Kodaira Vanishing Theorem And Embedding Theorem

G. L

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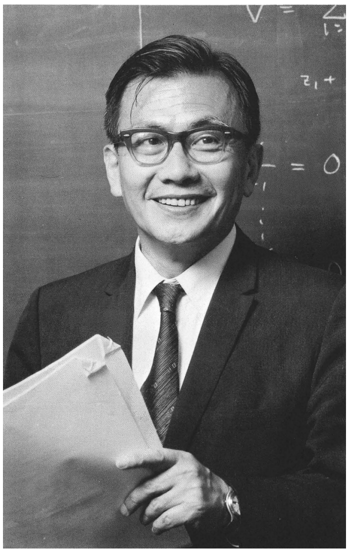
1 Introduction



K. Kodaira

Kunihiko Kodaira (March 16, 1915 - July 26, 1997) was one of the representatives of mathematics in the 20th century who made outstanding work in algebraic geometry and the theory of compact analytic surfaces. He is the first Asian Fields Medalist (in 1954, share with J. P. Serre) and one of the few mathematicians to also win the Wolf Medal (in 1984/1985, share with H. Lewy), whose reasons for the award respectively are

- Fields Medal. "Achieved major results in the theory of harmonic integrals and numerous applications to Kählerian and more specifically to algebraic varieties. He demonstrated, by sheaf cohomology, that **such varieties are Hodge manifolds**."
- Wolf Prize. "for his outstanding contributions to the study of complex manifolds and algebraic varieties." And the most remarkable works of K. Kodaira are **the projective imbedding theorem**, deformations of complex structures (with D. C. Spencer), and the classification of complex analytic surfaces.

The bolded part above is exactly Theorem 1.2 below, whose original statement (See [K54]) is

- A compact complex analytic variety V is (bi-regularly equivalent to) a nonsingular algebraic variety imbedded in a projective space if and only if V can carry a Hodge metric.

Combined with Chow's Theorem on $\mathbb{P}_n$, we know such manifolds are indeed algebraic. K. Kodaira showed how harmonic analysis and Hodge theory were integrated into the development of complex algebraic geometry, which had a profound and lasting impact.

M. F. Atiyah once commented this work (See [A]):

> In addition to the general theorems, which combined sheaf theory and harmonic forms to provide the new foundations, Kodaira made two specific and notable contributions ... *But perhaps his most striking individual achievement was in the general characterisation of projective algebraic varieties in [K54].*

Moreover, at the end of the letter which K. Kodaira sent to A. Weil on November 4, 1953, he wrote (See [H]):

> Recently I could prove that every Hodge variety (i.e. a Kähler variety whose fundamental form $i\sum g_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$ is homologous to an integral co-cycle) is an algebraic variety imbedded in a projective space. *I believe that my proof is correct; however, I am afraid that my result is too good.* I would appreciate very much your comment on this result.

Hence the Theorem 1.2 is absolutely a remarkable result in the history of algebraic geometry. We piont out that the core tool used by K. Kodaira to prove the remarkable theorem is the Blow Up Technology and the Kodaira Vanishing Theorem (Theorem 1.1) which was obtained by himself in 1953 (See [K53]) and was later slightly improved. The Blow Up Technology, in some sense, is routine if compared with Theorem 1.1.

Theorem 1.1 (Kodaira-Akizuki-Nakano vanishing theorem).
Suppose $E \to X$ be a positive holomorphic line bundle over a compact complex manifold X, then
$$H^q(X, \Omega^p(E)) = 0, \quad p + q > n$$
In particular, $H^q(X, \mathcal{O}(K_X \otimes E)) = 0$ for $q > 0$.

Theorem 1.2 (Kodaira embedding theorem).
A compact manifold X is projective algebraic if and only if it is a Hodge manifold.

In this note, we will describe the proof of Theorem 1.1, Theorem 1.2 in detail. The process is supplemented by classic examples and relevant important results. The general layout of each section is as follows.

- In section 2, we briefly introduce the theory of cohomology of sheaves, and Lemma 2.5 allows us to compute the cohomology group and get the important Example 3 and 4.
- In section 3, we introduce the concepts and properties about Hermitian metric, connection and curvature, and the Cherns form and Chern classes. We will see Chern forms appear naturally and beautifully and the description of Chern classes of complex line bundles plays an important role in the proof of Theorem 1.1.
- In section 4, we introduce the elliptic operator theory and get the fundamental theorem 4.22 concerning elliptic complexes, which converts cohomology classes into harmonic differential forms and therefore harmonic analysis and operator theory can show their power in algebraic geometry. Perhaps one can directly look at Example 9 and Example 10 if one just care about the proof of Theorem 1.1, because section 4 is lengthy and the specific details will not be used elsewhere in this note.
- In section 5, we restrict our attention to compact complex manifold and get Poincaré duality theorem 5.6 and Kodaira-Serre duality theorem 5.7. If we particularly consider compact Kähler manifolds, then will see on which there various operators have extraordinary relationships (Such as $\Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$, see Corollary 5.10 and Theorem 5.11), and this allows us to unify the previous examples and section 4 to obtain the famous Hodge's decomposition theorem 5.13 on compact Kähler manifolds.
- In section 6, we explain the positive condition of line bundle and use results from previous sections to deduce Theorem 1.1 by so called Bochner technique which analyzes the difference between different Laplacian operators. In subsection 6.3, we piont out the essential part of Theorem 1.2 lies in the vanishing of two specific cohomology groups, and demonstrate two ways,although just sketch, to slove it.

Throughout the entire note, we only admit a few results about sheaf theory and classical theorems of functional analysis (Proposition 4.1 by Sobolev, Proposition 4.2 by Rellich, the fundamental theorem 4.10 of pseudodifferential operators, and Riesz-Fredholm theorem 4.15) in Section 4, so this note is basically self-contained.

The main reference materials of this note are [GH], [S24], [W], K. Kodaira's original papers [K53] and [K54], and two excellent Chinese books [M], [ZY]. Moreover, we try our best to intersperse specific examples and related famous theorems (although some of them will not be proven) as Remark or Example in the text to enrich the content of the entire note.

2 The cohomology of sheaf

Sheaf theory was developed by Leray, Cartan and Serre originally in the 1950s. Once it came out, it become a whole new machinery with which to tackle global problems and see a great flowering of complex algebraic geometry.

We use $\mathcal{S}$ to denote one of the three structures on manifolds: $\mathcal{E}$ (differentiable), $\mathcal{A}$ (real analytic) and $\mathcal{O}$ (holomorphic), the undergroud field usually be $\mathbb{C}$ or $\mathbb{R}$. And we use $\mathcal{S}(U, E)$ to denote the sections of a sheaf E over an open set U, occasionally write $\Gamma(U, E)$ provided that there is no confusion as to which category we are dealing with.

The following Proposition show the relationship between vector bundles and locally free sheaves. And most of the sheaves we shall be dealing with actually are locally free sheaves arising from vector bundles.

Proposition 2.1. *Let $X = (X, \mathcal{S})$ be a connected $\mathcal{S}$-manifold. Then there is a one-to-one correspondence between (isomorphism classes of) $\mathcal{S}$-bundles over X and (isomorphism classes of) locally free sheaves of $\mathcal{S}$-modules over X.*

Proof. By definition the Proposition is easy. See Theorem II.1.13 of [W] for detailed proof. □

For a presheaf $\mathcal{F}$, we usually consider $\widetilde{\mathcal{F}} := \bigcup_{x \in X} \mathcal{F}_x$ and a natural projection $\pi : \widetilde{\mathcal{F}} \to X$. For each $s \in \mathcal{F}(U)$, we define $\tilde{s} : U \to \widetilde{\mathcal{F}}, x \mapsto s_x$ and $\{\tilde{s}(U) \mid U \text{ is open in } X\}$ as a basis for the topology of $\widetilde{\mathcal{F}}$. Hence we see π is a continuous and surjective map, moreover a local homeomorphism (Generally we say $\widetilde{\mathcal{F}}$ is a *étale space* over $\mathcal{F}$), and all $\tilde{s}$ are continuous. So we have associated a sheaf to $\mathcal{F}$, i.e. the sheaf of sections of $\widetilde{\mathcal{F}}$, denoted by $\overline{\mathcal{F}}$, called the sheaf generated by $\mathcal{F}$. And naturally there is a presheaf morphism

$$\tau : \mathcal{F} \to \overline{\mathcal{F}} \ ; \ \tau_U : \mathcal{F}(U) \to \overline{\mathcal{F}}(U) = \Gamma(U, \widetilde{\mathcal{F}}), s \mapsto \tilde{s}$$

One can show that if $\mathcal{F}$ is a sheaf, τ will be a sheaf morphism. So the étale space is useful is to pass from a presheaf to a sheaf and if $\mathcal{F}$ is a sheaf, then $\overline{\mathcal{F}}$ contains the same amount of information. Hence we usually consider this kind of sheaf and use $\mathcal{F}(U)$ and $\Gamma(U, \mathcal{F})$ interchangeably.

Now we give some example about the resolutionsequence of sheaves and then consider cohomology of sheaves.

Example 1. *Let X be a complex manifold of dimension n and $\mathcal{E}^{p,q}$ be the sheaf of (p, q) forms on X. Consider the sequence of sheaves, for fixed $p \geq 0$*

$$0 \longrightarrow \Omega^p \xrightarrow{i} \mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1} \xrightarrow{\overline{\partial}} \cdots \longrightarrow \mathcal{E}^{p,n} \longrightarrow 0 \tag{1}$$

where Ω^p is defined as the kernel sheaf of the mapping $\mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}$, which is the sheaf of holomorphic differential forms of type $(p, 0)$ (and we usually say holomorphic forms of degree p) i.e., in local coordinates, $\varphi \in \Omega^p(U)$ if and only if

$$\varphi = \sum_{|I|=p} \varphi_I dz^I, \quad \varphi_I \in \mathcal{O}(U),$$

Since $\overline{\partial}^2 = 0$, the sequence is actually a resolution of the sheaf Ω^p, by virtue of the Grothendieck version of the Poincaré lemma for the $\overline{\partial}$-operator whose proof can be obtained directly by the multivariate Cauchy integral formula.

And we also have a resolution of the constant sheaf $\mathbb{C}$*.*

$$0 \longrightarrow \mathbb{C} \longrightarrow \Omega^0 \xrightarrow{\partial} \Omega^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n \longrightarrow 0$$

Example 2. *Let* X *be a differentiable manifold of real dimension* m *and* $\mathcal{E}_X^p$ *be the sheaf of real-valued differential forms of degree* p*. Then there is a resolution of the constant sheaf* $\mathbb{R}$ *given by*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}_X^0 \xrightarrow{d} \mathcal{E}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}_X^m \longrightarrow 0$$

The classical Poincaré lemma asserts that on a star-shaped domain U*, the induced mapping* d_x *on the stalks is exact. Hence we could use star-shaped coverings to see the sequence is actually a resolution of the constant sheaf* $\mathbb{R}$*.*

And if X *is a topological manifold and* G *is an abelian group. We define* $\mathcal{S}^p(U, G)$ *be the group of singular cochains on* U *with coefficients in* G *and* δ *be the coboundary operator. Then we have a resolution of the constant sheaf* G*:*

$$0 \longrightarrow G \to \mathcal{S}^0(G) \xrightarrow{\delta} \mathcal{S}^1(G) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{S}^n(G) \longrightarrow \cdots$$

where $\mathcal{S}^p(G)$ *is the sheaf generated by the presheaf* $\mathcal{S}^p(-, G)$*. If* X *is a differentiable manifold, we usually consider* C^∞ *cochains and have a resolution of the constant sheaf* G*:*

$$0 \longrightarrow G \longrightarrow \mathcal{S}^*_\infty(G)$$

We start to define a canonical resolution of a sheaf $\mathcal{S}$ by using the sheaf of sections of the étale space $\tilde{\mathcal{S}}$ of $\mathcal{S}$. Let

$$\mathcal{C}^0(\mathcal{S})(U) = \{f : U \to \tilde{\mathcal{S}} \mid \pi \circ f = 1_U\}$$

and call $\mathcal{C}^0(\mathcal{S})$ *the sheaf of discontinuous sections of* $\mathcal{S}$ *over* X (which is introduced by Roger Godement). Now define $\mathcal{F}^1(\mathcal{S}) = \mathcal{C}^0(\mathcal{S})/\mathcal{S}$ and $\mathcal{C}^1(\mathcal{S}) = \mathcal{C}^0(\mathcal{F}^1(\mathcal{S}))$, by induction we define $\mathcal{F}^p(\mathcal{S}) = \mathcal{C}^{p-1}(\mathcal{S})/\mathcal{F}^{p-1}(\mathcal{S})$ and $\mathcal{C}^p(\mathcal{S}) = \mathcal{C}^0(\mathcal{F}^p(\mathcal{S}))$. From the exact image

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{S} \arrow[r] & \mathcal{C}^0(\mathcal{S}) \arrow[r] \arrow[dr, dashed] & \mathcal{F}^1(\mathcal{S}) \arrow[r] \arrow[d] & 0 & \\
 & & & \mathcal{C}^1(\mathcal{S}) \arrow[d] \arrow[dr, dashed] & & \\
 & & 0 \arrow[r] & \mathcal{F}^2(\mathcal{S}) \arrow[r] \arrow[d] & \mathcal{C}^2(\mathcal{S}) \arrow[r] \arrow[dr, dashed] & \cdots \\
 & & & 0 & & \cdots
\end{tikzcd}$$

we see the long exact sequence of sheaves which we call *the canonical resolution of* $\mathcal{S}$

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{C}^*(\mathcal{S})$$

A sheaf $\mathcal{S}$ is called *flasque/flabby* if $\mathcal{S}(X) \to \mathcal{S}(U)$ is surjective for any open set $U \subset X$, and called *soft* if $\mathcal{S}(X) \to \mathcal{S}(V)$ is surjective for any closed set $V \subset U$. And we call $\mathcal{S}$ is *fine* if for any locally finite open covering $\{U_\alpha\}$ of X(Sure we require X is paracompact), there exists a partition of unity $\{\rho_\alpha\}$ of $\mathcal{S}$ subordinate to $\{U_\alpha\}$. Many sheaves we encounter are fine (e.g. $\mathcal{E}_X, \mathcal{E}_X^{p,q}$ and $\mathcal{S}-$bundles) but may not be flabby. But we point out $\mathcal{O}_{\mathbb{C}}$ is not soft and constant sheaves are neither fine or soft. Below are some useful properties.

Proposition 2.2. *Fine or flabby sheaves are soft.*

Theorem 2.3. *Let* $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$ *is a short exact sequence of sheaves. then*

1 If $\mathcal{A}$ *is soft, then the induced sequence* $0 \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{B}(X) \longrightarrow \mathcal{C}(X) \longrightarrow 0$ *is exact.*

2 If $\mathcal{A}$ *and* $\mathcal{B}$ *are soft, then* $\mathcal{C}$ *is soft.*

Proof. See Theorem II.3.2 and Corollary II.3.8 of [W]. □

Definition 2.4 (Cohomology)**.**
Let $\mathcal{S}$ *be a sheaf over a space* X *and* $C^*(X,\mathcal{S}) = \Gamma(X, \mathcal{C}^*(\mathcal{S}))$. *We define the cohomology group of* $\mathcal{S}$ *over* X *of degree* q, $H^q(X,\mathcal{S})$ *as* $H^q(C^*(X,S))$ *which is the q-th cohomology group of the complex* $\mathcal{C}^*(X,\mathcal{S})$ *by setting* $C^{-1} = 0$.

We see $\mathcal{C}^0(\mathcal{S})$ is soft so $H^0(X,\mathcal{S}) = S(X)$ and if $\mathcal{S}$ is soft, then one can show that $H^{\geq 1}(X,\mathcal{S}) \equiv 0$. We say a resolution of $\mathcal{S}$: $0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{A}^*$ is *acyclic* if $H^q(X,\mathcal{A}^p) = 0$ for all $q > 0$ and $p \geq 0$ (So a soft resolution is acyclic, actually by Theorem 2.3). This term and the following lemma help us to compute $H^q(X,\mathcal{S})$.

Lemma 2.5 (The abstract de Rham theorem)**.**
Let $\mathcal{S}$ *be a sheaf over a space* X *and let* $0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{A}^*$ *be a resolution of* $\mathcal{S}$. *Then there is a natural homomorphism*

$$\gamma^p : H^p(\Gamma(X,\mathcal{A}^*)) \longrightarrow H^p(X,\mathcal{S})$$

where $H^p(\Gamma(X,\mathcal{A}^*))$ *is the pth derived group of the cochain complex* $\Gamma(X,\mathcal{A}^*)$.
Moreover, if $0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{A}^*$ *is acyclic,* γ^p *is an isomorphism.*

Proof. This is a standard result in cohomology theory. See Theorem II.3.13 of [W]. □

Example 3 (de Rham)**.**
By example 2, we see the map $I : \mathcal{E}^* \longrightarrow \mathcal{S}^*_\infty(\mathbb{R})$ *given by*

$$I_U : \mathcal{E}^*(U) \longrightarrow \mathcal{S}^*_\infty(U,\mathbb{R}), I_U(\varphi)(c) = \int_c \varphi$$

for a C^∞ *chain* c *is a sheaf morphism. And naturally induces a map* $I : H^p(\mathcal{E}^*(X)) \longrightarrow H^p(\mathcal{S}^*_\infty(X,\mathbb{R}))$. *We see the latter is an isomorphism, by noting* $\mathcal{E}^*$ *is fine (then soft) and each* $\mathcal{S}^p_\infty(\mathbb{R})$ *is soft which follows that each is a* $\mathcal{S}^0_\infty(\mathbb{R})-$ *module and* $\mathcal{S}^0_\infty(\mathbb{R})$ *is soft.*

Example 4 (Dolbeaule).

As in example 1, and the sequence (1) *is a resolution of* Ω^p *and is fine. So we have*

$$H^q(X,\Omega^p) \cong \frac{\mathrm{Ker}(\mathcal{E}^{p,q}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,q+1}(X))}{\mathrm{Im}(\mathcal{E}^{p,q-1}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,q}(X))}$$

If we consider forms with coefficients in a holomorphic vector bundle E *and denote* $\mathcal{O}(X, \wedge^p T^*(X) \otimes_{\mathbb{C}} E) =: \Omega^p(X,E)$*,* $\mathcal{E}(X, \wedge^{p,q} T^*(X) \otimes_{\mathbb{C}} E) =: \mathcal{E}^{p,q}(X,E)$*, then similarly we have*

$$H^q(X,\Omega^p(E)) \cong \frac{\mathrm{Ker}(\mathcal{E}^{p,q}(X,E) \xrightarrow{\overline{\partial}\otimes 1} \mathcal{E}^{p,q+1}(X,E))}{\mathrm{Im}(\mathcal{E}^{p,q-1}(X,E) \xrightarrow{\overline{\partial}\otimes 1} \mathcal{E}^{p,q}(X,E))}$$

3 Hermitian metric, connection and curvature

3.1 Basic concepts

Suppose that $E \to X$ is a *vector bundle* (Here we refer to a differentiable $\mathbb{C}$-vector bundle over a differentiable manifold) of rank r and $f = \{e_1, \cdots, e_r\}, e_j \in \mathcal{E}(U,E)$ is a local frame of E over an sufficiently small open set $U \subset X$. And locally we write $\xi \in \mathcal{E}(U,E) \cong \mathcal{E}(U, U \times \mathbb{C}^r)$ as

$$\xi = \xi(f) = (\xi^1, \cdots, \xi^r)^t,\ \xi^j \in \mathcal{E}(U)$$

Considering a change of frame $f \mapsto fg$ where $g : U \to \mathrm{GL}_r(\mathbb{C})$, one can see

$$\xi(fg) = g^{-1}\xi(f)$$

Moreover, if E is a holomorphic vector bundle, then we shall also have holomorphic frames $f = \{e_1, \cdots, e_r\}$ where $e_j \in \mathcal{O}(U,E)$ and $e_1 \wedge \cdots \wedge e_r(x) \neq 0$ for all $x \in U$.

In the remainder of this article, we assume that $f = \{e_1, \cdots, e_r\}$ represents a frame for vector bundle E on some open set $U \subset X$ unless otherwise specified.

Definition 3.1 (Hermitian metric).

Let $E \to X$ *be a vector bundle. A Hermitian metric* h *on* E *is an assignment of a Hermitian inner product* $\langle , \rangle_x$ *to each fibre* E_x *of* E *s.t. for any open set* $U \subset X$ *and* $\xi, \eta \in \mathcal{E}(U,E)$ *the function*

$$\langle \xi, \eta \rangle : U \longrightarrow \mathbb{C}$$

given by $\langle \xi, \eta \rangle(x) = \langle \xi(x), \eta(x) \rangle_x$ *which is* C^∞*. A vector bundle* E *equipped with a Hermitian metric* h *is called a Hermitian vector bundle.*

Locally we can represents h as a positive definite Hermitian symmetric matrix $h(f)$ whose entries are C^∞ by setting $h(f)_{\rho\sigma} = \langle e_\sigma, e_\rho \rangle$. For any $\xi, \eta \in \mathcal{E}(U,E)$, we see

$$\langle \xi, \eta \rangle = \langle \sum_\rho \xi^\rho(f) e_\rho, \sum_\sigma \eta^\sigma(f) e_\sigma \rangle = \overline{\eta(f)}^t h(f) \xi(f)$$

and easily get the transformation law for local representations of the Hermitian metric

$$h(fg) = \overline{g}^t h(f) g \tag{2}$$

Proposition 3.2. *Every vector bundle $E \to X$ admits a Hermitian metric.*

Proof. There exists a locally finite (X is locally compact) covering $\{U_\alpha\}$ of X and frames f_α defined on U_α. Define a Hermitian metric h_α on $E|_{U_\alpha}$ by setting, for $\xi, \eta \in E_x$, $x \in U_\alpha$,

$$\langle \xi, \eta \rangle_x^\alpha = \overline{\eta(f_\alpha)(x)} \cdot \xi(f_\alpha)(x)$$

Then take $\{\rho_\alpha\}$ as a C^∞ partition of unity subordinate to the covering $\{U_\alpha\}$ and let, for $\xi, \eta \in E_x$,

$$\langle \xi, \eta \rangle_x = \sum_\alpha \rho_\alpha(x) \langle \xi, \eta \rangle_x^\alpha$$

We can now verify that $\langle , \rangle$ so defined gives a Hermitian metric for $E \to X$. $\square$

We now consider differential forms with vector bundle coefficients by setting $\mathcal{E}^p(X, E) = \mathcal{E}(X, \wedge^p T^*(X) \otimes_{\mathbb{C}} E)$ be the differential forms of degree p on X with coefficients in E. By the sheaf isomorphism $\mathcal{E}^p \otimes_{\mathbb{C}} \mathcal{E}(E) \cong \mathcal{E}^p(E)$, we have a local representation for $\xi \in \mathcal{E}^p(U, E)$ given by

$$\xi = \sum_\rho \xi^\rho(f) e_\rho \text{ and } \xi(x) = \sum_{\rho,k} \phi_{k,\rho}(x) \omega_k(x) \otimes e_\rho(x)$$

where $\xi^\rho(f) \in \mathcal{E}^p(U), x \in U$ and $\{\omega_k\}$ is a local frame for $\wedge^p T^*(X) \otimes \mathbb{C}$ and $\phi_{k,\rho}$ are uniquely determined C^∞ functions defined near x. One can see the differential form ξ^ρ so determined is independent of the choice of frame $\{\omega_k\}$ and the transformation law $\xi(fg) = g^{-1}\xi(f)$.

Definition 3.3 (Connection)**.**
Let $E \to X$ be a vector bundle. Then a connection$D$ on $E \to X$ is a $\mathbb{C}$-linear mapping

$$D : \mathcal{E}(X, E) \longrightarrow \mathcal{E}^1(X, E),$$

which satisfies $D(\varphi\xi) = d\varphi \cdot \xi + \varphi D\xi$, where $\varphi \in \mathcal{E}(X)$ and $\xi \in \mathcal{E}(X, E)$.

If $E = X \times \mathbb{C}$ is the trivial line bundle, then the ordinary exterior differentiation $d : \mathcal{E}(X) \to \mathcal{E}^1(X)$ is a connection on E. Thus a connection is a generalization of exterior differentiation to vector-valued differential forms, and we can extend D to higher-order E-valued differential forms naturally.

Locally, we define the *connection matrix* $\theta(D, f)$ associated with the connection D and the frame f by setting

$$De_\sigma := \sum_\rho \theta_{\rho\sigma}(D, f) e_\rho,\ \theta_{\rho\sigma}(D, f) \in \mathcal{E}^1(U)$$

sometimes we just say θ if D, f are fixed. Now we have

$$D\xi(f) = D\left((e_1, \dots, e_r) \cdot (\xi^1, \cdots, \xi^r)^t\right) = (e_1, \cdots, e_r) \cdot [(d\xi^1, \cdots, d\xi^r)^t + \theta \cdot (\xi^1, \cdots, \xi^r)^t] = (d+\theta)\xi(f)$$

i.e. $D = d + \theta$.

We want to show that the connection D (as we shall see below, every vector bundle admits a connection, see Proposition 3.8) on E induces in a natural manner an element called *curvature tensor*

$$\Theta_E(D) \in \mathcal{E}^2(X, \mathrm{Hom}(E, E))$$

where $\mathrm{Hom}(E,E)$ be the vector bundle whose fibres are $\mathrm{Hom}(E_x,E_x)$.

First, we inspect the element $\chi \in \mathcal{E}^p(X,\mathrm{Hom}(E,E))$. Note

$$\mathcal{E}^p(U,\mathrm{Hom}(E,E)) \cong \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathcal{E}(U,\mathrm{Hom}(E,E)) \cong \mathcal{E}^p(U) \otimes_{\mathcal{E}(U)} \mathrm{M}_r(\mathbb{C})$$

thus we can represent χ locally as an $r\times r$ matrix $\chi(f)$ where $\chi_{\rho\sigma}(f) \in \mathcal{E}^p(U)$ and $[\chi(f)\xi(f)]^\rho = \sum_\sigma \chi(f)_{\rho\sigma}\xi^\sigma(f)$. And then

$$\chi(fg) = g^{-1}\chi(f)g \tag{3}$$

In turn, it can be verified that χ satisfying the above formula actually with entries in $\mathcal{E}(U)$ defines an element in $\mathcal{E}^p(U,\mathrm{Hom}(E,E))$.

Definition 3.4 (Curvature)**.**

Let D be a connection in a vector bundle $E \to X$. Then the curvature $\Theta_E(D)$ is defined to be that element $\Theta \in \mathcal{E}^2(X,\mathrm{Hom}(E,E))$ s.t. the $\mathbb{C}$-linear mapping $\Theta : \mathcal{E}(X,E) \longrightarrow \mathcal{E}^2(X,E)$ has the representation with respect to a frame f

$$\Theta(f) = \Theta(D,f) = d\theta(f) + \theta(f)\wedge\theta(f).$$

which is a $r\times r$ matrix of 2-forms and $\Theta_{\rho\sigma} = d\theta_{\rho\sigma} + \sum_k \theta_{\rho k}\wedge\theta_{k\sigma}$.

Remark 3.5 (Well-definability of Θ)**.**

From

$$(e_1,\cdots,e_r)g\theta(fg) = D((e_1,\cdots,e_r)g) = (e_1,\cdots,e_r)(d+\theta)g$$

we see

$$g\theta(fg) = dg + \theta(f)g \tag{4}$$

Take the exterior derivative of equation (4) and use the fact that $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^k\alpha\wedge d\beta$ when α is a k-form, we get $d\theta(f)g - \theta(f)dg = dg\theta(fg) + gd\theta(fg)$,so

$$g[\Theta(fg)] = g[d\theta(fg) + \theta(fg)\wedge\theta(fg)] = [d\theta(f) + \theta(f)\wedge\theta(f)]g = \Theta(f)g$$

by some simplification. Thus Θ is well-defined.

Proposition 3.6.

(i) $D(f)^2 = \Theta(f)$

(ii)(Bianchi identity) $d\Theta(f) = [\Theta(f),\theta(f)]$

Proof. We omit the notation f, and then for any $\xi \in \mathcal{E}(U,E)$, we have

$$D^2\xi = (d+\theta)^2\xi = \theta d\xi + d(\theta\xi) + \theta\wedge\theta\xi = (d\theta + \theta\wedge\theta)\xi = \Theta\xi$$

then gets (i). (ii) follows that

$$d\Theta = d\theta\wedge\theta - \theta\wedge d\theta = [d\theta + \theta\wedge\theta, \theta] = [\Theta,\theta]$$

□

Remark 3.7. *We now define a Lie product on the algebra*

$$\mathcal{E}^*(X, \mathrm{Hom}(E,E)) = \sum_p \mathcal{E}^p(X, \mathrm{Hom}(E,E))$$

If $\chi \in \mathcal{E}^p(X, \mathrm{Hom}(E,E))$ and f is a frame for E over the open set U, then $\chi(f) \in \mathfrak{M}_r \otimes_{\mathbb{C}} \mathcal{E}^p(U)$ by the fact that the differential forms in $\mathcal{E}^p(X, \mathrm{Hom}(E,E))$ are locally matrices of p-forms. Hence if $\psi \in \mathcal{E}^q(X, \mathrm{Hom}(E,E))$, we define

$$[\chi(f), \psi(f)] = \chi(f) \wedge \psi(f) - (-1)^{pq} \psi(f) \wedge \chi(f) \tag{5}$$

And we see from equation (3), $[\chi(fg), \psi(fg)] = g^{-1}[\chi(f), \psi(f)]g$ *if g is a change of frame, which shows the Lie bracket is well defined on $\mathcal{E}^*(X, \mathrm{Hom}(E,E))$ and one may check it satisfies the Jacobi identity. Thus $\mathcal{E}^*(X, \mathrm{Hom}(E,E))$ is a Lie algebra.*

At last of this subsection, we show that every vector bundle admits a connection.

Proposition 3.8. *Every Hermitian vector bundle $E \to X$ admits a connection D which is compatible with the Hermitian metric h.*

Proof. "Compatible" means that we always have

$$d\langle \xi, \eta \rangle = \langle D\xi, \eta \rangle + \langle \xi, D\eta \rangle$$

and one can vertify this are equivalent to say $dh(f) = h(f)\theta(f) + \overline{\theta(f)}^t h(f)$ for all frame. Note that we views $h : \mathcal{E}^p(X,E) \otimes \mathcal{E}^q(X,E) \to \mathcal{E}^{p+q}(X)$ by setting

$$\langle \omega \otimes \xi, \omega' \otimes \xi' \rangle_x = \omega \wedge \overline{\omega'} \langle \xi, \xi' \rangle_x$$

where $\omega \in \mathcal{E}^p(X)_x, \omega' \in \mathcal{E}^q(X)_x, \xi, \xi' \in \mathcal{E}(X,E)$. Then we can choose unitary frame f locally s.t. $h(f) = I$, and find a locally finite covering $\{U_\alpha\}$ of X and corresponding f_α. In this time we require $0 = \theta(f_\alpha) + \overline{\theta(f_\alpha)}^t$.

In each U_α, we can choose $\theta(f_\alpha) = 0$, and by 4 we require $\theta(f_\alpha g) = g^{-1}dg$, so we define θ as this. Now $dh(f_\alpha g) = d(\overline{g}^t \cdot g) = h(f_\alpha g)\theta(f_\alpha g) + \overline{\theta(f_\alpha g)}^t h(f_\alpha g)$, which vertifies the compatibility.

Hence we can glue these local connections to get a global connection by using a partition $\{\rho_\alpha\}$ of unity, as Proposition 3.2. □

3.2 The canonical D, Θ for a Hermitian holomorphic vector bundle

Now assume $E \to X$ is a holomorphic vector bundle equipped with a Hermitian metric h. We can choose a canonical connection D s.t. D is compatible with h and if $D = D' + D'' \in \mathcal{E}^{(1,0)}(X,E) \oplus \mathcal{E}^{(0,1)}(X,E)$ as a natural decomposition, then for $\xi \in \mathcal{O}(U,E)$, $D''\xi = 0$.

Specifically, let $\xi \in \mathcal{O}(U,E)$, then

$$D\xi(f) = (\partial + \theta^{(1,0)}(f))\xi(f) + \theta^{(0,1)}(f)\xi(f)$$

so we require $\theta^{(0,1)}(f) = 0$. Then from the compatible condition $dh(f) = h(f)\theta(f) + \overline{\theta(f)}^t h(f)$ we see $\partial h = h\theta, \overline{\partial} h = \overline{\theta}^t h$. Hence we can just define

$$\theta(f) = h^{-1}(f)\partial h(f)$$

and one can vertifies that this θ satisfies equation (4) and actually define a global connection $D = [\partial + \theta(f)] + \overline{\partial}$.

Proposition 3.9. *Canonical θ, D, Θ satisfies*

(i) $\theta(f)$ is a $(1,0)$-form and $\partial\theta(f) = -\theta(f) \wedge \theta(f)$.

(ii) $\Theta(f) = \overline{\partial}\theta(f)$ and $\partial\Theta(f) = [\Theta(f, \theta(f))]$.

Proof. We omit the notation f. Then by $\partial^2 = 0 = \overline{\partial}^2$

$$\partial\theta = \partial(h^{-1}\partial h) = -h^{-1} \cdot \partial h \cdot h^{-1} \wedge \partial h = -\theta \wedge \theta, \text{ then } \Theta = d\theta + \theta \wedge \theta = \overline{\partial}\theta = \overline{\partial}\partial \log h$$

and by the Bianchi identity we have $\partial\Theta = [\Theta, \theta]$. □

Example 5 (The Universal bundle over the Grassmannian manifold)**.**

Let $U_{r,n} \longrightarrow G_{r,n}$ be the universal bundle where $G(r,n)$ is the Grassmannian manifold and $U_{r,n}$ is the disjoint union of all r-planes in $\mathbb{C}^n$. And localize on a suitable open set $U \subset G_{r,n}$ we nuturally have a holomorphic frame $f = (e_1, \cdots, e_r)$ where $e_j : U \longrightarrow \mathbb{C}^n$ and $e_1 \wedge \cdots \wedge e_r \neq 0$. Then we could consider the nutural metric on $U_{r,n}$ be setting $h(f) = \overline{f}^t \cdot f$: $h(f)$ is positive definite and satisfies the transformation law 2, i.e. $h(fg) = \overline{g}^t \cdot h(f)g$, which is a well-defined Hermitian metric.

We now compute the canonical curvature Θ with respect to the nutural metric in the case $r = 1$. For $\varphi \in [\mathcal{E}^p(W)]^n, \psi \in [\mathcal{E}^q(W)]^n$ where $W \subset U$ is open, we define (which generalizes usual inner product on $\mathbb{C}^n$)

$$\langle \varphi, \psi \rangle = (-1)^{pq}\overline{\psi}^t \wedge \varphi$$

And see (note $h^{-1} = [\overline{f}^t f]^{-1}$),

$$\Theta(f) = \overline{\partial}(h^{-1}\partial h) = -\frac{\langle f, f\rangle\langle df, df\rangle - \langle df, f\rangle \wedge \langle d, df\rangle}{\langle f, f\rangle^2} = -\frac{|f|^2 \sum d\xi_i \wedge d\overline{\xi}_i - \sum \overline{\xi}_i \xi_j d\xi_i \wedge d\overline{\xi}_j}{|f|^4}$$

where $f = (\xi_i)$ is a holomorphic frame for $U_{1,n}$ and $\xi_i \in \mathcal{O}(W)$.

As for general r, One may see example III.2.4 of [W].

3.3 Chern classes: from a differential-geometric view

The approach here follows the exposition of Bott and Chern [BC], based on the original ideas of Chern and Weil. We shall see Chern classes

(†) can be realized as elements of $H^{2j}(X, \mathbb{R})$ having certain functorial properties;

(‡) will be the obstruction to giving global frames.

We denote $\mathfrak{M}_r$ as the set of all $r \times r$ complex matrices, and $\widetilde{I}_k(\mathfrak{M}_r)$ be the $\mathbb{C}$-vector space of all invariant k-linear forms on $\mathfrak{M}_r$, which implies if $\widetilde{\varphi} \in \widetilde{I}_k(\mathfrak{M}_r)$, then $\widetilde{\varphi}(gA_1g^{-1}, \cdots, gA_rg^{-1}) = \widetilde{\varphi}(A_1, \cdots, A_r)$ for $\forall g \in \mathrm{GL}(r, \mathbb{C}), A_i \in \mathfrak{M}_r$. We also note $\widetilde{\varphi}$ induces a $\varphi : \mathfrak{M}_r \longrightarrow \mathbb{C}$ by setting $\varphi(A) = \widetilde{\varphi}(A, \cdots, A)$, and actually they decide mutually (By a trick called polarization), so we omit the widetilde from now. Moreover, φ extend naturally on $\mathcal{E}^*(\mathrm{Hom}(E, E))$, see localizing on U and use a frame f, we set

$$\varphi(A_1 \otimes w_1, \cdots, A_k \otimes w_k) = w_1 \wedge \cdots \wedge w_k \varphi(A_1, \cdots, A_k) \text{ for } A_i \otimes w_i \in \mathfrak{M}_r(U) \otimes \mathcal{E}^p(U)$$

Note it is independent of f because $\xi(fg) = g^{-1}\xi(f)g$.

Theorem 3.10 (Due to A. Weil).
Let $E \to X$ be a differentiable $\mathbb{C}$-vector bundle, let D be a connection on E, and suppose that $\varphi \in I_k(\mathfrak{M}_r)$. Then
(a) $\varphi_X(\Theta_E(D))$ is closed.
(b) The image of $\varphi_X(\Theta_E(D))$ in $H^{2k}(X, \mathbb{C})$ is independent of the connection D.

Proof. From the definition of a k-linear form, we get

$$d\varphi_U(A_1, \cdots, A_k) = \sum_\alpha (-1)^{g(\alpha)} \varphi_U(A_1, \cdots, dA_\alpha, \cdots, A_k)$$

for $A_\alpha \in \mathfrak{M}_r \otimes \mathcal{E}^{p_\alpha}(U)$ and open set $U \subset X$, where $g(\alpha) = \sum_{\beta<\alpha} \deg A_\beta$. Specifically, for $A_j, B \in \mathfrak{M}_r$, we have

$$\sum_j \varphi(A_1, \cdots, [A_j, B], \cdots, A_k) = 0 \tag{6}$$

because $\varphi(e^{-tB}A_1e^{tB}, \cdots, e^{-tB}A_ke^{tB}) - \varphi(A_1, \cdots, A_k) = 0$ and the fact that $\frac{d}{dt}[e^{-tB}A_je^{tB}] \mid_{t=0}= [A_j, B]$.

Recalling the Lie product on $\mathfrak{M}_r \otimes \mathcal{E}^*$ in Remark 3.7, by equation (6) , we have

$$\sum_\alpha (-1)^{f(\alpha)} \varphi_U(A_1, \cdots, [A_\alpha, B], \cdots, A_k) = 0 \tag{7}$$

for all $A_\alpha \in \mathfrak{M}_r \otimes \mathcal{E}^{p_\alpha}(U)$ and $B \in \mathfrak{M}_r \otimes \mathcal{E}^q(U)$, where $f(\alpha) = \deg B \sum_{\beta\le\alpha} \deg A_\beta$.

Note for (a), it suffices to show that for a frame f over U, $d\varphi_U(\Theta(f)) = 0$. We omit the notation f and note Θ is a 2-form, immediately we have

$$d\varphi_U(\Theta) = d\varphi_U(\Theta, \cdots, \Theta) = \sum \varphi_U(\Theta, \cdots, d\Theta, \cdots, \Theta) = \sum \varphi_U(\Theta, \cdots, [\Theta, \theta], \cdots, \Theta) = 0$$

Here we use the Bianchi identity in Proposition 3.6 and equation (7). Thus $\varphi_X(\Theta_E)$ is a closed form.

For part (b), we shall show that for two connections D_1, D_2 on $E \to X$ there is a differential form α s.t. $\varphi(\Theta_E(D_1)) - \varphi(\Theta_E(D_2)) = d\alpha$. And this motivates us to consider one-parameter families of connections on $E \to X$. We define a C^∞ one-parameter family of connections on E to be a family of connections $\{D_t\}_{t\in\mathbb{R}}$ s.t. for a C^∞ frame f over U open in X the connection matrix $\theta_t(f) := \theta(D_t, f)$ has coefficients which are C^∞ one-parameter families of differential forms on E. We see

$$\frac{\partial}{\partial t} D_t \xi(f) = \frac{\partial}{\partial t}(d\xi(f) + \theta_t(f)\xi(f)) = \left(\frac{\partial}{\partial t}\theta_t(f)\right)\xi(f)$$

Moreover, since a change of frame is independent of t, this clearly defines a map, which is an element of $\mathcal{E}^1(X, \mathrm{Hom}(E, E))$, $\dot{D}_{t_0} : \mathcal{E}(X, E) \longrightarrow \mathcal{E}^1(X, E), \xi \mapsto \frac{\partial}{\partial t} D_t\xi|_{t_0}$ for each $t_0 \in \mathbb{R}$, and locally $\dot{\theta}_{t_0}(f) := \dot{D}_{t_0}(f) = \frac{\partial}{\partial t}\theta_t(f)|_{t_0}$.

Now we asserts that for any $\varphi \in I_k(\mathfrak{M}_r)$, we have

$$\varphi_X(\Theta_b) - \varphi_X(\Theta_a) = d\left(\int_a^b \varphi'(\Theta_t; \dot{D}_t)dt\right)$$

where Θ_t be the induced curvature, $\varphi'(\xi; \eta) = \sum_\alpha \varphi(\xi, \cdots, \xi, \underset{(\alpha)}{\eta}, \xi, \cdots, \xi)$, (α) denotes the αth argument, and $\xi, \eta \in \mathcal{E}^*(X, \mathrm{Hom}(E, E))$.

It suffices to show For a frame f over U, we have (Here $\Theta = \Theta_E(D_t, f)$, $\theta = \theta(D_t, f)$),

$$\dot{\varphi}_U(\Theta) = d\varphi'_U(\Theta; \dot{\theta})$$

where the dot denotes differentiation with respect to the parameter t, as above. We proceed by computing

$$\begin{aligned}
d\varphi'_U(\Theta;\dot{\theta}) &= d\Big(\sum_\alpha \varphi_U(\Theta, \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, \Theta)\Big) \\
&= \sum_\alpha \Bigg\{ \sum_{i<\alpha} \varphi_U(\Theta, \cdots, \underset{(i)}{d\Theta}, \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, \Theta) + \varphi_U(\Theta, \cdots, \underset{(\alpha)}{d\dot{\theta}}, \cdots, \Theta) \\
&\quad - \sum_{i>\alpha} \varphi_U(\Theta, \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, \underset{(i)}{d\Theta}, \cdots, \Theta) \Bigg\} \\
&= \sum_\alpha \varphi_U(\Theta, \cdots, \underset{(\alpha)}{\dot{\Theta}}, \cdots, \Theta) + \sum_\alpha \Bigg\{ \sum_{i<\alpha} \varphi_U(\Theta, \cdots, [\Theta, \underset{(i)}{\theta}], \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, \Theta) \\
&\quad - \varphi_U(\Theta, \cdots, [\dot{\theta}, \underset{(\alpha)}{\theta}], \cdots, \Theta) - \sum_{i>\alpha} \varphi_U(\Theta, \cdots, \underset{(\alpha)}{\dot{\theta}}, \cdots, [\Theta, \underset{(i)}{\theta}], \cdots, \Theta) \Bigg\} \\
&\overset{\text{equation(7)}}{=} \sum_\alpha \varphi_U(\Theta, \cdots, \dot{\Theta}, \cdots, \Theta) = \dot{\varphi}_U(\Theta)
\end{aligned}$$

where we use $\dot{\Theta} = d\dot{\theta} + [\dot{\theta}, \theta]$, $d\Theta = [\Theta, \theta]$ and equation (7). Thus we have shown this assertion.

Hence if D_1 and D_2 are two given connections, for $E \to X$,we let $D_t = tD_1 + (1-t)D_2$, we see

$$\varphi_X(\Theta_E(D_1)) - \varphi_X(\Theta_E(D_2)) = \varphi_X(\Theta_1) - \varphi_X(\Theta_2) = d\alpha \text{ where } \alpha = \int_0^1 \varphi'(\Theta_t; \dot{D}_t)\, dt$$

which shows (b).

□

Definition 3.11 (Chern class)**.**

Let $E \longrightarrow X$ be a differentiable vector bundle equipped with a connection D. Then the kth Chern form of E relative to the connection D is defined to be

$$c_k(E, D) = (\Phi_k)_X \left(\frac{i}{2\pi}\Theta_E(D)\right) \in \mathcal{E}^{2k}(X)$$

where $\Phi_k \in I_k(\mathfrak{M}_r)$ and satisfies for $\forall A \in \mathfrak{M}_r$, $\det(I + A) = \sum_{k=0}^r \Phi_k(A)$. The (total) Chern form of E relative to D is defined to be

$$c(E, D) = \sum_{k=0}^r c_k(E, D) \text{ where } r = \operatorname{rank} E$$

The kth Chern class of the vector bundle E, denoted by $c_k(E)$, is the cohomology class of $c_k(E, D)$ in the de Rham group $H^{2k}(X, \mathbb{C})$, and the total Chern class of E, denoted by $c(E)$, is the cohomology class of $c(E, D)$ in $H^*(X, \mathbb{C})$; i.e., $c(E) = \sum_{k=0}^r c_k(E)$.

Remark 3.12. *Chern classes are well-defined by Theorem 3.10. Moreover, $c(E, D)$ is actually a real form, we can see this by taking D in Proposition 3.8 and then (omit the local frame f as usual) $dh = h\theta + \overline{\theta}^t h$. Act d on both sides and one can get $0 = h\Theta + \overline{\Theta}^t h$. We choose f s.t. $h = I$, then $\Theta = -\overline{\Theta}^t$. Hence $c := c(E, D, f)$ satisfies*

$$c = \det\left(I + \frac{i}{2\pi}\Theta\right) = \det\left(I - \frac{i}{2\pi}\overline{\Theta}^t\right) = \det\left(I - \frac{i}{2\pi}\overline{\Theta}\right) = \overline{c}$$

The Chern class is like a symmetric polynomial of the eigenvalues of a matrix which can be said to be the most important invariant describing the matrix. We see

$$c_1(E, D) = \frac{i}{2\pi}\mathrm{tr}\Theta, \;\; c_2(E, D) = \frac{-1}{8\pi^2}\left((\mathrm{tr}\Theta)^2 - \mathrm{tr}\Theta^2\right)$$

Now let's fulfill the (†) promise stated at the beginning of this subsection.

Theorem 3.13. *Suppose that E and E' are differentiable $\mathbb{C}$-vector bundles over a differentiable manifold X. Then*

(a) If $\varphi : Y \longrightarrow X$ is a differentiable mapping where Y is a differentiable manifold, then $c(\varphi^* E) = \varphi^* c(E)$, where $\varphi^* E$ is the pullback vector bundle and $\varphi^* c(E)$ is the pullback of the cohomology class $c(E)$.

(b) $c(E \oplus E') = c(E) \cdot c(E')$ in the de Rham cohomology ring $H^*(X, \mathbb{R})$.

(c) $c(E)$ depends only on the isomorphism class of the vector bundle E.

(d) If E^* is the dual vector bundle to E, then $c_k(E^*) = (-1)^k c_k(E)$.

Proof. For (a), we consider all notations with superscripts $*$ as pullbacks. For any connection D on E, define $\theta^*(f^*) = \varphi^*\theta(f)$ and easily see $g^*\theta^*(f^*g^*) = \theta^*(f^*)g^* + dg^*$, then θ^* defines a global connection on h^*E. Hence $\Theta(D^*, f^*) = \varphi^*(\Theta(D, f))$ and we get (a). For (b), recall that for $[\varphi], [\varphi'] \in H^*(X, \mathbb{R})$, the wedge/cup product is defined as $[\varphi] \cdot [\varphi'] := [\varphi \wedge \varphi']$. Hence it is sufficient to find a $D^\oplus$ on $E \oplus E'$ s.t. $c(E \oplus E', D^\oplus) = c(E, D) \wedge c(E', D')$, but see locally this is obvious. For (c), the argument is similar in (a).

For (d), define $\langle , \rangle$ as the duality between E and E^*, and f, f^* are dual frames locally. We set a D^* on E^* by setting $\theta^* = -\theta^t(D, f)$ which indeed is a connection matrix (one need to vertify equation (4)). Hence the associated $\Theta^* = -\Theta^t$ and

$$c_k(E^*, D^*) = \Phi_k\left(-\frac{i}{2\pi}\Theta^t\right) = (-1)^k \Phi_k\left(\frac{i}{2\pi}\Theta^t\right) = (-1)^k c_k(E, D)$$

□

Now let's fulfill the (‡) promise, i.e. the obstruction-theoretic properties of Chern classes.

Proposition 3.14. *Let $E \longrightarrow X$ be a differentiable vector bundle of rank r. Then*

(a) $c_0(E) = 1$.

(b) If $E \cong X \times \mathbb{C}^r$ is trivial, then $c_j(E) = 0$, $j = 1, \cdots, r$; i.e., $c(E) = 1$.

(c) If $E \cong E' \oplus T_s$ where T_s is a trivial vector bundle of rank s, then $c_j(E) = 0$, $j = r - s + 1, \cdots, r$.

Proof. (b) follows $\theta = 0$ in this time. (c) follows $c(E) = c(E')$ and E' is rank $r - s$. □

3.4 Chern classes: restrict on complex line bundles

In this subsection we concentrate on the case E is a complex line bundle over a complex manifold X, whose results has important applications when we prove Kodaira's vanishing and embedding theorem.

Proposition 3.15. *Let $E \to X$ be a differentiable $\mathbb{C}$-vector bundle of rank r. Then there is an integer $N > 0$ and a differentiable mapping $\Phi : X \to G_{r,N}(\mathbb{C})$ s.t. $\Phi^*(U_{r,N}) \cong E$, where $U_{r,N} \to G_{r,N}$ is the universal bundle.*

Proof. By find a finite open cover and a simple partition of unity E^*, we see that there exists a finite number of global sections $\xi_1, \dots, \xi_N \in \mathcal{E}(X, E^*)$, s.t. at any point $x \in X$ there are r sections $\{\xi_{\alpha_1}, \dots, \xi_{\alpha_r}\}$ which are linearly independent at x and hence in a neighborhood of x. Define $\Phi : X \longrightarrow G_{r,N}$ that $\Phi(x)$ is the r-dimensional subspace of $\mathbb{C}^N$ spaned by $M(f^*) = [\xi_1(f^*)(x), \dots, \xi_N(f^*)(x)]$ where f^* is a frame $x_0 \in X$ (Actually the map is independent of f). We now claim that $\Phi^* U_{r,N} \cong E$ and it is suffices to define a bundle morphism $\widetilde{\Phi}$

$$\begin{array}{ccc} E & \xrightarrow{\widetilde{\Phi}} & U_{r,N} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\Phi} & G_{r,N} \end{array}$$

which commutes with the mapping Φ and which is injective on each fibre. We define

$$\widetilde{\Phi}(x, v) = (\langle v, \xi_1(x) \rangle, \dots, \langle v, \xi_N(x) \rangle) \text{ for } x \in X, v \in E_x$$

where $\langle , \rangle$ denotes the bilinear pairing between E and E^*. Thus $\widetilde{\Phi}|_{E_x}$ is a $\mathbb{C}$-linear mapping into $\mathbb{C}^N$. We see the coefficients of $M(f^*)$ are C^∞ functions near x and hence

$$\widetilde{\Phi}(x, v) = v(f)^t \cdot M(f^*),$$

Hence $\widetilde{\Phi}|_{E_x}$ is injective and $\widetilde{\Phi}(E_x) = \pi^{-1}(\Phi(x))$, the Proposition is proved. □

Proposition 3.16. *Let $E \longrightarrow X$ be a complex line bundle. Then $c_1(E) \in \widetilde{H}^2(X, \mathbb{Z})$.*

Proof. Here we use $\widetilde{H}^2(X, \mathbb{Z})$ to represent the nutural image of $H^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{R})$. we see that it suffices to show $c_1(U_{1,N}) \in H^2(\mathbb{P}_{N-1}, \mathbb{Z})$ because $c_1(E) = \Phi^*(c_1(U_{1,N}))$ and . By Example **??**,

$$\alpha := c_1(U_{1,N}, D(h)) = \frac{i}{2\pi}\Theta = \frac{1}{2\pi i} \frac{|f|^2 \sum d\xi_j \wedge d\bar{\xi}_j - \sum \bar{\xi}_j \xi_k d\xi_j \wedge d\bar{\xi}_k}{|f|^4},$$

where $f = (\xi_1, \dots, \xi_N)$ is a frame for $U_{1,N}$. We see

$$H^q(\mathbb{P}_n(\mathbb{C}), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & q \text{ even and } q \le 2n \\ 0, & q \text{ odd or } q > 2n \end{cases}$$

which can be shown easily using singular cohomology and in fact there is a cell decomposition $\mathbb{P}_0 \subset \mathbb{P}_1 \subset \cdots \subset \mathbb{P}_{N-1}$ where $\mathbb{P}_{j-1} \subset \mathbb{P}_j$ is a linear hyperplane, and $\mathbb{P}_j - \mathbb{P}_{j-1} \cong \mathbb{C}^j$. The submanifold $\mathbb{P}_j \subset \mathbb{P}_{N-1}$ is a generator for $H_{2j}(\mathbb{P}_N, \mathbb{Z})$, and there are no torsion elements.

Hence a closed differential form φ of degree $2j$ will be a representative of an integral cohomology class in $H^{2j}(\mathbb{P}_{N-1}, \mathbb{Z})$ if and only if $\int_{\mathbb{P}_j} \varphi \in \mathbb{Z}$. We take $\mathbb{P}_1 \subset \mathbb{P}_{N-1}$ as $\{(z_1, \dots, z_N) : z_j = 0, j = 3, \dots, N\}$ and consider the frame f on $W = \{z : z_1 \neq 0\}$, given by $f([1, \xi_2, \dots, \xi_N]) = (1, \xi_2, \dots, \xi_N)$. We see $f \mid_{W \cap \mathbb{P}_1}$ is given by $f([1, \xi_2, 0, \dots, 0]) = (1, \xi_2, 0, \dots, 0)$. Hence

$$\int_{\mathbb{P}_1} \alpha = \int_{\mathbb{P}_1 \cap W} \alpha = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} = -\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{dx \wedge dy}{(1 + |z|^2)^2} = -2 \int_0^\infty \frac{r dr}{(1 + r^2)^2} = -1.$$

which shows that $c_1(U_{1,N}) \in H^2(\mathbb{P}_{N-1}, \mathbb{Z})$ and hence that $c_1(E) \in \widetilde{H}^2(X, \mathbb{Z})$. $\square$

Recall $\mathcal{O}$ be the structure sheaf of X and let $\mathcal{O}^*$ be the sheaf of nonvanishing holomorphic functions on X.

Lemma 3.17. *There is a one-to-one correspondence between the equivalence classes of holomorphic line bundles on X and the elements of the cohomology group* $H^1(X, \mathcal{O}^*) =: \mathrm{Pic}(X)$.

Proof. We shall represent $H^1(X, \mathcal{O}^*)$ by means of Čech cohomology. First we see there is an open covering $\{U_\alpha\} = \mathcal{U}$ and holomorphic transition functions $\{g_{\alpha\beta}\}$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(1, \mathbb{C}) = \mathbb{C} - \{0\}$ and satisfy $g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$ and $g_{\alpha\alpha} = 1$ on U_α. Hence the data $\{g_{\alpha\beta}\}$ define a cocycle $g \in Z^1(\mathcal{U}, \mathcal{O}^*)$ and hence a cohomology class in the direct limit $H^1(X, \mathcal{O}^*)$ which will be same for equivalent line bundles. Conversely, given any cohomology class $\xi \in H^1(X, \mathcal{O}^*)$, it can be represented by a cocycle $g = \{g_{\alpha\beta}\}$ on some covering $\mathcal{U} = \{U_\alpha\}$. By means of the functions $\{g_{\alpha\beta}\}$ one can construct a holomorphic line bundle having these transition functions (Just like Proposition 2.1). And the rest are routine. $\square$

Recall there is an exact sequence of sheaves $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$ and the induced cohomology sequence

$$\begin{tikzcd} H^1(X, \mathcal{O}) \arrow[r] & H^1(X, \mathcal{O}^*) \arrow[r, "\delta"] \arrow[dr, dashed] & H^2(X, \mathbb{Z}) \arrow[r] \arrow[d, hook, "j"] & H^2(X, \mathcal{O}) \\ & & H^2(X, \mathbb{R}) & \end{tikzcd}$$

where j is the natural homomorphism and δ is the Bockstein operator. We assert that there is a commutative diagram

$$\begin{tikzcd} H^1(X, \mathcal{O}^*) \arrow[r, "\delta"] \arrow[dr, "c_1"'] & H^2(X, \mathbb{Z}) \arrow[d, "j"] \\ & H^2(X, \mathbb{R}) \end{tikzcd}$$

Note there is a isomorphism

$$\widetilde{H}^2(X, \mathbb{R})_{(\text{Čech})} \longrightarrow H^2(X, \mathbb{R})_{(\text{de Rham})}, \tag{4.6}$$

thus for a cocycle $\xi \in Z^2(\mathcal{U}, \mathbb{R})$ we have associated a closed differential form $\varphi(\xi) \in \mathcal{E}^2(X)$. One can choose $\tau \in C^1(\mathcal{U}, \mathbb{Z})$ s.t. $\delta\tau = \xi$ and $\mu \in C^0(\mathcal{U}, \mathcal{E}^1)$ s.t. $\delta\mu = d\tau \in Z^1(\mathcal{U}, \mathcal{E}^1)$. Then we see $d\mu \in Z^0(\mathcal{U}, \mathcal{E}^2) = \mathcal{E}^2(X)$ and choose $\varphi(\xi) = -d\mu$.

We now use a Specifical $\mathcal{U}$ which has the property that any intersection of elements of the covering is a cell (in particular is simply connected) to describe the Bockstein operator $\delta : H^1(X, \mathcal{O}^*) \longrightarrow H^2(X, \mathbb{Z})$. If $g = \{g_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$, we define $\sigma = \{\sigma_{\alpha\beta}\}$ by$\sigma_{\alpha\beta} = \frac{1}{2\pi i}\log g_{\alpha\beta} \in C^1(\mathcal{U}, \mathcal{O})$ where we choose any branch of the logarithm. Hence $\delta\sigma \in Z^2(\mathcal{U}, \mathcal{O})$ (note $\delta^2 = 0$). One see $g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$, then

$$(\delta\sigma)_{\alpha\beta\gamma} = \frac{1}{2\pi i}(\log g_{\beta\gamma} - \log g_{\alpha\gamma} + \log g_{\alpha\beta}),$$

which is integer-valued, thus $\delta\sigma \in Z^2(\mathcal{U}, \mathbb{Z})$ and is a representative for $\delta(g) \in H^2(X, \mathbb{Z})$.

Now let $g = \{g_{\alpha\beta}\}$ be the transition functions of $E \to X$ and set frames f_α for E over U_α. Donote $h_\alpha = h(f_\alpha)$ which is a positive C^∞ function defined in U_α. Thus $c_1(E, h) = \frac{1}{2\pi i}\partial\bar{\partial}\log h_\alpha$. We let, as above, $\delta\sigma \in Z^2(\mathcal{U}, \mathbb{Z})$ where $\sigma_{\alpha\beta} = \frac{1}{2\pi i}\log g_{\alpha\beta}$ be the $\delta-$image of $\{g_{\alpha\beta}\}$ in $H^2(X, \mathbb{Z})$. Choose $\tau = \sigma$ and $\mu = \{\mu_\alpha\}$ where $\mu_\alpha = \frac{1}{2\pi i}\partial\log h_\alpha$ in the construction of $\varphi(\xi)$, then

$$(\delta\mu)_{\alpha\beta} = \mu_\beta - \mu_\alpha = \frac{1}{2\pi i}\partial\log\frac{h_\beta}{h_\alpha} = \frac{1}{2\pi i}\partial\log g_{\alpha\beta}\bar{g}_{\alpha\beta} = \frac{1}{2\pi i}d\log g_{\alpha\beta} = d\sigma_{\alpha\beta} = d\tau_{\alpha\beta}$$

where use the transformation law (2) (note rank$E = 1$) and $\partial\log\bar{g}_{\beta\alpha} = \overline{\partial}\log g_{\beta\alpha} = 0$ since $g_{\beta\alpha}$ is holomorphic. Thus the closed 2-form associated with the cocycle $\delta\sigma = \delta\tau$ (where $= \xi$ in previous context) is given by

$$\varphi = -d\mu = d\left(\frac{i}{2\pi}\partial\log h_\alpha\right) = \frac{i}{2\pi}\bar{\partial}\partial\log h_\alpha = c_1(E, h)$$

Hence we have shown the diagram above is commutative.

Remark 3.18. *Similarly for the exact sequence*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^* \longrightarrow 0$$

on a differentiable manifold X, we have (note $\mathcal{E}$ is fine)

$$0 = H^1(X, \mathcal{E}) \longrightarrow H^1(X, \mathcal{E}^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{E}) = 0$$

so there is an isomorphism $H^1(X, \mathcal{E}^*) \xrightarrow{\delta, \cong} H^2(X, \mathbb{Z})$ which asserts that all differentiable complex line bundles are determined by their Chern class in $H^2(X, \mathbb{Z})$.

Back to the previous case, we want to characterize the image of c_1 in the above commutative diagram, and namely the following theorem.

Theorem 3.19. $c_1(H^1(X, \mathcal{O}^*)) = \widetilde{H}^2_{1,1}(X, \mathbb{Z})$, *where $\widetilde{H}^2_{1,1}(X, \mathbb{Z}) \subset \widetilde{H}^2(X, \mathbb{Z})$ which admits a d-closed differential form of type* $(1, 1)$ *as a representative.*

Proof. If we denote $H^2_{1,1}(X, \mathbb{Z}) = j^{-1}(\widetilde{H}^2_{1,1}(X, \mathbb{Z})) \subset H^2(X, \mathbb{Z})$, then see it suffices to show that

$$\delta(H^1(X, \mathcal{O}^*)) = H^2_{1,1}(X, \mathbb{Z})$$

which suffices to show that the image of $H^2_{1,1}(X,\mathbb{Z})$ in $H^2(X,\mathcal{O})$ is zero. But see there is a homomorphism of resolutions of sheaves

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathcal{E}^0 & \xrightarrow{d} & \mathcal{E}^1 & \xrightarrow{d} & \mathcal{E}^2 & \xrightarrow{d} & \cdots \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \pi_{0,1}} & & \downarrow{\scriptstyle \pi_{0,2}} & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{E}^{0,0} & \xrightarrow{\bar\partial} & \mathcal{E}^{0,1} & \xrightarrow{\bar\partial} & \mathcal{E}^{0,2} & \xrightarrow{\bar\partial} & \cdots \end{array}$$

where $\pi_{0,q}:\mathcal{E}^q\to\mathcal{E}^{0,q}$ is the projection on the submodule of forms of type $(0,q)$. Therefore the image of $H^2_{1,1}(X,\mathbb{C})$ in $H^2(X,\mathcal{O})$ is zero and we prove the theorem. ☐

Remark 3.20 (Divisor on a complex manifold)**.**
We have the following exact sequence,

$$0\longrightarrow\mathcal{O}^*\longrightarrow\mathcal{M}^*\longrightarrow\mathcal{D}:=\mathcal{M}^*/\mathcal{O}^*\longrightarrow 0$$

where $\mathcal{M}^*$ is the sheaf of non-trivial meromorphic functions on X and $\mathcal{D}$ is called the sheaf of divisors on X because a section of $\mathcal{D}$ is called a divisor. If $D\in H^0(X,\mathcal{D})=:\mathrm{Div}(X)$, then there is a covering $\mathfrak{U}=\{U_\alpha\}$ and sections of $\mathcal{M}^*$, say f_α defined in U_α s.t.

$$\frac{f_\beta}{f_\alpha}=g_{\alpha\beta}\in\mathcal{O}^*(U_\alpha\cap U_\beta)$$

We see $g_{\alpha\beta}\cdot g_{\beta\gamma}\cdot g_{\gamma\alpha}=1$ on $U_\alpha\cap U_\beta\cap U_\gamma$. And we have

$$H^0(X,\mathcal{M}^*)\longrightarrow H^0(X,\mathcal{D})=:\mathrm{Div}(X)\longrightarrow H^1(X,\mathcal{O}^*)=:\mathrm{Pic}(X)\xrightarrow{\delta}H^2(X,\mathbb{Z})$$

where two different divisors have same image in $\mathrm{Pic}(X)$ *if they " differ by" a global meromorphic function (this is called linear equivalence in algebraic geometry exactly the case we are similar).*

4 Ellpitic operators and Elliptic complexes

Let E be a Hermitian differentiable vector bundle over X, a compact differentiable manifold with a strictly positive smooth measure μ. We define $\mathcal{E}_k(X,E)$ be the *kth order differentiable sections of E over X* for $0\le k\le\infty$ and set $\mathcal{E}_\infty(X,E)=\mathcal{E}(X,E)$. We shall denote *the compactly supported sections* by $\mathcal{D}(X,E)\subset\mathcal{E}(X,E)$ and *the compactly supported functions* by $\mathcal{D}(X)\subset\mathcal{E}(X)$. We define a L^2-norm by setting $\|\xi\|_0=(\xi,\xi)^{1/2}$ where $(\xi,\eta):=\int_X\langle\xi(x),\eta(x)\rangle_E\,d\mu$, and let $W^0(X,E)$ be the completion of $\mathcal{E}(X,E)$.

If we find a finite trivializing covering of X, say $\{U_\alpha,\varphi_\alpha\}$ and $\{\rho_\alpha\}$ a partition of unity subordinate to $\{U_\alpha\}$, then we can define the Sobolev norm (can be defined for all $s\in\mathbb{R}$) for $\xi\in\mathcal{E}(X,E)$ by setting

$$\|\xi\|_{s,E}=\sum_\alpha\|\varphi_\alpha^*\rho_\alpha\xi\|_{s,\mathbb{R}^n},$$

Here $\|\ \|_{s,\mathbb{R}^n}$ is the Sobolev norm for a compactly supported differentiable function $f:\mathbb{R}^n\longrightarrow\mathbb{C}^m$ defined (for a scalar-valued function, once could extend this to a vector-valued function) by

$$\|f\|^2_{s,\mathbb{R}^n}=\int|\hat f(y)|^2(1+|y|^2)^s\,dy\text{ where }\hat f(y)=(2\pi)^{-n}\int e^{-i(x,y)}f(x)\,dx$$

We see on $\mathcal{D}(\mathbb{R}^n)$ the norm $\|\cdot\|_{s,\mathbb{R}^n}$ is equivalent to $\left[\sum_{|\alpha|\leq s}\int_{\mathbb{R}^n}|D^\alpha f|^2\,dx\right]^{1/2}$, which essentially follows $\widehat{D^\alpha f}(y)=y^\alpha\hat{f}(y)$ and $||f||_0=||\hat{f}||_0$. Here $y^\alpha=y_1^{\alpha_1}\cdots y_n^{\alpha_n}$,$D^\alpha=(-i)^{|\alpha|}D_1^{\alpha_1}\cdots D_n^{\alpha_n}$ and $D_j=\partial/\partial x_j$. Note that norm $\|\cdot\|_s$ defined on E depends on the choice of $\{U_\alpha,\varphi_\alpha,\rho_\alpha\}$. We let $W^s(X,E)$ be the completion of $\mathcal{E}(X,E)$ with respect to the norm $\|\cdot\|_s$. Then there is a fact that the topology on $W^s(X,E)$ is independent of the choices made; i.e., any two such norms are equivalent. Note that for $s=0$ we have made two different choices of norms, one using the local trivializations and one using the Hermitian structure on E, and that these two L^2-norms are also equivalent.

Now there is a sequence of inclusions of the Hilbert spaces $W^s(X,E)$:

$$\cdots\supset W^s\supset W^{s+1}\supset\cdots\supset W^{s+j}\supset\cdots.$$

If we let H^* denote the conjugate-linear continuous functionals of a topological vector space H over $\mathbb{C}$, then it can be shown that (Thus we could defined W^{-s} in this manner)

$$(W^s)^*\cong W^{-s}\quad(s\geq 0)$$

We now state some important results concerning $W^s(X,E)$ without proof.

Proposition 4.1 (Sobolev).
Let $n=\dim_{\mathbb{R}}X$, and suppose that $s>[n/2]+k+1$. Then $W^s(X,E)\subset\mathcal{E}_k(X,E)$.

Proposition 4.2 (Rellich–Kondrachov theorem).
The natural inclusion $j:W^s(X,E)\subset W^t(X,E)$ for $t<s$ is a completely continuous (i.e. compact, means the image of a closed ball is relatively compact) linear map.

4.1 Differentiable operators and symbols

Let E and F be differentiable $\mathbb{C}$-vector bundles over a differentiable manifold X and $L:\mathcal{E}(X,E)\longrightarrow\mathcal{E}(X,F)$ be a $\mathbb{C}$-linear map. We say L is a *differential operator* if under any choice of local coordinates and local trivializations, we could represent L as a linear partial differential operator $\widetilde{L}$ where

$$\widetilde{L}(f)_i=\sum_{\substack{j=1\\|\alpha|\leq k}}^{p}a_\alpha^{ij}D^\alpha f_j,\ i=1,\cdots,q\text{ for }f=(f_1,\cdots,f_p)\in[\mathcal{E}(U)]^p$$

A differential operator is said to be of *order k* if there are no derivatives of order $\geq k+1$ appearing in a local representation. Let $\mathrm{Diff}_k(E,F)$ denote the vector space of all differential operators of order k mapping $\mathcal{E}(X,E)$ to $\mathcal{E}(X,F)$.

If X is moreover compact, we define $\mathrm{OP}_k(E,F)$ as the vector space of $\mathbb{C}$-linear mappings $T:\mathcal{E}(X,E)\longrightarrow\mathcal{E}(X,F)$ s.t. there is a continuous extension of T, say $\overline{T}_s:W^s(X,E)\longrightarrow W^{s-k}(X,F)$ for all s. We see they the *operators of order k* mapping E to F. And actually we have

$$\mathrm{Diff}_k(E,F)\subset\mathrm{OP}_k(E,F)$$

where follows $\widehat{D^\alpha f}(\xi)=\xi^\alpha\hat{f}(\xi)$ if we locally views.

We say *the adjoint operator* of a $\mathbb{C}-$linear map $L:\mathcal{E}(X,E)\longrightarrow\mathcal{E}(X,F)$ is a $\mathbb{C}-$linear map $L^*:\mathcal{E}(X,F)\longrightarrow\mathcal{E}(X,E)$ s.t. $(L\xi,\eta)=(\xi,L^*\eta)$ for all $\xi\in\mathcal{E}(X,E)$ and $\eta\in\mathcal{E}(X,F)$. And we see if L^* exists, then it is unique, if we note $\mathcal{E}(X,E)$ is dense in $W^0(X,E)$.

Proposition 4.3 (Existence of adjoints on $\mathrm{OP}_*(E,F)$).
Let $L \in \mathrm{OP}_k(E,F)$. Then L^* exists, and moreover $L^* \in \mathrm{OP}_k(F,E)$, and the extension $(\overline{L^*})_s : W^s(X,F) \longrightarrow W^{s-k}(X,E)$ is given by the adjoint map $(\overline{L}_{k-s})^* : W^s(X,F) \longrightarrow W^{s-k}(X,E)$.

Proof. The prove is direct since one has a candidate $(\overline{L}_{k-s})^*$ (for each s) which gives the desired adjoint when restricted to $\mathcal{E}(X,F)$ in a suitable manner, and one just uses the uniqueness of the adjoint. □

Now we define the *symbol* of a differential operator, which is used for the classification of differential operators into various types. Let $T^*(X)$ be the real cotangent bundle to a differentiable manifold X and $T'(X)$ denote $T^*(X)$ with the zero cross section deleted (i.e. the bundle of nonzero cotangent vectors). Let $T'(X) \xrightarrow{\pi} X$ denote the projection mapping and π^*E is the pullbacks of E over $T'(X)$. We set, for any $k \in \mathbb{Z}$,

$$\mathrm{Smbl}_k(E,F) := \{S \in \mathrm{Hom}(\pi^*E, \pi^*F) : S(x,\rho v) = \rho^k \sigma(x,v), (x,v) \in T'(X), \rho > 0\}$$

We now define a linear map, called the k-symbol of the differential operator L as

$$\sigma_k : \mathrm{Diff}_k(E,F) \longrightarrow \mathrm{Smbl}_k(E,F)$$

where for $(x,v) \in T'(X)$, $e \in E_x$ be given, we find $g \in \mathcal{E}(X)$ and $f \in \mathcal{E}(X,E)$ s.t. $dg_x = v$, and $f(x) = e$, then

$$\sigma_k(L)(x,v)e = L\left(\frac{i^k}{k!}(g - g(x))^k f\right)(x) \in F_x.$$

This defines a linear mapping $\sigma_k(L)(x,v) : E_x \longrightarrow F_x$, which shows $\sigma_k(L) \in \mathrm{Smbl}_k(E,F)$. The following proposition shows that what we have done just now.

Proposition 4.4. *With notations as above, and j is the natural inclusion, then*

$$0 \longrightarrow \mathrm{Diff}_{k-1}(E,F) \xrightarrow{j} \mathrm{Diff}_k(E,F) \xrightarrow{\sigma_k} \mathrm{Smbl}_k(E,F)$$

Proof. We may consider all locally and see, if $L = \sum_{|\nu| \le k} A_\nu D^\nu : [\mathcal{E}(U)]^p \longrightarrow [\mathcal{E}(U)]^q$ where $\{A_\nu\}$ are $q \times p$ matrices of C^∞ functions on an open set U, then

$$\sigma_k(L)(x,v) = \sum_{|\nu|=k} A_\nu(x)\xi^\nu$$

Here $v = \xi_1 dx_1 + \dots + \xi_n dx_n$ and $\sigma_k(L)(x,v)$ is a linear mapping from $x \times \mathbb{C}^p \to x \times \mathbb{C}^q$ given by matrix multiplication. The rest of the verification is straightforward. □

We now look at some examples.

Example 6. *If $L : [\mathcal{E}(\mathbb{R}^n)]^p \to [\mathcal{E}(\mathbb{R}^n)]^q$ is an element of $\mathrm{Diff}_k(\mathbb{R}^n \times \mathbb{C}^p, \mathbb{R}^n \times \mathbb{C}^q)$, then*

$$\sigma_k(L)(x,v) = \sum_{|\nu|=k} A_\nu(x)\xi^\nu,$$

where $L = \sum_{|\nu| \le k} A_\nu D^\nu, v = \sum_{j=1}^n \xi_j dx_j$ adn the $\{A_\nu\}$ being $q \times p$ matrices of differentiable functions in $\mathbb{R}^n$.

Example 7. *Consider the de Rham complex given in Example 2, and for $T^* = T^*(X) \otimes \mathbb{C}$ we have*

$$\mathcal{E}(X, \wedge^0 T^*) \xrightarrow{d} \mathcal{E}(X, \wedge^1 T^*) \xrightarrow{d} \cdots,$$

and we want to compute the associated 1-symbol mappings,

$$\wedge^0 T_x^* \xrightarrow{\sigma_1(d)(x,\nu)} \wedge^1 T_x^* \xrightarrow{\sigma_1(d)(x,\nu)} \wedge^2 T_x^* \longrightarrow \cdots.$$

Actually for $e \in \wedge^p T_x^*$, we have $\sigma_1(d)(x,\nu)e = i\nu \wedge e$. Moreover, one can see the sequence of associated 1-symbol mappings above is an exact sequence of vector spaces.

Example 8. *Consider the Dolbeault complex on a complex manifold X,*

$$\mathcal{E}^{p,0}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,1}(X) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{E}^{p,n}(X) \longrightarrow 0$$

and its associated 1-symbol sequence

$$\longrightarrow \wedge^{p,q-1} T_x^*(X) \xrightarrow{\sigma_1(\bar\partial)(x,\nu)} \wedge^{p,q} T_x^*(X) \xrightarrow{\sigma_1(\bar\partial)(x,\nu)} \wedge^{p,q+1} T_x^*(X) \longrightarrow \cdots$$

We have $\nu \in T_x^*(X)$, considered as a real cotangent bundle. Consequently, $\nu = \nu^{1,0} + \nu^{0,1}$, given by the injection $0 \longrightarrow T_x^*(X) \longrightarrow T_x^*(X) \otimes_{\mathbb{R}} \mathbb{C} = \wedge^{1,0} T_x^*(X) \oplus \wedge^{0,1} T_x^*(X)$. Then we claim that

$$\sigma_1(\bar\partial)(x,\nu)e = i\nu^{0,1} \wedge e,$$

and the above 1-symbol sequence is exact, the computations are straightforward.

We also consder $E \longrightarrow X$ be a holomorphic vector bundle over a complex manifold X, and the differentiable (p,q)-forms with coefficients in E, namely $\mathcal{E}^{p,q}(X,E)$, see Example 1. We have the complex ($\bar\partial_E := \bar\partial \otimes 1$),

$$\longrightarrow \mathcal{E}^{p,q}(X,E) \xrightarrow{\bar\partial_E} \mathcal{E}^{p,q+1}(X,E) \longrightarrow \cdots$$

and the associated 1-symbol sequence

$$\longrightarrow \wedge^{p,q} T_x^* \otimes E_x \xrightarrow{\sigma_1(\bar\partial_E)(x,\nu)} \wedge^{p,q+1} T_x^* \otimes E_x \longrightarrow \cdots$$

We let $\nu = \nu^{1,0} + \nu^{0,1}$, as before, and we have for $f \otimes e \in \wedge^{p,q} T_x^* \otimes E$,

$$\sigma_1(\bar\partial)(x,\nu) f \otimes e = (i\nu^{0,1} \wedge f) \otimes e,$$

and the 1-symbol sequence is again exact.

Proposition 4.5 (Existence of adjoints on $\mathrm{Diff}_k(E,F)$)**.**

Let $L \in \mathrm{Diff}_k(E,F)$. Then L^* exists and $L^* \in \mathrm{Diff}_k(F,E)$. Moreover, $\sigma_k(L^*) = \sigma_k(L)^*$, where $\sigma_k(L)^*$ is the adjoint of the linear map $\sigma_k(L)(x,\nu) : E_x \longrightarrow F_x$.

Proof. Let $L \in \mathrm{Diff}_k(E,F)$ and h_E and h_F are Hermitian metrics on E and F. Then for any $\xi, \eta \in \mathcal{D}(X,E)$, $(\xi,\eta) = \int_{\mathbb{R}^n} {}^t\bar\eta(x) h_E(x) \xi(x) \rho(x)\, dx$ where $\rho(x)$ is a density and $\eta(x) = \eta(f)(x)$ for some local frame f on E. Suppose that $L : \mathcal{D}(X,E) \to \mathcal{D}(X,F)$ is a linear differential operator of order k, and assume that the sections have support in a trivializing neighborhood U which gives local coordinates for X near some point. Then

$$(L\xi,\tau) = \int_{\mathbb{R}^n} {}^t\bar{\tau}(x)h_F(x)(M(x,D)\xi(x))\rho(x)\,dx$$

where $M(x,D) = \sum_{|\alpha|\le k} C_\alpha(x)D^\alpha$ and $C_\alpha(x)$ is an matrix of C^∞ functions in $\mathbb{R}^n$. By integration by parts, we have

$$(L\xi,\tau) = \int_{\mathbb{R}^n} {}^t(\sum_{|\alpha|\le k} \widetilde{C}_\alpha(x)D^\alpha\tau(x))h_E(x)\xi(x)\rho(x)\,dx$$

where $\widetilde{C}_\alpha(x)$ are matrices of smooth functions defined by the formula

$${}^t(\sum_{|\alpha|\le k} \widetilde{C}_\alpha D^\alpha\tau) = \sum_{|\alpha|\le k}(-1)^{|\alpha|}D^\alpha({}^t\bar{\tau}\rho h_F C_\alpha)h_E^{-1}\rho^{-1}$$

Hence the formula define a linear differential operator $L^* : \mathcal{D}(X,F) \to \mathcal{D}(X,E)$, which has automatically the property of being the adjoint of L. The remainder of the discussion is direct. $\square$

4.2 Pseudodifferential operators

In this subsection we want to introduce an important generalization of differential operators called pseudodifferential operators which can be used in solving differential equations "invert" Laplacian operators on compact Riemannian manifolds, then leads to the theory of harmonic differential forms introduced by Hodge in his study of algebraic geometry. If $U \subset \mathbb{R}^n$ is open and $p(x,\xi)$ is a polynomial in ξ of degree m, with coefficients being C^∞ functions in the variable $x \in U$, then we can obtain the most general linear partial differential operators in U by letting $P = p(x,D)$ be the differential operator obtained by replacing the vector $\xi = (\xi_1,\cdots,\xi_n)$ by $(-iD_1,\cdots,-iD_n)$. By using the Fourier transform we see

$$Pu(x) = p(x,D)u(x) = \int p(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}\,d\xi \text{ for } u \in \mathcal{D}(U) \tag{8}$$

$P(x,D) : \mathcal{D}(U) \to \mathcal{D}(U)$, since differential operators preserve supports. Hence to define the generalization of differential operators, we consider the properties $p(x,\xi)$ appears in the integrand. To do this, we shall define classes of functions which possess, axiomatically, several important properties of the polynomials considered above.

Definition 4.6. *Let U be an open set in $\mathbb{R}^n$ and let m be any integer.*

(a) Let $\widetilde{S}^m(U)$ be the class of C^∞ functions $p(x,\xi)$ defined on $U\times\mathbb{R}^n$, satisfying the following properties. For any compact set K in U, and for any multiindices α,β, there exists a constant $C_{\alpha,\beta,K}$, depending on α,β,K, and p so that

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha,\beta,K}(1+|\xi|)^{m-|\alpha|}, \quad x \in K, \xi \in \mathbb{R}^n.$$

(b) Let $S^m(U)$ denote the set of $p \in \widetilde{S}^m(U)$ s.t.

$$\text{The limit } \sigma_m(p)(x,\xi) = \lim_{\lambda\to\infty} \frac{p(x,\lambda\xi)}{\lambda^m} \text{ exists for } \xi \neq 0,$$

Moreover, $p(x,\xi) - \psi(\xi)\sigma_m(p)(x,\xi) \in \widetilde{S}^{m-1}(U)$ where $\psi \in C^\infty(\mathbb{R}^n)$ is a cut-off function with $\psi(\xi) \equiv 0$ near $\xi = 0$ and $\psi(\xi) \equiv 1$ outside the unit ball.

(c) Let $\widetilde{S}^m_0(U)$ denote the class of $p \in \widetilde{S}^m(U)$ s.t. there is a compact set $K \subset U$, so that for any $\xi \in \mathbb{R}^n$, the function $p(x,\xi)$, considered as a function of $x \in U$, has compact support in K [i.e., $p(x,\xi)$ has uniform compact support in the x-variable]. Let $S^m_0(U) = S^m(U) \cap \widetilde{S}^m_0(U)$.

And we define the prototype (local form) of our pseudodifferential operator by using equation (8) i.e. we set for any $p \in \widetilde{S}^m(U)$ and $u \in \mathcal{D}(U)$,

$$L(p)u(x) = \int p(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}\, d\xi,$$

and we call $L(p)$ a *canonical pseudodifferential operator of order m*. We now want to define pseudodifferential operators in general.

Definition 4.7 (Pseudodifferential operators)**.**

- *We say a linear mapping $L : \mathcal{D}(X) \to \mathcal{E}(X)$ a pseudodifferential operator on X if and only if for any coordinate chart $U \subset X$ and any open set $U' \subset\subset U$ there exists a $p \in S^m_0(U)$ s.t. if $u \in \mathcal{D}(U')$, then (extending u by zero to be in $\mathcal{D}(X)$)*

$$Lu = L(p)u$$

- *We say a linear mapping $L : \mathcal{D}(X,E) \to \mathcal{E}(X,F)$ is a pseudodifferential operator on X if and only if for any coordinate chart U with trivializations of E and F over U and for any open set $U' \subset\subset U$ there exists an $r \times p$ matrix (p^{ij}) and $p^{ij} \in S^m_0(U)$, s.t. the induced map*

$$L_U : \mathcal{D}(U')^p \longrightarrow \mathcal{E}(U)^r$$

with $u \in \mathcal{D}(U')^p \xrightarrow{L_U} Lu$, extending u by zero to be an element of $\mathcal{D}(X,E)$ (where $p = \operatorname{rank} E$, $r = \operatorname{rank} F$) is a matrix of canonical pseudodifferential operators $L(p^{ij})$, $i = 1, \cdots, r$, $j = 1, \cdots, p$.

Definition 4.8 (pseudodifferential operator of order m)**.**

Let X be a differentiable manifold. A pseudodifferential operator $L : \mathcal{D}(X) \to \mathcal{E}(X)$ is said to be a pseudodifferential operator of order m on X if, for any choice of local coordinates chart $U \subset X$, the corresponding canonical pseudodifferential operator L_U is of order m; i.e., $L_U = L(p)$, where $p \in S^m(U)$. The class of all pseudodifferential operators on X of order m is denoted by $\mathrm{PDiff}_m(X)$.

Now we state one of the fundamental results: Theorem 4.10 in the theory of pseudodifferential operators on manifolds. Before this we state a proposition.

Proposition 4.9. *Let E and F be vector bundles over a differentiable manifold X. There exists a canonical linear map*

$$\sigma_m : \mathrm{PDiff}_m(E,F) \longrightarrow \mathrm{Smbl}_m(E,F),$$

which is defined locally in a coordinate chart $U \subset X$ by $\sigma_m(L_U)(x,\xi) = [\sigma_m(p^{ij})(x,\xi)]$ where $L_U = [L(p^{ij})]$ is a matrix of canonical pseudodifferential operators, and where $(x,\xi) \in U \times (\mathbb{R}^n - \{0\})$ is a point in $T^*(U)$ expressed in the local coordinates of U.

Proof. See Proposition IV.3.15 of [W]. □

Theorem 4.10. *Let E and F be vector bundles over a differentiable manifold X. Then the following sequence is exact,*

$$0 \longrightarrow K_m(E,F) \xrightarrow{j} \mathrm{PDiff}_m(E,F) \xrightarrow{\sigma_m} \mathrm{Smbl}_m(E,F) \longrightarrow 0$$

where σ_m is the canonical symbol map given by Proposition 4.9, $K_m(E,F)$ is the kernel of σ_m, and j is the natural injection. Moreover, $K_m(E,F) \subset \mathrm{OP}_{m-1}(E,F)$ if X is compact.

Proof. See Theorem IV.3.16 of [W]. □

From now we centralize our attention on operators which generalize the classic Laplacian operator in Euclidean space and its inverse, which will be called elliptic operators.

Definition 4.11 (Elliptic operator)**.**
Let E and F be vector bundles over a differentiable manifold X.
(a) Let $s \in \mathrm{Smbl}_k(E,F)$. Then s is said to be elliptic if and only if for any $(x,\xi) \in T'(X)$, the linear map $s(x,\xi) : E_x \longrightarrow F_x$ is an isomorphism.
(b) Let $L \in \mathrm{PDiff}_k(E,F)$. Then L is said to be elliptic (of order k) if and only if $\sigma_k(L)$ is an elliptic symbol.

Definition 4.12 (Pseudoinverse/Parametrix)**.**
Let $L \in \mathrm{PDiff}(E,F)$, then a pseudoinverse *or* parametrix *for L is an operator $Q \in \mathrm{PDiff}(F,E)$ s.t. $QL - I_E \in \mathrm{OP}_{-1}(E)$ and $LQ - I_F \in \mathrm{OP}_{-1}(F)$.*

Theorem 4.13 (Existence of pseudoinverse on a compact manifold)**.**
Let k be any integer and let $L \in \mathrm{PDiff}_k(E,F)$ be elliptic. If X is compact, then there exists a parametrix for L.

Proof. Let $s = \sigma_k(L)$, then by definition s^{-1} exists as a linear transformation, $s^{-1}(x,\xi) : F_x \longrightarrow E_x$ and $s^{-1} \in \mathrm{Smbl}_{-k}(F,E)$. By Theorem 4.10, there is $\widetilde{L} \in \mathrm{PDiff}_{-k}(F,E)$ s.t. $\sigma_{-k}(\widetilde{L}) = s^{-1}$. We have then that

$$\sigma_0(L \circ \widetilde{L} - I_F) = \sigma_0(L \circ \widetilde{L}) - \sigma_0(I_F) = \sigma_k(L) \cdot \sigma_{-k}(\widetilde{L}) - \sigma_0(I_F) = 0$$

By Theorem 4.10 again, we see that $L \circ \widetilde{L} - I_F \in \mathrm{OP}_{-1}(F,F)$. Similarly, $\widetilde{L} \circ L - I_E$ is also in $\mathrm{OP}_{-1}(E,E)$. □

This theorem tells us that modulo smoothing operators (We call any operator $L \in \mathrm{OP}_{-1}(E,F)$ a *smoothing operator*) we have an inverse for a given elliptic operator. And S is called a smoothing operator precisely because of the role it plays in the proof of Lemma 4.17. We shall see it smooths out the weak solution $\xi \in W^s(E)$.

4.3 Elliptic operators

Our main theorem is Theorem 4.20, before that we need Theorem 4.18 and Theorem 4.19. Again before the latter two theorems we need some Propositions. Let X be a compact differentiable manifold and suppose that $L \in \mathrm{OP}_m(E,F)$. Then we say that L is *compact* (or *completely continuous*) if for every s the extension $L_s : W^s(E) \to W^{s-m}(F)$ is a compact operator as a mapping of Banach spaces.

Proposition 4.14. *Let X be a compact manifold and let $S \in \mathrm{OP}_{-1}(E)$. Then S is a compact operator of order* 0*.*

Proof. We have for any s the following commutative diagram,

$$\begin{tikzcd} W^s(E) \arrow[r, "\widetilde{S}_s"] \arrow[dr, "S_s"'] & W^s(E) \\ & W^{s+1}(E) \arrow[u, "j"] \end{tikzcd}$$

where S_s is the extension of S to a mapping $W^s \to W^{s+1}$, given since $S \in \mathrm{OP}_{-1}(E,E)$, and $\widetilde{S}_s$ is the extension of S, as a mapping $W^s \to W^s$, given by the fact that $\mathrm{OP}_{-1}(E,E) \subset \mathrm{OP}_0(E,E)$. Since j is a compact operator (by Proposition 4.2), then $\widetilde{S}_s$ must also be compact. $\square$

Theorem 4.15 ((Part of) Riesz-Fredholm theorem)**.**
Let X be a Banach space and A be a compact operator on X, $T = I - A$, then $\mathrm{Im}(T) = \mathrm{Ker}(T^*)^{\perp}$ *and* $\mathrm{codim}(\mathrm{Im}(T)) = \dim \mathrm{Ker}(T) < \infty$*.*

Proof. See Theorem 3.2.1 of [ZY]. $\square$

In the remainder of this section we shall let E and F be fixed Hermitian vector bundles over a compact differentiable manifold X. For $L \in \mathrm{Diff}_m(E,F)$, we define

$$\mathcal{H}_L = \{u \in \mathcal{E}(X,E) \mid Lu = 0\} \text{ and } \mathcal{H}_L^{\perp} = \{v \in W^0(E) \mid (u,v)_E = 0,\ \forall u \in \mathcal{H}_L\}$$

Proposition 4.16. *Let $L \in \mathrm{PDiff}_m(E,F)$ be an elliptic pseudodifferential operator. Then there exists a parametrix P for L so that $L \circ P$ and $P \circ L$ have continuous extensions as Fredholm operators: $W^s(F) \to W^s(F)$ and $W^s(E) \to W^s(E)$ respectively, for each integer s.*

Proof. Recall that an operator T on a Banach space is called a Fredholm operator if T has finite-dimensional kernel and cokernel. Hence the proposition directly follows Theorem 4.13, Proposition 4.14 and Theorem 4.15. $\square$

Lemma 4.17. *Suppose that $L \in \mathrm{Diff}_m(E,F)$ is elliptic, and $\xi \in W^s(E)$ has the property that $L_s\xi = \sigma \in \mathcal{E}(X,F)$. Then $\xi \in \mathcal{E}(X,E)$.*

Proof. If P is a parametrix for L, then $P \circ L - I = S \in \mathrm{OP}_{-1}(E)$. Note we have $(P \circ L)\xi \in \mathcal{E}(X,E)$ and $\xi = (P \circ L - S)\xi$. And $\xi \in W^s(E)$, $(P \circ L)\xi \in \mathcal{E}(X,E)$ and $S\xi \in W^{s+1}(E)$, then $\xi \in W^{s+1}(E)$. Repeating this process, we see that $\xi \in W^{s+k}(E)$ for all $k > 0$.

But by Proposition 4.1, it follows that $\xi \in \mathcal{E}_l(X,E)$, for all $l > 0$. Hence $\xi \in \mathcal{E}(X,E) = \mathcal{E}_\infty(X,E)$. $\square$

Theorem 4.18 (Finiteness of Kernel theorem for elliptic differential operators)**.**
Let $L \in \mathrm{Diff}_k(E,F)$ be elliptic and $\mathcal{H}_{L_s} = Ker\ L_s : W^s(E) \to W^{s-k}(F)$, then

1. *$\mathcal{H}_{L_s} \subset \mathcal{E}(X,E)$ and hence $\mathcal{H}_{L_s} = \mathcal{H}_L$, all s.*
2. *$\dim \mathcal{H}_{L_s} = \dim \mathcal{H}_L < \infty$ and $\dim W^{s-k}(F)/L_s(W^s(E)) < \infty$.*

Proof. By Theorem 4.16, denote P be a parametrix for L and then $(P \circ L)_s : W^s(E) \longrightarrow W^s(F)$. From the following commutative diagram of Banach spaces:

$$\begin{tikzcd} W^s(E) \arrow[r, "(P\circ L)"] \arrow[dr, "L_s"'] & W^s(E) \\ & W^{s+1}(E), \arrow[u, "P_{s-k}"'] \end{tikzcd}$$

we see $\mathrm{Ker}\, L_s \subset \mathrm{Ker}(P \circ L)_s$, thus $\mathcal{H}_{L_s}$ is finite dimensional for all s (Again by Proposition 4.16). Similarly we see that L_s has a finite dimensional cokernel. Now just need to explain $\mathcal{H}_{L_s} \subset \mathcal{E}(X,E)$, which is known as the regularity of the homogeneous solutions of an elliptic differential equation. This is done by Lemma 4.17. □

Theorem 4.19 (Existence of solution theorem for elliptic differential operators)**.**
Let $L \in \mathrm{Diff}_m(E,F)$ be elliptic, and suppose that $\tau \in \mathcal{H}^{\perp}_{L^*} \cap \mathcal{E}(X,F)$. Then there exists a unique $\xi \in \mathcal{E}(X,E)$ s.t. $L\xi = \tau$ and s.t. ξ is orthogonal to $\mathcal{H}_L$ in $W^0(E)$.

Proof. Consider the following diagram of Banach spaces,

$$\begin{tikzcd} W^m(E) \arrow[r, "L_m"] \arrow[d, shift left] & W^0(F) \arrow[d, shift left] \\ W^{-m}(E) \arrow[u, shift left] & W^0(F) \arrow[l, "L_m^*"'] \arrow[u, shift left] \end{tikzcd}$$

where $(L_m)^* = (L^*)_0$. A well-known and elementary functional analysis result asserts that the closure of the range is perpendicular to the kernel of the adjoint. Thus $L_m(W^m(E))$ is dense in $\mathcal{H}^{\perp}_{L_m^*}$, which implies $L_m\xi = \tau$ has a solution $\xi \in W^m(E)$ by noting that L_m has finite dimensional cokernel and then L_m has closed range. By orthogonally projecting ξ along the closed subspace Ker L_m ($= \mathcal{H}_L$ by Theorem 4.18), we obtain a unique solution satisfied the conditions (one may use Lemma 4.17). □

Theorem 4.20 (Fundamental decomposition theorem for self-adjoint elliptic operators)**.**
Let $L \in \mathrm{Diff}_m(E)$ be self-adjoint and elliptic. Then there exist linear mappings $H_L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,E)$ and $G_L : \mathcal{E}(X,E) \longrightarrow \mathcal{E}(X,E)$ s.t.

(a) $H_L(\mathcal{E}(X,E)) = \mathcal{H}_L(E)$ and $\dim_{\mathbb{C}} \mathcal{H}_L(E) < \infty$.

(b) $L \circ G_L + H_L = G_L \circ L + H_L = I_E$, where I_E = identity on $\mathcal{E}(X,E)$.

(c) H_L and $G_L \in \mathrm{OP}_0(E)$, and, in particular, extend to bounded operators on $W^0(E)(= L^2(X,E))$.

(d) $\mathcal{E}(X,E) = \mathcal{H}_L(X,E) \oplus G_L \circ L(\mathcal{E}(X,E)) = \mathcal{H}_L(X,E) \oplus L \circ G_L(\mathcal{E}(X,E))$, and this decomposition is orthogonal with respect to the inner product in $W^0(E)$.

Proof. Let H_L be the orthogonal projection (in $W^0(E)$) onto the closed subspace $\mathcal{H}_L(E)$, which is finite dimensional by Theorem 4.18 . From the proof of Theorem 4.19, there is a bijective continuous mapping

$$L_m : W^m(E) \cap \mathcal{H}_L^{\perp} \longrightarrow W^0(E) \cap \mathcal{H}_L^{\perp}.$$

By the Banach open mapping theorem (See Theorem 2.3.9 of [ZY]), L_m has a continuous linear inverse which we denote by $G_0 : W^0(E) \cap \mathcal{H}_L^{\perp} \longrightarrow W^m(E) \cap \mathcal{H}_L^{\perp}$. We extend G_0 to all of $W^0(E)$ by letting $G_0(\xi) = 0$ if $\xi \in \mathcal{H}_L$. By $W^m(E) \subset W^0(E)$, we see $G_0 : W^0(E) \longrightarrow W^0(E)$. Moreover, we have $L_m \circ G_0 = I_E - H_L$ since $L_m \circ G_0 =$ identity on $\mathcal{H}_L^{\perp}$ and similarly $G_0 \circ L_m = I_E - H_L$. Since $G_0(\mathcal{E}(X,E)) \subset \mathcal{E}(X,E)$ by elliptic regularity (See Lemma 4.17), we can restrict G_0 and get $G_L = G_0|_{\mathcal{E}(X,E)}$, from which we see all of the conditions (a)–(d) are satisfied. □

4.4 Elliptic complexes and the fundamental theorem 4.22

Let $\{E_i\}_{i=0}^N$ be a sequence of differentiable vector bundles defined over a compact differentiable manifold X, and $\{L_j\}_{j=0}^{N-1}$ be a sequence of differential operators with some fixed order s.t. there is a sequence

$$\mathcal{E}(E_0) \xrightarrow{L_0} \mathcal{E}(E_1) \xrightarrow{L_1} \mathcal{E}(E_2) \longrightarrow \cdots \xrightarrow{L_{N-1}} \mathcal{E}(E_N) \tag{E}$$

and a corresponding sequence of symbols

$$0 \longrightarrow \pi^* E_0 \xrightarrow{\sigma_k(L_0)} \pi^* E_1 \xrightarrow{\sigma_k(L_1)} \pi^* E_2 \longrightarrow \cdots \xrightarrow{\sigma_k(L_{N-1})} \pi^* E_N \longrightarrow 0. \tag{9}$$

Note in this subsection we abbreviate $\mathcal{E}(X, E_j)$ by $\mathcal{E}(E_j)$, which not to be confused with the sheaf of sections of E_j.

Definition 4.21 (Complex and elliptic complex, due to Atiyah and Bott)**.**
Sequence E is called a complex if $L_i \circ L_{i-1} = 0$, $i = 1, \cdots, N-1$. Such a complex is called an elliptic complex if the associated symbol sequence 9 is exact.

We make a trivial extension to a complex larger at both ends and define *the q-th cohomology groups of the complex (E)* as

$$H^q(E) = \mathrm{Ker}(L_q)/\mathrm{Im}(L_{q-1}) = Z^q(E)/B^q(E)$$

Also define *the Laplacian operators* of the elliptic complex (E) by

$$\Delta_j = L_j^* L_j + L_{j-1} L_{j-1}^* : \mathcal{E}(E_j) \longrightarrow \mathcal{E}(E_j)$$

for $j = 0, \cdots, N$, which are well-defined self-adjoint and elliptic operators of order $2k$. By Theorem 4.20, we abbreviate some notations associated with Δ_j as follows

$$\mathcal{H}(E_j) := \mathcal{H}_{\Delta_j}(E_j) = \mathrm{Ker}\Delta_j,\ H_j := H_{\Delta_j} \text{ and } G_j := G_{\Delta_j}$$

We can also use $\mathcal{E}(E) = \oplus \mathcal{E}(E_j)$ to simplify the notations coming from Theorem 4.20 by "sum up" the graded components, and get Δ, L, L^*, G, H, I etc. Note we still have some formal relations like

$$\Delta = LL^* + L^*L \text{ and } I = H + G\Delta = H + \Delta G$$

Theorem 4.22 (Fundamental theorem concerning elliptic complexes)**.**
Let $(\mathcal{E}(E), L)$ be an elliptic complex equipped with an inner product. Then

(a) There is an orthogonal decomposition

$$\mathcal{E}(E) = \mathcal{H}(E) \oplus LL^*G\mathcal{E}(E) \oplus L^*LG\mathcal{E}(E),$$

(b) The following commutation relations are valid:

(1) $I = H + \Delta G = H + G\Delta$.
(2) $HG = GH = H\Delta = \Delta H = 0$.
(3) $L\Delta = \Delta L$, $L^*\Delta = \Delta L^*$.
(4) $LG = GL$, $L^*G = GL^*$.

(c) $\dim_{\mathbb{C}} \mathcal{H}(E) < \infty$*, and there is a canonical isomorphism*

$$\mathcal{H}(E_j) \cong H^j(E).$$

Proof. By Theorem 4.20, we have $\mathcal{E}(E) = \mathcal{H}(E) \oplus (LL^* + L^*L)G\mathcal{E}(E)$. But for $\xi, \eta \in \mathcal{E}(E)$,

$$(LL^*G\xi, L^*LG\eta) = (L^2L^*G\xi, LG\eta) = (0, LG\eta) = 0$$

So we obtain (a). For (b), one just need to show $LG = GL$, and it suffices to show that $LG = GL$ on $\mathcal{H}(E)^\perp$. Again by Theorem 4.20, each $\xi \in \mathcal{H}(E)^\perp$ has form $\xi = \Delta\varphi$ for some $\varphi \in \mathcal{E}(E)$. However,

$$L\varphi - L\varphi = (H + G\Delta)(L\varphi) - L(H + G\Delta)\varphi \implies (GL - LG)\Delta\varphi = (LH - HL)\varphi$$

Note we have a assertion:

[**Ass**] for $\xi \in \mathcal{E}(E)$, $\Delta\xi = 0$ if and only if $L\xi = L^*\xi = 0$, moreover, $LH = HL = L^*H = HL^* = 0$.

The if part is trivial and the only if part comes from $(\Delta\xi, \xi) = ||L^*\xi||^2 + ||L\xi||^2$. Hence $LH = L^*H = 0$. And we see H is self-adjoint, then $(HL\xi, \eta) = (\xi, L^*H\eta) = 0$, so $HL = 0$ and $HL^* = 0$ is proved in a similar manner.

Thus back to proof of Theorem 4.22 we see (b) is valid.

For (c), the finiteness assertion is again valid by Theorem 4.20. We can define

$$\Phi : Z^q(E) \longrightarrow \mathcal{H}(E_q), \quad \xi \to H(\xi)$$

By [**Ass**], $\forall \eta \in \mathcal{H}(E_q)$, we have $H\eta = \eta$ and $\eta \in Z^q(E)$, so Φ is a surjective linear mapping. Note $\forall \xi \in Z^q(E)$,

$$\xi = H\xi + LL^*G\xi + L^*LG\xi = H\xi + LL^*G\xi$$

Hence $\forall \xi \in \mathrm{Ker}\Phi, \xi = LL^*G\xi \in B^q(E)$ and by [**Ass**] $B^q(E) \subset \mathrm{Ker}\Phi$, so $\mathrm{Ker}\Phi = B^q(E)$ and we get the desired isomorphism. □

Now we give two very important example deduced by Theorem 4.22.

Example 9 (de Rham complex on a compact differentiable manifold X)**.**
Let $(\mathcal{E}^*(X), d)$ be the de Rham complex on a compact differentiable manifold X. Recall Example 2 and we have $H^r(X, \mathbb{C}) \cong H^r(\mathcal{E}^*(X))$. However, Example 7 say $(\mathcal{E}^*(X), d)$ is actually an elliptic complex, so by Theorem 4.22,

$$H^r(X, \mathbb{C}) \cong \mathcal{H}_{\Delta_d}(\mathcal{E}^r(X)) =: \mathcal{H}^r(X)$$

which means for $\forall c \in H^r(X, \mathbb{C})$ there exists a unique Δ_d−harmonic r−form φ representing this class c. And we see from Theorem 4.22 again, $\dim_{\mathbb{C}} H^q(X, \mathbb{C}) = \dim_{\mathbb{C}} \mathcal{H}^q(X) =: b_q < \infty$. *where $\{b_q\}_{q=0}^{\dim_{\mathbb{R}} X}$ are called* Betti numbers *of the compact manifold X. Recall $H^*(X, \mathbb{C})$ also be determined by the topological structure of X from the singular cohomology view, so b_q and* Euler characteristic $\chi(X) := \sum(-1)^q b_q$, *are topological invariants of X.*

Example 10 (Dolbeault complex on a compact complex manifold X)**.**
Let X be a compact complex manifold of complex dimension n, and from Example 8 we have a elliptic complex:

$$\cdots \xrightarrow{\bar\partial} \mathcal{E}^{p,q}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+1}(X) \xrightarrow{\bar\partial} \mathcal{E}^{p,q+2}(X) \xrightarrow{\bar\partial} \cdots$$

for fixed p, $0 \le p \le n$. From Example 4 and Theorem 4.22, we have

$$H^q(X, \Omega^p) \cong H^q(\mathcal{E}^{p,*}(X)) = \mathcal{H}_{\overline{\Box}}(\mathcal{E}^{p,q}(X)) =: \mathcal{H}^{p,q}(X)$$

where we denote the Laplacian of $\overline{\partial}$ by $\overline{\Box} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. Similarly, we represent the cohomology classes by the $\overline{\Box}$-harmonic (p, q)-forms.

We define the Hodge numbers *of X, for $0 \le p, q \le n$, by setting (Use Theorem 4.22) $h^{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^p) = \dim_{\mathbb{C}} \mathcal{H}^{p,q}(X) < \infty$. Note that $h^{p,q}$ are invariants of the complex structure of X and do not depend on the choice of metric.*

The following theorem show how the Hodge numbers and the Betti numbers are related, whose proof is a simple consequence of the fact that there is a spectral sequence (See [F] for more details. On Kähler case one can see Corollary 5.14) :

[F] *Let X be a compact complex manifold. Then*

$$\chi(X) = \sum_r (-1)^r b_r(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X).$$

Now we consider the elliptic complex of (p, q)-forms with coefficients in E where E is a holomorphic vector bundle over X. Similarly to the former case (from Example 4 and Example 8), we have

$$H^q(X, \Omega^p(E)) \cong H^q(\mathcal{E}^{p,*}(X, E)) = \mathcal{H}_{\overline{\Box}}(\mathcal{E}^{p,q}(X, E)) =: \mathcal{H}^{p,q}(X, E)$$

Here we define $\overline{\Box}$ to be the Laplacian of $\overline{\partial} \otimes 1$ if no confusion occurs. We see $\mathcal{H}^{p,q}(X, E)$ is the $\overline{\Box}$-harmonic E-valued (p, q)-forms in $\mathcal{E}^{p,q}(X, E)$ and we let the generalized Hodge numbers *be (Use Theorem 4.22)* $h^{p,q}(E) := \dim_{\mathbb{C}} \mathcal{H}^{p,q}(X,E) < \infty$*. Moreover we define the Euler characteristic of the holomorphic vector bundle* E *to be*

$$\chi(E) = \chi(X,E) = \sum_{q=0}^{n} (-1)^q h^{0,q}(E)$$

As before, $h^{p,q}(E)$ *and* $\chi(E)$ *depend only on the complex structures of* X *and* E*, since the dimensions are independent of the particular metric used.*

Remark 4.23 (Riemann-Roch-Hirzebruch Theorem)**.**

However, it is a remarkable fact that $\chi(E)$ *as above can be expressed in terms of topological invariants of the vector bundle* E *(its Chern classes) and of the complex manifold* X *itself (the Todd classes of the tangent bundle to* X*). Firstly we denote*

$$c(E) = \prod_{i=1}^{r} (1 + x_i)$$

where $x_i \in H^*(X, \mathbb{C})$ *ans something similar the eigenvalues of a matrix. Then we denote the Chern character of* E*,* $ch(E)$ *and the Todd class of the tangent bundle to* X*,* $\mathcal{T}(T(X))$ *by (Note the following are actually finite sums because* $H^{>2\dim_{\mathbb{C}} X}(X, \mathbb{C}) = 0$*)*

$$ch(E) = \sum_{i=0}^{r} \exp(x_i) \text{ and } \mathcal{T}(T(X)) = \prod_{i=1}^{n} \frac{x_i}{1 - \exp(-x_i)}$$

Then we have the following theorem

Theorem 4.24 *[Riemann-Roch-Hirzebruch, 1954]*

Let E *be a holomorphic vector bundle over a compact complex manifold* X*, then*

$$\chi(E) = \int_X ch(E) \cdot \mathcal{T}(T(X))$$

From this theorem we immediately see LHS depends only on the topological structure of X *, and RHS is an integer.*

It is worth to piont out that for curves, the Hirzebruch-Riemann-Roch theorem is actually the classical Riemann-Roch theorem. Recall that for each divisor D *on a curve there is an invertible sheaf* $\mathcal{O}(D)$ *(which corresponds to a line bundle) s.t. the linear system of* D *is more or less the space of sections of* $\mathcal{O}(D)$*. For curves the Todd class is* $1 + c_1(T(X))/2$*, and the Chern character of a sheaf* $\mathcal{O}(D)$ *is just* $1 + c_1(\mathcal{O}(D))$*, so the Hirzebruch–Riemann–Roch theorem states that*

$$h^{0,0}(\mathcal{O}(D)) - h^{0,1}(\mathcal{O}(D)) = \int_X [c_1(\mathcal{O}(D)) + c_1(T(X))/2]$$

But $h^{0,0}(\mathcal{O}(D))$ *is the dimension* $l(D)$ *of the linear system of* D *, and by Serre duality* $h^{0,1}(\mathcal{O}(D)) = h^{0,0}(\mathcal{O}(K - D)) = l(K - D)$ *where* K *is the canonical divisor. Moreover,* $c_1(\mathcal{O}(D))$ *integrated over* X *is* $\deg D$*, and* $c_1(T(X))$ *integrated over* X *is the Euler class* $2-2g$ *of the curve* X*, where* g *is the genus. So we get the classical Riemann Roch theorem*

$$l(D) - l(K - D) = \deg(D) + 1 - g$$

From the discussion above, we can also get Weil's Riemann Roch theorem for vector bundles V *over curves, by noting the Chern character is* $rank(V) + c_1(V)$*,*

$$h^0(V) - h^1(V) = c_1(V) + rank(V)(1 - g)$$

5 Compact complex manifolds

5.1 Basic Hermitian stracture on a complex vector space

Let E be a complex vector space of complex dimension n. Let E' be the real dual space to the underlying real vector space of E, and $F := E' \otimes_{\mathbb{R}} \mathbb{C}$. Then F has complex dimension $2n$, and we say that $\omega \in \wedge^p F$ is *real* if $\overline{\omega} = \omega$ (Note $\wedge F$ is equipped with a natural conjugation). We suppose there is a Hermitian inner product $\langle \, , \rangle$ on E, and locally, represented as

$$\langle u, v\rangle = h(u,v), \; u, v \in E \text{ where } h = \sum_{\mu,v} h_{\mu,v} z_\mu \otimes \overline{z}_v$$

where $\{z_\mu\}_{\mu=1}^n$ is a basis for $\wedge^{1,0}F$ and $(h_{\mu,v})$ is a positive definite Hermitian symmetric matrix. We can write $h = S + iA$ where S is a symmetric positive definite bilinear form and

$$A = \frac{1}{2i} \sum_{\mu,v} h_{\mu v}(z_\mu \otimes \overline{z}_v - \overline{z}_v \otimes z_\mu) = -i \sum_{\mu,v} h_{\mu,v} z_\mu \wedge \overline{z}_v$$

Definition 5.1 (fundamental 2-form)**.** *We define the fundamental 2-form Ω associated to the hermitian metric h as*

$$\Omega = \frac{i}{2} \sum_{\mu,v} h_{\mu,v} z_\mu \wedge \overline{z}_v$$

Hence $h = S - 2i\Omega$ and moreover Ω is a real 2-form of type $(1,1)$. If we choose $\{z_\mu\}$ s.t. $(h_{\mu,v}) = I$ and write $z_\mu = x_\mu + iy_\mu$ then

$$S = \sum_\mu (x_\mu \otimes x_\mu + y_\mu \otimes y_\mu), \; \Omega = \sum_\mu x_\mu \wedge y_\mu$$

Thus $\Omega^n = n! x_1 \wedge y_1 \wedge \cdots \wedge x_n \wedge y_n$ and there is a nutural volume form $\mathrm{Vol} = \frac{\Omega^n}{n!}$ and a Hodge $*$−operator $* : \wedge^p F \longrightarrow \wedge^{2n-p} F$ which is extended by complex-linearity from acting on E'. $*$ is defined s.t.

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta\rangle \mathrm{Vol}$$

And if we define

$$\omega = \sum (-1)^r \Pi_r$$

where $\Pi_r : \wedge E' \longrightarrow \wedge^r E'$ is nutural projection and extends by complex-linearity to $\wedge F$, then $** = \omega$. And let

$$\Pi_{p,q} : \wedge F \longrightarrow \wedge^{p,q} F$$

be the nutural projection and

$$J := \sum i^{p-q} \Pi_{p,q}$$

which represents the complex structure of the vector space F, satisfies $J^2 = \omega$.

Now we define

$$L : \wedge^{p,q} F \longrightarrow \wedge^{p+1,q+1} F, v \mapsto \Omega \wedge v$$

and its adjoint L^* (with respect to the Hermitian inner product) satisfies

$$\langle L\alpha, \beta\rangle \mathrm{Vol} = \Omega \wedge \alpha \wedge *\overline{\beta} = \alpha \wedge L * \overline{\beta} = \alpha \wedge (*\omega*) L * \overline{\beta} = \langle \alpha, \omega * L * \beta\rangle \mathrm{Vol}$$

because $*\omega* = \mathrm{id}$ and $*, \omega, L$ are real, hence $L^* = \omega * L *$.

Lemma 5.2. *With notations as above, we have*

(1) $*\,|_{\wedge^{p,q}F}\colon \wedge^{p,q}F \longrightarrow \wedge^{n-q,n-p}F$ *is an isomorphism.*

(2) $[L^*, L] = \sum_{p=0}^{2n}(n-p)\Pi_p$.

(3) $[L,\omega] = [L,J] = [L^*,\omega] = [L^*,J] = 0$.

Proof. The essential part of the proof of (1),(2) lies in the following formula:

$$*(z_A \wedge \bar{z}_B \wedge w_M) = \gamma(a,b,m) z_A \wedge \bar{z}_B \wedge w_{M'}$$

where A, B, and M are mutually disjoint increasing multiindices and $a = |A|$, $b = |B|$, $m = |M|$, $M' = N - (A \cup B \cup M)$,$\gamma(a,b,m)$ is a nonvanishing constant. Moreover,

$$\gamma(a,b,m) = i^{a-b}(-1)^{p(p+1)/2+m}(-2i)^{p-n}$$

where $p = a + b + 2m$ is the total degree of $z_A \wedge \bar{z}_B \wedge w_M$. This could be verified by direct computation (See Proposition V.1.1 and Lemma V.1.2 of [W]). (3) follows that L and L^* are homogeneous and real operators. □

Now we focus on a compact oriented Riemannian manifold X of m $\mathbb{R}-$dimensions. We see X carries a canonical volume element, i.e. a $m-$form $*(1) \in \mathcal{E}^m(X)$. Then define inner product on $\mathcal{E}^*(X) = \oplus_{p=0}^m \mathcal{E}^p(X)$ as

$$(\varphi, \psi) = \int_X \varphi \wedge *\overline{\psi}$$

One can see it is a positive definite, Hermitian symmetric, sesquilinear form. And there is an orthogonal direct sum decomposition $\mathcal{E}^r(X) = \sum_{p+q=r} \mathcal{E}^{p,q}(X)$.

On X we define $\overline{*} : \mathcal{E}^*(X) \longrightarrow \mathcal{E}^*(X)$ by setting $\overline{*}\varphi = *\overline{\varphi}$, which is a conjugate-linear isomorphism. We have the following Proposition.

Proposition 5.3. *Let X be an oriented compact Riemannian manifold of real dimension m and let $\Delta = dd^* + d^*d$, where d^* is the adjoint of d, then*

(a) $d^* = (-1)^{m+mp+1}\overline{*}d\overline{*} = (-1)^{m+mp+1} * d *$ *on* $\mathcal{E}^p(X)$.

(b) $*\Delta = \Delta *$, $\overline{*}\Delta = \Delta\overline{*}$.

Proof. For $\varphi \in \mathcal{E}^{p-1}(X)$ and $\psi \in \mathcal{E}^p(X)$, we have

$$\begin{aligned}(d\varphi, \psi) &= \int_X d\varphi \wedge \overline{*}\psi \overset{\text{Stokes thm}}{=} (-1)^p \int_X \varphi \wedge d\overline{*}\psi \\ &= (-1)^p \int_X \varphi \wedge \overline{*}(\overline{*}\omega d\overline{*})\psi = (-1)^{p+mp+1}(\varphi, \overline{*}d\overline{*}\psi)\end{aligned}$$

since $** = \overline{**} = \omega$ and $*, d$ are real (Note in this time $\omega = \sum(-1)^{mr+r}\Pi_r$). So we get (a). And (b) follows from (a) and the fact that $\omega d * d\varphi = d * d\omega\varphi$. □

If X is a Hermitian complex manifold and $E \longrightarrow X$ is a Hermitian vector bundle. Let $\tau : E \longrightarrow E^*$ be be a conjugate-linear bundle isomorphism of E onto its dual bundle E^*, then we define a conjugate-linear isomorphism of Hermitian vector bundles

$$\overline{*}_E : \mathcal{E}^p(X, E) \longrightarrow \mathcal{E}^{2m-p}(X, E^*), \varphi \otimes e \mapsto \overline{*}(\varphi) \otimes \tau(e)$$

for $\varphi \in \mathcal{E}^p(X)$ and $e \in E_x$. Actually, as Lemma 5.2,

$$\overline{*}_E : \mathcal{E}^{p,q}(X, E) \longrightarrow \mathcal{E}^{m-p,n-q}(X, E)$$

is a conjugate-linear isomorphism. We natrually extend Hodge inner product on $\mathcal{E}^*(X, E)$ by setting (In a sense we only consider functions with compact support)

$$(\varphi, \psi) = \int_X \varphi \wedge \overline{*}_E \psi$$

and $(s \otimes e) \wedge (s' \otimes e') = s \wedge s' \cdot \langle e, e' \rangle \in \mathcal{E}^{2m}(X)_x$ for $s \in \mathcal{E}^p(X)_x, s' \in \mathcal{E}^{2m-p}(X)_x$ and $e \in E_x, e' \in E^*$. Similarly to Proposition 5.3, we have

Proposition 5.4. *Let X be a Hermitian complex manifold and let $E \longrightarrow X$ be a Hermitian holomorphic vector bundle. Then (a) $\overline{\partial} : \mathcal{E}^{p,q}(X, E) \longrightarrow \mathcal{E}^{p,q+1}(X, E)$ has an adjoint $\overline{\partial}^*$ with respect to the Hodge inner product on $\mathcal{E}^{**}(X, E)$ given by*

$$\overline{\partial}^* = -\overline{*}_{E^*} \overline{\partial} \overline{*}_E$$

(b) If $\overline{\Box} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^* \overline{\partial}$ is the complex Laplacian acting on $\mathcal{E}^{**}(X, E)$, then

$$\overline{\Box} \overline{*}_E = \overline{*}_E \overline{\Box}$$

Proof. In this case we have

$$\overline{*}_E \overline{*}_E = w = \sum (-1)^r \Pi_r$$

since the real dimension of X is even. If $\varphi \in \mathcal{E}^{p,q-1}(X, E)$ and that $\psi \in \mathcal{E}^{p,q}(X, E)$, then $\varphi \wedge \overline{*}_E \psi$ is a scalar differential form of type $(n, n-1)$, and hence $\partial(\varphi \wedge \overline{*}_E \psi) = 0$. Moreover,

$$\overline{\partial}(\varphi \wedge \overline{*}_E \psi) = \overline{\partial}\varphi \wedge \overline{*}_E \psi + (-1)^{p+q-1} \varphi \wedge \overline{\partial} \overline{*}_E \psi.$$

Use Stokes' thm as in the proof of Proposition 5.3,

$$(\overline{\partial}\varphi, \psi) = (-1)^{p+q} \int \varphi \wedge \overline{\partial} \overline{*}_E \psi = (-1)^{p+q} \int \varphi \wedge \overline{*}_E (\omega \overline{*}_{E^*} \overline{\partial} \overline{*}_E \psi) = -\int \varphi \wedge \overline{*}_E (*_{E^*} \overline{\partial} \overline{*}_E \psi)$$

So we get (a).

And (b) follows from (a) and whose proof is similar to that of Proposition 5.3. □

Remark 5.5 (Why only consider $\overline{\partial}$ on $\mathcal{E}^*(X, E)$)**.**
We see only $\overline{\partial}$ natrually acts on $\mathcal{E}^{p,q}(X, E)$ for any holomorphic vector bundle $E \to X$ because $\overline{\partial}$ anihilates the transition functions of E whereas ∂ and d do not. If we just consider the scalar coefficients, then similarly $\partial^* = -\overline{*}\partial\overline{*}$ and $\Box = \partial\partial^* + \partial^*\partial$ will commutes with $\overline{*}$.

We will state the following two duality theorems, which are essentially deduced from Theorem 4.22, Proposition 5.3 and Proposition 5.4.

Theorem 5.6 (Poincaré duality)**.**

Let X be a compact m-dimensional orientable differentiable manifold. Then there is a conjugate linear isomorphism

$$\sigma : H^r(X, \mathbb{C}) \longrightarrow H^{m-r}(X, \mathbb{C}),$$

and hence $H^{m-r}(X, \mathbb{C})$ is isomorphic to the dual of $H^r(X, \mathbb{C})$.

Proof. Let $*$ be associated to a Riemannian metric on X, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{E}^r(X) \arrow[r, "\overline{*}"] \arrow[d, "H_\Delta"] & \mathcal{E}^{m-r}(X) \arrow[d, "H_\Delta"] \\
\mathcal{H}^r(X) \arrow[r, "\overline{*}"] \arrow[d, equal] & \mathcal{H}^{m-r}(X) \arrow[d, equal] \\
H^r(X, \mathbb{C}) \arrow[r, "\sigma"] & H^{m-r}(X, \mathbb{C})
\end{tikzcd}$$

where notation $H_\Delta, \mathcal{H}^r(X)$ comes from Example 9 and note $\Delta\overline{*} = \overline{*}\Delta$ in Proposition 5.3. Moreover, from Example 9 we know $H^r(X, \mathbb{C}) \cong \mathcal{H}^r(X)$, and then σ is the conjugate linear isomorphism induced by $\overline{*}$. □

Theorem 5.7 (Kodaira-Serre duality)**.**

Let X be a compact complex manifold of complex dimension n and let $E \to X$ be a holomorphic vector bundle over X. Then there is a conjugate linear isomorphism

$$\sigma : H^r(X, \Omega^p(E)) \longrightarrow H^{n-r}(X, \Omega^{n-p}(E^*))$$

and hence these spaces are dual to one another.

Proof. By introducing Hermitian metrics on X and E, we can define the $\overline{*}_E$ operator and obtain the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}^{p,q}(X, E) \arrow[r, "\overline{*}_E"] \arrow[d, "H_{\overline{\Box}}"] & \mathcal{E}^{n-p,n-q}(X, E^*) \arrow[d, "H_{\overline{\Box}}"] \\
\mathcal{H}^{p,q}(X, E) \arrow[r, "\overline{*}_E"] \arrow[d, "\cong"] & \mathcal{H}^{n-p,n-q}(X, E^*) \arrow[d, "\cong"] \\
H^{p,q}(X, E) \arrow[r] \arrow[d, "\cong"] & H^{n-p,n-q}(X, E^*) \arrow[d, "\cong"] \\
H^q(X, \Omega^p(E)) \arrow[r, "\sigma"] & H^{n-q}(X, \Omega^{n-p}(E^*))
\end{tikzcd}$$

where $\overline{*}_E$ maps harmonic forms to harmonic forms by Proposition 5.4 and the commutation comes from Theorem 4.22. Hence the result follows. □

5.2 The Kähler case: relationship between differential operators

Definition 5.8 (Kähler manifold)**.** *A Kähler manifold is a complex manifold X equipped with a Hermitian metric h and a complex structure J s.t. the fundamental 2-form Ω associated to h is closed, i.e. $d\Omega = 0$. Inthis time we call h is a Kähler metric. We say a complex manifold X is said to be of Kähler type if it admits at least one Kähler metric.*

Recall that on a complex manifold a Hermitian metric h can be expressed as

$$h = \sum_{\mu,v} h_{\mu v}(z) z_\mu \otimes \bar{z}_v \text{ where } h_{\mu v}(z) = h\left(\frac{\partial}{\partial z_\mu}, \frac{\partial}{\partial z_v}\right)(z)$$

and the fundamental 2-form Ω is given by $\frac{i}{2}\sum_{\mu,v} h_{\mu v}(z) dz_\mu \wedge d\bar{z}_v$. Now we give some examples of Kähler manifolds.

Example 11 ($\mathbb{C}^n$ and $\mathbb{C}^n/\Gamma$)**.**

On $\mathbb{C}^n$ the standard metric is $h = \sum_{\mu=1}^n dz_\mu \otimes d\bar{z}_\mu$ where $z_\mu = x_\mu + iy_\mu$, $\mu = 1, \dots, n$. Then $\Omega = \frac{i}{2}\sum_{\mu=1}^n dz_\mu \wedge d\bar{z}_\mu = \sum_{\mu=1}^n dx_\mu \wedge dy_\mu$ clearly satisfies $d\Omega = 0$, and hence h is a Kähler metric on $\mathbb{C}^n$.

If $\omega_1, \dots, \omega_{2n}$ be $2n$ vectors in $\mathbb{C}^n$ which are linearly independent over $\mathbb{R}$ and let Γ be the lattice consisting of all integral linear combinations of $\{\omega_1, \dots, \omega_{2n}\}$. Then we denote $X = \mathbb{C}^n/\Gamma$ and give X the usual quotient topology. X is nuturally a complex manifold and we called a complex torus. *The Kähler metric h on $\mathbb{C}^n$, given above, is invariant under the action of Γ on $\mathbb{C}^n$, so we can find a Hermitian metric $\tilde{h}$ on X s.t. if $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Gamma$ is the holomorphic projection mapping, then $\pi^*(\tilde{h}) = h$. Since π is a local diffeomorphism, and d commutes with $(\pi_U^{-1})^*$ in a neighborhood U of a point $z \in \mathbb{C}^n$, we have*

$$d\widetilde{\Omega}|_{\pi(U)} = (\pi_U^{-1})^* d\Omega|_U = 0$$

Namely, $\tilde{h}$ defined on X is a Kähler metric, and all such X are then necessarily of Kähler type.

Example 12 ($\mathbb{P}_n$)**.**

One of the most important manifolds of Kähler type is $\mathbb{P}_n$. We let $(\xi_0, \dots, \xi_n)$ be homogeneous coordinates for $\mathbb{P}_n$, and consider the differential form $\widetilde{\Omega}$ defined as

$$\widetilde{\Omega} = \frac{i}{2}\frac{\sum_{\mu=0}^n d\xi_\mu \wedge d\bar{\xi}_\mu - \sum_{\mu,\nu=0}^n \bar{\xi}_\mu \xi_\nu d\xi_\mu \wedge d\bar{\xi}_\nu}{|\xi|^4}.$$

$\widetilde{\Omega}$ defines a d-closed differential form Ω on $\mathbb{P}_n$ of type $(1,1)$ whose local coordinates in a particular coordinate system, for example, say $w_j = \frac{\xi_j}{\xi_0}$, $\quad j = 1, \dots, n$, is

$$\Omega(w) = \frac{i}{2}\frac{(1+|w|^2)\sum_{\mu=1}^n dw_\mu \wedge d\bar{w}_\mu - \sum_{\mu,\nu=1}^n \bar{w}_\mu w_\nu dw_\mu \wedge d\bar{w}_\nu}{(1+|w|^2)^2}.$$

Thus the associated tensor

$$h = \left(\sum_{\mu,\nu} h_{\mu\nu}(w) dw_\mu \otimes d\bar{w}_\nu\right)(1+|w|^2)^{-2}$$

has coefficients (ignoring the positive denominator above) $h_{\mu\nu}(w) = (1+|w|^2)\delta_{\mu\nu} - \bar{w}_\mu w_\nu, \quad \mu, \nu = 1, \dots, n$. *We see that* $\tilde{h} = [h_{\mu\nu}]$ *is Hermitian symmetric and positive definite, because*

$$ {}^t\bar{u}\tilde{h}u = \sum_{\mu,\nu} h_{\mu\nu} u_\mu \bar{u}_\nu = \sum_{\mu,\nu}(1+|w|^2)\delta_{\mu\nu}u_\mu\bar{u}_\nu - \left(\sum_\mu \bar{w}_\mu u_\mu\right)\left(\sum_\nu w_\nu \bar{u}_\nu\right) \geq |u|^2 \text{ for } u \in \mathbb{C}^n $$

where $(\cdot,\cdot)$ *denote the standard inner product in* $\mathbb{C}^n$. *Hence* h *defines a Hermitian metric, and even a Kähler metric on* $\mathbb{P}_n$, *which is called the Fubini-Study metric classically.*

Example 13 (Compact Riemann surfaces)**.**

We claim that every complex manifold X *of complex dimension 1 (a Riemann surface) is of Kähler type. Let* h *be an arbitrary Hermitian metric on* X, *we see it is indeed a Kähler metric. But this is trivial, since the associated fundamental form* Ω *is of type* $(1,1)$ *and therefore of total degree 2 on* X. *However* X *has two real dimensions, so we must have* $d\Omega = 0$.

Suppose more that X *is a compact Riemann surface. Then by the Hodge decomposition theorem for Kähler manifolds (See Theorem 4.20 below), we have* $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$. *Moreover,* $h^{1,0}(X) = h^{0,1}(X)$, *and then* $2h^{1,0}(X) = b_1(X)$. *Thus* $h^{1,0}(X)$ *is a topological invariant of* X, *called the genus of the Riemann surface, usually denoted by* g.

Example 14 (Submanifold of a Kähler manifold)**.**

We claim that a Kähler metric h *on a Kähler manifold* X *will induce a Kähler metric on any complex submanifold* M *of* X. *Let* $j : M \to X$ *be the injection mapping, then* $h_M = j^*h$ *defines a metric on* M, $j^*\Omega = \Omega_M$ *is the associated fundamental form to* h_M *on* M. *Since* $d\Omega_M = j^*d\Omega = 0$, *it is clear that* Ω_M *is also a Kähler fundamental form.*

We also define $d_c = J^{-1}dJ, d_c^* = J^{-1}d^*J$, so for a function φ we have

$$ d_c\varphi = \omega J dJ\varphi = (-1)J(\partial\varphi + \overline{\partial}\varphi) = -i(\partial - \overline{\partial})\varphi $$

Hence $d_c = -i(\partial - \overline{\partial})$. And then

$$ dd_c = 2i\partial\overline{\partial} $$

, which is a real operator of type $(1,1)$ acting on differential forms in $\mathcal{E}^*(X)$. Recall $L = \Omega \wedge -$ and $L^* = \omega * L *$.

Proposition 5.9. *Let* X *be a Kähler manifold. Then*

$$ [L,d] = 0 = [L^*,d^*] \text{ and } [L,d^*] = d_c, [L^*,d] = -d_c^* $$

Proof. $[L,d] = 0$ comes from $d\Omega = 0$ and Ω is $(1,1)$ type. And $[L^*,d^*] = d_c$ is the adjoint form of $[L,d] = 0$. Similarly, we just need to verify $[L^*,d] = -d_c^* = -i(\partial^* - \overline{\partial}^*)$, and to deduce this, by noting L is a real operator, we just need to show $[L^*,\partial] = i\overline{\partial}^*$.

We first prove the identity in $\mathbb{C}^n$. Let $\omega = \frac{\sqrt{-1}}{2}\sum_i dz_i \wedge d\bar{z}_i$ be the standard Kähler form on $\mathbb{C}^n$. We introduce operators $e_k, \overline{e}_k$ by

$$ e_k(\eta) := dz_k \wedge \eta, \quad \overline{e}_k(\eta) := d\overline{z}_k \wedge \eta. $$

Their adjoints are denoted by i_k and $\overline{i}_k$ respectively. Recall that $|dz_k|^2 = |dx|^2 + |dy|^2 = 2$, so we conclude that $i_k = 2\iota_{\frac{\partial}{\partial z_k}}$, where $\iota_{\frac{\partial}{\partial z_k}}$ is the "interior product" operator, defined by

$$ \iota_{\sum_r f_r \frac{\partial}{\partial x_r}}(dx_1 \wedge \cdots \wedge dx_n) = \sum_r (-1)^{r-1} f_r dx_1 \wedge \cdots \wedge \widehat{dx_r} \wedge \cdots \wedge dx_n $$

Similarly, $\bar{i}_k = 2\iota_{\frac{\partial}{\partial \bar{z}_k}}$. It is easy to check that

$$i_l e_k + e_k i_l = 2\delta_{kl} = \bar{i}_l \bar{e}_k + \bar{e}_k \bar{i}_l$$

We also define the degree-preserving linear maps $\partial_k, \bar{\partial}_k$ by

$$\partial_k\left(\sum_{I,J}\eta_{IJ}dz_I\wedge d\bar{z}_J\right) := \sum_{I,J}\frac{\partial\eta_{IJ}}{\partial z_k}dz_I\wedge d\bar{z}_J, \quad \bar{\partial}_k\left(\sum_{I,J}\eta_{IJ}dz_I\wedge d\bar{z}_J\right) := \sum_{I,J}\frac{\partial\eta_{IJ}}{\partial \bar{z}_k}dz_I\wedge d\bar{z}_J$$

Note that ∂_k and $\bar{\partial}_k$ commute with $e_k, \bar{e}_k$ and hence also $i_k, \bar{i}_k$. Also an "integration by part" trick gives us the relation $\partial_k^* = -\bar{\partial}_k, \bar{\partial}_k^* = -\partial_k$. Now we have

$$\begin{aligned}
\partial &= \sum_k \partial_k e_k = \sum_k e_k \partial_k, \quad \partial^* = -\sum_k \bar{\partial}_k i_k = -\sum_k i_k \bar{\partial}_k \\
\bar{\partial} &= \sum_k \bar{\partial}_k \bar{e}_k = \sum_k \bar{e}_k \bar{\partial}_k, \quad \bar{\partial}^* = -\sum_k \partial_k \bar{i}_k = -\sum_k \bar{i}_k \partial_k. \\
L &= \frac{\sqrt{-1}}{2}\sum_k e_k \bar{e}_k, \quad L^* = -\frac{\sqrt{-1}}{2}\sum_k \bar{i}_k i_k
\end{aligned}$$

So we can compute

$$\begin{aligned}
L^*\partial &= -\frac{\sqrt{-1}}{2}\sum_k \bar{i}_k i_k \partial_l e_l = -\frac{\sqrt{-1}}{2}\sum_{k,l}\partial_l \bar{i}_k i_k e_l \\
&= -\frac{\sqrt{-1}}{2}\sum_k (\partial_k \bar{i}_k)(2 - e_k i_k) - \frac{\sqrt{-1}}{2}\sum_{k\neq l}\partial_l e_l \bar{i}_k i_k = \sqrt{-1}\bar{\partial}^* + \partial L^*
\end{aligned}$$

and this is what we want. For the general compact Kähler case, one can use Kähler normal coordinates to reduce the computations to our $\mathbb{C}^n$ case. The key point is that only first order derivatives are involved.

One can see a direct but cumbersome prove in Theorem V.4.8 of [W]. □

Corollary 5.10. *With notation as above,*

$$\begin{aligned}
&[L,\partial^*] = i\bar{\partial}, \quad [L,\bar{\partial}^*] = -i\partial, \quad [L^*,\partial] = i\bar{\partial}^*, \quad [L^*,\bar{\partial}] = -i\partial^*. \\
&[L,d_c] = 0 = [L^*,d_c^*], \quad [L,d_c^*] = -d, \quad [L^*,d_c] = d^* \\
&d^*d_c = -d_c d^* = d^*Ld^* = -d_c L^* d_c, \quad dd_c^* = -d_c^* d = d_c^* L d_c^* = -dL^*d \\
&\partial\bar{\partial}^* = -\bar{\partial}^*\partial = -i\bar{\partial}^* L\bar{\partial}^* = -i\partial L^*\partial, \quad \bar{\partial}\partial^* = -\partial^*\bar{\partial} = i\partial^* L\partial^* = i\bar{\partial}L^*\bar{\partial}.
\end{aligned}$$

Proof. The first line is clear. And the second line comes that the operator J commutes with the real operators L, L^*, and $J^{-1}d_cJ = -d, J^{-1}d_c^*J = -d^*$. The third line follows $[L^*, d_c] = [L^*, -i(\partial - \bar{\partial})] = -i(i\bar{\partial}^* + i\partial^*) = d^*$ and then

$$d^*d_c = [L^*d_c]d_c = -d_cL^*d_c, \quad -d_cd^* = -d_cL^*d_c = d^*d_c$$

Similarly for the others, by using $d_c = [L, d^*]$, etc. □

On a Kähler manifold, those operators we define before have some nice relationship and helps us to derive some useful results. Recall $\Box = \partial\partial^* + \partial^*\partial$.

Theorem 5.11. *Let X be a Kähler manifold. If these operators $d, d^*, \partial, \overline{\partial}, \partial^*, \overline{\partial}^*, \Box, \overline{\Box}$ are defined with respect to the Kähler metric on X, then $\Delta = dd^* + d^*d$ commutes with $*, d$, and L, and*

$$\Delta = 2\Box = 2\overline{\Box}$$

In particular,

1. *$\Box$ and $\overline{\Box}$ are real operators.*
2. *$\Delta|_{\mathcal{E}_{p,q}} : \mathcal{E}_{p,q} \to \mathcal{E}_{p,q}$.*

Proof. It is clear that $[\Delta, d] = 0$ and by Proposition **??** $[\Delta, *] = 0$. Note $[L, d] = 0$ and $dd_c = 2i\partial\overline{\partial} = -d_c d$, then

$$-[L, \Delta] = -d[L, d^*] - [L, d^*]d = -dd_c - d_c d = 0$$

Moreover, $2\partial = d + id_c, 2\overline{\partial} = d - id_c$, then

$$4\Box = (d + id_c)(d^* - id_c^*) + (d^* - id_c^*)(d + id_c) = dd^* + d^*d + d_c d_c^* + d_c^* d_c = \Delta + J^{-1}\Delta J$$

But $J^{-1}\Delta J = J^{-1}(d[L^*, d_c] + [L^*, d_c]d)J = \Delta$ since $[L, J] = 0 = [L^*,]J$ and $d_c d = -dd_c$. Hence $\Delta = 2\Box = 2\overline{\Box}$. And note Δ is real and $\Box$ is type $(0, 0)$, so follows the rest. $\Box$

Corollary 5.12. *On a Kähler manifold, the self-adjoint and real operator Δ commutes with J, L, d, ∂.*

5.3 The Hodge decomposition on compact Kähler manifolds

Recall the Dolbeault groups on X, $H^{p,q}(X)$ which is exactly $H^q(X, \Omega^p)$, represented by $\overline{\partial}$-closed(p, q)−forms and these vector spaces are finite dimensional. We now state the following decomposition theorem of Hodge (as amplified by Kodaira).

Theorem 5.13 (The Hodge Decomposition Theorem on Compact Kähler Manifolds)**.**
Let X be a compact complex manifold of Kähler type. Then there is a direct sum decomposition

$$H^r(X, \mathbb{C}) = \sum_{p+q=r} H^{p,q}(X),$$

moreover $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Proof. It is sufficiently to show (See Example 9 and Example 10)

$$\mathcal{H}^r(X) = \sum_{p+q=r} \mathcal{H}^{p,q}(X),$$

Suppose that $\varphi \in \mathcal{H}^r(X)$. Then $\overline{\Box}\varphi = 2\Delta\varphi = 0$. But

$$\overline{\Box}\varphi = \overline{\Box}\varphi^{r,0} + \cdots + \overline{\Box}\varphi^{0,r} = 0$$

and $\overline{\Box}$ preserves bidegree, so each component is zero, and therefore there is a injective mapping

$$\tau : \mathcal{H}^r(X) \longrightarrow \sum_{p+q=r} \mathcal{H}^{p,q}(X),\ \varphi \longmapsto (\varphi^{r,0}, \dots, \varphi^{0,r})$$

The mapping is clearly injective, and is surjective again by $\Delta = 2\overline{\Box}$. The latter assertion follows immediately from the fact that $\overline{\Box}$ is real and the conjugation is an isomorphism from $\mathcal{E}^{p,q}(X)$ to $\mathcal{E}^{q,p}(X)$. $\Box$

Not every complex manifold is of Kähler manifold, because there are some basic restrictions, e.g. $H^2(X, \mathbb{C})$ must be nontrivial (If not, Ω is exact and $\int_X \Omega^n = 0$, but $\int_X \Omega^n = n!\mathrm{Vol} > 0$, a contradiction).

And we now state more. Recall we have the Betti number $b_r(X) = \dim_{\mathbb{C}} H^r(X, \mathbb{C})$ and the Hodge number $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$. There are topological restrictions on a compact complex manifold admitting a Kähler metric, although such manifold always admits a Hermitian metric (See Proposition 3.2).

Corollary 5.14. *Let X be a compact Kähler manifold. Then*

1. $b_r(X) = \sum_{p+q=r} h^{p,q}(X)$.
2. $h^{p,q}(X) = h^{q,p}(X)$.
3. $b_q(X)$ *is even for odd* q.
4. $h^{1,0}(X) = \frac{1}{2} b_1(X)$ *is a topological invariant.*

Proof. immediately follows from Theorem 5.13. $\Box$

Example 15 (Hopf surface: a non-Kähler compact complex manifold)**.**

We see if Γ is a discontinuous group of automorphism on a complex manifold X, possibly with fixed points, then X/Γ can still be given a complex structure as a complex space (a generalization of a complex manifold) with singularities at the image of the fixed points. We proceed as follows to construct an example of a Hopf surface. Consider the 3-sphere $\mathbb{S}^3 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$, and then we observe that there is a diffeomorphism

$$f : S^3 \times \mathbb{R} \xrightarrow{\approx} \mathbb{C}^2 - \{0\},\ (z_1, z_2, t) \mapsto (e^t z_1, e^t z_2)$$

And we see $\mathbb{Z}$ acts on $S^3 \times \mathbb{R}$ nuturally by

$$(z_1, z_2, t) \longrightarrow (z_1, z_2, t + m) \textit{ for } m \in \mathbb{Z}$$

Clearly $(S^3 \times \mathbb{R})/\mathbb{Z} \cong S^3 \times S^1$. Under the diffeomorphism f we can see the action of $\mathbb{Z}$ on $\mathbb{C}^2 - \{0\}$ is properly discontinuous without fixed points. Thus $X := (\mathbb{C}^2 - \{0\})/\mathbb{Z}$ is a complex manifold which is diffeomorphic to $S^3 \times S^1$ (and this is compact). We then see

$$b_0(X) = b_1(X) = b_3(X) = b_4(X) = 1 \textit{ and } b_2(X) = 0$$

Where one may use the Künneth theorem which says $H_n(X \times Y) = \sum_{p+q=n} H_p(Y) \otimes H_q(X)$ and Poincaré duality about Orientable n-dimensional manifold which says $H^q(X) \cong H_{n-q}(X)$. In particular, $b_1(X) = 1$, and hence X cannot be Kähler, since odd degree Betti numbers must be even on Kähler manifolds. Such manifolds X are called Hopf surfaces. The Hopf surface is the simplest example of a compact complex manifold that cannot be embedded in projective space of any dimension.

For another application of Theorem 5.11 and Theorem 4.22, we state the so called $\partial\overline{\partial}-$Lemma, which is very useful in Kähler geometry. One may see a simple case of this lemma in Lemma II.2.15 of [W].

Lemma 5.15 ($\partial\overline{\partial}-$Lemma)**.**

If η is any d-closed (p,q)-form on a compact Kähler manifold X, and η is d- or ∂- or $\bar{\partial}$-exact, then

$$\eta = \partial\bar{\partial}\gamma$$

for some $(p-1,q-1)$-form γ. When $p=q$ and η is real, then we can take $\gamma = \sqrt{-1}\xi$ for a real $(p-1,q-1)$-form ξ.

Proof. Recall that in the Kähler case we have $\Delta_d = 2\square = 2\overline{\square}$. Hence by the orthogonal decomposition (a) of Theorem 4.22, where one may write, for example, $\mathcal{E}(X) = \mathcal{H}_\Delta \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(d^*)$ to see clearly, we see any ξ satisfied it is d- or ∂- or $\overline{\partial}$-exact, then its harmonic projection (Note $d, \partial, \overline{\partial}$ have same harmonic form) must be zero.

Now we see $H_{\overline{\square}}\eta = 0$, so

$$\eta = \overline{\square} G_{\overline{\square}}\eta = \bar{\partial}\bar{\partial}^* G_{\overline{\square}}\eta$$

where we use $[\bar{\partial}, G_{\overline{\square}}] = 0$ and $d\eta = 0 \implies \partial\eta = \overline{\partial}\eta = 0$. Moreover, we note

$$\partial(\bar{\partial}^* G_{\overline{\square}}\eta) = -\bar{\partial}^*\partial G_{\square}\eta = -\bar{\partial}^* G_{\square}\partial\eta = 0$$

where use $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$ in Corollary 5.10 and $G_{\overline{\square}} = G_{\square}$. Hence it is also orthogonal to harmonic forms and we can use Hodge decomposition for $\square$:

$$\bar{\partial}^* G_{\overline{\square}}\eta = \square G_{\square}\bar{\partial}^* G_{\overline{\square}}\eta = \partial\partial^* G_{\square}\bar{\partial}^* G_{\overline{\square}}\eta$$

where again use $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$. Hence we get (one may use Theorem 4.22 again)

$$\eta = \bar{\partial}\partial\partial^* G_{\square}\bar{\partial}^* G_{\overline{\square}}\eta = \partial\bar{\partial}\left(-\partial^* G_{\square}\bar{\partial}^* G_{\overline{\square}}\eta\right) = \partial\bar{\partial}\left(-\partial^*\bar{\partial}^* G_{\overline{\square}}^2\eta\right).$$

$\square$

Remark 5.16 (Kähler metrics and non-linear PDEs)**.**

The most often used case is about $(1,1)$-class. Let $[\omega] = [\tilde{\omega}] \in H^2(X,\mathbb{R})$. Then by the $\partial\bar{\partial}$-lemma above we have

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi.$$

where $\varphi \in C^\infty(X;\mathbb{R})$ and φ is unique up to a constant. We point out if $\varphi \in C^\infty(X;\mathbb{R})$ satisfies $\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$, then it actually defines a Kähler metric with the same Kähler class. Hence we conclude that the space of Kähler metrics within the same cohomology class $[\omega]$ is isomorphic to

$$\{\varphi \in C^\infty(X;\mathbb{R}) \mid \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}/\mathbb{R}.$$

One of the most important problems in Kähler geometry is the existence of canonical metrics in a given Kähler class. Through the $\partial\bar{\partial}$-lemma, we can reduce the problem to a (usually non-linear) partial differential equation for φ. This is the starting point of using non-linear PDEs to solve problems in Kähler geometry.

6 Kodaira vanishing and embedding theorem

6.1 Hodge manifolds and positive line bundles

If $\dim_{\mathbb{C}} X = n$, we say $K_X = \wedge^n T^*(X)$ is the *canonical bundle* of X (sometimes omit the subscript X), and $\mathcal{O}_X(K_X) = \Omega_X^n$ is the sheaf of holomorphic n-forms on X. We want to consider a restricted class of Kähler manifolds whose Ω is *integral*, i.e. its cohomology class $[\Omega]$ is actually in $H^*(X, \mathbb{Z})$. In this time we say Ω is a *Hodge form*, X is a *Hodge manifold* (due to A. Weil) and h is a *Hodge metric*. Here are some examples of Hodge manifolds.

Example 16 (compact projective algebraic manifold)**.**
Let X be a compact projective algebraic manifold which means X is a submanifold of $\mathbb{P}_N$ for some N. Let Ω be the fundamental form associated with the Fubini-Study metric on $\mathbb{P}_N$ (See Example 12) and Ω is the negative (omit a positive constant) of the Chern form for the universal bundle $U_{1,N+1} \to \mathbb{P}_N$, it follows that Ω is a Hodge form on $\mathbb{P}_N$ (See Propositions 3.16 and Theorem 3.19). The restriction of Ω (as a differential form) to X will also be a Hodge form, and hence X is a Hodge manifold, where we use the fact that a complex submanifold of a Hodge manifold is again a Hodge manifold.

Example 17 (compact connected Riemann surface)**.**
We claim a compact connected Riemann surface X is a Hodge manifold. Note $\dim_{\mathbb{R}} X = 2$, *then by Poincaré duality (See Theorem 5.6) that* $\mathbb{C} = H^0(X, \mathbb{C}) \cong H^2(X, \mathbb{C}) = H^{1,1}(X)$. *Let $\widetilde{\Omega}$ be the fundamental form on X associated with a Hermitian metric. Then $\widetilde{\Omega}$ is a closed form of type* $(1,1)$ *which is a basis element for the one-dimensional de Rham group* $H^2(X, \mathbb{C})$. *Let* $c = \int_X \widetilde{\Omega} \neq 0$, *and then* $\Omega = c^{-1}\widetilde{\Omega}$ *will be an integral positive form on X of type* $(1,1)$. *Hence X is Hodge.*

We can generalize this case to the assertion that any Kähler manifold X with the property that $\dim_{\mathbb{C}} H^{1,1}(X) = 1$ *is necessarily Hodge. This follows from the fact that multiplication by an appropriate constant will make the Kähler form on X integral, as above.*

A differential form φ of type $(1,1)$ on X is said to be *positive* if locally (at $p \in X$)

$$\varphi = i \sum_{\mu,\nu} \varphi_{\mu\nu}(z) dz_\mu \wedge d\overline{z}_\nu,$$

and the matrix $[\varphi_{\mu\nu}(z)]$ is a positive definite Hermitian symmetric matrix for each fixed point z near p, and we denote this condition by $\varphi > 0$.

Definition 6.1. *Let $E \to X$ be a holomorphic line bundle and let $c_1(E)$ be the first Chern class of E considered as an element of the de Rham group $H^2(X, \mathbb{R})$. Then E is said to be* positive *if there is a real closed differential form ψ of type $(1,1)$ s.t. $\psi \in c_1(E)$ and ψ is a positive differential form. E is said to be* negative *if E^* is positive.*

And we see a equivalent definition of positive line bundle when X is a compact complex manifold.

Proposition 6.2. *Let $E \longrightarrow X$ be a holomorphic line bundle over a compact complex manifold X. Then E is positive if and only if there is a Hermitian metric h on E s.t. $i\Theta_E$ is a positive differential form, where Θ_E is the curvature of E with respect to the canonical connection induced by h.*

Proof. From the differential-geometric definition of $c_1(E)$ (See Remark 3.12), we see $i\Theta_E$ positive for some metric h will imply that E is positive (Note rank$E = 1$ so in this time $c_1(E) = \frac{i}{2\pi}\Theta_E$).

Conversely, suppose that E is positive and that $\varphi \in c_1(E)$ is a positive differential form. Let h be any metric on E, and then with respect to a local frame f we have (Set $h = h(f)$ we know the canonical curvature associated with h has form $\Theta_E = \overline{\partial}\partial \log h$)

$$\varphi_0 = \frac{i}{2\pi}\bar{\partial}\partial \log h \in c_1(E),$$

and hence $\varphi - \varphi_0 = d\eta$ for some $\eta \in \mathcal{E}^1(X)$. Moreover, φ is a Kähler form on X (we may denote the fundamental 2-form as $\Omega = \varphi/2$), and then X becomes a Kähler manifold. Then we may apply the harmonic theory, with notation as Theorem 4.22 and Theorem 5.11, then

$$d\eta = d(H + \Delta G)\eta = dH\eta + d\Delta G\eta = \Delta G d\eta = 2(\bar{\partial}\overline{\partial}^* G + 2\bar{\partial}^* G\bar{\partial})d\eta$$

Where we use $dH = 0$, $[d, \Delta] = [d, G] = 0 = [\partial, G] = \frac{1}{2}[\partial, G_{\overline{\Box}}]$ (See Theorem 4.22) and $\Delta = 2\Box = 2\overline{\Box}$. We see $\bar{\partial}d\eta = \partial d\eta = 0$ which follows from the fact that $\bar{\partial}\varphi = \bar{\partial}\varphi_0 = \partial\varphi = \partial\varphi_0 = 0$. Thus we obtain

$$d\eta = 2\bar{\partial}\overline{\partial}^* G d\eta = 2i\bar{\partial}\partial L^* G d\eta$$

where $i\bar{\partial}^* = L^*\partial - \partial L^*$ from Corollary 5.10, and use $[\partial, G] = 0 = \partial d\eta$. Therefore we set $r = 2L^* G d\eta$, and by letting $h' = h \cdot e^{2\pi r}$ be a new metric for E, we obtain

$$\frac{i}{2\pi}\bar{\partial}\partial \log h' = \frac{i}{2\pi}\bar{\partial}\partial \log h + i\bar{\partial}\partial r = \varphi_0 + d\eta = \varphi$$

Hence we have $\Theta_{E,h'} = \overline{\partial}\partial \log h'$ has property that $i\Theta_{E,h'}$ is a positive differential form. □

Example 18 (the negative and positive bundles on $\mathbb{P}_n$)**.**

We have the following three basic line bundles over $X = \mathbb{P}_n$,

(a) *The hyperplane section bundle: $H \longrightarrow \mathbb{P}_n$.*

(b) *The universal bundle: $U \longrightarrow \mathbb{P}_n$ ($U = U_{1,n+1}$).*

(c) *The canonical bundle: $K = \wedge^n T^*(\mathbb{P}_n) \longrightarrow \mathbb{P}_n$.*

Here H is the line bundle associated to the divisor of a hyperplane in $\mathbb{P}_n$ (See Remark 3.20), e.g., $[t_0 = 0]$, (note all such line bundle are isomorphic) and then the divisor is defined by $\{t_0/t_\alpha\}$ in $U_\alpha = \{t_\alpha \neq 0\}$, the line bundle H has transition functions

$$h_{\alpha\beta} = \left(\frac{t_0}{t_\alpha}\right)\left(\frac{t_0}{t_\beta}\right)^{-1} = \frac{t_\beta}{t_\alpha} \quad \text{in } U_\alpha \cap U_\beta$$

We see the universal bundle (See Example 5) has transition functions $u_{\alpha\beta} = \frac{t_\alpha}{t_\beta}$ in $U_\alpha \cap U_\beta$ and thus $H^* = U$. Now we compute the transition functions for the canonical bundle K on $\mathbb{P}_n$. If we let $\zeta_j^\beta = t_j/t_\beta$, $j \neq \beta$, the usual coordinates in U_β. And we have $\zeta_j^\alpha = \zeta_j^\beta \cdot \zeta_\beta^\alpha$ in $U_\alpha \cap U_\beta$ which is the (nonlinear) change of coordinates for $\mathbb{P}_n$ from U_α to U_β. Hence a basis of $K|_{U_\alpha}$ given by Φ_α can be written as

$$\Phi_\alpha := (-1)^\alpha d\zeta_0^\alpha \wedge \cdots \wedge d\zeta_{\alpha-1}^\alpha \wedge d\zeta_{\alpha+1}^\alpha \wedge \cdots \wedge d\zeta_n^\alpha = (\zeta_\beta^\alpha)^{n+1}\Phi_\beta$$

Now we see that these transition functions for the frames $\{\Phi_\alpha\}$ induce transition functions $\{k_{\alpha\beta}\}$ for the canonical bundle K which are given by $k_{\alpha\beta}([t_0,\dots,t_n]) = \left(\frac{t_\alpha}{t_\beta}\right)^{n+1}$ and then

$$K = \wedge^n T^*(\mathbb{P}_n) = U^{n+1} = (H^*)^{n+1}$$

.

Moreover, the universal bundle $U \to \mathbb{P}_n$ has the curvature form

$$\Theta = 2i\Omega$$

(See Example 5), where Ω is a positive differential form expressed in homogeneous coordinates (See Example 12)

$$\Omega = \frac{i}{2}\frac{\sum\limits_{\mu=0}^{n} |t|^2 dt_\mu \wedge d\bar{t}_\mu - \sum\limits_{\mu,\nu=0}^{n} \bar{t}_\mu t_\nu dt_\mu \wedge d\bar{t}_\nu}{|t|^4}$$

Namely, Ω is the canonical Kähler form on $\mathbb{P}_n$ associated with the Fubini-Study metric. Hence $i\Theta_U = -2\Omega$ is a negative differential form on $\mathbb{P}_n$, thus $H^* = U$, and $K = U^{n+1}$ are negative line bundles over $\mathbb{P}_n$, and the hyperplane section bundle $H \to \mathbb{P}_n$ is positive.

Remark 6.3 (Chern classes and holomorphic line bundles on $\mathbb{P}_n$).

From the Hodge decomposition theorem 5.13 we see

$$0 = H^1(\mathbb{P}_n, \mathbb{C}) = H^{1,0}(\mathbb{P}_n) \oplus H^{0,1}(\mathbb{P}_n)$$

and

$$\mathbb{C} \cong H^2(\mathbb{P}_n, \mathbb{C}) = H^{2,0}(\mathbb{P}_n) \oplus H^{1,1}(\mathbb{P}_n) \oplus H^{0,2}(\mathbb{P}_n)$$

And we see $H^{1,1}(\mathbb{P}_n) = \mathbb{C}[\Omega]$, where Ω is the fundamental form on $\mathbb{P}_n$. Hence

$$H^1(\mathbb{P}_n, \mathcal{O}) = H^2(\mathbb{P}_n, \mathcal{O}) = 0$$

Actually, one may continue this to show $H^q(\mathbb{P}_n, \Omega^p) = 0$ when $p \neq q$ and $H^p(\mathbb{P}_n, \Omega^p) = H^{2p}(\mathbb{P}_n, \mathbb{C})$.

Now consider the short exact sequence $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0$ on $\mathbb{P}_n$ and the induced cohomology sequence

$$0 = H^1(\mathbb{P}_n, \mathcal{O}) \longrightarrow H^1(\mathbb{P}_n, \mathcal{O}^*) \xrightarrow{c_1} H^2(\mathbb{P}_n, \mathbb{Z}) \longrightarrow H^2(\mathbb{P}_n, \mathcal{O}) = 0$$

Let $\mathbb{P}_1 \subset \mathbb{P}_n$ be a generator for $H^2(\mathbb{P}_n, \mathbb{Z}) = \mathbb{Z}$. We find

$$c_1(H) = c_1(U^*) = -c_1(U) = 1$$

(See Proposition 3.16), and since c_1 is an isomorphism of abelian groups, it follows that every holomorphic line bundle $L \longrightarrow \mathbb{P}_n$ is a power of H, i.e. $L = H^m$, and $c_1(L)(\mathbb{P}_1) = m$.

Thus the holomorphic line bundles on $\mathbb{P}_n$ are completely classified in this manner by their Chern classes.

6.2 Kodaira vanishing theorem

Theorem 6.4 (Kodaira-Akizuki-Nakano vanishing theorem).

Suppose $E \to X$ be a positive holomorphic line bundle over a compact complex manifold X,then

$$H^q(X, \Omega^p(E)) = 0, \quad p+q>n$$

In particular, $H^q(X, \mathcal{O}(K_X \otimes E)) = 0$ for $q>0$.

Proof (By Akizuki-Nakano).

By the proof of Proposition 6.2, we see X is a Kähler manifold, and we denote the corresponding Hermitian metric which allows $i\Theta$ to be positive by h. We also consider the canonical connection

$$D = D(h) = D' + D'' \text{ where } D' = \partial + \theta, D'' = \overline{\partial}$$

Let $\Delta_F := F^*F + FF^*$ for any operator F acting on differential forms. The proof of this theorem essentially use the so called *Bochner technique*, which is a method to estimate the Laplacian of a differential form. Specifically, here we show

$$\Delta_{\overline{\partial}} - \Delta_{D'} = [L, L^*] \tag{10}$$

First we see

$$(D')^* = -*\overline{\partial}* = \partial^*, \quad (D'')^* = -*\partial*+\omega*\theta* = \overline{\partial}^* + \omega*\theta*$$

then by Corollary 5.10

$$[L^*, \overline{\partial}] = -i(D')^*, \quad [L^*, D'] = i\overline{\partial}^*$$

Again by Corollary 5.10, we have (note $D'D' = 0 = \overline{\partial}\overline{\partial}$) the Kodaira-Nakano identity

$$i\Delta_{\overline{\partial}} - i\Delta_{D'} = \overline{\partial}[L^*, D'] + [L^*, D']\overline{\partial} + D'[L^*, \overline{\partial}] + [L^*, \overline{\partial}]D' = L^*\Theta - \Theta L^*$$

Note

$$\Omega = i\Theta$$

by the choice of h, then we have the equation (10). But by Lemma 5.2 we have

$$[L, L^*] = (p+q-n)\mathrm{id}$$

when acting on $\mathcal{E}^{p,q}(E)$. So for $\forall s \in \mathcal{H}^{p,q}(X,E)$ and $p+q>n$,

$$0 \leq (p+q-n)||s||^2 = (\Delta_{\overline{\partial}}s - \Delta_{D'}s, s) = -(\Delta_{D'}s, s) = -(||D's||^2 + ||(D')^*s||^2) \leq 0$$

Hence we must have $s=0$ and then $\mathcal{H}^{p,q}(X,E)$ vanishes when $p+q>n$. And by

$$H^q(X, \Omega^p(E)) \cong \mathcal{H}^{p,q}(X,E)$$

we get this theorem. In particular,

$$H^q(X, \mathcal{O}(K_X \otimes E)) = H^q(X, \Omega^n(E))$$

which will vanish for $q>0$. □

Remark 6.5. *Moreover, we have silimar results for the negative line bundle by using Kodaira-Serre duality (Theorem 5.7), which is exactly the Theorem VI.2.4 of [W], stated as follows:*

Suppose that X is a compact complex manifold.

1. *Let $E \longrightarrow X$ be a holomorphic line bundle with the property that $E \otimes K^*$ is a positive line bundle. Then*
$$H^q(X, \mathcal{O}(E)) = 0, \quad q > 0.$$
2. *Let $E \longrightarrow X$ be a negative line bundle. Then*
$$H^q(X, \Omega^p(E)) = 0, \quad p + q < n.$$

The theorem above is first proved by Kodaira (the case (a) and $p = 0$ in (b), see [K53]). One may find that in Kodaira's original paper, he used Chern classes instead of the "positive" portray for line boundles.

Remark 6.6 (Analog on Stein manifolds: Theorem B)**.**

Theorem 6.4 plays a role in the theory of compact complex manifolds analogous to the well-known (due to Cartan and Serre) Theorem B of Stein manifold (e.g. a connected Riemann surface is a Stein manifold if and only if it is not compact) theory, which says

- *if F is a coherent sheaf on a Stein manifold, then $H^q(X, F) = 0$ for $q \geq 1$.*

Note $H^q(X, \mathcal{O}(E))$, $q \geq 1$, do not need to vanish for all holomorphic vector bundles E over a compact complex manifold X (E.g. $H^n(X, K_X) = \mathbb{C}$ by the Theorem 5.7), which would be the case for Stein manifolds.

Remark 6.7 (Algebraic version of Kodaira vanishing theorem)**.**

We now want to remark the algebraic version of Theorem 6.4, which says

- *If $\mathcal{L}$ is an ample line bundle on a smooth projective variety X over a field k of characteristic zero, then*
$$H^p(X, \Omega^q \otimes \mathcal{L}) = 0 \quad \text{for } p + q > n$$
Here Ω^q is the qth wedge power of the sheaf of differentials $\Omega_{X/k}$.

We may first assume $k = \mathbb{C}$. By Serre's GAGA principle (See [S56]), there is an associated analytic space X^{an}, which is a complex manifold as X is smooth and there is an analytification functor from coherent sheaves on X to coherent analytic sheaves on X^{an}, moreover, this functor is an equivalence of categories, and the sheaf cohomologies are the same. This functor also sends the sheaf $\Omega^q = \bigwedge^q \Omega_{X/k}$ to the sheaf of holomorphic q-differentials on X^{an}, which we denote by Ω^q as above. Hence it is sufficient to show some power of $\mathcal{L}^{an}$ is positive, so we can assume that $\mathcal{L}$ is very ample. But then $\mathcal{L}$ is the pull-back of $\mathcal{O}(1)$ on projective space, which then is a positive line bundle. Thus, the analytic version of the vanishing theorem completes the proof.

Now let k be an arbitrary field of characteristic zero. one may use the Lefschetz principle *to believe the result is valid.*

Remark 6.8 (Raynaud surface: Kodaira vanishing thm fails for characteristic $p > 0$)**.**

A Raynaud surface is named after M. Raynaud and is introduced by William E. Lang in 1979. Such surface is constructed to show that Theorem 6.4 does not always hold for cases where the characteristics of the base domain are non-zero. One can see [L] for more details.

6.3 Kodaira embedding theorem

We call a compact complex manifold X which admits an embedding into $\mathbb{P}_n(\mathbb{C})$ (for some n) a *projective algebraic manifold*. A remarkable theorem of Chow shows every analytic compact subvariety of $\mathbb{P}_n(\mathbb{C})$ is algebraic, i.e. an algebraic subvariety.

J. P. Serre's famous paper [S56] proves general results that relate classes of algebraic varieties, regular morphisms and sheaves with classes of analytic spaces, holomorphic mappings and sheaves.

Theorem 6.9 (Kodaira embedding theorem)**.**
A compact manifold X is projective algebraic if and only if it is a Hodge manifold.

Before showing the sketch of the proof of Theorem 6.9, we narrate some corollaries of it.

Corollary 6.10. *Any compact Riemann surface is projective algebraic.*

Proof. By example 17. And one can show any compact Riemann surface with genus g (whose definition is given in example 13) can be embedded holomorphically into $\mathbb{CP}^{g+1}$, see Theorem 5.5.4 of [M]. □

Corollary 6.11. *For any compact complex manifold X, it admits a positive line bundle $E \longrightarrow X$ if and only if it is projective algebraic.*

Proof. By example 18 and the property of $c_1(E)$. And one can find the generalization of the theorem in this form, which includes the case where X admits singularities in [G]. □

The third corollary needs some preparation. If $L \to X$ be a holomorphic line bundle, s.t. $H^0(X, \mathcal{O}(L)) \neq 0$. Then we can take a basis of $H^0(X, \mathcal{O}(L))$, say $s_0, \cdots, s_N$, and define a "map"

$$\iota_L : X \longrightarrow \mathbb{CP}^N, \quad x \mapsto [s_0(x), \cdots, s_N(x)]$$

We may understand this by locally and see τ is independent of the local trivialization we choose. The problem is, τ is not well-defined on the "base locus" of L, $Bs(L) := \{x \in X \mid s(x) = 0, \forall s \in H^0(X, \mathcal{O}(L))\}$

What Corollary 6.11 (or Theorem 6.9) actually says is : If $L \to X$ is a positive line bundle on a compact complex manifold, then we can find a large integer $m_0 > 0$ s.t. for all $m > m_0$,

- $L^{\otimes m}$ is *base point free* , i.e. $Bs(L^{\otimes m}) = \emptyset$;
- if one choose a basis of $H^0(X, \mathcal{O}(L^{\otimes m}))$, say $s_0, \cdots, s_{N_m}$, then the *Kodaira map*

  $$\iota_{L^m} : X \to \mathbb{CP}^{N_m}, \quad x \mapsto [s_0(x), \cdots, s_{N_m}(x)]$$

  is a holomorphic embedding.

Now we give some relevant definitions.

Definition 6.12. *With notation as above. If there is an integer $m_0 > 0$ s.t. for all $m > m_0$, $L^{\otimes m}$ is base point free, then we say L is* semi-ample*; If L is base point free and the Kodaira map ι_L is a holomorphic embedding, then we say L is* very ample*; If there is an integer $m_0 > 0$ s.t. for all $m > m_0$, $L^{\otimes m}$ is very ample, then we say L is* ample*.*

Corollary 6.13. *On a compact complex manifold, a holomorphic line bundle is ample if and only if it is positive.*

Proof. In fact, if L is positive, then it is ample by Theorem 6.9. On the other hand, if L is ample, we can find $m \in \mathbb{N}$ s.t. ι_{L^m} is a holomorphic embedding. Then the pullback of the hyperplane bundle is isomorphic to $L^{\otimes m}$, and the induced metric has positive curvature. The corresponding metric on L also has positive curvature. □

Corollary 6.14 (Kodaira-Nakano)**.**

If $E \longrightarrow M$ is any line bundle and $L \longrightarrow M$ a positive line bundle, then for $k >> 0$, the bundle $L^{\otimes k} \otimes E$ is very ample.

This is analogous to a theorem of J. P. Serre in the algebraic setting: if $X \to SpecA$ is a projective scheme, and $\mathcal{L}$ an ample line bundle on X, then for any coherent sheaf $\mathcal{F}$ on X, we have $\mathcal{F} \otimes \mathcal{L}^{\otimes r}$ is spanned by its global sections and has no cohomology in positive degrees.

Proof. immediately from the proof of the Kodaira embedding theorem by Blow Up way (See below or section 4 of Chapter 1 of [GH]). □

Remark 6.15 (Fujita's conjecture (1985))**.**

Fujita's conjecture is named after Takao Fujita, and is formulated it in 1985, whose statement is as follows:

> *For a positive holomorphic line bundle L on a compact complex manifold X, the line bundle $K_X \otimes L^{\otimes m}$ is*
>
> - *spanned by sections when $m \geq \dim_{\mathbb{C}} X + 1$;*
> - *very ample when $m \geq \dim_{\mathbb{C}} X + 2$,*

Fujita conjecture provides an explicit bound on m, which is optimal for projective spaces. This conjecture is still open in general, but has been proved for some special cases, e.g. when X is a surface or a threefold.

Remark 6.16 (The (Strongest) Whitney embedding theorem)**.** *This is a similar work to Kodaira embedding theorem, but H. Whitney's famous work in the field of differential manifolds, which says*

> *Any smooth manifold of dimension m can be immersed into $\mathbb{R}^{2m-1}$ and embedded into $\mathbb{R}^{2m}$.*

There also have many refinement about this, e.g. Any compact orientable surface embeds to $\mathbb{R}^3$; For $m \neq 2^k$, any smooth m-manifold embeds to $\mathbb{R}^{2m-1}$. (But if $m = 2^k$, $\mathbb{RP}^m$ cannot be embedded into $\mathbb{R}^{2m-1}$).

***Outline of the proof of Kodaira embedding theorem 6.9**.*

We know that on Hodge manifolds, there always has a positive holomorphic line bundle L because $c_1(H^1(X, \mathcal{O}^*)) = H^2_{1,1}(X, \mathbb{Z})$ by Theorem 3.19. Here we prove that $\iota_{L^{\otimes m}}$ is an embedding for a sufficiently large m, and need show:

- $L^{\otimes m}$ is base point free when m large enough. Note x is compact, we only need to show for $\forall p \in X$, $\exists m_p \in \mathbb{N}$ s.t. for all $m \geq m_p$, we can find a $s \in H^0(X, \mathcal{O}(L^{\otimes m}))$ s.t. $s(p) \neq 0$. That is enough to show the linear map $r_p : H^0(X, \mathcal{O}(L^{\otimes m})) \to H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{O}/\mathfrak{m}_p)$ is surjective.

- For m large, global sections of $L^{\otimes m}$ separate points. We only need to prove that $\forall p \neq q$ in X, $r_{p,q} : H^0(X, \mathcal{O}(L^{\otimes m})) \to H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{O}/\mathfrak{m}_{p,q})$ is surjective for m sufficiently large.

- For m large, ι_{L^m} is an immersion. That is, for any point $p \in X$, global sections of $L^{\otimes m}$ separate tangent directions at p. We only need to show the linear map $r_{p,p}$: $H^0(X, \mathcal{O}(L^{\otimes m})) \to H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{O}/\mathfrak{m}_p^2)$ is surjective for m sufficiently large.

One may note that $\mathcal{O}_p/\mathfrak{m}_p^2 \cong \mathbb{C} \oplus T_p^*(X)$, and $\mathcal{O}/\mathfrak{m}_p$ actually is called *the skyscraper sheaf* at p. It is enough to prove the latter two. Note that if we denote by $\mathfrak{m}_p$ the ideal sheaf of holomorphic germs vanishing at p and $\mathfrak{m}_{p,q}$ the ideal sheaf of holomorphic germs vanishing at p and q, then what we need prove is that

$$H^0(X, \mathcal{O}(L^{\otimes m})) \to H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{O}/\mathfrak{m}_{p,q})$$

and

$$H^0(X, \mathcal{O}(L^{\otimes m})) \to H^0(X, \mathcal{O}(L^{\otimes m}) \otimes \mathcal{O}/\mathfrak{m}_p^2)$$

are both surjective when m is large enough. Now we use short exact sequences of sheaves:

$$0 \to \mathfrak{m}_{p,q} \to \mathcal{O} \to \mathcal{O}/\mathfrak{m}_{p,q} \to 0 \text{ and } 0 \to \mathfrak{m}_p^2 \to \mathcal{O} \to \mathcal{O}/\mathfrak{m}_p^2 \to 0$$

Tensor with the locally free sheaf $\mathcal{O}(L^{\otimes m})$ and consider the induced long exact sequences. One can see that all are reduced to prove the vanishing of

$$H^1(X, \mathcal{O}(L^{\otimes m}) \otimes \mathfrak{m}_{p,q}) \text{ and } H^1(X, \mathcal{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2)$$

We found that the main problem is that $\mathfrak{m}_{p,q}$ and $\mathfrak{m}_p^2$ are not line bundles , only are coherent analytic sheaves (We see in the category correspondence between Affine Scheme and $(\mathrm{CRing})^{\mathrm{op}}$, the coherent $\mathcal{O}_{\mathrm{Spec}(A)}-$module corresponds to the finite generated $A-$module and the locally free $\mathcal{O}_{\mathrm{Spec}(A)}-$module corresponds to the finite generated projective $A-$module). So the Kodaira vanishing theorem 6.4 could not be applied. There are two ways to continue the proof :

- Kodaira's Way. Blow up the points p appropriately, then $\mathfrak{m}_p^2$ and $\mathfrak{m}_{pq}$ become locally free on the blown up complex manifold $\widetilde{X}$. In this time we can applies the theory of harmonic differential forms (See Proposition 2.1) to give the desired vanishing theorems. And one needs to show that vanishing upstairs implies vanishing downstairs. The completed proof can be found in Section 3,4 of Chapter VI of [W], or section 4 of Chapter 1 of [GH], or one directly see the remarkable paper [K54].

- Grauert's Way. Prove the generalized Kodaira vanishing theorem about coherent analytic sheaves, which says if E is a positive line bundle and $\mathcal{F}$ is any coherent analytic sheaf, then there is an integer $\mu_0 > 0$ s.t. $H^q(X, \mathcal{O}(E^\mu) \otimes \mathcal{F}) = 0$ for $\mu \geq \mu_0$ and $q \geq 1$. The completed proof can be found in [G].

$\square$

Continue proving Theorem 6.9 by Kodaira's way.

Recall the blow up $\widetilde{X} \xrightarrow{\pi} X$ of X at $p \in X$. Donote U as a neighborhood of p, with coordinates $z = (z_1, \dots, z_n)$, where $z = 0$ corresponds to the point p and

$$W = \{(z,t) \in U \times \mathbb{P}_{n-1} : t_\alpha z_\beta - t_\beta z_\alpha = 0, \alpha, \beta = 1, \dots, n\}$$

Thus $\pi : W \longrightarrow U$ given by $\pi(z,t) = z$ and we can set identity outside U to get $\widetilde{X}$. One may say $S = \pi^{-1}(0) = \{0\} \times \mathbb{P}_{n-1}$ *the exceptional plane* and see $\pi|_{W-S}$ is a biholomorphism. We write π_p to emphisize the the blow up at point p.

$S \cong \mathbb{P}_{n-1}$ so we can comsider the hyperplane section bundle H (More specifically, we consider the projection $\sigma : W \longrightarrow \mathbb{P}_{n-1}$ and $\sigma^* H$) as in Example 18, moreover, we also consider the line bundle $L \longrightarrow \widetilde{X}$ associated to the divisor S. If we set $V_\alpha = \{t_\alpha \neq 0\} \subset \mathbb{P}_{n-1}$, then the transition functions of H are given by $h_{\alpha\beta} = t_\beta / t_\alpha$ in $V_\alpha \cap V_\beta$. And $S \cap (U \times V_\alpha) \cap W$ is defined by $z_\alpha = 0$, hence the transition functions of L are given by $l_{\alpha\beta} = z_\alpha / z_\beta = t_\alpha / t_\beta$ in $U \times (V_\alpha \cap V_\beta) \cap W$. So we get

$$L_{|W} = \sigma^* H^*$$

We set $\mathcal{I}_S$ be the ideal sheaf of $S \subset \widetilde{X}$ and $\widetilde{\mathcal{O}} = \mathcal{O}_{\widetilde{X}}$, $\widetilde{F} = \pi^* F$, where $F = E^{\otimes m}$ for some m and E is a positive line bundle on X (Note X is Hodge manifold with fundamental form Ω, we can find such E s.t. Ω is a representative of $c_1(E)$). We see the mapping π_p induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F}) \otimes \mathcal{I}_S^2) \arrow[r] \arrow[d, dashed, bend right, "\alpha"'] & \Gamma(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F})) \arrow[r] \arrow[d, dashed, bend right, "\beta"'] & \Gamma(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F}) \otimes \widetilde{\mathcal{O}}/\mathcal{I}_S^2) \\
0 \arrow[r] & \Gamma(X, \mathcal{O}(F) \otimes \mathfrak{m}_p^2) \arrow[r] \arrow[u, "\pi_1^*"'] & \Gamma(X, \mathcal{O}(F)) \arrow[r] \arrow[u, "\pi^*"'] & \Gamma(X, \mathcal{O}(F) \otimes \mathcal{O}/\mathfrak{m}_p^2) \arrow[u, "\pi_2^*"']
\end{tikzcd}$$

where π_1^* and π_2^* are induced by π^*, which are well defined mappings and π_2^* is injective. α and β are isomorphisms whose existence we explain below. Actually we just construct β. For $n = 1$ it is obvious. For $n > 1$, note $\pi_p : \widetilde{X} \to X$ is biholomorphic on the complement of S, we set

$$\tilde{\beta}(\xi) = (\pi_p^{-1})^* \xi \text{ for } \xi \in \Gamma(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F}))$$

which is a well-defined element of $\Gamma(X - \{p\}, \mathcal{O}(F))$. By Hartogs' theorem (The original proof used Cauchy's integral formula for functions of several complex variables, given by F. Hartogs in 1906), we find a unique element $\tilde{\beta}(\xi) \in \Gamma(X, \mathcal{O}(F))$ and we define as $\beta(\xi)$. Note $\beta^{-1} = \pi_p^*$ and hence β is an isomorphism. Moreover, as noted above $\beta^{-1}(\eta) \in \Gamma(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F}) \otimes \mathcal{I}_S^2)$ if and only if $\eta \in \Gamma(X, \mathcal{O}(F) \otimes \mathfrak{m}_p^2)$, so we also find α.

Hence we find

$$H^1(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F}) \otimes \mathcal{I}_S^2) = 0 \implies H^1(X, \mathcal{O}(F) \otimes \mathfrak{m}_p^2) = 0$$

so we just need to prove the vanish of the fromer. We note that $\mathcal{I}_S$ is the ideal sheaf of a divisor S in $\widetilde{X}$, thus it is a locally free sheaf of rank 1. Actually,

$$\mathcal{I}_S \cong \widetilde{\mathcal{O}}(L^*)$$

Then $\mathcal{I}_S^2 \cong \mathcal{O}((L^*)^2)$, and we have

$$H^1(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F}) \otimes \mathcal{I}_S^2) = H^1(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F} \otimes (L^*)^2))$$

Hence by Kodaira's vanishing theorem 6.4, we find all reduced to prove the line bundle

$$K_{\widetilde{X}}^* \otimes \widetilde{F} \otimes (L^*)^2$$

is a positive line bundle.

We claim the following equation

$$K_{\widetilde{X}} = \pi_p^* K_X \otimes L_p^{n-1} \tag{11}$$

With notations above, we see $(z_1, t_2/t_1, \dots, t_n/t_1)$ are holomorphic coordinates for $U \times V_1 \cap W$. Thus there is a holomorphic frame for $K_{\widetilde{X}}$ on V_1

$$f_1 = dz_1 \wedge d\left(\frac{t_2}{t_1}\right) \wedge \cdots \wedge d\left(\frac{t_n}{t_1}\right) = z_1^{1-n} dz_1 \wedge \cdots \wedge dz_n$$

where the latter equality follows from the defining equations for W. Similarly, we have a frame for $K_{\widetilde{X}}$ over $U \times V_\alpha \cap W$, which is

$$f_\alpha = z_\alpha^{1-n} dz_1 \wedge \cdots \wedge dz_n$$

So the transition functions for the line bundle $K_{\widetilde{X}}|_W$ are given by $g_{\alpha\beta} = f_\alpha / f_\beta = (z_\alpha / z_\beta)^{n-1}$, which implies that

$$K_{\widetilde{X}}|_W = L^{n-1}|_W \cong L^{n-1} \otimes \pi_p^* K_X|_W$$

since K_X is trivial on U. And $L|_{\widetilde{X}-W}$ is trivial, π_p is biholomorphic on $\widetilde{X} - W$, hence $K_{\widetilde{X}}|_{\widetilde{X}-W} \cong K_{\widetilde{X}} \otimes L^{n-1}|_{\widetilde{X}-W}$. Thus we have the equation (11).

By Proposition 6.2, for show $K_{\widetilde{X}}^* \otimes \widetilde{F} \otimes (L^*)^2$ is a positive line bundle, we converts to analysis its canonical curvature. Firstly note

$$\Theta_{E \otimes F} = \Theta_E + \Theta_F$$

which comes from $\Theta_E = \overline{\partial}\partial \log h(f)$ for a given Hermitian metric h on E and a frame f of E. We now need to construct appropriate metrics on L_p and $K_{\widetilde{X}}$.

- L_p. Let U' is open and $0 \in U' \subset\subset U$, and choose $\rho \in \mathcal{D}(U)$ s.t. $\rho \geq 0$ in U and $\rho \equiv 1$ on U'. Let
$$\Theta_H = \bar{\partial}\partial \log \frac{|t_\alpha|^2}{|t_1|^2 + \cdots + |t_n|^2} \quad (\text{in } V_\alpha \subset \mathbb{P}_{n-1})$$
be the curvature of $H \to \mathbb{P}_{n-1}$ with respect to the natural metric, called the Fubini-Study metric h_0 (See Example 12). Hence we can give $L|_W = \sigma^* H^*$ the metric $h_1 = \sigma^* h_0$. Note $L^*|_{X-U'}$ is trivial, and we can equip it with a constant metric h_2. Then, the metric $h = \rho h_1 + (1-\rho) h_2$ defines a metric on $L^* \to \widetilde{X}$. Moreover, $h = h_1$ in $W' = U' \times \mathbb{P}_{n-1} \cap W$, so $\Theta_{L^*} = \Theta_{\sigma^* H}$ in W' and $\Theta_{L^*} \equiv 0$ in $\widetilde{X} - W'$.

- K_X. We give K_X an arbitrary Hermitian metric, and then by equation (11) we have
$$\Theta_{K_{\widetilde{X}}} = \Theta_{\pi^* K_X} + (n-1)\Theta_L$$

Therefore we have (Recall $F = E^{\otimes m}$)

$$\Theta_{K^*_{\widetilde{X}} \otimes \widetilde{F} \otimes (L^*)^2} = m\Theta_{\pi^* E} + (n+1)\Theta_{L^*} + \Theta_{\pi^* K^*_{\widetilde{X}}}$$

We note Θ_E is positive (Here we may omit the coefficient i compared with the true definition of "positive"), then we can choose $m_p \in \mathbb{N}$ sufficiently large s.t. $m > m_p$ implies that $\Theta_{K^*_{\widetilde{X}} \otimes \widetilde{F} \otimes (L^*)^2}$ is positive form. Here one may note on $U' \times \mathbb{P}_{n-1}$, $\Theta_{\pi^* E}$ depends only on the z-variable and $\Theta_{\sigma^* H}$ depends only on the t-variable, and their coefficient matrices are each positive definite, so is their sum restricts on W'. And X is compact, so $m\Theta_E + \Theta_{K^*_X} > 0$ for sufficiently large m. If one consider the points near p, then the positive will remain by continuity, so one just use a finite number of such neighborhoods to cover X and find a universal m_0 to satisfy the requirement. Hence by Kodaira vanishing theorem 6.4, one get $H^1(X, \mathcal{O}(L^{\otimes m}) \otimes \mathfrak{m}_p^2) = 0$ for $m > m_0$ and $p \in X$. One may similarly prove

$$\pi_p^* E^\mu \otimes L_p^* \otimes K^*_{\widetilde{X}} \text{ and } \pi^*_{p,q} E^\mu \otimes L^*_{p,q} \otimes K^*_{\widetilde{((\widetilde{X})_q)_p}}$$

are all positive line bundles for sufficiently large μ. Then there also have a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F}) \otimes \mathcal{J}_S^2) \arrow[r] \arrow[d, dashed, bend right, "\alpha"'] & \Gamma(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F})) \arrow[r] \arrow[d, dashed, bend right, "\beta"'] & \Gamma(\widetilde{X}, \widetilde{\mathcal{O}}(\widetilde{F}) \otimes \widetilde{\mathcal{O}}/\mathcal{J}_S^2) \\
0 \arrow[r] & \Gamma(X, \mathcal{O}(F) \otimes \mathfrak{m}_{pq}) \arrow[r] \arrow[u, "\pi^*_{(p,q)_1}"'] & \Gamma(X, \mathcal{O}(F)) \arrow[r] \arrow[u, "\pi^*_{p,q}"'] & \Gamma(X, \mathcal{O}(F) \otimes \mathcal{O}/\mathfrak{m}_{pq}) \arrow[u, "\pi^*_{(p,q)_2}"']
\end{tikzcd}$$

where in this time $S = \pi^{-1}_{p,q}(\{p\} \cup \{q\})$ and $\mathcal{J}_S \cong \widetilde{\mathcal{O}}(L^*_{pq})$, α and β are silimar as above. Quite similarly, one could prove the necessarily conditions for $\iota_{E^{\otimes m}}$ is an embedding, which deduces the Kodaira embedding theorem 6.9.

$\square$

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