

## Electromagnetic transitions and $\beta$ -decay

Morten Hjorth-Jensen, National Superconducting Cyclotron  
Laboratory and Department of Physics and Astronomy,  
Michigan State University, East Lansing, MI 48824, USA &  
Department of Physics, University of Oslo, Oslo, Norway

Spring 2015

## Electromagnetic multipole moments and transitions

The reduced transition probability  $B$  is defined in terms of reduced matrix elements of a one-body operator by

$$B(i \rightarrow f) = \frac{\langle J_f || \mathcal{O}(\lambda) || J_i \rangle^2}{(2J_i + 1)}.$$

With our definition of the reduced matrix element,

$$\langle J_f || \mathcal{O}(\lambda) || J_i \rangle^2 = \langle J_i || \mathcal{O}(\lambda) || J_f \rangle^2,$$

the transition probability  $B$  depends upon the direction of the transition by the factor of  $(2J_i + 1)$ . For electromagnetic transitions  $J_i$  is that for the higher-energy initial state. But in Coulomb excitation the initial state is usually taken as the ground state, and it is normal to use the notation  $B(\uparrow)$  for transitions from the ground state.

## Electromagnetic multipole moments and transitions

The one-body operators  $\mathcal{O}(\lambda)$  represent a sum over the operators for the individual nucleon degrees of freedom  $i$

$$\mathcal{O}(\lambda) = \sum_i \mathcal{O}(\lambda, i).$$

The electric transition operator is given by

$$\mathcal{O}(E\lambda) = r^\lambda Y_\mu^\lambda(\hat{r}) e_q e,$$

where  $Y_\mu^\lambda$  are the spherical harmonics and  $q$  stands for proton  $q = p$  or neutron  $q = n$ .

## Electromagnetic multipole moments and transitions

Gamma transitions with  $\lambda = 0$  are forbidden because the photon must carry off at least one unit of angular momentum. The  $e_q$  are the electric charges for the proton and neutron in units of  $e$ . For the free-nucleon charge we would take  $e_p = 1$  and  $e_n = 0$ , for the proton and neutron, respectively. Although the bare operator acts upon the protons, we will keep the general expression in terms of  $e_q$  in order to incorporate the **effective charges** for the proton and neutron, which represent the center-of-mass corrections and the average effects of the renormalization from wavefunction admixtures outside the model space.

## Electromagnetic multipole moments and transitions

The magnetic transition operator is given by:

$$\begin{aligned} \mathcal{O}(M\lambda) &= \left[ l \frac{2g_q^l}{(\lambda+1)} + s g_q^s \right] \nabla [r^\lambda Y_\mu^\lambda(\hat{r})] \mu_N \\ &= \sqrt{\lambda(2\lambda+1)} \left[ [Y^{\lambda-1}(\hat{r}) \otimes l]_\mu^\lambda \frac{2g_q^l}{(\lambda+1)} + [Y^{\lambda-1}(\hat{r}) \otimes s]_\mu^\lambda g_q^s \right] r^{\lambda-1} \mu_N, \end{aligned}$$

where  $\mu_N$  is the nuclear magneton,

$$\mu_N = \frac{e\hbar}{2m_p c} = 0.105 \text{ efm},$$

and where  $m_p$  is the mass of the proton.

## Electromagnetic multipole moments and transitions

The g-factors  $g_q^l$  and  $g_q^s$  are the orbital and spin g-factors for the proton and neutron, respectively. The free-nucleon values for the g-factors are  $g_p^l = 1$ ,  $g_n^l = 0$ ,  $g_p^s = 5.586$  and  $g_n^s = -3.826$ . We may use effective values for these g-factors to take into account the truncation of the model space.

## Electromagnetic multipole moments and transitions

The most common types of transitions are  $E1$ ,  $E2$  and  $M1$ . The  $E1$  transition operator is given by  $\lambda=1$ :

$$O(E1) = r Y_{\mu}^{(1)}(\hat{r}) e_q e = \sqrt{\frac{3}{4\pi}} \mathbf{r} e_q e.$$

The  $E2$  transition operator with  $\lambda=2$ :

$$O(E2) = r^2 Y_{\mu}^{(2)}(\hat{r}) e_q e,$$

## Electromagnetic multipole moments and transitions

The  $M1$  transition operator with  $\lambda = 1$  and with

$$Y^0 = 1/\sqrt{4\pi},$$

we have

$$O(M1) = \sqrt{\frac{3}{4\pi}} [g_q' + s g_d^s] \mu_N.$$

## Electromagnetic multipole moments and transitions

The selection rules are given by the triangle condition for the angular momenta,  $\Delta(J_i, J_f, \lambda)$ . The electromagnetic interaction conserves parity, and the elements of the operators for  $E\lambda$  and  $M\lambda$  can be classified according to their transformation under parity change

$$\hat{P} \hat{O} \hat{P}^{-1} = \pi_O \hat{O},$$

where we have  $\pi_O = (-1)^\lambda$  for  $Y^\lambda$ ,  $\pi_O = -1$  for the vectors  $\mathbf{r}$ ,  $\nabla$  and  $\mathbf{p}$ , and  $\pi_O = +1$  for the pseudo vectors  $\mathbf{l} = \mathbf{r} \times \mathbf{p}$  and  $\sigma$ . For a given matrix element we have:

$$\langle \Psi_f | O | \Psi_i \rangle = \langle \Psi_f | P^{-1} O P^{-1} | \Psi_i \rangle = \pi_i \pi_f \pi_O \langle \Psi_f | O | \Psi_i \rangle.$$

The matrix element will vanish unless  $\pi_i \pi_f \pi_O = +1$ .

## Electromagnetic multipole moments and transitions

The transitions are divided into two classes, those which do not change parity change  $\pi_i \pi_f = +1$  which go by the operators with  $\pi_O = +1$ :

$$\pi_i \pi_f = +1 \text{ for } M1, E2, M3, E4 \dots,$$

and the ones which do change parity change  $\pi_i \pi_f = -1$  which go by the operators with  $\pi_O = -1$ :

$$\pi_i \pi_f = -1 \text{ for } E1, M2, E3, M4 \dots$$

## Electromagnetic multipole moments and transitions

The electromagnetic moment operator can be expressed in terms of the electromagnetic transition operators. By the parity selection rule of the moments are nonzero only for  $M1$ ,  $E2$ ,  $M3$ ,  $E4$ , ... The most common are:

$$\mu = \sqrt{\frac{4\pi}{3}} \langle J, M = J | O(M1) | J, M = J \rangle = \sqrt{\frac{4\pi}{3}} \begin{Bmatrix} J & 1 & J \\ -J & 0 & J \end{Bmatrix} \langle J || O(M1) || J \rangle$$

and

$$Q = \sqrt{\frac{16\pi}{5}} \langle J, M = J | O(E2) | J, M = J \rangle = \sqrt{\frac{16\pi}{5}} \begin{Bmatrix} J & 2 & J \\ -J & 0 & J \end{Bmatrix} \langle J || O(E2) || J \rangle$$

## Electromagnetic multipole moments and transitions

Electromagnetic transitions and moments depend upon the reduced nuclear matrix elements  $\langle f || O(\lambda) || i \rangle$ . These can be expressed as a sum over one-body transition densities (OBTD) times single-particle matrix elements

$$\langle f || O(\lambda) || i \rangle = \sum_{k_\alpha k_\beta} \text{OBTD}(f k_\alpha k_\beta \lambda) \langle k_\alpha || O(\lambda) || k_\beta \rangle,$$

where the OBTD is given by

$$\text{OBTD}(f k_\alpha k_\beta \lambda) = \frac{\langle f || [a_{k_\alpha}^\dagger \otimes \tilde{a}_{k_\beta}]^\lambda || i \rangle}{\sqrt{(2\lambda + 1)}}.$$

The labels  $i$  and  $f$  are a short-hand notation for the initial and final state quantum numbers ( $n\omega_i J_i$ ) and ( $n\omega_f J_f$ ), respectively. Thus the problem is divided into two parts, one involving the nuclear structure dependent one-body transition densities OBTD, and the other involving the reduced single-particle matrix elements (SPME).

## Electromagnetic multipole moments and transitions

The SPME for the  $E\lambda$  operator is given by:

$$\langle k_a || O(E\lambda) || k_b \rangle = (-1)^{j_a+1/2} \frac{[1 + (-1)^{l_a+\lambda+l_b}]}{2} \\ \times \sqrt{\frac{(2j_a+1)(2\lambda+1)(2j_b+1)}{4\pi}} \begin{pmatrix} j_a & \lambda & j_b \\ 1/2 & 0 & -1/2 \end{pmatrix} \langle k_a | r^\lambda | k_b \rangle e_q e.$$

## Electromagnetic multipole moments and transitions

The SPME for the spin part of the magnetic operator is

$$\langle k_a || O(M\lambda, s) || k_b \rangle = \\ = \sqrt{\lambda(2\lambda+1)} \langle j_a || [Y^{\lambda-1}(\hat{r}) \otimes s]^\lambda || j_b \rangle \langle k_a | r^{\lambda-1} | k_b \rangle g_q^s \mu_N, \\ = \sqrt{\lambda(2\lambda+1)} \sqrt{(2j_a+1)(2j_b+1)(2\lambda+1)} \begin{Bmatrix} l_a & 1/2 & j_a \\ l_b & 1/2 & j_b \\ \lambda-1 & 1 & \lambda \end{Bmatrix} \\ \times \langle l_a || Y^{\lambda-1}(\hat{r}) || l_b \rangle \langle || s || s \rangle \langle k_a | r^{\lambda-1} | k_b \rangle g_q^s \mu_N, \\ \text{where} \\ \langle || s || s \rangle = \sqrt{3/2}.$$

## Electromagnetic multipole moments and transitions

The SPME for the orbital part of the magnetic operator is:

$$\langle k_a || O(M\lambda, l) || k_b \rangle = \\ = \frac{\sqrt{\lambda(2\lambda+1)}}{\lambda+1} \langle j_a || [Y^{\lambda-1}(\hat{r}) \otimes l]^\lambda || j_b \rangle \langle k_a | r^{\lambda-1} | k_b \rangle g_q^l \mu_N \\ = \frac{\sqrt{\lambda(2\lambda+1)}}{\lambda+1} (-1)^{l_a+1/2+j_b+\lambda} \sqrt{(2j_a+1)(2j_b+1)} \\ \times \begin{Bmatrix} l_a & l_b & \lambda \\ j_b & j_a & 1/2 \end{Bmatrix} \langle l_a || [Y^{\lambda-1}(\hat{r}) \otimes l]^\lambda || l_b \rangle \langle k_a | r^{\lambda-1} | k_b \rangle g_q^l \mu_N,$$

## Electromagnetic multipole moments and transitions

where we have defined

$$\langle l_a || [Y^{\lambda-1}(\hat{r}) \otimes l]^\lambda || l_b \rangle = (-1)^{\lambda+l_a+l_b} \sqrt{(2\lambda+1)l_b(l_b+1)(2l_b+1)} \\ \times \begin{Bmatrix} \lambda-1 & 1 & \lambda \\ l_b & l_a & l_b \end{Bmatrix} \langle l_a || Y^{\lambda-1}(\hat{r}) || l_b \rangle,$$

with

$$\langle l_a || Y^{\lambda-1}(\hat{r}) || l_b \rangle = (-1)^{l_a} \sqrt{\frac{(2l_a+1)(2l_b+1)(2\lambda-1)}{4\pi}} \begin{pmatrix} l_a & \lambda-1 & l_b \\ 0 & 0 & 0 \end{pmatrix}$$

## Electromagnetic multipole moments and transitions

For the  $M1$  operator the radial matrix element is

$$\langle k_a | r^0 | k_b \rangle = \delta_{n_a, n_b},$$

and the SPME simplify to:

$$\langle k_a || O(M1, s) || k_b \rangle = \sqrt{\frac{3}{4\pi}} \langle j_a || s || j_b \rangle \delta_{n_a, n_b} g_q^s \mu_N \\ = \sqrt{\frac{3}{4\pi}} (-1)^{l_a+j_a+3/2} \sqrt{(2j_a+1)(2j_b+1)} \begin{Bmatrix} 1/2 & 1/2 & 1 \\ j_b & j_a & l_a \end{Bmatrix} \\ \times \langle s || s || s \rangle \delta_{l_a, l_b} \delta_{n_a, n_b} g_q^s \mu_N,$$

## Electromagnetic multipole moments and transitions

where we have

$$\langle s || s || s \rangle = \sqrt{3/2},$$

and

$$\langle k_a || O(M1, l) || k_b \rangle = \sqrt{\frac{3}{4\pi}} \langle j_a || l || j_b \rangle \delta_{n_a, n_b} g_q^l \mu_N \\ = \sqrt{\frac{3}{4\pi}} (-1)^{l_a+j_b+3/2} \sqrt{(2j_a+1)(2j_b+1)} \begin{Bmatrix} l_a & l_b & 1 \\ j_b & j_a & 1/2 \end{Bmatrix} \\ \times \langle l_a || l || l_b \rangle \delta_{n_a, n_b} g_q^l \mu_N,$$

where

$$\langle l_a || l || l_b \rangle = \delta_{l_a, l_b} \sqrt{l_a(l_a+1)(2l_a+1)}.$$

Thus the  $M1$  operator connects only those orbitals which have the same  $n$  and  $l$  values.

### Electromagnetic multipole moments and transitions

For further reading, see Suhonen's chapters 6-7 and Alex Brown's chapters 23, 28 and 29. Before we proceed with  $\beta$ -decay, we need to say something about so-called core-polarization effects. To do this, we have to introduce elements from many-body perturbation theory. We assume here that we are only interested in the ground state of the system and expand the exact wave function in terms of a series of Slater determinants

$$|\Psi_0\rangle = |\Phi_0\rangle + \sum_{m=1}^{\infty} C_m |\Phi_m\rangle,$$

where we have assumed that the true ground state is dominated by the solution of the unperturbed problem, that is

$$\hat{H}_0 |\Phi_0\rangle = W_0 |\Phi_0\rangle.$$

The state  $|\Psi_0\rangle$  is not normalized, rather we have used an intermediate normalization  $\langle \Phi_0 | \Psi_0 \rangle = 1$  since we have  $\langle \Phi_0 | \Phi_0 \rangle = 1$ .

### Electromagnetic multipole moments and transitions

The Schrödinger equation is

$$\hat{H} |\Psi_0\rangle = E |\Psi_0\rangle,$$

and multiplying the latter from the left with  $\langle \Phi_0 |$  gives

$$\langle \Phi_0 | \hat{H} |\Psi_0\rangle = E \langle \Phi_0 | \Psi_0 \rangle = E,$$

and subtracting from this equation

$$\langle \Psi_0 | \hat{H}_0 | \Phi_0 \rangle = W_0 \langle \Psi_0 | \Phi_0 \rangle = W_0,$$

and using the fact that the both operators  $\hat{H}$  and  $\hat{H}_0$  are hermitian results in

$$\Delta E = E - W_0 = \langle \Phi_0 | \hat{H}_I | \Psi_0 \rangle,$$

which is an exact result. We call this quantity the correlation energy.

### Electromagnetic multipole moments and transitions

This equation forms the starting point for all perturbative derivations. However, as it stands it represents nothing but a mere formal rewriting of Schrödinger's equation and is not of much practical use. The exact wave function  $|\Psi_0\rangle$  is unknown. In order to obtain a perturbative expansion, we need to expand the exact wave function in terms of the interaction  $\hat{H}_I$ . Here we have assumed that our model space defined by the operator  $\hat{P}$  is one-dimensional, meaning that

$$\hat{P} = |\Phi_0\rangle \langle \Phi_0|,$$

and

$$\hat{Q} = \sum_{m=1}^{\infty} |\Phi_m\rangle \langle \Phi_m|.$$

### Electromagnetic multipole moments and transitions

We can thus rewrite the exact wave function as

$$|\Psi_0\rangle = (\hat{P} + \hat{Q}) |\Psi_0\rangle = |\Phi_0\rangle + \hat{Q} |\Psi_0\rangle.$$

Going back to the Schrödinger equation, we can rewrite it as, adding and subtracting a term  $\omega |\Psi_0\rangle$  as

$$(\omega - \hat{H}_0) |\Psi_0\rangle = (\omega - E + \hat{H}_I) |\Psi_0\rangle,$$

where  $\omega$  is an energy variable to be specified later.

### Electromagnetic multipole moments and transitions

We assume also that the resolvent of  $(\omega - \hat{H}_0)$  exists, that is it has an inverse which defines the unperturbed Green's function as

$$(\omega - \hat{H}_0)^{-1} = \frac{1}{(\omega - \hat{H}_0)}.$$

We can rewrite Schrödinger's equation as

$$|\Psi_0\rangle = \frac{1}{\omega - \hat{H}_0} (\omega - E + \hat{H}_I) |\Psi_0\rangle,$$

and multiplying from the left with  $\hat{Q}$  results in

$$\hat{Q} |\Psi_0\rangle = \frac{\hat{Q}}{\omega - \hat{H}_0} (\omega - E + \hat{H}_I) |\Psi_0\rangle,$$

which is possible since we have defined the operator  $\hat{Q}$  in terms of the eigenfunctions of  $\hat{H}$ .

### Electromagnetic multipole moments and transitions

These operators commute meaning that

$$\hat{Q} \frac{1}{(\omega - \hat{H}_0)} \hat{Q} = \hat{Q} \frac{1}{(\omega - \hat{H}_0)} = \frac{\hat{Q}}{(\omega - \hat{H}_0)}.$$

With these definitions we can in turn define the wave function as

$$|\Psi_0\rangle = |\Phi_0\rangle + \frac{\hat{Q}}{\omega - \hat{H}_0} (\omega - E + \hat{H}_I) |\Psi_0\rangle.$$

This equation is again nothing but a formal rewrite of Schrödinger's equation and does not represent a practical calculational scheme. It is a non-linear equation in two unknown quantities, the energy  $E$  and the exact wave function  $|\Psi_0\rangle$ . We can however start with a guess for  $|\Psi_0\rangle$  on the right hand side of the last equation.

## Electromagnetic multipole moments and transitions

The most common choice is to start with the function which is expected to exhibit the largest overlap with the wave function we are searching after, namely  $|\Phi_0\rangle$ . This can again be inserted in the solution for  $|\Psi_0\rangle$  in an iterative fashion and if we continue along these lines we end up with

$$|\Psi_0\rangle = \sum_{i=0}^{\infty} \left\{ \frac{\hat{Q}}{\omega - \hat{H}_0} (\omega - E + \hat{H}_I) \right\}^i |\Phi_0\rangle,$$

for the wave function and

$$\Delta E = \sum_{i=0}^{\infty} \langle \Phi_0 | \hat{H}_I \left\{ \frac{\hat{Q}}{\omega - \hat{H}_0} (\omega - E + \hat{H}_I) \right\}^i | \Phi_0 \rangle,$$

which is now a perturbative expansion of the exact energy in terms of the interaction  $\hat{H}_I$  and the unperturbed wave function  $|\Psi_0\rangle$ .

## Electromagnetic multipole moments and transitions

In our equations for  $|\Psi_0\rangle$  and  $\Delta E$  in terms of the unperturbed solutions  $|\Phi_i\rangle$  we have still an undetermined parameter  $\omega$  and a dependency on the exact energy  $E$ . Not much has been gained thus from a practical computational point of view.

In Brillouin-Wigner perturbation theory it is customary to set  $\omega = E$ . This results in the following perturbative expansion for the energy  $\Delta E$

$$\Delta E = \sum_{i=0}^{\infty} \langle \Phi_0 | \hat{H}_I \left\{ \frac{\hat{Q}}{\omega - \hat{H}_0} (\omega - E + \hat{H}_I) \right\}^i | \Phi_0 \rangle =$$

$$\langle \Phi_0 | \left( \hat{H}_I + \hat{H}_I \frac{\hat{Q}}{E - \hat{H}_0} \hat{H}_I + \hat{H}_I \frac{\hat{Q}}{E - \hat{H}_0} \hat{H}_I \frac{\hat{Q}}{E - \hat{H}_0} \hat{H}_I + \dots \right) | \Phi_0 \rangle.$$

## Electromagnetic multipole moments and transitions

$$\Delta E = \sum_{i=0}^{\infty} \langle \Phi_0 | \hat{H}_I \left\{ \frac{\hat{Q}}{\omega - \hat{H}_0} (\omega - E + \hat{H}_I) \right\}^i | \Phi_0 \rangle =$$

$$\langle \Phi_0 | \left( \hat{H}_I + \hat{H}_I \frac{\hat{Q}}{E - \hat{H}_0} \hat{H}_I + \hat{H}_I \frac{\hat{Q}}{E - \hat{H}_0} \hat{H}_I \frac{\hat{Q}}{E - \hat{H}_0} \hat{H}_I + \dots \right) | \Phi_0 \rangle.$$

This expression depends however on the exact energy  $E$  and is again not very convenient from a practical point of view. It can obviously be solved iteratively, by starting with a guess for  $E$  and then solve till some kind of self-consistency criterion has been reached. Actually, the above expression is nothing but a rewrite again of the full Schrödinger equation.

Defining  $e = E - \hat{H}_0$  and recalling that  $\hat{H}_0$  commutes with  $\hat{Q}$  by construction and that  $\hat{Q}$  is an idempotent operator  $\hat{Q}^2 = \hat{Q}$ . Using this equation in the above expansion for  $\Delta E$  we can write the denominator

$$\hat{Q} \frac{1}{\omega - \hat{H}_0} =$$

## Electromagnetic multipole moments and transitions

Inserted in the expression for  $\Delta E$  leads to

$$\Delta E = \langle \Phi_0 | \hat{H}_I + \hat{H}_I \hat{Q} \frac{1}{E - \hat{H}_0 - \hat{Q} \hat{H}_I \hat{Q}} \hat{Q} \hat{H}_I | \Phi_0 \rangle.$$

In RS perturbation theory we set  $\omega = W_0$  and obtain the following expression for the energy difference

$$\Delta E = \sum_{i=0}^{\infty} \langle \Phi_0 | \hat{H}_I \left\{ \frac{\hat{Q}}{W_0 - \hat{H}_0} (\hat{H}_I - \Delta E) \right\}^i | \Phi_0 \rangle =$$

$$\langle \Phi_0 | \left( \hat{H}_I + \hat{H}_I \frac{\hat{Q}}{W_0 - \hat{H}_0} (\hat{H}_I - \Delta E) + \hat{H}_I \frac{\hat{Q}}{W_0 - \hat{H}_0} (\hat{H}_I - \Delta E) \frac{\hat{Q}}{W_0 - \hat{H}_0} (\hat{H}_I - \Delta E) + \dots \right) | \Phi_0 \rangle.$$

## Electromagnetic multipole moments and transitions

Recalling that  $\hat{Q}$  commutes with  $\hat{H}_0$  and since  $\Delta E$  is a constant we obtain that

$$\hat{Q} \Delta E | \Phi_0 \rangle = \hat{Q} \Delta E | \hat{Q} \Phi_0 \rangle = 0.$$

Inserting this results in the expression for the energy results in

$$\Delta E = \langle \Phi_0 | \left( \hat{H}_I + \hat{H}_I \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H}_I + \hat{H}_I \frac{\hat{Q}}{W_0 - \hat{H}_0} (\hat{H}_I - \Delta E) \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H}_I + \dots \right) | \Phi_0 \rangle.$$

## Electromagnetic multipole moments and transitions

We can now this expression in terms of a perturbative expression in terms of  $\hat{H}_I$  where we iterate the last expression in terms of  $\Delta E$

$$\Delta E = \sum_{i=1}^{\infty} \Delta E^{(i)}.$$

We get the following expression for  $\Delta E^{(i)}$

$$\Delta E^{(1)} = \langle \Phi_0 | \hat{H}_I | \Phi_0 \rangle,$$

which is just the contribution to first order in perturbation theory,

$$\Delta E^{(2)} = \langle \Phi_0 | \hat{H}_I \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H}_I | \Phi_0 \rangle,$$

which is the contribution to second order.

## Electromagnetic multipole moments and transitions

$$\Delta E^{(3)} = \langle \Phi_0 | \hat{H}_I \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H}_I \frac{\hat{Q}}{W_0 - \hat{H}_0} \hat{H}_I \Phi_0 \rangle - \langle \Phi_0 | \hat{H}_I \frac{\hat{Q}}{W_0 - \hat{H}_0} \langle \Phi_0 | \hat{H}_I | \Phi_0 \rangle$$

being the third-order contribution.

## Electromagnetic multipole moments and transitions

Let us analyse a given contribution to first order in perturbation, that is, the contribution includes (more material to come)

## $\beta$ -decay

We will now focus on allowed  $\beta$ -decay. Suhonen's chapter 7 and Alex Brown's chapter 29 cover much of the material to be discussed on  $\beta$ -decay.

## $\beta$ -decay

The allowed beta decay rate  $W$  between a specific set of initial and final states is given by

$$W_{i,f} = (f/K_o) [g_V^2 B_{i,f}(F_{\pm}) + g_A^2 B_{i,f}(GT_{\pm})],$$

where  $f$  is dimensionless three-body phase-space factor which depends upon the beta-decay  $Q$  value, and  $K_o$  is a specific combination of fundamental constants

$$K_o = \frac{2\pi^3 \hbar^7}{m_e^5 c^4} = 1.8844 \times 10^{-94} \text{ erg}^2 \text{ cm}^6 \text{ s}.$$

The  $\pm$  sign refer to  $\beta_{\pm}$  decay of nucleus  $(A_i, Z_i)$  into nucleus  $(A_i, Z_i \mp 1)$ . The weak-interaction vector ( $V$ ) and axial-vector ( $A$ ) coupling constants for the decay of neutron into a proton are denoted by  $g_V$  and  $g_A$ , respectively.

## $\beta$ -decay

The total decay rate for a given initial state is obtained by summing the partial rates over all final states

$$W = \sum_f W_{if},$$

with the branching fraction to a specific final state given by

$$b_{if} = \frac{W_{if}}{W}.$$

Beta decay lifetime are usually given in terms of the half-life with a total half-life of

$$T_{1/2} = \frac{\ln(2)}{W}.$$

The partial half-life for a particular final state will be denoted by  $t_{1/2}$

$$t_{1/2} = \frac{T_{1/2}}{b_{if}}.$$

## $\beta$ -decay

Historically one combines the partial half-life for a particular decay with the calculated phase-space factor  $f$  to obtain an  $ft$  value given by

$$ft_{1/2} = \frac{C}{[B(F_{\pm}) + (g_A/g_V)^2 B(GT_{\pm})]}$$

where

$$C = \frac{\ln(2) K_o}{(g_V)^2}.$$

## $\beta$ -decay

One often compiles the allowed beta decay in terms of a **logft** which stands for  $\log_{10}$  of the  $ft_{1/2}$  value. The values of the coupling constants for Fermi decay,  $g_V$ , and Gamow-Teller decay,  $g_A$  are obtained as follows. For a  $0^+ \rightarrow 0^+$  nuclear transition  $B(GT) = 0$ , and for a transition between  $T = 1$  analogue states with  $B(F) = 2$  we find

$$C = 2t_{1/2}f.$$

The partial half-lives and  $Q$  values for several  $0^+ \rightarrow 0^+$  analogue transitions have been measured to an accuracy of about one part in 10000. With phase space factors one obtains

$$C = 6170(4)$$

This result, together with the value of  $K_0$ , can be used to obtain  $g_V$ .

## $\beta$ -decay

At the quark level  $g_V = -g_A$ . But for nuclear structure we use the value obtained from the neutron to proton beta decay

$$|g_A/g_V| = 1.261(8).$$

## $\beta$ -decay

The operator for Fermi beta decay in terms of sums over the nucleons is

$$\mathcal{O}(F_{\pm}) = \sum_k t_{k\pm}.$$

The matrix element is

$$B(F) = |\langle f | T_{\pm} | i \rangle|^2,$$

where

$$T_{\pm} = \sum_k t_{\pm}$$

is the total isospin raising and lowering operator for total isospin constructed out of the basic nucleon isospin raising and lowering operators

$$t_- |n\rangle = |p\rangle \quad t_- |p\rangle = 0,$$

and

$$t_+ |p\rangle = |n\rangle, \quad t_+ |n\rangle = 0.$$

## $\beta$ -decay

The matrix elements obey the triangle conditions  $J_f = J_i$  ( $\Delta J = 0$ ). The Fermi operator has  $\pi_O = +1$ , and thus the initial and final nuclear states must have  $\pi_i \pi_f = +1$  for the matrix element to be nonzero under the parity transform. When isospin is conserved the Fermi matrix element must obey the isospin triangle condition  $T_f = T_i$  ( $\Delta T = 0$ ), and the Fermi operator can only connect isobaric analogue states.

## $\beta$ -decay

For  $\beta_-$  decay

$$\begin{aligned} & T_- |\omega_i, J_i, M_i, T_i, T_{zi}\rangle \\ &= \sqrt{(T_i(T_i+1) - T_{zi}(T_{zi}-1))} |\omega_i, J_i, M_i, T_i, T_{zi}-1\rangle, \end{aligned}$$

and

$$\begin{aligned} B(F_-) &= |\langle \omega_f, J_f, M_f, T_f, T_{zi}-1 | T_- | \omega_i, J_i, M_i, T_i, T_{zi} \rangle|^2 \\ &= [T_i(T_i+1) - T_{zi}(T_{zi}-1)] \delta_{\omega_f, \omega_i} \delta_{J_f, J_i} \delta_{M_f, M_i} \delta_{T_f, T_i}. \end{aligned}$$

## $\beta$ -decay

For  $\beta_+$  we have

$$\begin{aligned} B(F_+) &= |\langle \omega_f, J_f, M_f, T_f, T_{zi}+1 | T_+ | \omega_i, J_i, M_i, T_i, T_{zi} \rangle|^2 \\ &= [T_i(T_i+1) - T_{zi}(T_{zi}+1)] \delta_{\omega_f, \omega_i} \delta_{J_f, J_i} \delta_{M_f, M_i} \delta_{T_f, T_i}. \end{aligned}$$

For neutron-rich nuclei ( $N_i > Z_i$ ) we have  $T_i = T_{zi}$  and thus

$$B(F_-)(N_i > Z_i) = 2T_{zi} = (N_i - Z_i) \delta_{\omega_f, \omega_i} \delta_{J_f, J_i} \delta_{M_f, M_i} \delta_{T_f, T_i},$$

and

$$B(F_+)(N_i > Z_i) = 0.$$

## $\beta$ -decay

The reduced single-particle matrix elements are given by

$$\langle k_a, p | \sigma t_- | k_b, n \rangle = \langle k_a, n | \sigma t_+ | k_b, p \rangle = 2 \langle k_a | s | k_b \rangle,$$

where the matrix elements of  $s$  are given by

$$\langle k_a | s | k_b \rangle = \langle j_a | s | j_b \rangle \delta_{n_a, n_b} \\ = (-1)^{j_a + j_b + 3/2} \sqrt{(2j_a + 1)(2j_b + 1)} \begin{Bmatrix} 1/2 & 1/2 & 1 \\ j_b & j_a & l_a \end{Bmatrix} \langle s | s | s \rangle \delta_{\ell_a, \ell_b} \delta_{n_a, n_b}$$

with

$$\langle s | s | s \rangle = \sqrt{3/2}.$$

## $\beta$ -decay

The matrix elements of  $s$  has the selection rules  $\delta_{\ell_a, \ell_b}$  and  $\delta_{n_a, n_b}$ . Thus the orbits which are connected by the GT operator are very selective; they are those in the same major oscillator shell with the same  $\ell$  value. The matrix elements such as  $1s_{1/2} - 0d_{3/2}$  which have the allowed  $\Delta j$  coupling but are zero due to the  $\Delta \ell$  coupling are called  $\ell$ -forbidden matrix elements.

## $\beta$ -decay

Sum rules for Fermi and Gamow-Teller matrix elements can be obtained easily.

The sum rule for Fermi is obtained from the sum

$$\sum_f [B_F(F_-) - B_F(F_+)] = \sum_f [|\langle f | T_- | i \rangle|^2 - |\langle f | T_+ | i \rangle|^2]$$

The final states  $f$  in the  $T_-$  matrix element go with the  $Z_f = Z_i + 1$  nucleus and those in the  $T_+$  matrix element to with the  $Z_f = Z_i - 1$  nucleus. One can explicitly sum over the final states to obtain

$$\sum_f [\langle i | T_+ | f \rangle \langle f | T_- | i \rangle - \langle i | T_- | f \rangle \langle f | T_+ | i \rangle] \\ = \langle i | T_+ T_- - T_- T_+ | i \rangle = \langle i | 2 T_z | i \rangle = (N_i - Z_i).$$

## $\beta$ -decay

The sum rule for Gamow-Teller is obtained as follows

$$\sum_{f, \mu} |\langle f | \sum_k \sigma_{k, \mu} t_{k-} | i \rangle|^2 - \sum_{f, \mu} |\langle f | \sum_k \sigma_{k, \mu} t_{k+} | i \rangle|^2 \\ = \sum_{f, \mu} \langle i | \sum_k \sigma_{k, \mu} t_{k+} | f \rangle \langle f | \sum_{k'} \sigma_{k', \mu} t_{k'-} | i \rangle \\ - \sum_{f, \mu} \langle i | \sum_k \sigma_{k, \mu} t_{k-} | f \rangle \langle f | \sum_{k'} \sigma_{k', \mu} t_{k'+} | i \rangle \\ = \sum_{\mu} \left[ \langle i | \left( \sum_k \sigma_{k, \mu} t_{k+} \right) \left( \sum_{k'} \sigma_{k', \mu} t_{k'-} \right) - \left( \sum_k \sigma_{k, \mu} t_{k-} \right) \left( \sum_{k'} \sigma_{k', \mu} t_{k'+} \right) | i \rangle \right] \\ = \sum_{\mu} \langle i | \sum_k \sigma_{k, \mu}^2 [t_{k+} t_{k-} - t_{k-} t_{k+}] | i \rangle = 3 \langle i | \sum_k [t_{k+} t_{k-} - t_{k-} t_{k+}] | i \rangle \\ = 3 \langle i | T_+ T_- - T_- T_+ | i \rangle = 3 \langle i | 2 T_z | i \rangle = 3(N_i - Z_i).$$

## $\beta$ -decay

We have used the fact that  $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$ . When  $k \neq k'$  the operators commute and cancel. Thus

$$\sum_f [B_F(F_-) - B_F(F_+)] = (N_i - Z_i),$$

and

$$\sum_f [B_F(GT_-) - B_F(GT_+)] = 3(N_i - Z_i).$$

The sum-rule for the Fermi matrix elements applies even when isospin is not conserved.

## $\beta$ -decay

For  $N > Z$  we usually have  $T_i = T_{zi}$  which means that  $B(F_+) = 0$ . For  $N = Z$  ( $T_{zi} = 0$ ) and  $T_i = 0$  we get  $B(F_+) = B(F_-) = 0$ , and for  $T_i = 1$  we have  $B(F_+) = B(F_-) = 2$ . Fermi transitions which would be zero if isospin is conserved are called isospin-forbidden Fermi transitions.

When  $N > Z$  there are some situations where one has  $B(GT_+) = 0$ , and then we obtain  $B(GT_-) = 3(N_i - Z_i)$ . In particular for the  $\beta_-$  decay of the neutron we have  $B(F_-) = 1$  and  $B(GT_-) = 3$ .