vectors $r_1, r_2, \ldots, r_m$

$$\mathbf{r}_1.\mathbf{v} = \mathbf{r}_2.\mathbf{v} = \ldots = \mathbf{r}_m\mathbf{v} = 0$$

$$\begin{bmatrix} \mathbf{r}_1.\mathbf{v} \\ \mathbf{r}_2.\mathbf{v} \\ \cdots \\ \mathbf{r}_m.\mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

LS $A\mathbf{x} = \mathbf{0}$

v is element of Null(A)

Null(A) and Row (A) are Orthogonal Complement to each other

Proof:

1. Let there be any vector **v** that is orthogonal to every vector in Row(A), show that $A\mathbf{v} = \mathbf{0}$, where **v** is an element of Null(A)
2. If $A\mathbf{v} = \mathbf{0}$, show that **v** is orthogonal to every vector in Row(A).

$A\mathbf{v} = 0$, then $\mathbf{r}_1.\mathbf{v} = \mathbf{r}_2.\mathbf{v} = \ldots = \mathbf{r}_m\mathbf{v} = 0$

If **k** is any vector in Row(A), then **k** can be written as a linear combination of the vectors in Row(A): $\mathbf{k} = c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + \cdots + c_m\mathbf{r}_m$

then
$$\begin{aligned}\mathbf{k}.\mathbf{v} &= (c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + \cdots + c_m\mathbf{r}_m).\mathbf{v}\\ &= c_1(\mathbf{r}_1.\mathbf{v}) + c_2(\mathbf{r}_2.\mathbf{v}) + \ldots + c_m(\mathbf{r}_m.\mathbf{v})\\ &= c_1.0 + c_2.0 + \ldots + c_m.0 = 0\end{aligned}$$

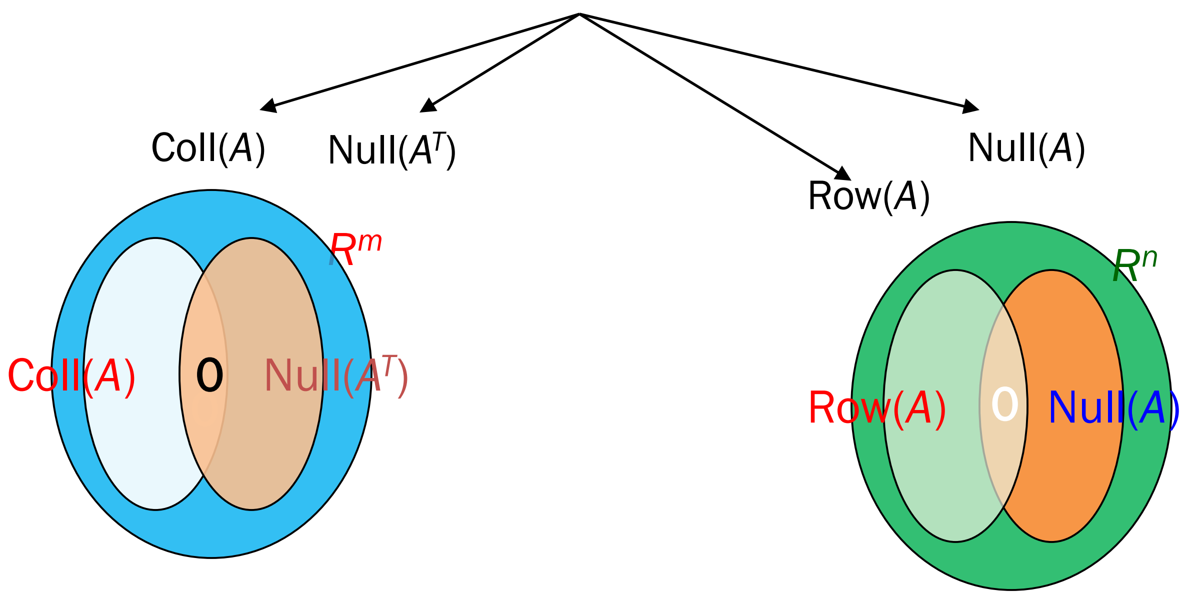
v is orthogonal to every vector in Row(A)

Null(A) and Row (A) are Orthogonal Complement to each other

- Coll(A^T) = Row(A)
- Based on the previous explanation, substituting A with A^T, we obtain that Coll(A) and (Null(A^T)) are orthogonal complement to each other

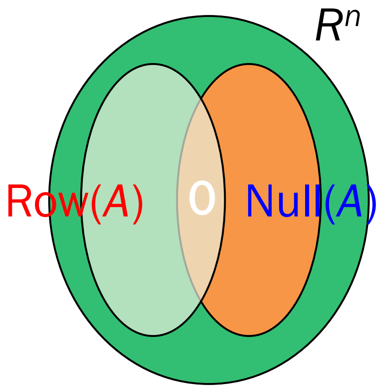
Row(A), Coll(A), Null(A)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1..n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2..n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3..n} \\ \dots & \dots & \dots & \dots & a_{4..n} \\ a_{n..1} & a_{n..2} & a_{n..3} & \dots & a_{m..n} \end{pmatrix}$$



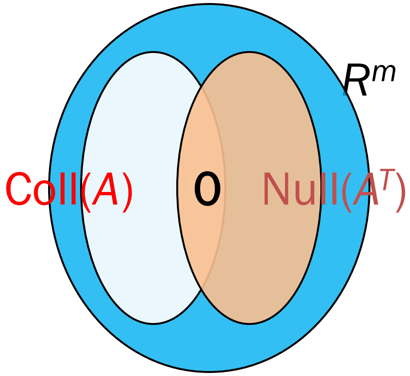
Relation between Row(A), Coll(A), Null(A)

Null(A) and Row(A) are orthogonal complements in R^n

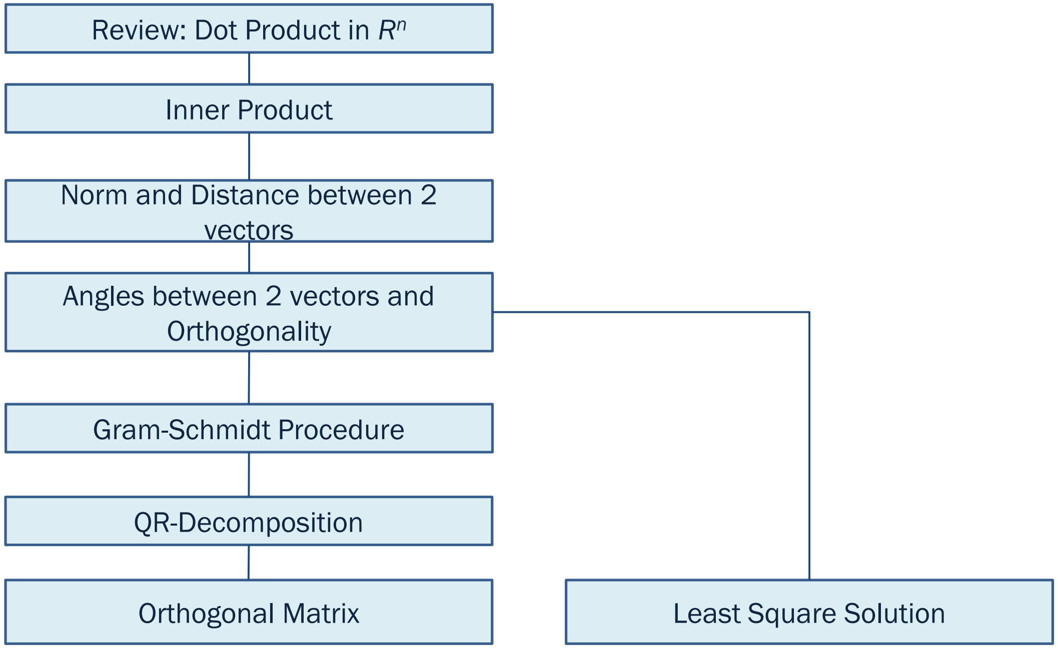


Null(A) = [Row(A)]$^{\perp}$

Null(A^T) and Coll(A) are orthogonal complements in R^m



Null(A^T) = [Coll(A)]$^{\perp}$



6.4 Gram-Schmidt Procedure

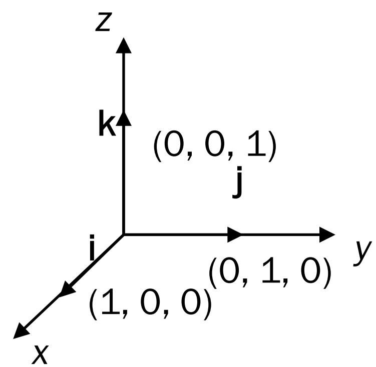
Orthogonal and Orthonormal Basis

Definition 6.7.a.: Orthogonal Set

A set *S* in the inner product space *V* is said to be orthogonal if and only if every pair of its different vectors are orthogonal.

Definition 6.7.b: Orthogonal and Orthonormal Basis

V is a vector space, *B* is a basis of *V*. *B* is an orthogonal basis if *B* is an orthogonal set. If B is orthogonal and the *norm* of every vector in *B* is 1, then B is an orthonormal basis.



Example 8:

1. { **i, j, k** } orthogonal basis in R^3

2. $\{1, x, x^2, x^3\}$ Orthogonal basis in P^3

Given a basis of a finite-dimensional vector space, we can obtain an orthonormal basis using the Gram-Schmidt procedure.

Example 9: orthonormal basis in P^3

Let there be an orthonormal basis of P^3, i.e. $B = \{1, x, x^2, x^3\}$ where its inner product is:

$$\langle \mathbf{p}, \mathbf{q} \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$

- Proof:

$$B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

$$\mathbf{v}_1 = 1 + 0x + 0x^2 + 0x^3$$

$$\mathbf{v}_2 = 0 + 1x + 0x^2 + 0x^3$$

$$\mathbf{v}_3 = 0 + 0x + 1x^2 + 0x^3$$

$$\mathbf{v}_4 = 0 + 0x + 0x^2 + 1x^3$$

Every pair of its vector is orthogonal:

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 1.0 + 0.1 + 0.0 + 0.0 = 0$$

$$\langle \mathbf{v}_1, \mathbf{v}_3 \rangle = 1.0 + 0.0 + 0.1 + 0.0 = 0$$

$$\langle \mathbf{v}_1, \mathbf{v}_4 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \langle \mathbf{v}_2, \mathbf{v}_4 \rangle = \langle \mathbf{v}_3, \mathbf{v}_4 \rangle = 0$$

The norm of every vector = 1

$$\|\mathbf{v}_1\| = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1.1 + 0 + 0 + 0 = 1 \qquad \|\mathbf{v}_3\| = \|\mathbf{v}_4\| = 1$$

$$\|\mathbf{v}_2\| = \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = 0 + 1.1 + 0 + 0 = 1$$

Coordinate relative to an orthonormal basis

$$\mathbf{u}_1 = \left(0, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$$

$$\mathbf{u}_2 = \left(\frac{1}{3}\sqrt{6}, -\frac{1}{6}\sqrt{6}, \frac{1}{6}\sqrt{6}\right)$$

$$\mathbf{u}_3 = \left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}\right)$$

$$\mathbf{a} = (1,1,1)$$

B = {$\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$}
write **a** as a coordinate relative to B

Coordinate relative to an orthonormal basis

Theorem 6.4:

If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthonormal basis of vector space *V*, and **a** is an element of *V*, then

$$\mathbf{a} = \langle \mathbf{a}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{a}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \ldots + \langle \mathbf{a}, \mathbf{u}_n \rangle \mathbf{u}_n$$

Every vector in *V* can be expressed as a linear combination of the vectors in basis where the coefficients are the inner product of that vector with the vectors in the basis.

Example 10:

$$\mathbf{u}_1 = \left(0, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right)$$

$$\mathbf{u}_2 = \left(\frac{1}{3}\sqrt{6}, -\frac{1}{6}\sqrt{6}, \frac{1}{6}\sqrt{6}\right)$$

$$\mathbf{u}_3 = \left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}\right)$$

$$\mathbf{a} = (1,1,1)$$

B = {$\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$} is an orthonormal basis of *V*, **a** is a vector in *V* then **a** can be expressed as a linear combination of the vectors in basis.

$$\mathbf{a} = \sqrt{2}\mathbf{u}_1 + \frac{1}{3}\sqrt{6}\mathbf{u}_2 + \frac{1}{3}\sqrt{3}\mathbf{u}_3$$

Coordinate matrix **a** relative to *B*: $[\mathbf{a}]_B = \left(\sqrt{2}, \frac{1}{3}\sqrt{6}, \frac{1}{3}\sqrt{3}\right)$

Norm, Angle and Distance Formula

Theorem 6.5:

If *B* is an orthonormal basis of an inner product space, and if

$$[\mathbf{u}]_B = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n) \text{ and } [\mathbf{v}]_B = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$$

then

(a) $\|\mathbf{u}\| = \sqrt{\mathbf{u}_1^{\ 2} + \mathbf{u}_2^{\ 2} + \ldots + \mathbf{u}_n^{\ 2}}$

(b) $d(\mathbf{u}, \mathbf{v}) = \sqrt{(\mathbf{u}_1 - \mathbf{v}_1)^2 + (\mathbf{u}_2 - \mathbf{v}_2)^2 + \ldots + (\mathbf{u}_n - \mathbf{v}_n)^2}$

(c) $< \mathbf{u}, \mathbf{v} > = \mathbf{u}_1\mathbf{v}_1 + \mathbf{u}_2\mathbf{v}_2 + \ldots + \mathbf{u}_n\mathbf{v}_n$

Example 11: determine the norm of **a** = (1, 1, 1).

Option 1. using dot product

$$\|\mathbf{a}\| = \langle \mathbf{a}.\mathbf{a} \rangle^{\frac{1}{2}} \qquad \|\mathbf{a}\| = \sqrt{1+1+1} = \sqrt{3}$$

Option 2. using coordinate vector relative to the orthonormal basis

$$[\mathbf{a}]_B = \left(\sqrt{2}, \frac{1}{3}\sqrt{6}, \frac{1}{3}\sqrt{3}\right) \qquad \|\mathbf{a}\| = \sqrt{\mathbf{u}_1^{\ 2} + \mathbf{u}_2^{\ 2} + \mathbf{u}_3^{\ 2}} = \sqrt{2 + \frac{6}{9} + \frac{3}{9}} = \sqrt{3}$$

Orthogonal Basis

Theorem 6.6.:
If $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ is an orthogonal basis of the inner product space *V*, and **a** is an element of *V*, then

$$\mathbf{a} = \frac{\langle \mathbf{a}, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\langle \mathbf{a}, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 + \ldots + \frac{\langle \mathbf{a}, \mathbf{u}_n \rangle}{\|\mathbf{u}_n\|^2}\mathbf{u}_n$$

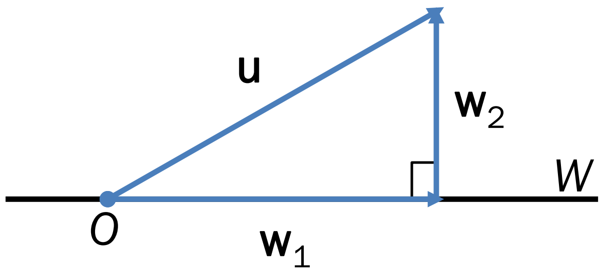
Example 12:

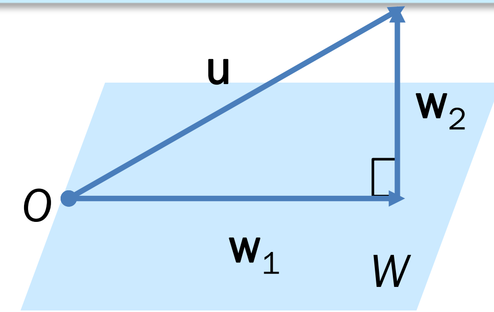
$$B = \left\{(0,1,1), \left(1, -\frac{1}{2}, \frac{1}{2}\right), \left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right)\right\} \qquad \mathbf{a} = (1,1,1)$$

B is an orthogonal basis of *V*, **a** is a vector in *V* then **a** can be expressed as a linear combination of vectors of basis.

$$\mathbf{a} = 1(0,1,1) + \frac{2}{3}\left(1, -\frac{1}{2}, \frac{1}{2}\right) + \frac{1}{2}\left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right) \qquad [\mathbf{a}]_B = \left(1, \frac{2}{3}, \frac{1}{2}\right)$$

Projection Theorem





If W is a line or plane that goes through the main axis, then every vector **u** in the euclidean vector space can be expressed as: $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$
where $\mathbf{w}_1$ is along (inside of) W, and $\mathbf{w}_2$ is perpendicular to W

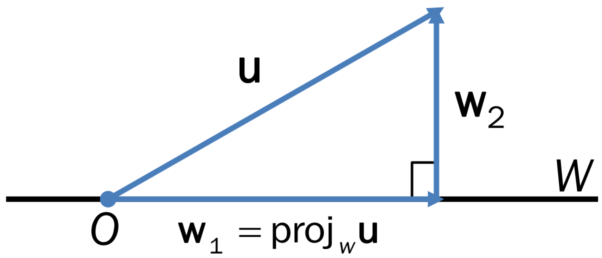
Theorem 6.7.: Projection Theorem

If W is a finite-dimensional subspace of an inner product space V, then every vector **u** in V can be expressed in exactly one way as:

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ is in W, and $\mathbf{w}_2$ is in $W^{\perp}$.
$\mathbf{w}_1$ is the orthogonal projection of **u** on W ($\mathrm{proj}_W \mathbf{u}$), $\mathbf{w}_2$ is the component of **u** orthogonal to W.

Orthogonal Projection in a Subspace



Theorem 6.8: Orthogonal Projection in a Subspace

Let W be a finite-dimensional subspace of an inner product space V.

(a) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r$$

(b) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W, and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \ldots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2} \mathbf{v}_r$$

Finding Orthogonal and Orthonormal Basis

Orthonormalization proses using Gram-Schmidt:

$B = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ is a basis for inner product space V

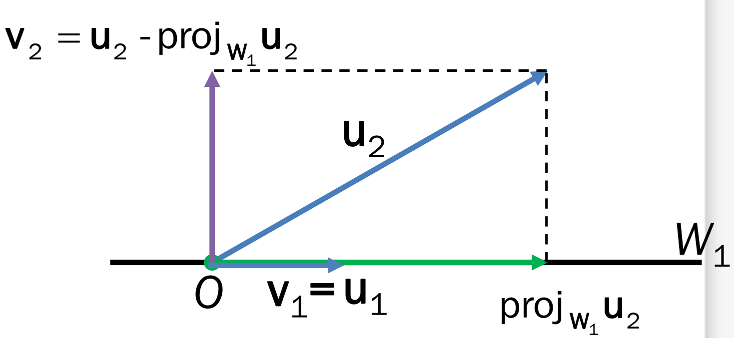
Jika $\mathbf{v}_1 = \mathbf{u}_1$

$$\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

$$\vdots$$

$$\mathbf{v}_n = \mathbf{u}_n - \text{proj}_{W_{n-1}}\mathbf{u}_n = \mathbf{u}_n - \sum_{i=1}^{n-1}\frac{\langle \mathbf{v}_n, \mathbf{v}_i \rangle}{\|\mathbf{v}_i\|^2}\mathbf{v}_i$$



$\mathbf{w}_1 = span(\{\mathbf{v}_1\})$

$\mathbf{w}_2 = span(\{\mathbf{v}_1, \mathbf{v}_2\})$

$\Rightarrow B' = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ Is an orthogonal basis

$\Rightarrow B'' = \{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \cdots, \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}\}$ Is an orthonormal basis

Example 13: Gram-Schmidt Process

Given a basis of R^3 $B = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$. Change the basis into an orthonormal basis using Gram-Schmidt process.

1. Take $\mathbf{v}_1 = \mathbf{u}_1$

$$\mathbf{v}_1 = (0,1,1)$$

2. $\mathbf{v}_2 = \mathbf{u}_2 - \text{proj}_{W_1}\mathbf{u}_2 = \mathbf{u}_2 - \dfrac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$

$$\mathbf{v}_2 = (1,0,1) - \frac{0.1 + 1.0 + 1.1}{0^2 + 1^2 + 1^2}(0,1,1)$$

$$\mathbf{v}_2 = (1,0,1) - \frac{1}{2}(0,1,1)$$

$$\mathbf{v}_2 = \left(1, -\frac{1}{2}, \frac{1}{2}\right)$$

3. $\mathbf{v}_3 = \mathbf{u}_3 - \text{proj}_{W_2}\mathbf{u}_3 = \mathbf{u}_3 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \dfrac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$

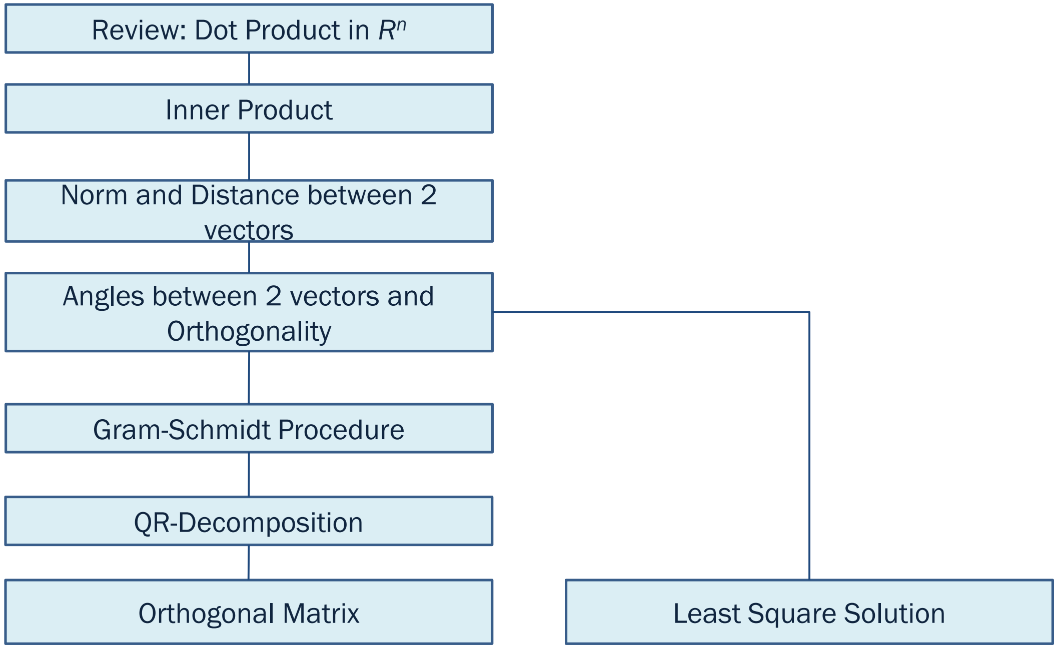
$$\mathbf{v}_3 = (1,1,0) - \frac{1}{2}(0,1,1) - \frac{1}{3/2}\left(1, -\frac{1}{2}, \frac{1}{2}\right)$$

$$\mathbf{v}_3 = \left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

4. Orthogonal basis is obtained : $B' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

5. Orthonormal basis : $B'' = \left\{\dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \dfrac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \dfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|}\right\}$

$$B'' = \left\{ \begin{array}{l} \left(0, \frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}\right), \left(\frac{1}{3}\sqrt{6}, -\frac{1}{6}\sqrt{6}, \frac{1}{6}\sqrt{6}\right), \\ \left(\frac{1}{3}\sqrt{3}, \frac{1}{3}\sqrt{3}, -\frac{1}{3}\sqrt{3}\right) \end{array} \right\}$$



6.5 QR-Decomposition

Gram-Schmidt Process on matrices with linearly independent columns

- Let matrix A have linearly independent columns, such that it forms a basis for Coll(A):

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$$

- After applying Gram-Schmidt proses, orthonormal basis is obtained

$$B^* = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$$

- Express every column vector as a linear combination of orthonormal basis

$$\begin{aligned}
\mathbf{u}_1 &= \langle \mathbf{u}_1, \mathbf{p}_1 \rangle \mathbf{p}_1 + \langle \mathbf{u}_1, \mathbf{p}_2 \rangle \mathbf{p}_2 + \ldots + \langle \mathbf{u}_1, \mathbf{p}_n \rangle \mathbf{p}_n \\
\mathbf{u}_2 &= \langle \mathbf{u}_2, \mathbf{p}_1 \rangle \mathbf{p}_1 + \langle \mathbf{u}_2, \mathbf{p}_2 \rangle \mathbf{p}_2 + \ldots + \langle \mathbf{u}_2, \mathbf{p}_n \rangle \mathbf{p}_n \\
&\vdots \\
\mathbf{u}_n &= \langle \mathbf{u}_n, \mathbf{p}_1 \rangle \mathbf{p}_1 + \langle \mathbf{u}_n, \mathbf{p}_2 \rangle \mathbf{p}_2 + \ldots + \langle \mathbf{u}_n, \mathbf{p}_n \rangle \mathbf{p}_n
\end{aligned}$$

G-S (cont'd)

$$\mathbf{u}_1 = \langle \mathbf{u}_1, \mathbf{p}_1 \rangle \mathbf{p}_1 + \langle \mathbf{u}_1, \mathbf{p}_2 \rangle \mathbf{p}_2 + \ldots + \langle \mathbf{u}_1, \mathbf{p}_n \rangle \mathbf{p}_n$$

$$\mathbf{u}_2 = \langle \mathbf{u}_2, \mathbf{p}_1 \rangle \mathbf{p}_1 + \langle \mathbf{u}_2, \mathbf{p}_2 \rangle \mathbf{p}_2 + \ldots + \langle \mathbf{u}_2, \mathbf{p}_n \rangle \mathbf{p}_n$$

$$\vdots$$

$$\mathbf{u}_n = \langle \mathbf{u}_n, \mathbf{p}_1 \rangle \mathbf{p}_1 + \langle \mathbf{u}_n, \mathbf{p}_2 \rangle \mathbf{p}_2 + \ldots + \langle \mathbf{u}_n, \mathbf{p}_n \rangle \mathbf{p}_n$$

Can be written as:

$$A = [\mathbf{u}_1 | \mathbf{u}_2 | \cdots | \mathbf{u}_n] = [p_1 | p_2 | \cdots | p_n] \begin{bmatrix} \langle \mathbf{u}_1, \mathbf{p}_1 \rangle & \langle \mathbf{u}_2, \mathbf{p}_1 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{p}_1 \rangle \\ \langle \mathbf{u}_1, \mathbf{p}_2 \rangle & \langle \mathbf{u}_2, \mathbf{p}_2 \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{p}_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \mathbf{u}_1, \mathbf{p}_n \rangle & \langle \mathbf{u}_2, \mathbf{p}_n \rangle & \cdots & \langle \mathbf{u}_n, \mathbf{p}_n \rangle \end{bmatrix}$$

$A = QR$, R is an upper triangular matrix, because $\mathbf{u}_j$ is orthogonal to $\mathbf{p}_k$ for $k>j$, the inner product is 0.

QR-Decomposition

Theorem 6.9:

If A is a matrix with linearly independent column vectors, then A can be decomposed into a product of two matrices, i.e. $A = QR$; where Q is a matrix with orthonormal columns and R is an invertible upper triangular matrix.

QR-Decomposition can be applied on numerical algorithms; e.g. to find an eigenvalue of a large matrix.

Example 14: *QR*-Decomposition

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{Gram-Schmidt process \& normalization}} \mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} -\frac{3}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 0 \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix},$$

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Determine *R* (where its inner product is the dot product).

Find QQ^T and Q^TQ

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} \frac{1}{2} & \frac{-3}{\sqrt{12}} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$Q^TQ = I$$

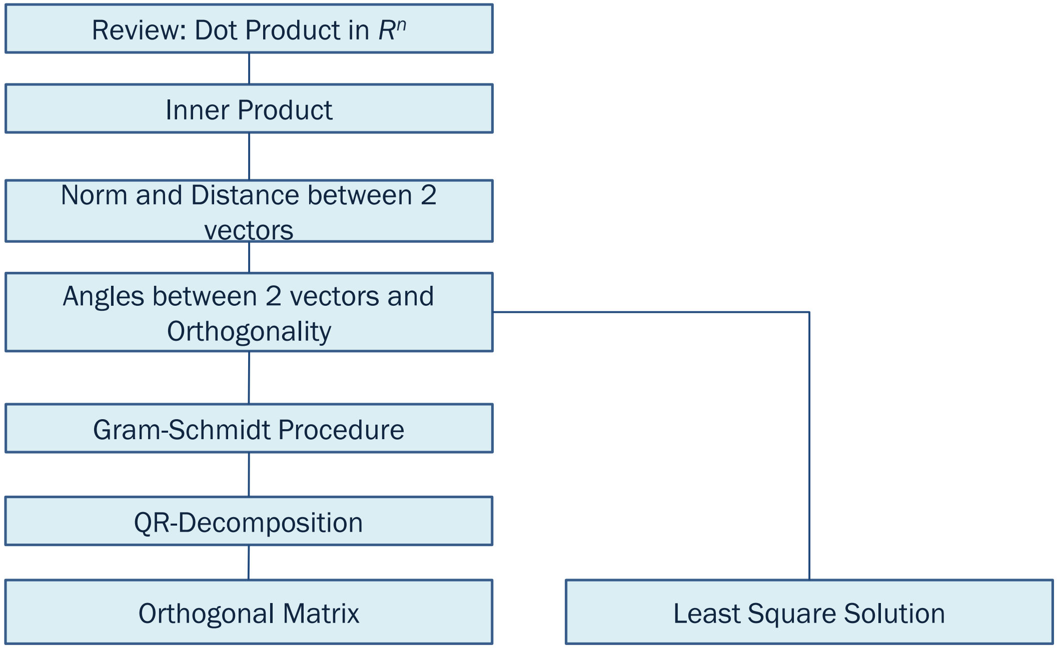
$$QQ^T = I$$

$$R = IR = Q^TQR = Q^T(QR) = Q^TA$$

$$R = \begin{bmatrix} 2 & \frac{3}{2} & 1 \\ 0 & \frac{3}{\sqrt{12}} & \frac{2}{\sqrt{12}} \\ 0 & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$$

QR-Decomposition Procedure

- Let A have linearly independent columns.
- B = basis of Coll(A) which consists of columns of A
- Apply Gram-Schmidt process on B and normalize to obtain orthonormal basis of Coll(A)
- Let Q be the matrix with orthonormal columns
- R is an invertible upper triangular matrix, i.e. $R = Q^T A$



6.6 Orthogonal Matrix

Gram-Schmidt Process on Invertible Square Matrices

- A is an invertible square matrix
- A set which consists of columns of A, i.e. $\{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ forms a basis for Coll(A)
- Coll(A) = R^n
- After applying the G-S process and normalize B, then we obtain the normal standard basis for R^n

Orthogonal Matrix

Definisi 6.8:

A *nxn* matrix *A* is orthogonal if and only if $A^T = A^{-1}$

Which means

$A^TA = I$ $\qquad A = [\, \mathbf{c}_1 \;\; \mathbf{c}_2 \;\; \ldots \; \mathbf{c}_n]$

$(\mathrm{I})ij = \mathbf{c}_i^T \mathbf{c}_j = \mathbf{c}_{i.}\mathbf{c}_j$

So $\mathbf{c}_i.\mathbf{c}_j = 0$ if $i \neq j$

$\mathbf{c}_i.\mathbf{c}_j = 1$ if $i = j$, where $\mathbf{c}_i.\mathbf{c}_i = \|\mathbf{c}_i\|^{2.}$

Conclusion:

A set which consists of column vectors of *A* is orthonormal; a set which consist of the rows of *A* is also orthonormal.

Orthogonal Matrix Property

If A is an orthogonal matrix, then

- $\mathrm{Det}(A) = \pm 1$
- A^{-1} is orthogonal
- If $\mathbf{x}$ a vector in R^n then $\|\mathbf{x}\| = \|A\mathbf{x}\|$

  (vector $\mathbf{x}$ and its transformation has the same norm)

Proof:

- $\mathrm{Det}(A) = \det(A^T) = \det(A^{-1}) = 1/\det(A)$. Then $\det(A) = 1$ or -1
- $(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T$
- $\|A\mathbf{x}\| = \sqrt{(A\mathbf{x})^T A\mathbf{x}} = \sqrt{\mathbf{x}^T A^T A\mathbf{x}} = \sqrt{\mathbf{x}^T I\mathbf{x}} = \sqrt{\mathbf{x}^T\mathbf{x}} = \|\mathbf{x}\|$

Example 15: Orthogonal Matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

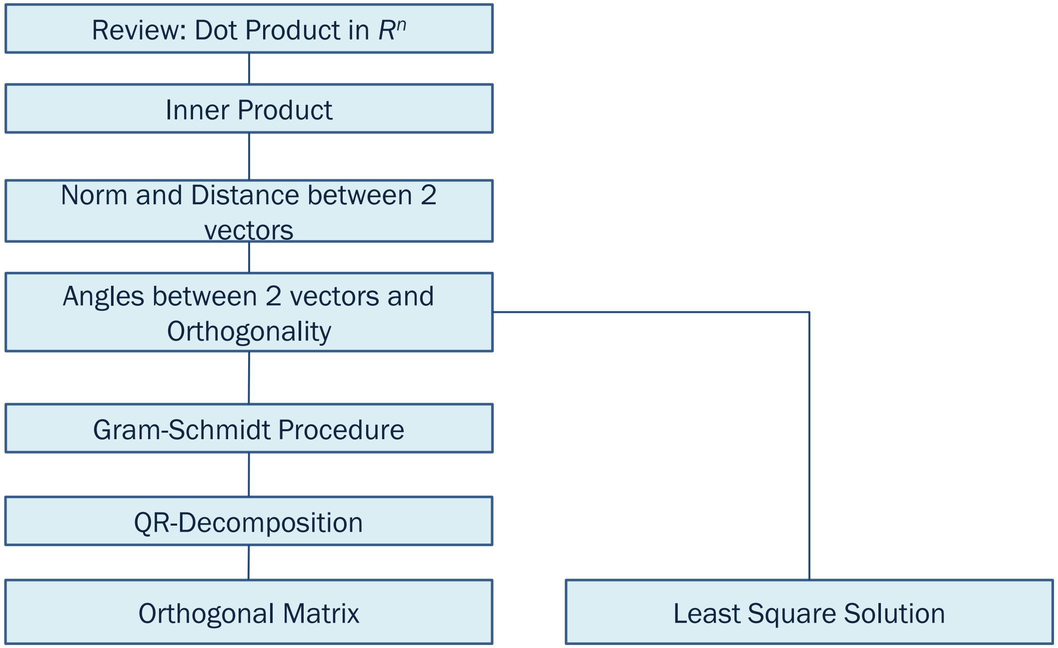
$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Properties of Orthogonal Matrices

1. The inverse of an orthogonal matrix is also an orthogonal matrix.
2. A product of orthogonal matrices will result in an orthogonal matrix.
3. If A is orthogonal, then det(A) = 1 or det(A) = -1.
4. Orthogonal matrices have orthonormal columns (prove it!)

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n]$$

$$[I]_{ij} = \left[A^T A\right]_{ij} = \mathbf{a}_i{}^T \mathbf{a}_j = \begin{cases} 1, \text{ if } \ i = j \\ 0, \text{ if } \ i \neq j \end{cases}$$

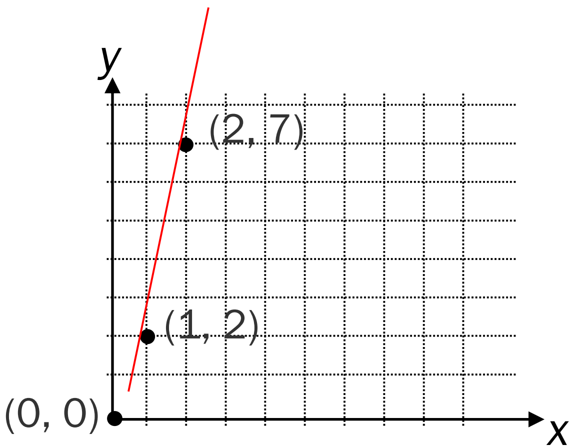


6.7 Least Square Solution

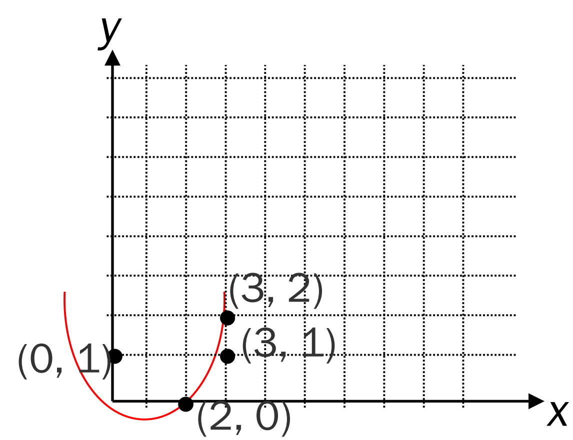
Match a graph on a data

One of the most frequent problems we encounter is finding the relation between two factors (e.g. two variables x and y) and express it into an equation $y = f(x)$ that corresponds to (x, y) which are raw data we obtain beforehand, e.g. (x_1, y_1), (x_2, y_2), (x_3, y_3), …, (x_n, y_n)

Straight Line



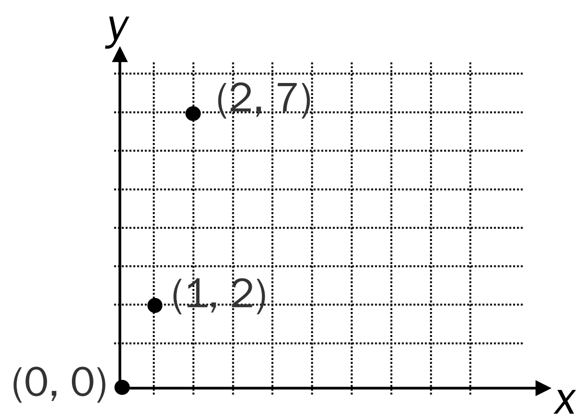
$$y = ax + b$$



$$y = ax^2 + bx + c$$

Match a straight line on a data

Let there be points on a plane. Find a line equation that is most suitable to describe the data.



We obtain the following linear system

$$0 = a.0 + b$$
$$2 = a.\,1 + b$$
$$7 = a.\,2 + b$$

where its matrix equation is:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$$

The linear system does not have a solution (inconsistent) because the points are not on the same line. Question: **how to find the most suitable line that describes the data?**

Brief Review

Two principles are needed in this process.

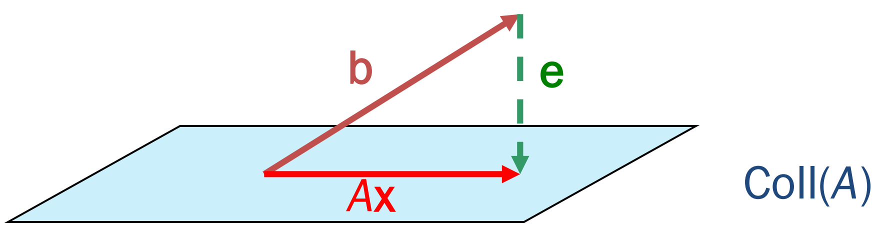
1. Consistency Theorem: LS $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

2. $\text{Null}(A^T) = (\text{Coll}(A))^{\perp}$

Least Square Solution

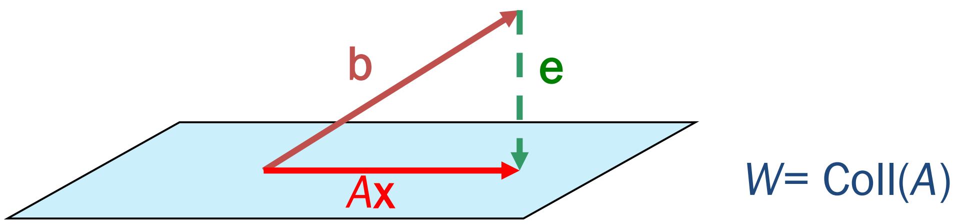
Least Square Solution:

Let $A\mathbf{x} = \mathbf{b}$ be a linear system that has *m* equations with *n unknowns*, determine $\mathbf{x}$ (if possible) that minimizes $||A\mathbf{x} - \mathbf{b}||$ in the euclidean n-space. Vector $\mathbf{x}$ is called the least square solution of $A\mathbf{x} = \mathbf{b}$.



Least square solution $\mathbf{x}$ produces the vector $A\mathbf{x}$ on Coll(A) that is the closest to $\mathbf{b}$.

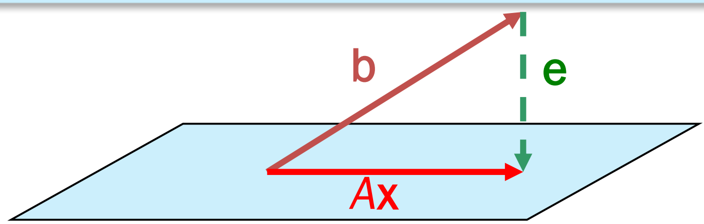
Finding the best approximation



Finding the solution of an approximation is finding $\mathbf{x}^*$ such that $A\mathbf{x}^* = \text{proy}_{\text{Coll}(A)}\, \mathbf{b}$

Problem : A and **b** is known, find **x** such that $A\mathbf{x} = \text{proy}_{\text{Coll(A)}}\, \mathbf{b}$**.** It is not easy to find $\text{proy}_{\text{Coll}(A)}\, \mathbf{b}$

Normal System



W= Coll(A)

Given A and **b**, finding the best approximation with the least error means finding **x** that satisfies: $A\mathbf{x} = \text{proy}_{\text{Coll}(A)}\mathbf{b}$

$A\mathbf{x} = \text{proy}_{\text{Coll}(A)}\mathbf{b}$. (shown in the picture above)

$\mathbf{b} - A\mathbf{x} = \mathbf{b} - \text{proy}_{\text{Coll}(A)}\mathbf{b}$

$\mathbf{b} - A\mathbf{x}$ orthogonal to W, and $W^{\perp} = \text{Null}(A^T)$, the set of solutions of $A^T\mathbf{x} = \mathbf{0}$. $\mathbf{b}-A\mathbf{x}$ is inside of the nullspace of A^T.

Hence, $A^T(\mathbf{b}-A\mathbf{x}) = \mathbf{0}$ or $A^T\mathbf{b} = A^TA\mathbf{x}$ which is known as the normal system of $A\mathbf{x} = \mathbf{b}$.

Such that, finding the approximation of the solution of $A\mathbf{x} = \mathbf{b}$, is the same as finding the solution of the consistent linear system, i.e. the normal system.

Definition: Least Square Solution

If A is a mxn matrix, the least square solution of $A\mathbf{x} = \mathbf{b}$ is the solution $\mathbf{x}^*$ from the normal system $A^TA\mathbf{x}^* = A^T\mathbf{b}$

Normal System (cont'd)

Given a linear system $A\mathbf{x} = \mathbf{b}$ with m equations with n *unknowns* each, then:

$A^T\mathbf{b} = A^TA\mathbf{x}$ is the normal system of said linear system.

1. The normal system consist of n equations with n *unknowns*.
2. The normal system is guaranteed to be consistent and the solution of a normal system is called as the least square solution.
3. If vector x is the least square solution, then: $\text{proy}_{\text{Coll}(A)}\mathbf{b} = A\mathbf{x}$
4. The normal system can have infinitely many solutions

If the columns of A are linearly independent, then the least square solution is exactly one.

If the columns of A is linearly independent, then A^TA has an inverse, such that the least square solution is as follows

$$A^TA\mathbf{x} = A^T\mathbf{b} \qquad \text{(multiply with } (A^TA)^{-1}\text{)}$$

$$(A^TA)^{-1}A^TA\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} \qquad \text{(associative)}$$

$$\mathbf{x} = (A^TA)^{-1}A^T\mathbf{b} \qquad \text{(least square solution)}$$

Example 16: a straight line (1)

Given points (0, 0), (1, 2), and (2, 7). We will find the line equation that most represents these data. We construct a linear system from the data we have.

A line equation has the general form of $y = ax + b$. We need to find a and b.

If the points are all on the line, then a and b satisfy the following linear system:

$0 = a.\,0 + b$

$2 = a.\,1 + b$ where the matrix equation is: $\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$

$7 = a.\,2 + b$

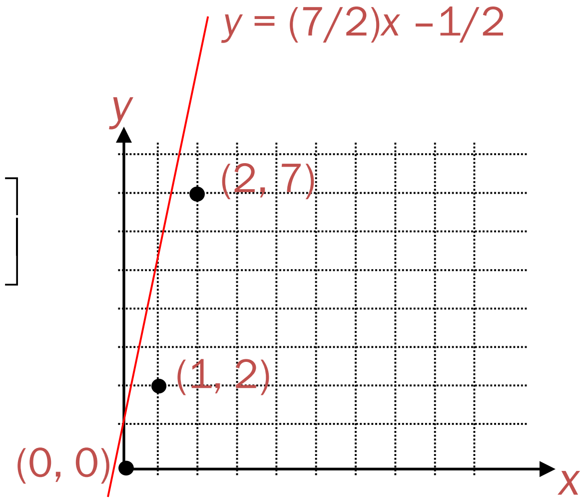
Give notations to the matrices

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$$

$$M^T M = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \qquad M^T\mathbf{b} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 16 \\ 9 \end{bmatrix}$$

The normal system:

$$M^T M\mathbf{v} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 16 \\ 9 \end{bmatrix}$$



Solution: $a = 7/2$, $b = -1/2$. Such that, $y = 7/2x - 1/2$

A Straight Line

Let there be n data points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)$

And we want to match a straight line $y = ax + b$ that will represent the data above.

If the points are on the line, then a and b satisfy

$$\begin{aligned} y_1 &= ax_1 + b \\ y_2 &= ax_2 + b \\ y_3 &= ax_3 + b \\ \vdots & \quad \vdots \\ y_n &= ax_n + b \end{aligned} \qquad \text{written} \quad \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

The normal system

$$M \mathbf{u} = \mathbf{y}$$

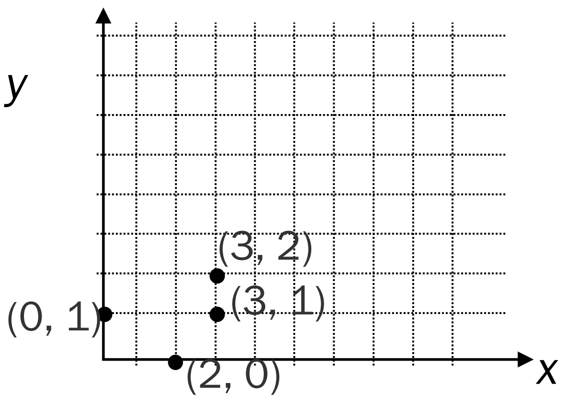
$$M^T M.\mathbf{u} = M^T \mathbf{y}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ \ldots & \ldots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \ldots \\ y_n \end{bmatrix}$$

Finding the solution to the normal system, we obtain the values for the line equation.

Example 17: parabola (1)

Given data points (0, 1), (2, 0), (3, 1), and (3, 2)



The general form of a parabola is: $y = ax^2 + bx + c$

Substitute the variables in the equation based on the data points we have to form a linear system, then find the normal system.

Solution: parabola (cont'd)

$$\begin{aligned} y_1 &= ax_1^2 + bx_1 + c \\ y_2 &= ax_2^2 + bx_2 + c \\ y_3 &= ax_3^2 + bx_3 + c \\ y_4 &= ax_4^2 + bx_4 + c \end{aligned} \longrightarrow \begin{aligned} 1 &= a(0)^2 + b(0) + c \\ 0 &= a(2)^2 + b(2) + c \\ 1 &= a(3)^2 + b(3) + c \\ 2 &= a(3)^2 + b(3) + c \end{aligned}$$

Written as: $\begin{pmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

Normal system:

$$\begin{pmatrix} 0 & 4 & 9 & 9 \\ 0 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 & 4 & 9 & 9 \\ 0 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$$

The solution of normal system:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \left(\begin{pmatrix} 0 & 4 & 9 & 9 \\ 0 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \\ 9 & 3 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 & 4 & 9 & 9 \\ 0 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -11/6 \\ 1 \end{pmatrix}$$

Parabola (cont'd)

Given data points (0, 1), (2, 0), (3, 1), and (3, 2)

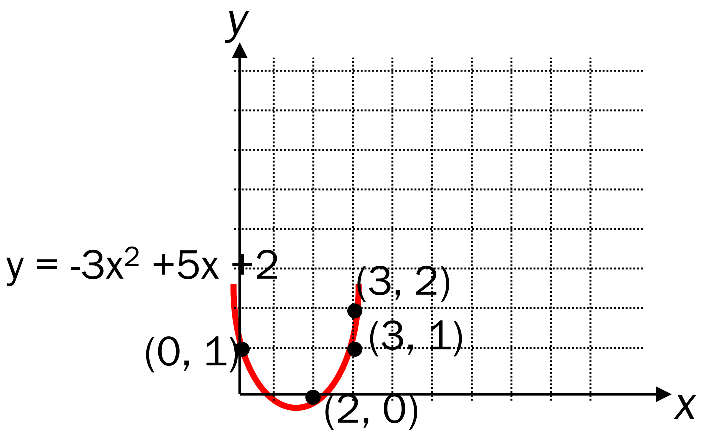
The general form of a parabola is $y = ax^2 + bx + c$

The solution:

$a = 2/3$

$b = -11/6$

$c = 1$



The parabola equation:

$y = -2x^2/3 -11x/6 +1$

Exercise 6

- What is the least square solution of LS $A\mathbf{x} = \mathbf{b}$ if the linear system itself is consistent?

  Answer:

  It is the same as the solution of said linear system.

- Let $A\mathbf{x} = \mathbf{b}$ be an inconsistent linear system. When can we determine the least square solution by calculating $\mathbf{x} = \left(A^T A\right)^{-1}\left(A^T\mathbf{b}\right)$?

  Answer:

  When columns of A are linearly independent, or when $(A^T A)^{-1}$ exists

Reflection

1. Summarize the material you just learned.
2. List 5 new things that you learned from this module where you have not learned it before anywhere else.
3. Find 2 real world problems that you can solve by using the method you just learned in this module.

Summary

- Check your summary, Does it contain the following key concepts?
  - ❑ Inner product
  - ❑ Weighted inner product
  - ❑ Norm of a vector and distance between two vectors
  - ❑ Angle between two vectors
  - ❑ The generalized pythagorean theorem
  - ❑ Orthogonal basis
  - ❑ Orthonormal basis
  - ❑ Cauchy-Swartz inequality
  - ❑ Orthogonal complement and its properties
  - ❑ Relation (orthogonal complement) between row space, column space, null space
  - ❑ Gram-Schmidt Process
  - ❑ QR-Decomposition
  - ❑ Least square solution
  - ❑ Normal system

Post-test module

MODUL E-LEARNING ALJABAR LINIER
FAKULTAS ILMU KOMPUTER, UNIVERSITAS INDONESIA

Post-test

- Answer the following questions.

Complete the following sentences.

1. The inner product space is.....................
2. In the Euclidean vector space R^2, and given vectors **a**= (-1, 2), **b** = (4, -2),
   a. Draw the vectors in that space
   b. determine <**a, b**>
3. If $\mathbf{p} = a_0 + a_1x + a_2x^2$ and $\mathbf{q} = b_0 + b_1x + b_2x^2$ is any vector in P^2.
   A function is defined as $f(\mathbf{p},\mathbf{q}) = \langle \mathbf{p},\mathbf{q} \rangle = a_0b_0 + a_1b_1 + a_2b_2$

   The angle between x^2 and $2x$ in P^2 is ....

Determine whether these statements are True/False.

1. The least square solution of $A\mathbf{x} = \mathbf{b}$ is $(A^TA)^{-1}A^T\mathbf{b} = A^{-1}\mathbf{b}$
2. If $||\mathbf{a} + \mathbf{b}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2$, then **a** and **b** orthogonal.
3. A consistent linear system does not have a least square solution.

Congratulations, you have finished module 6.
Prepare for the next module