§6.2.

1(3). $(1-x^2)y'' - 2xy' + 2y = 0$. $y_1 = x$.

$\Leftrightarrow$ $y'' - \underbrace{\frac{2x}{1-x^2}}_{p(x)} y' + \frac{2}{1-x^2} y = 0$

由书 P240,

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} e^{-\int_{x_0}^{x} p\, dx}$$

$$= x \int \frac{1}{x^2} \exp\left(\int \frac{2t}{1-t^2} dt \right) dx$$

$$= x \int \frac{1}{x^2} \exp\left(-\int \left(\frac{1}{t-1} + \frac{1}{t+1} \right) dt \right) dx$$

$$= x \int \frac{1}{x^2} \exp\left(-\ln|x^2-1| + c \right) dx$$

取 $c_0 = 0$, $y_2(x) = x \int \frac{1}{x^2} \frac{1}{x^2-1} dx$

$$= x \int \left(\frac{1}{2} \left(\frac{1}{x-1} - \frac{1}{x+1} \right) - \frac{1}{x^2} \right) dx$$

$$= \frac{1}{2} x \ln\left| \frac{x-1}{x+1} \right| + 1$$

$y_2(x)$ 为一与 $y_1(x)$ 线性无关的解.

$\Rightarrow$ 通解为 $y(x) = c_1 y_1(x) + c_2 y_2(x)$

$$= c_1 x + c_2 \left(\frac{1}{2} x \ln\left| \frac{x-1}{x+1} \right| + 1 \right).$$

$g := y' \Rightarrow (1+x^2)g' + 2xg - 6x^2 - 2 = 0$. ①

齐次的 $g_1 = 2x$, 考虑 $(1+x^2)g' + 2xg = 0$. ②

$$\Leftrightarrow \frac{1}{g}dg = -\frac{2x}{1+x^2}dx$$

$$\Leftrightarrow \ln|g| = \int -\frac{2x}{1+x^2}dx = -\int \frac{d(1+x^2)}{1+x^2}$$
$$= -\ln(1+x^2) + C$$

$$\Leftrightarrow g = \frac{C}{1+x^2}$$

$\Rightarrow$ ① 的通解为 $g = 2x + \frac{C}{1+x^2}$.

$y'(-1) = g(-1) = 0 \Rightarrow -2 + \frac{C}{2} = 0 \Rightarrow C = 4$

$y'(x) = 2x + \frac{4}{1+x^2}$, $y(x) = \int (2x + \frac{4}{1+x^2})\,dx$

$= x^2 + 4\arctan x + C_2'$

$y(-1) = 0 \Rightarrow 1 + 4\cdot(-\frac{\pi}{4}) + C_2 = 0$

$\Rightarrow C_2 = \pi - 1$

$\Rightarrow y(x) = x^2 + 4\arctan x + \pi - 1$

4. (2). $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \begin{cases} \lambda_1 = -1+i \\ \lambda_2 = -1-i \end{cases}$

$\Rightarrow$ 齐特的 $y_1 = e^{-x}\cos x$, $y_2 = e^{-x}\sin x$

$\Rightarrow$ 通解为 $y = c_1 y_1 + c_2 y_2$

$= (c_1\cos x + c_2 \sin x)\cdot e^{-x}$.

5.(1) $y'' + y = 2\sin\frac{x}{2}$.

设 $y = \lambda \cdot \sin\frac{x}{2}$,

则: $\lambda \cdot \left(1 - \left(\frac{1}{2}\right)^2\right) \sin\left(\frac{x}{2}\right) = 2\sin\frac{x}{2}$

$\Leftrightarrow \quad \frac{3}{4}\lambda = 2 \quad \Leftrightarrow \quad \lambda = \frac{8}{3}$.

$y = \frac{8}{3}\sin\frac{x}{2}$ 为原方程一特解.

9 (2). $x''' - 2x'' + x' - 2x = 0$

$\lambda^3 - 2\lambda^2 + \lambda - 2 = 0$.

$\Leftrightarrow \quad (\lambda^2+1)(\lambda-2) = 0$

$\Rightarrow \begin{cases} \lambda_1 = 2 \\ \lambda_2 = +i \\ \lambda_3 = -i \end{cases}$

通解为 $y = C_1 \cdot e^{2x} + C_2 \cdot \cos x + C_3 \cdot \sin x$.