Quantum numbers and nuclear forces

Morten Hjorth-Jensen, National Superconducting Cyclotron Laboratory and Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824, USA & Department of Physics, University of Oslo, Oslo, Norway

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Outline

- Discussion of single-particle and two-particle quantum numbers, uncoupled and coupled schemes
- Discussion of nuclear forces and models thereof (this material will not be covered in depth this spring)

For quantum numbers, chapter 1 on angular momentum and chapter 5 of Suhonen. See also chapters 5, 12 and 13 of Alex Brown. For models of the nuclear forces, see present lectures and pdf files. For a good discussion of isospin, see Alex Brown's lecture notes chapter 12, 13 and 19.

Single-particle and two-particle quantum numbers

In order to understand the basics of the nucleon-nucleon interaction, we need to define the relevant quantum numbers and how we build up a single-particle state and a two-body state.

- For the single-particle states, due to the fact that we have the spin-orbit force, the quantum numbers for the projection of orbital momentum l, that is m_l, and for spin s, that is m_s, are no longer so-called good quantum numbers. The total angular momentum j and its projection m_j are then so-called *good quantum numbers*.
- This means that the operator $\hat{J}^2$ does not commute with $\hat{L}_z$ or $\hat{S}_z$.
- We also start normally with single-particle state functions defined using say the harmonic oscillator. For these functions, we have no explicit dependence on j. How can we introduce single-particle wave functions which have j and its projection m_j as quantum numbers?

Single-particle and two-particle quantum numbers, brief review on angular momenta etc

We have that the operators for the orbital momentum are given by

$$L_x = -i\hbar(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}) = yp_z - zp_y,$$

$$L_y = -i\hbar(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}) = zp_x - xp_z,$$

$$L_z = -i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) = xp_y - yp_x.$$

Single-particle and two-particle quantum numbers, brief review on angular momenta etc

Since we have a spin orbit force which is strong, it is easy to show that the total angular momentum operator

$$\hat{J} = \hat{L} + \hat{S}$$

does not commute with $\hat{L}_z$ and $\hat{S}_z$. To see this, we calculate for example

$$\begin{aligned} [\hat{L}_z, \hat{J}^2] &= [\hat{L}_z, (\hat{L} + \hat{S})^2] \\ &= [\hat{L}_z, \hat{L}^2 + \hat{S}^2 + 2\hat{L}\hat{S}] \\ &= [\hat{L}_z, \hat{L}\hat{S}] = [\hat{L}_z, \hat{L}_x\hat{S}_x + \hat{L}_y\hat{S}_y + \hat{L}_z\hat{S}_z] \neq 0, \end{aligned} \tag{1}$$

since we have that $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$ and $[\hat{L}_z, \hat{L}_y] = i\hbar\hat{L}_x$.

Single-particle and two-particle quantum numbers, brief review on angular momenta etc

We have also

$$|\hat{J}| = \hbar\sqrt{J(J+1)},$$

with the the following degeneracy

$$M_J = -J, -J+1, \ldots, J-1, J.$$

With a given value of L and S we can then determine the possible values of J by studying the z component of $\hat{J}$. It is given by

$$\hat{J}_z = \hat{L}_z + \hat{S}_z.$$

The operators $\hat{L}_z$ and $\hat{S}_z$ have the quantum numbers $L_z = M_L\hbar$ and $S_z = M_S\hbar$, respectively, meaning that

$$M_J\hbar = M_L\hbar + M_S\hbar,$$

or

Single-particle and two-particle quantum numbers, brief review on angular momenta etc

For nucleons we have that the maximum value of $M_S = m_s = 1/2$, yielding

$$(m_j)_{\max} = l + \frac{1}{2}.$$

Using this and the fact that the maximum value of $M_J = m_j$ is j we have

$$j = l + \frac{1}{2}, l - \frac{1}{2}, l - \frac{3}{2}, l - \frac{5}{2}, \dots$$

To decide where this series terminates, we use the vector inequality

$$|\hat{L} + \hat{S}| \geq \left| |\hat{L}| - |\hat{S}| \right|.$$

Single-particle and two-particle quantum numbers, brief review on angular momenta etc

Using $\hat{J} = \hat{L} + \hat{S}$ we get

$$|\hat{J}| \geq |\hat{L}| - |\hat{S}|,$$

or

$$|\hat{J}| = \hbar\sqrt{J(J+1)} \geq |\hbar\sqrt{L(L+1)} - \hbar\sqrt{S(S+1)}|.$$

Single-particle and two-particle quantum numbers, brief review on angular momenta etc

If we limit ourselves to nucleons only with $s = 1/2$ we find that

$$|\hat{J}| = \hbar\sqrt{j(j+1)} \geq |\hbar\sqrt{l(l+1)} - \hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1)}|.$$

It is then easy to show that for nucleons there are only two possible values of j which satisfy the inequality, namely

$$j = l + \frac{1}{2} \text{ or } j = l - \frac{1}{2},$$

and with $l = 0$ we get

$$j = \frac{1}{2}.$$

Single-particle and two-particle quantum numbers, brief review on angular momenta etc

Let us study some selected examples. We need also to keep in mind that parity is conserved. The strong and electromagnetic Hamiltonians conserve parity. Thus the eigenstates can be broken down into two classes of states labeled by their parity $\pi = +1$ or $\pi = -1$. The nuclear interactions do not mix states with different parity.

For nuclear structure the total parity originates from the intrinsic parity of the nucleon which is $\pi_{\mathrm{intrinsic}} = +1$ and the parities associated with the orbital angular momenta $\pi_l = (-1)^l$. The total parity is the product over all nucleons

$\pi = \prod_i \pi_{\mathrm{intrinsic}}(i)\pi_l(i) = \prod_i (-1)^{l_i}$

The basis states we deal with are constructed so that they conserve parity and have thus a definite parity.

Note that we do have parity violating processes, more on this later although our focus will be mainly on non-parity viloating processes

Single-particle and two-particle quantum numbers

Consider now the single-particle orbits of the $1s0d$ shell. For a $0d$ state we have the quantum numbers $l = 2$, $m_l = -2, -1, 0, 1, 2$, $s + 1/2$, $m_s = \pm 1/2$, $n = 0$ (the number of nodes of the wave function). This means that we have positive parity and

$$j = \frac{3}{2} = l - s \qquad m_j = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}.$$

and

$$j = \frac{5}{2} = l + s \qquad m_j = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}.$$

Single-particle and two-particle quantum numbers

Our single-particle wave functions, if we use the harmonic oscillator, do however not contain the quantum numbers j and m_j. Normally what we have is an eigenfunction for the one-body problem defined as

$$\phi_{nlm_l sm_s}(r,\theta,\phi) = R_{nl}(r)Y_{lm_l}(\theta,\phi)\xi_{sm_s},$$

where we have used spherical coordinates (with a spherically symmetric potential) and the spherical harmonics

$$Y_{lm_l}(\theta,\phi) = P(\theta)F(\phi) = \sqrt{\frac{(2l+1)(l-m_l)!}{4\pi(l+m_l)!}}P_l^{m_l}(cos(\theta))\exp(im_l\phi),$$

with $P_l^{m_l}$ being the so-called associated Legendre polynomials.

Single-particle and two-particle quantum numbers

Examples are

$$Y_{00} = \sqrt{\frac{1}{4\pi}},$$

for $l = m_l = 0$,

$$Y_{10} = \sqrt{\frac{3}{4\pi}} cos(\theta),$$

for $l = 1$ and $m_l = 0$,

$$Y_{1\pm1} = \sqrt{\frac{3}{8\pi}} sin(\theta) exp(\pm i\phi),$$

for $l = 1$ and $m_l = \pm 1$,

$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3cos^2(\theta) - 1)$$

for $l = 2$ and $m_l = 0$ etc.

Single-particle and two-particle quantum numbers

How can we get a function in terms of j and m_j? Define now

$$\phi_{nlm_l sm_s}(r,\theta,\phi) = R_{nl}(r)Y_{lm_l}(\theta,\phi)\xi_{sm_s},$$

and

$$\psi_{njm_j;lm_l sm_s}(r,\theta,\phi),$$

as the state with quantum numbers jm_j. Operating with

$$\hat{j}^2 = (\hat{l}+\hat{s})^2 = \hat{l}^2 + \hat{s}^2 + 2\hat{l}_z\hat{s}_z + \hat{l}_+\hat{s}_- + \hat{l}_-\hat{s}_+,$$

on the latter state we will obtain admixtures from possible $\phi_{nlm_l sm_s}(r,\theta,\phi)$ states.

Single-particle and two-particle quantum numbers

To see this, we consider the following example and fix

$$j = \frac{3}{2} = l - s \qquad m_j = \frac{3}{2}.$$

and

$$j = \frac{5}{2} = l + s \qquad m_j = \frac{3}{2}.$$

It means we can have, with $l = 2$ and $s = 1/2$ being fixed, in order to have $m_j = 3/2$ either $m_l = 1$ and $m_s = 1/2$ or $m_l = 2$ and $m_s = -1/2$. The two states

$$\psi_{n=0j=5/2m_j=3/2;l=2s=1/2}$$

and

$$\psi_{n=0j=3/2m_j=3/2;l=2s=1/2}$$

will have admixtures from $\phi_{n=0l=2m_l=2s=1/2m_s=-1/2}$ and $\phi_{n=0l=2m_l=1s=1/2m_s=1/2}$. How do we find these admixtures? Note that we don't specify the values of m_l and m_s in the functions ψ

Single-particle and two-particle quantum numbers

We operate with

$$\hat{j}^2 = (\hat{l} + \hat{s})^2 = \hat{l}^2 + \hat{s}^2 + 2\hat{l}_z\hat{s}_z + \hat{l}_+\hat{s}_- + \hat{l}_-\hat{s}_+$$

on the two jm_j states, that is

$$\hat{j}^2\psi_{n=0j=5/2m_j=3/2;l=2s=1/2} = \alpha\hbar^2[l(l+1)+\frac{3}{4}+2m_l m_s]\phi_{n=0l=2m_l=2s=1/2m_s}$$

$$\beta\hbar^2\sqrt{l(l+1)-m_l(m_l-1)}\phi_{n=0l=2m_l=1s=1/2m_s=1/2},$$

and

$$\hat{j}^2\psi_{n=0j=3/2m_j=3/2;l=2s=1/2} = \alpha\hbar^2[l(l+1)+\frac{3}{4}+2m_l m_s]+\phi_{n=0l=2m_l=1s=1/2}$$

$$\beta\hbar^2\sqrt{l(l+1)-m_l(m_l+1)}\phi_{n=0l=2m_l=2s=1/2m_s=-1/2}.$$

Single-particle and two-particle quantum numbers

This means that the eigenvectors $\phi_{n=0l=2m_l=2s=1/2m_s=-1/2}$ etc are not eigenvectors of $\hat{j}^2$. The above problems gives a 2×2 matrix that mixes the vectors $\psi_{n=0j=5/2m_j3/2;l=2m_ls=1/2m_s}$ and $\psi_{n=0j=3/2m_j3/2;l=2m_ls=1/2m_s}$ with the states $\phi_{n=0l=2m_l=2s=1/2m_s=-1/2}$ and $\phi_{n=0l=2m_l=1s=1/2m_s=1/2}$. The unknown coefficients α and β results from eigenvectors of this matrix. That is, inserting all values m_l, l, m_s, s we obtain the matrix

$$\begin{bmatrix} 19/4 & 2 \\ 2 & 31/4 \end{bmatrix}$$

whose eigenvectors are the columns of

$$\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$$

These numbers define the so-called Clebsch-Gordan coupling coefficients (the overlaps between the two basis sets). We can thus write

Clebsch-Gordan coefficients

The Clebsch-Gordan coeffficients $\langle lm_l sm_s | jm_j \rangle$ have some interesting properties for us, like the following orthogonality relations

$$\sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | JM \rangle \langle j_1 m_1 j_2 m_2 | J' M' \rangle = \delta_{J,J'} \delta_{M,M'},$$

$$\sum_{JM} \langle j_1 m_1 j_2 m_2 | JM \rangle \langle j_1 m_1' j_2 m_2' | JM \rangle = \delta_{m_1,m_1'} \delta_{m_2,m_2'},$$

$$\langle j_1 m_1 j_2 m_2 | JM \rangle = (-1)^{j_1+j_2-J} \langle j_2 m_2 j_1 m_1 | JM \rangle,$$

and many others. The latter will turn extremely useful when we are going to define two-body states and interactions in a coupled basis.

Quantum numbers and the Schroeodinger equation in relative and CM coordinates

Summing up, for for the single-particle case, we have the following eigenfunctions

$$\psi_{njm_j;ls} = \sum_{m_l m_s} \langle lm_l sm_s | jm_j \rangle \phi_{nlm_l sm_s},$$

where the coefficients $\langle lm_l sm_s | jm_j \rangle$ are the so-called Clebsch-Gordan coeffficients. The relevant quantum numbers are n (related to the principal quantum number and the number of nodes of the wave function) and

$$\hat{j}^2 \psi_{njm_j;ls} = \hbar^2 j(j+1)\psi_{njm_j;ls},$$

$$\hat{j}_z \psi_{njm_j;ls} = \hbar m_j \psi_{njm_j;ls},$$

$$\hat{l}^2 \psi_{njm_j;ls} = \hbar^2 l(l+1)\psi_{njm_j;ls},$$

$$\hat{s}^2 \psi_{njm_j;ls} = \hbar^2 s(s+1)\psi_{njm_j;ls},$$

Quantum numbers and the Schroedinger equation in relative and CM coordinates

For a two-body state where we couple two angular momenta j_1 and j_2 to a final angular momentum J with projection M_J, we can define a similar transformation in terms of the Clebsch-Gordan coeffficients

$$\psi_{(j_1 j_2)JM_J} = \sum_{m_{j_1} m_{j_2}} \langle j_1 m_{j_1} j_2 m_{j_2} | JM_J \rangle \psi_{n_1 j_1 m_{j_1}; l_1 s_1} \psi_{n_2 j_2 m_{j_2}; l_2 s_2}.$$

We will write these functions in a more compact form hereafter, namely,

$$|(j_1 j_2)JM_J\rangle = \psi_{(j_1 j_2)JM_J},$$

and

$$|j_i m_{j_i}\rangle = \psi_{n_i j_i m_{j_i}; l_i s_i},$$

where we have skipped the explicit reference to l, s and n. The spin of a nucleon is always $1/2$ while the value of l can be deduced from the parity of the state. It is thus normal to label a state with

Quantum numbers and the Schroedinger equation in relative and CM coordinates

Our two-body state can thus be written as

$$|(j_1 j_2) J M_J\rangle = \sum_{m_{j_1} m_{j_2}} \langle j_1 m_{j_1} j_2 m_{j_2} | J M_J\rangle |j_1 m_{j_1}\rangle |j_2 m_{j_2}\rangle.$$

Due to the coupling order of the Clebsch-Gordan coefficient it reads as j_1 coupled to j_2 to yield a final angular momentum J. If we invert the order of coupling we would have

$$|(j_2 j_1) J M_J\rangle = \sum_{m_{j_1} m_{j_2}} \langle j_2 m_{j_2} j_1 m_{j_1} | J M_J\rangle |j_1 m_{j_1}\rangle |j_2 m_{j_2}\rangle,$$

and due to the symmetry properties of the Clebsch-Gordan coefficient we have

$$|(j_2 j_1) J M_J\rangle = (-1)^{j_1+j_2-J} \sum_{m_{j_1} m_{j_2}} \langle j_1 m_{j_1} j_2 m_{j_2} | J M_J\rangle |j_1 m_{j_1}\rangle |j_2 m_{j_2}\rangle = (-1)^{j_1+}$$

Quantum numbers

We have thus the coupled basis

$$|(j_1 j_2) J M_J\rangle = \sum_{m_{j_1} m_{j_2}} \langle j_1 m_{j_1} j_2 m_{j_2} | J M_J\rangle |j_1 m_{j_1}\rangle |j_2 m_{j_2}\rangle.$$

and the uncoupled basis

$$|j_1 m_{j_1}\rangle |j_2 m_{j_2}\rangle.$$

The latter can easily be generalized to many single-particle states whereas the first needs specific coupling coefficients and definitions of coupling orders. The *m*-scheme basis is easy to implement numerically and is used in most standard shell-model codes. Our coupled basis obeys also the following relations

$$\hat{J}^2 |(j_1 j_2) J M_J\rangle = \hbar^2 J(J+1) |(j_1 j_2) J M_J\rangle$$

$$\hat{J}_z |(j_1 j_2) J M_J\rangle = \hbar M_J |(j_1 j_2) J M_J\rangle,$$

Components of the force and isospin

The nuclear forces are almost charge independent. If we assume they are, we can introduce a new quantum number which is conserved. For nucleons only, that is a proton and neutron, we can limit ourselves to two possible values which allow us to distinguish between the two particles. If we assign an isospin value of $\tau = 1/2$ for protons and neutrons (they belong to an isospin doublet, in the same way as we discussed the spin $1/2$ multiplet), we can define the neutron to have isospin projection $\tau_z = +1/2$ and a proton to have $\tau_z = -1/2$. These assignements are the standard choices in low-energy nuclear physics.

Isospin

This leads to the introduction of an additional quantum number called isospin. We can define a single-nucleon state function in terms of the quantum numbers n, j, m_j, l, s, τ and τ_z. Using our definitions in terms of an uncoupled basis, we had

$$\psi_{njm_j;ls} = \sum_{m_l m_s} \langle lm_l sm_s | jm_j \rangle \phi_{nlm_l sm_s},$$

which we can now extend to

$$\psi_{njm_j;ls}\xi_{\tau\tau_z} = \sum_{m_l m_s} \langle lm_l sm_s | jm_j \rangle \phi_{nlm_l sm_s}\xi_{\tau\tau_z},$$

with the isospin spinors defined as

$$\xi_{\tau=1/2\tau_z=+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and

$$\xi_{\tau=1/2\tau_z=-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Isospin

We can in turn define the isospin Pauli matrices (in the same as we define the spin matrices) as

$$\hat{\tau}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\hat{\tau}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and

$$\hat{\tau}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and operating with $\hat{\tau}_z$ on the proton state function we have

$$\hat{\tau}_z \psi^p(\mathbf{r}) = -\frac{1}{2}\psi^p(\mathbf{r}),$$

and for neutrons we have

$$\hat{\tau}_z \psi^n(\mathbf{r}) = \frac{1}{2}\psi^n(\mathbf{r}).$$

Isospin

We can now define the so-called charge operator as

$$\frac{\hat{Q}}{e} = \frac{1}{2}(1 - \hat{\tau}_z) = \begin{Bmatrix} 0 & 0 \\ 0 & 1 \end{Bmatrix},$$

which results in

$$\frac{\hat{Q}}{e}\psi^p(\mathbf{r}) = \psi^p(\mathbf{r}),$$

and

$$\frac{\hat{Q}}{e}\psi^n(\mathbf{r}) = 0,$$

as it should be.

Isospin

The total isospin is defined as

$$\hat{T} = \sum_{i=1}^{A} \hat{\tau}_i,$$

and its corresponding isospin projection as

$$\hat{T}_z = \sum_{i=1}^{A} \hat{\tau}_{z_i},$$

with eigenvalues $T(T+1)$ for $\hat{T}$ and $1/2(N-Z)$ for $\hat{T}_z$, where N is the number of neutrons and Z the number of protons.
If charge is conserved, the Hamiltonian $\hat{H}$ commutes with $\hat{T}_z$ and all members of a given isospin multiplet (that is the same value of T) have the same energy and there is no T_z dependence and we say that $\hat{H}$ is a scalar in isospin space.

Two-body matrix elements

Till now we have not said anything about the explicit calculation of two-body matrix elements. It is time to amend this deficiency. We have till now seen the following definitions of a two-body matrix elements. In m-scheme with quantum numbers $p = j_p m_p$ etc we have a two-body state defined as

$$|(pq)M\rangle = a_p^\dagger a_q^\dagger |\Phi_0\rangle,$$

where $|\Phi_0\rangle$ is a chosen reference state, say for example the Slater determinant which approximates ^{16}O with the $0s$ and the $0p$ shells being filled, and $M = m_p + m_q$. Recall that we label single-particle states above the Fermi level as $abcd\ldots$ and states below the Fermi level for $ijkl\ldots$. In case of two-particles in the single-particle states a and b outside ^{16}O as a closed shell core, say ^{18}O, we would write the representation of the Slater determinant as

$$|^{18}\mathrm{O}\rangle = |(ab)M\rangle = a_a^\dagger a_b^\dagger |^{16}\mathrm{O}\rangle = |\Phi^{ab}\rangle.$$

In case of two-particles removed from say ^{16}O, for example two

Two-body matrix elementsin m-scheme

For a one-hole-one-particle state we have

$$|^{16}\mathrm{O}\rangle_{1p1h} = |(ai)M\rangle = a_a^\dagger a_i |^{16}\mathrm{O}\rangle = |\Phi_i^a\rangle,$$

and finally for a two-particle-two-hole state we

$$|^{16}\mathrm{O}\rangle_{2p2h} = |(abij)M\rangle = a_a^\dagger a_b^\dagger a_j a_i |^{16}\mathrm{O}\rangle = |\Phi_{ij}^{ab}\rangle.$$

Two-body matrix elements in *m*-scheme

Let us go back to the case of two-particles in the single-particle states a and b outside ^{16}O as a closed shell core, say ^{18}O. The representation of the Slater determinant is

$$|^{18}\mathrm{O}\rangle = |(ab)M\rangle = a_a^\dagger a_b^\dagger |^{16}\mathrm{O}\rangle = |\Phi^{ab}\rangle.$$

The anti-symmetrized matrix element is detailed as

$$\langle (ab)M|\hat{V}|(cd)M\rangle = \langle (j_a m_a j_b m_b)M = m_a+m_b|\hat{V}|(j_c m_c j_d m_d)M = m_a+m$$

and note that anti-symmetrization means

$$\langle (ab)M|\hat{V}|(cd)M\rangle = -\langle (ba)M|\hat{V}|(cd)M\rangle = \langle (ba)M|\hat{V}|(dc)M\rangle,$$

$$\langle (ab)M|\hat{V}|(cd)M\rangle = -\langle (ab)M|\hat{V}|(dc)M\rangle.$$

This matrix element is the expectation value of

$$\langle ^{16}\mathrm{O}|a_b a_a \frac{1}{4}\sum \langle (pq)M|\hat{V}|(rs)M'\rangle a_p^\dagger a_q^\dagger a_s a_r a_c^\dagger a_c^\dagger |^{16}\mathrm{O}\rangle.$$

Two-body matrix elements in J-scheme

We have also defined matrix elements in the coupled basis, the so-called J-coupled scheme. In this case the two-body wave function for two neutrons outside ^{16}O is written as

$$|^{18}\mathrm{O}\rangle_J = |(ab)JM\rangle = \left\{a_a^\dagger a_b^\dagger\right\}_M^J |^{16}\mathrm{O}\rangle = N_{ab}\sum_{m_a m_b}\langle j_a m_a j_b m_b|JM\rangle|\Phi^{ab}\rangle,$$

with

$$|\Phi^{ab}\rangle = a_a^\dagger a_b^\dagger|^{16}\mathrm{O}\rangle.$$

We have now an explicit coupling order, where the angular momentum j_a is coupled to the angular momentum j_b to yield a final two-body angular momentum J. The normalization factor (to be derived below) is

$$N_{ab} = \frac{\sqrt{1+\delta_{ab}\times(-1)^J}}{1+\delta_{ab}}.$$

Two-body matrix elements in J-scheme

The implementation of the Pauli principle looks different in the J-scheme compared with the m-scheme. In the latter, no two fermions or more can have the same set of quantum numbers. In the J-scheme, when we write a state with the shorthand

$$|^{18}\mathrm{O}\rangle_J = |(ab)JM\rangle,$$

we do refer to the angular momenta only. This means that another way of writing the last state is

$$|^{18}\mathrm{O}\rangle_J = |(j_a j_b)JM\rangle.$$

We will use this notation throughout when we refer to a two-body state in J-scheme. The Kronecker δ function in the normalization factor refers thus to the values of j_a and j_b. If two identical particles are in a state with the same j-value, then only even values of the total angular momentum apply.

Two-body matrix elements in J-scheme

Note also that, using the anti-commuting properties of the creation operators, we obtain

$$N_{ab}\sum_{m_a m_b}\langle j_a m_a j_b m_b|JM> |\Phi^{ab}\rangle = -N_{ab}\sum_{m_a m_b}\langle j_a m_a j_b m_b|JM\rangle|\Phi^{ba}\rangle.$$

Furthermore, using the property of the Clebsch-Gordan coefficient

$$\langle j_a m_a j_b m_b|JM>=(-1)^{j_a+j_b-J}\langle j_b m_b j_a m_a|JM\rangle,$$

which can be used to show that

$$|(j_b j_a)JM\rangle = \left\{a_b^\dagger a_a^\dagger\right\}_M^J|^{16}\mathrm{O}\rangle = N_{ab}\sum_{m_a m_b}\langle j_b m_b j_a m_a|JM\rangle|\Phi^{ba}\rangle,$$

is equal to

$$|(j_b j_a)JM\rangle = (-1)^{j_a+j_b-J+1}|(j_a j_b)JM\rangle.$$

This relation is important since we will need it when using

Two-body matrix elements in J-scheme

The two-body matrix element is a scalar and since it obeys rotational symmetry, it is diagonal in J, meaning that the corresponding matrix element in J-scheme is

$$\langle (j_a j_b) JM | \hat{V} | (j_c j_d) JM \rangle = N_{ab} N_{cd} \sum_{m_a m_b m_c m_d} \langle j_a m_a j_b m_b | JM \rangle$$

$$\times \langle j_c m_c j_d m_d | JM \rangle \langle (j_a m_a j_b m_b) M | \hat{V} | (j_c m_c j_d m_d) M \rangle,$$

and note that of the four m-values in the above sum, only three are independent due to the constraint $m_a + m_b = M = m_c + m_d$. Since

$$|(j_b j_a) JM\rangle = (-1)^{j_a + j_b - J + 1} |(j_a j_b) JM\rangle,$$

the anti-symmetrized matrix elements need now to obey the following relations

$$\langle (j_a j_b) JM | \hat{V} | (j_c j_d) JM \rangle = (-1)^{j_a + j_b - J + 1} \langle (j_b j_a) JM | \hat{V} | (j_c j_d) JM \rangle,$$

$$\langle (j_a j_b) JM | \hat{V} | (j_c j_d) JM \rangle = (-1)^{j_c + j_d - J + 1} \langle (j_a j_b) JM | \hat{V} | (j_d j_c) JM \rangle$$

Two-body matrix elements in J-scheme

Using the orthogonality properties of the Clebsch-Gordan coefficients,

$$\sum_{m_a m_b} \langle j_a m_a j_b m_b | JM \rangle \langle j_a m_a j_b m_b | J' M' \rangle = \delta_{JJ'} \delta_{MM'},$$

and

$$\sum_{JM} \langle j_a m_a j_b m_b | JM \rangle \langle j_a m'_a j_b m'_b | JM \rangle = \delta_{m_a m'_a} \delta_{m_b m'_b},$$

we can also express the two-body matrix element in m-scheme in terms of that in J-scheme, that is, if we multiply with

$$\sum_{JMJ'M'} \langle j_a m'_a j_b m'_b | JM \rangle \langle j_c m'_c j_d m'_d | J' M' \rangle$$

from left in

$$\langle (j_a j_b) JM | \hat{V} | (j_c j_d) JM \rangle = N_{ab} N_{cd} \sum_{m_a m_b m_c m_d} \langle j_a m_a j_b m_b | JM \rangle \langle j_c m_c j_d m_d | JM$$

Two-body matrix elements in J-scheme

we obtain

$$\langle (j_a m_a j_b m_b) M | \hat{V} | (j_c m_c j_d m_d) M \rangle = \frac{1}{N_{ab} N_{cd}} \sum_{JM} \langle j_a m_a j_b m_b | JM \rangle \langle j_c m_c j_d m_d$$

$$\times \langle (j_a j_b) JM | \hat{V} | (j_c j_d) JM \rangle .$$

Phenomenology of nuclear forces

From Yukawa to Lattice QCD and Effective Field Theory

- Chadwick (1932) discovers the neutron and Heisenberg (1932) proposes the first Phenomenology (Isospin).
- Yukawa (1935) and his Meson Hypothesis
- Discovery of the pion in cosmic ray (1947) and in the Berkeley Cyclotron Lab (1948).
- Nobelprize awarded to Yukawa (1949). Rabi (1948) measures quadrupole moment of the deuteron.
- Taketani, Nakamura, Sasaki (1951): 3 ranges.
  One-Pion-Exchange (OPE): o.k.
- Multi-pion exchanges: Problems! Taketani, Machida, Onuma (1952);
- *Pion Theories* Brueckner, Watson (1953).

Phenomenology of nuclear forces

From Yukawa to Lattice QCD and Effective Field Theory

- Many pions = multi-pion resonances: $\sigma(600)$, $\rho(770)$, $\omega(782)$ etc. One-Boson-Exchange Model.
- Refined Meson Theories
- Sophisticated models for two-pion exchange:
  - Paris Potential (Lacombe et al., Phys. Rev. C **21**, 861 (1980))
  - Bonn potential (Machleidt et al., Phys. Rep. **149**, 1 (1987))

*Quark cluster models. Begin of effective field theory studies.

Phenomenology of nuclear forces

From Yukawa to Lattice QCD and Effective Field Theory

- 1990's
  - 1993-2001: High-precision NN potentials: Nijmegen I, II, '93, Reid93 (Stoks et al. 1994),
  - Argonne V18 (Wiringa et al, 1995), CD-Bonn (Machleidt et al. 1996 and 2001.
  - Advances in effective field theory: Weinberg (1990); Ordonez, Ray, van Kolck and many more.
- 3rd Millenium
  - Another "pion theory"; but now right: constrained by chiral symmetry. Three-body and higher-body forces appear naturally at a given order of the chiral expansion.

Nucleon-nucleon interaction from Lattice QCD, final confirmation of meson hypothesis of Yukawa? See for example Ishii *et al*, PRL 2007

Phenomenology of nuclear forces

Features of the Nucleon-Nucleon (NN) Force

The aim is to give you an overview over central features of the nucleon-nucleon interaction and how it is constructed, with both technical and theoretical approaches.

- The existence of the deuteron with $J^\pi = 1^+$ indicates that the force between protons and neutrons is attractive at least for the ${}^3S_1$ partial wave. Interference between Coulomb and nuclear scattering for the proton-proton partial wave ${}^1S_0$ shows that the NN force is attractive at least for the ${}^1S_0$ partial wave.
- It has a short range and strong intermediate attraction.
- Spin dependent, scattering lengths for triplet and singlet states are different,
- Spin-orbit force. Observation of large polarizations of scattered nucleons perpendicular to the plane of scattering.

Phenomenology of nuclear forces

- Strongly repulsive core. The s-wave phase shift becomes negative at ≈ 250 MeV implying that the singlet S has a hard core with range $0.4 - 0.5$ fm.
- Charge independence (almost). Two nucleons in a given two-body state always (almost) experience the same force. Modern interactions break charge and isospin symmetry lightly. That means that the pp, neutron-neutron and pn parts of the interaction will be different for the same quantum numbers.
- Non-central. There is a tensor force. First indications from the quadrupole moment of the deuteron pointing to an admixture in the ground state of both $I = 2$ ($^3D_1$) and $I = 0$ ($^3S_1$) orbital momenta.

Phenomenology of nuclear forces

Short Range Evidence

Comparison of the binding energies of ^{2}H (deuteron), ^{3}H (triton), ^{4}He (alpha - particle) show that the nuclear force is of finite range ($1 - 2$ fm) and very strong within that range.

For nuclei with $A > 4$, the energy saturates: Volume and binding energies of nuclei are proportional to the mass number A (as we saw from exercise 1).

Nuclei are also bound. The average distance between nucleons in nuclei is about 2 fm which must roughly correspond to the range of the attractive part.

Phenomenology of nuclear forces

Charge Dependence

- After correcting for the electromagnetic interaction, the forces between nucleons (pp, nn, or np) in the same state are almost the same.
- *Almost the same*: Charge-independence is slightly broken.
- Equality between the pp and nn forces: Charge symmetry.
- Equality between pp/nn force and np force: Charge independence.
- Better notation: Isospin symmetry, invariance under rotations in isospin

Phenomenology of nuclear forces

Charge Dependence, ${}^1S_0$ Scattering Lengths

Charge-symmetry breaking (CSB), after electromagnetic effects have been removed:

- $a_{pp} = -17.3 \pm 0.4\mathrm{fm}$
- $a_{nn} = -18.8 \pm 0.5\mathrm{fm}$. Note however discrepancy from *nd* breakup reactions resulting in $a_{nn} = -18.72 \pm 0.13 \pm 0.65\mathrm{fm}$ and $\pi^- + d \to \gamma + 2n$ reactions giving $a_{nn} = -18.93 \pm 0.27 \pm 0.3\mathrm{fm}$.

Charge-independence breaking (CIB)

- $a_{pn} = -23.74 \pm 0.02\mathrm{fm}$

Symmetries of the Nucleon-Nucleon (NN) Force

- Translation invariance
- Galilean invariance
- Rotation invariance in space
- Space reflection invariance
- Time reversal invariance
- Invariance under the interchange of particle 1 and 2
- Almost isospin symmetry

A typical form of the nuclear force

Here we display a typical way to parametrize (non-relativistic expression) the nuclear two-body force in terms of some operators, the central part, the spin-spin part and the central force.

$$V(\mathbf{r}) = \left\{ C_c + C_\sigma \sigma_1 \cdot \sigma_2 + C_T \left(1 + \frac{3}{m_\alpha r} + \frac{3}{(m_\alpha r)^2} \right) S_{12}(\hat{r}) \right.$$

$$\left. + C_{SL} \left(\frac{1}{m_\alpha r} + \frac{1}{(m_\alpha r)^2} \right) \mathbf{L} \cdot \mathbf{S} \right\} \frac{e^{-m_\alpha r}}{m_\alpha r}$$

How do we derive such terms? (Note: no isospin dependence and that the above is an approximation)

Nuclear forces

To derive the above famous form of the nuclear force using field theoretical concepts, we will need some elements from relativistic quantum mechanics. These derivations will be given below. The material here gives some background to this. I know that many of you have not taken a course in quantum field theory. I hope however that you can see the basic ideas leading to the famous non-relativistic expressions for the nuclear force.

Furthermore, when we analyze nuclear data, we will actually try to explain properties like spectra, single-particle energies etc in terms of the various terms of the nuclear force. Moreover, many of you will hear about these terms at various talks, workshops, seminars etc. Then, it is good to have an idea of what people actually mean!!

Dramatis Personae

Baryons	Mass (MeV)	Mesons	Mass (MeV)
p, n	938.926	π	138.03
Λ	1116.0	η	548.8
Σ	1197.3	σ	≈ 550.0
Δ	1232.0	ρ	770
		ω	782.6
		δ	983.0
		K	495.8
		$K^\star$	895.0

Components of the force and quantum numbers

But before we proceed, we will look into specific quantum numbers of the relative system and study expectation vaues of the various terms of

$$V(\mathbf{r}) = \left\{ C_c + C_\sigma \sigma_1 \cdot \sigma_2 + C_T \left(1 + \frac{3}{m_\alpha r} + \frac{3}{(m_\alpha r)^2} \right) S_{12}(\hat{r}) \right.$$

$$\left. + C_{SL} \left(\frac{1}{m_\alpha r} + \frac{1}{(m_\alpha r)^2} \right) \mathbf{L} \cdot \mathbf{S} \right\} \frac{e^{-m_\alpha r}}{m_\alpha r}$$

Relative and CoM system, quantum numbers

When solving the scattering equation or solving the two-nucleon problem, it is convenient to rewrite the Schroedinger equation, due to the spherical symmetry of the Hamiltonian, in relative and center-of-mass coordinates. This will also define the quantum numbers of the relative and center-of-mass system and will aid us later in solving the so-called Lippman-Schwinger equation for the scattering problem.
We define the center-of-mass (CoM) momentum as

$$\mathbf{K} = \sum_{i=1}^{A} \mathbf{k}_i,$$

with $\hbar = c = 1$ the wave number $k_i = p_i$, with p_i the pertinent momentum of a single-particle state. We have also the relative momentum

$$\mathbf{k}_{ij} = \frac{1}{2}(\mathbf{k}_i - \mathbf{k}_j).$$

We will below skip the indices ij and simply write $\mathbf{k}$

Relative and CoM system, quantum numbers

In a similar fashion we can define the CoM coordinate

$$\mathbf{R} = \frac{1}{A}\sum_{i=1}^{A}\mathbf{r}_i,$$

and the relative distance

$$\mathbf{r}_{ij} = (\mathbf{r}_i - \mathbf{r}_j).$$

Relative and CoM system, quantum numbers

With the definitions

$$\mathbf{K} = \sum_{i=1}^{A} \mathbf{k}_i,$$

and

$$\mathbf{k}_{ij} = \frac{1}{2}(\mathbf{k}_i - \mathbf{k}_j).$$

we can rewrite the two-particle kinetic energy (note that we use $\hbar = c = 1$ as

$$\frac{\mathbf{k}_1^2}{2m_n} + \frac{\mathbf{k}_2^2}{2m_n} = \frac{\mathbf{k}^2}{m_n} + \frac{\mathbf{K}^2}{4m_n},$$

where m_n is the average of the proton and the neutron masses.

Relative and CoM system, quantum numbers

Since the two-nucleon interaction depends only on the relative distance, this means that we can separate Schroedinger's equation in an equation for the center-of-mass motion and one for the relative motion.
With an equation for the relative motion only and a separate one for the center-of-mass motion we need to redefine the two-body quantum numbers.
Previously we had a two-body state vector defined as $|(j_1j_2)JM_J\rangle$ in a coupled basis. We will now define the quantum numbers for the relative motion. Here we need to define new orbital momenta (since these are the quantum numbers which change). We define

$$\hat{l}_1 + \hat{l}_2 = \hat{\lambda} = \hat{l} + \hat{L},$$

where $\hat{l}$ is the orbital momentum associated with the relative motion and $\hat{L}$ the corresponding one linked with the CoM. The total spin S is unchanged since it acts in a different space. We have thus that

Phenomenology of nuclear forces

The total two-nucleon state function has to be anti-symmetric. The total function contains a spatial part, a spin part and an isospin part. If isospin is conserved, this leads to in case we have an s-wave with spin $S = 0$ to an isospin two-body state with $T = 1$ since the spatial part is symmetric and the spin part is anti-symmetric.
Since the projections for T are $T_z = -1, 0, 1$, we can have a pp, an nn and a pn state.
For $l = 0$ and $S = 1$, a so-called triplet state, ${}^3S_1$, we must have $T = 0$, meaning that we have only one state, a pn state. For other partial waves, the following table lists states up to f waves. We can systemize this in a table as follows, recalling that
$|\mathbf{l} - \mathbf{S}| \leq |\mathbf{J}| \leq |\mathbf{l} + \mathbf{S}|$,

${}^{2S+1}l_J$	J	l	S	T	$\|pp\rangle$	$\|pn\rangle$	$\|nn\rangle$
${}^1S_0$	0	0	0	1	yes	yes	yes
${}^3S_1$	1	0	1	0	no	yes	no
${}^3P_0$	0	1	1	1	yes	yes	yes
${}^1P_1$	1	1	0	0	no	yes	no
${}^3P_1$	1	1	1	1	yes	yes	yes
${}^3P_2$	2	1	1	1	yes	yes	yes

Components of the force and quantum numbers

The tensor force is given by

$$S_{12}(\hat{r}) = \frac{3}{r^2}\left(\sigma_1 \cdot \mathbf{r}\right)\left(\sigma_2 \cdot \mathbf{r}\right) - \sigma_1 \cdot \sigma_2$$

where the Pauli matrices are defined as

$$\sigma_x = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix},$$

$$\sigma_y = \begin{Bmatrix} 0 & -\imath \\ \imath & 0 \end{Bmatrix},$$

and

$$\sigma_z = \begin{Bmatrix} 1 & 0 \\ 0 & -1 \end{Bmatrix},$$

with the properties $\sigma = 2\mathbf{S}$ (the spin of the system, being $1/2$ for nucleons), $\sigma_x^2 = \sigma_y^2 = \sigma_z = \mathbf{1}$ and obeying the commutation and anti-commutation relations $\{\sigma_x, \sigma_y\} = 0$ $[\sigma_x, \sigma_y] = \imath\sigma_z$ etc.

Components of the force and quantum numbers

When we look at the expectation value of $\langle \sigma_1 \cdot \sigma_2 \rangle$, we can rewrite this expression in terms of the spin $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$, resulting in

$$\langle \sigma_1 \cdot \sigma_2 \rangle = 2(S^2 - s_1^2 - s_2^2) = 2S(S+1) - 3,$$

where we $s_1 = s_2 = 1/2$ leading to

$$\begin{cases} \langle \sigma_1 \cdot \sigma_2 \rangle = 1 & \text{if } S = 1 \\ \langle \sigma_1 \cdot \sigma_2 \rangle = -3 & \text{if } S = 0 \end{cases}$$

Components of the force and quantum numbers

Similarly, the expectation value of the spin-orbit term is

$$\langle \mathbf{IS} \rangle = \frac{1}{2} \left(J(J+1) - I(I+1) - S(S+1) \right),$$

which means that for s-waves with either $S = 0$ and thereby $J = 0$ or $S = 1$ and $J = 1$, the expectation value for the spin-orbit force is zero. With the above phenomenological model, the only contributions to the expectation value of the potential energy for s-waves stem from the central and the spin-spin components since the expectation value of the tensor force is also zero.

Components of the force and quantum numbers

For $s = 1/2$ spin values only for two nucleons, the expectation value of the tensor force operator is

	l'		
l	$J+1$	J	$J-1$
$J+1$	$-\frac{2J(J+2)}{2J+1}$	0	$\frac{6\sqrt{J(J+1)}}{2J+1}$
J	0	2	0
$J-1$	$\frac{6\sqrt{J(J+1)}}{2J+1}$	0	$-\frac{2(2J+1)}{2J+1}$

We will derive these expressions after we have discussed the Wigner-Eckart theorem.

Components of the force and isospin

If we now add isospin to our simple V_4 interaction model, we end up with 8 operators, popularly dubbed V_8 interaction model. The explicit form reads

$$V(\mathbf{r}) = \left\{ C_c + C_\sigma \sigma_1 \cdot \sigma_2 + C_T \left(1 + \frac{3}{m_\alpha r} + \frac{3}{(m_\alpha r)^2} \right) S_{12}(\hat{r}) \right.$$

$$\left. + C_{SL} \left(\frac{1}{m_\alpha r} + \frac{1}{(m_\alpha r)^2} \right) \mathbf{L} \cdot \mathbf{S} \right\} \frac{e^{-m_\alpha r}}{m_\alpha r}$$

$$+ \left\{ C_{c\tau} + C_{\sigma\tau} \sigma_1 \cdot \sigma_2 + C_{T\tau} \left(1 + \frac{3}{m_\alpha r} + \frac{3}{(m_\alpha r)^2} \right) S_{12}(\hat{r}) \right.$$

$$\left. + C_{SL\tau} \left(\frac{1}{m_\alpha r} + \frac{1}{(m_\alpha r)^2} \right) \mathbf{L} \cdot \mathbf{S} \right\} \tau_1 \cdot \tau_2 \frac{e^{-m_\alpha r}}{m_\alpha r}$$

Phenomenology of nuclear forces

References for Various Phenomenological Interactions

From 1950 till approximately 2000: One-Boson-Exchange (OBE) models dominate. These are models which typically include several low-mass mesons, that is with masses below 1 GeV. Potentials which are based upon the standard non-relativistic operator structure are called "Phenomenological Potentials" Some historically important examples are

- Gammel-Thaler potential (Phys. Rev. **107**, 291, 1339 (1957) and the
- Hamada-Johnston potential, Nucl. Phys. **34**, 382 (1962)), both with a hard core.
- Reid potential (Ann. Phys. (N.Y.) **50**, 411 (1968)), soft core.
- Argonne V_{14} potential (Wiringa et al., Phys. Rev. C **29**, 1207 (1984)) with 14 operators and the Argonne V_{18} potential (Wiringa et al., Phys. Rev. C **51**, 38 (1995)), uses 18 operators
- A good historical reference: R. Machleidt, Adv. Nucl. Phys.

Phenomenology of nuclear forces

The total two-nucleon state function has to be anti-symmetric. The total function contains a spatial part, a spin part and an isospin part. If isospin is conserved, this leads to in case we have an s-wave with spin $S = 0$ to an isospin two-body state with $T = 1$ since the spatial part is symmetric and the spin part is anti-symmetric.
Since the projections for T are $T_z = -1, 0, 1$, we can have a pp, an nn and a pn state.
For $l = 0$ and $S = 1$, a so-called triplet state, ${}^3S_1$, we must have $T = 0$, meaning that we have only one state, a pn state. For other partial waves, see exercises below.

Phenomenology of nuclear forces

Phenomenology of one-pion exchange

The one-pion exchange contribution (see derivation below), can be written as

$$V_\pi(\mathbf{r}) = -\frac{f_\pi^2}{4\pi m_\pi^2}\tau_1\cdot\tau_2\frac{1}{3}\left\{\sigma_1\cdot\sigma_2 + \left(1+\frac{3}{m_\pi r}+\frac{3}{(m_\pi r)^2}\right)S_{12}(\hat{r})\right\}\frac{e^{-m_\pi r}}{m_\pi r}$$

Here the constant $f_\pi^2/4\pi \approx 0.08$ and the mass of the pion is $m_\pi \approx 140$ MeV/c^2.

Phenomenology of nuclear forces

Let us look closer at specific partial waves for which one-pion exchange is applicable. If we have $S = 0$ and $T = 0$, the orbital momentum has to be an odd number in order for the total anti-symmetry to be obeyed. For $S = 0$, the tensor force component is zero, meaning that the only contribution is

$$V_\pi(\mathbf{r}) = \frac{3f_\pi^2}{4\pi m_\pi^2} \frac{e^{-m_\pi r}}{m_\pi r},$$

since $\langle \sigma_1 \cdot \sigma_2 \rangle = -3$, that is we obtain a repulsive contribution to partial waves like ${}^1P_0$.

Phenomenology of nuclear forces

Since $S = 0$ yields always a zero tensor force contribution, for the combination of $T = 1$ and then even l values, we get an attractive contribution

$$V_\pi(\mathbf{r}) = -\frac{f_\pi^2}{4\pi m_\pi^2}\frac{e^{-m_\pi r}}{m_\pi r}.$$

With $S = 1$ and $T = 0$, l can only take even values in order to obey the anti-symmetry requirements and we get

$$V_\pi(\mathbf{r}) = -\frac{f_\pi^2}{4\pi m_\pi^2}\left(1 + (1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r))^2})S_{12}(\hat{r})\right)\frac{e^{-m_\pi r}}{m_\pi r},$$

while for $S = 1$ and $T = 1$, l can only take odd values, resulting in a repulsive contribution

$$V_\pi(\mathbf{r}) = \frac{1}{3}\frac{f_\pi^2}{4\pi m_\pi^2}\left(1 + (1 + \frac{3}{m_\pi r} + \frac{3}{(m_\pi r)^2})S_{12}(\hat{r})\right)\frac{e^{-m_\pi r}}{m_\pi r}.$$

Phenomenology of nuclear forces

The central part of one-pion exchange interaction, arising from the spin-spin term, is thus attractive for s-waves and all even l values. For p-waves and all other odd values it is repulsive. However, its overall strength is weak. This is discussed further in one of exercises below.

The Lippman-Schwinger equation for two-nucleon scattering

What follows now is a more technical discussion on how we can solve the two-nucleon problem. This will lead us to the so-called Lippman-Schwinger equation for the scattering problem and a rewrite of Schroedinger's equation in relative and center-of-mass coordinates.
Before we break down the Schroedinger equation into a partial wave decomposition, we derive now the so-called Lippman-Schwinger equation. We will do this in an operator form first. Thereafter, we rewrite it in terms of various quantum numbers such as relative momenta, orbital momenta etc. The Schrödinger equation in abstract vector representation is

$$\left(\hat{H}_0 + \hat{V}\right)|\psi_n\rangle = E_n|\psi_n\rangle.$$

In our case for the two-body problem $\hat{H}_0$ is just the kinetic energy. We rewrite it as

$$\left(\hat{H}_0 - E_n\right)|\psi_n\rangle = -\hat{V}|\psi_n\rangle.$$

The Lippman-Schwinger equation for two-nucleon scattering

The equation

$$|\psi_n\rangle = \frac{1}{\left(E_n - \hat{H}_0\right)} \hat{V}|\psi_n\rangle,$$

is normally solved in an iterative fashion. We assume first that

$$|\psi_n\rangle = |\phi_n\rangle,$$

where $|\phi_n\rangle$ are the eigenfunctions of

$$\hat{H}_0|\phi_n\rangle = \omega_n|\phi_n\rangle$$

the so-called unperturbed problem. In our case, these will simply be the kinetic energies of the relative motion.

The Lippman-Schwinger equation for two-nucleon scattering

Inserting $|\phi_n\rangle$ on the right-hand side of

$$|\psi_n\rangle = \frac{1}{(E_n - \hat{H}_0)}\hat{V}|\psi_n\rangle,$$

yields

$$|\psi_n\rangle = |\phi_n\rangle + \frac{1}{\left(E_n - \hat{H}_0\right)}\hat{V}|\phi_n\rangle,$$

as our first iteration. Reinserting again gives

$$|\psi_n\rangle = |\phi_n\rangle + \frac{1}{\left(E_n - \hat{H}_0\right)}\hat{V}|\phi_n\rangle + \frac{1}{(E_n - \hat{H}_0)}\hat{V}\frac{1}{\left(E_n - \hat{H}_0\right)}\hat{V}|\phi_n\rangle,$$

and continuing we obtain

$$|\psi_n\rangle = \sum_{i=0}^{\infty}\left[\frac{1}{(E_n - \hat{H}_0)}\hat{V}\right]^i|\phi_n\rangle.$$

The Lippman-Schwinger equation for two-nucleon scattering

It is easy to see that

$$|\psi_n\rangle = \sum_{i=0}^{\infty} \left[\frac{1}{(E_n - \hat{H}_0)} \hat{V} \right]^i |\phi_n\rangle,$$

can be rewritten as

$$|\psi_n\rangle = |\phi_n\rangle + \frac{1}{(E_n - \hat{H}_0)} \hat{V} \left(1 + \frac{1}{(E_n - \hat{H}_0)} \hat{V} + \frac{1}{(E_n - \hat{H}_0)} \hat{V} \frac{1}{(E_n - \hat{H}_0)} \hat{V}$$

which we rewrite as

$$|\psi_n\rangle = |\phi_n\rangle + \frac{1}{(E_n - \hat{H}_0)} \hat{V} |\psi_n\rangle.$$

The Lippman-Schwinger equation for two-nucleon scattering

In operator form we have thus

$$|\psi_n\rangle = |\phi_n\rangle + \frac{1}{(E_n - \hat{H}_0)}\hat{V}|\psi_n\rangle.$$

We multiply from the left with $\hat{V}$ and $\langle\phi_m|$ and obtain

$$\langle\phi_m|\hat{V}|\psi_n\rangle = \langle\phi_m|\hat{V}|\phi_n\rangle + \langle\phi_m|\hat{V}\frac{1}{(E_n - \hat{H}_0)}\hat{V}|\psi_n\rangle.$$

We define thereafter the so-called T-matrix as

$$\langle\phi_m|\hat{T}|\phi_n\rangle = \langle\phi_m|\hat{V}|\psi_n\rangle.$$

We can rewrite our equation as

$$\langle\phi_m|\hat{T}|\phi_n\rangle = \langle\phi_m|\hat{V}|\phi_n\rangle + \langle\phi_m|\hat{V}\frac{1}{(E_n - \hat{H}_0)}\hat{T}|\phi_n\rangle.$$

The Lippman-Schwinger equation for two-nucleon scattering

The equation

$$\langle \phi_m | \hat{T} | \phi_n \rangle = \langle \phi_m | \hat{V} | \phi_n \rangle + \langle \phi_m | \hat{V} \frac{1}{(E_n - \hat{H}_0)} \hat{T} | \phi_n \rangle,$$

is called the Lippman-Schwinger equation. Inserting the completeness relation

$$\mathbf{1} = \sum_n |\phi_n\rangle\langle\phi_n|, \;\; \langle \phi_n | \phi_{n'} \rangle = \delta_{n,n'}$$

we have

$$\langle \phi_m | \hat{T} | \phi_n \rangle = \langle \phi_m | \hat{V} | \phi_n \rangle + \sum_k \langle \phi_m | \hat{V} | \phi_k \rangle \frac{1}{(E_n - \omega_k)} \langle \phi_k | \hat{T} | \phi_n \rangle,$$

which is (when we specify the state $|\phi_n\rangle$) an integral equation that can actually be solved by matrix inversion easily! The unknown quantity is the T-matrix.

The Lippman-Schwinger equation for two-nucleon scattering

Now we wish to introduce a partial wave decomposition in order to solve the Lippman-Schwinger equation. With a partial wave decomposition we can reduce a three-dimensional integral equation to a one-dimensional one.
Let us continue with our Schroedinger equation in the abstract vector representation

$$(T + V)\,|\psi_n\rangle = E_n|\psi_n\rangle$$

Here T is the kinetic energy operator and V is the potential operator. The eigenstates form a complete orthonormal set according to

$$\mathbf{1} = \sum_n |\psi_n\rangle\langle\psi_n|, \;\; \langle\psi_n|\psi_{n'}\rangle = \delta_{n,n'}$$

The Lippman-Schwinger equation for two-nucleon scattering

The most commonly used representations are the coordinate and the momentum space representations. They define the completeness relations

$$\mathbf{1} = \int d\mathbf{r}\, |\mathbf{r}\rangle\langle\mathbf{r}|, \;\; \langle\mathbf{r}|\mathbf{r}'\rangle = \delta(\mathbf{r}-\mathbf{r}')$$

$$\mathbf{1} = \int d\mathbf{k}\, |\mathbf{k}\rangle\langle\mathbf{k}|, \;\; \langle\mathbf{k}|\mathbf{k}'\rangle = \delta(\mathbf{k}-\mathbf{k}')$$

Here the basis states in both **r**- and **k**-space are dirac-delta function normalized. From this it follows that the plane-wave states are given by,

$$\langle\mathbf{r}|\mathbf{k}\rangle = \left(\frac{1}{2\pi}\right)^{3/2} \exp\left(i\mathbf{k}\cdot\mathbf{r}\right)$$

which is a transformation function defining the mapping from the abstract $|\mathbf{k}\rangle$ to the abstract $|\mathbf{r}\rangle$ space.

The Lippman-Schwinger equation for two-nucleon scattering

That the **r**-space basis states are delta-function normalized follows from

$$\delta(\mathbf{r}-\mathbf{r}') = \langle \mathbf{r}|\mathbf{r}'\rangle = \langle \mathbf{r}|\mathbf{1}|\mathbf{r}'\rangle = \int d\mathbf{k}\langle \mathbf{r}|\mathbf{k}\rangle\langle \mathbf{k}|\mathbf{r}'\rangle = \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{k} e^{i\mathbf{k}(\mathbf{r}-\mathbf{r}')}$$

and the same for the momentum space basis states,

$$\delta(\mathbf{k}-\mathbf{k}') = \langle \mathbf{k}|\mathbf{k}'\rangle = \langle \mathbf{k}|\mathbf{1}|\mathbf{k}'\rangle = \int d\mathbf{r}\langle \mathbf{k}|\mathbf{r}\rangle\langle \mathbf{r}|\mathbf{k}'\rangle = \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{r} e^{i\mathbf{r}(\mathbf{k}-\mathbf{k}')}$$

The Lippman-Schwinger equation for two-nucleon scattering

Projecting on momentum states, we obtain the momentum space Schroedinger equation as

$$\frac{\hbar^2}{2\mu}k^2\psi_n(\mathbf{k}) + \int d\mathbf{k}' V(\mathbf{k},\mathbf{k}')\psi_n(\mathbf{k}') = E_n\psi_n(\mathbf{k}) \qquad (2)$$

Here the notation $\psi_n(\mathbf{k}) = \langle \mathbf{k}|\psi_n\rangle$ and $\langle \mathbf{k}|V|\mathbf{k}'\rangle = V(\mathbf{k},\mathbf{k}')$ has been introduced. The potential in momentum space is given by a double Fourier-transform of the potential in coordinate space, i.e.

$$V(\mathbf{k},\mathbf{k}') = \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{r} \int d\mathbf{r}' \exp -i\mathbf{k}\mathbf{r}\, V(\mathbf{r},\mathbf{r}') \exp i\mathbf{k}'\mathbf{r}'$$

The Lippman-Schwinger equation for two-nucleon scattering

Here it is assumed that the potential interaction does not contain any spin dependence. Instead of a differential equation in coordinate space, the Schroedinger equation becomes an integral equation in momentum space. This has many tractable features. Firstly, most realistic nucleon-nucleon interactions derived from field-theory are given explicitly in momentum space. Secondly, the boundary conditions imposed on the differential equation in coordinate space are automatically built into the integral equation. And last, but not least, integral equations are easy to numerically implement, and convergence is obtained by just increasing the number of integration points. Instead of solving the three-dimensional integral equation, an infinite set of 1-dimensional equations can be obtained via a partial wave expansion.

The Lippman-Schwinger equation for two-nucleon scattering

The wave function $\psi_n(\mathbf{k})$ can be expanded in a complete set of spherical harmonics, that is

$$\psi_n(\mathbf{k}) = \sum_{lm} \psi_{nlm}(k) Y_{lm}(\hat{k}) \qquad \psi_{nlm}(k) = \int d\hat{k} Y^*_{lm}(\hat{k}) \psi_n(\mathbf{k}).,$$

(3)

By inserting equation (??) in equation (??), and projecting from the left $Y_{lm}(\hat{k})$, the three-dimensional Schroedinger equation (??) is reduced to an infinite set of 1-dimensional angular momentum coupled integral equations,

$$\left(\frac{\hbar^2}{2\mu} k^2 - E_{nlm}\right) \psi_{nlm}(k) = -\sum_{l'm'} \int_0^\infty dk' k'^2 V_{lm,l'm'}(k,k') \psi_{nl'm'}(k')$$

(4)

where the angular momentum projected potential takes the form,

$$V_{lm,l'm'}(k,k') = \int d\hat{k} \int d\hat{k}' Y^*_{lm}(\hat{k}) V(\mathbf{k},\mathbf{k}') Y_{l'm'}(\hat{k}') \qquad (5)$$

The Lippman-Schwinger equation for two-nucleon scattering

Note that we discuss only the orbital momentum, we will include angular momentum and spin later . The

The Lippman-Schwinger equation for two-nucleon scattering

Since the plane waves depend only on the absolute values of position and momentum, $|\mathbf{k}|$ and $|\mathbf{r}|$, and the angle between them, θ_{kr}, they may be expanded in terms of bipolar harmonics of zero rank, i.e.

$$\exp\left(i\mathbf{k}\cdot\mathbf{r}\right) = 4\pi\sum_{l=0}^{\infty} i^l j_l(kr)\left(Y_l(\hat{k})\cdot Y_l(\hat{r})\right) = \sum_{l=0}^{\infty}(2l+1)i^l j_l(kr)P_l(\cos\theta_{kr}$$

where the addition theorem for spherical harmonics has been used in order to write the expansion in terms of Legendre polynomials. The spherical Bessel functions, $j_l(z)$, are given in terms of Bessel functions of the first kind with half integer orders,

$$j_l(z) = \sqrt{\frac{\pi}{2z}}J_{l+1/2}(z).$$

The Lippman-Schwinger equation for two-nucleon scattering

Inserting the plane-wave expansion into the brackets of equation (??) yields,

$$\begin{aligned}
\int d\hat{k}\, Y^*_{lm}(\hat{k})\langle \mathbf{k}|\mathbf{r}\rangle &= \left(\frac{1}{2\pi}\right)^{3/2} 4\pi i^{-l} j_l(kr)\, Y^*_{lm}(\hat{r}), \\
\int d\hat{k}'\; Y_{lm}(\hat{k}')\langle \mathbf{r}'|\mathbf{k}'\rangle &= \left(\frac{1}{2\pi}\right)^{3/2} 4\pi i^{l'} j_{l'}(k'r')\, Y_{l'm'}(\hat{r}).
\end{aligned}$$

The Lippman-Schwinger equation for two-nucleon scattering

The connection between the momentum- and position space angular momentum projected potentials are then given,

$$V_{lm,l'm'}(k,k') = \frac{2}{\pi} i^{l'-l} \int_0^\infty dr r^2 \int_0^\infty dr' r'^2 j_l(kr) V_{lm,l'm'}(r,r') j_{l'}(k'r')$$

which is known as a double Fourier-Bessel transform. The position space angular momentum projected potential is given by

$$V_{lm,l'm'}(r,r') = \int d\hat{r} \int d\hat{r}' Y^*_{lm}(\hat{r}) V(\mathbf{r},\mathbf{r}') Y_{l'm'}(\hat{r}').$$

The Lippman-Schwinger equation for two-nucleon scattering

No assumptions of locality/non-locality and deformation of the interaction has so far been made, and the result in equation (??) is general. In position space the Schroedinger equation takes form of an integro-differential equation in case of a non-local interaction, in momentum space the Schroedinger equation is an ordinary integral equation of the Fredholm type, see equation (??). This is a further advantage of the momentum space approach as compared to the standard position space approach. If we assume that the interaction is of local character, i.e.

$$\langle \mathbf{r}|V|\mathbf{r}'\rangle = V(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') = V(\mathbf{r})\frac{\delta(r-r')}{r^2}\delta(\cos\theta - \cos\theta')\delta(\varphi-\varphi'),$$

then equation (??) reduces to

$$V_{lm,l'm'}(r,r') = \frac{\delta(r-r')}{r^2}\int d\hat{r}\; Y^*_{lm}(\hat{r})V(\mathbf{r})Y_{l'm'}(\hat{r}), \qquad (7)$$

The Lippman-Schwinger equation for two-nucleon scattering

and equation (??) reduces to

$$V_{lm,l'm'}(k,k') = \frac{2}{\pi} i^{l'-l} \int_0^\infty dr r^2 j_l(kr) V_{lm,l'm'}(r) j_{l'}(k'r) \qquad (8)$$

where

$$V_{lm,l'm'}(r) = \int d\hat{r} Y^*_{lm}(\hat{r}) V(\mathbf{r}) Y_{l'm'}(\hat{r}), \qquad (9)$$

The Lippman-Schwinger equation for two-nucleon scattering

In the case that the interaction is central, $V(\mathbf{r}) = V(r)$, then

$$V_{lm,l'm'}(r) = V(r) \int d\hat{r} Y^*_{lm}(\hat{r}) Y_{l'm'}(\hat{r}) = V(r)\delta_{l,l'}\delta_{m,m'}, \qquad (10)$$

and

$$V_{lm,l'm'}(k,k') = \frac{2}{\pi}\int_0^\infty dr r^2 j_l(kr) V(r) j_{l'}(k'r)\delta_{l,l'}\delta_{m,m'} = V_l(k,k')\delta_{l,l'}\delta_{m,m} \qquad (11)$$

where the momentum space representation of the interaction finally reads,

$$V_l(k,k') = \frac{2}{\pi}\int_0^\infty dr\, r^2\, j_l(kr) V(r) j_l(k'r). \qquad (12)$$

The Lippman-Schwinger equation for two-nucleon scattering

For a local and spherical symmetric potential, the coupled momentum space Schrödinger equations given in equation (??) decouples in angular momentum, giving

$$\frac{\hbar^2}{2\mu}k^2\psi_{nl}(k) + \int_0^\infty dk' k'^2 V_l(k,k')\psi_{nl}(k') = E_{nl}\psi_{nl}(k) \qquad (13)$$

Where we have written $\psi_{nl}(k) = \psi_{nlm}(k)$, since the equation becomes independent of the projection m for spherical symmetric interactions. The momentum space wave functions $\psi_{nl}(k)$ defines a complete orthogonal set of functions, which spans the space of functions with a positive finite Euclidean norm (also called l^2-norm), $\sqrt{\langle\psi_n|\psi_n\rangle}$, which is a Hilbert space. The corresponding normalized wave function in coordinate space is given by the Fourier-Bessel transform

$$\phi_{nl}(r) = \sqrt{\frac{2}{\pi}}\int dk k^2 j_l(kr)\psi_{nl}(k)$$

The Lippman-Schwinger equation for two-nucleon scattering

We will thus assume that the interaction is spherically symmetric and use the partial wave expansion of the plane waves in terms of spherical harmonics. This means that we can separate the radial part of the wave function from its angular dependence. The wave function of the relative motion is described in terms of plane waves as

$$\exp(\imath \mathbf{kr}) = \langle \mathbf{r}|\mathbf{k}\rangle = 4\pi \sum_{lm} \imath^l j_l(kr)\, Y^*_{lm}(\hat{\mathbf{k}})\, Y_{lm}(\hat{\mathbf{r}}),$$

where j_l is a spherical Bessel function and Y_{lm} the spherical harmonics.

The Lippman-Schwinger equation for two-nucleon scattering

In terms of the relative and center-of-mass momenta $\mathbf{k}$ and $\mathbf{K}$, the potential in momentum space is related to the nonlocal operator $V(\mathbf{r}, \mathbf{r}')$ by

$$\langle \mathbf{k}'\mathbf{K}'|V|\mathbf{k}\mathbf{K}\rangle = \int d\mathbf{r}d\mathbf{r}' \exp -(i\mathbf{k}'\mathbf{r}')V(\mathbf{r}', \mathbf{r}) \exp i\mathbf{k}\mathbf{r}\delta(\mathbf{K}, \mathbf{K}').$$

We will assume that the interaction is spherically symmetric. Can separate the radial part of the wave function from its angular dependence. The wave function of the relative motion is described in terms of plane waves as

$$\exp(i\mathbf{k}\mathbf{r}) = \langle \mathbf{r}|\mathbf{k}\rangle = 4\pi \sum_{lm} i^l j_l(kr) Y^*_{lm}(\hat{\mathbf{k}}) Y_{lm}(\hat{\mathbf{r}}),$$

where j_l is a spherical Bessel function and Y_{lm} the spherical harmonic.

The Lippman-Schwinger equation for two-nucleon scattering

This partial wave basis is useful for defining the operator for the nucleon-nucleon interaction, which is symmetric with respect to rotations, parity and isospin transformations. These symmetries imply that the interaction is diagonal with respect to the quantum numbers of total relative angular momentum $\mathcal{J}$, spin S and isospin T (we skip isospin for the moment). Using the above plane wave expansion, and coupling to final $\mathcal{J}$ and S and T we get

$$\langle \mathbf{k}'|V|\mathbf{k}\rangle = (4\pi)^2 \sum_{STll'm_l m_{l'}\mathcal{J}} i^{l+l'} Y^*_{lm}(\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{k}}')$$

$$\langle lm_l Sm_S|\mathcal{J}M\rangle\langle l'm_{l'}Sm_S|\mathcal{J}M\rangle\langle k'l'S\mathcal{J}M|V|klS\mathcal{J}M\rangle,$$

where we have defined

$$\langle k'l'S\mathcal{J}M|V|klS\mathcal{J}M\rangle = \int j_{l'}(k'r')\langle l'S\mathcal{J}M|V(r',r)|lS\mathcal{J}M\rangle j_l(kr) r'^2 dr' r^2$$

We have omitted the momentum of the center-of-mass motion $\mathbf{K}$ and the corresponding orbital momentum L, since the interaction is