

# Singular Value Decomposition

# SVD - Overview

A technique for handling matrices (sets of equations) that do not have an inverse. This includes square matrices whose determinant is zero and all rectangular matrices.

Common usages include computing the least-squares solutions, rank, range (column space), null space and pseudoinverse of a matrix.

# SVD - Basics

The SVD of a *m-by-n* matrix  $\mathbf{A}$  is given by the formula :

$$\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^T$$

Where :

$\mathbf{U}$  is a *m-by-n* matrix of the orthonormal eigenvectors of  $\mathbf{A}\mathbf{A}^T$

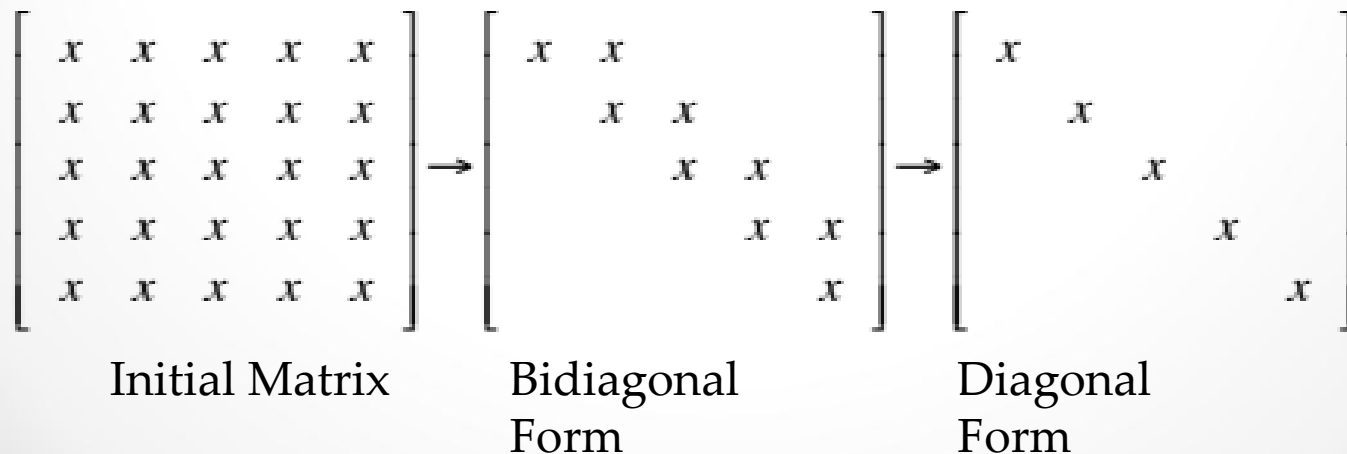
$\mathbf{V}^T$  is the transpose of a *n-by-n* matrix containing the orthonormal eigenvectors of  $\mathbf{A}^T\mathbf{A}$

$\mathbf{W}$  is a *n-by-n* Diagonal matrix of the *singular values* which are the square roots of the eigenvalues of  $\mathbf{A}^T\mathbf{A}$

# The Algorithm

Derivation of the SVD can be broken down into two major steps [2] :

1. Reduce the initial matrix to bidiagonal form using Householder transformations
2. Diagonalize the resulting matrix using QR transformations



# Householder Transformations

A Householder matrix is defined as :

$$H = I - 2ww^T$$

Where  $w$  is a unit vector with  $|w|^2 = 1$ .

It ends up with the following properties :

$$H = H^T$$

$$H^{-1} = H^T$$

$$H^2 = I \text{ (Identity Matrix)}$$

If multiplied by another matrix, it results in a new matrix with zero'ed elements in a selected row / column based on the values chosen for  $w$ .

# Applying Householder

To derive the bidiagonal matrix, we apply successive Householder matrices :

$$P_1 \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \xrightarrow{M_1} \begin{bmatrix} x & x & x & x & x \\ & x & x & x & x \\ & x & x & x & x \\ & x & x & x & x \\ & x & x & x & x \end{bmatrix} \xrightarrow{S_1} P_2 \begin{bmatrix} x & x & & & \\ & x & x & x & x \\ & x & x & x & x \\ & x & x & x & x \\ & x & x & x & x \end{bmatrix} \xrightarrow{M_2}$$

$$\begin{bmatrix} x & x & & & \\ & x & x & x & x \\ & & x & x & x \\ & & x & x & x \\ & & x & x & x \end{bmatrix} \xrightarrow{S_2} \dots \xrightarrow{P_m} \begin{bmatrix} x & x & & & \\ & x & x & x & x \\ & & x & x & x \\ & & & x & x \\ & & & x & x \end{bmatrix} \xrightarrow{M_m}$$

$$\begin{bmatrix} x & x & & & \\ & x & x & & \\ & & x & x & x \\ & & & x & x \\ & & & x & x \end{bmatrix} \xrightarrow{S_n} \begin{bmatrix} x & x & & & \\ & x & x & & \\ & & x & x & \\ & & & x & x \\ & & & & x \end{bmatrix}$$

$M_n$   $B$

# Application con't

From here we see :

$$P_1 M = M_1$$

$$M_1 S_1 = M_2$$

$$P_2 M_2 = M_3$$

....

$$M_N S_N = B \text{ [If } M > N, \text{ then } P_M M_M = B]$$

This can be re-written in terms of  $M$  :

$$M = P_1^T M_1 = P_1^T M_2 S_1^T = P_1^T P_2^T M_3 S_1^T = \dots = P_1^T \dots P_M^T B S_N^T \dots S_1^T = P_1 \dots P_M B S_N \dots S_1$$

(Because  $H^T = H$ )

# Householder Derivation

Now that we've seen how Householder matrices are used, how do we get one? Going back to its definition :  $H = I - 2ww^T$

Which is defined in terms of  $w$  - which is defined as

$$w = \frac{(x - y)}{\|x - y\|} \quad \text{and} \quad Hx = y \quad \text{and} \quad \|x\| = \|y\|$$

To make the Householder matrix useful,  $w$  must be derived from the column (or row) we want to transform.

This is accomplished by setting  $x$  to row / column to transform and  $y$  to desired pattern.



# Householder Example

To derive  $P_I$  for the given matrix  $M$ :

$$M = \begin{bmatrix} 4 & 3 & 0 & 2 \\ 2 & 1 & 2 & 1 \\ 4 & 4 & 0 & 3 \end{bmatrix}$$

We would have :

$$x = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} \quad y = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad \text{With : } \|x\| = \sqrt{4^2 + 2^2 + 4^2} = \sqrt{36} = 6$$

$$\|y\| = \sqrt{6^2 + 0^2 + 0^2} = \sqrt{36} = 6$$

This leads to :

$$w = \frac{(x - y)}{\|x - y\|} = \frac{(4 - 6), (2 - 0), (4 - 0)}{\|(4 - 6), (2 - 0), (4 - 0)\|} = \frac{[-2, 2, 4]}{\sqrt{-2^2 + 2^2 + 4^2}} = \left[ \frac{-2}{\sqrt{24}}, \frac{2}{\sqrt{24}}, \frac{4}{\sqrt{24}} \right]$$

Simplifying :

$$w = \left[ \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right]$$

Then :

$$P_1 = I - 2ww^T = I - 2 \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \end{bmatrix} = I - 2 \begin{bmatrix} \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = I - \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

# Example con't

Finally :

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

With that :

$$M_1 = P_1 M = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 & 3 & 0 & 2 \\ 2 & 1 & 2 & 1 \\ 4 & 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 5 & \frac{2}{3} & \frac{11}{3} \\ 0 & -1 & \frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{4}{3} & -\frac{1}{3} \end{bmatrix}$$

Which we can see zero'ed the first column.

$P_1$  can be verified by performing the

reverse operation :

$$M = P_1 M_1 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 6 & 5 & \frac{2}{3} & \frac{11}{3} \\ 0 & -1 & \frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{4}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 4 & 3 & 0 & 2 \\ 2 & 1 & 2 & 1 \\ 4 & 4 & 0 & 3 \end{bmatrix}$$

# Example con't

Likewise the calculation of  $S_1$  for :

Would have :

$$x = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 3 \\ 11 \\ 3 \end{bmatrix} \quad y = \begin{bmatrix} 6 \\ \sqrt{350} \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 6 & 5 & \frac{2}{3} & \frac{11}{3} \\ 0 & -1 & \frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{4}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\text{With : } \|x\| = \sqrt{6^2 + 5^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{11}{3}\right)^2} = \sqrt{\frac{674}{9}} \approx 8.6538366$$

$$\|y\| = \sqrt{6^2 + \left(\frac{\sqrt{350}}{3}\right)^2 + 0^2 + 0} = \sqrt{\frac{674}{9}} \approx 8.6538366$$

This leads to :

$$w = \frac{(x-y)}{\|x-y\|} = [0, -0.314814, 0.169789, 0.933843]$$

Then :

$$S_1 = I - 2ww^T = I - 2 \begin{bmatrix} 0 \\ -0.314814 \\ 0.169789 \\ 0.933843 \end{bmatrix} [0, -0.314814, 0.169789, 0.933843]$$

# Example con't

Finally :

$$S_1 = I - 2 \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.099108 & -0.053452 & -0.293987 \\ 0.0 & -0.053452 & 0.028828 & 0.158556 \\ 0.0 & -0.293987 & 0.158556 & 0.872063 \end{bmatrix} = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.801784 & 0.106904 & 0.587974 \\ 0.0 & 0.106904 & 0.942344 & -0.317112 \\ 0.0 & 0.587974 & -0.317112 & -0.744126 \end{bmatrix}$$

With that :

$$M_2 = M_1 S_1 = \begin{bmatrix} 6 & 5 & \frac{2}{3} & \frac{11}{3} \\ 0 & -1 & \frac{4}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{4}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.801784 & 0.106904 & 0.587974 \\ 0.0 & 0.106904 & 0.942344 & -0.317112 \\ 0.0 & 0.587974 & -0.317112 & -0.744126 \end{bmatrix}$$

$$\approx \begin{bmatrix} 6 & 5 & 0.0 & 0.0 \\ 0 & -1.05 & 1.36 & -0.51 \\ 0 & -0.34 & -1.15 & 0.67 \end{bmatrix}$$

Which we can see zero'ed the first row.

# The QR Algorithm

As seen, the initial matrix is placed into bidiagonal form which results in the following decomposition :

$$M = PBS \text{ with } P = P_I \dots P_N \text{ and } S = S_N \dots S_I$$

The next step takes B and converts it to the final diagonal form using successive QR transformations.

# QR Decompositions

The QR decomposition is defined as :

$$M = QR$$

Where  $Q$  is an orthogonal matrix (such that  $Q^T = Q^{-1}$ ,  $Q^T Q = Q Q^T = I$ )

And  $R$  is an upper triangular matrix :

$$R = \begin{bmatrix} x & x & x & x \\ & x & x & x \\ & & x & x \\ & & & x \end{bmatrix}$$

It has the property such that  $RQ = M_1$  to which another decomposition can be performed. Hence  $M_1 = Q_1 R_1$ ,  $R_1 Q_1 = M_2$  and so on. In practice, after enough decompositions,  $M_x$  will converge to the desired SVD diagonal matrix –  $W$ .

# QR Decomposition con't

Because  $Q$  is orthogonal (meaning  $QQ^T = Q^TQ = I$ ), we can redefine  $M_x$  in terms of  $Q_{x-1}$  and  $M_{x-1}$  only :

$$R_{x-1}Q_{x-1} = M_x \rightarrow Q_{x-1}R_{x-1}Q_{x-1} = Q_{x-1}M_x \rightarrow Q_{x-1}^T Q_{x-1} R_{x-1} Q_{x-1} = M_x \rightarrow Q_{x-1}^T M_x Q_{x-1} = M_{x-1}$$

Which can be written as :  $M_{x-1} = Q_{x-1}^T M_x Q_{x-1}$

Starting with  $M_0 = M$ , we can describe the entire decomposition of  $W$  as :

$$M_0 = Q_0^T M_1 Q_0 = Q_0^T Q_1^T M_2 Q_1 Q_0 = \dots = Q_0^T Q_1^T \dots Q_w^T W Q_w \dots Q_1 Q_0$$

One question remains – How do we derive  $Q$ ?

Multiple methods exist for QR decompositions – including Householder Transformations, Hessenberg Transformations, Given's Rotations, Jacobi Transformations, etc.

Unfortunately the algorithm from book is not explicit on its chosen methodology – possibly Givens as it is used by reference material.

# QR Decomposition using Givens rotations

A Givens rotation is used to rotate a plane about two coordinates axes and can be used to zero elements similar to the householder reflection.

It is represented by a matrix of the form :

$$G(i, j, \theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{ii} & s_{ji} & 0 \\ 0 & -s_{ij} & c_{jj} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{aligned} c &= \cos(\theta) \\ s &= \sin(\theta) \end{aligned}$$

The multiplication  $G^T A$ \* effects only the rows  $i$  and  $j$  in  $A$ .

Likewise the multiplication  $AG$  only effects the columns  $i$  and  $j$ .

[1] Shows transpose on pre-multiply – but examples do not appear to be transposed (i.e.  $-s$  is still located  $i,j$ ).



# Givens rotation

The zeroing of an element is performed by computing the  $c$  and  $s$  in the following system.

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sqrt{a^2 + b^2} \\ 0 \end{bmatrix}$$

Where  $b$  is the element being zeroed and  $a$  is next to  $b$  in the preceding column / row.

This is results in :

$$c = \frac{a}{\sqrt{a^2 + b^2}} \quad s = \frac{b}{\sqrt{a^2 + b^2}}$$

# Givens rotation and the Bidiagonal matrix

The application of Givens rotations on a bidiagonal matrix looks like the following and results in its implicit QR decomposition.

$$\begin{array}{c}
 \begin{array}{c} \left[ \begin{array}{cccc} x & x & & \\ & x & x & \\ & & x & x \\ & & & x \end{array} \right] \\ B_1 \end{array} \xrightarrow{J_1} \begin{array}{c} \left[ \begin{array}{cccc} x & x & & \\ + & x & x & \\ & & x & x \\ & & & x \end{array} \right] \\ B_2 \end{array} \xrightarrow{J_3} \begin{array}{c} \left[ \begin{array}{cccc} x & x & + & \\ & x & x & \\ & & x & x \\ & & & x \end{array} \right] \\ B_3 \end{array} \xrightarrow{J_4} \begin{array}{c} \left[ \begin{array}{cccc} x & x & & \\ & x & x & \\ + & x & x & \\ & & x & x \end{array} \right] \\ B_4 \end{array} \xrightarrow{J_5} \begin{array}{c} \left[ \begin{array}{cccc} x & x & & \\ & x & x & + \\ & & x & x \\ & & & x \end{array} \right] \\ B_5 \end{array} \xrightarrow{J_6} \begin{array}{c} \left[ \begin{array}{cccc} x & x & & \\ & x & x & \\ & & x & x \\ & & + & x \end{array} \right] \\ B_6 \end{array} \xrightarrow{J_7} \begin{array}{c} \left[ \begin{array}{cccc} x & x & & \\ & x & x & \\ & & x & x \\ & & & x \end{array} \right] \\ B_7 \end{array} \xrightarrow{J_8} \begin{array}{c} \left[ \begin{array}{cccc} x & x & & \\ & x & x & \\ & & x & x \\ & & & x \end{array} \right] \\ B_8 \end{array} \xrightarrow{J_9} \begin{array}{c} \left[ \begin{array}{cccc} x & x & & \\ & x & x & \\ & & x & x \\ & & & x \end{array} \right] \\ B' \end{array}
 \end{array}$$

# Givens and Bidiagonal

With the exception of  $J_1$ ,  $J_x$  is the Givens matrix computed from the element being zeroed.

$J_1$  is computed from the following :

$$\begin{bmatrix} c & s \\ -s & s \end{bmatrix} \begin{bmatrix} d_1^2 - \lambda \\ d_1 f_1 \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Which is derived from B and the smallest eigenvalue ( $\lambda$ ) of T

$$B = \begin{bmatrix} d_1 & f_1 & & & \\ & d_2 & f_2 & & \\ & & \dots & f_{n-2} & \\ & & & d_{n-1} & f_{n-1} \\ & & & & d_n \end{bmatrix} \quad T = \begin{bmatrix} d_{n-1}^2 + f_{n-2}^2 & d_{n-1} f_{n-1} \\ d_{n-1} f_{n-1} & d_n^2 + f_{n-1}^2 \end{bmatrix}$$

# Bidiagonal and QR

This computation of  $J_1$  causes the implicit formation of  $B^T B$   
which causes :

$$B' = J_i \dots J_4 J_2 B J_1 J_3 \dots J_k = U_i \dots U_4 U_2 B^T B J_1 J_3 \dots J_k \approx Q^T B^T B Q$$

# QR Decomposition Given's rotation example

$$\text{Let } \mathbf{A}^{(0)} = \mathbf{A} = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ \boxed{1} & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

1. Use  $a_{31}$  to eliminate  $a_{41}$ .  $r_{3,4} = \sqrt{1^2 + 1^2} = \sqrt{2}$ .  $\begin{cases} \cos \theta_{3,4} = a_{31}/r = 1/\sqrt{2}, \\ \sin \theta_{3,4} = a_{41}/r = 1/\sqrt{2}. \end{cases}$

$$\mathbf{G}_{3,4}^{(1)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos \theta_{3,4} & \sin \theta_{3,4} \\ & & -\sin \theta_{3,4} & \cos \theta_{3,4} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\sqrt{2} & 1/\sqrt{2} \\ & & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$


$$\mathbf{A}^{(1)} = \mathbf{G}_{3,4}^{(1)} \mathbf{A}^{(0)} = \begin{pmatrix} 1 & -1 & 4 \\ \boxed{1} & 4 & -2 \\ \sqrt{2} & 3/\sqrt{2} & \sqrt{2} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix}.$$



# QR Decomposition Given's rotation example

2. Use  $a_{21}$  to eliminate  $a_{31}$ .  $r_{2,3} = \sqrt{1^2 + \sqrt{2}^2} = \sqrt{3}$ .  $\begin{cases} \cos \theta_{2,3} = a_{21}/r = \sqrt{2}/\sqrt{3}, \\ \sin \theta_{2,3} = a_{31}/r = 1/\sqrt{3}. \end{cases}$

$$\mathbf{G}_{2,3}^{(1)} = \begin{pmatrix} 1 & & & \\ & \cos \theta_{2,3} & \sin \theta_{2,3} & \\ & -\sin \theta_{2,3} & \cos \theta_{2,3} & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & \sqrt{2}/\sqrt{3} & \\ & -\sqrt{2}/\sqrt{3} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^{(2)} &= \mathbf{G}_{2,3}^{(1)} \mathbf{A}^{(1)} = \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & \sqrt{2}/\sqrt{3} & \\ & -\sqrt{2}/\sqrt{3} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ \sqrt{2} & 3/\sqrt{2} & \sqrt{2} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 & 4 \\ \sqrt{3} & 7/\sqrt{3} & 0 \\ 0 & -5/\sqrt{6} & \sqrt{6} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \end{aligned}$$


# QR Decomposition Given's rotation

3. Use  $a_{11}$  to eliminate  $a_{21}$ .  $r_{1,2} = \sqrt{1^2 + \sqrt{3}^2} = 2$ .  $\begin{cases} \cos \theta_{1,2} = a_{11}/r = 1/2, \\ \sin \theta_{1,2} = a_{21}/r = \sqrt{3}/2. \end{cases}$

$$\mathbf{G}_{1,2}^{(1)} = \begin{pmatrix} \cos \theta_{1,2} & \sin \theta_{1,2} & & \\ -\sin \theta_{1,2} & \cos \theta_{1,2} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 & & \\ -\sqrt{3}/2 & 1/2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$


$$\begin{aligned} \mathbf{A}^{(3)} &= \mathbf{G}_{1,2}^{(1)} \mathbf{A}^{(2)} = \begin{pmatrix} 1/2 & \sqrt{3}/2 & & \\ -\sqrt{3}/2 & 1/2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 4 \\ \sqrt{3} & 7/\sqrt{3} & 0 \\ 0 & -5/\sqrt{6} & \sqrt{6} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5/\sqrt{3} & -2\sqrt{3} \\ 0 & -5/\sqrt{6} & \sqrt{6} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \end{aligned}$$

# QR Decomposition Given's rotation example

4. Use  $a_{32}$  to eliminate  $a_{42}$ .  $r_{3,4} = \sqrt{(-5/\sqrt{6})^2 + (-5/\sqrt{2})^2} = 10/\sqrt{6}$ .

$$\begin{cases} \cos \theta_{3,4} = a_{32}/r = -1/2, \\ \sin \theta_{3,4} = a_{42}/r = -\sqrt{3}/2. \end{cases}$$

$$\mathbf{G}_{3,4}^{(2)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos \theta_{3,4} & \sin \theta_{3,4} \\ & & -\sin \theta_{3,4} & \cos \theta_{3,4} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1/2 & -\sqrt{3}/2 \\ & & \sqrt{3}/2 & -1/2 \end{pmatrix}$$


$$\begin{aligned} \mathbf{A}^{(4)} &= \mathbf{G}_{3,4}^{(2)} \mathbf{A}^{(3)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1/2 & -\sqrt{3}/2 \\ & & \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5/\sqrt{3} & -2\sqrt{3} \\ 0 & -5/\sqrt{6} & \sqrt{6} \\ 0 & -5/\sqrt{2} & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5/\sqrt{3} & -2\sqrt{3} \\ 0 & 10/\sqrt{6} & 0 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \end{aligned}$$




# QR Decomposition Given's rotation example

5. Use  $a_{22}$  to eliminate  $a_{32}$ .  $r_{2,3} = \sqrt{(10/\sqrt{6})^2 + (5/\sqrt{3})^2} = 5$ .  $\begin{cases} \cos \theta_{2,3} = a_{22}/r = 1/\sqrt{3}, \\ \sin \theta_{2,3} = a_{32}/r = 2/\sqrt{6}. \end{cases}$


$$\mathbf{G}_{2,3}^{(2)} = \begin{pmatrix} 1 & & & \\ & \cos \theta_{2,3} & \sin \theta_{2,3} & \\ & -\sin \theta_{2,3} & \cos \theta_{2,3} & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & 2/\sqrt{6} & \\ & -2/\sqrt{6} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^{(5)} &= \mathbf{G}_{2,3}^{(2)} \mathbf{A}^{(4)} = \begin{pmatrix} 1 & & & \\ & 1/\sqrt{3} & 2/\sqrt{6} & \\ & -2/\sqrt{6} & 1/\sqrt{3} & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5/\sqrt{3} & -2\sqrt{3} \\ 0 & 10/\sqrt{6} & 0 \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \end{aligned}$$


# QR Decomposition Given's rotation example

6. Use  $a_{33}$  to eliminate  $a_{43}$ .  $r_{3,4} = \sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = 4$ .  $\begin{cases} \cos \theta_{3,4} = a_{33}/r = 1/\sqrt{2}, \\ \sin \theta_{3,4} = a_{43}/r = 1/\sqrt{2}. \end{cases}$

$$\mathbf{G}_{3,4}^{(3)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos \theta_{3,4} & \sin \theta_{3,4} \\ & & -\sin \theta_{3,4} & \cos \theta_{3,4} \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\sqrt{2} & 1/\sqrt{2} \\ & & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$\begin{aligned} \mathbf{A}^{(6)} &= \mathbf{G}_{3,4}^{(3)} \mathbf{A}^{(5)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1/\sqrt{2} & 1/\sqrt{2} \\ & & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 2\sqrt{2} \\ 0 & 0 & 2\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$


# QR Decomposition Given's rotation example

7. The R-factor is  $\begin{pmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$ ; the Q-factor is

$$\begin{aligned} Q &= \left( G_{3,4}^{(3)} G_{2,3}^{(2)} G_{3,4}^{(2)} G_{1,2}^{(1)} G_{2,3}^{(1)} G_{3,4}^{(1)} \right)^{-1} \\ &= G_{3,4}^{(1)T} G_{2,3}^{(1)T} G_{1,2}^{(1)T} G_{3,4}^{(2)T} G_{2,3}^{(2)T} G_{3,4}^{(3)T} \\ &= \begin{pmatrix} 1/2 & -1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix} = Q \end{aligned}$$

# Putting it together - SVD

Starting from the beginning with a matrix  $M$ , we want to derive -  $UWV^T$

Using Householder transformations :  $M = PBS$  [Step 1]

Using QR Decompositions :  $B = Q_0^T Q_1^T \dots Q_w^T W Q_w \dots Q_1 Q_0$  [Step 2]

Substituting step 2 into 1 :  $M = P Q_0^T Q_1^T \dots Q_w^T W Q_w \dots Q_1 Q_0 S$

With  $U$  being derived from :  $U = P Q_0^T Q_1^T \dots Q_w^T$

And  $V^T$  being derived from :  $V^T = Q_w \dots Q_1 Q_0 S$

Which results in the final SVD :  $M = UWV^T$

# SVD Applications

Calculation of the (pseudo) inverse :

[1] : Given  $M = U W V^T$

[2] : Multiply by  $M^{-1}$   $M^{-1} M = M^{-1} U W V^T \rightarrow 1 = M^{-1} U W V^T$

[3] : Multiply by  $V$   $V = M^{-1} U W V^T V \rightarrow V = M^{-1} U W$

[4]\* : Multiply by  $W^{-1}$   $V W^{-1} = M^{-1} U W W^{-1} \rightarrow V W^{-1} = M^{-1} U$

[5] : Multiply by  $U^T$   $V W^{-1} U^T = M^{-1} U U^T \rightarrow V W^{-1} U^T = M^{-1}$

[6] : Rearranging  $M^{-1} = V W^{-1} U^T$

\*Note – Inverse of a diagonal matrix is  $\text{diag}(a_1, \dots, a_n)^{-1} = \text{diag}(1/a_1, \dots, 1/a_n)$

# SVD Applications con't

Solving a set of homogenous linear equations i.e.  $Mx = b$

Case 1 :  $b = 0$

$x$  is known as the nullspace of  $M$  which is defined as the set of all vectors that satisfy the equation  $Mx = 0$ . This is any column in  $V^T$  associated with a singular value (in  $W$ ) equal to 0.

Case 2 :  $b \neq 0$

Then we have :  $Mx = b$

Which can be re-written as :  $M^{-1}Mx = M^{-1}b \rightarrow x = M^{-1}b$

From the previous slide we know :  $M^{-1} = VW^{-1}U^T$

Hence :  $x = VW^{-1}U^T b$  which is easily solvable

# SVD Applications con't

## Rank, Range, and Null space

- The **rank** of matrix **A** can be calculated from SVD by the number of nonzero singular values.
- The **range** of matrix **A** is The left singular vectors of **U** corresponding to the non-zero singular values.
- The **null space** of matrix **A** is The right singular vectors of **V** corresponding to the zeroed singular values.

$$A = U W V^T$$

Range

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Rank

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Null Space

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

# SVD Applications con't

## Condition number

- SVD can tell How close a square matrix  $A$  is to be singular.
- The ratio of the largest singular value to the smallest singular value can tell us how close a matrix is to be singular:

$$A = U \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sigma_k \end{bmatrix} V^T \quad c = \frac{\sigma_1}{\sigma_k}$$

- $A$  is singular if  $c$  is infinite.
- $A$  is ill-conditioned if  $c$  is too large (machine dependent).



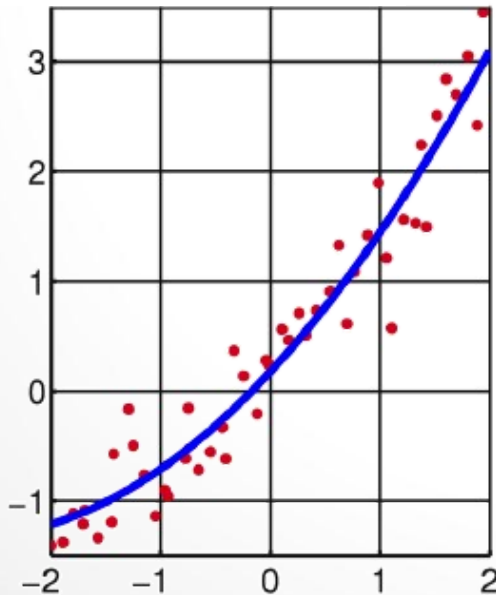
# SVD Applications con't

## Data Fitting Problem

$$y = ax^2 + bx + c$$

$$\underbrace{\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ \vdots & \vdots & \vdots \\ x_N^2 & x_N & 1 \end{bmatrix}}_S \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_a = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix}}_y$$

$$a = V \cdot [\text{diag}(1/\sigma_i)] \cdot (U^T y)$$

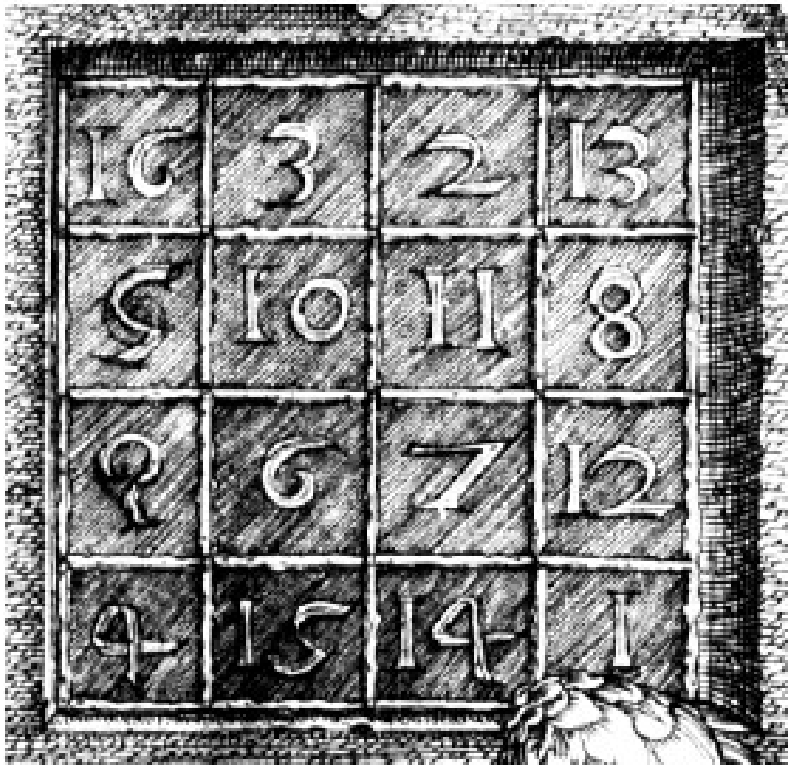


# SVD Applications con't

Image processing

$$[U, W, V] = \text{svd}(A)$$

$$\text{NewImg} = U(:, 1) * W(1, 1) * V(:, 1)'$$



# SVD Applications con't

## Digital Signal Processing (DSP)

- SVD is used as a method for noise reduction.
- Let a matrix  $A$  represent the noisy signal:
  - compute the SVD,
  - and then discard small singular values of  $A$ .
- It can be shown that the small singular values mainly represent the noise, and thus the rank- $k$  matrix  $A_k$  represents a filtered signal with less noise.

# Additional References

1. Golub & Van Loan – Matrix Computations; 3<sup>rd</sup> Edition, 1996
2. Golub & Kahan – Calculating the Singular Values and Pseudo-Inverse of a Matrix; SIAM Journal for Numerical Analysis; Vol. 2, #2; 1965
3. An Example of QR Decomposition, Che-Rung Lee, November 19, 2008