fowdr-checkpoint

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FOWDR — Computing the Dispersion Relations of the Fast Flavor Oscillation Waves in Dense Neutrino Gases

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1 Introduction

`fowdr` is a Python package for computing the dispersion relations (DRs) of the fast flavor oscillation waves in dense neutrino gases. After the pioneering work by Ignacio Izaguirre, Georg Raffelt, and Irene Tamborra, we studied the critical points of the DRs of the neutrino gases that satisfy the following conditions [PRD 99, 063005]:

- The neutrino mixing is dominated by two flavors;
- The densities of the neutrinos are high enough so that the mass splitting of the neutrino can be ignored;
- The neutrinos are almost entirely in the weak-interaction states;
- The neutrino gas has exact translational symmetries along the x and y directions and approximate translational and axial symmetries about the z direction;
- The angular distribution of the neutrino electron lepton number (ELN) has no crossing or only one crossing.

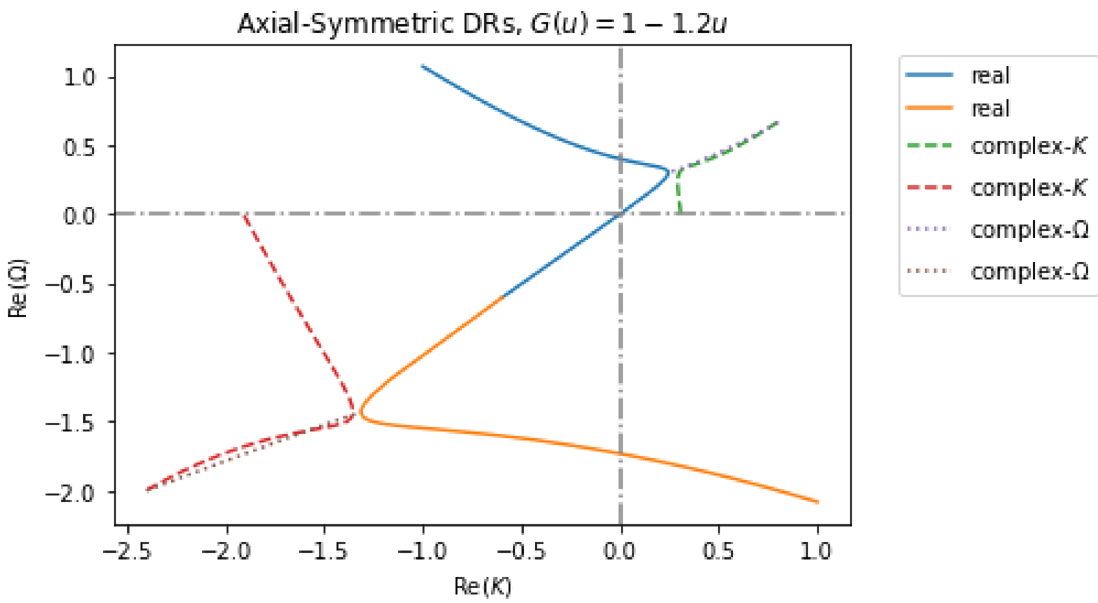
This work forms the foundation of `fowdr` which will be referred to as REF in the rest of this document. Please cite REF if you find `fowdr` useful.

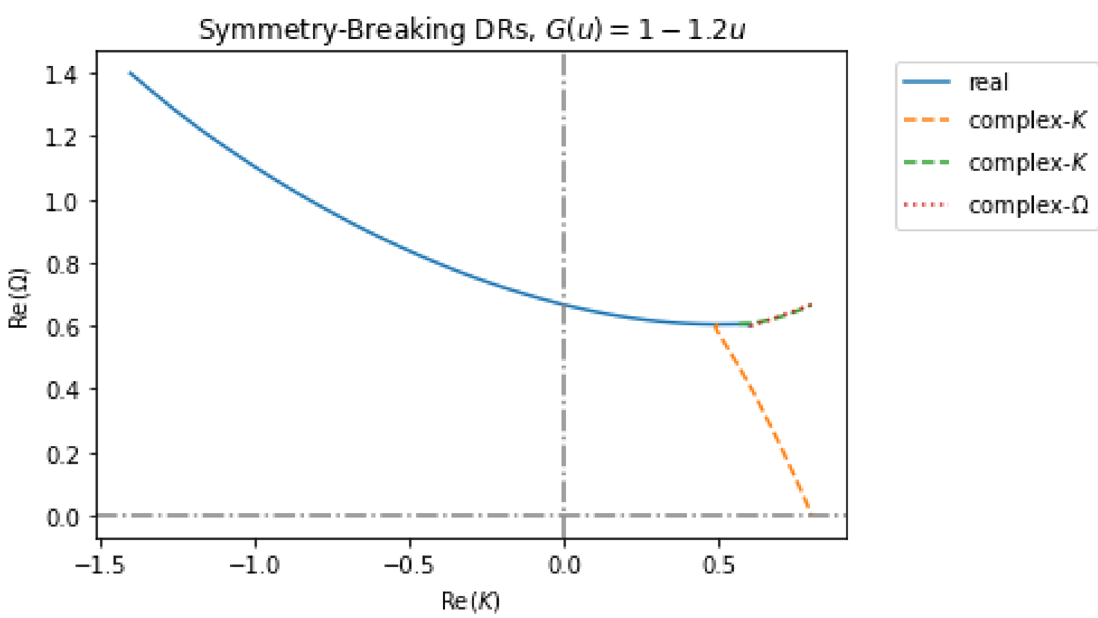
`fowdr` depends on NumPy and SciPy.

2 Usage

`fowdr` has two modules `asdr` and `sbdr` which compute the DRs that preserve and break the axial symmetry, respectively. Both modules provide functions `DR_real(G, ...)`, `DR_complexK(G, ...)`, and `DR_complexOmega(G, ...)` to compute the real DR branches (with both real wave number K and frequency Ω), complex-K DR branches (with real Ω), and complex-Ω DR branches (with real K), respectively. The first argument of these functions `G` is a function with a single real argument that gives the ELN distribution $G(u)$, where u is the z component of the neutrino velocity along z. All these functions return a list of tuples `(K, Omega)` where `K` and `Omega` are one-dimensional NumPy arrays of the same length that contain a list of (K, Ω) on the corresponding DR. Below is an example using a simple linear ELN distribution $G(u) = a - bu$, where a and b are two positive constants.

[1]:
```python
import matplotlib.pyplot as plt
from fowdr import asdr, sbdr

a = 1; b = 1.2
G = lambda u: a - b*u # ELN distribution function G(u) = a - b*u

# plot the symmetry-preserving DRs
plt.figure()
for kk, ww in asdr.DR_real(G): # real DR
    plt.plot(kk, ww, '-', label='real')
for kk, ww in asdr.DR_complexK(G): # complex-K DR
    plt.plot(kk.real, ww, '--', label='complex-$K$')
for kk, ww in asdr.DR_complexOmega(G): # complex-Omega DR
    plt.plot(kk, ww.real, ':', label='complex-$\Omega$')

plt.axhline(0, ls='-.', color='grey') # x axis
plt.axvline(0, ls='-.', color='grey') # y axis
plt.legend(bbox_to_anchor=(1.05, 1), loc="upper left")
plt.xlabel('Re($K$)')
plt.ylabel('Re($\Omega$)')
plt.title(f'Axial-Symmetric DRs, $G(u)={a}-{b:.1f}u$')
plt.show()

# plot the symmetry-breaking DRs
for kk, ww in sbdr.DR_real(G): # real DR
    plt.plot(kk, ww, '-', label='real')
for kk, ww in sbdr.DR_complexK(G): # complex-K DR
    plt.plot(kk.real, ww, '--', label='complex-$K$')
for kk, ww in sbdr.DR_complexOmega(G): # complex-Omega DR
    plt.plot(kk, ww.real, ':', label='complex-$\Omega$')

plt.axhline(0, ls='-.', color='grey')
plt.axvline(0, ls='-.', color='grey')
plt.legend(bbox_to_anchor=(1.05, 1), loc="upper left")
plt.xlabel('Re($K$)')
plt.ylabel('Re($\Omega$)')
plt.title(f'Symmetry-Breaking DRs, $G(u)={a}-{b:.1f}u$')
plt.show()
```





Besides the ELN distribution `G`, the DR producing functions also accept the following optional arguments:

- `num_pts` is the number of points to compute on each DR branch. The default value is 100.
- `int_opts` is a dictionary of optional keyword arguments that will be passed on to `scipy.integrate.quad` which computes various integrals. It is empty by default.
- `DR_complexK` and `DR_complexOmega` in both modules accepts `rt_opts` which is a dictionary

of optional keyword arguments that will be passed on to `scipy.optimize.root`. It is empty by default.

- `shift` indicates whether K and Ω are the shifted values as defined in Eq. (21) of REF. It is `True` by default.
- It can be numerically challenging to calculate certain properties on a few special points. `DR_complexK` and `DR_complexOmega` in both modules and `asdr.DR_real` accept a small number `eps` as an optional argument which adjusts the numerical behaviors of some of the underlying functions. Its default value is 10^{-5}. It can be loosely understood as the uncertainty in the refractive index $n = K/\Omega$.
- Because the real axially symmetric DRs span from $-\infty$ to $+\infty$, `asdr.DR_real` accepts two optional arguments `maxK` and `minK` which specify the maximum and minimum K values to compute. Their default values are $+1$ and -1, respectively.

For simplicity, `fowdr` assumes that $G(-1) > 0$. One can always flip the z axis if this is not the case. `fowdr` assumes no presence of ordinary. A uniform matter background simply shifts Ω and K [see Eq. (21) of REF].

More examples of how to use `fowdr` can be found in `examples.ipynb`.

3 Method

Below is a brief description of the method used by `fowdr`. Please see the source code and REF for details.

3.1 Real Branches

The symmetry-breaking DR branches obey the equation

$$\mathfrak{D}_{\mathrm{SB}}(\Omega, K) = J_0 - J_2 - 2 = 0,$$

where

$$J_p(\Omega, K) = \int_{-1}^{1} G(u)\, \frac{u^p}{\Omega - Ku}\, \mathrm{d}u$$

is denoted as I_p in REF. Therefore, they can be simply computed as

$$\Omega(n) = \frac{1}{2} \int_{-1}^{1} G(u)\, \frac{1 - u^2}{1 - nu}\, \mathrm{d}u$$

and $K(n) = n\Omega(n)$, where the refractive index $n = \Omega/K$ is in the range $[-1, -1]$.

Similarly, the axially symmetric DRs obey the equation

$$\mathfrak{D}_{\mathrm{AS}}(\Omega, K) = (J_0 + 1)(J_2 - 1) - J_1^2 = 0$$

which yields

$$\Omega_{\pm}(n) = \frac{I_2 - I_0 \pm \sqrt{\Delta}}{2},$$

where

$$I_p(n) = \int_{-1}^{1} G(u)\, \frac{u^p}{1 - nu}\, \mathrm{d}u$$

is denoted as $\tilde{I}_p$ in REF, and

$$\Delta = (I_2 - I_0)^2 + 4(I_2 I_0 - I_1^2).$$

If there is no crossing, $\Omega_\pm(n)$ and $K_\pm(n) = n\Omega_\pm(n)$ give two distinct real DRs for $n \in (-1, 1)$. If there is a crossing (indicated by $G(1) < 0$), then $\Delta < 0$, and $\Omega_\pm(n)$ and $K_\pm(n)$ with $n \in (-1, n_*]$ give two parts of a single DR that joins at $n = n_*$ where $\Delta(n_*) = 0$. Neverless, `as.DR_real` will still return these two parts were two distinct branches. As the crossing deepens, the axially symmetric real DR disappears when n_* reaches -1.

3.2 Complex-K Branches

For a given real Ω on a complex-K DR branch, one can solve K from the DR equation $\mathfrak{D}_{\mathrm{SB}} = 0$ or $\mathfrak{D}_{\mathrm{AS}} = 0$ by using the root finding functions of SciPy. The difficult part of this calculation is to know the range of Ω and a relative good initial guess of K. Fortunately, the end points of the complex-K branches can be calculated [REF].

For the symmetry-breaking case, $\Omega = 0$ is one of the end points of a complex-K branch, and the corresponding wave number K_0 can be calculated from the DR equation at $\Omega \to 0^+$ by using the Sokhotski-Plemelj theorem. When there is no ELN crossing, the other end of the complex-K branch is a turning point at $\Omega(n_{\mathrm{c}})$ on the real branch where

$$\left.\frac{\mathrm{d}\Omega}{\mathrm{d}n}\right|_{n=n_{\mathrm{c}}} = 0.$$

(This point is labeled as Ω_{b} in REF.) A new critical point $n_{\mathrm{x}} = 1/u_{\mathrm{x}}$ appears when there is an ELN crossing at u_{x} where $G(u_{\mathrm{x}}) = 0$. (This point is labeled as Ω_{c} in REF.) The wave number $K(n_{\mathrm{x}})$ can also be computed from the DR equation by using the Sokhotski-Plemelj theorem. If the crossing is shallow, there is a second turning point n'_{c} on the real branch, and a complex-K branch runs from $\Omega(n_{\mathrm{x}})$ to $\Omega(n'_{\mathrm{c}})$ besides the one from $\Omega(n_{\mathrm{c}})$ to $\Omega = 0$. If the crossing is deep, there is no turning point on the real branch, and there is only one complex-K branch that runs from $\Omega(n_{\mathrm{x}})$ to $\Omega = 0$.

The axially symmetric case is like a pair of symmetry-breaking cases. For example, there two complex-K branches that approach $\Omega = 0$ from the positive and negative sides, respectively, and there are two critical points that correspond to n_{x} when there is crossing. The plus and minus parts of the real branch can each have zero or two turning points. Therefore, there can be 2, 3, or 4 complex-K branches in the axially symmetric case.

3.3 Complex-Ω Branches

There is no complex-Ω DR branch if there is no ELN crossing. When there is crossing, the complex-Ω branches can also be solved from the DR equations by root finding. The range of the real K of a complex-Ω branch is again determined by the critical points [REF]. For the symmetry-breaking case, one of the end points is at $K(n_{\mathrm{x}})$, and the other end is on a turning point $K(n''_{\mathrm{c}})$ on the real branch where

$$\left.\frac{\mathrm{d}K}{\mathrm{d}n}\right|_{n=n''_{\mathrm{c}}} = 0.$$

(This point is labeled as K_{t} in REF.) The axially symmetric case is again like double the symmetry-breaking case except when there is no real DR branch. In that case, there is only one complex-Ω branch that runs from $K_+(n_{\mathrm{x}})$ to $K_-(n_{\mathrm{x}})$.

4 Acknowledgements

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