4.1

1. (4). $\int \tan^2 x\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx = \int \frac{1}{\cos^2 x}dx - \int dx = \tan x - x + C$

(6). $\int \frac{1+\cos^2 x}{1+\cos 2x}dx = \int \frac{1+\cos^2 x}{2\cos^2 x}dx = \frac{1}{2}\tan x + \frac{1}{2}x + C$

2. (2). $\int \frac{1}{x^2}\sin\frac{1}{x}\,dx = -\int \sin\frac{1}{x}\,d(\frac{1}{x}) = \cos\frac{1}{x} + C$

(5). $\int x\sqrt{1-x^2}\,dx \overset{x^2=t}{=\!=} \frac{1}{2}\int \sqrt{1-t}\,dt = -\frac{1}{3}(1-t)^{\frac{3}{2}} + C = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C$

(9). $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx = \frac{1}{2}x - \frac{1}{4}\sin 2x$

3. (5). $\int \frac{1}{1+\sqrt{x+1}}dx \overset{t=\sqrt{x+1}}{=\!=} \int \frac{2t}{1+t}dt = 2t - 2\ln(1+t) = \cdots$

(6). $\int \frac{x\ln x}{(1+x^2)^{3/2}}dx = -\int \ln x\, d(\frac{1}{\cancel{2}\sqrt{1+x^2}}) = -\frac{\ln x}{\cancel{2}\sqrt{1+x^2}} + \cancel{\#}\int \frac{dx}{x\sqrt{1+x^2}}$ $(x=\tan t)$

$= -\frac{\ln x}{\cancel{2}\sqrt{1+x^2}} + \cancel{\#}\int \frac{\frac{1}{\cos^2 t}dt}{\tan t\cdot\frac{1}{\cos t}} \rightarrow \int \frac{1}{\sin t}dt$

$\int \frac{1}{\sin t}dt = \int \frac{\sin t}{\sin^2 t}dt = \int \frac{d(\cos t)}{\cos^2 t - 1} = \frac{1}{2}\int (\frac{1}{\cos t - 1} - \frac{1}{\cos t + 1})\,d(\cos t)$

$= \frac{1}{2}\ln\frac{1-\cos t}{1+\cos t}$

$\cancel{(11).\ \int \frac{x-1}{x^2\sqrt{x^2-1}}dx \overset{x=\frac{1}{\cos t}}{=\!=} \int \frac{\frac{1}{\cos t}-1}{\frac{1}{\cos^2 t}\cdot\frac{\sin t}{\cos t}}\cdot\frac{\sin t}{\cos^2 t}dt = \int (1-\cos t)\,dt = \cdots}$

~~4.~~ (11). $\int \frac{x-1}{x^2\sqrt{x^2-1}}dx \overset{x=\frac{1}{t}}{=\!=} -\int \frac{\frac{1}{t}-1}{\sqrt{\frac{1}{t^2}-1}}dt = -\int \frac{1}{\sqrt{1-t^2}}dt + \frac{1}{2}\int \frac{dt^2}{\sqrt{1-t^2}}$ $(t>0)$

$= \arccos\frac{1}{x} + \sqrt{1-\frac{1}{x^2}} + C_1$ $(x>0)$

$t<0$时

$-\int \frac{\frac{1}{t}-1}{\sqrt{\frac{1}{t^2}-1}}dt$

$= \int \frac{1}{\sqrt{1-t^2}}dt - \frac{1}{2}\int \frac{dt^2}{\sqrt{1-t^2}}$

$= \arcsin\frac{1}{x} - \sqrt{1-\frac{1}{x^2}} + C_2$ $(x$

4. (2). $\int \max\{1, x^2\}\,dx = \begin{cases} \frac{1}{3}x^3 + C_1, & x<-1 \\ x + C_2, & -1\le x\le 1 \\ \frac{1}{3}x^3 + C_3, & x>1 \end{cases}$

$\begin{cases} \frac{1}{3} + C_1 = -1 + C_2 \\ 1 + C_2 = \frac{1}{3} + C_3 \end{cases}$

5. (3). $\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$

$= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$

$\Rightarrow \int \cos(\ln x)\,dx = \frac{x}{2}(\sin(\ln x) + \cos(\ln x)) + C.$

(6). $\int x^2 e^x\,dx = x^2e^x - 2\int xe^x\,dx$

$= x^2e^x - 2xe^x + 2\int e^x\,dx$

$= x^2e^x - 2xe^x + 2e^x + C.$

(8). $\int x(\arctan x)^2\,dx = \frac{1}{2}\int(\arctan x)^2\,d(x^2+1)$

$= \frac{1}{2}(x^2+1)(\arctan x)^2 - \int \arctan x\,dx$

$\int \arctan x\,dx = x\arctan x - \int \frac{x}{x^2+1}\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$

6. (2). $I_n = \int x^n e^x\,dx$

$= x^ne^x - \int nx^{n-1}e^x\,dx = x^ne^x - nI_{n-1}.$

7. (3). $\int \frac{1}{x^4+x^6}\,dx = \int\left(\frac{1}{x^4} - \frac{1}{x^2(1+x^2)}\right)dx = \int\frac{1}{x^4}\,dx - \int\frac{1}{x^2}\,dx + \int\frac{1}{1+x^2}\,dx$

$= -\frac{1}{3}x^{-3} + x^{-1} + \arctan x + C.$

(7). $\int e^x\sin x\,dx = e^x\sin x - \int e^x\cos x\,dx$

$= e^x\sin x - e^x\cos x - \int e^x\sin x\,dx$

$\Rightarrow \begin{cases} \int e^x\sin x\,dx = \frac{1}{2}[e^x\sin x - e^x\cos x] \\ \int e^x\cos x\,dx = \frac{1}{2}[e^x\sin x + e^x\cos x] \end{cases}$

$I = \int xe^x\sin x\,dx = xe^x\sin x - \int e^x(\sin x + x\cos x)\,dx$

$= xe^x\sin x - \int e^x\sin x\,dx - \int e^x x\cos x\,dx$

$= xe^x\sin x - \int e^x\sin x\,dx - \left(xe^x\cos x - \int e^x(\cos x - x\sin x)\,dx\right)$

解出 I 即可.

(8). $$\int \frac{1}{(1+\tan x)\sin^2 x}dx = -\int \frac{1}{1+\frac{1}{\cot x}}d(\cot x)$$

$$= -\int \frac{\cot x}{\cot x+1}d(\cot x)$$

$$= -\cot x + \ln|\cot x+1| + C$$

4.2

1. (2). $$\frac{x^4}{x^2+1} = \frac{x^4-1+1}{x^2+1} = \frac{(x^2-1)(x^2+1)+1}{x^2+1} = x^2-1+\frac{1}{x^2+1}$$

(7). $$\int \frac{x^5-x}{x^8+1}dx \overset{t=x^2}{=\!=} \frac{1}{2}\int \frac{t^2-1}{t^4+1}dt = \frac{1}{2}\int \frac{1-\frac{1}{t^2}}{(t+\frac{1}{t})^2-2}dt$$

$$= \frac{1}{2}\int \frac{d(t+\frac{1}{t})}{(t+\frac{1}{t})^2-2} = \cdots$$

2. $t=\tan\frac{x}{2}$,

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2t}{t^2+1}$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$$

$$dx = \frac{2dt}{1+t^2}$$

(1)、(4) 代入即可

(10). 可用下法: $A = \int \frac{\cos x\, dx}{a\sin x + b\cos x}$　　$B = \int \frac{\sin x\, dx}{a\sin x + b\cos x}$

$$\Rightarrow \begin{cases} aB+bA = \int dx = x \\ aA-bB = \int \frac{a\cos x - b\sin x}{a\sin x + b\cos x}dx = \ln|a\sin x + b\cos x| \end{cases}$$

解出 A, B.