Analysis of Categorical Data

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1

Outline

- Analysis of Categorical Data
  - Introduction
  - Homogeneity test
  - Independence test

2

Introduction

- A study of data in categories
- Case: Population I of interest; Each population is separated into J categories
  - Example: 3 department stores vs. 5 payment methods (cash, check, store credit card, Visa, Mastercard)
- Homogeneity (Hypothesis) Test
  - Proportions of all categories in each population are the same

3

Introduction (cont.)

- In general, data are put in the table
- Let n_{ij} = number of samples in (i,j) category
- Table contains $\{n_{ij}\}$'s is called two-way contingency table

	1	2	…	j	…	J
1	n_{11}	n_{12}	…	n_{1j}	…	n_{1J}
2	n_{21}					
…	…					
i	n_{i1}			n_{ij}		
…	…					
I	n_{I1}					n_{IJ}

4

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5

Homogeneity Test

- Population I of interest; Each population is separated into J categories

- Let
  - n_{ij} = number of samples in (i,j) category
  - n_j = number of samples in j category = $\sum_i n_{ij}$
  - n_i = number of samples in i population = $\sum_j n_{ij}$
  - n = number of all samples = $\sum_i \sum_j n_{ij}$
  - p_{ij} = proportions of samples in (i,j) category
- Hypothesis test
  - Null hypothesis (H_0): $p_{1j} = p_{2j} = \ldots = p_{Ij}$
    - Proportion of samples in j category for each population is the same
  - Alternative hypothesis (H_a): H_0 is not true

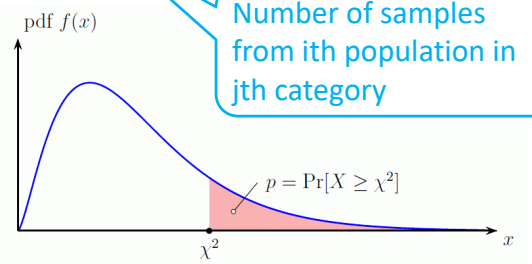
6

Homogeneity Test (cont.)

P(samples in category j)

- Let $\hat{e}_{ij}$ = expected number of samples = $n_i p_j = n_i \frac{n_j}{n}$
- Test statistic
  - $\chi^2 = \sum_i \sum_j \frac{(n_{ij} - \hat{e}_{ij})^2}{\hat{e}_{ij}}$
- Rejection region
  - $\chi^2 \geq \chi^2_{\alpha,(I-1)(J-1)}$

Number of samples from ith population in jth category



- In each row i, there are J cells but $n_i = \sum_j n_{ij}$ is fixed. Hence, d.f.per row = J-1. There are I rows. Thus, sum of d.f. from all rows = I(J-1)
- In addition, we estimate $p_1, p_2, \ldots, p_J$ with $\sum_i p_i = 1$. There are J-1 parameters to estimate.
- At the end, resulting d.f. = I(J-1) - (J-1) = (I-1)(J-1)

Example

- A can food company have three product sizes; each size is produced at different production lines
- Test in nonconformity of cans at significance level 0.5
  - Blemish, Crack, Improper pull tab location, Missing pull tab, Others

		Nonconformity					
		Blemish	Crack	Location	Missing	Others	Sample size
Production line	1	34	65	17	21	13	150
	2	23	52	25	19	6	125
	3	32	28	16	14	10	100
	Total	89	145	58	54	29	375

n_i

n_J

Example (cont.)

- Hypothesis
  - H_0: All production lines are homogeneous in term of nonconformity categories (Blemish, Crack, Improper pull tab location, Missing pull tab, Others)
    - I = number of production lines = 3
    - J = types of nonconformity = 5
    - That is we test whether $p_{1j} = p_{2j} = p_{3j}$ for j = 1, 2, …, 5
  - H_a: Production lines are not homogeneous

Example (cont.)

- Find $\hat{e}_{ij}$ = expected number of samples = $n_i \frac{n_j}{n}$

		$\hat{e}_{ij}$					
		Blemish	Crack	Location	Missing	Others	Sample size
Production line	1	$\frac{150(89)}{375}$ =35.60	$\frac{150(145)}{375}$ =58.00	$\frac{150(58)}{375}$ =23.20	$\frac{150(54)}{375}$ =21.60	$\frac{150(29)}{375}$ =11.60	150
	2	$\frac{125(89)}{375}$ = 29.67	48.33	19.33	18.00	9.67	125
	3	$\frac{100(89)}{375}$ = 23.73	38.7	15.47	14.40	7.73	100
	Total	89	145	58	54	29	375

n_i

n_J

Example (cont.)

- Find test statistic = $\sum_i \sum_j \frac{(n_{ij}-\hat{e}_{ij})^2}{\hat{e}_{ij}}$

		$\frac{(n_{ij}-\hat{e}_{ij})^2}{\hat{e}_{ij}}$				
		Blemish	Crack	Location	Missing	Others
Production line	1	$\frac{(34-35.60)^2}{35.60}$ = 0.072	$\frac{(65-58.00)^2}{58.00}$ = 0.845	$\frac{(17-23.20)^2}{23.20}$ = 1.657	$\frac{(21-21.60)^2}{21.60}$ = 0.017	$\frac{(13-11.60)^2}{11.60}$ = 0.169
	2	$\frac{(23-29.67)^2}{29.67}$ =1.498	0.278	1.661	0.056	1.391
	3	$\frac{(32-23.73)^2}{23.73}$ = 2.879	2.943	0.018	0.011	0.664

- Test statistic = $\sum_i \sum_j \frac{(n_{ij}-\hat{e}_{ij})^2}{\hat{e}_{ij}} = 14.159$

Example (cont.)

- Test statistic = $\sum_i \sum_j \frac{(n_{ij}-\hat{e}_{ij})^2}{\hat{e}_{ij}} = 14.159$
- Find rejection region:
  - Degree of freedom = (I-1) (J-1) = (3-1)(5-1) = (2)(4) = 8
  - $\chi^2_{0.05,8}$= 15.507
- Thus, we do not reject hypothesis at α = 0.05
- At significance level = 0.05, all production lines are homogeneous in term of nonconformity categories

Example (cont.)

- Test statistic = $\sum_i \sum_j \frac{(n_{ij} - \hat{e}_{ij})^2}{\hat{e}_{ij}} = 14.159$
- Find p-value
  - Degree of freedom = (I-1) (J-1) = (3-1)(5-1) = (2)(4) = 8
  - P-Value = 0.077
- Thus, we do not reject hypothesis at since p-value > α = 0.05
- At significance level = 0.05, all production lines are homogeneous in term of nonconformity categories

```
from scipy.stats import chi2

1-chi2.cdf(14.159,8)

0.07771412238511499
```


Example 2

- Compare two books whether they were written by the same author or not

- How to compare these two books?

Image Source: http://www.clipartpanda.com/categories/school-book-clipart

Example 2

- Compare whether the frequencies of words in three of Austen's works are the same

Word	Sense and Sensibility	Emma	Sandition
a	147	186	101
an	25	26	11
this	32	39	15
that	94	105	37
with	59	74	28
without	18	10	10

- Test homogeneity
- Let p_{ij} = probability of word j appeared in work i
- Hypothesis
  - H_0: $p_{1j} = p_{2j} = p_{3j}$ for j = 1,2,…,6
  - H_a: probability of word j appeared in work i is not the same

Example 2 (cont.)

n_{ij} = # of times that word j appeared in work i

- Compare whether the frequencies of words in three of Austen's works are the same

Word	*Sense and Sensibility* (by Austen)	*Emma* (by Austen)	*Sandition* (by Austen)
a	147	186	101
an	25	26	11
this	32	39	15
that	94	105	37
with	59	74	28
without	18	10	10

- Find n_i and n_j

Word	*Sense and Sensibility*	*Emma*	*Sanditi on (by Austen)*	nj
a	147	186	101	434
an	25	26	11	62
this	32	39	15	86
that	94	105	37	236
with	59	74	28	161
without	18	10	10	38
ni	375	440	202	1017

Example 2 (cont.)

- Find $\hat{e}_{ij}$ = expected number of samples = $n_i \frac{n_j}{n}$

Word	*Sense and Sensibility*	*Emma*	*Sanditi on (by Austen)*	nj
a	147	186	101	434
an	25	26	11	62
this	32	39	15	86
that	94	105	37	236
with	59	74	28	161
without	18	10	10	38
ni	375	440	202	1017

$\hat{e}_{ij} = n_i \frac{n_j}{n}$

Word	*Sense and Sensibility*	*Emma*	*Sanditi on (by Austen)*
a	160.0295	187.7679	86.20256
an	22.86136	26.82399	12.31465
this	31.71091	37.20747	17.08161
that	87.02065	102.1042	46.87512
with	59.36578	69.65585	31.97837
without	14.0118	16.44051	7.547689

Example 2 (cont.)

- Find test statistic = $\sum_i \sum_j \frac{(n_{ij} - \hat{e}_{ij})^2}{\hat{e}_{ij}}$

n_{ij}

Word	*Sense and Sensibility*	*Emma*	*Sanditi on (by Austen)*
a	147	186	101
an	25	26	11
this	32	39	15
that	94	105	37
with	59	74	28
without	18	10	10

$\hat{e}_{ij} = n_i \frac{n_j}{n}$

Word	*Sense and Sensibility*	*Emma*	*Sanditi on (by Austen)*
a	160.0295	187.7679	86.20256
an	22.86136	26.82399	12.31465
this	31.71091	37.20747	17.08161
that	87.02065	102.1042	46.87512
with	59.36578	69.65585	31.97837
without	14.0118	16.44051	7.547689

Word	*Sense and Sensibility*	*Emma*	*Sanditi on (by Austen)*
a	1.060853	0.016646	2.540114
an	0.200067	0.025312	0.140346
this	0.002635	0.086358	0.253671
that	0.559768	0.082127	2.08038
with	0.002254	0.270927	0.494941
without	1.135168	2.523047	0.796777

Test statistic = 12.27

Example (cont.)

- Test statistic = $\sum_i \sum_j \frac{(n_{ij} - \hat{e}_{ij})^2}{\hat{e}_{ij}} = 12.27$
- Given α = 0.1, find rejection region
  - Degree of freedom = (6-1)(3-1) = 10
  - $\chi^2_{0.1,10}$= 15.987
- Thus, we do not reject hypothesis at α = 0.1
- Portportions of words in three of Austen's works are the same

Outline

- Analysis of Categorical Data
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  - Homogeneity test
  - Independence test

Introduction

- A study of data in categories
- Case: Single population with two factors; One factor with I categories, and the other factor with J categories
  - Example: One department store, 6 departments (male clothes, female clothes, children, cosmetics, shoes, grocery) vs. 5 payment methods (cash, check, store credit card, Visa, Mastercard)
- Independence Test
  - Two factors occur independently

Introduction (cont.)

- In general, data are put in the table
- Let n_{ij} = number of samples in (i,j) category
- Table contains $\{n_{ij}\}$'s is called two-way contingency table

	1	2	…	j	…	J
1	n_{11}	n_{12}	…	n_{1j}	…	n_{1J}
2	n_{21}					
…	…					
i	n_{i1}			n_{ij}		
…	…					
I	n_{I1}					n_{IJ}

Independence Test

- Single population with two factors; One factor with I categories, and the other factor with J categories

- Let
  - n_{ij} = number of samples in (i,j) category
  - n_j = number of samples in j category = $\sum_i n_{ij}$
  - n_i = number of samples in i category = $\sum_j n_{ij}$
  - n = number of all samples = $\sum_i \sum_j n_{ij}$
  - p_{ij} = proportions of samples in (i,j) category
- Hypothesis test
  - Null hypothesis (H_0): $p_{ij} = p_i p_j$
    - Proportion of samples in categories i and j are independent
  - Alternative hypothesis (H_a): H_0 is not true

Independence Test (cont.)

If two factors are independent, $p_{ij} = p_i p_j$

- Let $\hat{e}_{ij}$ = expected number of samples = $np_{ij} = np_ip_j = n\frac{n_i}{n}\frac{n_j}{n} = \frac{n_i n_j}{n}$
- Test statistic
  - $\chi^2 = \sum_i \sum_j \frac{(n_{ij} - \hat{e}_{ij})^2}{\hat{e}_{ij}}$
- Rejection region
  - $\chi^2 \geq \chi^2_{\alpha,(I-1)(J-1)}$

Derivation of $\hat{e}_{ij}$ is different from Homogeneity test

Same $\hat{e}_{ij}$ as Homogeneity Test

Example

- Study of gasoline station condition and aggressiveness in gasoline pricing
- Two factors: gasoline station condition (modern, standard, sub-standard) vs. aggressiveness in pricing (aggressive, neutral, nonaggressive)
- Test whether two factors are independent of each other at significance level = 0.01

		Aggressiveness in pricing			
		Aggressive	Neutral	Non Aggressive	Sample Size
Condition	Substandard	24	15	17	56
	Standard	52	73	80	205
	Modern	58	86	36	180
	Total	134	174	133	441

Example (cont.)

- Hypothesis
  - H_0: Gasoline station condition and aggressiveness in pricing are independent
    - I = number of conditions = 3
    - J = levels of pricing aggressiveness = 3
    - We test on $p_{ij} = p_i\, p_j$
  - H_a: Gasoline station condition and aggressiveness in pricing are not independent

Example (cont.)

- Find $\hat{e}_{ij}$ = expected number of samples = $\frac{n_i n_j}{n}$

		$\hat{e}_{ij}$			
		Aggressive	Neutral	Non Aggressive	Sample Size
Condition	Substandard	$\frac{56(134)}{441}$ =17.02	$\frac{56(174)}{441}$ =22.10	$\frac{56(133)}{441}$ =16.89	56
	Standard	$\frac{205(134)}{441}$ =62.29	80.88	61.83	205
	Modern	$\frac{180(134)}{441}$ =54.69	71.02	54.29	180
	Total	134	174	133	441

Example (cont.)

- Find test statistic = $\sum_i \sum_j \frac{(n_{ij}-\hat{e}_{ij})^2}{\hat{e}_{ij}}$

		$\frac{(n_{ij}-\hat{e}_{ij})^2}{\hat{e}_{ij}}$		
		Aggressive	Neutral	Non Aggressive
Condition	Substandard	$\frac{(24-17.02)^2}{17.02}$ = 2.867	$\frac{(15-22.10)^2}{22.10}$ = 2.278	$\frac{(17-16.89)^2}{16.89}$ = 0.001
	Standard	$\frac{(52-62.29)^2}{62.29}$ = 1.700	0.769	5.343
	Modern	$\frac{(58-54.69)^2}{54.69}$ = 0.200	3.160	6.160

- Test statistic = $\sum_i \sum_j \frac{(n_{ij}-\hat{e}_{ij})^2}{\hat{e}_{ij}}$ = 22.476

Example (cont.)

- Test statistic = $\sum_i \sum_j \frac{(n_{ij} - \hat{e}_{ij})^2}{\hat{e}_{ij}} = 22.476$
- Given α = 0.01, find p-value:
  - Degree of freedom = (I-1) (J-1) = (3-1)(3-1) = 4
- P-value = 0.00016
- P-value < α = 0.01 => Null hypothesis is rejected
- Gasoline station condition and aggressiveness in pricing are dependent

$\chi^2_{0.01,4}$=13.277

Rejection region

$\chi^2 \geq 13.277$

```
from scipy.stats import chi2
1-chi2.cdf(22.476,4)
0.0001611050155756466
```


Example 2

- Is there a relationship between marital status and educational level?

Education	Married once	Married more than once
College degree	550	61
No college degree	681	144

- Test independency
- Let p_{ij} = probability of person with education i has marriage type j
- Hypothesis
  - H_0: Education and marriage type are independent
    - I = number of education type = 2
    - J = number of marriage type = 2
    - We test $p_{ij} = p_i p_j$
  - H_a: Education and marriage type are not independent

Example 2 (cont.)

- Is there a relationship between marital status and educational level?

Education	Married once	Married more than once
College degree	550	61
No college degree	681	144

- Find n_i and n_j

Education	Married once	Married more than once	nj
College	550	61	611
No college	681	144	825
ni	1231	205	1436

Example 2 (cont.)

- Find $\hat{e}_{ij}$ = expected number of samples = $\frac{n_i n_j}{n}$

Education	Married once	Married more than once	nj
College	550	61	611
No college	681	144	825
ni	1231	205	1436

$\hat{e}_{ij} = \frac{n_i n_j}{n}$

Education	Married once	Married more than once
College	523.7751	87.22493
No college	707.2249	117.7751

Example 2 (cont.)

- Find test statistic = $\sum_i \sum_j \frac{(n_{ij} - \hat{e}_{ij})^2}{\hat{e}_{ij}}$

n_{ij}

Education	Married once	Married more than once
College	550	61
No college	681	144

$\hat{e}_{ij} = \frac{n_i n_j}{n}$

Education	Married once	Married more than once
College	523.7751	87.22493
No college	707.2249	117.7751

Education	Married once	Married more than once
College	1.31306	7.88475
No college	0.97246	5.83950

Test statistic = 16.01

Example 2 (cont.)

- Test statistic = $\sum_i \sum_j \frac{(n_{ij} - \hat{e}_{ij})^2}{\hat{e}_{ij}}$ = 16.001
- Given α = 0.01
  - Degree of freedom = (2-1)(2-1) = 1
- P-value = 6.33×10^{-5}
- Thus, we reject null hypothesis at α = 0.01
- Education and marriage are dependent.

$\chi^2_{0.01,1} = 6.64$

$Rejection\ region$

$\chi^2 \geq 6.64$

```
1-chi2.cdf(16.001,1)
6.330903499540685e-05
```

References

1. J.L. Devore and K.N.Berk, Modern Mathematical Statistics with Applications, Springer, 2012.
2. J.A. Rice, Mathematical Statistics and Data Analysis, Duxbury Press, 1995.