Error correction and error mitigation

FRIB-TA Summer School

June 22, 2022

Ryan LaRose

Recap of the gate model of quantum computing

- Qubits
  - Complex vectors

$$|0\rangle := [1,\ 0]^T \quad |1\rangle := [0,\ 1]^T$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

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A bit of history

- Shor (1994): “Check out this poly-time algorithm for factoring.”

Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer*

Peter W. Shor†

Abstract

A digital computer is generally believed to be an efficient universal computing device; that is, it is believed able to simulate any physical computing device with an increase in computation time by at most a polynomial factor. This may not be true when quantum mechanics is taken into consideration. This paper considers factoring integers and finding discrete logarithms, two problems which are generally

A bit of history

- Community (1994): “Cool! But there’s no way this could ever be done in practice. Quantum states are very fragile.”

Maintaining coherence in quantum computers

W. G. Unruh*
Canadian Institute for Advanced Research, Cosmology Program, Department of Physics, University of British Columbia, Vancouver, Canada V6T 1Z1
(Received 10 June 1994)

The effects of the inevitable coupling to external degrees of freedom of a quantum computer are examined. It is found that for quantum calculations (in which the maintenance of coherence over a large number of states is important), not only must the coupling be small, but the time taken in the quantum calculation must be less than the thermal time scale $\hbar/k_B T$. For longer times the condition on the strength of the coupling to the external world becomes much more stringent.

A bit of history

- Shor (1995): “Check out this quantum error correcting code.”

Scheme for reducing decoherence in quantum computer memory

Peter W. Shor*
AT&T Bell Laboratories, Room 2D-149, 600 Mountain Avenue, Murray Hill, New Jersey 07974
(Received 17 May 1995)

Recently, it was realized that use of the properties of quantum mechanics might speed up certain computations dramatically. Interest has since been growing in the area of quantum computation. One of the main difficulties of quantum computation is that decoherence destroys the information in a superposition of states contained in a quantum computer, thus making long computations impossible. It is shown how to reduce the effects of decoherence for information stored in quantum memory, assuming that the decoherence process acts independently on each of the bits stored in memory. This involves the use of a quantum analog of error-correcting codes.

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  - Example: Pauli errors

$$X|0\rangle = |1\rangle$$
$$X|1\rangle = |0\rangle$$

Bit flip

$$Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

Phase flip

$$Y|0\rangle = i|1\rangle = iXZ|0\rangle$$
$$Y|1\rangle = -i|0\rangle = iXZ|1\rangle$$

Bit & phase flip

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  - This is because Paulis (+ identity) span $\mathbb{C}^{2\times 2}$

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- But coefficients *e_i* could still be (in principle) infinitesimal.
  - **How do we deal with this?**

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  - For example, we could choose M_i such that $E|\psi\rangle \mapsto \eta_i \sigma_i |\psi\rangle$
  - The $\sigma_i \in \{I, X, Y, Z\}$ is now a discrete error which can be corrected.
  - The $\eta_i \in \mathbb{C}$ is continuous **but is a global phase, so doesn't matter**.

Emphasis on Pauli errors

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- **Further, we can only consider bit flip and phase flip errors WLOG.**
  - Paulis + identity span $\mathbb{C}^{2\times 2}$
  - Y = i XZ and global phase doesn't matter
  - (Identity is not an error!)

Classical error correction : The repetition code

- A key concept in error correction is adding redundancy.
- For example, given a bit, we can make three copies of it:
  - 0 -> 000
  - 1 -> 111
- This is known as the (classical) **repetition code**.
- The idea is very simple: If an error occurs on one bit only, we can correct it by looking at the other two bits and taking a majority vote.

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Analyzing the repetition code

- Suppose each bit flips independently with probability *p*. For which *p* is the repetition code beneficial?
  - The probability of an error without the encoding is p.
  - With the encoding, the probability of an error is prob(> 1 bit flips) which is

$$p_e := 3p^2(1-p) + p^3 = 3p^2 - 2p^3$$

  - By setting $p_e < p$ we find that the repetition code is better provided that

$$p < 1/2$$

- This is a simple example, but the spirit is representative of the thinking you do in error correction.

QEC: Important point about adding redundancy

- Given the classical repetition code, we might try to do the same with qubits, i.e. map

$$|\psi\rangle \mapsto |\psi\rangle|\psi\rangle|\psi\rangle$$

- This is not possible in general, as expressed by the "no cloning theorem"

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- By taking the inner product of these equations, we can see there can only exist such a *U* if the states |psi> and |phi> are orthogonal

Aside: Remark about no cloning

- Note in the previous proof the only properties we used were tensor products and linearity.
- In this respect no cloning is also a classical theorem.
- Specifically: No linear *stochastic* map (not necessarily unitary map) can clone arbitrary classical probability distributions in tensor product.
    - See http://info.phys.unm.edu//~crosson/Phys572/QI-572-L9.pdf for more. (The proof is the same, but there is a longer, interesting discussion.)

QEC: Can we add any redundancy?

- From no cloning we cannot make copies of our state as in the classical repetition code. Can we copy anything?
- Claim: We can “copy basis information” in the following sense:

$$\alpha|0\rangle + \beta|1\rangle \mapsto \alpha|000\rangle + \beta|111\rangle$$

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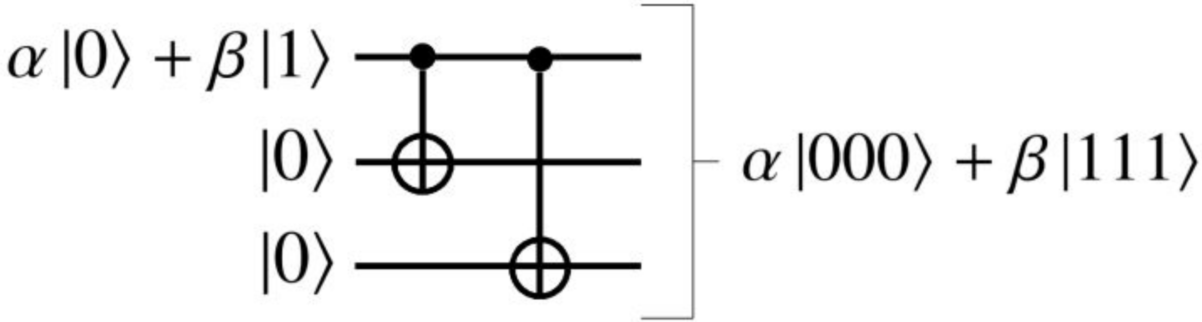

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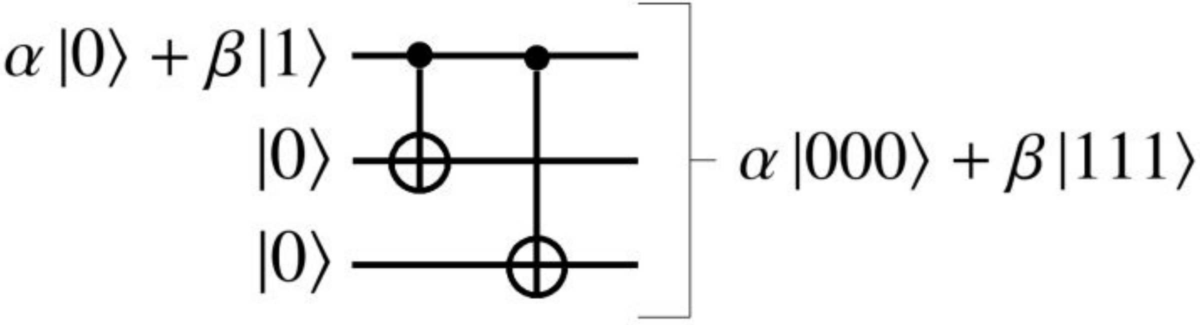
- Note that this encoding circuit entangles the “input” qubit with two other qubits.



- Since errors in quantum computers are due to (for the most part) qubits entangling with their environment, we can understand a quote from John Preskill:
- *“We have learned that it is possible to fight entanglement with entanglement.”*

Repetition code for bit flip errors

- The encoding a|0> + b|1> -> a |000> + b|111> gives us redundancy. Now what?
- We need to check which errors (if any) occured in the encoded state.
- We do this by (projective) measurements. What projections should we apply to find out what happened?
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Understanding the projectors: More detail

- Measurements
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Do this with the other 3 projectors!

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Turning the table

- By measuring these operators, we learn what errors (if any) occurred.
- Since we know which error occurred, we can correct it.

Syndrome measurement	**Meaning**	**Correction operator**
$P_0 = \|000\rangle\langle 000\| + \|111\rangle\langle 111\|$	No qubit was flipped.	I
$P_1 = \|100\rangle\langle 100\| + \|011\rangle\langle 011\|$	The first qubit was flipped.	X_0
$P_2 = \|010\rangle\langle 010\| + \|101\rangle\langle 101\|$	The second qubit was flipped.	X_1
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Repetition code for phase flip errors

- We can now correct bit flip errors. What about phase flip errors?
  - These are related by a change of basis

$$HXH = Z$$

  - Thus we can do the encoding:
    - |0> -> (|0> + |1>)(|0> + |1>)(|0> + |1>)
    - |1> -> (|0> - |1>)(|0> - |1>)(|0> - |1>)
    - And update the syndrome measurements in a similar way.
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What about both bit flip and phase flip errors?

Shor's 9-qubit code

- This is formed by **concatenating** the bit flip and phase flip codes.
  - Concatenation is an important, often used concept in error correction.
  - The idea is simply to combine the two codes.
- Step 1: Apply bit flip code to physical qubit.
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$$|1\rangle \mapsto |111\rangle \mapsto (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) =: |\bar{1}\rangle$$

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$$|0\rangle \mapsto |000\rangle \mapsto (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) =: |\bar{0}\rangle$$

$$|1\rangle \mapsto |111\rangle \mapsto (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) =: |\bar{1}\rangle$$

$$|\bar{\psi}\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$$

Shor’s 9-qubit code

- This is formed by **concatenating** the bit flip and phase flip codes.
    - Concatenation is an important, often used concept in error correction.
    - The idea is simply to combine the two codes.
- Step 1: Apply bit flip code to physical qubit.
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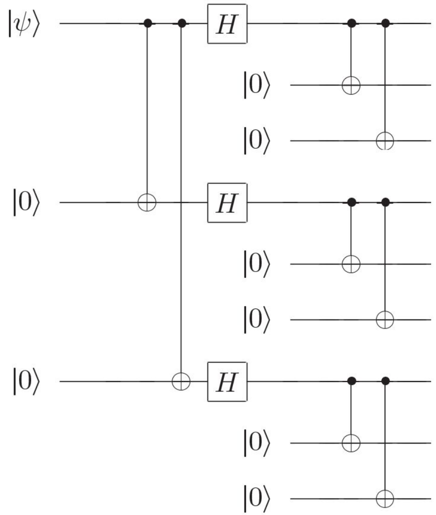

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Note 1: Error correction vs. fault tolerance

- **Error correction**:
  - Theory in which some components do not have errors (by assumption)
  - E.g., state preparation is perfect, errors occur only during gates
  - This is “easier” than fault tolerance (simplifying assumptions)

- **Fault tolerance**:
  - Theory in which all components have errors
  - State preparation, gates, measurements, …
  - This is “harder” than error correction (no simplifying assumptions)

Note 2: Redundancy vs. partitioning

Blue = good basis vector (*codeword*)

Red = bad basis vector (*error state*)

■ |000>	■ |100>
■ |001>	■ |101>
■ |010>	■ |110>
■ |011>	■ |111>

From projections to stabilizers

- Remember the four projectors for the bit-flip code?

Syndrome measurement	Meaning	Correction operator
$P_0 = \|000\rangle\langle 000\| + \|111\rangle\langle 111\|$	No qubit was flipped.	I
$P_1 = \|100\rangle\langle 100\| + \|011\rangle\langle 011\|$	The first qubit was flipped.	X_0
$P_2 = \|010\rangle\langle 010\| + \|101\rangle\langle 101\|$	The second qubit was flipped.	X_1
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- **There's a more succinct way to determine which errors occurred**.

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- Consider measuring the operator $Z_1 Z_2 \equiv ZZI$

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$$ZZ = \underbrace{\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)}_{\text{+1 eigenspace. Bits are the same.}} - \underbrace{\left(|01\rangle\langle 01| + |10\rangle\langle 10|\right)}_{\text{-1 eigenspace. Bits are different.}}$$

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We get +1 if the bits are the same and -1 if the bits are different.

Thus, measuring <ZZ> is asking:
Are these bits the same or different?

From projections to stabilizers

- Just as Z1 Z2 asks if the first two bits are the same/different, Z2 Z3 asks if the second two bits are the same/different.
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Q: Could we do the same with Z1 Z2 and Z1 Z3?

From projections to stabilizers

Syndrome measurement outcome $(\langle Z_1 Z_2 \rangle, \langle Z_2 Z_3 \rangle)$	**Meaning**	**Correction operator**
(1, 1)	No qubit was flipped.	I
(-1, 1)	The first qubit was flipped.	X_0
(-1, -1)	The second qubit was flipped.	X_1
(1, -1)	The third qubit was flipped.	X_2

The operators Z1 Z2 and Z2 Z3 are known as **stabilizer elements**.

Stabilizer elements? Elements of what?

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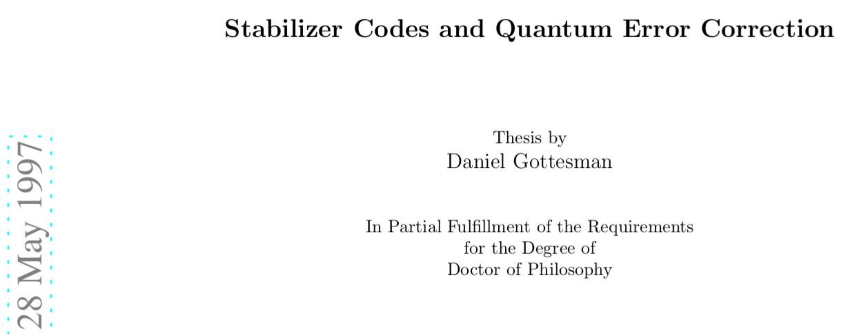
- This is a subgroup of P3 (the Pauli group on 3 qubits).
- The subspace of P3 **stabilized** by S is spanned by |000> and |111>.
    - These are the codewords for the bit-flip code.

Why the stabilizer formalism?

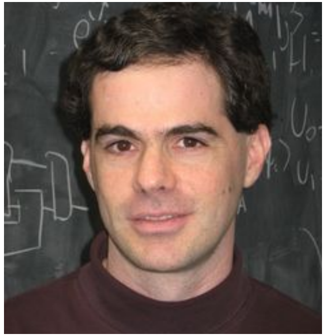
- Describing codewords themselves is cumbersome with more complicated codes.
  - Stabilizers offer a more succinct representation.
  - Namely, via the generator representation of a group.
- Very convenient abstraction that allows for generalization.
  - Many codes can be described in the stabilizer formalism.
  - Pick a stabilizer and you have your very own code!
- First introduced by [Gottesman in his 1996 PhD thesis](#).


Stabilizer Codes and Quantum Error Correction

Thesis by
Daniel Gottesman

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

28 May 1997



Quantum error mitigation intro

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Expectation

$$\langle\psi|O|\psi\rangle$$

Reality

$$\mathrm{Tr}\left[\mathcal{E}(|\psi\rangle\langle\psi|)O\right]$$

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Idea

$$\langle \psi | O | \psi \rangle \overset{?}{\approx} \sum_{i,j} c_{ij} \mathrm{Tr}\left[\mathcal{E}_i(|\psi_j\rangle\langle\psi_j|)O\right]$$

Quantum error mitigation intro: Zero-noise extrapolation

$$\mathcal{E}_p \rightarrow \mathcal{E}_{\lambda_1 p}, \mathcal{E}_{\lambda_2 p}, \ldots, \mathcal{E}_{\lambda_k p}$$

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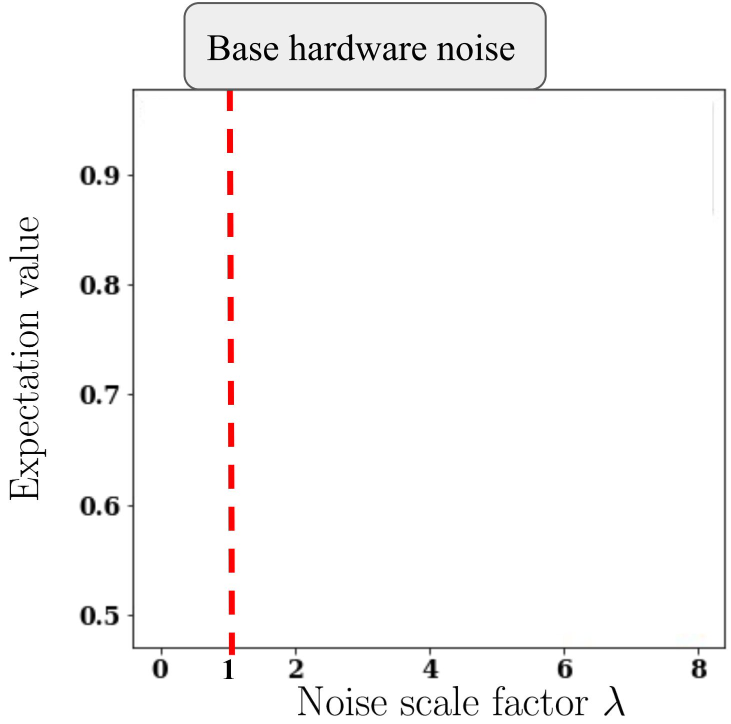

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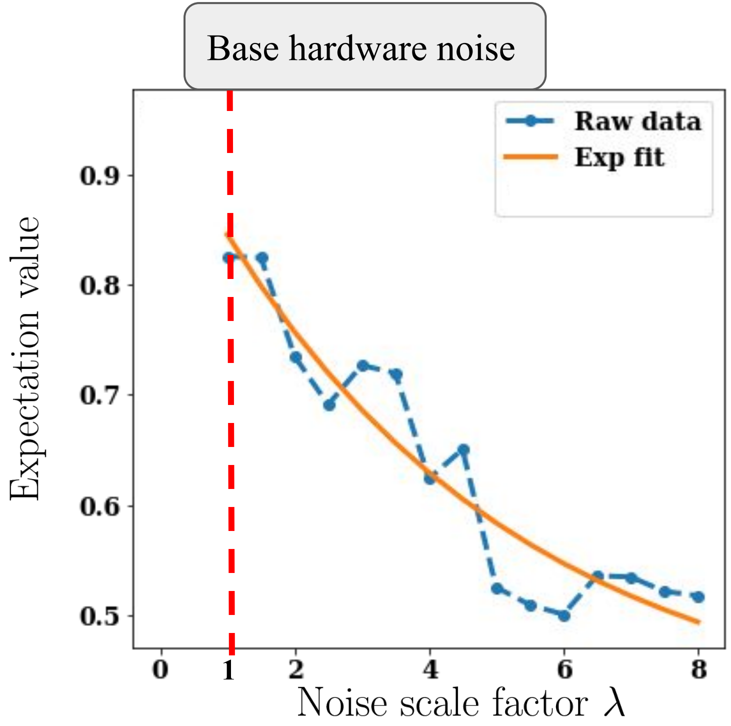

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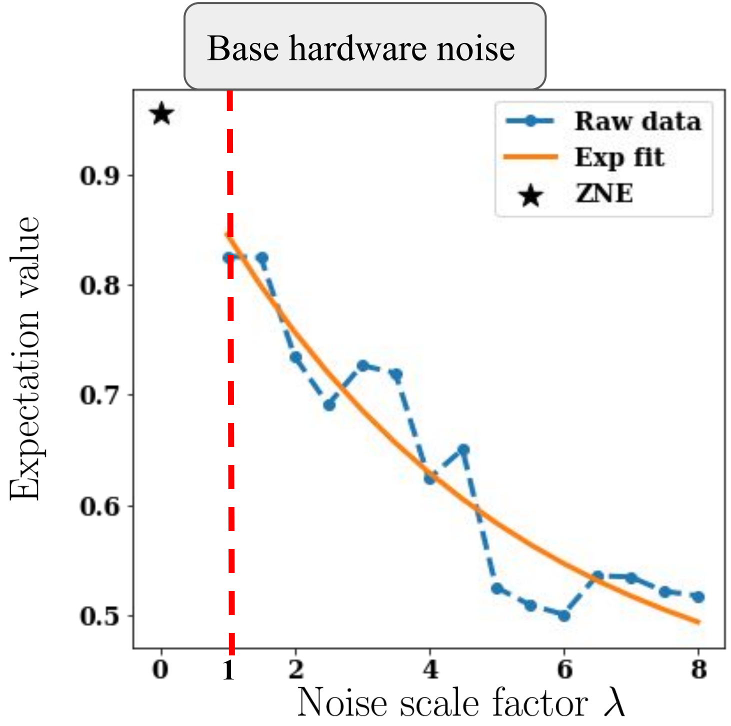

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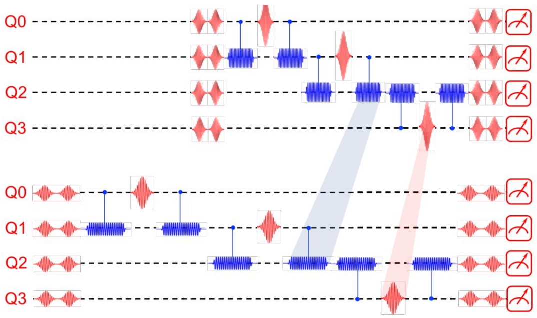
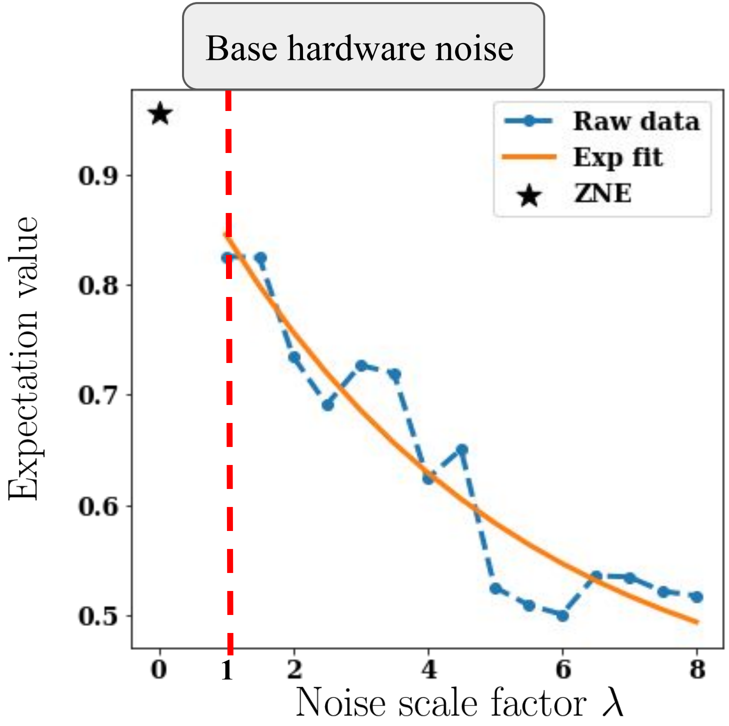
Quantum error mitigation intro: Zero-noise extrapolation

Noise can be scaled by:

- Pulse stretching [Kandala 2018]





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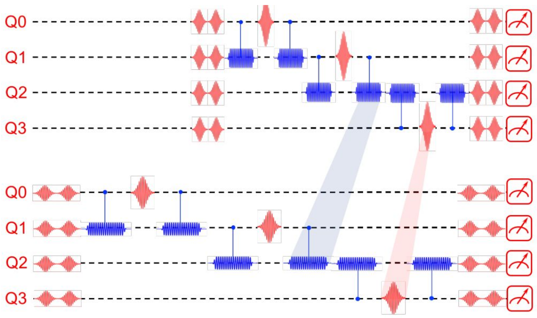

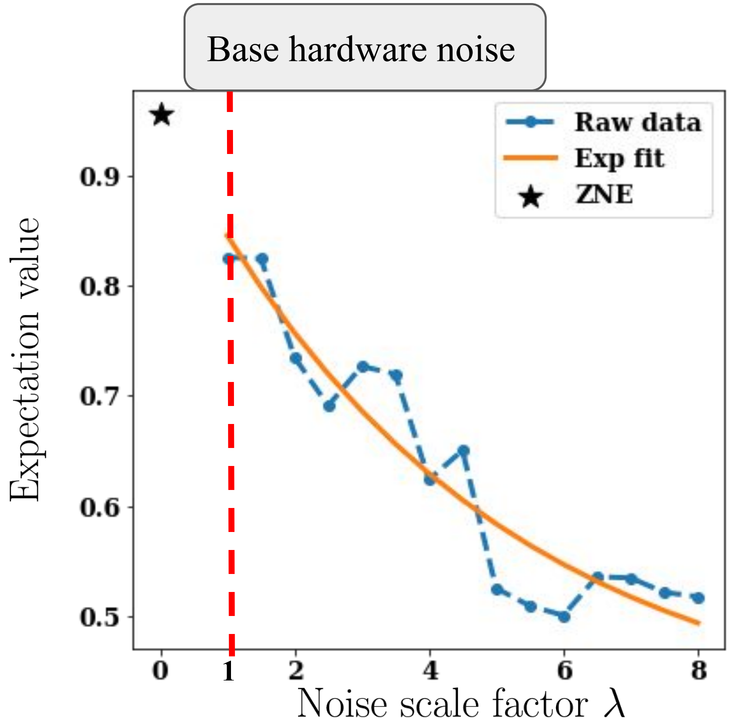

Noise can be scaled by:

- Unitary folding

$$G \mapsto GG^\dagger G$$

```
Original circuit:
0: ───H───@───
          │
1: ───────X───
```

```
Folded circuit:
0: ───H───@───@───H───H───@───
          │   │           │
1: ───────X───X───────────X───
```

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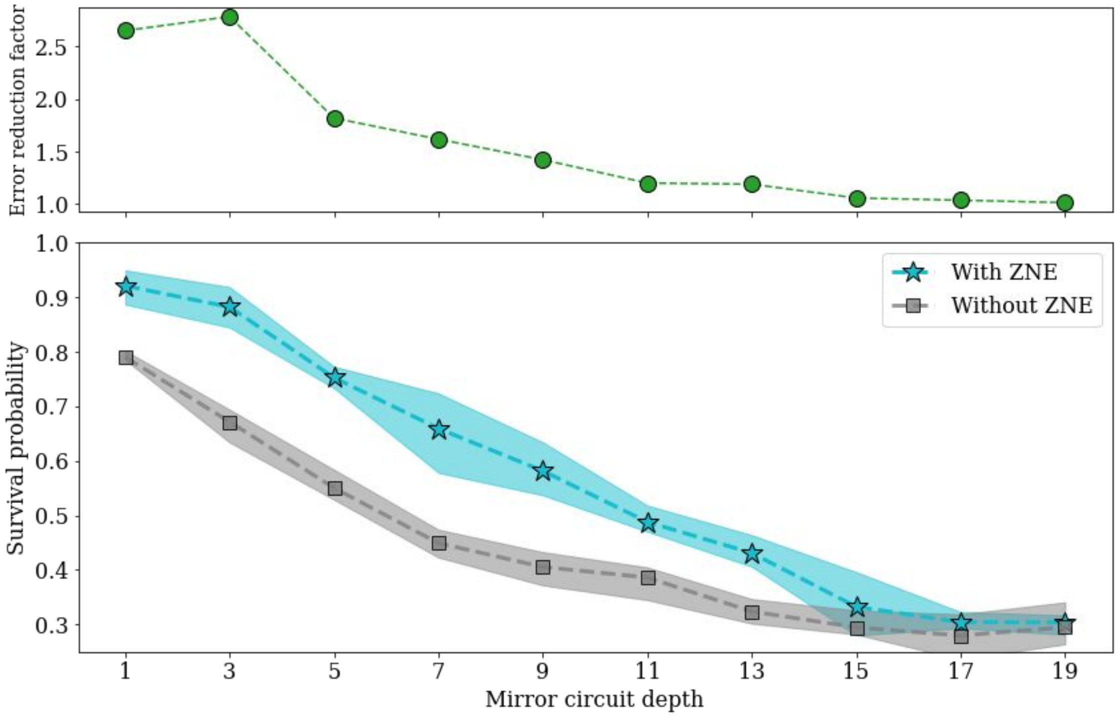
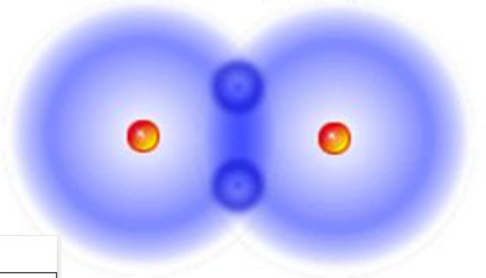
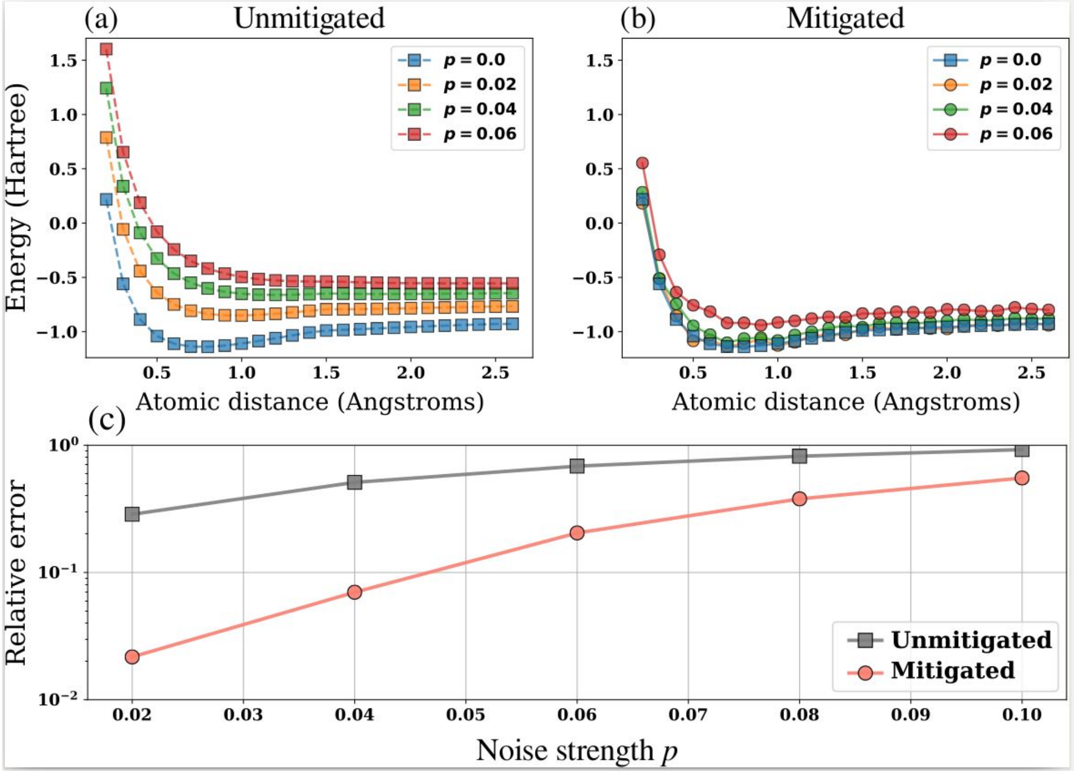
Example: Mirror circuits with ZNE on AWS Braket



https://mitiq.readthedocs.io/en/latest/examples/braket_mirror_circuit.html

Example: VQE on H2 with ZNE





https://arxiv.org/abs/2009.04417