Analysis of Variance
Two Factors

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Outline

- Two-Factor ANOVA
  - Additive Factors
  - Interaction Factors
  - Interaction Plot
- Additions

Two-Factor ANOVA : Additive Factors

- Compare two or more populations on <u>two</u> factors
  - Factor A has I level of treatments
  - Factor B has J level of treatments
  - Example: test washing detergents on pens
    - A = Brand of pens, I = 3
    - B = Washing detergent, J = 4

		Washing detergent			
		1	2	3	4
Brand of pens	1	0.97	0.48	0.48	0.46
	2	0.77	0.14	0.22	0.25
	3	0.67	0.39	0.57	0.19

Additive Two-Factor ANOVA : Effects Model

$X_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij}$ $\qquad \sum \alpha_i = 0 \qquad \sum \beta_j = 0$

- X_{ij} = random sample j of treatment i
- μ = overall mean of treatment
- α_i = effect due to factor A at level i
- β_j = effect due to factor B at level j
- ε_{ij} = random error from sample j of treatment i
  - Assumed to be independent and normally distributed with mean = 0, variance = σ^2
- Factor A is independent of factor B

Additive Two-Factor ANOVA : Effects Model (cont.)

$\boxed{E(X_{ij}) = \mu + \alpha_i + \beta_j}$ $\boxed{\sum \alpha_i = 0}$ $\boxed{\sum \beta_j = 0}$

- If $\alpha_i = 0$ and $\beta_j = 0$, then all treatments have the same response

$\boxed{E(X_{ij}) = \mu}$

- Thus, null hypotheses for two additive factor ANOVA
  - Hypothesis on A : factor A at any level i has no effect on overall mean.
    - H_{0A}: $\alpha_1 = \alpha_2 = \ldots = \alpha_I = 0$
  - Hypothesis on B : factor B at any level j has no effect on overall mean.
    - H_{0B}: $\beta_1 = \beta_2 = \ldots = \beta_J = 0$

Two-Factor ANOVA : Additive Factors (cont.)

- Let
  - μ_{ij} = mean of treatment i of factor A and treatment j of factor B
  - I = number of treatments from factor A
  - J = number of treatments from factor B
- Hypothesis on A : factor A at any level i has no effect on true mean.
  - H_{0A}: $\alpha_1 = \alpha_2 = \ldots = \alpha_I = 0$
  - H_{aA}: Not all α_i's are equal (Factor A has effect.)
- Hypothesis on B : factor B at any level j has no effect on true mean.
  - H_{0B}: $\beta_1 = \beta_2 = \ldots = \beta_J = 0$
  - H_{aB}: Not all β_j's are equal (Factor B has effect.)

Two-Factor ANOVA : Additive Factors (cont.)

- Test statistic (cont.):
  - I = Number of treatments from factor A
  - J = Number of treatments from factor B

SST = $\sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X})^2$ df = $IJ - 1$

SSA = $\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{X}_i - \bar{X})^2$ df = $I - 1$

SSB = $\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{X}_j - \bar{X})^2$ df = $J - 1$

SSE = $\sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X}_i - \bar{X}_j + \bar{X})^2$ df = $(I - 1)(J - 1)$

$$\boxed{SST = SSA + SSB + SSE}$$

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Two-Factor ANOVA : Additive Factors (cont.)

- Test statistic (cont.):
  - I = Number of treatments from factor A
  - J = Number of treatments from factor B

Another option: Sample-based computation

SST = $\sum_{i=1}^{I}\sum_{j=1}^{J}X_{ij}^2 - \frac{1}{IJ}(\sum_{i=1}^{I}\sum_{j=1}^{J}X_{ij})^2$ df = $IJ - 1$

SSA = $\frac{1}{J}\sum_{i=1}^{I}(\sum_{j=1}^{J}X_{ij})^2 - \frac{1}{IJ}(\sum_{i=1}^{I}\sum_{j=1}^{J}X_{ij})^2$ df = $I - 1$

SSB = $\frac{1}{I}\sum_{j=1}^{J}(\sum_{i=1}^{I}X_{ij})^2 - \frac{1}{IJ}(\sum_{i=1}^{I}\sum_{j=1}^{J}X_{ij})^2$ df = $J - 1$

SSE = SST – SSA - SSB df = $(I - 1)(J - 1)$

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Two-Factor ANOVA : Additive Factors (cont.)

- Test statistic (cont.):

Hypothesis	Mean Square (MS)	Test statistic (f)	Rejection region
H_{0A} vs. H_{aA}	SSA/(I-1)	f_A = MSA / MSE	$f_A > F_{\alpha, I-1, (I-1)(J-1)}$
H_{0B} vs. H_{aB}	SSB/(J - 1)	f_B = MSB / MSE	$f_B > F_{\alpha, J-1, (I-1)(J-1)}$

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Example

- Test 4 washing detergents on 3 brands of pens at significance level = 0.05

		Washing detergent				
		1	2	3	4	
Brand of pens	1	0.97	0.48	0.48	0.46	0.598
	2	0.77	0.14	0.22	0.25	0.345
	3	0.67	0.39	0.57	0.19	0.455
		0.803	0.337	0.423	0.300	0.466

- A = Brand of pens, I = 3
- B = Washing detergents, J = 4
- SST* = 0.6947, df = 11
- SSA* = 0.1282, df = 2
- SSB* = 0.4797, df = 3
- SSE = 0.6947 – 0.1282 – 0.4797 = 0.0868, df = 11 – 2 – 3 = 6

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Example (cont.)

- Test 4 washing detergents on 3 brands of pens (cont.):
  - SST* = 0.6947, df = 11
  - SSA* = 0.1282, df = 2
  - SSB* = 0.4797, df = 3
  - SSE = 0.0868, df = 6

Hypothesis	Mean Square (MS = SS/df)	Test statistic (f)	Rejection region
H_{0A} vs. H_{aA}	0.1282/2 = 0.0641	f_A = 4.43	$F_{0.05, 2, 6} = 5.14$
H_{0B} vs. H_{aB}	0.4797 / 3 = 0.1599	f_B = 11.05	$F_{0.05, 3, 6} = 4.76$
Error	0.0868/6 =0.0144		

Example (cont.)

- Test 4 washing detergents on 3 brands of pens (cont.):

Hypothesis	Mean Square (MS)	Test statistic (f)	Rejection region
H_{0A} vs. H_{aA}	0.1282/2 = 0.0641	f_A = 4.43	$F_{0.05, 2, 6} = 5.14$
H_{0B} vs. H_{aB}	0.4797 / 3 = 0.1599	f_B = 11.05	$F_{0.05, 3, 6} = 4.76$
Error	0.0868/6 =0.0144		

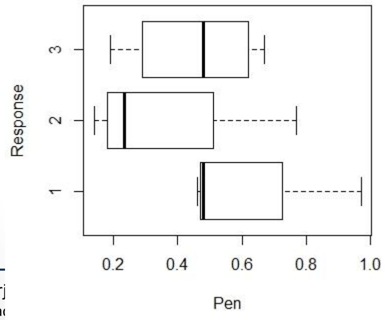
- H_{0A} is not rejected. Factor A (brand of pens) has no effect on mark removal
- H_{0B} is rejected. Factor B (washing detergents) has effect on mark removal

Example

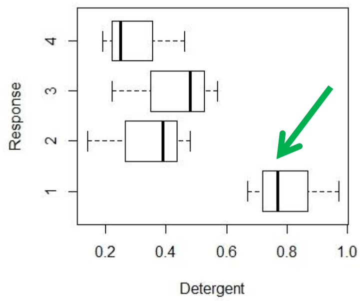
- Factor A (brand of pens) has no effect on mark removal
- Factor B (washing detergents) has effect on mark removal

		Washing detergent				
		1	2	3	4	
Brand of pens	1	0.97	0.48	0.48	0.46	0.598
	2	0.77	0.14	0.22	0.25	0.345
	3	0.67	0.39	0.57	0.19	0.455
		0.803	0.337	0.423	0.300	0.466

Brand of pens



Washing detergent



Multiple Comparisons

- To specify which mean is different from others
  - Fix on factor A or B
  - Find Tukey's Honestly Significant Difference (HSD) using the following formula

    Factor A: $w_A = q_{\alpha,I,(I-1)(J-1)}\sqrt{\frac{MSE}{J}}$

    Factor B: $w_B = q_{\alpha,\, J,(I-1)(J-1)}\sqrt{\frac{MSE}{I}}$

    - $q_{\alpha,\, m,\, n}$ = q-value from studentized range distribution with 2 degrees of freedom m, n
  - Follow the same steps as finding other HSD

Example

Factor A: $w_A = q_{\alpha,I,(I-1)(J-1)}\sqrt{\frac{MSE}{J}}$

- Test 4 washing detergents on 3 brands of pens at significance level = 0.05

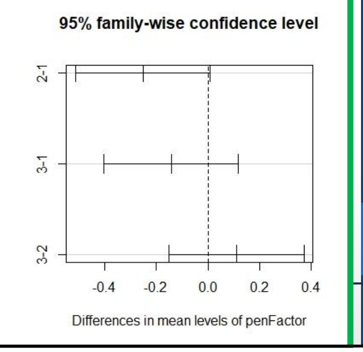
		Washing detergents				
		1	2	3	4	
Brand of pens	1	0.97	0.48	0.48	0.46	0.598
	2	0.77	0.14	0.22	0.25	0.345
	3	0.67	0.39	0.57	0.19	0.455
		0.803	0.337	0.423	0.300	0.466

Hypothesis	Mean Square (MS)	Test statistic (f)	Rejection region
H_{0A} vs. H_{aA}	0.1282/2 = 0.0641	f_A = 4.43	$F_{0.05,2,6}$ = 5.14
H_{0B} vs. H_{aB}	0.4797 / 3 = 0.1599	f_B = 11.05	$F_{0.05,3,6}$ = 4.76
Error	0.0868/6 =0.0144		

- Fixed at brands of pens:

$$w_A = q_{0.05,3,6}\sqrt{\frac{MSE}{J}} = 4.34\sqrt{\frac{0.0144}{4}}=0.261$$

- Sort factor-A sample means: $\underline{\overset{\bar{x}_2}{0.345},\ \overset{\bar{x}_3}{0.455},\ \overset{\bar{x}_1}{0.598}}$
- 1 group of means: $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$



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Example (cont.)

Factor B: $w_B = q_{\alpha,J,(I-1)(J-1)}\sqrt{\frac{MSE}{I}}$

- Test 4 washing detergents on 3 brands of pens at significance level = 0.05

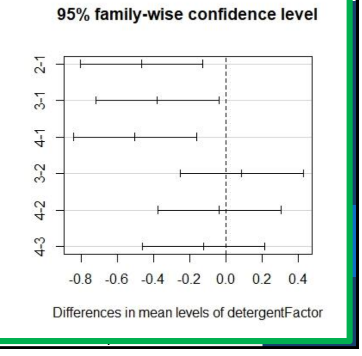
		Washing detergents				
		1	2	3	4	
Brand of pens	1	0.97	0.48	0.48	0.46	0.598
	2	0.77	0.14	0.22	0.25	0.345
	3	0.67	0.39	0.57	0.19	0.455
		0.803	0.337	0.423	0.300	0.466

Hypothesis	Mean Square (MS)	Test statistic (f)	Rejection region
H_{0A} vs. H_{aA}	0.1282/2 = 0.0641	f_A = 4.43	$F_{0.05,2,6}$ = 5.14
H_{0B} vs. H_{aB}	0.4797 / 3 = 0.1599	f_B = 11.05	$F_{0.05,3,6}$ = 4.76
Error	0.0868/6 =0.0144		

- Fixed at detergent factor:

$$w_B = q_{0.05,4,6}\sqrt{\frac{MSE}{I}} = 4.90\sqrt{\frac{0.0144}{3}}=0.340$$

- Sort factor-B sample means: $\underline{\overset{\bar{x}_4}{0.300},\ \overset{\bar{x}_2}{0.337},\ \overset{\bar{x}_3}{0.423}},\ \overset{\bar{x}_1}{0.803}$
- 2 groups of means: $\{\bar{x}_4, \bar{x}_2, \bar{x}_3\}$ and $\bar{x}_1$



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Example 2

- Test 4 coatings on 3 soil type for corrosion at significance level = 0.05

		Soil Type (B)			
		1	2	3	$\bar{x}_j$
Coating (A)	1	64	49	50	54.33
	2	53	51	48	50.67
	3	47	45	50	47.33
	4	51	43	52	48.67
	$\bar{x}_i$	53.75	47.00	50.00	50.25

- A = Coatings, I = 4, B = Soil Types, J = 3
- SST = 242.063, df = 11 $\quad$ SST = $\sum_{i=1}^{I}\sum_{j=1}^{J}(X_{ij} - \bar{X})^2$
- SSA = 83.583, df = 3 $\quad$ SSA = $\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{X}_i - \bar{X})^2$
- SSB = 91.500, df = 2 $\quad$ SSB = $\sum_{i=1}^{I}\sum_{j=1}^{J}(\bar{X}_j - \bar{X})^2$
- SSE = SST – SSA – SSB = 66.979, df = 11 – 3 – 2 = 6

Example 2 (cont.)

- Test 4 coatings on 3 soil types for corrosion (cont.):

Hypothesis	Mean Square (MS)	Test statistic (f)	Rejection region
H_{0A} vs. H_{aA}	83.583/3 = 27.861	f_A = 2.495	$F_{0.05, 3, 6}$ = 4.7571
H_{0B} vs. H_{aB}	91.500 / 2 = 45.750	f_B = 4.098	$F_{0.05, 2, 6}$ = 5.1433
Error	66.97/6 =11.163		

- H_{0A} is not rejected. Factor A (coatings) has no effect on corrosion
- H_{0B} is not rejected. Factor B (soil types) has no effect on corrosion

Outline

- Two-Factor ANOVA
  - Additive Factors
  - Interaction Factors
  - Interaction Plot
- Additions

Two-Factor ANOVA : Interaction Factors

- Compare two or more populations on <u>two interaction</u> factors
  - Factor A has I level of treatments
  - Factor B has J level of treatments
  - K Samples from treatment of factors and B are collected
  - Example: 3 varieties of tomatoes on 4 planting density

	Planting Density											
Variety	10,000			20,000			30,000			40,000		
H	10.5	9.2	7.9	12.8	11.2	13.3	12.1	12.6	14.0	10.8	9.1	12.5
Ife	8.1	8.6	10.1	12.7	13.7	11.5	14.4	15.4	13.7	11.3	12.5	14.5
P	16.1	15.3	17.5	16.6	19.2	18.5	20.8	18.0	21.0	18.4	18.9	17.2

Interaction Two-Factor ANOVA : Effects Model

$$\boxed{X_{ijk} = \mu \quad + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}} \quad \boxed{\sum \alpha_i = 0} \quad \boxed{\sum \beta_j = 0}$$

- X_{ij} = random sample j of treatment i
- μ = overall mean of treatment i on factor j
- α_i = effect due to factor A at level i
- β_j = effect due to factor B at level j
- Υ_{ij} = interaction parameter between factors A and B
- ε_{ij} = random error from sample j of treatment i
  - Assumed to be independent and normally distributed with mean = 0, variance = σ^2
- Factor A is not independent of factor B
  - If Υ_{ij} 's are all zeros, factors A and B are independent.

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Interaction Two-Factor ANOVA : Effects Model (cont.)

$$\boxed{X_{ijk} = \mu \quad + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}} \quad \boxed{\sum \alpha_i = 0} \quad \boxed{\sum \beta_j = 0}$$

- If $\alpha_i = 0$, $\beta_j = 0$, and $\gamma_{ij} = 0$, then all treatments have the same response

$$\boxed{E(X_{ijk}) = \mu}$$

- Thus, null hypotheses for interaction factor ANOVA
  - <u>Hypothesis on A and B</u> : factors A and B at any level i has no effect on overall mean.
    - H_{0AB}: $\Upsilon_{ij} = 0$ for all i,j ⟸ Test first If reject, no need to test H_{0A}, H_{0B}
  - <u>Hypothesis on A</u> : factor A at any level i has no effect on overall mean.
    - H_{0A}: $\alpha_1 = \alpha_2 = \ldots = \alpha_I = 0$
  - <u>Hypothesis on B</u> : factor B at any level j has no effect on overall mean.
    - H_{0B}: $\beta_1 = \beta_2 = \ldots = \beta_J = 0$

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Two-Factor ANOVA : Interaction Factors (cont.)

- Let
  - μ_{ijk} = mean of treatment i of factor A and treatment j of factor B
  - I = number of treatments from factor A
  - J = number of treatments from factor B
  - K = number of samples per treatments from factors A and B
- Hypothesis :
  - H_{0AB}: $\Upsilon_{ij} = 0$ for all i,j
  - H_{aAB}: At least $\Upsilon_{ij} \neq 0$
  - H_{0A}: $\alpha_1 = \alpha_2 = \ldots = \alpha_I = 0$
  - H_{aA}: At least $\alpha_i \neq 0$
  - H_{0B}: $\beta_1 = \beta_2 = \ldots = \beta_I = 0$
  - H_{aB}: At least $\beta_j \neq 0$

Two-Factor ANOVA : Interaction Factors

- Test statistic (cont.):

SST = $\sum_i \sum_j \sum_k (X_{ijk} - \bar{X})^2$ df = $IJK - 1$

SSA = $\sum_i \sum_j \sum_k (\bar{X}_i - \bar{X})^2$ df = $I - 1$

SSB = $\sum_i \sum_j \sum_k (\bar{X}_j - \bar{X})^2$ df = $J - 1$

SSAB = $\sum_i \sum_j \sum_k (\bar{X}_{ij} - \bar{X}_i - \bar{X}_j + \bar{X})^2$ df = $(I-1)(J-1)$

SSE = $\sum_i \sum_j \sum_k (X_{ijk} - \bar{X}_{ij})^2$ df = $IJ(K-1)$

$$\boxed{SST = SSA + SSB + SSAB + SSE}$$

Two-Factor ANOVA : Additive Factors (cont.)

- Test statistic (cont.):

Another option:
Sample-based computation

SST = $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} X_{ijk}^2 - \frac{1}{IJK}(\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} X_{ijk})^2$ df = $IJK-1$

SSA = $\frac{1}{JK}\sum_{i=1}^{I}(\sum_{j=1}^{J}\sum_{k=1}^{K} X_{ijk})^2 - \frac{1}{IJK}(\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} X_{ijk})^2$ df = $I-1$

SSB = $\frac{1}{IK}\sum_{j=1}^{J}(\sum_{i=1}^{I}\sum_{k=1}^{K} X_{ijk})^2 - \frac{1}{IJK}(\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} X_{ijk})^2$ df = $J-1$

SSE = $\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} X_{ijk}^2 - \frac{1}{K}\sum_{i=1}^{I}\sum_{j=1}^{J}[(\sum_{k=1}^{K} X_{ijk})^2]$ df = $(I-1)(J-1)$

SSAB =SST – SSA – SSB - SSE

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Two-Factor ANOVA : Interaction Factors (cont.)

- Test statistic (cont.):
  - I = Number of treatments from factor A
  - J = Number of treatments from factor B
  - K = number of samples per treatments from factors A and B

Hypothesis	Mean Square (MS)	Test statistic (f)	Rejection region
H_{0A} vs. H_{aA}	SSA/(**I**-1)	f_A = MSA / MSE	$f_A > F_{\alpha, I-1, IJ(K-1)}$
H_{0B} vs. H_{aB}	SSB/(**J** - 1)	f_B = MSB / MSE	$f_B > F_{\alpha, J-1, IJ(K-1)}$
H_{0AB} vs. H_{aAB}	$\frac{\text{SSAB}}{(\mathbf{I}-1)(\mathbf{J}-1)}$	f_{AB} = MSAB / MSE	$f_{AB} > F_{\alpha, (I-1)(J-1), IJ(K-1)}$

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Example

- Test 3 varieties of tomatoes on 4 planting density at significance level = 0.01

	Planting Density											
Variety	10,000			20,000			30,000			40,000		
H	10.5	9.2	7.9	12.8	11.2	13.3	12.1	12.6	14.0	10.8	9.1	12.5
Ife	8.1	8.6	10.1	12.7	13.7	11.5	14.4	15.4	13.7	11.3	12.5	14.5
P	16.1	15.3	17.5	16.6	19.2	18.5	20.8	18.0	21.0	18.4	18.9	17.2

- A = Varieties of tomatoes, I = 3
- B = Planting densities, J = 4
- K = Number of samples per factors A and B = 3
- IJK = 36

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Example (cont.)

- Test 3 varieties of tomatoes on 4 planting density at significance level = 0.01

	Planting Density											
Variety	10,000			20,000			30,000			40,000		
H	10.5	9.2	7.9	12.8	11.2	13.3	12.1	12.6	14.0	10.8	9.1	12.5
Ife	8.1	8.6	10.1	12.7	13.7	11.5	14.4	15.4	13.7	11.3	12.5	14.5
P	16.1	15.3	17.5	16.6	19.2	18.5	20.8	18.0	21.0	18.4	18.9	17.2

- SST = 460.36, df = 35 (IJK -1)
- SSA = 327.60, df = 2 (I-1)
- SSB = 86.69, df = 3 (J-1)
- SSE = 38.04, df = 24 (IJ(K-1)) -> MSE = 38.04 / 24 = 1.59
- SSAB = 460.36 – 327.60 – 86.69 – 38.04 = 8.03, df = 35 – 2 – 3 - 24 = 6

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Example (cont.)

- Test statistic :

Hypothesis	Mean Square (MS)	Test statistic (f)	Rejection region
H_{0A} vs. H_{aA}	163.8	f_A = 103.02	$F_{0.01, 2, 24} = 5.61$
H_{0B} vs. H_{aB}	28.9	f_B = 18.18	$F_{0.01, 3, 24} = 4.72$
H_{0AB} vs. H_{aAB}	1.34	f_{AB} = 0.84	$F_{0.01, 6, 24} = 3.67$

- H_{0AB} is not rejected. Interaction has no effect.
- H_{0A} is rejected. Factor A (varieties of tomatoes) has effect on average product
- H_{0B} is rejected. Factor B (planting densities) has effect on average product

Multiple Comparisons

- When interaction is rejected and one or both factors has the effect, we can perform multiple comparisons.
- To specify which mean is different from others
  - Fix on factor A or B
  - Find Tukey's Honestly Significant Difference (HSD) using the following formula

    Factor A: $w_A = q_{\alpha,I,IJ(K-1)}\sqrt{\frac{MSE}{JK}}$

    Factor B: $w_B = q_{\alpha,\,J,IJ(K-1)}\sqrt{\frac{MSE}{IK}}$
  - $q_{\alpha, m, n}$ = q-value from studentized range distribution with 2 degrees of freedom *m, n*
- Follow the same steps as finding other HSD

Example (cont.)

I = 3, J = 4, K = 3

IJ(K-1) = 24

- Test 3 varieties of tomatoes on 4 planting density at α = 0.01

	Planting Density												
Variety	10,000			20,000			30,000			40,000			$\bar{x}_i$
H	10.5	9.2	7.9	12.8	11.2	13.3	12.1	12.6	14.0	10.8	9.1	12.5	11.33
Ife	8.1	8.6	10.1	12.7	13.7	11.5	14.4	15.4	13.7	11.3	12.5	14.5	12.21
P	16.1	15.3	17.5	16.6	19.2	18.5	20.8	18.0	21.0	18.4	18.9	17.2	18.13

Factor A: $w_A = q_{\alpha,I,IJ(K-1)}\sqrt{\frac{MSE}{JK}}$

$$w_A = q_{0.01,3,24}\sqrt{\frac{MSE}{JK}} = 4.55\sqrt{\frac{1.59}{12}} = 1.66$$

- Sort sample means ($\bar{x}_i$) : $\underline{\overset{\bar{x}_1}{11.33},\ \overset{\bar{x}_2}{12.21}},\ \overset{\bar{x}_3}{18.13}$
- 2 groups of means: $\{\bar{x}_1, \bar{x}_2\}$ and $\bar{x}_3$

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Example (cont.)

I = 3, J = 4, K = 3

IJ(K-1) = 24

- Test 3 varieties of tomatoes on 4 planting density at α = 0.01

	Planting Density											
Variety	10,000			20,000			30,000			40,000		
H	10.5	9.2	7.9	12.8	11.2	13.3	12.1	12.6	14.0	10.8	9.1	12.5
Ife	8.1	8.6	10.1	12.7	13.7	11.5	14.4	15.4	13.7	11.3	12.5	14.5
P	16.1	15.3	17.5	16.6	19.2	18.5	20.8	18.0	21.0	18.4	18.9	17.2
$\bar{x}_j$	11.48			14.39			15.78			13.91		

Factor B: $w_B = q_{\alpha,J,IJ(K-1)}\sqrt{\frac{MSE}{IK}}$

$$w_B = q_{0.01,4,24}\sqrt{\frac{MSE}{IK}} = 4.91\sqrt{\frac{1.59}{9}} = 2.06$$

- Sort sample means ($\bar{x}_i$) : $\overset{\bar{x}_1}{11.48},\ \underline{\overset{\bar{x}_4}{13.91},\ \overset{\bar{x}_2}{14.39},\ \overset{\bar{x}_3}{15.78}}$
- 2 groups of mean: $\bar{x}_1$ and $\{\bar{x}_4, \bar{x}_2, \bar{x}_3\}$

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Example 2

- Test 2 varieties of Iron (Fe) on 3 concentration doses
- 18 samples per category

Fe^{2+}			Fe^{3+}		
10.2	**1.2**	**0.3**	**10.2**	**1.2**	**0.3**
0.71	2.20	2.25	2.20	4.04	2.71
1.66	2.93	3.93	2.69	4.16	5.43
2.01	3.08	5.08	3.54	4.42	6.38
2.16	3.49	5.82	3.75	4.93	6.38
2.42	4.11	5.84	3.83	5.49	8.32
2.42	4.95	6.89	4.08	5.77	9.04
2.56	5.16	8.50	4.27	5.86	9.56
2.60	5.54	8.56	4.53	6.28	10.01
3.31	5.68	9.44	5.32	6.97	10.08
3.64	6.25	10.52	6.18	7.06	10.62
3.74	7.25	13.46	6.22	7.78	13.80
3.74	7.90	13.57	6.33	9.23	15.99
4.39	8.85	14.76	6.97	9.34	17.90
4.50	11.96	16.41	6.97	9.91	18.25
5.07	15.54	16.96	7.52	13.46	19.32
5.26	15.89	17.56	8.36	18.40	19.87
8.15	18.30	22.82	11.65	23.89	21.60
8.24	18.59	29.13	12.45	26.39	22.25

  - A = Forms of Irons, I = 2, B = Concentration of doses, J = 3
  - K = Number of samples per factors A and B = 18, IJK = 108

Example 2 (cont.)

- Test 2 varieties of Iron (Fe) on 3 concentration doses

$\bar{x} = 8.64$

$\bar{x}_{Fe2+} = 7.88$
$\bar{x}_{Fe3+} = 9.40$

$\bar{x}_{10.2} = 4.82$
$\bar{x}_{1.2} = 8.92$
$\bar{x}_{0.3} = 12.19$

Fe^{2+}			Fe^{3+}		
10.2	**1.2**	**0.3**	**10.2**	**1.2**	**0.3**
0.71	2.20	2.25	2.20	4.04	2.71
1.66	2.93	3.93	2.69	4.16	5.43
2.01	3.08	5.08	3.54	4.42	6.38
2.16	3.49	5.82	3.75	4.93	6.38
2.42	4.11	5.84	3.83	5.49	8.32
2.42	4.95	6.89	4.08	5.77	9.04
2.56	5.16	8.50	4.27	5.86	9.56
2.60	5.54	8.56	4.53	6.28	10.01
3.31	5.68	9.44	5.32	6.97	10.08
3.64	6.25	10.52	6.18	7.06	10.62
3.74	7.25	13.46	6.22	7.78	13.80
3.74	7.90	13.57	6.33	9.23	15.99
4.39	8.85	14.76	6.97	9.34	17.90
4.50	11.96	16.41	6.97	9.91	18.25
5.07	15.54	16.96	7.52	13.46	19.32
5.26	15.89	17.56	8.36	18.40	19.87
8.15	18.30	22.82	11.65	23.89	21.60
8.24	18.59	29.13	12.45	26.39	22.25

  - SST = 3992.37, df = 107 (IJK -1)
  - SSA = 62.26, df = 1 (*I*-1)
  - SSB = 983.62, df = 2 (J-1)
  - SSE = 2938.20, df = 102 (*IJ*(K-1))
  - SSAB = SST – SSA – SSB – SSE = 8.29, df = 2

Example 2(cont.)

- SST = 3992.37, df = 107
- SSA = 62.26, df = 1
- SSB = 983.62, df = 2
- SSE = 2938.20, df = 102
- SSAB = 8.29, df = 2

MSE = 2938.20 / 102 = 28.81

- Test statistic :

Hypothesis	Mean Square (MS)	Test statistic (f)	Rejection region	
H_{0A} vs. H_{aA}	**62.26/1= 62.26**	f_A = **62.26/28.81 = 2.16**	$F_{0.01, 1, 102}$= 6.89 $F_{0.05, 1, 102}$= 3.93 $F_{0.1, 1, 102}$= 2.76	Accept at 0.1
H_{0B} vs. H_{aB}	**983.62/2 = 491.81**	f_B = **491.81/28.81 = 17.07**	$F_{0.01, 2, 102}$= 4.82 $F_{0.05, 2, 102}$= 3.09 $F_{0.1, 2, 102}$= 2.36	Reject at 0.01
H_{0AB} vs. H_{aAB}	**8.29/2 = 4.15**	f_{AB} = **4.15/28.81 = 0.14**	$F_{0.01, 2, 102}$= 4.82 $F_{0.05, 2, 102}$= 3.09 $F_{0.1, 2, 102}$= 2.36	Accept at 0.1

- H_{0AB} is not rejected. Interaction has no effect.
- H_{0A} is not rejected. Factor A (forms of Iron) has no effect on % of retained iron
- H_{0B} is rejected. Factor B (Concentration) has effect on % of retained iron

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Example 2(cont.)

- SST = 3992.37, df = 107
- SSA = 62.26, df = 1
- SSB = 983.62, df = 2
- SSE = 2938.20, df = 102
- SSAB = 8.29, df = 2

MSE = 2938.20 / 102 = 28.81

- Test statistic :

Hypothesis	Mean Square (MS)	Test statistic (f)	P-value
H_{0A} vs. H_{aA}	**62.26/1= 62.26**	f_A = **62.26/28.81 = 2.16**	0.14 Accept at 0.1
H_{0B} vs. H_{aB}	**983.62/2 = 491.81**	f_B = **491.81/28.81 = 17.07**	4.02×10^{-7} Reject at 0.01
H_{0AB} vs. H_{aAB}	SSAB **8.29/2 = 4.15**	f_{AB} = **4.15/28.81 = 0.14**	0.87 Accept at 0.1

- H_{0AB} is not rejected. Interaction has no effect.
- H_{0A} is not rejected. Factor A (forms of Iron) has no effect on % of retained iron
- H_{0B} is rejected. Factor B (Concentration) has effect on % of retained iron

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Outline

- Two-Factor ANOVA
  - Additive Factors
  - Interaction Factors
  - Interaction Plot
- Addition

Additive Two-Factor ANOVA : Effects Model

$$X_{ij} = \mu + \boxed{\alpha_i + \beta_j} + \varepsilon_{ij} \qquad \sum \alpha_i = 0 \qquad \sum \beta_j = 0$$

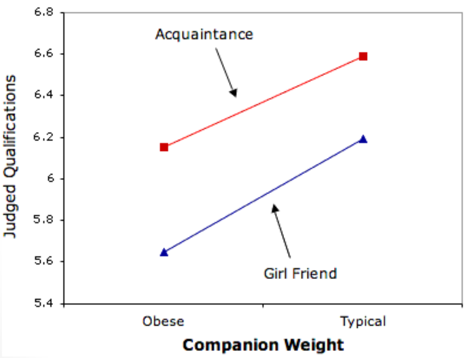
Adding effect of factors A and B

- X_{ij} = random sample j of treatment i
- μ = overall mean of treatment
- α_i = effect due to factor A at level i
- β_j = effect due to factor B at level j
- ε_{ij} = random error from sample j of treatment i
  - Assumed to be independent and normally distributed with mean = 0, variance = σ^2
- Factor A is independent of factor B

Interaction Plot : Example 1

$X_{ij} = \mu + \boxed{\alpha_i + \beta_j} + \varepsilon_{ij}$ $\sum \alpha_i = 0$ $\sum \beta_j = 0$

Adding effect of factors A and B



2 factors affect job applicants
A. Weight
B. Relationship type

Both lines (means) are (almost) parallel

Two lines are not crossing -> Change in one factor does not affect the other factor

When value of one factor changes, mean changes

Image source: http://onlinestatbook.com/2/analysis_of_variance/multiway.html

Interaction Plot : Example 3

2 factors affect on students' scores
A. Typing experience: none, some, lots (x-axis)
B. Exam method: book, computer (lines)

Check whether each factor affects on students' scores or not
both factors interact with each other or not

2 factors: A and B → 8 possibilities of factor effects

How many possibilities of factor affects?

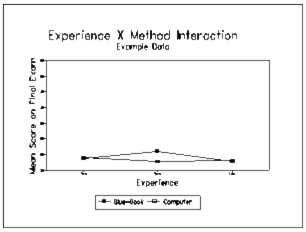
Source: http://www.psychstat.missouristate.edu/multibook/mlt09.htm

Interaction Plot : Example 3 (cont.)

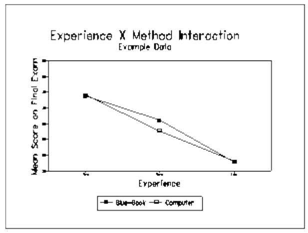
2 factors affect on students' scores

A. Typing experience: none, some, lots (x-axis)

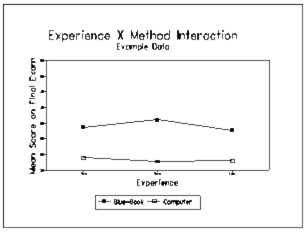
B. Exam method: book, computer (lines)



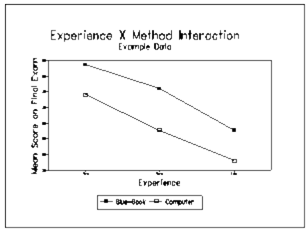
No effect from either A or B



Effect from A only



Effect from B only



Effect on A and B, but not AxB

Source: http://www.psychstat.missouristate.edu/multibook/mlt09.htm

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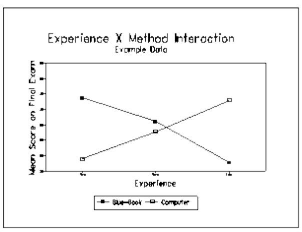
Interaction Plot : Example 3 (cont.)

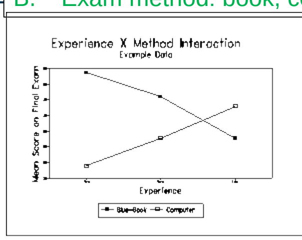
2 factors affect on students' scores

A. Typing experience: none, some, lots (x-axis)

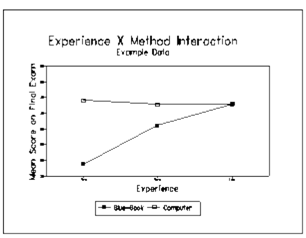
B. Exam method: book, computer (lines)



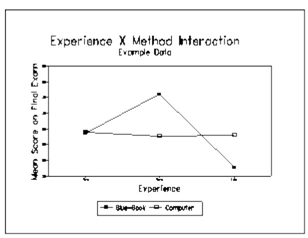
Effect on AxB only



Effect from A and AxB



Effect from B and AxB



Effect on A, B, and AxB

Source: http://www.psychstat.missouristate.edu/multibook/mlt09.htm

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Interaction Plot : Example 5

- Test 3 varieties of tomatoes on 4 planting density at significance level = 0.01

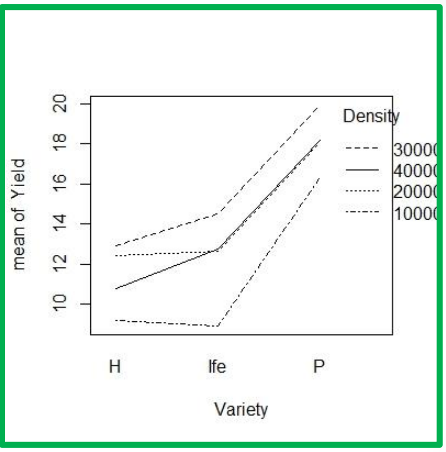
Variety	Planting Density											
	10,000			20,000			30,000			40,000		
H	10.5	9.2	7.9	12.8	11.2	13.3	12.1	12.6	14.0	10.8	9.1	12.5
Ife	8.1	8.6	10.1	12.7	13.7	11.5	14.4	15.4	13.7	11.3	12.5	14.5
P	16.1	15.3	17.5	16.6	19.2	18.5	20.8	18.0	21.0	18.4	18.9	17.2

- A = Varieties of tomatoes, I = 3
- B = Planting densities, J = 4
- K = Number of samples per factors A and B = 3
- IJK = 36

Interaction Plot : Example 5 (cont.)

- Test statistic :

Hypothesis	Mean Square (MS)	Test statistic (f)	Reje
H_{0A} vs. H_{aA}	163.8	f_A = 103.02	$F_{0.0}$
H_{0B} vs. H_{aB}	28.9	f_B = 18.18	$F_{0.0}$
H_{0AB} vs. H_{aAB}	1.34	f_{AB} = 0.84	$F_{0.0}$



- H_{0AB} is not rejected. Interaction has no effect.
- H_{0A} is rejected. Factor A (varieties of tomatoes) has effect on average product
- H_{0B} is rejected. Factor B (planting densities) has effect on average product

Interaction Plot : Example 6

- Test 2 varieties of Iron (Fe) on 3 concentration doses
- 18 samples on % retainment per category

Fe^{2+} 10.2	Fe^{2+} 1.2	Fe^{2+} 0.3	Fe^{3+} 10.2	Fe^{3+} 1.2	Fe^{3+} 0.3
0.71	2.20	2.25	2.20	4.04	2.71
1.66	2.93	3.93	2.69	4.16	5.43
2.01	3.08	5.08	3.54	4.42	6.38
2.16	3.49	5.82	3.75	4.93	6.38
2.42	4.11	5.84	3.83	5.49	8.32
2.42	4.95	6.89	4.08	5.77	9.04
2.56	5.16	8.50	4.27	5.86	9.56
2.60	5.54	8.56	4.53	6.28	10.01
3.31	5.68	9.44	5.32	6.97	10.08
3.64	6.25	10.52	6.18	7.06	10.62
3.74	7.25	13.46	6.22	7.78	13.80
3.74	7.90	13.57	6.33	9.23	15.99
4.39	8.85	14.76	6.97	9.34	17.90
4.50	11.96	16.41	6.97	9.91	18.25
5.07	15.54	16.96	7.52	13.46	19.32
5.26	15.89	17.56	8.36	18.40	19.87
8.15	18.30	22.82	11.65	23.89	21.60
8.24	18.59	29.13	12.45	26.39	22.25

- A = Forms of Irons, I = 2, B = Concentration of doses, J = 3
- K = Number of samples per factors A and B = 18, IJK = 108

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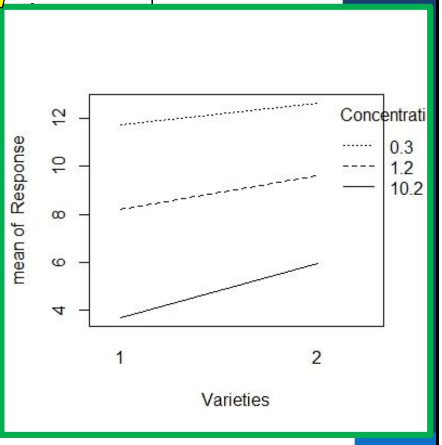
Interaction Plot : Example 6 (cont.)

Use interaction plot as GUIDELINE,
not actual decision on rejection in ANOVA

- Test statistic :

Hypothesis	Mean Square (MS)	Test statistic (f)	Rejection r
H_{0A} vs. H_{aA}	62.26/1= 62.26	f_A = 62.26/28.81 = 2.16	$F_{0.01,1,102}$= 6 $F_{0.05,1,102}$= 3 $F_{0.1,1,102}$= 2.
H_{0B} vs. H_{aB}	983.62/2 = 491.81	f_B = 491.81/28.81 = 17.07	$F_{0.01,2,102}$= 4. $F_{0.05,2,102}$= 3. $F_{0.1,2,102}$= 2.
H_{0AB} vs. H_{aAB}	8.29/2 = 4.15	f_{AB} = 4.15/28.81 = 0.14	$F_{0.01,2,102}$= 4 $F_{0.05,2,102}$= 3 $F_{0.1,2,102}$= 2.



- H_{0AB} is not rejected. Interaction has no effect.
- H_{0A} is not rejected. Factor A (forms of Iron) has no effect on % of retained iron
- H_{0B} is rejected. Factor B (Concentration) has effect on % of retained iron

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Outline

- Two-Factor ANOVA
  - Additive Factors
  - Interaction Factors
  - Interaction Plots
- Additions

More on Multiple Comparison

- Is it possible that H_{0A} (or H_{0B}) is not rejected but result to multiple group of means?
- Is it also possible that H_{0A} (or H_{0B}) is rejected but have one group of means?

Measured by MSE

ANOVA tests on ALL means whether they are identical

Multiple comparison tests on PAIRWISE means

Not measured by MSE

ANOVA detects variability among all means

ANOVA test is more sensitive than Multiple comparison

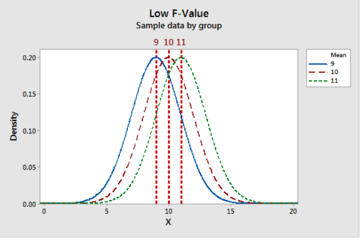
More on Multiple Comparison (cont.)

ANOVA tests on ALL means whether they are identical

Multiple comparison tests on PAIRWISE means

ANOVA detects lower variability among all means

ANOVA test is more sensitive than Multiple comparison



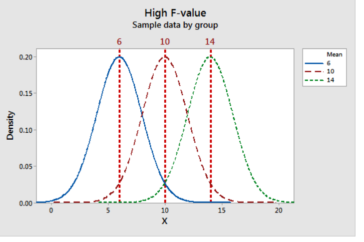


Image source: http://blog.minitab.com/blog/adventures-in-statistics-2/understanding-analysis-of-variance-anova-and-the-f-test

Interaction Two-Factor ANOVA : Effects Mode

$$\boxed{X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}}$$ $$\boxed{\sum \alpha_i = 0}$$ $$\boxed{\sum \beta_j = 0}$$

- If $\alpha_i = 0$, $\beta_j = 0$, and $\gamma_{ij} = 0$, then all treatments have the same response

  $$\boxed{E(X_{ijk}) = \mu}$$

- Thus, null hypotheses for interaction factor ANOVA
  - Hypothesis on A and B : factors A and B at any level i has no effect on overall mean.
    - H_{0AB}: $\Upsilon_{ij} = 0$ for all i,j ⇐ Test first If reject, no need to test H_{0A}, H_{0B}
  - Hypothesis on A : A interacts with B

    factor A at any level i has no effect on overall mean. Effect of A depends on B Thus, A and B both affect.
    - H_{0A}: $\alpha_1 = \alpha_2 = \ldots = \alpha_I = 0$
  - Hypothesis on B : factor B at any level j has no effect on overall mean.
    - H_{0B}: $\beta_1 = \beta_2 = \ldots = \beta_J = 0$

More on ANOVA

- When H_{0AB} is rejected and H_{0A} (or H_{0B}) is not rejected, what does it mean?

A and B interact.

A and B both affect.

That's why all means are equal and H_{0A} is not rejected.

A has effect but its effect (that depends on B) is the same for all responses

References

1. J.L. Devore and K.N. Berk, Modern Mathematical Statistics with Applications, Springer, 2012.
2. J.A. Rice, Mathematical Statistics and Data Analysis, Duxbury Press, 1995.