

Supplementary Notes: B-Splines

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Note: this is pretty much taken shamelessly from Appendix C of [Tibshirani \(2022\)](#).

To parametrize the space of k^{th} degree splines with knots at t_1, \dots, t_r , a simple choice is the *truncated power basis*, g_1, \dots, g_{r+k+1} , which recall is defined as

$$\begin{aligned} g_j(x) &= \frac{1}{(j-1)!} x^{j-1}, \quad j = 1, \dots, k+1, \\ g_{j+k+1}(x) &= \frac{1}{k!} (x - t_j)_+^k, \quad j = 1, \dots, r, \end{aligned} \tag{1}$$

Here $x_+ = \max\{x, 0\}$ denotes the positive part of x .

Though the truncated power basis (1) is the simplest basis for splines, the B-spline basis is just as fundamental, and it was “there at the very beginning”, appearing in Schoenberg’s original paper on splines ([Schoenberg, 1946](#)). Here we are quoting [de Boor \(1976\)](#), who gives a masterful survey of the history and properties of B-splines (and points out that the name “B-spline” is derived from Schoenberg’s use of the term “basic spline”, to further advocate for the idea that B-splines can be seen as *the* basis for splines).

Peano representation. There are different ways to construct B-splines; here we cover a construction based on what is called the *Peano representation* for B-splines. If f is a $k+1$ times differentiable function f on an interval $[a, b]$ (and its $(k+1)^{\text{st}}$ derivative is integrable), then by Taylor expansion

$$f(z) = \sum_{i=0}^k \frac{1}{i!} (D^i f)(a)(z-a)^i + \int_a^z \frac{1}{k!} (D^{k+1} f)(x)(z-x)^k dx.$$

Note that we can rewrite this as

$$f(z) = \sum_{i=0}^k \frac{1}{i!} (D^i f)(a)(z-a)^i + \int_a^b \frac{1}{k!} (D^{k+1} f)(x)(z-x)_+^k dx. \tag{2}$$

Next we will take a particular divided difference on both sides of the above display. First we recall the definition of a divided difference: with respect to two centers z_1, z_2 , it is defined by

$$f[z_1, z_2] = \frac{f(z_2) - f(z_1)}{z_2 - z_1},$$

and more generally, with respect to $k+1$ centers z_1, \dots, z_{k+1} , for an integer $k \geq 1$, it is defined by

$$f[z_1, \dots, z_{k+1}] = \frac{f[z_2, \dots, z_{k+1}] - f[z_1, \dots, z_k]}{z_{k+1} - z_1}.$$

(For this to reduce to the definition with two centers, when $k = 1$, we take by convention $f[z] = f(z)$.)

Returning back to our main thread, we take a divided difference in the Taylor expansion (2) with respect to arbitrary centers $z_1, \dots, z_{k+2} \in [a, b]$, where we assume without a loss of generality that $z_1 < \dots < z_{k+2}$, and then we use linearity to exchange divided differencing with integration, yielding

$$k! \cdot f[z_1, \dots, z_{k+2}] = \int_a^b (D^{k+1} f)(x) \underbrace{(\cdot - x)_+^k [z_1, \dots, z_{k+2}]}_{P^k(x; z_{1:(k+2)})} dx, \tag{3}$$

where we have also used the fact that a $(k+1)^{\text{st}}$ order divided difference (with respect to any $k+2$ centers) of a k^{th} degree polynomial is zero, and we multiplied both sides by $k!$. To be clear, the notation $(\cdot - x)_+^k[z_1, \dots, z_{k+2}]$ means that we are taking the divided difference of the function $z \mapsto (z - x)_+^k$ with respect to centers z_1, \dots, z_{k+2} .

B-spline definition. The result in (3) shows that the $(k+1)^{\text{st}}$ divided difference of any (smooth enough) function f can be written as a weighted average of its $(k+1)^{\text{st}}$ derivative, in a local neighborhood around the corresponding centers, where the weighting is given by a universal kernel $P^k(\cdot; z_{1:(k+2)})$ (that does not depend on f), which is called the *Peano kernel* formulation for the B-spline; to be explicit, this is

$$P^k(x; z_{1:(k+2)}) = (\cdot - x)_+^k[z_1, \dots, z_{k+2}].$$

Since

$$(z - x)_+^k - (-1)^{k+1}(x - z)_+^k = (z - x)^k,$$

and any $(k+1)^{\text{st}}$ order divided difference of the k^{th} degree polynomial $z \mapsto (z - x)^k$ is zero, we can rewrite the second-to-last display as

$$P^k(x; z_{1:(k+2)}) = (-1)^{k+1}(x - \cdot)_+^k[z_1, \dots, z_{k+2}].$$

The function $P^k(\cdot; z_{1:(k+2)})$ is called a k^{th} degree *B-spline* with knots $z_{1:(k+2)}$. It is a linear combination of k^{th} degree truncated power functions and is hence indeed a k^{th} degree spline.

It is often more convenient to deal with the *normalized B-spline*:

$$M^k(x; z_{1:(k+2)}) = (-1)^{k+1}(z_{k+2} - z_1)(x - \cdot)_+^k[z_1, \dots, z_{k+2}].$$

It is easy to show that

$$M^k(\cdot; z_{1:(k+2)}) \text{ is supported on } [z_1, z_{k+2}], \text{ and } M^k(x; z_{1:(k+2)}) > 0 \text{ for } x \in (z_1, z_{k+2}).$$

To see the support result, note that for $x > z_{k+2}$, we are taking a divided difference of all zeros, which of course zero, and for $x < z_1$, we are taking a $(k+1)^{\text{st}}$ order divided difference of a polynomial of degree k , which is again zero. To see the positivity result, we can, for example, appeal to induction on k and the recursion to come later.

B-spline basis. To build a local basis the space of k^{th} degree splines with knots t_1, \dots, t_r , which we assume lie in the interior of $[a, b]$, we first define boundary knots

$$t_{-k} < \dots < t_{-1} < t_0 = a, \quad \text{and} \quad b = t_{r+1} < t_{r+2} < \dots < t_{r+k+1}.$$

(Any such values for t_{-k}, \dots, t_0 and $t_{r+1}, \dots, t_{r+k+1}$ will suffice to produce a basis; in fact, setting $t_{-k} = \dots = t_0$ and $t_{r+1} = \dots = t_{r+k+1}$ would suffice, though this would require us to understand how to properly interpret divided differences with repeated centers; as in Definition 2.49 of [Schumaker \(2007\)](#).) We then define the normalized B-spline basis M_j^k , $j = 1, \dots, r+k+1$

$$M_j^k = M^k(\cdot; t_{(j-k-1):j}) \Big|_{[a,b]}, \quad j = 1, \dots, r+k+1.$$

It is clear that each M_j^k , $j = 1, \dots, r+k+1$ is a k^{th} degree spline with knots in t_1, \dots, t_r ; hence to verify that they are a basis for this space we only need to show their linear independence, which is straightforward using the structure of their supports.

For concreteness, we note that the 0^{th} degree normalized B-splines basis are simply indicator functions,

$$M_j^0 = 1_{I_j}, \quad j = 1, \dots, r+1.$$

Here $I_0 = [t_0, t_1]$ and $I_i = (t_i, t_{i+1}]$, $i = 1, \dots, r$, and we use $t_{r+1} = b$ for notational convenience. We note that this particular choice for the half-open intervals (left- versus right-side open) is arbitrary, but consistent with our definition of the truncated power basis (1) when $k = 0$.

Figure 1 shows example normalized B-splines of degrees 0 through 3.

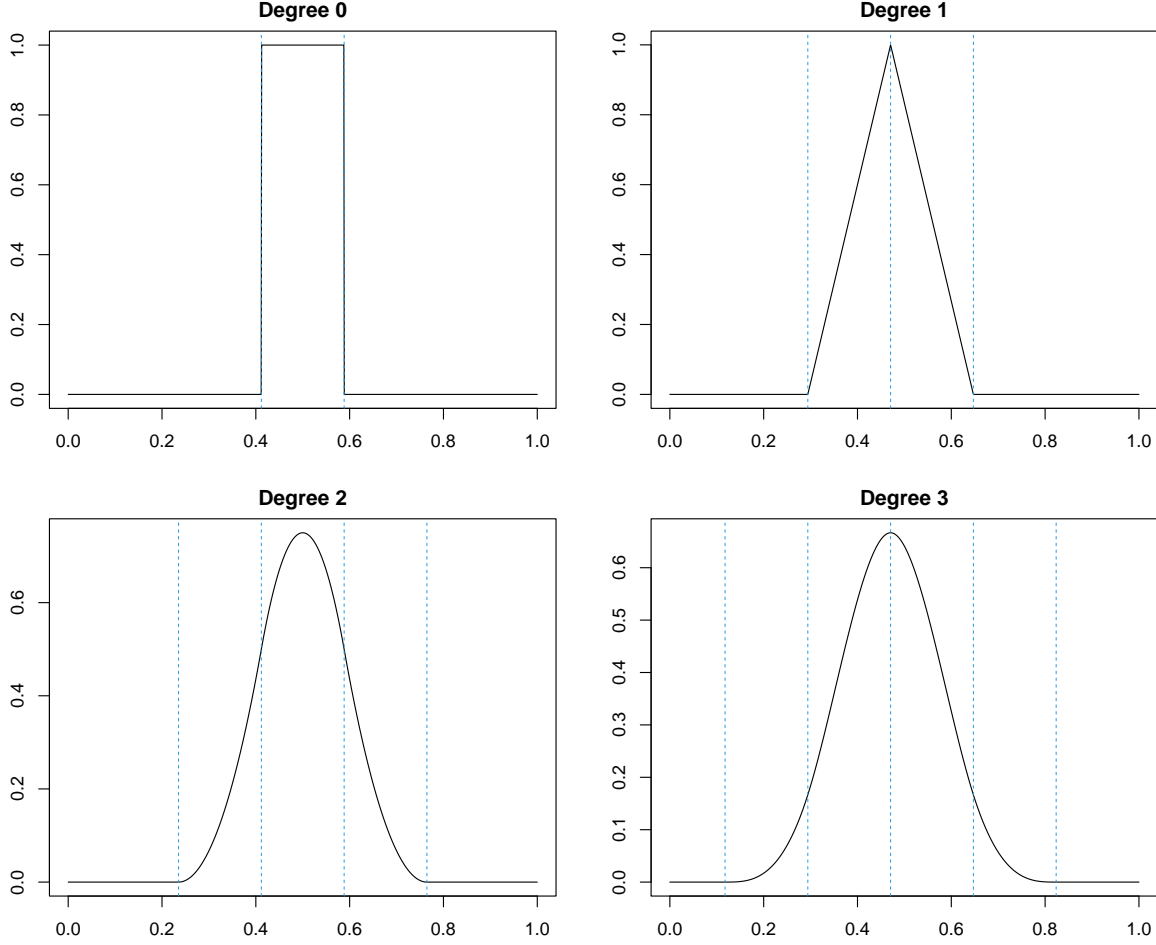


Figure 1: *B-splines of degrees 0 through 3. The knot points are marked by dashed blue vertical lines.*

Recursive formulation. B-splines satisfy a recursion relation that can be seen directly from the recursive nature of divided differences: for any $k \geq 1$ and centers $z_1 < \dots < z_{k+2}$,

$$\begin{aligned} (x - \cdot)_+^k[z_1, \dots, z_{k+2}] &= \frac{(x - \cdot)_+^k[z_2, \dots, z_{k+2}] - (x - \cdot)_+^k[z_1, \dots, z_{k+1}]}{z_{k+2} - z_1} \\ &= \frac{(x - z_{k+2})(x - \cdot)_+^{k-1}[z_2, \dots, z_{k+2}] - (x - z_1)(x - \cdot)_+^{k-1}[z_1, \dots, z_{k+1}]}{z_{k+2} - z_1}, \end{aligned}$$

where in the second line we applied the Leibniz rule for divided differences

$$fg[z_1, \dots, z_{k+1}] = \sum_{i=1}^{k+1} f[z_1, \dots, z_i]g[z_i, \dots, z_{k+1}]$$

to conclude that

$$\begin{aligned} (x - \cdot)_+^k[z_1, \dots, z_{k+1}] &= (x - z_1) \cdot (x - \cdot)_+^{k-1}[z_1, \dots, z_{k+1}] \\ (x - \cdot)_+^k[z_2, \dots, z_{k+2}] &= (x - \cdot)_+^{k-1}[z_2, \dots, z_{k+2}] \cdot (x - z_{k+2}). \end{aligned}$$

Translating the above recursion over to normalized B-splines, we get

$$M^k(x; z_{1:(k+2)}) = \frac{x - z_1}{z_{k+1} - z_1} \cdot M^{k-1}(x; z_{1:(k+1)}) + \frac{z_{k+2} - x}{z_{k+2} - z_2} \cdot M^{k-1}(x; z_{2:(k+2)}),$$

which means that for the normalized basis,

$$M_j^k(x) = \frac{x - t_{j-k-1}}{t_{j-1} - t_{j-k-1}} \cdot M_{j-1}^{k-1}(x) + \frac{t_j - x}{t_j - t_{j-k}} \cdot M_j^{k-1}(x), \quad j = 1, \dots, r + k + 1.$$

Above, we naturally interpret $M_0^{k-1} = M^{k-1}(\cdot; t_{-k:0})|_{[a,b]}$ and $M_{r+k+1}^{k-1} = M^{k-1}(\cdot; t_{(r+1):(r+k+1)})|_{[a,b]}$.

The above recursions are very important, both for verifying numerous properties of B-splines and for computational purposes. In fact, many authors prefer to use recursion to define a B-spline basis in the first place.

References

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