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15. 注：f'' 不一定存在.

$$\left(\frac{f(x)}{x}\right)' = \frac{xf'(x)-f(x)}{x^2} = \frac{1}{x}\left(f'(x)-\frac{f(x)}{x}\right)$$

$$= \frac{1}{x}\left(f'(x)-f'(\xi)\right) \quad \left(\exists\, \xi\in(0,1)\right).$$

$$>0.$$

16. 18. 19. 略.

17. 考虑 $F(x) = e^x(f'(x)-f(x))$.

20. (2) 略.

(4) $f(x) = (1+x)\ln(1+x) - \arctan x$

$$f'(x) = 1+\underbrace{\ln(1+x)}_{>0} - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} \geq 0, \quad \forall x\geq 0.$$

$$\Rightarrow f(x) > f(0) = 0, \quad \forall x>0.$$

(7). $LHS = \left(1-\frac{1}{x^2}\right)^{x-1}\left(1+\frac{1}{x}\right)^2 < \left(1+\frac{1}{x}\right)^2 < 4.$

22. $f(x) := \begin{cases} \dfrac{x}{1-e^{-x}} - a, & x \neq 0. \\ 1-a, & x = 0 \end{cases}$

$f'(x) > 0 \Rightarrow f \nearrow$.

$(f(x) - x)' < 0 \Rightarrow (f(x) - x) \searrow$.

$\lim\limits_{x\to-\infty} (f(x) - x) = +\infty$, $\lim\limits_{x\to+\infty} (f(x)-x) = -\infty$.

$\Rightarrow$ $(f(x) - x)$ 有唯一零点，记为 x_0

$$(f(b_1) - b_1) = f(1-a) - (1-a) = \frac{e^{a-1}-a}{1-e^{a-1}} > \frac{(1+(a-1))-a}{1-e^{a-1}} = 0$$

$\Rightarrow b_1 < x_0$. $\Rightarrow b_2 = f(b_1) < f(x_0) = x_0 \overset{归纳}{\Rightarrow} b_n < x_0$, $n \in \mathbb{N}^*$.

$b_n < x_0 \Rightarrow (f(b_n) - b_n) > (f(x_0) - x_0) = 0$

$\Leftrightarrow b_{n+1} > b_n$.

即 $b_n \nearrow$ 有上界 x_0.

在 $b_{n+1} = f(b_n)$ 两边取极限知 $\lim\limits_{n\to\infty} b_n = x_0$.

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1. 略

3. Cauchy 中值. 对 $f(x)$, x^2 即.

4. Cauchy 中值. 对 $x_1 = \frac{1}{b}$, $x_2 = \frac{1}{a}$, $f(x)$, $\frac{1}{x}$ 即.

5. (2)
$$= \lim_{x\to0} \frac{\left(1+\binom{n}{1}mx+\binom{n}{2}(mx)^2\right) - \left(1+\binom{m}{1}nx+\binom{m}{2}(nx)^2\right) + o(x^2)}{x^2}$$
$$= \lim_{x\to0} \frac{\left(\frac{1}{2}n(n-1)m^2 - \frac{1}{2}m(m-1)n^2\right)x^2}{x^2}$$
$$= \frac{1}{2}mn(n-m) \quad (n, m \geq 2 \text{ 时}).$$

其余情况自行讨论.

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5. (6). $= \lim_{x\to 0} \frac{a^x\left(\left(1+\frac{x}{a}\right)^x-1\right)}{x^2} = \lim_{x\to 0} \frac{\left(1+\frac{x}{a}\right)^x-1}{x^2}$ $(*)$

$x\to 0$ 时，上际 $\to 0$

$$\Rightarrow \left(\underbrace{\left(1+\frac{x}{a}\right)^x-1}_{\to \cdots}\right) \sim \ln\left(1+\left(\ \cdots\ \right)\right)$$

$$= x\ln\left(1+\frac{x}{a}\right)$$

$$\sim x\cdot\frac{x}{a} = \frac{1}{a}\cdot x^2 \quad \text{as } x\to 0.$$

$$\Rightarrow (*) = \lim_{x\to 0}\frac{\frac{1}{a}\cdot x^2}{x^2} = \frac{1}{a}.$$

(11) $\lim_{x\to 0}\left(\frac{1}{\arctan^2 x}-\frac{1}{x^2}\right) = \lim_{x\to 0}\frac{(x-\arctan x)(x+\arctan x)}{\arctan^2 x\cdot x^2}$

$$= \lim_{x\to 0}\frac{(x-\arctan x)(x+\arctan x)}{x^4}$$

$$= \lim_{x\to 0}\frac{x-\arctan x}{x^3}\cdot\lim_{x\to 0}\frac{x+\arctan x}{x}$$

$$= 2\lim_{x\to 0}\frac{1-\frac{1}{1+x^2}}{3x^2} = \frac{2}{3}.$$

(12). $\lim_{x\to 1^-}\ln x\ln(1-x)$

$$= \lim_{y\to 0^+}\underbrace{\ln(1-y)}\ln y$$

$$= \lim_{y\to 0^+}\underbrace{-y}\ln y \overset{t=\frac{1}{y}}{=} \lim_{t\to +\infty}\frac{\ln t}{t} = 0.$$

(13). $\lim_{x\to\frac{\pi}{2}^-}(\tan x)^{2x-\pi} = \lim_{y\to 0^+}\left(\frac{1}{\tan y}\right)^{2(\frac{\pi}{2}-y)-\pi}$

$$= \lim_{y\to 0^+}(\tan y)^{2y} = \lim_{y\to 0^+} e^{2y\ln(\tan y)}$$

$$= e^{\lim_{y\to 0^+} 2y\ln(\tan y)} = e^{\lim_{t\to 0^+} 2(\arctan t)\cdot\ln t}$$

$$= e^{\lim_{t\to 0^+} 2t\ln t} = e^0 = 1.$$

(20) $= \lim_{x\to+\infty}\frac{\ln x}{x^k} = \lim_{x\to+\infty}\frac{\frac{1}{x}}{k x^{k-1}} = 0$

6.

(1) $x_{n+1} = f(x_n) < x_n$.

$\Rightarrow x_n \searrow$，有下界 0.

设 $\lim\limits_{n\to\infty} x_n = x^*$.

$\Rightarrow f(x^*) = x^* \Rightarrow x^* = 0$. $f(0) = 0$.

(2).

$$\lim_{n\to\infty} n x_n = \lim_{n\to\infty} \frac{n}{\frac{1}{x_n}} \overset{☆}{=} \lim_{n\to\infty} \frac{(n+1) - n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}}$$

$$= \lim_{n\to\infty} \frac{x_n \cdot f(x_n)}{x_n - f(x_n)} \overset{☆}{=} \lim_{x\to 0} \frac{x f(x)}{x - f(x)}$$

$$\overset{☆}{=} \lim_{x\to 0} \frac{f(x) + x f'(x)}{1 - f'(x)} \quad \left(\frac{0}{0}\text{ 型}\right)$$

$$\overset{☆}{=} \lim_{x\to 0} -\frac{f'(x) + x f''(x) + f'(x)}{f''(x)}$$

$$\overset{☆}{=} -\frac{\lim\limits_{x\to 0}\left(2f'(x) + x f''(x)\right)}{\lim\limits_{x\to 0} f''(x)}$$

$$= -\frac{2}{f''(0)}.$$

（☆只在最后结果存在的情况下成立）