DC Point Restoration

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1 Summary

One is provide with s-parameters that are missing DC along other early points. This algorithm assumes that they are evenly spaced. This means that we want frequencies, for $N+1$ frequency points, for $n \in 0 \ldots N$

$$f_{[n]} = \frac{n}{N} \cdot Fe,$$

but are missing the first M points in a vector $\mathbf{X}$. The goal here is to recover them.

Given a time-domain vector $\mathbf{x}$ that is K points long, we know that the frequency content vector $\mathbf{X}$ is related to $\mathbf{x}$ through the DFT:

$$X_{[n]} = \sum_{k=0}^{K-1} x_k \cdot e^{-j \cdot 2\pi \cdot \frac{n \cdot k}{K}}.$$

This can be written in matrix form using the Fourier matrix, defined as

$$\mathcal{F}_{k,n} = \frac{1}{K} \cdot e^{-j \cdot 2\pi \cdot \frac{n \cdot k}{K}},$$

and therefore

$$\mathbf{X} = \mathcal{F} \cdot \mathbf{x}.$$

Given frequency points $\mathbf{X}$, one goes in the other direction as

$$\mathbf{x} = \mathbf{F}^{-1} \cdot \mathbf{X}.$$

One defines the matrix $\mathbf{Fi} = \mathcal{F}^{-1}$ and partition the system as

$$\begin{pmatrix} \underset{1\times 1}{\mathbf{Fi_{11}}} & \underset{1\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{12}}} & \underset{1\times 1}{\mathbf{Fi_{13}}} & \underset{1\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{14}}} \\ \underset{\left(\frac{K}{2}-1\right)\times 1}{\mathbf{Fi_{21}}} & \underset{\left(\frac{K}{2}-1\right)\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{22}}} & \underset{\left(\frac{K}{2}-1\right)\times 1}{\mathbf{Fi_{23}}} & \underset{\left(\frac{K}{2}-1\right)\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{24}}} \\ \underset{1\times 1}{\mathbf{Fi_{31}}} & \underset{1\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{32}}} & \underset{1\times 1}{\mathbf{Fi_{33}}} & \underset{1\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{34}}} \\ \underset{\left(\frac{K}{2}-1\right)\times 1}{\mathbf{Fi_{41}}} & \underset{\left(\frac{K}{2}-1\right)\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{42}}} & \underset{\left(\frac{K}{2}-1\right)\times 1}{\mathbf{Fi_{43}}} & \underset{\left(\frac{K}{2}-1\right)\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{44}}} \end{pmatrix} \cdot \begin{pmatrix} \underset{1}{\mathbf{X_0}} \\ \underset{\frac{K}{2}-1}{\mathbf{X_f}} \\ \underset{1}{\mathbf{X_N}} \\ \underset{\frac{K}{2}-1}{\mathbf{X_r}} \end{pmatrix} = \begin{pmatrix} \underset{1}{\mathbf{x_0}} \\ \underset{\frac{K}{2}-1}{\mathbf{x_f}} \\ \underset{1}{\mathbf{x_N}} \\ \underset{\frac{K}{2}-1}{\mathbf{x_r}} \end{pmatrix}.$$

The last equation can be written as

$$\mathbf{Fi_{41}} \cdot \mathbf{X_0} + \mathbf{Fi_{42}} \cdot \mathbf{X_f} + \mathbf{Fi_{43}} \cdot \mathbf{X_N} + \mathbf{Fi_{44}} \cdot \mathbf{X_r} = \mathbf{x_r}$$

or

$$\mathbf{Fi_{41}} \cdot \mathbf{X_0} + \begin{pmatrix} \mathbf{Fi_{42}} & \mathbf{Fi_{43}} & \mathbf{Fi_{44}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{X_f} \\ \mathbf{X_N} \\ \mathbf{X_r} \end{pmatrix} = \mathbf{x_r}.$$

These equations have been partitioned for the following reasons denoted by the subscripts on the frequency and time vector.

- The 0 subscript represents the DC point in the frequency data and zero time in the time vector.
- The f subscript represents the forward, positive frequency points and the time points after zero time in the time vector.
- The N subscript represents the Nyquist rate point in the frequency data and the most negative time point in the time vector.
- The r subscript represents the negative frequency points (going in reverse) and the negative time points in the time vector.

For causality to be enforced, the negative time points must be zero, and therefore $\mathbf{x_r} = \mathbf{0}$. Therefore, the equation is solved in a least squares sense as

$$\mathbf{Fi_{41}} \cdot \mathbf{X_0} = -\begin{pmatrix} \mathbf{Fi_{42}} & \mathbf{Fi_{43}} & \mathbf{Fi_{44}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{X_f} \\ \mathbf{X_N} \\ \mathbf{X_r} \end{pmatrix}$$

and

$$\mathbf{X_0} = -\mathbf{Fi_{41}}^{\dagger} \cdot \begin{pmatrix} \mathbf{Fi_{42}} & \mathbf{Fi_{43}} & \mathbf{Fi_{44}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{X_f} \\ \mathbf{X_N} \\ \mathbf{X_r} \end{pmatrix}.$$

For M initial points missing, the system is partitioned prior to solving as

$$\begin{pmatrix} \underset{1\times M}{\mathbf{Fi_{11}}} & \underset{1\times\left(\frac{K}{2}-M\right)}{\mathbf{Fi_{12}}} & \underset{1\times 1}{\mathbf{Fi_{13}}} & \underset{1\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{14}}} \\ \underset{\left(\frac{K}{2}-1\right)\times M}{\mathbf{Fi_{21}}} & \underset{\left(\frac{K}{2}-1\right)\times\left(\frac{K}{2}-M\right)}{\mathbf{Fi_{22}}} & \underset{\left(\frac{K}{2}-1\right)\times 1}{\mathbf{Fi_{23}}} & \underset{\left(\frac{K}{2}-1\right)\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{24}}} \\ \underset{1\times M}{\mathbf{Fi_{31}}} & \underset{1\times\left(\frac{K}{2}-M\right)}{\mathbf{Fi_{32}}} & \underset{1\times 1}{\mathbf{Fi_{33}}} & \underset{1\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{34}}} \\ \underset{\left(\frac{K}{2}-1\right)\times M}{\mathbf{Fi_{41}}} & \underset{\left(\frac{K}{2}-1\right)\times\left(\frac{K}{2}-M\right)}{\mathbf{Fi_{42}}} & \underset{\left(\frac{K}{2}-1\right)\times 1}{\mathbf{Fi_{43}}} & \underset{\left(\frac{K}{2}-1\right)\times\left(\frac{K}{2}-1\right)}{\mathbf{Fi_{44}}} \end{pmatrix} \cdot \begin{pmatrix} \underset{M}{\mathbf{X_0}} \\ \underset{\frac{K}{2}-M}{\mathbf{X_f}} \\ \underset{1}{\mathbf{X_N}} \\ \underset{\frac{K}{2}-1}{\mathbf{X_r}} \end{pmatrix} = \begin{pmatrix} \underset{M}{\mathbf{x_0}} \\ \underset{\frac{K}{2}-M}{\mathbf{x_f}} \\ \underset{1}{\mathbf{x_N}} \\ \underset{\frac{K}{2}-1}{\mathbf{x_r}} \end{pmatrix}.$$

Thus, it has been shown that the DC point and even other missing frequency points can be solved for given the other known frequency points, given an assumption of causality on the result.

2 Better Version

We have, in the frequency domain, the missing DC point $\mathbf{X_0}$ along with $M-1$ more missing frequency point $\mathbf{X_{mf}}$, which of course has corresponding negative frequency points $\mathbf{X_{mr}}$. By applying the inverse Fourier matrix such that $\mathbf{Fi}\cdot\mathbf{X} = \mathbf{x}$, we partition the problem for the known frequency points $\mathbf{X_K}$ and the other unknown ones, and the positive time points $\mathbf{x_p}$ and the negative time points $\mathbf{x_n}$[1], one writes

$$\begin{pmatrix} \underset{\frac{K}{2}\times 1}{\mathbf{Fi_{11}}} & \underset{\frac{K}{2}\times(M-1)}{\mathbf{Fi_{12}}} & \underset{\frac{K}{2}\times\sim}{\mathbf{Fi_{13}}} & \underset{\frac{K}{2}\times(M-1)}{\mathbf{Fi_{14}}} \\ \underset{\frac{K}{2}\times 1}{\mathbf{Fi_{21}}} & \underset{\frac{K}{2}\times(M-1)}{\mathbf{Fi_{22}}} & \underset{\frac{K}{2}\times\sim}{\mathbf{Fi_{23}}} & \underset{\frac{K}{2}\times(M-1)}{\mathbf{Fi_{24}}} \end{pmatrix} \cdot \begin{pmatrix} \underset{1}{\mathbf{X_0}} \\ \underset{M-1}{\mathbf{X_{mf}}} \\ \underset{\sim}{\mathbf{X_K}} \\ \underset{M-1}{\mathbf{X_{mr}}} \end{pmatrix} = \begin{pmatrix} \underset{\frac{K}{2}}{\mathbf{x_p}} \\ \underset{\frac{K}{2}}{\mathbf{x_n}} \end{pmatrix}.$$

The bottom equation is the one of interest here where it is enforced that $\mathbf{x_n} = \mathbf{0}$ (i.e. that the system is causal). It is

$$\mathbf{Fi_{21}}\cdot\mathbf{X_0} + \mathbf{Fi_{22}}\cdot\mathbf{X_{mf}} + \mathbf{Fi_{23}}\cdot\mathbf{X_K} + \mathbf{Fi_{24}}\cdot\mathbf{X_{mr}} = \mathbf{x_n} = \mathbf{0}. \tag{1}$$

Now, the negative frequency points $\mathbf{X_{mr}}$ are related to the positive frequency points $\mathbf{X_{mf}}$ in one is a flipped over vector of the other, and contain points that are the complex conjugate of the other. Therefore, there is a trivial permutation matrix that flips the vector over such that

$$(\mathbf{P}\cdot\mathbf{X_{mf}})^* = \mathbf{X_{mr}}. \tag{2}$$

Substituting equation (2) into equation (1) results in

$$\mathbf{Fi_{21}}\cdot\mathbf{X_0} + \mathbf{Fi_{22}}\cdot\mathbf{X_{mf}} + \mathbf{Fi_{23}}\cdot\mathbf{X_K} + \mathbf{Fi_{24}}\cdot\mathbf{P}\cdot{\mathbf{X_{mf}}}^* = \mathbf{x_n} = \mathbf{0},$$

which can be rewritten by separating out the real and imaginary parts as

$$\begin{aligned} &\mathbf{Fi_{21}}\cdot\mathbf{X_0} \\ &\quad + \mathbf{Fi_{22}}\cdot\left[\mathrm{Re}\left(\mathbf{X_{mf}}\right) + j\cdot\mathrm{Im}\left(\mathbf{X_{mf}}\right)\right] \\ &\quad + \mathbf{Fi_{23}}\cdot\mathbf{X_K} + \mathbf{Fi_{24}}\cdot\mathbf{P}\cdot\left[\mathrm{Re}\left(\mathbf{X_{mf}}\right) - j\cdot\mathrm{Im}\left(\mathbf{X_{mf}}\right)\right] \\ &\qquad = \mathbf{x_n} = \mathbf{0}. \end{aligned}$$

These block matrices can be aggregated through augmentation and stacking to write

[1]There is the understanding that $\mathbf{x_n}$ doesn't have to be strictly all of the negative time points, but rather the negative time points for which it is desired to enforce that they are zero.

$$\begin{pmatrix} \mathbf{Fi_{21}} & \mathbf{Fi_{22}} + \mathbf{Fi_{24}} \cdot \mathbf{P} & j \cdot (\mathbf{Fi_{22}} - \mathbf{Fi_{24}} \cdot \mathbf{P}) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{X_0} \\ \mathrm{Re}\left(\mathbf{X_{mf}}\right) \\ \mathrm{Im}\left(\mathbf{X_{mf}}\right) \end{pmatrix} = -\mathbf{Fi_{23}} \cdot \mathbf{X_K},$$

and solved as[2]

$$\begin{pmatrix} \mathbf{X_0} \\ \mathrm{Re}\left(\mathbf{X_{mf}}\right) \\ \mathrm{Im}\left(\mathbf{X_{mf}}\right) \end{pmatrix} = -\begin{pmatrix} \mathbf{Fi_{21}} & \mathbf{Fi_{22}} + \mathbf{Fi_{24}} \cdot \mathbf{P} & j \cdot (\mathbf{Fi_{22}} - \mathbf{Fi_{24}} \cdot \mathbf{P}) \end{pmatrix}^{\dagger} \cdot \mathbf{Fi_{23}} \cdot \mathbf{X_K}.$$

Note that while the matrix to which the † is applied looks to be fat on the left-hand side (indicating a minimum norm solution), it is, in practice, actually skinny. This leads to a least-squares solution. Note that one could quite easily pose this as a weighted least-squares solution, which could enforce, for example, more negative-in-time points to be more zero than the less negative-in-time points.

3 Enforcement of Realness on the result

Because of how the problem has been posed in the previous section, there is no enforcement of realness on the final result. In other words, the result

$$\begin{pmatrix} \mathbf{X_0} \\ \mathrm{Re}\left(\mathbf{X_{mf}}\right) \\ \mathrm{Im}\left(\mathbf{X_{mf}}\right) \end{pmatrix},$$

while looking real, are not actually enforced to be.

To solve this, recognize that

$$\mathbf{x} = \mathbf{F}^{-1} \cdot \mathbf{X}$$

can be written as

$$\mathbf{x} = \left[\mathrm{Re}\left(\mathbf{F}^{-1}\right) + j \cdot \mathrm{Im}\left(\mathbf{F}^{-1}\right)\right] \cdot \left[\mathrm{Re}\left(\mathbf{X}\right) + j \cdot \mathrm{Im}\left(\mathbf{X}\right)\right],$$

which can be expanded to

$$\mathbf{x} = \mathrm{Re}\left(\mathbf{F}^{-1}\right) \cdot \mathrm{Re}\left(\mathbf{X}\right) + j \cdot \mathrm{Re}\left(\mathbf{F}^{-1}\right) \cdot \mathrm{Im}\left(\mathbf{X}\right) + j \cdot \mathrm{Im}\left(\mathbf{F}^{-1}\right) \cdot \mathrm{Re}\left(\mathbf{X}\right) - \mathrm{Im}\left(\mathbf{F}^{-1}\right) \cdot \mathrm{Im}\left(\mathbf{X}\right).$$

It turns out that:

$$\mathbf{x} = j \cdot \mathrm{Re}\left(\mathbf{F}^{-1}\right) \cdot \mathrm{Im}\left(\mathbf{X}\right) + j \cdot \mathrm{Im}\left(\mathbf{F}^{-1}\right) \cdot \mathrm{Re}\left(\mathbf{X}\right) = \mathbf{0},$$

and therefore, we have

$$\mathbf{x} = \mathrm{Re}\left(\mathbf{F}^{-1}\right) \cdot \mathrm{Re}\left(\mathbf{X}\right) - \mathrm{Im}\left(\mathbf{F}^{-1}\right) \cdot \mathrm{Im}\left(\mathbf{X}\right).$$

[2] The matrix $\mathbf{P}$ multiplied from the right serves to flip the columns of the matrix to the left.

This allows the partitioned matrix problem to be rewritten as

$$\mathrm{Re}\begin{pmatrix} \mathbf{Fi_{11}} & \mathbf{Fi_{12}} & \mathbf{Fi_{13}} & \mathbf{Fi_{14}} \\ \mathbf{Fi_{21}} & \mathbf{Fi_{22}} & \mathbf{Fi_{23}} & \mathbf{Fi_{24}} \end{pmatrix} \cdot \mathrm{Re}\begin{pmatrix} \mathbf{X_0} \\ \mathbf{X_{mf}} \\ \mathbf{X_K} \\ \mathbf{X_{mr}} \end{pmatrix} - \mathrm{Im}\begin{pmatrix} \mathbf{Fi_{11}} & \mathbf{Fi_{12}} & \mathbf{Fi_{13}} & \mathbf{Fi_{14}} \\ \mathbf{Fi_{21}} & \mathbf{Fi_{22}} & \mathbf{Fi_{23}} & \mathbf{Fi_{24}} \end{pmatrix} \cdot \mathrm{Im}\begin{pmatrix} \mathbf{X_0} \\ \mathbf{X_{mf}} \\ \mathbf{X_K} \\ \mathbf{X_{mr}} \end{pmatrix} = \begin{pmatrix} \mathbf{x_p} \\ \mathbf{x_n} \end{pmatrix}$$

and since $\mathbf{x_n} = \mathbf{0}$,

$$\mathrm{Re}\begin{pmatrix} \mathbf{Fi_{21}} & \mathbf{Fi_{22}} & \mathbf{Fi_{23}} & \mathbf{Fi_{24}} \end{pmatrix} \cdot \mathrm{Re}\begin{pmatrix} \mathbf{X_0} \\ \mathbf{X_{mf}} \\ \mathbf{X_K} \\ \mathbf{X_{mr}} \end{pmatrix} - \mathrm{Im}\begin{pmatrix} \mathbf{Fi_{21}} & \mathbf{Fi_{22}} & \mathbf{Fi_{23}} & \mathbf{Fi_{24}} \end{pmatrix} \cdot \mathrm{Im}\begin{pmatrix} \mathbf{X_0} \\ \mathbf{X_{mf}} \\ \mathbf{X_K} \\ \mathbf{X_{mr}} \end{pmatrix} = \mathbf{0},$$

and

$$\begin{pmatrix} \mathbf{Fi_{21}} & \mathrm{Re}\left(\mathbf{Fi_{24}}\right) \cdot \mathbf{P} + \mathrm{Re}\left(\mathbf{Fi_{22}}\right) & \mathrm{Im}\left(\mathbf{Fi_{24}}\right) \cdot \mathbf{P} - \mathrm{Im}\left(\mathbf{Fi_{22}}\right) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{X_0} \\ \mathrm{Re}\left(\mathbf{X_{mf}}\right) \\ \mathrm{Im}\left(\mathbf{X_{mf}}\right) \end{pmatrix} = \mathrm{Im}\left(\mathbf{Fi_{23}}\right) \cdot \mathrm{Im}\left(\mathbf{X_K}\right) - \mathrm{Re}\left(\mathbf{F}\right.$$

Therefore, the result, with realness enforced, becomes

$$\begin{pmatrix} \mathbf{X_0} \\ \mathrm{Re}\left(\mathbf{X_{mf}}\right) \\ \mathrm{Im}\left(\mathbf{X_{mf}}\right) \end{pmatrix} = \begin{pmatrix} \mathbf{Fi_{21}} & \mathrm{Re}\left(\mathbf{Fi_{24}}\right) \cdot \mathbf{P} + \mathrm{Re}\left(\mathbf{Fi_{22}}\right) & \mathrm{Im}\left(\mathbf{Fi_{24}}\right) \cdot \mathbf{P} - \mathrm{Im}\left(\mathbf{Fi_{22}}\right) \end{pmatrix}^{\dagger} \cdot \left[\mathrm{Im}\left(\mathbf{Fi_{23}}\right) \cdot \mathrm{Im}\left(\mathbf{X_K}\right) - \mathrm{Re}\right.$$