一.

1. a_n 有界, $a_n > 0 \Rightarrow \exists M > 0$ s.t. $0 < a_n < M$, $\forall n \in \mathbb{N}^*$.

$a_n > 0 \Rightarrow b_n := \sum_{k=1}^{n} a_n$, $\{b_n\}_{n=1}^{\infty}$ 单调递增.

$\Rightarrow b_n$ 无界或收敛.

①. b_n 无界.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}^*$ s.t. $b_N > \frac{M}{\varepsilon}$

$\Rightarrow b_n \geq b_N > \frac{M}{\varepsilon}$, $\forall n > N$

$\Rightarrow \left|\frac{a_n}{b_n}\right| < \varepsilon$, $\forall n > N$

$\Rightarrow \lim\limits_{n\to\infty} \frac{a_n}{b_n} = 0$.

②. b_n 收敛. $\lim\limits_{n\to\infty} b_n = b$.

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to+\infty} b_n - \lim\limits_{n\to+\infty} b_{n-1} = b - b = 0$

$\Rightarrow \lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = 0$.

由 ① ② 知 $\lim\limits_{n\to\infty} \frac{a_n}{a_1 + \cdots + a_n} = 0$.

Remark: 1°. 夫这、取 min、取 inf 做错了 …… 得1~2分.

2°. 未指出 b_n 递增 …… 扣1分.

3°. 其余基本正确, 只有小跳步, …… 满分6分.

2.

参考过程1: 解: $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+3x+2}-\left(x+\frac{3}{2}\right)\right)$

$$=\lim_{x\to+\infty}\frac{(x^2+3x+2)-\left(x+\frac{3}{2}\right)^2}{\sqrt{x^2+3x+2}+\left(x+\frac{3}{2}\right)}$$

$$=\lim_{x\to+\infty}\frac{-\frac{1}{4}}{\sqrt{x^2+3x+2}+\left(x+\frac{3}{2}\right)}=0.$$

$$\Rightarrow 0=\lim_{x\to+\infty}\left(\sqrt{x^2+3x+2}+ax+b\right)$$

$$=\lim_{x\to+\infty}\left(\sqrt{x^2+3x+2}-\left(x+\frac{3}{2}\right)\right)+\lim_{x\to+\infty}\left((a+1)x+\left(b+\frac{3}{2}\right)\right)$$

$$=\lim_{x\to+\infty}\left((a+1)x+\left(b+\frac{3}{2}\right)\right)$$

$$\Rightarrow\begin{cases}a+1=0\\ b+\frac{3}{2}=0\end{cases}\Rightarrow\begin{cases}a=-1\\ b=-\frac{3}{2}.\end{cases}$$

参考 2: 解: $\sqrt{x^2+3x+2}=x\cdot\sqrt{1+\frac{3}{x}+\frac{2}{x^2}}$

$$=x\cdot\left(1+\frac{1}{2}\left(\frac{3}{x}+\frac{2}{x^2}\right)+o\left(\left(\frac{3}{x}+\frac{2}{x^2}\right)\right)\right)$$

$$=x\cdot\left(1+\frac{3}{2x}+o\left(\frac{1}{x}\right)\right)$$

$$=x+\frac{3}{2}+o(1),\quad x\to+\infty \text{ 时}.$$

$$\Rightarrow 0=\lim_{x\to+\infty}\left(\sqrt{x^2+3x+2}+ax+b\right)$$

$$=\lim_{x\to+\infty}\left(x+\frac{3}{2}+o(1)+ax+b\right)$$

$$=\lim_{x\to+\infty}\left((a+1)x+b+\frac{3}{2}\right)$$

$$\Rightarrow\begin{cases}a+1=0\\ b+\frac{3}{2}=0\end{cases}\Rightarrow\begin{cases}a=-1\\ b=-\frac{3}{2}\end{cases}$$

Remark:

*. 变为 $\lim\limits_{x\to+\infty}\frac{(\cdots)x^2+(\cdots)x+(\cdots)}{\sqrt{x^2+3x+2}-(ax+b)}$ 不讨论分子直接得 $\lim\limits_{x\to+\infty}$ 分母 $=0$ 的 扣 2~3 分.

*. 出现 $\lim\sqrt{x^2+3x+2}=\lim\left(x+\frac{3}{2}\right)$ 后面均由该式得出的，~~[illegible]~~ 类似严重错误，只有 (0~3分).

3. $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + o((x-x_0)^2)$

$$\lim_{x\to x_0}\left(\frac{1}{f(x)-f(x_0)} - \frac{1}{(x-x_0)f'(x_0)}\right)$$

$$= \lim_{x\to x_0}\left(\frac{1}{f'(x_0)(x-x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2+o((x-x_0)^2)} - \frac{1}{(x-x_0)f'(x_0)}\right)$$

$$= \lim_{x\to x_0} - \frac{\frac{1}{2}f''(x_0)(x-x_0)^2 + o((x-x_0)^2)}{\left(f'(x_0)(x-x_0)+o((x-x_0))\right)(x-x_0)f'(x_0)}$$

$$= \lim_{x\to x_0} - \frac{\frac{1}{2}f''(x_0)+o(1)}{\left(f'(x_0)+o(1)\right)f'(x_0)} = -\frac{f''(x_0)}{2(f'(x_0))^2}$$

Remark: ① 用导数定义但出现 $\lim \frac{\cdots - \cdots}{\cdots} = \lim \frac{\lim\cdots - \lim\cdots}{\cdots}$ 之类严重错误，至多得答案分 1分.

② 直接洛两次/Taylor 用 Lagrange 余项无视 $f''(x)$ 存在性的，答案对时得 3分.

③ 洛 + 导数定义，正确也得全分.

4. 解 $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2t}{1+t^2}}{\frac{1}{1+t^2}} = 2t$

$= 2\tan x$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = (2\tan x)' = \frac{2}{\cos^2 x}$$

$= 2+2t^2.$

Remark: $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ 各 3分.

5.

$$(f(x))^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \cdot (x^2)^{(k)} \cdot (\ln(1-x^2))^{(n-k)} \quad \cdots \text{2分}$$

$$= x^2 \cdot (\ln(1-x^2))^{(n)} + n \cdot 2x \cdot (\ln(1-x^2))^{(n-1)} + \cancel{2} \cdot n(n-1) \cdot (\ln(1-x^2))^{(n-2)}$$

$$\Rightarrow f^{(n)}(0) = n(n-1) \cdot (\ln(1-x^2))^{(n-2)}\Big|_{x=0} \quad \cdots \text{1分}$$

后面对也得
此分.

$$\ln(1-x^2) = \ln(1-x) + \ln(1+x).$$

$$(\ln(1+x))' = (1+x)^{-1}. \Rightarrow (\ln(1+x))^{(n)} = (-1)^{n-1} \cdot (n-1)!\,(1+x)^{-n}$$

$$(\ln(1-x))' = -(1-x)^{-1} \Rightarrow (\ln(1-x))^{(n)} = (-1) \cdot (n-1)!\,(1-x)^{-n} \quad \cdots \text{2分}$$

$$\Rightarrow \cancel{(\ln(1+x))^{(n)}}$$

$$(\ln(1-x^2))^{(n-2)}\Big|_{x=0} = \left((-1)^{n-1} + (-1)\right)(n-3)!$$

$$\Rightarrow f^{(n)}(0) = n(n-1)\left((-1)^{n-1} - 1\right) \cdot (n-3)!$$

$$= \begin{cases} \dfrac{2n!}{2-n}, & n = 2k+2 \\ 0, & n = 2k+1 \end{cases} \quad (k \in N^*)$$

$\cdots$ 1分

6.

$\sin x = x - \frac{x^3}{3!} + o(x^4)$ $= x - \frac{x^3}{6} + o(x^4)$，$x \to 0$ 时. ... 1分，

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)$

$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$，$x \to 0$ 时. ... 1分.

$\cos(\sin x) = 1 - \frac{1}{2}(x - \frac{x^3}{6} + o(x^4))^2 + \frac{1}{24}(x - \frac{x^3}{6} + o(x^4))^4 + o(x^4)$

$= 1 - \frac{1}{2}(x^2 + 2\cdot(-\frac{1}{6})x^4 + o(x^4)) + \frac{1}{24}x^4 + o(x^4)$ 1 + 1

$= 1 - \frac{x^2}{2} + (\frac{1}{24} + \frac{1}{6})x^4 + o(x^4)$，$x \to 0$ 时. ... 2分.

$\Rightarrow \cos(\sin x) - \cos x = \frac{x^4}{6} + o(x^4)$，$x \to 0$ 时

$\Rightarrow \lim_{x\to 0} \frac{\cos(\sin x) - \cos x}{x^4} = \lim_{x\to 0} \frac{\frac{x^4}{6} + o(x^4)}{x^4} = \frac{1}{6}$

... 2分.