$$\left\| \frac{1}{m} \sum_{k=1}^{m} |a_k(1)|^2 \begin{bmatrix} |a_k(1)|^2 & a_k(1)\tilde{\boldsymbol{a}}_k^* \\ a_k(1)\tilde{\boldsymbol{a}}_k & \tilde{\boldsymbol{a}}_k\tilde{\boldsymbol{a}}_k^* \end{bmatrix} - (\boldsymbol{e}_l\boldsymbol{e}_l^* + \boldsymbol{I}) \right\|$$

$$\leq \left| \frac{1}{m} \sum_{k=1}^{m} (|a_k(1)|^4 - 2) \right| + \left\| \frac{1}{m} \sum_{k=1}^{m} |a_k(1)|^2 \begin{bmatrix} 0 & a_k(1)\tilde{\boldsymbol{a}}_k^* \\ a_k(1)\tilde{\boldsymbol{a}}_k & \boldsymbol{0} \end{bmatrix} \right\|$$

$$+ \left\| \frac{1}{m} \sum_{k=1}^{m} |a_k(1)|^2 (\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_k^* - \boldsymbol{I}_{n-1}) \right\| + \left| \frac{1}{m} \sum_{k=1}^{m} (|a_k(1)|^2 - 1) \right|.$$

By Chebyshev's inequality, we have with probability at least $1 - c_1\delta^{-2}m^{-1}$,

$$\left| \frac{1}{m} \sum_{k=1}^{m} (|a_k(1)|^4 - 2)| \leq \frac{\delta}{4} and \left| \frac{1}{m} \sum_{k=1}^{m} (|a_k(1)|^2 - 1) \right| \leq \frac{\delta}{4}.$$

To bound the second term, we note that

$$\left\| \frac{1}{m} \sum_{k=1}^{m} |a_k(1)|^2 \begin{bmatrix} 0 & a_k(1)\tilde{\boldsymbol{a}}_k^* \\ a_k(1)\tilde{\boldsymbol{a}}_k & \boldsymbol{0} \end{bmatrix} \right\| = \left\| \frac{1}{m} \sum_{k=1}^{m} |a_k(1)|^2 a_k(1)\tilde{\boldsymbol{a}}_k^* \right\|$$

$$= \sup_{\boldsymbol{w}\in\mathbb{C}^{n-1}:||\boldsymbol{w}||=1} \frac{1}{m} \sum_{k=1}^{m} |a_k(1)|^2 a_k(1)\tilde{\boldsymbol{a}}_k^*\boldsymbol{w}.$$

For all $\boldsymbol{w}$ and all $k \in [m]$, $\tilde{\boldsymbol{a}}_k^*\boldsymbol{w}$ is distributed as $\mathcal{CN}(1)$ that is independent of the $\{a_k(1)\}$ sequence. So for one realization of $\{a_k(1)\}$, the Hoeffding-type inequality of Lemma 29 implies

$$\mathbb{P}\left[\frac{1}{m} \sum_{k=1}^{m} |a_k(1)|^2 a_k(1)\tilde{\boldsymbol{a}}_k^*\boldsymbol{w} > t \right] \leq e \exp\left(-\frac{c_2 m^2 t^2}{\sum_{k=1}^{m} |a_k(1)|^6} \right).$$

for any $\boldsymbol{w}$ with $||\boldsymbol{w}|| = 1$ and any $t > 0$. Taking $t = \delta/8$, together with a union bound on a $1/2$-net on the sphere, we obtain

$$\mathbb{P}\left[\left\| \frac{1}{m} \sum_{k=1}^{m} |a_k(1)|^2 a_k(1)\tilde{\boldsymbol{a}}_k^* \right\| > \delta/4 \right] \leq e \exp\left(-\frac{c_2 m^2 t^2}{64 \sum_{k=1}^{m} |a_k(1)|^6} + 12(n-1) \right).$$

Now an application of Chebyshev's inequality give that $\sum_{k=1}^{m} |a_k(1)|^6 \leq 20m$ with probability at least $1 - c_3 m^{-1}$. Substituting this into the above, we concluded that whenever $m \geq C_4\delta^{-2}n$ for some sufficiently large C_4,

$$\left\| \frac{1}{m} \sum_{k=1}^{m} |a_k(1)|^2 a_k(1)\tilde{\boldsymbol{a}}_k^* \right\| \leq \delta/4$$

with probability at least $1 - c_3 m - \exp(-c_5\delta^2 m)$.

The previous results are required to show that the interleavings are in a sense closed under composition, e.g., composing two left interleavings (with a suitable ordering of terms), yields a left interleaving (with an additive approximation factor). Now, we show the more interesting property: Most combinations of the different notions of interleavings do not interact, i.e., composition yields the maximum of the two rather than an additive factor. First, we show that if compositions is **not** in the natural order as in Propositions 4.11 and 4.10, the interleavings do not yield an additive factor.

Propositions 4.12 *Suppose one of the following two possible holds,*

$$M \overset{2\epsilon}{\underset{L}{\sim}} N \quad and \quad P \overset{2\epsilon}{\underset{L}{\sim}} N, \quad or \quad M \overset{2\epsilon}{\underset{R}{\sim}} N \quad and \quad P \overset{2\epsilon}{\underset{R}{\sim}} N,$$

then $M \overset{2\epsilon}{\sim} P$.

Proof In the first case, we have the following two exact sequences:

$$0 \to M \overset{i}{\to} N \overset{f}{\to} X \to 0 \quad and \quad 0 \to P \overset{g}{\to} N \overset{j}{\to} Y \to 0$$

Similarly in the second case, we have:

$$0 \to X \overset{f}{\to} M \overset{i}{\to} N \to 0 \quad and \quad 0 \to Y \overset{j}{\to} P \overset{g}{\to} N \to 0$$

In both cases, by assumption $X \overset{\epsilon}{\sim} 0$ and $Y \overset{\epsilon}{\sim} 0$. By propositions 4.1 and 4.2, for both cases there are 2ϵ-interleavings (ϕ, ψ) of M and N and (η, θ) of P and N. These fit into the following commutative diagram, where the horizontal arrows are ordinary morphisms and all other arrows are 2ϵ-morphisms.

By inspection of this diagram, we see that $(\theta i, \psi g)$ is a 2ϵ-interleaving of M and P $\quad\square$.

Propositions 4.13 *Suppose one of the following two possibilities holds,*

$$N \overset{2\epsilon}{\underset{L}{\sim}} M \quad and \quad N \overset{2\epsilon}{\underset{L}{\sim}} P, \quad or \quad N \overset{2\epsilon}{\underset{R}{\sim}} M \quad and \quad N \overset{2\epsilon}{\underset{R}{\sim}} P,$$