Chern Class: Initial Definition and Some Results

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提要

20 世纪 40 年代左右，陈省身（Shiing-Shen Chern, 1911.10.26 —2004.12.3）先生用了三种不同的方法对复向量丛来定义陈类，即障碍理论、舒伯特胞腔和在丛上连络的曲率形式，同时建立了三者的等价性。本文沿着最后一条路径，来定义陈形式与陈类，并尽可能地补充具体例子与相关定理，以辅理解。

陈类对整个数学，甚至数学物理都产生了广泛而深刻的影响[①]，重要性不言而喻，从中我们也可略窥复代数几何的美妙与简洁。最后附上杨先生于 1975 年赠与陈先生的诗，以纪念这一数学大师：

赞陈氏级

杨振宁

天衣岂无缝，匠心剪接成。浑然归一体，广邃妙绝伦。
造化爱几何，四力纤维能。千古存心事，欧高黎嘉陈。

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1 Chern Class

1.1 Hermitian metric, connection and curvature

Suppose that $E \to X$ is a *vector bundle*[②] (Here we refer to a differentiable $\mathbb{C}$-vector bundle over a differentiable manifold) of rank r and $f = \{e_1, \cdots, e_r\}, e_j \in \mathcal{E}(U, E)$ is a local frame of E over an sufficiently small open set $U \subset X$. And locally we write $\xi \in \mathcal{E}(U, E) \cong \mathcal{E}(U, U \times \mathbb{C}^r)$ as

$$\xi = \xi(f) = (\xi^1, \cdots, \xi^r)^t, \ \xi^j \in \mathcal{E}(U)$$

Considering a change of frame $f \mapsto fg$ where $g : U \to \mathrm{GL}_r(\mathbb{C})$, one can see

$$\xi(fg) = g^{-1}\xi(f)$$

Moreover, if E is a holomorphic vector bundle, then we shall also have holomorphic frames $f = \{e_1, \cdots, e_r\}$ where $e_j \in \mathcal{O}(U, E)$ and $e_1 \wedge \cdots \wedge e_r(x) \neq 0$ for all $x \in U$.

In the remainder of this article, we assume that $f = \{e_1, \cdots, e_r\}$ represents a frame for vector bundle E on some open set $U \subset X$ unless otherwise specified.

[①]陈先生于 1983 年获得 Wolf 奖 (Shared with Paul Erdos)，对其工作的介绍为：“Professor Shiing-Shen Chern has been the leading figure in global differential geometry. His earlier work on integral geometry, especially on the kinematic formula, was the source of most later work. **His ground-breaking discovery of characteristic classes (now known as Chern classes) was the turning point that set global differential geometry on a course of tumultuous development.** The field has blossomed under his leadership, and his results, together with those of his numerous students, have influenced the development of topology, algebraic geometry, complex manifolds, and most recently of gauge theories in mathematical physics.”

[②]There is a one-to-one correspondence between (isomorphism classes of) $\mathbb{C}$-bundles over X and (isomorphism classes of) locally free sheaves of $\mathcal{O}_X$-modules over X.

Definition 1.1 (Hermitian metric).

Let $E \to X$ be a vector bundle. A Hermitian metric h on E is an assignment of a Hermitian inner product $\langle , \rangle_x$ to each fibre E_x of E s.t. for any open set $U \subset X$ and $\xi, \eta \in \mathcal{E}(U, E)$ the function

$$\langle \xi, \eta \rangle : U \longrightarrow \mathbb{C}$$

given by $\langle \xi, \eta \rangle(x) = \langle \xi(x), \eta(x) \rangle_x$ which is C^∞. A vector bundle E equipped with a Hermitian metric h is called a Hermitian vector bundle.

Locally we can represents h as a positive definite Hermitian symmetric matrix $h(f)$ whose entries are C^∞ by setting $h(f)_{\rho\sigma} = \langle e_\sigma, e_\rho \rangle$. For any $\xi, \eta \in \mathcal{E}(U, E)$, we see

$$\langle \xi, \eta \rangle = \langle \sum_\rho \xi^\rho(f) e_\rho, \sum_\sigma \eta^\sigma(f) e_\sigma \rangle = \overline{\eta(f)}^t h(f) \xi(f)$$

and easily get the transformation law for local representations of the Hermitian metric

$$h(fg) = \overline{g}^t h(f) g \tag{1}$$

Proposition 1.2. *Every vector bundle $E \to X$ admits a Hermitian metric.*

Proof. There exists a locally finite (X is locally compact) covering $\{U_\alpha\}$ of X and frames f_α defined on U_α. Define a Hermitian metric h_α on $E|_{U_\alpha}$ by setting, for $\xi, \eta \in E_x$, $x \in U_\alpha$,

$$\langle \xi, \eta \rangle_x^\alpha = \overline{\eta(f_\alpha)(x)} \cdot \xi(f_\alpha)(x)$$

Then take $\{\rho_\alpha\}$ as a C^∞ partition of unity subordinate to the covering $\{U_\alpha\}$ and let, for $\xi, \eta \in E_x$,

$$\langle \xi, \eta \rangle_x = \sum_\alpha \rho_\alpha(x) \langle \xi, \eta \rangle_x^\alpha$$

We can now verify that $\langle , \rangle$ so defined gives a Hermitian metric for $E \to X$. □

We now consider differential forms with vector bundle coefficients by setting $\mathcal{E}^p(X, E) = \mathcal{E}(X, \wedge^p T^*(X) \otimes_{\mathbb{C}} E)$ be the differential forms of degree p on X with coefficients in E. By the sheaf isomorphism $\mathcal{E}^p \otimes_{\mathbb{C}} \mathcal{E}(E) \cong \mathcal{E}^p(E)$, we have a local representation for $\xi \in \mathcal{E}^p(U, E)$ given by

$$\xi = \sum_\rho \xi^\rho(f) e_\rho \text{ and } \xi(x) = \sum_{\rho,k} \phi_{k,\rho}(x) \omega_k(x) \otimes e_\rho(x)$$

where $\xi^\rho(f) \in \mathcal{E}^p(U), x \in U$ and $\{\omega_k\}$ is a local frame for $\wedge^p T^*(X) \otimes \mathbb{C}$ and $\phi_{k,\rho}$ are uniquely determined C^∞ functions defined near x. One can see the differential form ξ^ρ so determined is independent of the choice of frame $\{\omega_k\}$ and the transformation law $\xi(fg) = g^{-1}\xi(f)$.

Definition 1.3 (Connection).

Let $E \to X$ be a vector bundle. Then a connection D on $E \to X$ is a $\mathbb{C}$-linear mapping

$$D : \mathcal{E}(X, E) \longrightarrow \mathcal{E}^1(X, E),$$

which satisfies $D(\varphi\xi) = d\varphi \cdot \xi + \varphi D\xi$, where $\varphi \in \mathcal{E}(X)$ and $\xi \in \mathcal{E}(X, E)$.

If $E = X \times \mathbb{C}$ is the trivial line bundle, then the ordinary exterior differentiation $d : \mathcal{E}(X) \to \mathcal{E}^1(X)$ is a connection on E. Thus a connection is a generalization of exterior differentiation to vector-valued differential forms, and we can extend D to higher-order E-valued differential forms naturally.

Locally, we define the *connection matrix* $\theta(D, f)$ associated with the connection D and the frame f by setting

$$De_\sigma := \sum_\rho \theta_{\rho\sigma}(D, f)e_\rho,\ \theta_{\rho\sigma}(D, f) \in \mathcal{E}^1(U)$$

sometimes we just say θ if D, f are fixed. Now we have

$$D\xi(f) = D\left((e_1, \dots, e_r) \cdot (\xi^1, \cdots, \xi^r)^t\right) = (e_1, \cdots, e_r)\cdot[(d\xi^1, \cdots, d\xi^r)^t + \theta\cdot(\xi^1, \cdots, \xi^r)^t] = (d+\theta)\xi(f)$$

i.e. $D = d + \theta$.

We want to show that the connection D (as we shall see below, every vector bundle admits a connection, see Proposition 1.8) on E induces in a natural manner an element called *curvature tensor*

$$\Theta_E(D) \in \mathcal{E}^2(X, \mathrm{Hom}(E, E))$$

where $\mathrm{Hom}(E, E)$ be the vector bundle whose fibres are $\mathrm{Hom}(E_x, E_x)$.

First, we inspect the element $\chi \in \mathcal{E}^p(X, \mathrm{Hom}(E, E))$. Note

$$\mathcal{E}^p(U, \mathrm{Hom}(E, E)) \cong \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U, \mathrm{Hom}(E, E)) \cong \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathrm{M}_r(\mathbb{C})$$

thus we can represent χ locally as an $r\times r$ matrix $\chi(f)$ where $\chi_{\rho\sigma}(f) \in \mathcal{E}^p(U)$ and $[\chi(f)\xi(f)]^\rho = \sum_\sigma \chi(f)_{\rho\sigma}\xi^\sigma(f)$. And then

$$\chi(fg) = g^{-1}\chi(f)g \tag{2}$$

In turn, it can be verified that χ satisfying the above formula actually with entries in $\mathcal{E}^p(U)$ defines an element in $\mathcal{E}^p(U, \mathrm{Hom}(E, E))$.

Definition 1.4 (Curvature).

Let D be a connection in a vector bundle $E \to X$. Then the curvature $\Theta_E(D)$ is defined to be that element $\Theta \in \mathcal{E}^2(X, \mathrm{Hom}(E, E))$ s.t. the $\mathbb{C}$-linear mapping $\Theta : \mathcal{E}(X, E) \longrightarrow \mathcal{E}^2(X, E)$ has the representation with respect to a frame f

$$\Theta(f) = \Theta(D, f) = d\theta(f) + \theta(f) \wedge \theta(f).$$

which is a $r \times r$ matrix of 2-forms and $\Theta_{\rho\sigma} = d\theta_{\rho\sigma} + \sum_k \theta_{\rho k} \wedge \theta_{k\sigma}$.

Remark 1.5 (Well-definability of Θ).

From

$$(e_1, \cdots, e_r)g\theta(fg) = D((e_1, \cdots, e_r)g) = (e_1, \cdots, e_r)(d + \theta)g$$

we see

$$g\theta(fg) = dg + \theta(f)g \tag{3}$$

Take the exterior derivative of equation (3) and use the fact that $d(\alpha\wedge\beta) = d\alpha\wedge\beta+(-1)^k\alpha\wedge d\beta$ when α is a k-form, we get $d\theta(f)g - \theta(f)dg = dg\theta(fg) + gd\theta(fg)$,so

$$g[\Theta(fg)] = g[d\theta(fg) + \theta(fg) \wedge \theta(fg)] = [d\theta(f) + \theta(f) \wedge \theta(f)]g = \Theta(f)g$$

by some simplification. Thus Θ is well-defined.

Proposition 1.6.

(i) $D(f)^2 = \Theta(f)$

(ii)(Bianchi identity) $d\Theta(f) = [\Theta(f), \theta(f)]$

Proof. We omit the notation f, and then for any $\xi \in \mathcal{E}(U, E)$, we have

$$D^2\xi = (d+\theta)^2\xi = \theta d\xi + d(\theta\xi) + \theta \wedge \theta\xi = (d\theta + \theta \wedge \theta)\xi = \Theta\xi$$

then gets (i). (ii) follows that

$$d\Theta = d\theta \wedge \theta - \theta \wedge d\theta = [d\theta + \theta \wedge \theta, \theta] = [\Theta, \theta]$$

□

Remark 1.7. *We now define a Lie product on the algebra*

$$\mathcal{E}^*(X, \mathrm{Hom}(E, E)) = \sum_p \mathcal{E}^p(X, \mathrm{Hom}(E, E))$$

If $\chi \in \mathcal{E}^p(X, \mathrm{Hom}(E, E))$ *and* f *is a frame for* E *over the open set* U*, then* $\chi(f) \in \mathfrak{M}_r \otimes_{\mathcal{E}(U)} \mathcal{E}^p(U)$ *by the fact that the differential forms in* $\mathcal{E}^p(X, \mathrm{Hom}(E, E))$ *are locally matrices of p-forms. Hence if* $\psi \in \mathcal{E}^q(X, \mathrm{Hom}(E, E))$*, we define*

$$[\chi(f), \psi(f)] = \chi(f) \wedge \psi(f) - (-1)^{pq}\psi(f) \wedge \chi(f) \tag{4}$$

And we see from equation (2)*,* $[\chi(fg), \psi(fg)] = g^{-1}[\chi(f), \psi(f)]g$ *if* g *is a change of frame, which shows the Lie bracket is well defined on* $\mathcal{E}^*(X, \mathrm{Hom}(E, E))$ *and one may check it satisfies the Jacobi identity. Thus* $\mathcal{E}^*(X, \mathrm{Hom}(E, E))$ *is a Lie algebra.*

At last of this subsection, we show that every vector bundle admits a connection.

Proposition 1.8. *Every Hermitian vector bundle* $E \to X$ *admits a connection* D *which is compatible with the Hermitian metric* h*.*

Proof. "Compatible" means that we always have

$$d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle$$

and one can vertify this are equivalent to say $dh(f) = h(f)\theta(f) + \overline{\theta(f)}^t h(f)$ for all frame. Note that we views $h : \mathcal{E}^p(X, E) \otimes \mathcal{E}^q(X, E) \to \mathcal{E}^{p+q}(X)$ by setting

$$\langle \omega \otimes \xi, \omega' \otimes \xi' \rangle_x = \omega \wedge \overline{\omega'} \langle \xi, \xi' \rangle_x$$

where $\omega \in \mathcal{E}^p(X)_x, \omega' \in \mathcal{E}^q(X)_x, \xi, \xi' \in \mathcal{E}(X, E)$. Then we can choose unitary frame f locally s.t. $h(f) = I$, and find a locally finite covering $\{U_\alpha\}$ of X and corresponding f_α. In this time we require $0 = \theta(f_\alpha) + \overline{\theta(f_\alpha)}^t$.

In each U_α, we can choose $\theta(f_\alpha) = 0$, and by equation (3) we require $\theta(f_\alpha g) = g^{-1}dg$, so we define θ as this. Now $dh(f_\alpha g) = d(\overline{g}^t \cdot g) = h(f_\alpha g)\theta(f_\alpha g) + \overline{\theta(f_\alpha g)}^t h(f_\alpha g)$, which vertifies the compatibility.

Hence we can glue these local connections to get a global connection by using a partition $\{\rho_\alpha\}$ of unity, as Proposition 1.2. □

1.2 The canonical D, Θ for a Hermitian holomorphic vector bundle

Now assume $E \to X$ is a holomorphic vector bundle equipped with a Hermitian metric h. We can choose a canonical connection D s.t. D is compatible with h and if $D = D' + D'' \in \mathcal{E}^{(1,0)}(X,E) \oplus \mathcal{E}^{(0,1)}(X,E)$ as a natural decomposition, then for $\xi \in \mathcal{O}(U,E)$, $D''\xi = 0$.

Specifically, let $\xi \in \mathcal{O}(U,E)$, then

$$D\xi(f) = (\partial + \theta^{(1,0)}(f))\xi(f) + \theta^{(0,1)}(f)\xi(f)$$

so we require $\theta^{(0,1)}(f) = 0$. Then from the compatible condition $dh(f) = h(f)\theta(f) + \overline{\theta(f)}^t h(f)$ we see $\partial h = h\theta, \overline{\partial} h = \overline{\theta}^t h$. Hence we can just define

$$\theta(f) = h^{-1}(f)\partial h(f)$$

and one can vertifies that this θ satisfies equation (3) and actually define a global connection $D = [\partial + \theta(f)] + \overline{\partial}$.

Proposition 1.9. *Canonical θ, D, Θ satisfies*
(i) $\theta(f)$ is a $(1,0)$-form and $\partial\theta(f) = -\theta(f) \wedge \theta(f)$.
(ii) $\Theta(f) = \overline{\partial}\theta(f)$ and $\partial\Theta(f) = [\Theta(f), \theta(f)]$.

Proof. We omit the notation f. Then by $\partial^2 = 0 = \overline{\partial}^2$ and $\partial(hh^{-1}) = \partial h \cdot h^{-1} + h \cdot \partial h^{-1} = \partial I = 0$,

$$\partial\theta = \partial(h^{-1}\partial h) = -h^{-1} \cdot \partial h \cdot h^{-1} \wedge \partial h = -\theta \wedge \theta, \text{ then } \Theta = d\theta + \theta \wedge \theta = \overline{\partial}\theta = \overline{\partial}\partial \log h$$

and by the Bianchi identity we have $\partial\Theta = [\Theta, \theta]$. □

Example 1 (The Universal bundle over the Grassmannian manifold).
Let $U_{r,n} \longrightarrow G_{r,n}$ be the universal bundle where $G(r,n)$ is the Grassmannian manifold and $U_{r,n}$ is the disjoint union of all r-planes in $\mathbb{C}^n$. And localize on a suitable open set $U \subset G_{r,n}$ we nuturally have a holomorphic frame $f = (e_1, \cdots, e_r)$ where $e_j : U \longrightarrow \mathbb{C}^n$ and $e_1 \wedge \cdots \wedge e_r \neq 0$. Then we could consider the nutural metric on $U_{r,n}$ be setting $h(f) = \overline{f}^t \cdot f$: $h(f)$ is positive definite and satisfies the transformation law (1), i.e. $h(fg) = \overline{g}^t \cdot h(f)g$, which is a well-defined Hermitian metric.

We now compute the canonical curvature Θ with respect to the nutural metric in the case $r = 1$. For $\varphi \in [\mathcal{E}^p(W)]^n, \psi \in [\mathcal{E}^q(W)]^n$ where $W \subset U$ is open, we define (which generalizes usual inner product on $\mathbb{C}^n$)

$$\langle \varphi, \psi \rangle = (-1)^{pq}\overline{\psi}^t \wedge \varphi$$

And see (note $h^{-1} = [\overline{f}^t f]^{-1}$),

$$\Theta(f) = \overline{\partial}(h^{-1}\partial h) = -\frac{\langle f, f\rangle\langle df, df\rangle - \langle df, f\rangle \wedge \langle d, df\rangle}{\langle f, f\rangle^2} = -\frac{|f|^2 \sum d\xi_i \wedge d\overline{\xi}_i - \sum \overline{\xi}_i \xi_j d\xi_i \wedge d\overline{\xi}_j}{|f|^4}$$

where $f = (\xi_i)$ is a holomorphic frame for $U_{1,n}$ and $\xi_i \in \mathcal{O}(W)$.
As for general r, One may see example III.2.4 of [W].

1.3 Chern classes: from a differential-geometric view

The approach here follows the description of Bott and Chern [BC], based on the original ideas of Shiing-Shen Chern and A. Weil. We shall see Chern classes

(†) can be realized as elements of $H^{2j}(X,\mathbb{R})$ having certain functorial properties;

(‡) will be the obstruction to giving global frames.

We denote $\mathfrak{M}_r$ as the set of all $r\times r$ complex matrices, and $\tilde{I}_k(\mathfrak{M}_r)$ be the $\mathbb{C}$-vector space of all invariant k-linear forms on $\mathfrak{M}_r$, which implies if $\widetilde{\varphi}\in\tilde{I}_k(\mathfrak{M}_r)$, then $\widetilde{\varphi}(gA_1g^{-1},\cdots,gA_kg^{-1})=\widetilde{\varphi}(A_1,\cdots,A_k)$ for $\forall g\in\mathrm{GL}(r,\mathbb{C}),A_i\in\mathfrak{M}_r$. We also note $\widetilde{\varphi}$ induces a $\varphi:\mathfrak{M}_r\longrightarrow\mathbb{C}$ by setting $\varphi(A)=\widetilde{\varphi}(A,\cdots,A)$, and actually they decide mutually (By a trick called polarization), so we omit the widetilde from now. Moreover, φ extend naturally on $\mathcal{E}^*(\mathrm{Hom}(E,E))$, see localizing on U and use a frame f, we set

$$\varphi(A_1\otimes w_1,\cdots,A_k\otimes w_k)=w_1\wedge\cdots\wedge w_k\varphi(A_1,\cdots,A_k)\text{ for }A_i\otimes w_i\in\mathfrak{M}_r(U)\otimes\mathcal{E}^p(U)$$

Note it is independent of f because $\xi(fg)=g^{-1}\xi(f)g$.

Theorem 1.10 (Due to A. Weil)**.**
Let $E\to X$ be a differentiable $\mathbb{C}$-vector bundle, let D be a connection on E, and suppose that $\varphi\in I_k(\mathfrak{M}_r)$. Then
(a) $\varphi_X(\Theta_E(D))$ is closed.
(b) The image of $\varphi_X(\Theta_E(D))$ in $H^{2k}(X,\mathbb{C})$ is independent of the connection D.

Proof. From the definition of a k-linear form, we get

$$d\varphi_U(A_1,\cdots,A_k)=\sum_\alpha(-1)^{g(\alpha)}\varphi_U(A_1,\cdots,dA_\alpha,\cdots,A_k)$$

for $A_\alpha\in\mathfrak{M}_r\otimes\mathcal{E}^{p_\alpha}(U)$ and open set $U\subset X$, where $g(\alpha)=\sum_{\beta<\alpha}\deg A_\beta$. Moreover, for $A_j,B\in\mathfrak{M}_r$, we have

$$\sum_j\varphi(A_1,\cdots,[A_j,B],\cdots,A_k)=0 \tag{5}$$

because $\varphi(e^{-tB}A_1e^{tB},\cdots,e^{-tB}A_ke^{tB})-\varphi(A_1,\cdots,A_k)=0$ and the fact that $\frac{d}{dt}[e^{-tB}A_je^{tB}]\mid_{t=0}=[A_j,B]$.

Recalling the Lie product on $\mathfrak{M}_r\otimes\mathcal{E}^*$ in Remark 1.7, by equation (5) , we have

$$\sum_\alpha(-1)^{f(\alpha)}\varphi_U(A_1,\cdots,[A_\alpha,B],\cdots,A_k)=0 \tag{6}$$

for all $A_\alpha\in\mathfrak{M}_r\otimes\mathcal{E}^{p_\alpha}(U)$ and $B\in\mathfrak{M}_r\otimes\mathcal{E}^q(U)$, where $f(\alpha)=\deg B\sum_{\beta\le\alpha}\deg A_\beta$.

Note for (a), it suffices to show that for a frame f over U, $d\varphi_U(\Theta(f))=0$. We omit the notation f and note Θ is a 2-form, immediately we have

$$d\varphi_U(\Theta)=d\varphi_U(\Theta,\cdots,\Theta)=\sum\varphi_U(\Theta,\cdots,d\Theta,\cdots,\Theta)=\sum\varphi_U(\Theta,\cdots,[\Theta,\theta],\cdots,\Theta)=0$$

Here we use the Bianchi identity in Proposition 1.6 and equation (6). Thus $\varphi_X(\Theta_E)$ is a closed form.

For part (b), we shall show that for two connections D_1, D_2 on $E \to X$ there is a differential form α s.t. $\varphi(\Theta_E(D_1)) - \varphi(\Theta_E(D_2)) = d\alpha$. And this motivates us to consider one-parameter families of connections on $E \to X$. We define a C^∞ one-parameter family of connections on E to be a family of connections $\{D_t\}_{t\in\mathbb{R}}$ s.t. for a C^∞ frame f over U open in X the connection matrix $\theta_t(f) := \theta(D_t, f)$ has coefficients which are C^∞ one-parameter families of differential forms on E. We see

$$\frac{\partial}{\partial t} D_t \xi(f) = \frac{\partial}{\partial t}(d\xi(f) + \theta_t(f)\xi(f)) = \left(\frac{\partial}{\partial t}\theta_t(f)\right)\xi(f)$$

Moreover, since a change of frame is independent of t, this clearly defines a map, which is an element of $\mathcal{E}^1(X, \mathrm{Hom}(E,E))$, $\dot{D}_{t_0} : \mathcal{E}(X,E) \longrightarrow \mathcal{E}^1(X,E), \xi \mapsto \frac{\partial}{\partial t} D_t \xi|_{t_0}$ for each $t_0 \in \mathbb{R}$, and locally $\dot{\theta}_{t_0}(f) := \dot{D}_{t_0}(f) = \frac{\partial}{\partial t}\theta_t(f)|_{t_0}$.

Now we asserts that for any $\varphi \in I_k(\mathfrak{M}_r)$, we have

$$\varphi_X(\Theta_b) - \varphi_X(\Theta_a) = d\left(\int_a^b \varphi'(\Theta_t; \dot{D}_t) dt\right)$$

where Θ_t be the induced curvature, $\varphi'(\xi;\eta) = \sum_\alpha \varphi(\xi, \cdots, \xi, \underset{(\alpha)}{\eta}, \xi, \cdots, \xi)$, (α) denotes the αth argument, and $\xi, \eta \in \mathcal{E}^*(X, \mathrm{Hom}(E,E))$.

It suffices to show for a frame f over U, we have (Here $\Theta = \Theta_E(D_t, f)$, $\theta = \theta(D_t, f)$),

$$\dot{\varphi}_U(\Theta) = d\varphi'_U(\Theta; \dot{\theta})$$

where the dot denotes differentiation with respect to the parameter t, as above. We proceed by computing

$$\begin{aligned}
d\varphi'_U(\Theta;\dot{\theta}) &= d\Big(\sum_\alpha \varphi_U(\Theta, \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, \Theta)\Big) \\
&= \sum_\alpha \Bigg\{ \sum_{i<\alpha} \varphi_U(\Theta, \cdots, \underset{(i)}{d\Theta}, \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, \Theta) + \varphi_U(\Theta, \cdots, \underset{(\alpha)}{d\dot{\theta}}, \cdots, \Theta) \\
&\quad - \sum_{i>\alpha} \varphi_U(\Theta, \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, \underset{(i)}{d\Theta}, \cdots, \Theta) \Bigg\} \\
&= \sum_\alpha \varphi_U(\Theta, \cdots, \underset{(\alpha)}{\dot{\Theta}}, \cdots, \Theta) + \sum_\alpha \Bigg\{ \sum_{i<\alpha} \varphi_U(\Theta, \cdots, [\Theta, \underset{(i)}{\theta}], \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, \Theta) \\
&\quad - \varphi_U(\Theta, \cdots, [\dot{\theta}, \underset{(\alpha)}{\theta}], \cdots, \Theta) - \sum_{i>\alpha} \varphi_U(\Theta, \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, [\Theta, \underset{(i)}{\theta}], \cdots, \Theta) \Bigg\} \\
&\overset{\text{equation(6)}}{=} \sum_\alpha \varphi_U(\Theta, \cdots, \dot{\Theta}, \cdots, \Theta) = \dot{\varphi}_U(\Theta)
\end{aligned}$$

where we use $\dot{\Theta} = d\dot{\theta} + \dot{\theta} \wedge \theta + \theta \wedge \dot{\theta} = d\dot{\theta} + [\dot{\theta}, \theta]$, $d\Theta = [\Theta, \theta]$, the fact that $\theta, \dot{\theta}$ are 1-forms and equation (6). Thus we have shown this assertion.

Hence if D_1 and D_2 are two given connections, for $E \to X$,we let $D_t = tD_1 + (1-t)D_2$, we see

$$\varphi_X(\Theta_E(D_1)) - \varphi_X(\Theta_E(D_2)) = \varphi_X(\Theta_1) - \varphi_X(\Theta_2) = d\alpha \text{ where } \alpha = \int_0^1 \varphi'(\Theta_t; \dot{D}_t)\, dt$$

which shows part (b).

□

Definition 1.11 (Chern class)**.**

Let $E \longrightarrow X$ be a differentiable vector bundle equipped with a connection D. Then the kth Chern form of E relative to the connection D is defined to be

$$c_k(E,D) = (\Phi_k)_X\left(\frac{i}{2\pi}\Theta_E(D)\right) \in \mathcal{E}^{2k}(X)$$

where $\Phi_k \in I_k(\mathfrak{M}_r)$ and satisfies for $\forall A \in \mathfrak{M}_r$, $\det(I+A) = \sum_{k=0}^r \Phi_k(A)$. The (total) Chern form of E relative to D is defined to be

$$c(E,D) = \sum_{k=0}^{r} c_k(E,D) \text{ where } r = \operatorname{rank} E$$

The kth Chern class of the vector bundle E, denoted by $c_k(E)$, is the cohomology class of $c_k(E,D)$ in the de Rham group $H^{2k}(X,\mathbb{C})$, and the total Chern class of E, denoted by $c(E)$, is the cohomology class of $c(E,D)$ in $H^*(X,\mathbb{C})$; i.e., $c(E) = \sum_{k=0}^r c_k(E)$.

Remark 1.12. *Chern classes are well-defined by Theorem 1.10. Moreover, $c(E,D)$ is actually a real form, we can see this by taking D in Proposition 1.8 and then (omit the local frame f as usual) $dh = h\theta + \overline{\theta}^t h$. Act d on both sides and one can get*

$$0 = dh \wedge \theta + hd\theta + d\overline{\theta}^t h - \overline{\theta}^t \wedge dh = h\Theta + \overline{\Theta}^t h$$

We choose f s.t. $h = I$, then $\Theta = -\overline{\Theta}^t$. Hence $c := c(E,D,f)$ satisfies

$$c = \det\left(I + \frac{i}{2\pi}\Theta\right) = \det\left(I - \frac{i}{2\pi}\overline{\Theta}^t\right) = \det\left(I - \frac{i}{2\pi}\overline{\Theta}\right) = \overline{c}$$

The Chern class is like a symmetric polynomial of the eigenvalues of a matrix which can be said to be the most important invariant describing the matrix. We see

$$c_1(E,D) = \frac{i}{2\pi}\mathrm{tr}\Theta,\ c_2(E,D) = \frac{-1}{8\pi^2}\left((\mathrm{tr}\Theta)^2 - \mathrm{tr}\Theta^2\right)$$

Now let's fulfill the (†) promise stated at the beginning of this subsection.

Theorem 1.13. *Suppose that E and E' are differentiable $\mathbb{C}$-vector bundles over a differentiable manifold X. Then*

(a) If $\varphi : Y \longrightarrow X$ is a differentiable mapping where Y is a differentiable manifold, then $c(\varphi^*E) = \varphi^*c(E)$, where φ^*E is the pullback vector bundle and $\varphi^*c(E)$ is the pullback of the cohomology class $c(E)$.

(b) $c(E \oplus E') = c(E) \cdot c(E')$ in the de Rham cohomology ring $H^*(X,\mathbb{R})$.

(c) $c(E)$ depends only on the isomorphism class of the vector bundle E.

(d) If E^* is the dual vector bundle to E, then $c_k(E^*) = (-1)^k c_k(E)$.

Proof. For (a), we consider all notations with superscripts $*$ as pullbacks. For any connection D on E, define $\theta^*(f^*) = \varphi^*\theta(f)$ and easily see $g^*\theta^*(f^*g^*) = \theta^*(f^*)g^* + dg^*$, then θ^* defines a global connection on h^*E. Hence $\Theta(D^*, f^*) = \varphi^*(\Theta(D, f))$ and we get (a).

For (b), recall that for $[\varphi], [\varphi'] \in H^*(X, \mathbb{R})$, the wedge/cup product is defined as $[\varphi] \cdot [\varphi'] := [\varphi \wedge \varphi']$. Hence it is sufficient to find a $D^\oplus$ on $E \oplus E'$ s.t. $c(E \oplus E', D^\oplus) = c(E, D) \wedge c(E', D')$, but locally we have

$$c(E \oplus E', \begin{bmatrix} D & \\ & D' \end{bmatrix}) = \det \begin{bmatrix} I + \frac{i}{2\pi}\Theta & \\ & I' + \frac{i}{2\pi}\Theta' \end{bmatrix} = c(E, D) \wedge c(E', D').$$

For (c), the argument is similar in (a).

For (d), define $\langle , \rangle$ as the duality between E and E^*, and f, f^* are dual frames locally. We set a D^* on E^* by setting $\theta^* = -\theta^t(D, f)$ which indeed is a connection matrix. To see this, one need to vertify equation (3). Note if we denote $g^* = (g^{-1})^t$, then

$$\theta^*(f^*g^*) = -\theta^t(fg) = -(g^{-1}dg - g^{-1}\theta(f)g)^t = (g^*)^{-1}(dg^* + \theta^*(f^*)g^*)$$

where we use $dg^{-1} = -g^{-1}dgg^{-1}$. Hence the associated $\Theta^* = -\Theta^t$ and

$$c_k(E^*, D^*) = \Phi_k\left(-\frac{i}{2\pi}\Theta^t\right) = (-1)^k\Phi_k\left(\frac{i}{2\pi}\Theta^t\right) = (-1)^k c_k(E, D)$$

□

Remark 1.14 (Whitney's formula).

There is a general result of (b) of Theorem 1.13 which is called Whitney's formula:

> *If* $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$ *is a short exact sequence of vector bundles on* X*, then we have*
> $$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}).$$

This is exactly Proposition 4.2 of [BC],which shows that there is a form ξ *satisfies* $c(\mathcal{F}) - c(\mathcal{E})c(\mathcal{G}) = \bar{\partial}\partial\xi$.

The theorem is very powerful if we combine it with the Splitting principle(See Theorem 5.11 of [3264]):

> *Any identity among Chern classes of bundles that is true for bundles that are direct sums of line bundles is true in general.*

Now let's fulfill the (‡) promise, i.e. the obstruction-theoretic properties of Chern classes.

Proposition 1.15. *Let* $E \longrightarrow X$ *be a differentiable vector bundle of rank* r*. Then*

(a) $c_0(E) = 1$.

(b) If $E \cong X \times \mathbb{C}^r$ *is trivial, then* $c_j(E) = 0$, $j = 1, \cdots, r$*; i.e.,* $c(E) = 1$.

(c) If $E \cong E' \oplus T_s$ *where* T_s *is a trivial vector bundle of rank* s*, then* $c_j(E) = 0$, $j = r - s + 1, \cdots, r$.

Proof. (b) follows $\theta = 0$ in this time. (c) follows $c(E) = c(E')$ and E' is rank $r - s$. □

1.4 Chern classes: restrict on complex line bundles

In this subsection we concentrate on the case E is a complex line bundle over a complex manifold X.

Proposition 1.16. *Let $E \to X$ be a differentiable $\mathbb{C}$-vector bundle of rank r. Then there is an integer $N > 0$ and a differentiable mapping $\Phi : X \to G_{r,N}(\mathbb{C})$ s.t. $\Phi^*(U_{r,N}) \cong E$, where $U_{r,N} \to G_{r,N}$ is the universal bundle.*

Proof. By find a finite open cover and a simple partition of unity E^*, we see that there exists a finite number of global sections $\xi_1, \dots, \xi_N \in \mathcal{E}(X, E^*)$, s.t. at any point $x \in X$ there are r sections $\{\xi_{\alpha_1}, \dots, \xi_{\alpha_r}\}$ which are linearly independent at x and hence in a neighborhood of x. Define $\Phi : X \longrightarrow G_{r,N}$ that $\Phi(x)$ is the r-dimensional subspace of $\mathbb{C}^N$ spaned by $M(f^*) = [\xi_1(f^*)(x), \dots, \xi_N(f^*)(x)]$ where f^* is a frame $x_0 \in X$ (Actually the map is independent of f). We now claim that $\Phi^* U_{r,N} \cong E$ and it is suffices to define a bundle morphism $\widetilde{\Phi}$

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{\Phi}} & U_{r,N} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\Phi} & G_{r,N} \end{array}$$

which commutes with the mapping Φ and which is injective on each fibre. We define

$$\widetilde{\Phi}(x, v) = (\langle v, \xi_1(x) \rangle, \dots, \langle v, \xi_N(x) \rangle) \text{ for } x \in X, v \in E_x$$

where $\langle , \rangle$ denotes the bilinear pairing between E and E^*. Thus $\widetilde{\Phi}|_{E_x}$ is a $\mathbb{C}$-linear mapping into $\mathbb{C}^N$. We see the coefficients of $M(f^*)$ are C^∞ functions near x and hence

$$\widetilde{\Phi}(x, v) = v(f)^t \cdot M(f^*),$$

Hence $\widetilde{\Phi}|_{E_x}$ is injective and $\widetilde{\Phi}(E_x) = \pi^{-1}(\Phi(x))$, the Proposition is proved. $\square$

Remark 1.17. *Actually, the different isomorphism classes of differentiable vector bundles over X are classified by homotopy classes of maps into the Grassmannian $G_{r,N}$.*

Proposition 1.18. *Let $E \longrightarrow X$ be a complex line bundle. Then $c_1(E) \in H^2(X, \mathbb{Z})$.*

Proof. We see that it suffices to show $c_1(U_{1,N}) \in H^2(\mathbb{P}_{N-1}, \mathbb{Z})$ because $c_1(E) = \Phi^*(c_1(U_{1,N}))$ and . By Example 1,

$$\alpha := c_1(U_{1,N}, D(h)) = \frac{i}{2\pi}\Theta = \frac{1}{2\pi i} \frac{|f|^2 \sum d\xi_j \wedge d\bar{\xi}_j - \sum \bar{\xi}_j \xi_k d\xi_j \wedge d\bar{\xi}_k}{|f|^4},$$

where $f = (\xi_1, \dots, \xi_N)$ is a frame for $U_{1,N}$. We see

$$H^q(\mathbb{P}_n(\mathbb{C}), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q \text{ even and } q \leq 2n \\ 0, & q \text{ odd or } q > 2n \end{cases}$$

which can be shown easily using singular cohomology and in fact there is a cell decomposition $\mathbb{P}_0 \subset \mathbb{P}_1 \subset \cdots \subset \mathbb{P}_{N-1}$ where $\mathbb{P}_{j-1} \subset \mathbb{P}_j$ is a linear hyperplane, and $\mathbb{P}_j - \mathbb{P}_{j-1} \cong \mathbb{C}^j$. The submanifold $\mathbb{P}_j \subset \mathbb{P}_{N-1}$ is a generator for $H_{2j}(\mathbb{P}_N, \mathbb{Z})$, and there are no torsion elements.

Hence a closed differential form φ of degree $2j$ will be a representative of an integral cohomology class in $H^{2j}(\mathbb{P}_{N-1}, \mathbb{Z})$ if and only if $\int_{\mathbb{P}_j} \varphi \in \mathbb{Z}$. We take $\mathbb{P}_1 \subset \mathbb{P}_{N-1}$ as $\{(z_1, \dots, z_N) : z_j = 0, j = 3, \dots, N\}$ and consider the frame f on $W = \{z : z_1 \neq 0\}$, given by $f([1, \xi_2, \dots, \xi_N]) = (1, \xi_2, \dots, \xi_N)$. We see $f\mid_{W \cap \mathbb{P}_1}$ is given by $f([1, \xi_2, 0, \dots, 0]) = (1, \xi_2, 0, \dots, 0)$. Hence

$$\int_{\mathbb{P}_1} \alpha = \int_{\mathbb{P}_1 \cap W} \alpha = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = -\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{dx \wedge dy}{(1+|z|^2)^2} = -2 \int_0^\infty \frac{r dr}{(1+r^2)^2} = -1.$$

which shows that $c_1(U_{1,N}) \in H^2(\mathbb{P}_{N-1}, \mathbb{Z})$ and hence that $c_1(E) \in H^2(X, \mathbb{Z})$. □

Recall $\mathcal{O}$ be the structure sheaf of X and let $\mathcal{O}^*$ be the sheaf of nonvanishing holomorphic functions on X.

Lemma 1.19. *There is a one-to-one correspondence between the equivalence classes of holomorphic line bundles on X and the elements of the cohomology group $H^1(X, \mathcal{O}^*) =: \mathrm{Pic}(X)$.*

Proof. We shall represent $H^1(X, \mathcal{O}^*)$ by means of Čech cohomology. First we see there is an open covering $\{U_\alpha\} = \mathcal{U}$ and holomorphic transition functions $\{g_{\alpha\beta}\}$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(1, \mathbb{C}) = \mathbb{C} - \{0\}$ and satisfy $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ and $g_{\alpha\alpha} = 1$ on U_α. Hence the data $\{g_{\alpha\beta}\}$ define a cocycle $g \in Z^1(\mathcal{U}, \mathcal{O}^*)$ and hence a cohomology class in the direct limit $H^1(X, \mathcal{O}^*)$ which will be same for equivalent line bundles. Conversely, given any cohomology class $\xi \in H^1(X, \mathcal{O}^*)$, it can be represented by a cocycle $g = \{g_{\alpha\beta}\}$ on some covering $\mathcal{U} = \{U_\alpha\}$. By means of the functions $\{g_{\alpha\beta}\}$ one can construct a holomorphic line bundle having these transition functions. And the rest are routine. □

Recall there is an exact sequence of sheaves $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$ and the induced cohomology sequence

$$\begin{tikzcd} H^1(X, \mathcal{O}) \arrow[r] & H^1(X, \mathcal{O}^*) \arrow[r, "\delta"] \arrow[dr, dashed] & H^2(X, \mathbb{Z}) \arrow[r] \arrow[d, hook, "j"] & H^2(X, \mathcal{O}) \\ & & H^2(X, \mathbb{R}) & \end{tikzcd}$$

where j is the natural homomorphism and δ is the Bockstein operator. We assert that there is a commutative diagram

$$\begin{tikzcd} H^1(X, \mathcal{O}^*) \arrow[r, "\delta"] \arrow[dr, "c_1"'] & H^2(X, \mathbb{Z}) \arrow[d, "j"] \\ & H^2(X, \mathbb{R}) \end{tikzcd}$$

Note there is a isomorphism

$$H^2(X, \mathbb{R})_{(\text{Čech})} \longrightarrow H^2(X, \mathbb{R})_{(\text{de Rham})}, \tag{4.6}$$

thus for a cocycle $\xi \in Z^2(\mathcal{U}, \mathbb{R})$ we have associated a closed differential form $\varphi(\xi) \in \mathcal{E}^2(X)$. One can choose $\tau \in C^1(\mathcal{U}, \mathbb{Z})$ s.t. $\delta\tau = \xi$ and $\mu \in C^0(\mathcal{U}, \mathcal{E}^1)$ s.t. $\delta\mu = d\tau \in Z^1(\mathcal{U}, \mathcal{E}^1)$. Then we see $d\mu \in Z^0(\mathcal{U}, \mathcal{E}^2) = \mathcal{E}^2(X)$ and choose $\varphi(\xi) = -d\mu$.

We now use a Specifical $\mathcal{U}$ which has the property that any intersection of elements of the covering is a cell (in particular is simply connected) to describe the Bockstein operator $\delta : H^1(X, \mathcal{O}^*) \longrightarrow H^2(X, \mathbb{Z})$. If $g = \{g_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$, we define $\sigma = \{\sigma_{\alpha\beta}\}$ by$\sigma_{\alpha\beta} = \frac{1}{2\pi i} \log g_{\alpha\beta} \in C^1(\mathcal{U}, \mathcal{O})$ where we choose any branch of the logarithm. Hence $\delta\sigma \in Z^2(\mathcal{U}, \mathcal{O})$ (note $\delta^2 = 0$). One see $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$, then

$$(\delta\sigma)_{\alpha\beta\gamma} = \frac{1}{2\pi i}(\log g_{\beta\gamma} - \log g_{\alpha\gamma} + \log g_{\alpha\beta}),$$

which is integer-valued, thus $\delta\sigma \in Z^2(\mathcal{U}, \mathbb{Z})$ and is a representative for $\delta(g) \in H^2(X, \mathbb{Z})$.

Now let $g = \{g_{\alpha\beta}\}$ be the transition functions of $E \to X$ and set frames f_α for E over U_α. Donote $h_\alpha = h(f_\alpha)$ which is a positive C^∞ function defined in U_α. Thus $c_1(E, h) = \frac{1}{2\pi i}\partial\bar{\partial} \log h_\alpha$. We let, as above, $\delta\sigma \in Z^2(\mathcal{U}, \mathbb{Z})$ where $\sigma_{\alpha\beta} = \frac{1}{2\pi i} \log g_{\alpha\beta}$ be the $\delta-$image of $\{g_{\alpha\beta}\}$ in $H^2(X, \mathbb{Z})$. Choose $\tau = \sigma$ and $\mu = \{\mu_\alpha\}$ where $\mu_\alpha = \frac{1}{2\pi i}\partial \log h_\alpha$ in the construction of $\varphi(\xi)$, then

$$(\delta\mu)_{\alpha\beta} = \mu_\beta - \mu_\alpha = \frac{1}{2\pi i}\partial \log \frac{h_\beta}{h_\alpha} = \frac{1}{2\pi i}\partial \log g_{\alpha\beta}\bar{g}_{\alpha\beta} = \frac{1}{2\pi i} d \log g_{\alpha\beta} = d\sigma_{\alpha\beta} = d\tau_{\alpha\beta}$$

where use the transformation law (1) (note rank$E = 1$) and $\partial \log \bar{g}_{\beta\alpha} = \overline{\bar{\partial} \log g_{\beta\alpha}} = 0$ since $g_{\beta\alpha}$ is holomorphic. Thus the closed 2-form associated with the cocycle $\delta\sigma = \delta\tau$ (where $= \xi$ in previous context) is given by

$$\varphi = -d\mu = d\left(\frac{i}{2\pi}\partial \log h_\alpha\right) = \frac{i}{2\pi}\bar{\partial}\partial \log h_\alpha = c_1(E, h)$$

Hence we have shown the diagram above is commutative.

Remark 1.20 (this classification of differentiable line bundles by the first Chern class)**.** *Similarly for the exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^* \longrightarrow 0$$

on a differentiable manifold X, we have (note $\mathcal{E}$ is fine)

$$0 = H^1(X, \mathcal{E}) \longrightarrow H^1(X, \mathcal{E}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{E}) = 0$$

so there is an isomorphism $H^1(X, \mathcal{E}^*) \xrightarrow{\delta, \cong} H^2(X, \mathbb{Z})$ which asserts that all differentiable line bundles are determined by their Chern class in $H^2(X, \mathbb{Z})$.

Back to the previous case, we want to characterize the image of c_1 in the above commutative diagram, and namely the following theorem.

Theorem 1.21. $c_1(H^1(X, \mathcal{O}^*)) = H^2_{1,1}(X, \mathbb{Z})$*, where* $H^2_{1,1}(X, \mathbb{Z}) \subset H^2(X, \mathbb{Z})$ *which admits a d-closed differential form of type* $(1, 1)$ *as a representative.*

Proof. It suffices to show that

$$\delta(H^1(X,\mathcal{O}^*)) = H^2_{1,1}(X,\mathbb{Z})$$

which suffices to show that the image of $H^2_{1,1}(X,\mathbb{Z})$ in $H^2(X,\mathcal{O})$ is zero. But see there is a homomorphism of resolutions of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d, "i"] & \mathcal{E}^0 \arrow[r, "d"] \arrow[d, "i"] & \mathcal{E}^1 \arrow[r, "d"] \arrow[d, "\pi_{0,1}"] & \mathcal{E}^2 \arrow[r, "d"] \arrow[d, "\pi_{0,2}"] & \cdots \\
0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{E}^{0,0} \arrow[r, "\bar{\partial}"] & \mathcal{E}^{0,1} \arrow[r, "\bar{\partial}"] & \mathcal{E}^{0,2} \arrow[r, "\bar{\partial}"] & \cdots
\end{tikzcd}$$

where $\pi_{0,q} : \mathcal{E}^q \to \mathcal{E}^{0,q}$ is the projection on the submodule of forms of type $(0,q)$. Therefore the image of $H^2_{1,1}(X,\mathbb{C})$ in $H^2(X,\mathcal{O})$ is zero since a class in $H^2_{1,1}(X,\mathbb{C})$ is represented by a d-closed (1,1)-form, and we prove the theorem. $\square$

Remark 1.22 (Divisor on a complex manifold)**.**
We have the following exact sequence,

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{M}^* \longrightarrow \mathcal{D} := \mathcal{M}^*/\mathcal{O}^* \longrightarrow 0$$

where $\mathcal{M}^*$ is the sheaf of non-trivial meromorphic functions on X and $\mathcal{D}$ is called the sheaf of divisors on X i.e. $H^0(X,\mathcal{D}) =: \mathrm{Div}(X)$. And we have

$$H^0(X,\mathcal{M}^*) \longrightarrow H^0(X,\mathcal{D}) =: \mathrm{Div}(X) \longrightarrow H^1(X,\mathcal{O}^*) =: \mathrm{Pic}(X) \xrightarrow{\delta} H^2(X,\mathbb{Z})$$

where two different divisors have same image in $\mathrm{Pic}(X)$ *if they " differ by" a global meromorphic function (this is called linear equivalence in algebraic geometry exactly the case we are similar).*

2 Some Results

2.1 The Hodge Decomposition Theorem on Compact Kähler Manifolds

Theorem 2.1 (Fundamental theorem concerning elliptic complexes)**.**
Let $(\mathcal{E}(E), L)$ be an elliptic complex equipped with an inner product. Then

(a) There is an orthogonal decomposition

$$\mathcal{E}(E) = \mathcal{H}(E) \oplus LL^*G\mathcal{E}(E) \oplus L^*LG\mathcal{E}(E),$$

(b) The following commutation relations are valid:

(1) $I = H + \Delta G = H + G\Delta$.

(2) $HG = GH = H\Delta = \Delta H = 0$.

(3) $L\Delta = \Delta L$, $L^*\Delta = \Delta L^*$.

(4) $LG = GL$, $L^*G = GL^*$.

(c) $\dim_{\mathbb{C}} \mathcal{H}(E) < \infty$*, and there is a canonical isomorphism*

$$\mathcal{H}(E_j) \cong H^j(E).$$

We suppose there is a Hermitian inner product $\langle\, , \rangle$ on E, and locally, represented as

$$\langle u, v\rangle = h(u, v),\ u, v \in E \text{ where } h = \sum_{\mu,v} h_{\mu,v} z_\mu \otimes \overline{z}_v$$

where $(h_{\mu,v})$ is a positive definite Hermitian symmetric matrix. We can write $h = S + iA$ where S is a symmetric positive definite bilinear form and

$$A = \frac{1}{2i}\sum_{\mu,v} h_{\mu v}(z_\mu \otimes \overline{z}_v - \overline{z}_v \otimes z_\mu) = -i\sum_{\mu,v} h_{\mu,v} z_\mu \wedge \overline{z}_v$$

Definition 2.2 (fundamental 2-form)**.**
We define the fundamental 2-form Ω *associated to the hermitian metric* h *as*

$$\Omega = \frac{i}{2}\sum_{\mu,v} h_{\mu,v} z_\mu \wedge \overline{z}_v$$

Hence $h = S - 2i\Omega$ and moreover Ω is a real 2-form of type $(1, 1)$. If we choose $\{z_\mu\}$ s.t. $(h_{\mu,v}) = I$ and write $z_\mu = x_\mu + iy_\mu$ then

$$S = \sum_\mu (x_\mu \otimes x_\mu + y_\mu \otimes y_\mu),\ \Omega = \sum_\mu x_\mu \wedge y_\mu$$

Thus $\Omega^n = n! x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n$ and there is a nutural volume form $\mathrm{Vol} = \frac{\Omega^n}{n!}$ and a Hodge $*-$operator is defined s.t.

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta\rangle \mathrm{Vol}.$$

We define d^* and $\overline{\partial}^*$ as the adjoint operators of d and $\overline{\partial}$.

Now we give two very important example deduced by Theorem 2.1.

Example 2 (de Rham complex on a compact differentiable manifold X)**.**
Let $(\mathcal{E}^*(X), d)$ *be the de Rham complex on a compact differentiable manifold* X*:*

$$\mathbb{C} \to \mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \mathcal{E}^2(X) \xrightarrow{d} \cdots$$

*which is a FINE*③ *resolution and a ELLPITIC complex. Hence we see*

$$H^r(X, \mathbb{C}) = H^r(\mathcal{E}^*(X)) \cong \mathcal{H}_{\Delta_d}(\mathcal{E}^r(X)) =: \mathcal{H}^r(X)$$

which means for $\forall c \in H^r(X, \mathbb{C})$ *there exists a unique* Δ_d-*harmonic r-form* φ *representing this class c. And we see from Theorem 2.1 again,* $\dim_{\mathbb{C}} H^q(X, \mathbb{C}) = \dim_{\mathbb{C}} \mathcal{H}^q(X) =: b_q < \infty$*. where* $\{b_q\}_{q=0}^{\dim_{\mathbb{R}} X}$ *are called Betti numbers of the compact manifold* X*. Recall* $H^*(X, \mathbb{C})$ *also be determined by the topological structure of* X *from the singular cohomology view, so* b_q *and Euler characteristic* $\chi(X) := \sum(-1)^q b_q$*, are topological invariants of* X*.*

③A sheaf $\mathcal{S}$ is called *flasque/flabby* if $\mathcal{S}(X) \to \mathcal{S}(U)$ is surjective for any open set $U \subset X$, and called *soft* if $\mathcal{S}(X) \to \mathcal{S}(V)$ is surjective for any closed set $V \subset U$. And we call $\mathcal{S}$ is *fine* if for any locally finite open covering $\{U_\alpha\}$ of X(Sure we require X is paracompact), there exists a partition of unity $\{\rho_\alpha\}$ of $\mathcal{S}$ subordinate to $\{U_\alpha\}$. Many sheaves we encounter are fine (e.g. $\mathcal{E}_X, \mathcal{E}_X^{p,q}$ and $\mathcal{S}-$bundles) but may not be flabby. But we point out $\mathcal{O}_{\mathbb{C}}$ is not soft and constant sheaves are neither fine or soft.

One can prove that fine or flabby resolution is soft and we can use a soft, then acyclic resolution to calculate the cohomology group.

Example 3 (Dolbeault complex on a compact complex manifold X)**.**

Let X be a compact complex manifold of complex dimension n, and we have a FINE resolution and a ELLPITIC complex[4]*:*

$$0 \to \Omega^p \xrightarrow{\bar{\partial}} \mathcal{E}^{p,0}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{p,2}(X) \xrightarrow{\bar{\partial}} \cdots$$

for fixed p, $0 \leq p \leq n$. From Theorem 2.1, we have

$$H^q(X, \Omega^p) = H^q(\mathcal{E}^{p,*}(X)) = \mathcal{H}_{\overline{\Box}}(\mathcal{E}^{p,q}(X)) =: \mathcal{H}^{p,q}(X)$$

where we denote the Laplacian of $\overline{\partial}$ by $\overline{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. Similarly, we represent the cohomology classes by the $\overline{\Box}$-harmonic (p,q)-forms.

We define the Hodge numbers *of X, for $0 \leq p, q \leq n$, by setting (Use Theorem 2.1) $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^p) = \dim_{\mathbb{C}} \mathcal{H}^{p,q}(X) < \infty$. Note that $h^{p,q}$ are invariants of the complex structure of X and do not depend on the choice of metric.*

The following theorem show how the Hodge numbers and the Betti numbers are related, whose proof is a simple consequence of the fact that there is a spectral sequence (On Kähler case one can see Corollary 2.6) :

[FACT] *Let X be a compact complex manifold. Then*

$$\chi(X) = \sum_r (-1)^r b_r(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X).$$

Now we consider the elliptic complex of (p,q)-forms with coefficients in E where E is a holomorphic vector bundle over X. Similarly to the former case, we have

$$H^q(X, \Omega^p(E)) = H^q(\mathcal{E}^{p,*}(X,E)) = \mathcal{H}_{\overline{\Box}}(\mathcal{E}^{p,q}(X,E)) =: \mathcal{H}^{p,q}(X,E)$$

Here we define $\overline{\Box}$ to be the Laplacian of $\overline{\partial} \otimes 1$ if no confusion occurs. We see $\mathcal{H}^{p,q}(X,E)$ is the $\overline{\Box}$-harmonic E-valued (p,q)-forms in $\mathcal{E}^{p,q}(X,E)$ and we let the generalized Hodge numbers be (Use Theorem 2.1) $h^{p,q}(E) := \dim_{\mathbb{C}} \mathcal{H}^{p,q}(X,E) < \infty$. Moreover we define the Euler characteristic of the holomorphic vector bundle E *to be*

$$\chi(E) = \chi(X,E) = \sum_{q=0}^{n} (-1)^q h^{0,q}(E)$$

As before, $h^{p,q}(E)$ and $\chi(E)$ depend only on the complex structures of X and E, since the dimensions are independent of the particular metric used.

---

[4] Here Ω^p is defined as the kernel sheaf of the mapping $\mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}$, which is the sheaf of holomorphic differential forms of type $(p,0)$ (and we usually say holomorphic forms of degree p) i.e., in local coordinates, $\varphi \in \Omega^p(U)$ if and only if

$$\varphi = \sum_{|I|=p} \varphi_I dz^I, \quad \varphi_I \in \mathcal{O}(U),$$

Since $\overline{\partial}^2 = 0$, the sequence is actually a resolution of the sheaf Ω^p, by virtue of the Grothendieck version of the Poincaré lemma for the $\overline{\partial}$-operator whose proof can be obtained directly by the multivariate Cauchy integral formula.

Remark 2.3 (Why only consider $\overline{\partial}$ on $\mathcal{E}^*(X,E)$).

We see only $\overline{\partial}$ natrually acts on $\mathcal{E}^{p,q}(X,E)$ for any holomorphic vector bundle $E \to X$ because $\overline{\partial}$ anihilates the transition functions of E whereas ∂ and d do not. If we just consider the scalar coefficients, then similarly $\partial^* = -\overline{*}\partial\overline{*}$ and $\Box = \partial\partial^* + \partial^*\partial$ will commutes with $\overline{*}$.

Definition 2.4 (Kähler manifold).

A Kähler manifold is a complex manifold X equipped with a Hermitian metric h and a complex structure J s.t. the fundamental 2-form Ω associated to h is closed, i.e. $d\Omega = 0$. Inthis time we call h is a Kähler metric. We say a complex manifold X is said to be of Kähler type if it admits at least one Kähler metric.

We see $\mathbb{C}^n, \mathbb{P}^n, \mathbb{C}^n/\Gamma$ and compact Riemann surfaces with their submanifold are all Kähler manifolds. And there is a fact that in this case we have

$$\Delta = 2\Box = 2\overline{\Box}$$

Then we have the following decomposition theorem of Hodge (amplified by Kodaira).

Theorem 2.5 (The Hodge Decomposition Theorem on Compact Kähler Manifolds).

Let X be a compact complex manifold of Kähler type. Then there is a direct sum decomposition

$$H^r(X,\mathbb{C}) = \sum_{p+q=r} H^{p,q}(X),$$

moreover $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Not every complex manifold is of Kähler manifold, because there are some basic restrictions, e.g. $H^2(X,\mathbb{C})$ must be nontrivial (If not, Ω is exact and $\int_X \Omega^n = 0$, but $\int_X \Omega^n = n!\mathrm{Vol} > 0$, a contradiction).

And we now state more. Recall we have the Betti number $b_r(X) = \dim_{\mathbb{C}} H^r(X,\mathbb{C})$ and the Hodge number $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$. There are topological restrictions on a compact complex manifold admitting a Kähler metric, although such manifold always admits a Hermitian metric (See Proposition 1.2).

Corollary 2.6. *Let X be a compact Kähler manifold. Then*

1. $b_r(X) = \sum_{p+q=r} h^{p,q}(X)$.
2. $h^{p,q}(X) = h^{q,p}(X)$.
3. $b_q(X)$ *is even for odd* q.
4. $h^{1,0}(X) = \frac{1}{2}b_1(X)$ *is a topological invariant.*

Proof. immediately follows from Theorem 2.5. $\Box$

Example 4 (Hopf surface: a non-Kähler compact complex manifold).

We see if Γ is a discontinuous group of automorphism on a complex manifold X, possibly with fixed points, then X/Γ can still be given a complex structure as a complex space (a generalization of a complex manifold) with singularities at the image of the fixed points. We *proceed as follows to construct an example of a Hopf surface. Consider the 3-sphere* $\mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 \colon |z_1|^2 + |z_2|^2 = 1\}$, *and then we observe that there is a diffeomorphism*

$$f \colon S^3 \times \mathbb{R} \xrightarrow{\approx} \mathbb{C}^2 - \{0\},\ (z_1, z_2, t) \mapsto (e^t z_1, e^t z_2)$$

And we see $\mathbb{Z}$ *acts on* $S^3 \times \mathbb{R}$ *nuturally by*

$$(z_1, z_2, t) \longrightarrow (z_1, z_2, t + m) \text{ for } m \in \mathbb{Z}$$

Clearly $(S^3 \times \mathbb{R})/\mathbb{Z} \cong S^3 \times S^1$. *Under the diffeomorphism* f *we can see the action of* $\mathbb{Z}$ *on* $\mathbb{C}^2 - \{0\}$ *is properly discontinuous without fixed points. Thus* $X := (\mathbb{C}^2 - \{0\})/\mathbb{Z}$ *is a complex manifold which is diffeomorphic to* $S^3 \times S^1$ *(and this is compact). We then see*

$$b_0(X) = b_1(X) = b_3(X) = b_4(X) = 1 \text{ and } b_2(X) = 0$$

Where one may use the Künneth theorem which says $H_n(X \times Y) = \sum_{p+q=n} H_p(Y) \otimes H_q(X)$ *and Poincaré duality about Orientable n-dimensional manifold which says* $H^q(X) \cong H_{n-q}(X)$. *In particular,* $b_1(X) = 1$, *and hence* X *cannot be Kähler, since odd degree Betti numbers must be even on Kähler manifolds.*

Such manifolds X *are called* Hopf surfaces. *The Hopf surface is the simplest example of a compact complex manifold that cannot be embedded in projective space of any dimension.*

2.2 Riemann-Roch-Hirzebruch Theorem

It is a remarkable fact that $\chi(E)$ as above can be expressed in terms of topological invariants of the vector bundle E (its Chern classes) and of the complex manifold X itself (the Todd classes of the tangent bundle to X). Firstly we denote

$$c(E) = \prod_{i=1}^{r} (1 + x_i)$$

where $x_i \in H^*(X, \mathbb{C})$ ans something similar the eigenvalues of a matrix. Then we denote the Chern character of E, $ch(E)$ and the Todd class of the tangent bundle to X, $\mathcal{T}(T(X))$ by (Note the following are actually finite sums because $H^{>2\dim_{\mathbb{C}} X}(X, \mathbb{C}) = 0$)

$$ch(E) = \sum_{i=0}^{r} \exp(x_i) \text{ and } \mathcal{T}(T(X)) = \prod_{i=1}^{n} \frac{x_i}{1 - \exp(-x_i)}$$

Then we have the following theorem

> **Theorem** [Riemann-Roch-Hirzebruch, 1954] Let E be a holomorphic vector bundle over a compact complex manifold X, then
>
> $$\chi(E) = \int_X ch(E) \cdot \mathcal{T}(T(X))$$

From this theorem we immediately see LHS depends only on the topological structure of X , and RHS is an integer.

It is worth to piont out that for curves, the Hirzebruch-Riemann-Roch theorem is actually the classical Riemann-Roch theorem. Recall that for each divisor D on a curve there is an invertible sheaf $\mathcal{O}(D)$ (which corresponds to a line bundle) s.t. $\mathcal{O}(D)(U) = \{f \mid (f) \geq -D \text{ on } U\}$. For curves the Todd class is $1 + c_1(T(X))/2$, and the Chern character of a sheaf $\mathcal{O}(D)$ is just $1 + c_1(\mathcal{O}(D))$, so the Hirzebruch–Riemann–Roch theorem states that

$$h^{0,0}(\mathcal{O}(D)) - h^{0,1}(\mathcal{O}(D)) = \int_X [c_1(\mathcal{O}(D)) + c_1(T(X))/2]$$

But $h^{0,0}(\mathcal{O}(D)) = l(D)$ is the dimension of the linear system of D, and by Serre duality⑤ which says there is a conjugate linear isomorphism

$$\sigma : H^r(X, \Omega^p(E)) \longrightarrow H^{n-r}(X, \Omega^{n-p}(E^*))$$

Hence

$$h^{0,1}(\mathcal{O}(D)) = \dim_{\mathbb{C}} H^1(X, \mathcal{O}(D)) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}(D)^* \otimes \Omega^1_X)^* = h^{0,0}(\mathcal{O}(K-D)) = l(K-D)$$

where K is the canonical divisor with $\deg K = 2g-2$. Moreover, $c_1(\mathcal{O}(D))$ integrated over X is $\deg D$, and $c_1(T(X))$ integrated over X is the Euler class $2-2g$ of the curve X (there is a generalized result which will be discussed in Remark 2.7), where $g := h^{1,0}(X) = l(K)$ is the genus. So we get the classical Riemann-Roch theorem

$$l(D) - l(K-D) = \deg(D) + 1 - g$$

From the discussion above, we can also get Weil's Riemann Roch theorem for vector bundles V over curves, by noting the Chern character is $\operatorname{rank}(V) + c_1(V)$,

$$h^{0,0}(V) - h^{0,1}(V) = \int_X (\operatorname{rank}(V) + c_1(V)) \cdot (1 + \frac{c_1(T(X))}{2}) = \deg(\det(V)) + \operatorname{rank}(V)(1-g)$$

Remark 2.7 (Tangent bundle over $\mathbb{P}_1$).

We consider $T(\mathbb{P}_1)$ and it is well known that The natural metric on it is defined by

$$h(z) = h\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{(1+|z|^2)^2}$$

which is stable under the transform $w = 1/z$. Thus we have

$$\Theta = \bar{\partial}\partial \log h(z) = \bar{\partial}\left(\frac{-2\bar{z}}{1+|z|^2} dz\right) = \frac{2}{(1+|z|^2)^2} dz \wedge d\bar{z}.$$

Therefore,

$$\int_{\mathbb{P}_1} c_1(E,h) = \int_{\mathbb{P}_1} \frac{i}{2\pi}\Theta = \int_{\mathbb{P}_1} \frac{2dx \wedge dy}{\pi(1+|z|^2)^2} = \frac{2}{\pi}\int_0^\infty \int_0^{2\pi} \frac{\rho d\rho d\theta}{(1+\rho^2)^2} = 2.$$

The closed differential form $c_1(E,h)$ cannot be exact, since its integral over the 2-cycle $\mathbb{P}_1$ is nonzero. Therefore $T(\mathbb{P}_1)$ is a nontrivial complex line bundle. More importantly, one may note $\chi(\mathbb{P}_1) = 2 = \int_{\mathbb{P}_1} c_1(E,h)$, which is true in general case:

For a compact n-dimensional complex manifold X, $\displaystyle\int_X c_n(T(X)) = \chi(X)$.

⑤This is clear if we know Theorem 2.5, $\Box \bar{*}_E = \bar{*}_E \Box$ and $\bar{*}_E : \mathcal{E}^{p,q}(X,E) \longrightarrow \mathcal{E}^{n-p,n-q}(X,E)$ is a conjugate-linear isomorphism. Similarly, one can prove that $\bar{*}\Delta = \Delta\bar{*}$ and $\bar{*}$ is a conjugate-linear isomorphism, thus we have the Poincaré duality which states there is a conjugate linear isomorphism $\sigma : H^r(X,\mathbb{C}) \to H^{n-r}(X,\mathbb{C})$, if X be a compact n-dimensional orientable differentiable manifold.

2.3 The Holomorphic Line Bundles on $\mathbb{P}_n$

Now we use Chern classes to classify holomorphic line bundles on $\mathbb{P}_n$.

Example 5 (Line bundles on $\mathbb{P}_n$).

We have the following basic line bundles over $X = \mathbb{P}_n$,
(a) The hyperplane section bundle: $H \longrightarrow \mathbb{P}_n$.
(b) The universal bundle: $U \longrightarrow \mathbb{P}_n$.
(c) The canonical bundle: $K = \wedge^n T^*(\mathbb{P}_n) \longrightarrow \mathbb{P}_n$.
(d) The the trivial line bundle (i.e. the structure sheaf) $\mathcal{O}$

Here H *is the line bundle associated to the divisor of a hyperplane in* $\mathbb{P}_n$ *(See Remark 1.22), e.g.,* $[t_0 = 0]$, *(note all such line bundle are isomorphic) and then the divisor is defined by* $\{t_0/t_\alpha\}$ *in* $U_\alpha = \{t_\alpha \neq 0\}$, *the line bundle* H *has transition functions*

$$h_{\alpha\beta} = \left(\frac{t_0}{t_\alpha}\right)\left(\frac{t_0}{t_\beta}\right)^{-1} = \frac{t_\beta}{t_\alpha} \quad \text{in } U_\alpha \cap U_\beta$$

We see the universal bundle (See Example 1) has transition functions $u_{\alpha\beta} = \frac{t_\alpha}{t_\beta}$ *in* $U_\alpha \cap U_\beta$ *and thus* $H^* = U$. *Now we compute the transition functions for the canonical bundle* K *on* $\mathbb{P}_n$. *If we let* $\zeta_j^\beta = t_j/t_\beta$, $j \neq \beta$, *the usual coordinates in* U_β. *And we have* $\zeta_j^\alpha = \zeta_j^\beta \cdot \zeta_\beta^\alpha$ *in* $U_\alpha \cap U_\beta$ *which is the (nonlinear) change of coordinates for* $\mathbb{P}_n$ *from* U_α *to* U_β. *Hence a basis of* $K\mid_{U_\alpha}$ *given by* Φ_α *can be written as*

$$\Phi_\alpha := (-1)^\alpha d\zeta_0^\alpha \wedge \cdots \wedge d\zeta_{\alpha-1}^\alpha \wedge d\zeta_{\alpha+1}^\alpha \wedge \cdots \wedge d\zeta_n^\alpha = (\zeta_\beta^\alpha)^{n+1}\Phi_\beta$$

Now we see that these transition functions for the frames $\{\Phi_\alpha\}$ *induce transition functions* $\{k_{\alpha\beta}\}$ *for the canonical bundle* K *which are given by* $k_{\alpha\beta}([t_0, \dots, t_n]) = \left(\frac{t_\alpha}{t_\beta}\right)^{n+1}$ *and then*

$$K = \wedge^n T^*(\mathbb{P}_n) = U^{n+1} = (H^*)^{n+1}.$$

$T^*(X)$ *is called the cotangent bundle of* X *and we can describe tangent bundle* $T(X)$ *more concretely, by the following Euler sequence of sheaves on* X:

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{n+1} \to T(\mathbb{P}_n) \to 0.$$

which comes from

$$T(\mathbb{P}_n) \oplus \mathcal{O} = \mathrm{Hom}(U, U^\perp) \oplus \mathrm{Hom}(U, U) = \mathrm{Hom}(U, \mathcal{O}^{n+1}) = H^{n+1} = \mathcal{O}(1)^{n+1}.$$

From the Hodge decomposition theorem 2.5 we see

$$0 = H^1(\mathbb{P}_n, \mathbb{C}) = H^{1,0}(\mathbb{P}_n) \oplus H^{0,1}(\mathbb{P}_n)$$

and

$$\mathbb{C} \cong H^2(\mathbb{P}_n, \mathbb{C}) = H^{2,0}(\mathbb{P}_n) \oplus H^{1,1}(\mathbb{P}_n) \oplus H^{0,2}(\mathbb{P}_n)$$

And we see $H^{1,1}(\mathbb{P}_n) = \mathbb{C}[\Omega]$, where Ω is the fundamental form on $\mathbb{P}_n$. Hence[6]

$$H^1(\mathbb{P}_n, \mathcal{O}) = H^2(\mathbb{P}_n, \mathcal{O}) = 0.$$

[6] Actually, one may continue this to show $H^q(\mathbb{P}_n, \Omega^p) = 0$ when $p \neq q$ and $H^p(\mathbb{P}_n, \Omega^p) = H^{2p}(\mathbb{P}_n, \mathbb{C})$.

Now consider the short exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$ on $\mathbb{P}_n$ and the induced cohomology sequence

$$0 = H^1(\mathbb{P}_n, \mathcal{O}) \longrightarrow H^1(\mathbb{P}_n, \mathcal{O}^*) \xrightarrow{c_1} H^2(\mathbb{P}_n, \mathbb{Z}) \longrightarrow H^2(\mathbb{P}_n, \mathcal{O}) = 0$$

Let $\mathbb{P}_1 \subset \mathbb{P}_n$ be a generator for $H^2(\mathbb{P}_n, \mathbb{Z}) = \mathbb{Z}$. We find

$$c_1(H) = c_1(U^*) = -c_1(U) = 1$$

(See Proposition 1.18), and since c_1 is an isomorphism of abelian groups, it follows that every holomorphic line bundle $L \longrightarrow \mathbb{P}_n$ is a power of H, i.e. $L = H^m$, and $c_1(L)(\mathbb{P}_1) = m$.

Thus the holomorphic line bundles on $\mathbb{P}_n$ are completely classified in this manner by their Chern classes.

Example 6. *We see $T(\mathbb{P}_1) = \mathcal{O}(2)$ by Remark 2.7. Or use $c(T(\mathbb{P}_n)) = c(\mathcal{O}(1))^{n+1}$, we see $c_1(T(\mathbb{P}_1)) = 2$.*

Remark 2.8 (Birkhoff-Grothendieck theorem)**.**

The theorem says Every holomorphic vector bundle E over $\mathbb{P}_1$ is isomorphic to a direct sum of holomorphic line bundles, i.e. $E \cong \bigoplus_{i=1}^r \mathcal{O}(d_i)$ and d_i is unique up to a permutation. The first discoverers of the theorem were in hindsight, maybe Dedekind and Weber, who proved the following theorem:

> *(Dedekind-Weber) Let L be a field, for x a variable, we consider the ring $L[x, x^{-1}]$. For a given matrix $A \in \mathrm{GL}(n, L[x, x^{-1}])$, there exists matrices $B \in \mathrm{GL}(n, L[x])$, $C \in \mathrm{GL}(n, L[x^{-1}])$ such that*
>
> $$BAC = \begin{pmatrix} x^{d_1} & & & \\ & x^{d_2} & & \\ & & \ddots & \\ & & & x^{d_n} \end{pmatrix}$$
>
> *is a diagonal matrix, and where $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$, $d_i \in \mathbb{Z}$ and the sequence d_i is unique.*

Remark 2.9 (Kodaira embedding theorem)**.**

We call a Kähler manifold X whose Ω is integral, i.e. its cohomology class $[\Omega]$ is actually in $H^*(X, \mathbb{Z})$, is a Hodge manifold (due to A. Weil). One see compact projective algebraic manifolds and compact connected Riemannian manifolds are Hodge manifolds⑦.

And let $E \to X$ be a holomorphic line bundle and let $c_1(E)$ be the first Chern class of E considered as an element of the de Rham group $H^2(X, \mathbb{R})$. Then E is said to be positive if there is a real closed differential form ψ of type $(1,1)$ s.t. $\psi \in c_1(E)$ and ψ is a positive differential form. E is said to be negative if E^* is positive. By $c_1(E) = \frac{i}{2\pi}\Theta_E$ and $\overline{\partial}\partial$-Lemma⑧, E is

⑦There are also some manifolds which is Kähler but not Hodge: some complex torus in complex dimension ≥ 2 (which are not abelian varieties).

⑧$\partial\overline{\partial}$–Lemma: If η is any d-closed (p,q)-form on a compact Kähler manifold X, and η is d- or ∂- or $\overline{\partial}$-exact, then $\eta = \partial\overline{\partial}\gamma$ for some $(p-1, q-1)$-form γ. When $p = q$ and η is real, then we can take $\gamma = \sqrt{-1}\xi$ for a real $(p-1, q-1)$-form ξ. This Lemma is important for (usually non-linear) PDEs in Kähler geometry.

positive if and only if there is a Hermitian metric h on E s.t. $i\Theta_E$ is a positive differential form, where Θ_E is the curvature of E with respect to the canonical connection induced by h.

For example, the universal bundle $U \to \mathbb{P}_n$ has the curvature form $\Theta = 2i\Omega$ (See Example 1), where Ω is a positive differential form expressed in homogeneous coordinates

$$\Omega = \frac{i}{2}\frac{\sum\limits_{\mu=0}^{n}|t|^2 dt_\mu \wedge d\bar{t}_\mu - \sum\limits_{\mu,\nu=0}^{n}\bar{t}_\mu t_\nu dt_\mu \wedge d\bar{t}_\nu}{|t|^4}$$

Namely, Ω is the canonical Kähler form on $\mathbb{P}_n$ associated with the Fubini-Study metric. Hence $i\Theta_U = -2\Omega$ is a negative differential form on $\mathbb{P}_n$, thus $H^* = U$, and $K = U^{n+1}$ are negative line bundles over $\mathbb{P}_n$, and the hyperplane section bundle $H \to \mathbb{P}_n$ is positive.

*The amazing **Kodaira embedding theorem** states that for a compact complex manifold X, the following four conditions are equivalent:*

$$\begin{tikzcd}
& \textit{Hodge manifold} \arrow[dl, Rightarrow, "c_1(H^1(X,\mathcal{O}^*))=H^2_{1,1}(X,\mathbb{Z})\ni\Omega"'] & \\
\textit{positive line bundle} \arrow[rr, dashed, "\textbf{Kodaira}\textit{, 1954}"] & & \textit{projective algebraic} \arrow[ul, Rightarrow, "\mathbb{P}^n \textit{ is Hodge}"'] \\
& \textit{ample line bundle} \arrow[ul, Rightarrow, "\Theta_{\varphi^*(\mathcal{O}(1))}=\Theta_{L^{\otimes m}}=m\Theta_L"] \arrow[ur, Leftrightarrow, "\textit{By definition}"'] &
\end{tikzcd}$$

Note that if we denote by $\mathfrak{m}_p$ the ideal sheaf of holomorphic germs vanishing at p and $\mathfrak{m}_{p,q}$ the ideal sheaf of holomorphic germs vanishing at p and q, then what we need prove is that

$$H^0(X, \mathcal{O}(L^{\otimes m})) \to H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{O}/\mathfrak{m}_{p,q})$$

and

$$H^0(X, \mathcal{O}(L^{\otimes m})) \to H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{O}/\mathfrak{m}_p^2)$$

are both surjective when m is large enough. Now we use short exact sequences of sheaves:

$$0 \to \mathfrak{m}_{p,q} \to \mathcal{O} \to \mathcal{O}/\mathfrak{m}_{p,q} \to 0 \text{ and } 0 \to \mathfrak{m}_p^2 \to \mathcal{O} \to \mathcal{O}/\mathfrak{m}_p^2 \to 0$$

Tensor with the locally free sheaf $\mathcal{O}(L^{\otimes m})$ and consider the induced long exact sequences. One can see that the proof of the theorem is reduced to prove the vanishing of

$$H^1(X, \mathcal{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q}) \text{ and } H^1(X, \mathcal{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2)$$

We found that the main problem is that $\mathfrak{m}_{p,q}$ and $\mathfrak{m}_p^2$ are not line bundles, only are coherent analytic sheaves[9]*. So the Kodaira-Akizuki-Nakano vanishing theorem (See [K53]) could not be applied. There are two ways to continue the proof:*

---

[9] We see in the category correspondence between **Affine Scheme** and $\textbf{(CRing)}^{\text{op}}$, the coherent $\mathcal{O}_{\text{Spec}(A)}-$module corresponds to the finite generated $A-$module and the locally free $\mathcal{O}_{\text{Spec}(A)}-$module corresponds to the finite generated projective $A-$module.

- *Kodaira's Way. Blow up the points p appropriately, then* $\mathfrak{m}_p^2$ *and* $\mathfrak{m}_{pq}$ *become locally free on the blown up complex manifold* $\widetilde{X}$. *In this time we can applies the theory of harmonic differential forms to give the desired vanishing theorems. And one needs to show that vanishing upstairs implies vanishing downstairs. The completed proof can be found in Section 3,4 of Chapter VI of [W], or section 4 of Chapter 1 of [GH], or one directly see the remarkable paper [K54].*

- *Grauert's Way. Prove the generalized Kodaira vanishing theorem about coherent analytic sheaves, which says if* E *is a positive line bundle and* $\mathcal{F}$ *is any coherent analytic sheaf, then there is an integer* $\mu_0 > 0$ *s.t.* $H^q(X, \mathcal{O}(E^\mu) \otimes \mathcal{F}) = 0$ *for* $\mu \geq \mu_0$ *and* $q \geq 1$. *The completed proof can be found in [G].*

2.4 About 27 Lines on Each Smooth Cubic Surface in $\mathbb{P}_3$

Here we give a proof of the following classical theorem, which follows [3264] and take the statement of Theorem 5.1 of it:

Theorem 2.10. *Each smooth cubic surface in* $\mathbb{P}_3$ *contains exactly 27 distinct lines.*

Fix a smooth cubic surface $X = \{F = 0\}$, where $F \in H^0(\mathbb{P}_3, \mathcal{O}(3))$ is a general cubic homogeneous polynomial (ensuring that X is smooth). Our goal is to calculate the number of lines contained in X. We know the set of all lines is parameterized by the Grassmannian $\mathbb{G} = \mathbb{G}(1,3)$, with dimension $2 \times (4-2) = 4$.

Define a vector bundle E on $\mathbb{G}$, whose fiber at a point L is

$$E_L := H^0(L, \mathcal{O}_L(3)) = H^0(L, \mathcal{O}_{\mathbb{P}_3}(3)|_L) \cong \mathrm{Sym}^3 \mathcal{S}^*,$$

where $\mathcal{S}$ is the universal bundle on $\mathbb{G}$ (of rank 2). This is because for the 2-dimensional subspace $V_L \subset \mathbb{C}^4$ corresponding to the line L, we have $H^0(L, \mathcal{O}_L(1)) \cong V_L^*$, thus $H^0(L, \mathcal{O}_L(3)) \cong \mathrm{Sym}^3(V_L^*)$. Hence rank $E = \dim H^0(\mathbb{P}_1, \mathcal{O}(3)) = 4$.

Define the sheaf homomorphism

$$\mathrm{res} : V := H^0(\mathbb{P}_3, \mathcal{O}(3)) \otimes \mathcal{O}_{\mathbb{G}} \to E \text{ and } \mathrm{res}_L : H^0(\mathbb{P}_3, \mathcal{O}(3)) \to H^0(L, \mathcal{O}_L(3)),$$

where V is a rank $\binom{4+3-1}{4-1} = 20$ trivial sheaf.

The element F induces a global section $s_F \in H^0(\mathbb{G}, E)$, defined as $s_F(L) = \mathrm{res}_L(F) = F|_L$. The zero set $\mathrm{Zero}(s_F) = \{L \in \mathbb{G} \mid F|_L = 0\}$ is exactly the set of lines contained in X, because $L \subset X$ if and only if $F|_L = 0$. For a general F, the zero set is finite (i.e., the number of lines is finite), and the section s_F intersects the zero section transversely, so the number of zeros equals the integral of the top Chern class $c_4(E)$ over $\mathbb{G}$:

$$|\{L \subset X\}| = \int_{\mathbb{G}} c_4(E).$$

Now we can use the theory of Chern class to calculate, combing with section 3.3.1: Schubert cycles in $\mathbb{G}_{1,3}$ of [3264].

Remark 2.11 (The Chow ring of $\mathbb{G}_{1,3}$).
We fix a flag $p \subset L \subset H \subset \mathbb{P}_3$, and define the Schubert cycles:

$$\Sigma_{0,0} = \mathbb{G}(1,3), \quad \Sigma_{1,0} = \{\Lambda \mid \Lambda \cap L \neq \emptyset\}, \quad \Sigma_{2,0} = \{\Lambda \mid p \in \Lambda\},$$
$$\Sigma_{1,1} = \{\Lambda \mid \Lambda \subset H\}, \quad \Sigma_{2,1} = \{\Lambda \mid p \in \Lambda \subset H\}, \quad \Sigma_{2,2} = \{\Lambda \mid \Lambda = L\}.$$

One can show $\Sigma_{a,b}$ has codimension $a+b$ and the class $[\Sigma_{a,b}] \in A^{a+b}(\mathbb{G}(1,3))$ does not depend on the choice of flag. Thus we denote $\sigma_{a,b} = [\Sigma_{a,b}] \in A^{a+b}(\mathbb{G}(1,3))$, called Schubert classes, giving a a basis for the Chow ring $A^*(\mathbb{G}(1,3))$.

And we can even give the structure of the ring by the following relations:

$$\sigma_1^2 = \sigma_{1,1} + \sigma_2, \sigma_1\sigma_{1,1} = \sigma_1\sigma_2 = \sigma_{2,1}, \sigma_1\sigma_{2,1} = \sigma_{2,2}, \sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2}, \sigma_{1,1}\sigma_2 = 0$$

And we see
$$A(\mathbb{G}(1,3)) = \mathbb{Z}[\sigma_1, \sigma_2]/(\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_1^2\sigma_2 - \sigma_2^2).$$

Back to our case, by the theory of Schubert cycles (See Section 5.6.2: Grassmannians of [3264]), We have
$$1 + c_1 + c_2 =: c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1}$$
By the Splitting principle(See Theorem 5.11 of [3264]), we may assume $\mathcal{S}^* = \mathcal{S}_1 \oplus \mathcal{S}_2$ for two line bundles $\mathcal{S}_1, \mathcal{S}_2$ of $\mathbb{G}$, and denote $\alpha = c_1(\mathcal{S}_1), \beta = c_1(\mathcal{S}_2)$, then

$$\begin{aligned} c(\mathrm{Sym}^3\mathcal{S}^*) &= \mathcal{S}_1^3 \oplus (\mathcal{S}_1^2 \otimes \mathcal{S}_2) \oplus (\mathcal{S}_1 \otimes \mathcal{S}_2^2) \oplus \mathcal{S}_2^3 \\ &= (1+3\alpha)(1+2\alpha+\beta)(1+\alpha+2\beta)(1+3\beta) \\ \Longrightarrow c_4(\mathrm{Sym}^3\mathcal{S}^*) &= 9\alpha\beta[2(\alpha+\beta)^2 + \alpha\beta] = 9\sigma_{1,1}(2\sigma_1^2 + \sigma_{1,1}) = 27\sigma_{2,2}. \end{aligned}$$

And by the ring structure of $A^*(\mathbb{G}(1,3))$, $\deg(\sigma_{2,2}) = 1$, then we see

$$|\{L \subset X\}| = \deg(c_4(E)) = 27.$$

For a general smooth cubic surface X, the zeros of the section s_F are transverse and correspond to distinct lines, it follows that X contains exactly 27 different lines. We "deduce" the conclusion holds for all smooth cubic surfaces by a certain connectivity of the moduli space of smooth cubic surfaces, and we can even count lines in cases where X is singular (See Section 6.7: Lines on a cubic with a double point of [3264]).

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