7.1

2. (14). $\int_3^{+\infty} \frac{1}{x\ln x(\ln\ln x)^k}dx = \int_{\ln 3}^{+\infty}\frac{1}{t(\ln t)^k}dt = \int_{\ln\ln 3}^{+\infty}\frac{1}{t^k}dt \quad \overset{k>1}{<+\infty}$

$k\leq 1$ 发散

一类级数：$\sum_{n=2}^{\infty}\frac{1}{n^\alpha(\ln n)^\beta}$ ~~★~~

$\alpha>1$ 时，取 $\alpha_0\in\mathbb{R}$，s.t. $\alpha>\alpha_0>1$，此时 $\sum_{n=2}^{\infty}\frac{1}{n^{\alpha_0}}$ 收敛

$\lim_{n\to\infty}\frac{\frac{1}{n^\alpha(\ln n)^\beta}}{\frac{1}{n^{\alpha_0}}} = \lim_{n\to\infty}\frac{1}{n^{\alpha-\alpha_0}(\ln n)^\beta} = 0 \Rightarrow \sum_{n=2}^{\infty}\frac{1}{n^\alpha(\ln n)^\beta}<+\infty$

$\alpha=1$ 时，$\beta>1$，~~$\ast<+\infty$~~ $\sum_{n=2}^{\infty}\frac{1}{n^\alpha(\ln n)^\beta}<+\infty$

$\alpha>1$ 时，取 $\alpha_1\in\mathbb{R}$，s.t. $\alpha<\alpha_1<1$．$\sum_{n=2}^{\infty}\frac{1}{n^{\alpha_1}}$ 发散

$\lim_{n\to\infty}\frac{\frac{1}{n^\alpha(\ln n)^\beta}}{\frac{1}{n^{\alpha_1}}} = \lim_{n\to\infty}\frac{n^{\alpha_1-\alpha}}{(\ln n)^\beta} = +\infty \Rightarrow \sum_{n=2}^{\infty}\frac{1}{n^\alpha(\ln n)^\beta} = +\infty$.

(15). $\sqrt[n]{a_n} = (\cos\frac{1}{n})^{n^2} \sim (1-\frac{1}{2n^2})^{n^2} = (1-\frac{1}{2n^2})^{-2n^2\cdot(-\frac{1}{2})} = \frac{1}{\sqrt{e}}<1$

(16). $\sqrt[n]{a_n} = \frac{an}{n+1} \to a$

$a=1$ 时，$a_n = (\frac{n}{n+1})^n = (1-\frac{1}{n+1})^n \to \frac{1}{e} \neq 0$

3. $a_n=(-1)^n$

若 $a_n>0$，记 $S_n=\sum_{k=1}^n a_k$，$T_n=\sum_{k=1}^n(a_k+a_{k+1})$

$S_n<T_n \Rightarrow S_n<+\infty$.

4. (1). $\lim_{n\to\infty}\frac{a_n}{\frac{1}{n}} = \lim_{n\to\infty} na_n = a\neq 0 \Rightarrow \sum a_n$ 与 $\sum\frac{1}{n}$ 同敛散.

(2). $\not\Rightarrow$：$a_n=(-1)^n\frac{1}{n}$ $\quad\not\Leftarrow$：$a_n=\frac{1}{n\ln n}$

但 $a_n>0$ 并单减时，反之正确：

由 Cauchy $\Rightarrow \lim_{n\to\infty}\sum_{k=n+1}^{2n}a_k=0$，于是 $na_{2n}\leq\sum_{k=n+1}^{2n}a_k\to 0$

$\Rightarrow \lim_{n\to\infty}2na_{2n}=0$

再由 $(n+1)a_{2n+1}\leq\sum_{k=n+1}^{2n+1}a_k\to 0 \Rightarrow \lim_{n\to\infty}(2n+1)a_{2n+1}\to 0$

$\Rightarrow \lim_{n\to\infty}na_n=0$.

(3). $\sum_{k=1}^n a_k = \underbrace{\sum_{k=1}^n k(a_k-a_{k+1})}_{<+\infty} + \underbrace{na_{n+1}}_{\to a} < +\infty$.

5. $\lim\limits_{n\to\infty}\frac{a_n^2}{a_n}=\lim\limits_{n\to\infty}a_n=0 \Rightarrow \sum\limits_{n=1}^{\infty}a_n^2<+\infty$

反之不成立：$a_n=\frac{1}{n}$.

6. $a_{n+1}<a_n+S_{n+1}-S_{n-1} \Rightarrow a_{n+1}-S_n<a_n-S_{n-1}$

$\{a_{n+1}-S_n\}_{n\geq1}\downarrow$ 且 $a_{n+1}-S_n>-S \Rightarrow \lim\limits_{n\to\infty}(a_{n+1}-S_n)$ 存在 $\Rightarrow \lim\limits_{n\to\infty}a_n$ √

7. $|a_nb_n|<\frac{a_n^2+b_n^2}{2}$　　$(a_n+b_n)^2<2(a_n^2+b_n^2)$

取 $b_n=\frac{1}{n}$

8. (1). $\sum\frac{1}{n^p}<+\infty \Rightarrow \lim\limits_{n\to\infty}\sum\limits_{k=n+1}^{2n}\frac{1}{k^p}=0$

(2). $\sum\frac{1}{p^n}<+\infty \Rightarrow$ √

9. $a_n\downarrow a$，若 $a=0$，则 $\sum\limits_{n=1}^{\infty}(-1)^na_n<+\infty$，矛盾 $\Rightarrow a>0$

再比 $\left(\frac{1}{a_n+1}\right)^n<\left(\frac{1}{a+1}\right)^n \Rightarrow \sum\frac{1}{(a_n+1)^n}<+\infty$.

10. $\lim\limits_{n\to\infty}\frac{S_n}{n}\overset{\text{Stolz}}{=}\lim\limits_{n\to\infty}a_n=0$

$\frac{S_{n+1}}{n+1}-\frac{S_n}{n}=\frac{na_{n+1}-(a_1+\cdots+a_n)}{n(n+1)}<0$

$\overset{\text{Dirichlet}}{\Rightarrow}\sum\cdots<+\infty$.

11. $|a_n+b_n|\leq|a_n|+|b_n|$

12. (10). $\left(1-\cos\frac{1}{n}\right)^p\sim\left(\frac{1}{2n^2}\right)^p=\frac{1}{2^pn^{2p}}$

$p>\frac{1}{2}$ 绝对收敛

$0<p\leq\frac{1}{2}$ 条件收敛

$p\leq0$ 发散

13. $S_n=S_n^+-S_n^- \Rightarrow \frac{S_n^+}{S_n^-}=\frac{\overset{\to\text{有限}}{S_n}}{\underset{\to\infty}{S_n^-}}+1\to1$

14. $\frac{|a_{n+1}|}{b_{n+1}}<\frac{|a_n|}{b_n} \Rightarrow \frac{|a_n|}{b_n}\to c$

$\Rightarrow\sum|a_n|<+\infty$

15. $\sum\frac{\sin n}{\sqrt{n}}<+\infty$

$\left(1+\frac{1}{n}\right)^n\uparrow e$

$\overset{\text{Abel}}{\Longrightarrow}\sum\frac{\sin n}{\sqrt{n}}\left(1+\frac{1}{n}\right)^n<+\infty$.

7.2

2. (1). $\sqrt[n]{a_n} = \sqrt[n]{n}\, e^{-x} \to e^{-x}$

$x>0$ 时, $\sum a_n < +\infty$

$x=0$, $\sum n = +\infty$

(6). $\sqrt[n]{a_n} \sim \sqrt[n]{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}\, \frac{x}{n} \to \frac{x}{e}$

$x\in(-e, e)$, $\sum a_n$ ~~发散~~ 收敛.

$x\leq -e$ 或 $x\geq e$ 时, $a_n \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(\frac{x}{n}\right)^n \not\to 0$.

(8). $x\in(-1,1)$, $\sqrt[n]{|a_n|} = \frac{|x|}{\sqrt[n]{1-x^n}} \to |x|$ 收敛

$x<-1$ or $x>1$, $a_n \not\to 0$.

3. $|u_n(x)+\cdots+u_{n+p}(x)| \leq \frac{1}{n} \to 0$

若有, $a_n \geq \frac{1}{n}$, $\sum \frac{1}{n} = +\infty$, 矛盾.

4. (3). $\sup |u_n(x)| = 1$

(4). $f_n(x) = x^2 e^{-nx}$, $f_n'(x) = \frac{2x - nx^2}{e^{nx}} = 0$ 时, $x = \frac{2}{n}$

$\Rightarrow f_n(x) \leq f_n\left(\frac{2}{n}\right) = \frac{4}{n^2}e^{2}$, $\sum \frac{4}{n^2} < +\infty \Rightarrow$ 一致收敛

(6). ①. $\sum_{k=n}^{2n} \frac{1}{k^x} > \frac{n}{(2n)^x}$

$\sup | \quad 1 \geq \frac{1}{2}$, 不一致收敛

②. 引理: $\sum_{n=1}^{\infty} u_n(x)$ 在 (a,b) 有定义, $u_n(x)\in C(a,b)$

$\lim_{x\to a^-} u_n(x)$ 存在, 若 $\sum_{n=1}^{\infty} u_n(a+)$ 发散, 则非一致收敛.

Proof: 只需考察

$\sup |u_n(x)+\cdots+u_{n+p}(x)| \geq \sup |u_n(a+)+\cdots+u_{n+p}(a+)| \not\to 0$.

这里 $\sum \frac{1}{n} = +\infty \Rightarrow$ 非一致收敛.

5. $\sum_{n=1}^{\infty} a_n$ 收敛

$\left|\frac{1}{e^{nx}}\right| \le 1$, $\forall x \ge 0$. $\{\frac{1}{e^{nx}}\}_{n\ge1}$ 单调 $\Big\} \xrightarrow{Abel} \sum \frac{a_n}{e^{nx}}$ 一致收敛.

6. ~~$\sum_{n=1}^{\infty} \frac{1}{n^x}$~~

$\forall \delta > 0$, 对 $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ 在 $[1+\delta, +\infty)$ 一致收敛 $\left(\frac{1}{n^x} \le \frac{1}{n^{1+\delta}} \right)$

因此 $\zeta(x)$ 在 $(1, +\infty)$ 连续. $\zeta'(x) = -\sum_{n=1}^{\infty} \frac{\ln n}{n^x}$

同理在 $[1+\delta, +\infty)$ 一致收敛 ..

7. 类似于 6

8. $\left| \frac{x^n \cos\frac{n\pi}{x}}{(1+2x)^n} \right| \le \left|\frac{x}{1+2x}\right|^n < \frac{1}{2^n} \Rightarrow$ 一致收敛

于是 $\lim_{x\to+\infty} f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

$\lim_{x\to1} f(x) = \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{3}\right)^n = -\frac{1}{4}$.

9. $\forall \delta > 0$, $f(x)$ 在 $[\delta, +\infty)$ 一致收敛.

11. (1). 设 $a_n = \sup_{x\in[a,b]} u_n(x)$ ~~≥ 0~~ $= u_n(x_n) > 0$

则 $a_{n+1} =$ ~~$f_{n+1}(x_{n+1})$~~ $u_{n+1}(x_{n+1}) \overset{单减}{\le} u_n(x_{n+1}) \le u_n(x_n) = a_n$

于是 $a_n \downarrow$, 设 $\lim_{n\to\infty} a_n = a_0$

$\{x_n\}_{n\ge1}$ 有子列 $\{x_{n_k}\}_{k\ge1}$ 收敛到 $x_0 \in [a,b]$

对 $\forall m \in \mathbb{N}$, 当 $n_k > m$ 时, 有 $a_{n_k} = u_{n_k}(x_{n_k}) \le u_m(x_{n_k})$

令 $n_k \to \infty$ (即 $k \to \infty$) $\Rightarrow a_0 \le u_m(x_0)$

令 $m \to \infty \Rightarrow a_0 = 0$.

(2). $f_n(x) = S(x) - \sum_{k=1}^{n} u_k(x) \downarrow 0$

f_n 非负且连续, 利用 Dini 即可

Dini 定理推广: 只需 $u_n(x)$ 连续, $u_n(x) \to u(x)$, $u(x)$ 连续

并且 $\forall x \in [a,b]$, $\{u_n(x)\}_{n\ge1}$ 单调 即可.

10. 利用 $f_{n+1}'(x)=f_n(x)f_{n+1}(x) \Rightarrow f_{n+1}(x)=e^{\int_0^x f_n(t)dt}$ ☆

归纳证明 $f_n(x)\le \frac{1}{1-x}$. $x\in[0,1)$

f_1 ✓

$$f_{n+1}(x)=e^{\int_0^x f_n(t)dt} \le e^{\int_0^x \frac{1}{1-t}dt} = e^{-\ln(1-x)}=\frac{1}{1-x}$$ ✓

归纳证明 $f_{n+1}(x)\ge f_n(x)$. $x\in[0,1)$ ，利用☆可得

于是由Dini定理，$f_n(x) \rightrightarrows f(x)$ 。

代入有 $f'(x)=f^2(x) \Rightarrow f(x)=\frac{1}{1-x}$.

Chapter 7:

1. $S_n=1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}-\underbrace{\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}_{\text{Stolz}\ \to 0} \to \frac{\pi^2}{6}$.

2. $\frac{2n+3}{(n+1)(n+2)}=\frac{1}{n+1}+\frac{1}{n+2}$.

3. "$\Rightarrow$" $S>\sum_{n=1}^{m}\frac{a_{n+1}-a_n}{a_n}=\sum_{n=1}^{m}\int_{a_n}^{a_{n+1}}\frac{1}{a_n}dx \ge \sum_{n=1}^{m}\int_{a_n}^{a_{n+1}}\frac{1}{x}=\ln a_{m+1}-\ln a_1$

$\Rightarrow \{a_n\}$ 有界

"$\Leftarrow$" $\lim_{n\to\infty}a_n=a>0$

$\sum_{n=1}^{\infty}(a_{n+1}-a_n)<\infty$. $\left\{\frac{1}{a_n}\right\}_{n\ge1}$ ↓ 有界

$\Rightarrow \sum\frac{a_{n+1}-a_n}{a_n}<\infty$.

4. $\sum_{k=1}^{n}\frac{a_{k+1}-a_k}{a_{k+1}^{\alpha+1}}=\sum_{k=1}^{n}\int_{a_k}^{a_{k+1}}\frac{dx}{a_{k+1}^{\alpha+1}}\le\sum_{k=1}^{n}\int_{a_k}^{a_{k+1}}\frac{dx}{x^{\alpha+1}}=\int_{a_1}^{a_{n+1}}\frac{dx}{x^{\alpha+1}}=\frac{1}{\alpha}(a_1^{-\alpha}-a_{n+1}^{-\alpha})<\frac{a_1^{-\alpha}}{\alpha}$

$\Rightarrow \sum_{k=1}^{\infty}\frac{a_{k+1}-a_k}{a_{k+1}^{\alpha+1}}<+\infty$

$\sum_{k=1}^{\infty}\frac{a_{k+1}-a_k}{a_{k+1}}\left(\frac{1}{a_k^{\alpha}}-\frac{1}{a_k^{\alpha+1}}\right)\le\sum_{k=1}^{\infty}\left(\frac{1}{a_k^{\alpha}}-\frac{1}{a_k^{\alpha+1}}\right)<+\infty$

记 $\frac{a_{k+1}-a_k}{a_{k+1}a_k^{\alpha}}=\frac{a_{k+1}-a_k}{a_{k+1}^{\alpha+1}}+\frac{a_{k+1}-a_k}{a_{k+1}}\left(\frac{1}{a_k^{\alpha}}-\frac{1}{a_{k+1}^{\alpha}}\right)$ 相成之

$a_{n+1} \le a_n + c_n a_n = (1+c_n)a_n \le \cdots \le a_1 \prod_{k=1}^{n}(1+c_k) \le a_1 e^{\sum_{k=1}^{n} c_k}$

记 $C = \sum_{n=1}^{\infty} c_k \Rightarrow a_{n+1} \le a_1 e^{C} = M$

于是 $\sum_{n=1}^{\infty} c_n a_n \le M \sum_{n=1}^{\infty} c_n < +\infty \Rightarrow \sum_{n=1}^{\infty} c_n a_n < +\infty$

利用 7.1.6 $\Rightarrow \lim_{n\to\infty} a_n = a$ 存在

若 $a \ne 0$，则 $\exists N$，$n \ge N$ 时，$\frac{a}{2} < a_n < 2a$

$a_{n+1} \le a_n - b_n\varphi(a_n) + c_n a_n \le \cdots \le a_N + c_N a_N + \cdots + c_n a_n - [b_N\varphi(a_N) + \cdots + b_n\varphi(a_n)]$

$< a_N + c_N a_N + 2a\sum_{k=N+1}^{n} c_k - \varphi(\frac{a}{2})\sum_{k=N+1}^{n} b_k - b_N\varphi(a_N)$

$< a_N + c_N a_N + 2aC - b_N\varphi(a_N) - \varphi(\frac{a}{2})\sum_{k=N+1}^{n} b_k$

$\to -\infty$

矛盾：

8. (1)、 $a_n = \underbrace{\sum_{k=2}^{n}(a_k - a_{k-1})}_{\text{收敛}} + a_1$

(2)、 $a_n = \frac{1}{n}$

9. 归纳证明 $f_n(x) \ge x$，~~f_n~~ √

归纳证明 $f_n(x) \le f_{n+1}(x)$，注意到 $f_n^2 - f_{n+1}^2 = (f_n + f_{n+1})(f_n - f_{n+1})$

$= x(f_{n-1} - f_n)$ 即可.

11. 设 $f_0(x)$ 在 $[0,a]$ 上最大值 M

归纳证明 $f_n(x) \le M\frac{x^n}{n!}$

$f_n(x) = \int_0^x f_{n-1}(u)\,du \le \int_0^x M\frac{u^{n-1}}{(n-1)!}\,du = M\frac{x^n}{n!}$ √

于是 $\sup_{[0,a]} f_n(x) \le M\frac{a^n}{n!} \to 0$.