泛函分析期末

G. Li

December 24, 2024

$\boxed{\textbf{3.12}}$ Let E be a Banach space and let $x_0 \in E$. Let $\varphi : E \to (-\infty, +\infty]$ be a convex l.s.c. function with $\varphi \not\equiv +\infty$.

1. Show that the following properties are equivalent:

   (A) $\exists R, \exists M < +\infty$ such that $\varphi(x) \leq M, \quad \forall x \in E$ with $\|x - x_0\| \leq R$

   (B) $\lim\limits_{\substack{f \in E^* \\ \|f\| \to \infty}} \{\varphi^*(f) - \langle f, x_0 \rangle\} = +\infty.$

2. Assuming (A) or (B) prove that

$$\inf_{f \in E^*} \{\varphi^*(f) - \langle f, x_0 \rangle\} \quad \text{is achieved.}$$

Hint: Use the weak $*$ topology $\sigma(E^*, E)$ or Theorem 1.12. What is the value of this inf?

Proof. (1) 对于 (A) $\Rightarrow$ (B) 方向. 注意到对于任意 $f \in E^*$,

$$\varphi^*(f) - \langle f, x_0 \rangle \geq \sup_{||x - x_0|| \leq R} \{\langle f, x - x_0 \rangle\} - M = R||f|| - M \to +\infty$$

反之对于 (B) $\Rightarrow$ (A) 方向. 通过平移我们总可假设 $\varphi^*(0) < \alpha < \infty$. 取 N 充分大使得 $\forall ||f|| \geq N$,

$$\varphi^*(f) - \langle f, x_0 \rangle \geq \alpha$$

由于 φ^* 于 E^* 上是凸且下半连续的（由定义直接推得）, 以及 $\varphi^{**} = \varphi$（Fenchel-Moreau 定理,Theorem 1.11）, 从而对于任意 $f \in E^*$, 由于 $||\frac{N}{||f||} f|| \geq N$, 故而

$$\frac{N}{||f||} \left(\varphi^*(f) - \langle f, x_0 \rangle - \varphi^*(0)\right) \geq \varphi^* \left(\frac{N}{||f||} f\right) - \langle \frac{N}{||f||} f, x_0 \rangle - \varphi^*(0) \geq \alpha - \varphi^*(0)$$

从而存在常数 $m > 0, n \in \mathbb{R}$ 使得 $\varphi^*(f) - \langle f, x_0 \rangle \geq m||f|| - n$ 对于任意的 $f \in E^*$. 故而对于 $||x - x_0|| \leq m$, 我们有

$$\varphi(x) = \varphi^{**}(x) = \sup_{f \in E^*} \{\langle f, x \rangle - \varphi^*(f)\} \leq \sup_{f \in E^*} \{\langle f, x - x_0 \rangle + n - m||f||\} \leq n$$

(2) $f \overset{\psi}{\mapsto} \varphi^*(f) - \langle f, x_0 \rangle$ 是凸的, 且在弱 $*$ 拓扑下是下半连续的. 为了说明后半句, 我们需要说明对于所有 $\alpha \in \mathbb{R}$,$S_\alpha = \{f : \varphi^*(f) > \alpha\}$ 在弱 $*$ 拓扑下是开集. 设 $f \in S_\alpha$. 那么存在

$\epsilon > 0$ 使得 $\varphi^*(f) \geq \alpha + 2\epsilon$, 这意味着存在 $x \in E$ 使得 $\langle f, x\rangle - \varphi(x) > \alpha + \epsilon$. 考虑

$$U = \left\{ g \in E^* : |\langle g, x\rangle - \langle f, x\rangle| < \frac{\epsilon}{2} \right\}$$

这是 f 在弱 $*$ 拓扑中的一个邻域. 那么对于所有 $g \in U$, 有

$$\varphi^*(g) \geq \langle g, x\rangle - \varphi(x) \geq \langle f, x\rangle - \varphi(x) - \epsilon/2 > \alpha + \epsilon/2 > \alpha$$

由 (B), $\{f \in E^* \mid \psi(f) \leq \lambda\}$ 对于任意 $\lambda \in \mathbb{R}$ 都是有界的, 由下半连续知其是闭的, 结合 Banach-Alaoglu-Bourbaki 定理（Theorem 3.16[1]）知其是弱 $*$ 紧的, 从而 $\inf_{E^*}$ 值能够达到[2]. 并且

$$\inf_{f \in E^*} \{\varphi^*(f) - \langle f, x_0\rangle\} = -\sup_{f \in E^*} \{-\varphi^*(f) + \langle f, x_0\rangle\} = -\varphi^{**}(x_0) = -\varphi(x_0)$$

□

3.28 Let E be a uniformly convex Banach space. Let F denote the (multivalued) duality map from E into E^*, see Remark 2 following Corollary 1.3 and also Exercise 1.1.
Prove that for every $f \in E^*$ there exists a unique $x \in E$ such that $f \in F(x)$.

Proof. 或者考察对偶映射 $F^* : E^* \longrightarrow E^{**}$, 由 Corollary 1.3[3]. $F^*(f)$ 非空, 从而结合 $E^{**} = E$ 知有这样的 x 使得 $f \in F(x)$.

关于唯一性, 由于 E 是一致凸的, 从而若 $f \in F(x_1), f \in F(x_2)$ 而 $x_1 \neq x_2$, 则 $||\frac{x_1+x_2}{2}|| < ||f||$, 故而

$$||f||^2 = \langle f, \frac{x_1 + x_2}{2}\rangle < ||f||^2$$

得到矛盾.

□

5.4 Let $K \subset H$ be a nonempty closed convex set. Let $f \in H$ and let $u = P_K f$. Prove that

$$|v - u|^2 \leq |v - f|^2 - |u - f|^2 \quad \forall v \in K.$$

Deduce that

$$|v - u| \leq |v - f| \quad \forall v \in K.$$

Give a geometric interpretation.

[1]Theorem 3.16: B_{E^*} 于 $\sigma(E^*, E)$ 下是紧的

[2]这来自于 Section 1.4 中关于下半连续函数的一些性质. 也可以直接看出来.

[3]Corollary 1.3：对于任一 $x_0 \in E$, 存在 $f_0 \in E^*$ 使得 $||f_0|| = ||x_0||$ 以及 $\langle f_0, x_0\rangle = ||x_0||^2$. 这是 Hahn-Banach 定理的直接推论.

Proof. 只需要注意到 $|v-u|^2 = |v-f|^2 - |u-f|^2 + 2(f-u, v-u)$. 几何解释也是直观的. $\square$

5.7: Projection onto a convex cone

Let $K \subset H$ be a convex cone with vertex at 0, i.e.,

$$0 \in K \quad \text{and} \quad \lambda u + \mu v \in K \quad \forall \lambda, \mu > 0, \quad \forall u, v \in K;$$

assume in addition that K is closed.

Given $f \in H$, prove that $u = P_K f$ is characterized by the following properties:

$$u \in K, \quad (f-u, v) \leq 0 \quad \forall v \in K \quad \text{and} \quad (f-u, u) = 0.$$

Proof. 一方面, 若 $u = P_K f$, 则有 $u \in K$ 以及对于 $\forall v \in K$, 都有 $(f-u, v-u) \leq 0$. 结合 K 的定义可知 u 满足题设的特征条件.

另一方面, 假设 v, u 都满足所有条件, 则由于

$$|v-u|^2 = (f-u+v-f, v-u) = (f-u, v-u) + (f-v, u-v) = (f-u, v) + (f-v, u) \leq 0$$

得到 $v = u$. $\square$

5.14

Let $a : H \times H \to \mathbb{R}$ be a bilinear continuous form such that $a(v,v) \geq 0 \quad \forall v \in H$.

Prove that the function $v \mapsto F(v) = a(v,v)$ is convex, of class C^1, and determine its differential.

Proof. F 的凸性来自于 $t(1-t)a(u-v, u-v) \geq 0$, 对于任意的 $u, v \in H$ 以及 $t \in [0,1]$.

考察线性算子 $A \in \mathcal{L}(H)$, $(Au, v) := a(u, v)$. 从而根据

$$F(u+h) - F(u) = (Au + A^*u, h) + a(h,h)$$

得到 $F'(u) = Au + A^*u$. $\square$

$\boxed{\textbf{5.20}}$ Assume that $S \in \mathcal{L}(H)$ satisfies $(Su, u) \geq 0$ for all $u \in H$.

1. Prove that $N(S) = R(S)^{\perp}$.

2. Prove that $I + tS$ is bijective for every $t > 0$.

3. Prove that

$$\lim_{t \to +\infty} (I + tS)^{-1} f = P_{N(S)} f \quad \forall f \in H.$$

Hint: Two methods are possible:
(a) Consider the cases $f \in N(S)$ and $f \in R(S)$.
(b) Use weak convergence.

Proof. (1) 若 $u \in N(S)$, 则 $(Sv - Su, v - u) = (Sv, v - u) \geq 0$, 对于任意 $v \in H$. 利用 $v \longrightarrow \mathbb{R}v$ 的技巧, 得到只能有 $(Sv, u) = 0$, 即 $u \in R(S)^{\perp}$. 反之, 如果 $u \in R(S)^{\perp}$, 则 $(Sv - Su, v) \geq 0$ 对于任意 $v \in H$ 成立, 同样只能得到 $(Su, v) = 0$, 即 $u \in N(S)$.

(2) 容易得到 $((I + tS)v, v) \geq |v|^2$, 从而为单射（因为是强制的）且 $R(I + TS)$ 是闭的（因为有 $|v| \leq |(I + tS)v|$）. 注意到 $R(I + tS)$ 是稠密的, 因为若 $((I + tS)u, v) = 0$ 对于任意 $u \in H$ 都成立, 则 $v = 0$, 这是因为由此可得 $\varphi(v) = 0$ 对于任意 $\varphi \in H^*$. 从而可知 $I + tS$ 为双射.[4]

(3) 我们采用提示中的两个方法.

方法（a）. 令 $u_t = (I = tS)^{-1} f$. 若 $f \in N(S)$, 则 $u_t = f$; 若 $f \in R(S)$, 则令 $f = Sv$, 得 $u_t + tSu_t - Sv = f - f = 0$, 故而 $(u_t, tu_t - v) \leq 0$, 因此 $|u_t| \leq \frac{1}{t}|v|$. 知此时 $u_t \longrightarrow 0$. 同样的, 利用 $(I + tS)$ 为连续线性算子可知, 对于 $f \in \overline{R(S)}$, 仍然有 $u_t \longrightarrow 0$. 对于一般的 $f \in E$, 令 $f = P_{N(S)} f + P_{\overline{R(S)}} f$, 可知结论.

方法（b）. 考虑 $u_t = (I + tS)^{-1} f$. 结合 $t(Su_t, u_t) \geq 0$ 得到 $|u_t| \leq |f|, |tSu_t| \leq |f|$. 由于 Hilbert 空间是一致凸的, 因此都是自反的（Proposition 5.1 以及 Milman-Pettis 定理, 即 Theorem 3.31[5]）, 由 Theorem 3.18[6] 可知 u_t 有弱收敛子列 $u_{t_k} \rightharpoonup u$ 并且 $Su = 0$. 由 (1) 可知 $t(Su_t, v) = (f - u_t, v) = 0$, 对于任意的 $v \in N(S)$, 取极限得 $(f - u, v) = 0$. 得到 $u = P_{N(S)} f$ 且由 $P_{N(S)}(f)$ 的唯一性得到 $u_t \rightharpoonup u$. 注意到 $(f - u_t, u_t) = t(Su_t, u_t) \geq 0$, 从而

$$\limsup_{t \to \infty} |u_t|^2 \leq (f, u) = |u|^2$$

从而知 $u_t \to u$, 即为强收敛.

□

[4]其实这就是 Lax-Milgram 定理（Corollary 5.8）的另一形式, 见 Proposition 11.29 或者 Chapter 5 的 Remark 8.

[5]Theorem 3.31: 一致凸的 Banach 空间都是自反的.

[6]Theorem 3.18：自反的 Banach 空间中的有界序列都有弱收敛的子列.

$\boxed{\textbf{6.7}}$ Let E and F be two Banach spaces, and let $T \in \mathcal{L}(E, F)$. Consider the following properties:

(P) For every weakly convergent sequence (u_n) in E, if $u_n \rightharpoonup u$, then $Tu_n \to Tu$ strongly in F.

(Q) T is continuous from E equipped with the weak topology $\sigma(E, E^*)$ into F equipped with the strong topology.

1. Prove that
$$(Q) \iff T \text{ is a finite-rank operator.}$$
2. Prove that $T \in \mathcal{K}(E, F) \implies (P)$.
3. Assume that either $E = \ell^1$ or $F = \ell^1$. Prove that every operator $T \in \mathcal{L}(E, F)$ satisfies (P).

Hint: Use a result of Problem 8.

In what follows we assume that E is reflexive.

4. Prove that $T \in \mathcal{K}(E, F) \iff (P)$.
5. Deduce that every operator $T \in \mathcal{L}(E, \ell^1)$ is compact.
6. Prove that every operator $T \in \mathcal{L}(c_0, E)$ is compact.

Hint: Consider the adjoint operator T^*.

Proof. (1) 假设 (Q) 条件成立, 则对于 $\forall \varepsilon > 0$, 于 E 中存在弱拓扑意义下的开邻域 $0 \in V$ 使得 $x \in V \implies ||Tx|| < \varepsilon$. 总可假设 V 形如 $\{x \in E \mid \langle f_i, x\rangle < \delta, \text{对于所有 } i\}$ 对于 $f_1, \cdots, f_n \in E^*$ 及某个 $\delta > 0$. 考察

$$M = \{x \in E \mid \langle f_i, x\rangle = 0\}$$

由于 $T(M) \subset B_\varepsilon$ 而 M 为一子空间, 故 $M \subset \ker T$. 但 M 有有限的余维数, 可令 $E = M + N$ 且 N 为有限维空间, 从而 $R(T) \subset T(N)$ 也为有限维的.

反之假如 T 是有限秩的, 则由 Theorem 3.10[7]可知 T 于 $(E, \sigma(E, E^*)) \longrightarrow (F, \sigma(F, F^*))$ 连续, 从而于 $(E, \sigma(E, E^*)) \longrightarrow (R(T), \sigma(R(T), R(T)^*))$ 连续. 但在有限维空间 $R(T)$ 上, 弱拓扑和强拓扑等价, 故而 T 于 $(E, \sigma(E, E^*)) \longrightarrow (R(T), \text{norm})$ 连续, 当然也成立 (Q).

[7]Theorem 3.10: 对于两 Banach 空间 E, F 之间的线性算子 $T : E \longrightarrow F$, 其连续当且仅当 $\sigma(E, E^*) \longrightarrow \sigma(F, F^*)$ 下连续. 一边是明显的, 另一边来自于闭图像定理以及弱闭的集合总是强闭的. 特别地, 当集合为凸集时, 可以利用 Hahn-Banach 定理证明强闭的集合也是弱闭的, 此即 Theorem 3.7 陈述的内容.

(2) 若 $u_n \rightharpoonup u$ 于 E, 则 $Tu_n \rightharpoonup Tu$ 于 F, 这是来自于 Theorem 3.10. 由于 $\{u_n\}$ 有界（共鸣定理）, 故 $T(u_n)$ 有紧闭包. 从而 $Tu_n \longrightarrow Tu$, 否则考察其强收敛子列 Tu_{n_k}, 再根据强收敛必定弱收敛以及弱收敛极限的唯一性可知可知矛盾.（后半部分即为 Exerxise 3.5）

(3) Problem 8 的结论告诉我们, 若 $(x^n) \subset \ell^1$ 使得对于任一 $f \in \ell^\infty$, 都有 $\langle f, x^n \rangle$ 收敛, 则 (x^n) 于 ℓ^1 中强收敛于某个 x. 从而结合 $u_n \rightharpoonup u$ 则有 $Tu_n \rightharpoonup Tu$ 的事实, 容易得到 $E = \ell^1$ 或者 $F = \ell^1$ 时, (P) 成立.

(4) 我们只需要在 E 自反的条件下证明 (P) $\Longrightarrow$ $T \in \mathcal{K}(E, F)$. 由 Theorem 3.18[8] , 此时若 x_n 有界, 则有子列 $x_{n_i} \rightharpoonup x$, 根据 (P) 我们有 $Tx_{n_i} \longrightarrow Tx$. 故而 T 为紧算子.

(5) 这是 (3) 和 (4) 的直接推论.

(6) 回忆 $c_0 = \{x \in \ell^\infty \mid \lim x = 0\}$, 而根据 Proppposition 11.19,$(c_0)^* = \ell^1$. 这启发我们考虑 $T^* \in \mathcal{L}(E^*, (c_0)^* = \ell^1)$. 由于 E^* 是自反的 (Corollary 3.21[9]), 从而由 (5) 知 T^* 是紧算子, 结合 Schauder 定理（Theorem 6.4）[10]知 T 是紧算子.

$\square$

附录

0.1 一些常用结论

不特殊说明,E 都表示一 Banach 空间, H 都表示一 Hilbert 空间.

Theorem 3.16 (Banach-Alaoglu-Bourbaki) B_{E^*} 于 $\sigma(E^*, E)$ 下是紧的.

Theorem 3.17 (Kakutani) E 自反 $\iff$ B_E 于 $\sigma(E, E^*)$ 下是紧的.

Lemma 3.4 (Goldstine) 典范嵌入 $J(B_E)$ 在 $(B_{E^{**}}, \sigma(E^{**}, E^*))$ 中稠密.

Theorem 3.18/3.19 (Eberlein-Smulian) E 自反 $\iff$ 其有界序列都有弱收敛子列.

Corollary 3.21 E 自反 $\iff E^*$ 自反.

Theorem 3.26 若 E^* 可分, 则 E 也可分.

Corollary 3.27 E 可分且自反 $\iff E^*$ 可分且自反.

Theorem 3.28 E 可分 $\iff B_{E^*}$ 对 $\sigma(E^*, E)$ 可度量化.

Theorem 3.29 若 E^* 可分, 则 E 可分 $\iff B_E$ 对 $\sigma(E, E^*)$ 可度量化.

Theorem 3.31 (Milman-Pettis) 一致凸的 Banach 空间都是自反的.

Theorem 5.5 (Riesz-Fréchet representation theorem) $\forall \varphi \in H^*$, 都存在唯一的 $f \in H$ 使得 $\langle \varphi, u \rangle = (f, u)$ 对于所有 $u \in H$ 都成立. 此外还有 $|f| = ||\varphi||_{H^*}$.

[8]Theorem 3.18 和 Theorem 3.19 统称为 Eberlein-Smulian 定理, 陈述了以下结论：Banach 空间 E 是自反 $\iff$ 其有界序列都有弱收敛的子列

[9]Corollary 3.21: 对于 Banach 空间 E, E 自反当且仅当 E^* 自反

[10]Theorem 6.4: $T \in \mathcal{K}(E, F) \iff T^* \in \mathcal{K}(F^*, E^*)$.

Corollary 5.8 (Lax-Milgram) 若 $a(u,v)$ 是 H 上连续且强制的双线性映射, 则任给 $\varphi \in H^*$, 都存在唯一的 $u \in H$ 使得 $a(u,v) = \langle \varphi, v \rangle$ 对于任意的 $v \in H$. 此外如果 a 是对称的, 则 u 被以下性质刻画：

$$u \in H, \quad \frac{1}{2}a(u,u) - \langle \varphi, u \rangle = \min_{v \in H} \left\{ \frac{1}{2}a(v,v) - \langle \varphi, v \rangle \right\}$$

From Section 11.4

	Reflexive	Separable	Dual Space
ℓ^p with $1 < p < \infty$	YES	YES	$\ell^{p'}$
ℓ^1	NO	YES	ℓ^∞
c_0	NO	YES	ℓ^1
c	NO	YES	$\ell^1 \times \mathbb{R}$
ℓ^∞	NO	NO	Strictly bigger than ℓ^1

0.2 PROBLEM 8

Weak convergence in ℓ^1. Schur's theorem.

Let $E = \ell^1$, so that $E^* = \ell^\infty$ (see Section 11.3). Given $x \in E$ write

$$x = (x_1, x_2, \dots, x_i, \dots) \quad \text{and} \quad \|x\|_1 = \sum_{i=1}^{\infty} |x_i|,$$

and given $f \in E^*$ write

$$f = (f_1, f_2, \dots, f_i, \dots) \quad \text{and} \quad \|f\|_\infty = \sup_i |f_i|.$$

Let (x^n) be a sequence in E such that $x^n \to 0$ weakly $\sigma(E, E^*)$. Our goal is to show that $\|x^n\|_1 \to 0$.

1. Given $f, g \in B_{E^*}$ (i.e., $\|f\|_\infty \leq 1$ and $\|g\|_\infty \leq 1$) set

$$d(f,g) = \sum_{i=1}^{\infty} \frac{1}{2^i} |f_i - g_i|.$$

   Check that d is a metric on B_{E^*} and that B_{E^*} is compact for the corresponding topology.

2. Given $\varepsilon > 0$ set

$$F_k = \{ f \in B_{E^*} : |\langle f, x^n \rangle| \leq \varepsilon \quad \forall n \geq k \}.$$

Prove that there exist some $f^0 \in B_{E^*}$, a constant $\rho > 0$, and an integer k_0 such that

$$[f \in B_{E^*} \text{ and } d(f, f^0) < \rho] \Rightarrow [f \in F_{k_0}].$$

[**Hint**: Use Baire category theorem.]

3. Fix an integer N such that $(1/2^{N-1}) < \rho$. Prove that

$$\|x^n\|_1 \leq \varepsilon + 2\sum_{i=1}^{N} |x_i^n| \quad \forall n \geq k_0.$$

4. Conclude.

5. Using a similar method prove that if (x^n) is a sequence in ℓ^1 such that for every $f \in \ell^\infty$ the sequence $(\langle f, x^n \rangle)$ converges to some limit, then (x^n) converges to a limit strongly in ℓ^1.

6. Consider $E = L^1(0,1)$, so that $E^* = L^\infty(0,1)$. Construct a sequence (u^n) in E such that $u^n \to 0$ weakly $\sigma(E, E^*)$ and such that $\|u^n\|_1 = 1$ $\forall n$.

Proof. **以下证明照抄至 Brezis 原书. 证明是直接的, 我们只需要第 5 问的结果.**

1. Let $\mathcal{T}$ be the topology corresponding to the metric d. Since B_{E^*} equipped with the topology $\sigma(E^*, E)$ is compact, it suffices to check that the canonical injection

$$(B_{E^*}, \sigma(E^*, E)) \to (B_{E^*}, \mathcal{T})$$

is continuous. This amounts to proving that for every $f_0 \in B_{E^*}$ and for every $\varepsilon > 0$ there exists a neighborhood $V(f^0)$ of f^0 for $\sigma(E^*, E)$ such that

$$V(f^0) \cap B_{E^*} \subset \{f \in B_{E^*}; d(f, f^0) < \varepsilon\}.$$

Let (e^i) be the canonical basis of ℓ^1. Choose

$$V(f^0) = \{f \in E^*; |\langle f - f^0, e^i \rangle| < \delta \ \forall i = 1, 2, \ldots, n\}$$

with $\delta + (1/2^{n-1}) < \varepsilon$.

2. Note that (B_{E^*}, d) is a complete metric space (since it is compact). The sets F_k are closed for the topology $\mathcal{T}$, and, moreover, $\bigcup_{k=1}^\infty F_k = B_{E^*}$ (since $\langle f, x^n \rangle \to 0$ for every $f \in E^*$). Baire's theorem says that there exists some integer k_0 such that $\mathrm{Int}(F_{k_0}) \neq \emptyset$.

3. Write $f^0 = (f_1^0, f_2^0, \ldots, f_i^0, \ldots)$ and consider the elements $f \in B_{E^*}$ of the form

$$f = (f_1^0, f_2^0, \ldots, f_N^0, \pm 1, \pm 1, \pm 1, \ldots),$$

so that $d(f, f^0) \leq \sum_{i=N+1}^{\infty} \frac{2}{2^i} < \rho$. Such f's belong to F_{k_0} and one has, for every $n \geq k_0$,

$$|\langle f, x^n \rangle| = \left| \sum_{i=1}^{\infty} f_i x_i^n \right| = \left| \sum_{i=1}^{N} f_i^0 x_i^n + \sum_{i=N+1}^{\infty} (\pm x_i^n) \right| \leq \varepsilon.$$

It follows that

$$\sum_{i=N+1}^{\infty} |x_i^n| \leq \varepsilon + \sum_{i=1}^{N} |f_i^0||x_i^n| \leq \varepsilon + \sum_{i=1}^{N} |x_i^n|,$$

and thus $\sum_{i=1}^{\infty} |x_i^n| \leq \varepsilon + 2 \sum_{i=1}^{N} |x_i^n| \quad \forall n \geq k_0$.

4. The conclusion is clear, since for each fixed i the sequence x_i^n tends to 0 as $n \to \infty$.

5. Given $\varepsilon > 0$, set

$$F_k = \{f \in B_{E^*}; |\langle f, x^n - x^m \rangle| \leq \varepsilon \ \forall m, n \geq k\}.$$

By the same method as above one finds integers k_0 and N such that

$$\|x^n - x^m\|_1 \leq \varepsilon + 2 \sum_{i=1}^{N} |x_i^n - x_i^m| \quad \forall m, n \geq k_0.$$

It follows that (x^n) is a Cauchy sequence in ℓ^1.

6. See Exercises 4.18 and 4.19. □