P140.

3.6

1. (1) $y=\dfrac{x^3+2x+1}{x-1}=x^2+x+3+\dfrac{4}{x-1}$

$\dfrac{1}{1-x}=1+\sum_{k=1}^{n}x^k+o(x^n)$ ，$x\to 0$ 时

$\Rightarrow y=x^2+x+3-4(1+\sum_{k=1}^{n}x^k)+o(x^n)$

$=-1-3x-3x^2-4\sum_{k=3}^{n}x^k+o(x^n)$，$x\to 0$ 时. $(n>3)$

3. $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+o(x^6)$

$=1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+o(x^6)$ ，$x\to 0$ 时

$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)$，$x\to 0$ 时.

$\Rightarrow \ln(1+\cos x)=\ln(1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+o(x^6))$

$=\left(-\frac{1}{2}x^2+\frac{1}{24}-\frac{1}{720}x^6+o(x^6)\right)-\frac{1}{2}(-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^4))^2+\frac{1}{3}(-\frac{1}{2}x^2+o(x^2))^3+o(x^6)$

$=\left(-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+o(x^6)\right)+\left(-\frac{1}{8}x^4+\frac{1}{48}x^6+o(x^6)\right)+\left(\frac{1}{24}x^6+o(x^6)\right)+o(x^6)$

$=-\frac{1}{2}x^2+\left(-\frac{1}{8}+\frac{1}{24}\right)x^4+\left(-\frac{1}{720}+\frac{1}{48}-\frac{1}{24}\right)x^6+o(x^6)$

$=-\frac{1}{2}x^2-\frac{1}{12}x^4-\frac{1}{45}x^6+o(x^6)$ ，$x\to 0$ 时.

5. $y=x^{-1}\Rightarrow y^{(n)}=(-1)^n\cdot n!\,x^{-(n+1)}.\Rightarrow \dfrac{y^{(n)}}{n!}=(-1)^n\cdot x^{-(n+1)}$.

$\left.\dfrac{y^{(n)}}{n!}\right|_{x=-1}=(-1)^n\cdot(-1)^{-(n+1)}=-1.$

$\left.\dfrac{y^{(n+1)}}{(n+1)!}\right|_{x=\xi}=(-1)^{n+1}\xi^{-(n+2)}$

$\Rightarrow y=-1-\sum_{k=1}^{n}(x+1)^k+(-1)^{n+1}\cdot\xi^{-(n+2)}\cdot(x+1)^{n+1}$ （ξ 介于 -1 与 x 之间），

$(x<0)$. ~~$x\to$ 时~~

~~$x<0$ 时 ξ 介于 x 与 -1~~

6. (1) $\lim\limits_{x\to 0} \dfrac{\cos x - e^{-\frac{1}{2}x^2}}{\sin^4 x}$

$$\overset{\sin x \sim x}{\underset{\text{Taylor展开}}{=\!=}} \lim_{x\to 0} \frac{\left(\cancel{1}-\cancel{\frac{x^2}{2!}}+\frac{x^4}{4!}+o(x^4)\right)-\left(\cancel{1}+\cancel{(-\frac{1}{2}x^2)}+\frac{1}{2!}(-\frac{1}{2}x^2)^2+o(x^4)\right)}{x^4}$$

$$= \frac{1}{4!} - \frac{1}{2!}\times\left(-\frac{1}{2}\right)^2 = -\frac{1}{12}.$$

(3) $\lim\limits_{x\to\infty} \left(x - x^2\ln(1+\frac{1}{x})\right)$

$= \lim\limits_{x\to\infty} \left(x - x^2\left(\frac{1}{x} - \frac{1}{2}(\frac{1}{x})^2 + o((\frac{1}{x})^2)\right)\right)$ (因为 $\frac{1}{x}\to 0$).

$= \lim\limits_{x\to\infty} \left(x - x + \frac{1}{2} + o(1)\right) = \frac{1}{2}.$

8 对 $f(x)$ 在 Taylor 展开后有

~~分别有~~

$f(0) = f(x) + f'(x)(-x) + \frac{1}{2}f''(\xi_1)\cdot x^2 \quad (0<\xi_1<x)$

$f(2) = f(x) + f'(x)\cdot(2-x) + \frac{1}{2}f''(\xi_2)(2-x)^2 \quad (x<\xi_2<2)$

$, x\in(0,2).$

$\Rightarrow$ 相减有 $2|f'(x)| = \left|f(0) - f(2) + \frac{1}{2}f''(\xi_1)\cdot x^2 - \frac{1}{2}f''(\xi_2)(2-x)^2\right|$

$\leq |f(0)| + |f(2)| + \frac{1}{2}\cdot|f''(\xi_1)|x^2 + \frac{1}{2}|f''(\xi_2)|\cdot(2-x)^2$

$\leq 2 + \frac{1}{2}\left(x^2 + (2-x)^2\right)$

≤ 4

$\Rightarrow |f'(x)| \leq 2, \ \forall x\in(0,2).$

由 f' 介值性 $\Rightarrow |f'(x)| \leq 2, \ \forall x\in[0,2].$

9. ① $\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} = \begin{cases} x^n, & x\in\mathbb{Q} \\ 0, & x\notin\mathbb{Q} \end{cases}$ $(x\neq 0)$

$0 \le \left|\frac{f(x)-f(0)}{x-0}\right| \le |x|^n$, $\lim_{x\to 0}|x|^n = 0$

$\xRightarrow{夹} \lim_{x\to 0}\left|\frac{f(x)-f(0)}{x-0}\right| = 0 \Rightarrow \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 0.$

即 $f'(0)=0$.

②. $\forall x_0\neq 0$, 下证 $f'(x_0)$ 不存在.

(1). $x_0\in\mathbb{Q}\setminus\{0\}$.

$\exists\{x_n\}$ s.t. $x_n\in\mathbb{R}\setminus\mathbb{Q}$ 且 $x_n\to x_0$, 当 $n\to\infty$ 时.

$$\Rightarrow \frac{f(x_n)-f(x_0)}{x_n-x_0} = \frac{0-f(x_0)}{x_n-x_0} = \frac{-x_0^{n+1}}{x_n-x_0} \to \infty, \text{ 当 } n\to\infty \text{ 时}$$

$\Rightarrow f'(x_0)$ 不存在.

(2). $x_0\in\mathbb{R}\setminus\mathbb{Q}$.

$\exists\{x_n\}$ s.t. $x_n\in\mathbb{Q}$ 且 $x_n\to x_0$, 当 $n\to\infty$ 时.

$$\Rightarrow \left|\frac{f(x_n)-f(x_0)}{x_n-x_0}\right| = \left|\frac{x_n^{n+1}-0}{x_n-x_0}\right| \to +\infty, \text{ 当 } n\to\infty \text{ 时.}$$

$\Rightarrow f'(x_0)$ 不存在.

$\Rightarrow f''(0)$ 不存在.

Chap 3.

2. $f(x) = -f(-x) \Rightarrow f'(x) = f'(-x) \Rightarrow f''(x) = -f''(-x)$

$\Rightarrow f(0) = f''(0) = 0. \xRightarrow{f可导从而连续} \lim_{x\to 0} f(x) = 0$

(1). 若 $\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{f(x)}{x} \overset{\frac{0}{0}}{\underset{\text{L'hospital}}{=}} \lim_{x\to 0} f'(x) \overset{f'连续}{=} f'(0)$

$a = f'(0)$.

3. 考虑 $f(x) = \frac{a_0}{n+1} x^{n+1} + \cdots + a_n \cdot x$

4. 考虑 $\varphi(x) = e^x \cdot f(x)$

6. 考虑 $\varphi(x) = (1-x) f(x)$.　　（方法不唯一）.