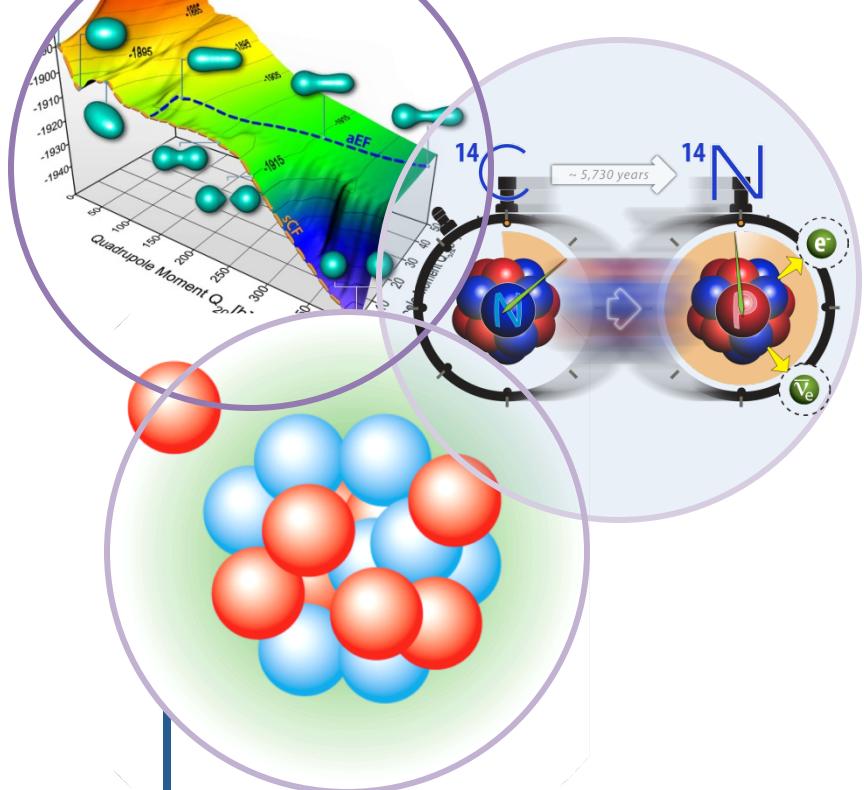


Infinite nucleonic matter from coupled-cluster theory

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Lecture 1-2, TALENT school

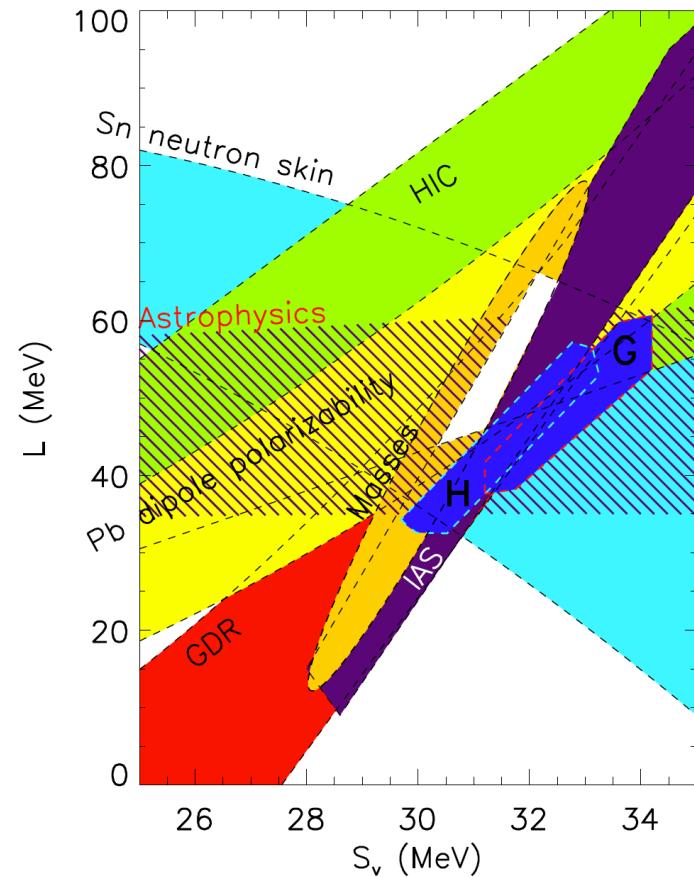
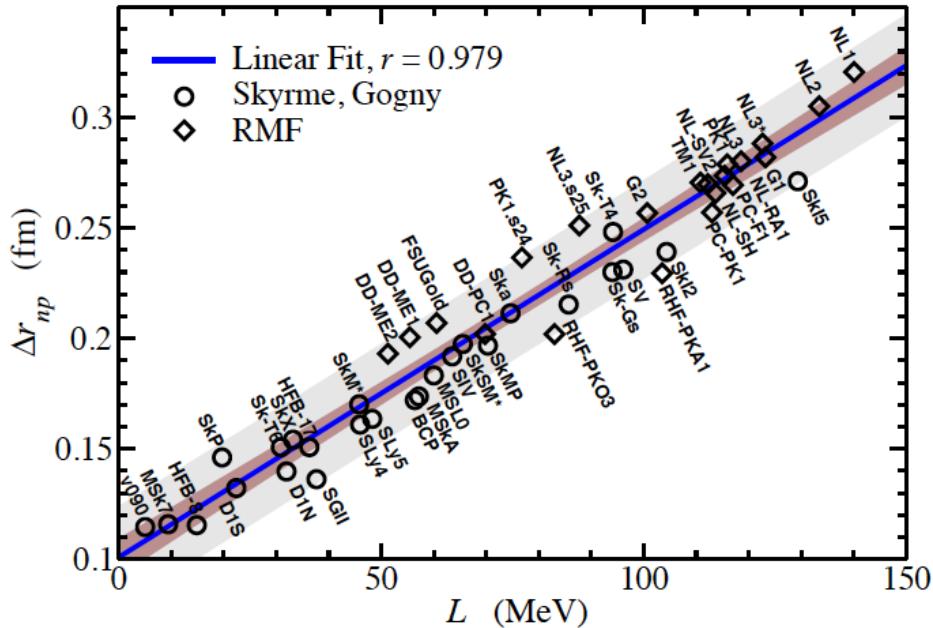
Outline

- Why study infinite nucleonic matter?
- Infinite nucleonic matter using periodic boundary conditions
- Setting up single-particle basis
- The Minnesota potential and computation of matrix elements in plane-wave basis
- Hartree-Fock for infinite matter using periodic boundary conditions
- Finite size effects and twist-averaging
- Going beyond mean-field
 - Many-body perturbation theory
 - Coupled-cluster in ladder approximation
 - Coupled-cluster doubles (CCD) approximation
- Recent applications of CCD to infinite nucleonic matter with chiral nucleon-nucleon and three-nucleon interactions

Why infinite nucleonic matter?

- Determines properties of supernova explosions and of neutron stars
- Equation of state links neutron stars to neutron radii in atomic nuclei
- The symmetry energy of nucleonic matter is related to the neutron skin

X. Viñas et al, Eur. Phys. J. A 50, 27 (2014)



Lattimer & Steiner, EPJA 50 (2014) 40

- The incompressibility is probed in giant dipole excitations in nuclei
- The nuclear matter saturation point from first principles is a challenge

Nucleonic matter from periodic boundary conditions

- Standard way to solve the infinite matter problem is to solve integral equations in relative center-of-mass frame using a partial wave expansion (e.g. Brueckner-Bethe-Goldstone theory)
 - Difficult to treat particle-hole correlations
 - Easy to extrapolate to thermodynamic limit
- Here we will focus on solving the infinite matter problem using periodic boundary conditions (PBC)
 - Straightforward to include all classes of correlations (e.g. particle-hole)
 - No partial wave expansion makes it easier to code
 - More difficult to extrapolate to thermodynamic limit
 - Finite size effects often non-negligible (can be minimized using twist-averaged periodic boundary conditions)

Periodic boundary conditions (PBC) in 3D

Schrodinger equation for plane waves in a cubic box

$$\frac{-\hbar^2}{2m} \nabla^2 \phi(\mathbf{x}) = \varepsilon \phi(\mathbf{x})$$

Periodic boundary conditions gives:

$$\phi(x, y, z) = \phi(x + L_x, y, z)$$

$$\phi(x, y, z) = \phi(x, y + L_y, z)$$

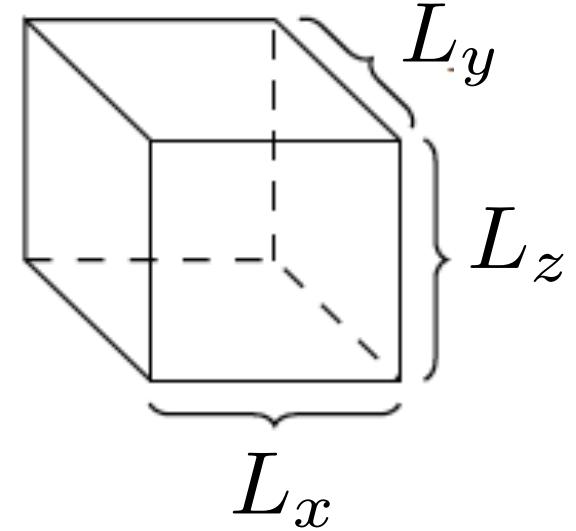
$$\phi(x, y, z) = \phi(x, y, z + L_z)$$

Assume box of equal sides: $L = L_x = L_y = L_z$

This yields normalized plane wave solutions with discrete momenta:

$$\phi_n(\mathbf{x}) = \frac{1}{\sqrt{V}} e^{i \mathbf{k}_n \cdot \mathbf{x}}$$

$$k_{n_i} = \frac{2\pi n_i}{L}$$
$$n_i = 0, \pm 1, \pm 2, \dots$$



Periodic boundary conditions in 3D

- The energy eigenvalues (ordered)

$$\varepsilon_n = \frac{\hbar^2}{2m} \left(k_{n_x}^2 + k_{n_y}^2 + k_{n_z}^2 \right)$$

yield closed shell “Fermi” spheres:

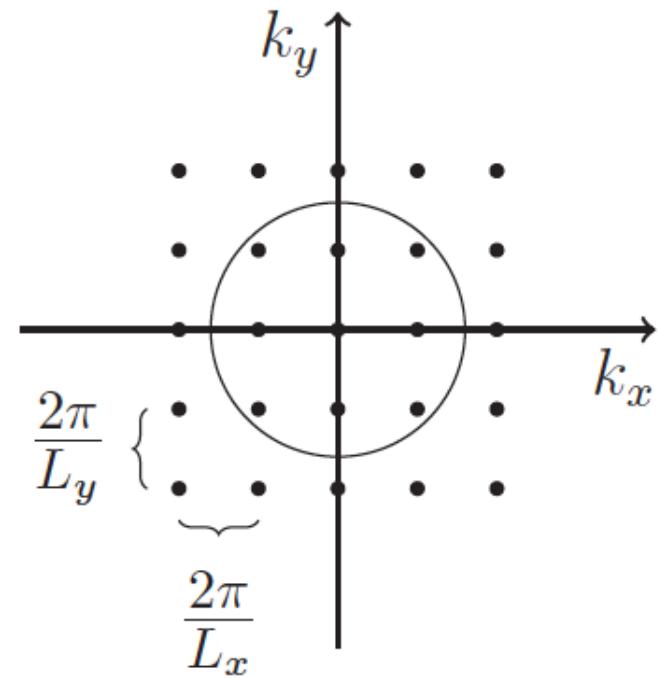
$$n = 1, 7, 19, 27, 33, 57, \dots$$

- Taking spin and isospin quantum numbers into account we have the single-particle basis

$$|\mathbf{k}_n, s_z, t_z\rangle = |\mathbf{k}\rangle$$

- Closed shells for nuclear matter is then:

Example in 2D:



$$n = 4, 28, 76, 108, 132, 228, \dots$$

Setting up the basis for nuclear matter with PBC

- Specify number nucleons in the reference state (only for closed shells: 4,28,76,... etc)
- Specify density $\rho = g_s k_F^3 / (6\pi^2)$ or Fermi-momentum k_F ($g_s = 2$ for neutron matter and $g_s = 4$ for nuclear matter)
- Compute the volume of the cubic box according to

$$V = L^3 = A/\rho$$

- From L we can compute the lattice spacing:

$$\Delta k = 2\pi/L$$

- Specify n_{\max} for the basis
- Generate a Fortran/python/C++ code that set up a single particle basis for PBC

Me: Run Fortran example in terminal window

Setting up the basis for nuclear matter with PBC

- Student task:
 - Code an algorithm for setting up a single-particle basis for neutron/nuclear matter for given density/Fermi momentum, number of nucleons, and n_{\max}

A simple central nucleon–nucleon potential:

The Minnesota potential: central Gaussian potential and diagonal in spin and isospin.

$$V^c(r) = \frac{1}{2} \left(V_R + \frac{1}{2}(1 + P_{12}^\sigma)V_T + \frac{1}{2}(1 - P_{12}^\sigma)V_S \right) (1 - P_{12}^\sigma P_{12}^\tau)$$

The spin and isospin exchange operators are given by:

$$P_{12}^\sigma = \frac{1}{2}(1 + (\sigma_1 \cdot \sigma_2)), \quad P_{12}^\tau = \frac{1}{2}(1 + (\tau_1 \cdot \tau_2))$$

The potential in r-space:

The parameters of the potential are given by:

$$V_R = V_{0R} \exp(-\kappa_R r_{12}^2)$$

$$V_{0R} = 200 \text{ MeV}, \quad \kappa_R = 1.487 \text{ fm}^{-2}$$

$$V_T = -V_{0T} \exp(-\kappa_T r_{12}^2)$$

$$V_{0T} = 178 \text{ MeV}, \quad \kappa_T = 0.639 \text{ fm}^{-2}$$

$$V_S = -V_{0S} \exp(-\kappa_S r_{12}^2)$$

$$V_{0S} = 91.85 \text{ MeV}, \quad \kappa_S = 0.465 \text{ fm}^{-2}$$

Second quantized normal-ordered Hamiltonian:

$$H_N = \sum_{pq} \langle \mathbf{k}_p | f | \mathbf{k}_q \rangle : a_p^\dagger a_q : + \frac{1}{4} \sum_{pqrs} \langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_r \mathbf{k}_s \rangle : a_p^\dagger a_q^\dagger a_s a_r :$$

The normal-ordered one-body part is given by:

$$\langle \mathbf{k}_p | f | \mathbf{k}_q \rangle = \langle \mathbf{k}_p | t | \mathbf{k}_q \rangle + \sum_i \langle \mathbf{k}_p \mathbf{k}_i | V_{NN} | \mathbf{k}_q \mathbf{k}_i \rangle$$

The normal-ordered two-body part is given by:

$$\langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_r \mathbf{k}_s \rangle = \langle \mathbf{k}_p \mathbf{k}_q | V_{NN} | \mathbf{k}_r \mathbf{k}_s \rangle$$

Note: all two-body matrix elements are here assumed to be anti-symmetric

Second quantized normal-ordered Hamiltonian:

Matrix elements of kinetic energy:

$$\langle \mathbf{k}_p | t | \mathbf{k}_q \rangle = \frac{\hbar^2}{2m} \vec{k}_p^2 \delta_{pq}$$

Note periodic boundary conditions ensures that momentum is conserved:

$$\vec{k}_p + \vec{k}_q = \vec{k}_r + \vec{k}_s$$

Matrix elements of a Gaussian nucleon-nucleon interaction are: $V_R = V_0 \exp(-\alpha r_{12}^2)$

$$\langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_r \mathbf{k}_s \rangle = \frac{V_0}{L^3} \left(\frac{\pi}{\alpha} \right)^{3/2} \exp(-q^2/4\alpha) \delta_{\vec{k}_p + \vec{k}_q, \vec{k}_r + \vec{k}_s}$$

Anti-symmetric matrix elements are then computed:

$$\langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_r \mathbf{k}_s \rangle_{AS} = \langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_r \mathbf{k}_s \rangle - \langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_s \mathbf{k}_r \rangle$$

Setting up the basis for nuclear matter with PBC

- Student task:
 - Derive a programmable expression for the Minnesota potential in a plane-wave basis for periodic boundary conditions:

$$\langle \mathbf{k}_p \mathbf{k}_q | v | \mathbf{k}_r \mathbf{k}_s \rangle$$

Hartree–Fock for infinite matter

Hartree-Fock computed using PBC:

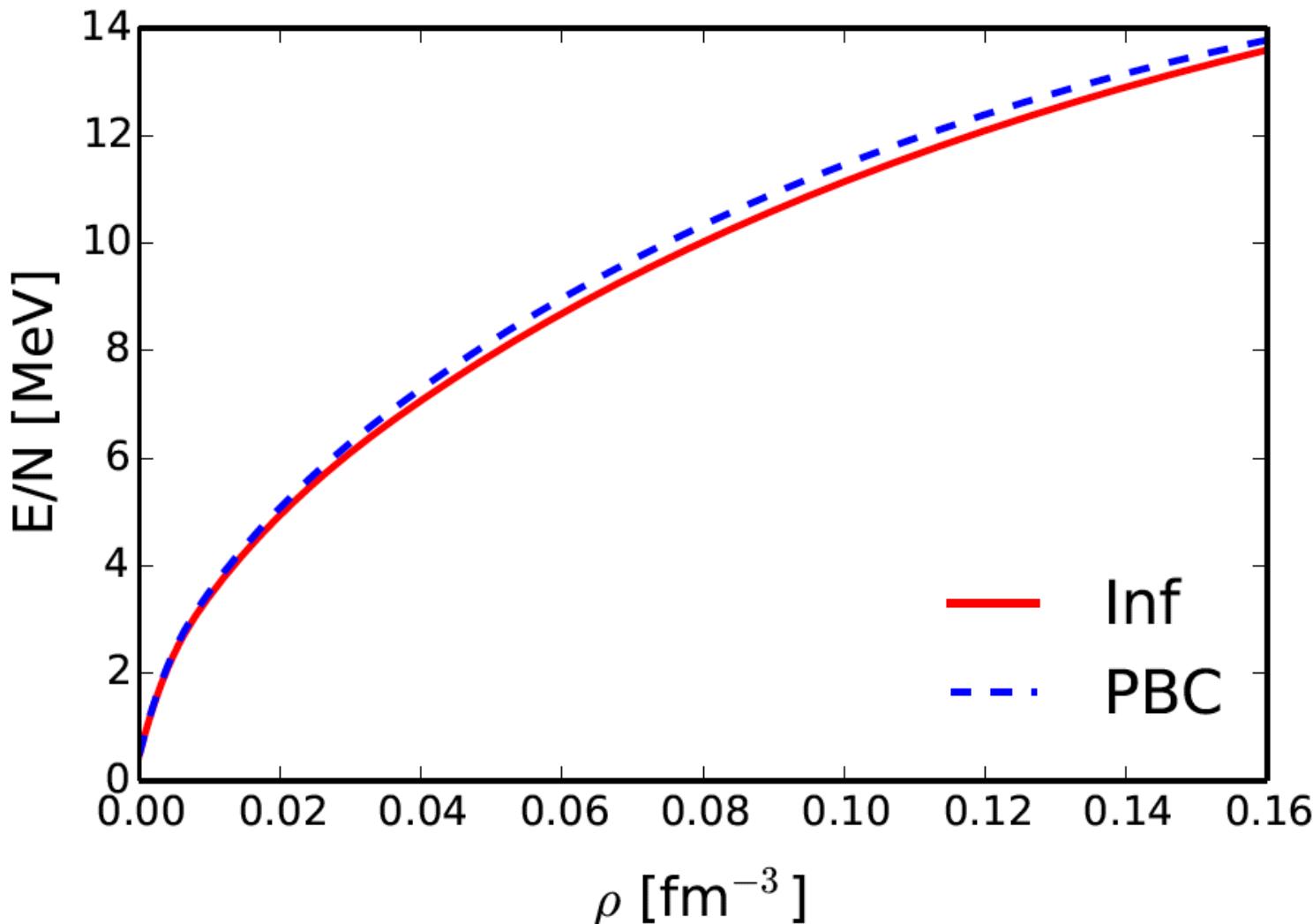
$$\begin{aligned} E_0 &= \langle \Phi_0 | H | \Phi_0 \rangle \\ &= \sum_i \langle \mathbf{k}_i | f | \mathbf{k}_i \rangle + \frac{1}{2} \sum_{i,j} \langle \mathbf{k}_i \mathbf{k}_j | v | \mathbf{k}_i \mathbf{k}_j \rangle \end{aligned}$$

Hartree-Fock computed in relative center-of-mass coordinates for a central potential:

$$\frac{E_0}{A} = \frac{3\hbar^2 k_F^2}{10m} + \frac{1}{(2\pi)^6} \frac{1}{\rho} \sum_{T_z, S_z} \int_0^{k_F} d\vec{p} \int_0^{k_F} d\vec{p}' \langle \vec{p} | v | \vec{p}' \rangle$$

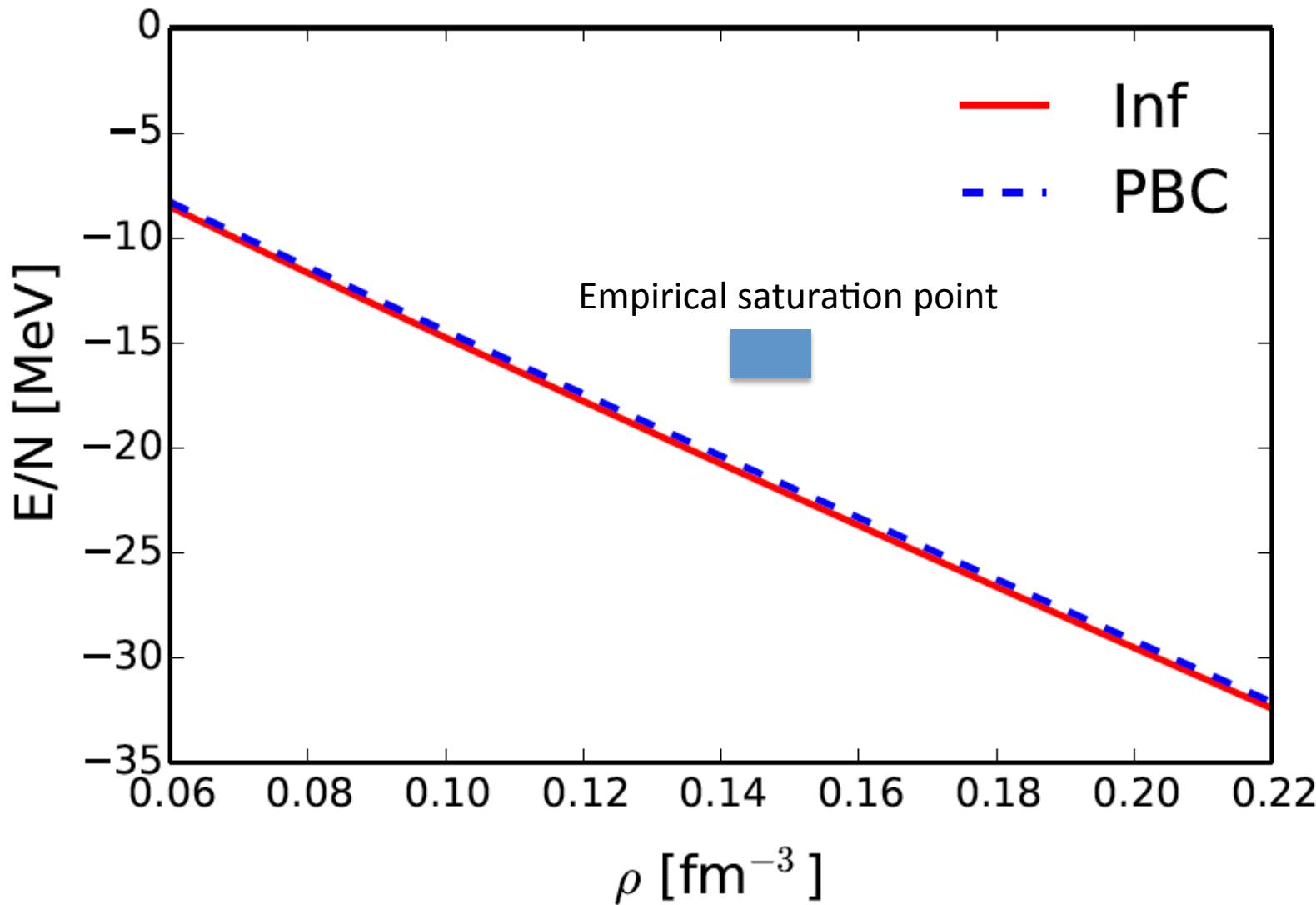
Hartree–Fock for infinite matter

Benchmark for Hartree-Fock energy in pure neutron matter for the Minnesota potential. PBC used 14 particles and $n_{\max} = 3$.



Hartree–Fock for infinite matter

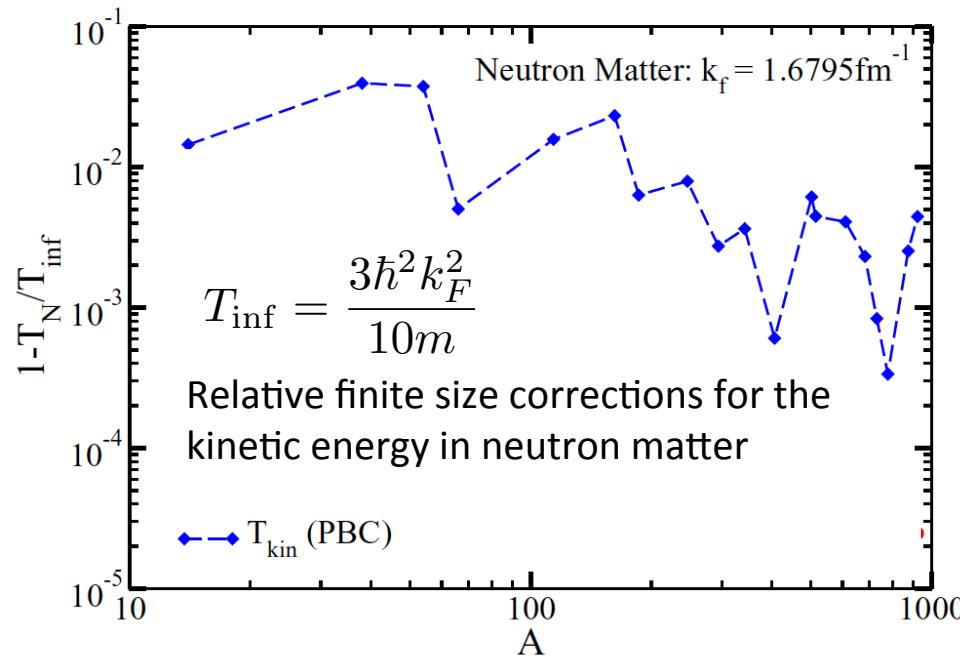
Benchmark for Hartree-Fock energy in symmetric neutron matter for the Minnesota potential. PBC used 28 particles and $n_{\max} = 3$.



Setting up the basis for nuclear matter with PBC

- Student task:
 - Code a program for computing the Hartree-Fock energy per nucleon in infinite neutron/nuclear matter for given density/Fermi momentum using the Minnesota nucleon-nucleon potential
 - Study how E/A changes with density/Fermi momentum
 - Study finite size effects by varying the particle number
 - Benchmark against Hartree-Fock computed in relative center-of-mass coordinates

Twist averaged periodic boundary conditions



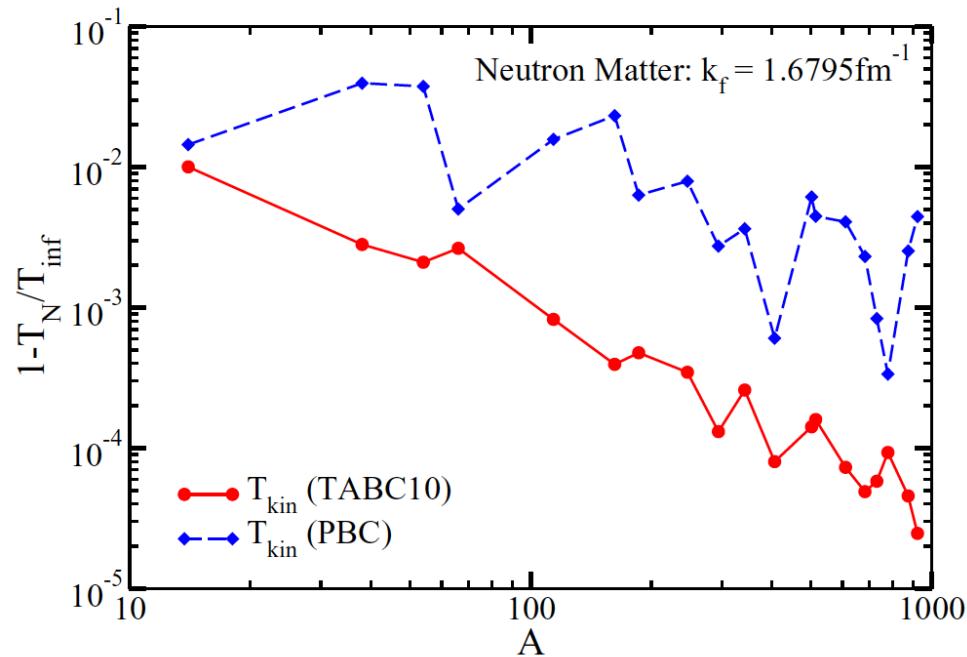
- How to quantify error from shell oscillations/finite size effects?
- By increasing number of nucleons we don't see a systematic and smooth convergence to the thermodynamic limit
- Comparing the kinetic energy computed with PBC and in the thermodynamic limit we can find an optimal number of particles
- Is there a more systematic way?
Yes: Use twist-averaged PBC.

“Twist averaged” boundary conditions: average over all possible phases of Bloch waves.
Exactly removes finite size effects for the free Fermi gas:

$$k_{n_i} = \frac{(2\pi n_i + \theta_i)}{L} \quad \psi_k(x + L) = e^{i\theta} \psi_k(x)$$

[Gros, Z. Phys. B 86, 359 (1992); Lin, Zong, and Ceperley, Phys. Rev. E 64, 016702 (2001)]
Discrete momenta are

Twist averaged periodic boundary conditions



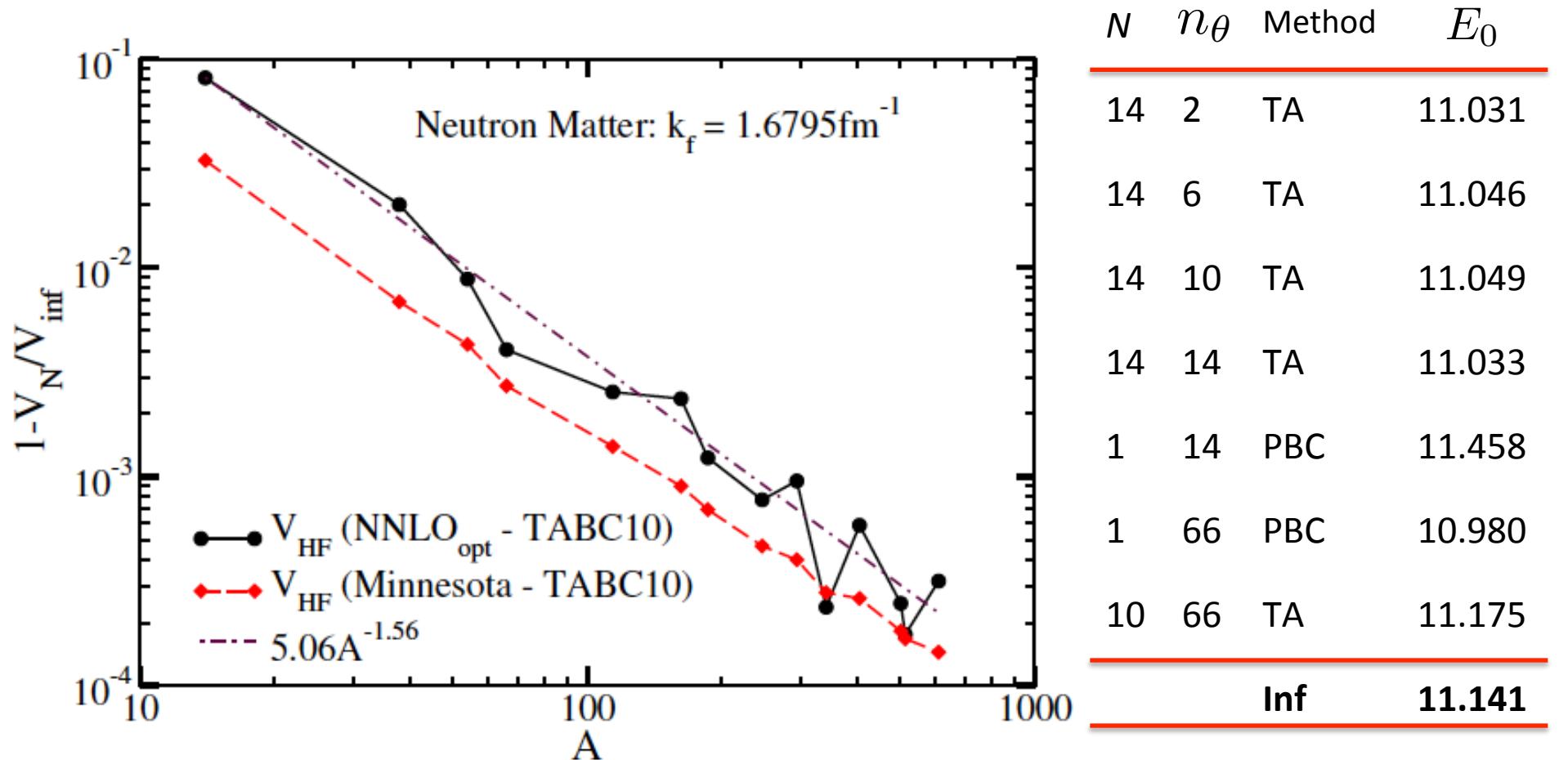
- Finite size effects for kinetic energy in neutron matter for PBC and TAPBC
- With TAPBC we obtain a smooth/systematic convergence to thermodynamic limit

- Choose a mesh (e.g. Gauss-Legendre) for the twist angles $\theta_{x,y,z}$ in the interval: $[0, \pi]$
- Loop over mesh in x,y,z and compute kinetic and hartree-fock energy
- Very costly: for 10 points you need to compute the energy at one density 10^3 times.
- The final energy computed with TAPBC is

$$E_0^{TA} = E_0^{TA} + \left(\frac{1}{\pi^3} \right) w_{\theta_x} w_{\theta_y} w_{\theta_z} E_0(\theta_x, \theta_y, \theta_z)$$

Twist averaged periodic boundary conditions

Relative finite size corrections for the Hartree-Fock energy of pure neutron matter with the Minnesota potential



$$\rho = 0.10 \text{ fm}^{-3}, n_{\max} = 3$$

Quantifying errors from finite size effects

- Student task:
 - Implement twist-averaged periodic boundary conditions in pure neutron matter for the kinetic energy (possibly HF)
 - Study convergence to thermodynamic limit with number of closed shell neutrons and number meshpoints for the twist angles

Going beyond mean-field: MBPT2

- Correlations beyond mean-field are important in nuclear physics (strongly correlated many-body problem)
- First attempt is many-body perturbation theory at second order

$$E_{\text{MBPT2}} = E_0 + \frac{1}{4} \sum_{ijab} \langle \mathbf{k}_i \mathbf{k}_j | v | \mathbf{k}_a \mathbf{k}_b \rangle \langle \mathbf{k}_a \mathbf{k}_b | t^0 | \mathbf{k}_i \mathbf{k}_j \rangle$$

Here a,b,c, are orbitals above the Fermi level (unoccupied) and i,j,k... are orbitals below the Fermi level (occupied)

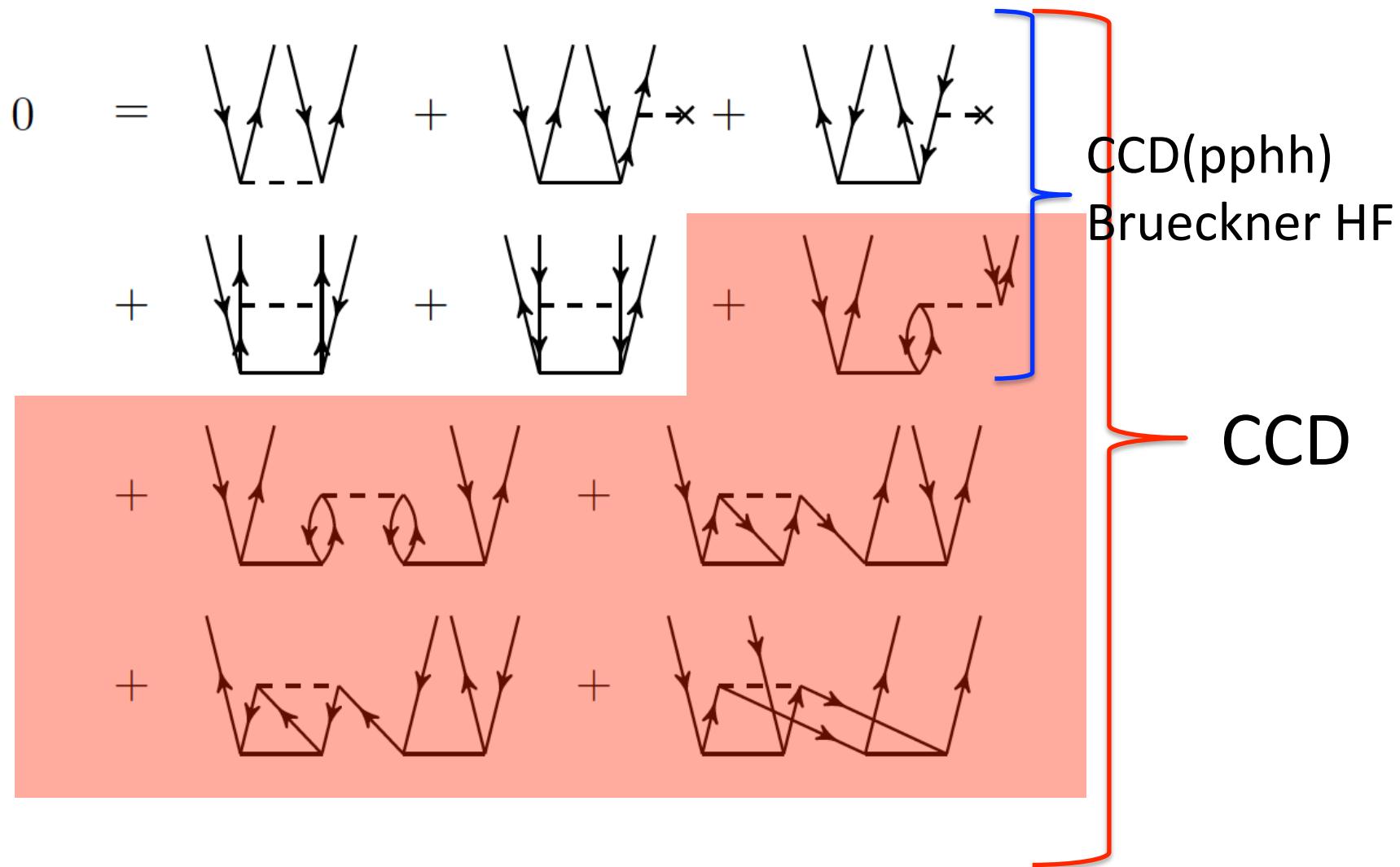
Hartree-Fock energy (1st order MBPT):

$$E_0 = \sum_i \langle \mathbf{k}_i | f | \mathbf{k}_i \rangle + \frac{1}{2} \sum_{ij} \langle \mathbf{k}_i \mathbf{k}_j | v | \mathbf{k}_i \mathbf{k}_j \rangle$$

The t^0 amplitudes are:

$$\langle \mathbf{k}_a \mathbf{k}_b | t^0 | \mathbf{k}_i \mathbf{k}_j \rangle = \frac{\langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_i \mathbf{k}_j \rangle}{f_{ii} + f_{jj} - f_{aa} - f_{bb}} \quad f_{pq} = \langle \mathbf{k}_p | f | \mathbf{k}_q \rangle$$

Going beyond mean-field with coupled-cluster method



Brief introduction to coupled-cluster diagrams

$$i \downarrow a \ b \uparrow j = \langle ab|t|ij\rangle \quad i \overline{a} \ \overline{b} \ j = \langle ij|v|ab\rangle$$

- Assign labels to particle and hole lines
- Matrix elements are in the order

$$\langle \text{left}_{\text{out}}, \text{right}_{\text{out}} | v | \text{left}_{\text{in}}, \text{right}_{\text{in}} \rangle$$

- Each diagram gets a factor $\frac{1}{2}$ for equivalent lines
- Each diagram gets the phase $(-1)^{h+l}$

$$\begin{array}{c} \text{Diagram 1: } \text{left}_{\text{out}} \uparrow \text{right}_{\text{out}} \downarrow + \text{left}_{\text{in}} \downarrow \text{right}_{\text{in}} \uparrow \\ \text{Diagram 2: } \text{left}_{\text{out}} \downarrow \text{right}_{\text{out}} \uparrow + \text{left}_{\text{in}} \uparrow \text{right}_{\text{in}} \downarrow \end{array} = \frac{1}{2} \sum_{cd} \langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_c \mathbf{k}_d \rangle \langle \mathbf{k}_c \mathbf{k}_d | t | \mathbf{k}_i \mathbf{k}_j \rangle + \frac{1}{2} \sum_{kl} \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_k \mathbf{k}_l \rangle \langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_i \mathbf{k}_j \rangle$$

Going beyond mean-field: CCD_{ladd}

The CCD equations are given by:

$$0 = \langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_i \mathbf{k}_j \rangle$$

$$+ P(ab) \sum_c \langle \mathbf{k}_b | f | \mathbf{k}_c \rangle \langle \mathbf{k}_a \mathbf{k}_c | t | \mathbf{k}_i \mathbf{k}_j \rangle$$

$$- P(ij) \sum_k \langle \mathbf{k}_k | f | \mathbf{k}_j \rangle \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_i \mathbf{k}_k \rangle$$

$$+ \frac{1}{2} \sum_{cd} \langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_c \mathbf{k}_d \rangle \langle \mathbf{k}_c \mathbf{k}_d | t | \mathbf{k}_i \mathbf{k}_j \rangle$$

$$+ \frac{1}{2} \sum_{kl} \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_k \mathbf{k}_l \rangle \langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_i \mathbf{k}_j \rangle$$

The most costly term to evaluate in the CCD approximation. Computational cost is:

$$n_u^4 n_o^2$$

Going beyond mean-field: CCD_{ladd}

$$D_{ij}^{ab} \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_i \mathbf{k}_j \rangle = \langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_i \mathbf{k}_j \rangle$$

$$D_{ij}^{ab} = f_{ii} + f_{jj} - f_{aa} - f_{bb}$$

$$\begin{aligned} &+ P(ab) \sum_{c \neq a} \langle \mathbf{k}_b | f | \mathbf{k}_c \rangle \langle \mathbf{k}_a \mathbf{k}_c | t | \mathbf{k}_i \mathbf{k}_j \rangle \\ &- P(ij) \sum_{k \neq i} \langle \mathbf{k}_k | f | \mathbf{k}_j \rangle \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_i \mathbf{k}_k \rangle \\ &+ \frac{1}{2} \sum_{cd} \langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_c \mathbf{k}_d \rangle \langle \mathbf{k}_c \mathbf{k}_d | t | \mathbf{k}_i \mathbf{k}_j \rangle \\ &+ \frac{1}{2} \sum_{kl} \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_k \mathbf{k}_l \rangle \langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_i \mathbf{k}_j \rangle \end{aligned}$$

Solve non-linear CCD equations iteratively starting from the MBPT2 ansatz for t^0

$$t^{(i)} = \alpha t^{(i)} + (1 - \alpha)t^{(i-1)}$$

Once the t -amplitudes are converged compute the CCD energy:

$$E_{\text{CCD}} = E_0 + \frac{1}{4} \sum_{ijab} \langle \mathbf{k}_i \mathbf{k}_j | v | \mathbf{k}_a \mathbf{k}_b \rangle \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_i \mathbf{k}_j \rangle$$

Solving the CCD_{ladd} equations efficiently

$$\frac{1}{2} \sum_{cd} \underbrace{\langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_c \mathbf{k}_d \rangle}_{A} \underbrace{\langle \mathbf{k}_c \mathbf{k}_d | t | \mathbf{k}_i \mathbf{k}_j \rangle}_{C} \underbrace{\langle \mathbf{k}_i \mathbf{k}_j | I | \mathbf{k}_i \mathbf{k}_j \rangle}_{I}$$

We can write the tensor contractions as a matrix-matrix multiplication:

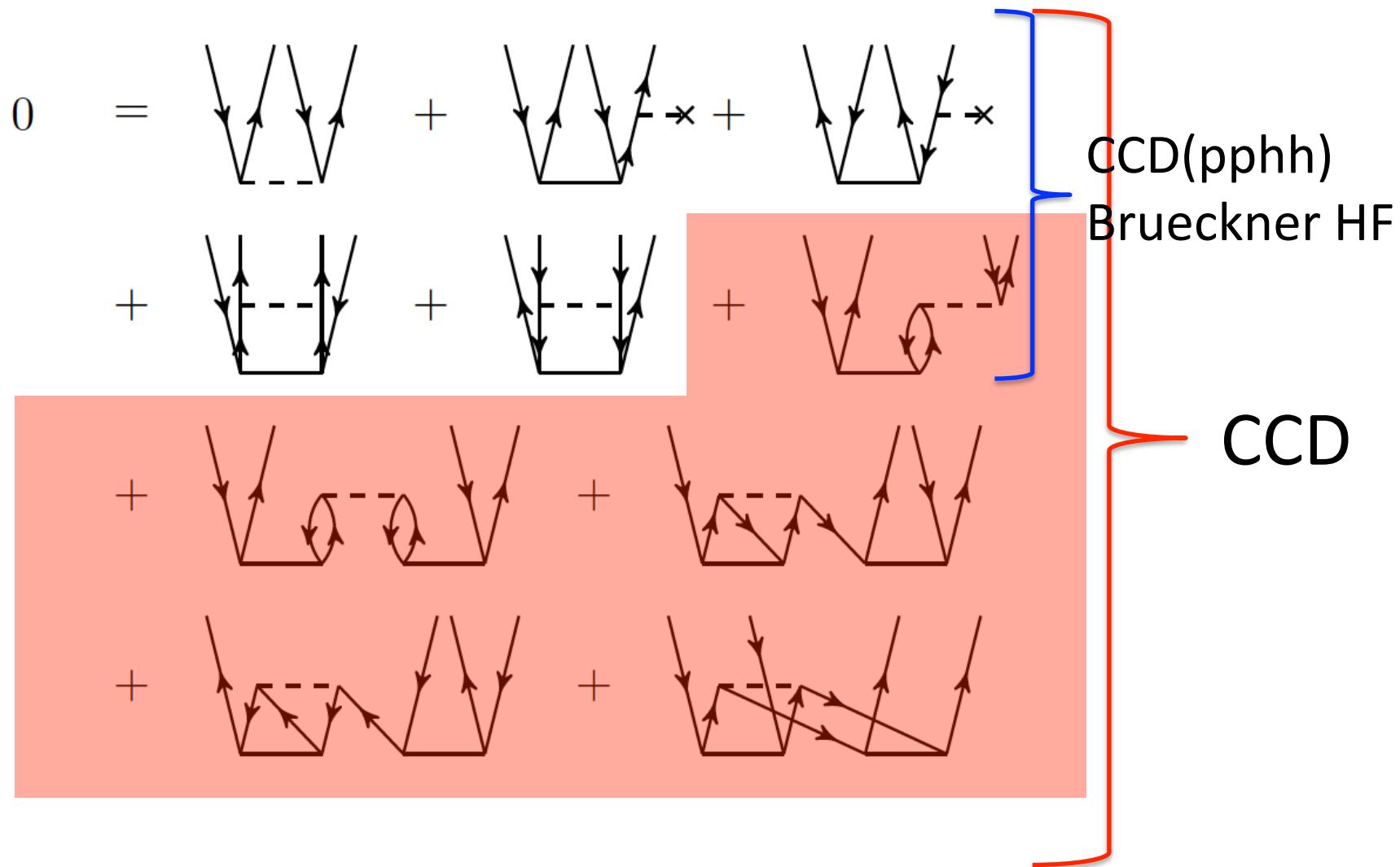
$$\frac{1}{2} \sum_C \langle A | v | C \rangle \langle C | t | I \rangle$$

Matrix-matrix multiplications can be efficiently solved by using linear algebra libraries such as LAPACK (Linear Algebra PACKage) and BLAS (Basic Linear Algebra Subprograms) routines.

Going beyond the mean-field

- Student task:
 - Implement MBPT2 for pure neutron matter and symmetric nuclear matter using PBC and the Minnesota potential
 - Code coupled-cluster for neutron matter and symmetric nuclear matter keeping only particle-particle and hole-hole ladder terms (CCD_{ladd}) and the Minnesota potential

Going beyond mean-field with coupled-cluster method



Going beyond mean-field: CCD

The full CCD equations in factorized form:

$$\begin{aligned} 0 = & \langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_i \mathbf{k}_j \rangle \\ & + P(ab) \sum_c \langle \mathbf{k}_b | \chi | \mathbf{k}_c \rangle \langle \mathbf{k}_a \mathbf{k}_c | t | \mathbf{k}_i \mathbf{k}_j \rangle \\ & - P(ij) \sum_k \langle \mathbf{k}_k | \chi | \mathbf{k}_j \rangle \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_i \mathbf{k}_k \rangle \\ & + \frac{1}{2} \sum_{cd} \langle \mathbf{k}_a \mathbf{k}_b | \chi | \mathbf{k}_c \mathbf{k}_d \rangle \langle \mathbf{k}_c \mathbf{k}_d | t | \mathbf{k}_i \mathbf{k}_j \rangle \\ & + \frac{1}{2} \sum_{kl} \langle \mathbf{k}_a \mathbf{k}_b | t | \mathbf{k}_k \mathbf{k}_l \rangle \langle \mathbf{k}_k \mathbf{k}_l | \chi | \mathbf{k}_i \mathbf{k}_j \rangle \\ & + P(ij)P(ab) \sum_{kc} \langle \mathbf{k}_a \mathbf{k}_c | t | \mathbf{k}_i \mathbf{k}_k \rangle \langle \mathbf{k}_k \mathbf{k}_b | \chi | \mathbf{k}_c \mathbf{k}_j \rangle \end{aligned}$$

Going beyond mean-field: CCD

One-body intermediates:

$$\begin{aligned} \langle \mathbf{k}_b | \chi | \mathbf{k}_c \rangle &= \langle \mathbf{k}_b | f | \mathbf{k}_c \rangle & \langle \mathbf{k}_k | \chi | \mathbf{k}_j \rangle &= \langle \mathbf{k}_k | f | \mathbf{k}_j \rangle \\ - \frac{1}{2} \sum_{kld} \langle \mathbf{k}_b \mathbf{k}_d | t | \mathbf{k}_k \mathbf{k}_l \rangle &\langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_c \mathbf{k}_d \rangle & + \frac{1}{2} \sum_{kcd} \langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_c \mathbf{k}_d \rangle &\langle \mathbf{k}_c \mathbf{k}_d | t | \mathbf{k}_j \mathbf{k}_l \rangle \end{aligned}$$

Two-body intermediates:

$$\begin{aligned} \langle \mathbf{k}_k \mathbf{k}_l | \chi | \mathbf{k}_i \mathbf{k}_j \rangle &= \langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_i \mathbf{k}_j \rangle & \langle \mathbf{k}_k \mathbf{k}_b | \chi | \mathbf{k}_c \mathbf{k}_j \rangle &= \langle \mathbf{k}_k \mathbf{k}_b | v | \mathbf{k}_c \mathbf{k}_j \rangle \\ + \frac{1}{2} \sum_{cd} \langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_c \mathbf{k}_d \rangle &\langle \mathbf{k}_c \mathbf{k}_d | t | \mathbf{k}_i \mathbf{k}_j \rangle & + \sum_{ld} \langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_c \mathbf{k}_d \rangle &\langle \mathbf{k}_c \mathbf{k}_d | t | \mathbf{k}_l \mathbf{k}_j \rangle \end{aligned}$$

$$\langle \mathbf{k}_a \mathbf{k}_b | \chi | \mathbf{k}_c \mathbf{k}_d \rangle = \langle \mathbf{k}_a \mathbf{k}_b | v | \mathbf{k}_c \mathbf{k}_d \rangle$$

Why use intermediates?

$$n_u^3 n_o^3 + n_u^2 n_o^2 + n_u^3 n_o^4 \quad n_u \gg n_o$$

$$\sum_{kc} \langle \mathbf{k}_a \mathbf{k}_c | t | \mathbf{k}_i \mathbf{k}_k \rangle \langle \mathbf{k}_k \mathbf{k}_b | \chi | \mathbf{k}_c \mathbf{k}_j \rangle$$
$$= \sum_{kc} \langle \mathbf{k}_a \mathbf{k}_c | t | \mathbf{k}_i \mathbf{k}_k \rangle \langle \mathbf{k}_k \mathbf{k}_b | v | \mathbf{k}_c \mathbf{k}_j \rangle$$
$$+ \frac{1}{2} \sum_{kcl d} \langle \mathbf{k}_a \mathbf{k}_c | t | \mathbf{k}_i \mathbf{k}_k \rangle \langle \mathbf{k}_k \mathbf{k}_l | v | \mathbf{k}_c \mathbf{k}_d \rangle \langle \mathbf{k}_b \mathbf{k}_d | t | \mathbf{k}_i \mathbf{k}_j \rangle$$
$$n_u^3 n_o^3 + n_u^4 n_o^4$$

Going beyond the mean-field

- Student task:
 - Derive the CCD equations from the CCD diagrams using the diagrams rules
 - Write the CCD equations in a factorized form using intermediates
 - Implement CCD for symmetric nuclear matter and pure neutron matter
 - Do a comparison between HF, MBPT2, CCD_{ladd} and CCD in pure neutron matter and nuclear matter around saturation density

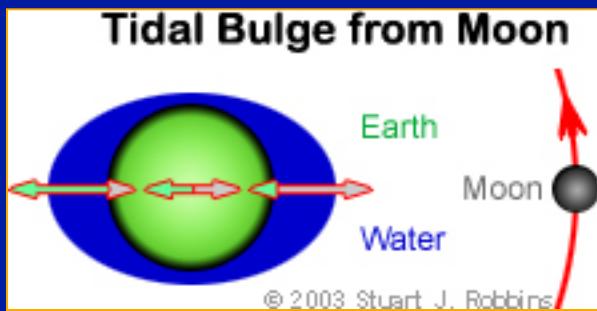
Three-nucleon forces – Why?



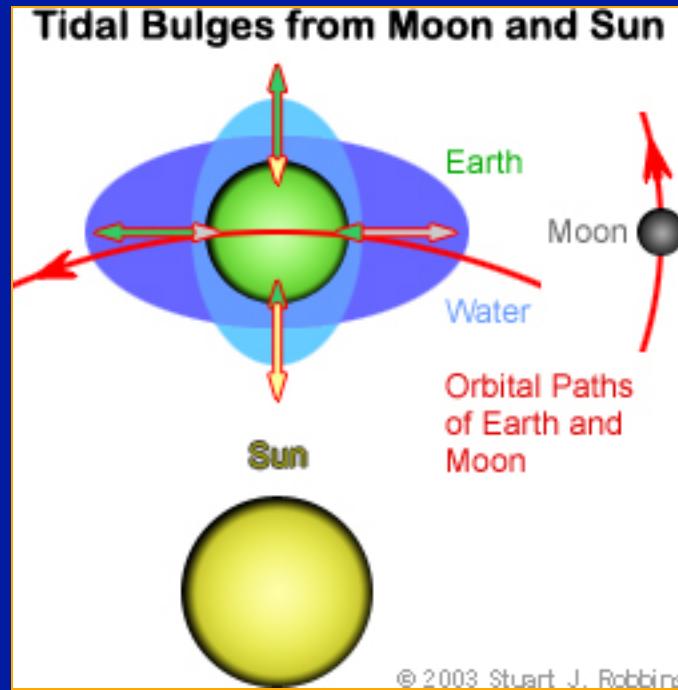
- Nucleons are not point particles (i.e. not elementary).
- We neglected some internal degrees of freedom (e.g. Δ -resonance, “polarization effects”, ...), and unconstrained high-momentum modes.

Example from celestial mechanics:

Earth-Moon system: point masses and modified two-body interaction



Other tidal effects cannot be included in the two-body interaction! Three-body force unavoidable for point masses.



The question is not: Do three-body forces enter the description?
The (only) question is: How large are three-body forces?

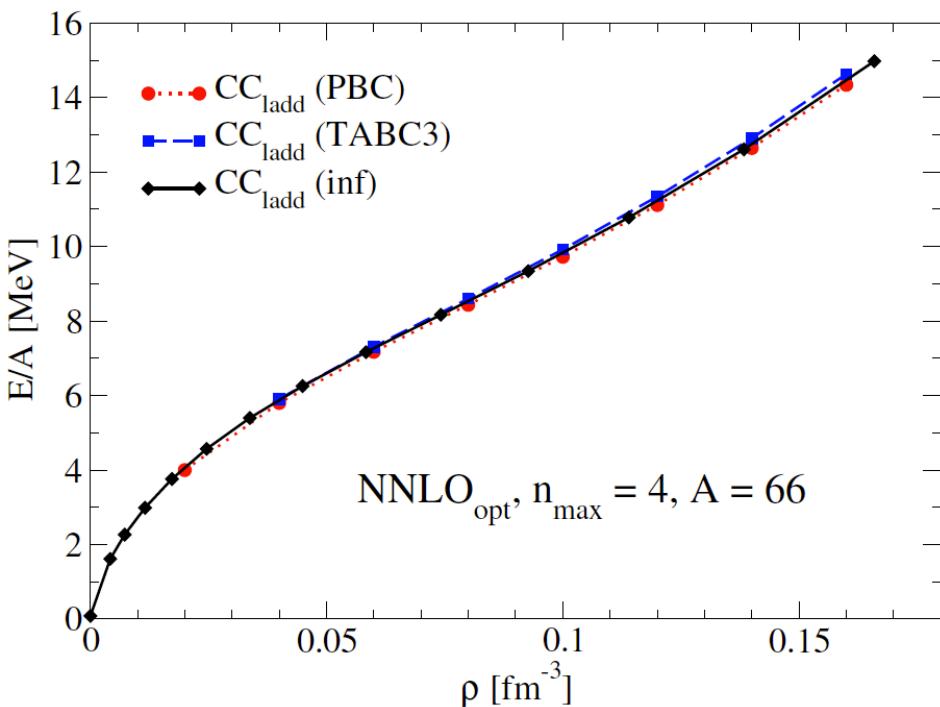
Nuclear forces from chiral effective field theory

[Weinberg; van Kolck; Epelbaum *et al.*; Entem & Machleidt; ...]

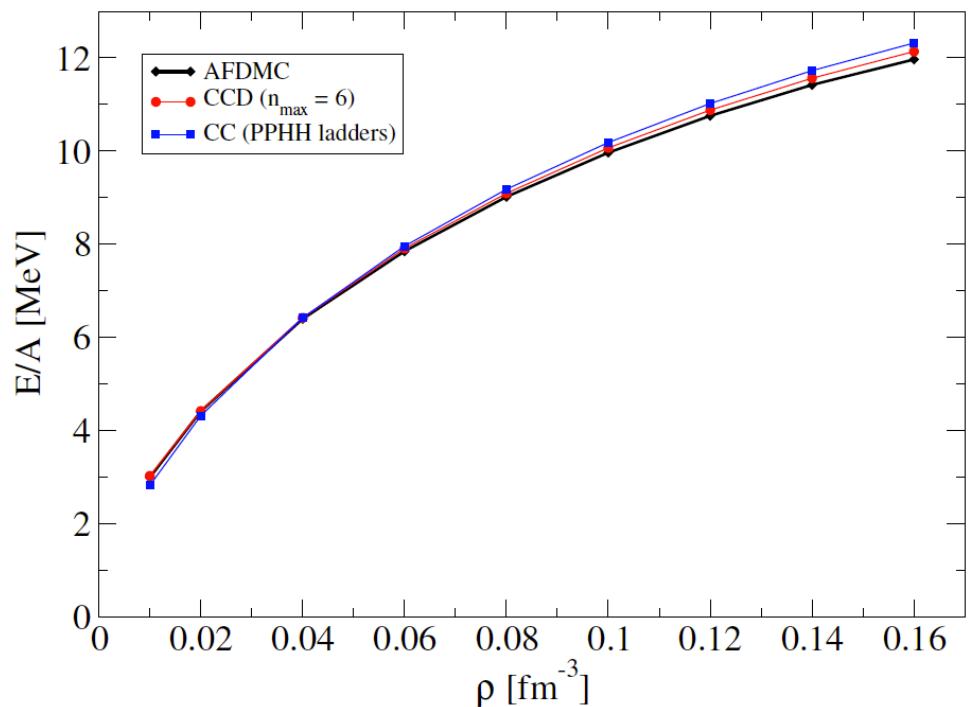
	NN	3N	4N
LO $\mathcal{O}\left(\frac{Q^0}{\Lambda^0}\right)$			—
NLO $\mathcal{O}\left(\frac{Q^2}{\Lambda^2}\right)$		—	—
N ² LO $\mathcal{O}\left(\frac{Q^3}{\Lambda^3}\right)$		—	—
N ³ LO $\mathcal{O}\left(\frac{Q^4}{\Lambda^4}\right)$		+ ...	+ ...

- developing higher orders and higher rank (3NF, 4NF) [Epelbaum 2006; Bernard et al 2007; Krebs et al 2012; Hebeler et al 2015; ...]
 - implemented in continuum and on lattice [Borasoy et al 2007]
 - local / non-local formulations [Gezerlis et al 2013]
 - propagation of uncertainties on horizon [Navarro Perez 2014]
 - different optimization protocols [Ekström et al 2013]
- Much improved understanding and handling via renormalization group transformations [Bogner et al 2003; Bogner et al 2007]

Benchmark calculations of neutron matter



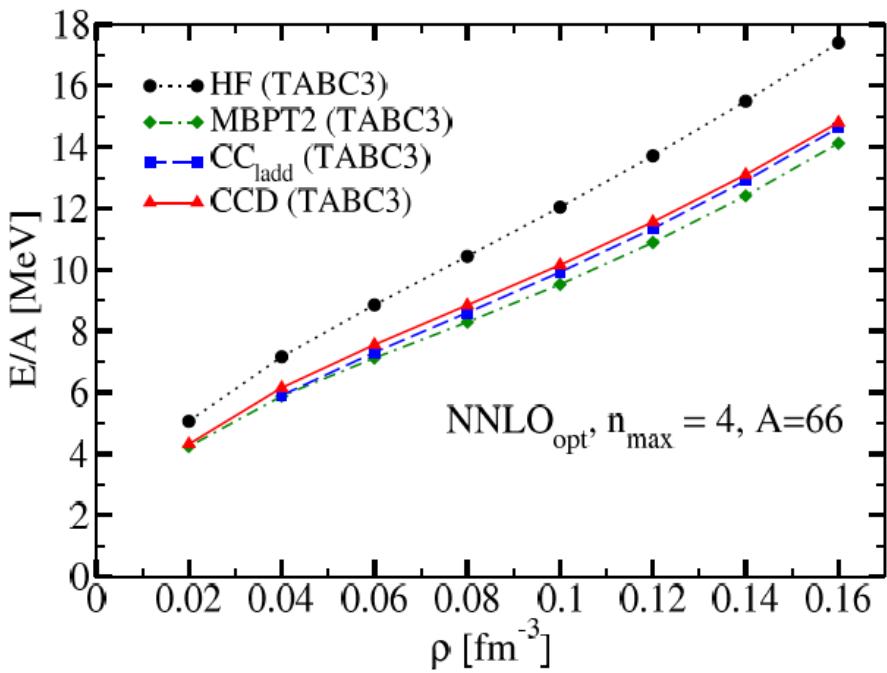
Comparison with continuum coupled-cluster method [Baardsen *et al.*, PRC (2013)]



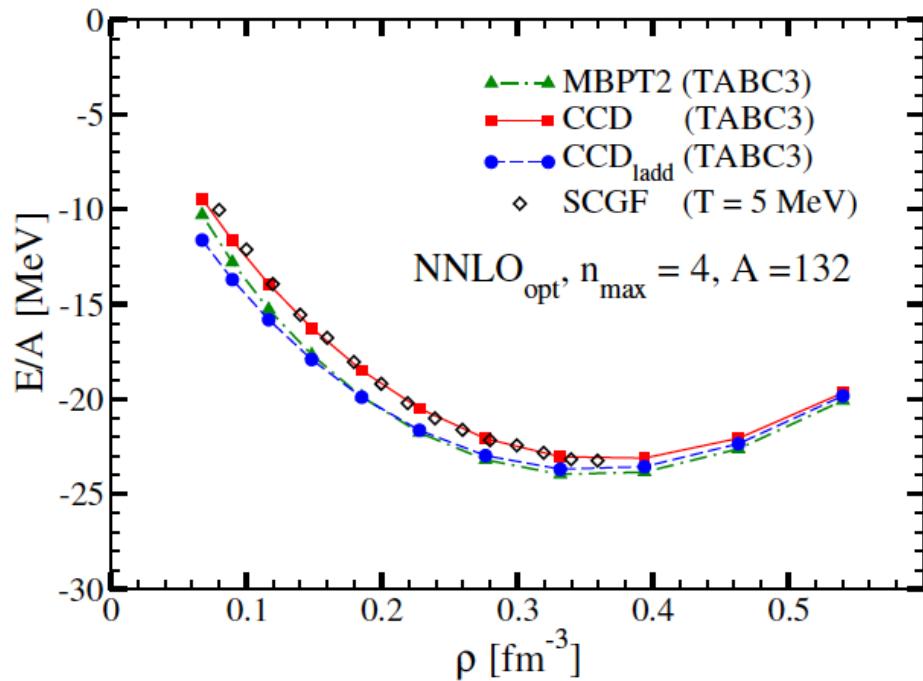
Comparison with auxiliary field diffusion Monte Carlo and Minnesota potential

Role of particle-hole excitations in nucleonic matter

Pure neutron matter (NN-only)



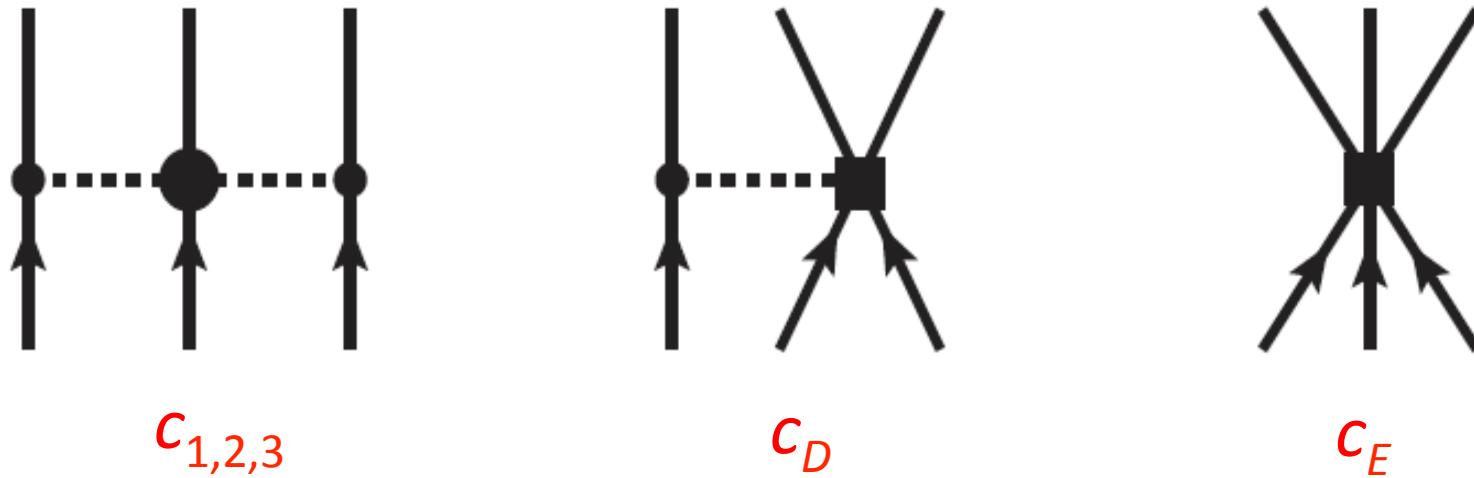
Symmetric nuclear matter (NN-only)



- Particle-hole and non-linear terms in CCD are small in pure neutron matter.
- MBPT2/ CCD_{ladd} /CCD results agree within 500keV/A. Indicates that PNM is perturbative
- Particle-hole and non-linear terms play a larger role in symmetric nuclear matter.
- CCD_{ladd} and CCD results differ by up to 1.5MeV/A around saturation density.

Three nucleon force (3NF) and regulator dependence

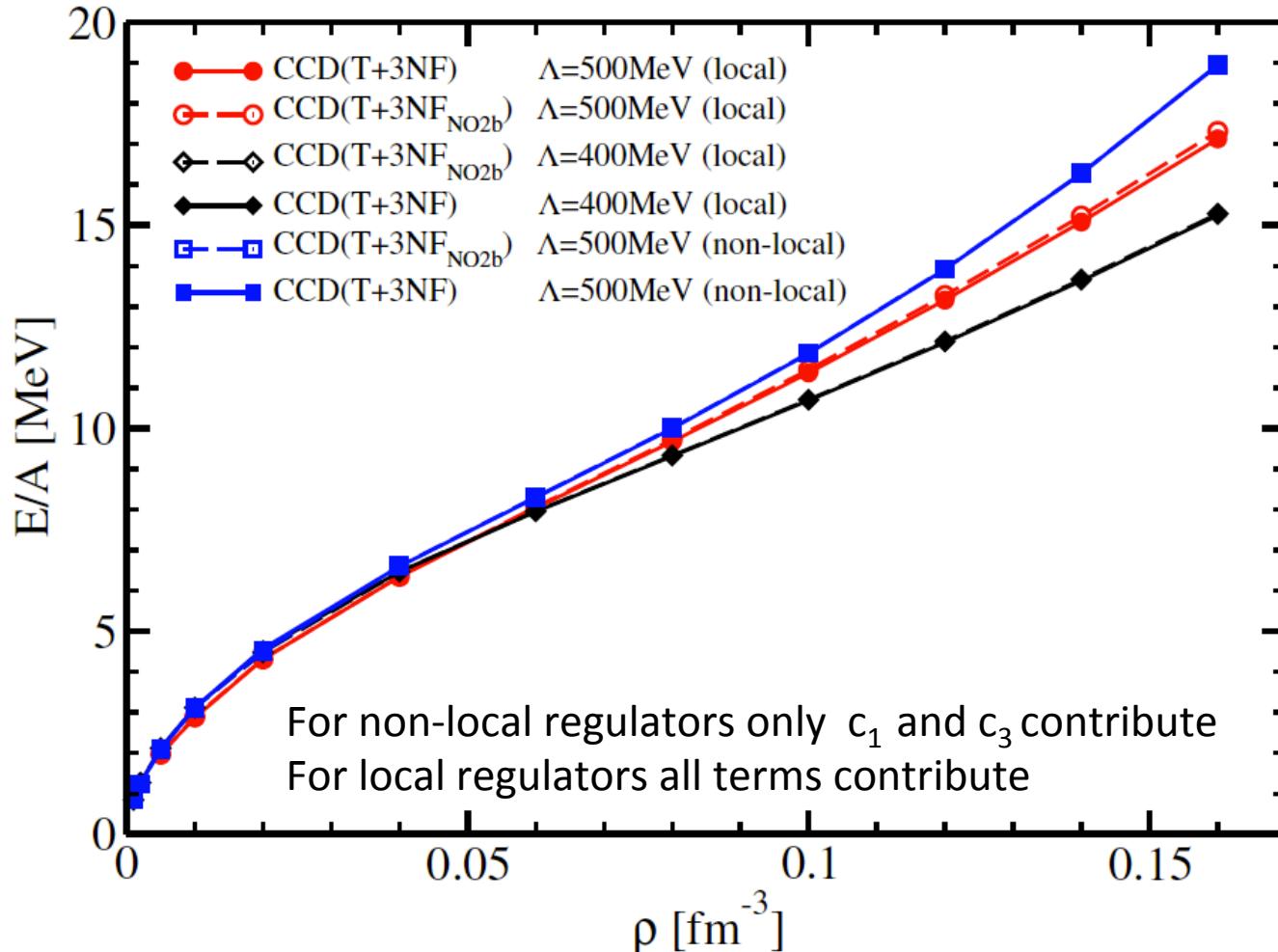
van Kolck, Phys. Rev. C 49, 2934 (1994); Epelbaum et al., Phys. Rev. C 66, 064001 (2002)



Nonlocal form of 3NF [Epelbaum et al. PRC (2002)]: Cutoff is in Jacobi momenta
 $\Lambda=500$ MeV: $c_D=-2$, $c_E=-0.791$ (from $A=3$ binding energies)

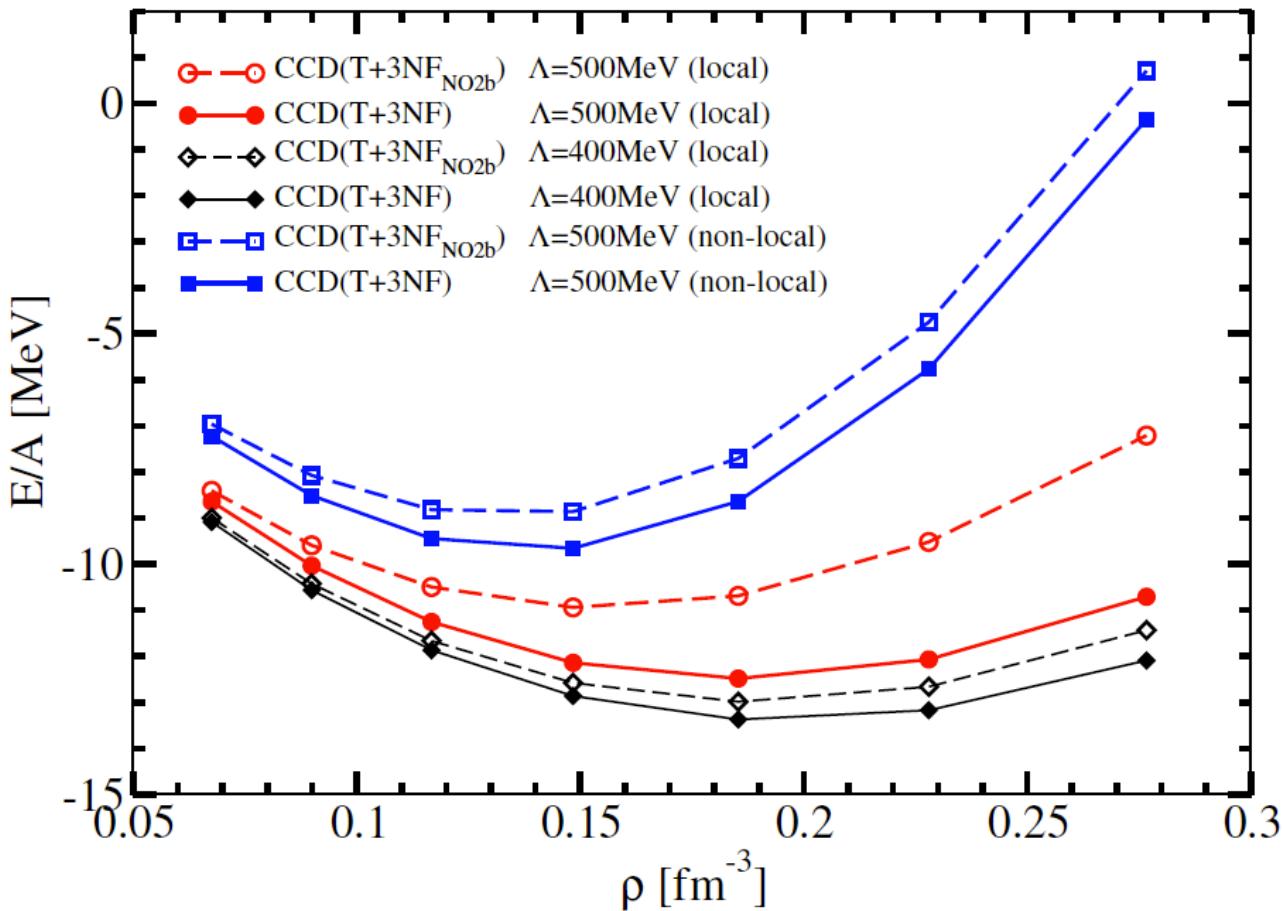
Local form of 3NF [Navratil, Few Body Syst. (2007)]: Cutoff is in the momentum transfer
 $\Lambda=500$ MeV: $c_D=-0.39$, $c_E=-0.389$ (from $A=3,4$ binding and ${}^3\text{H}$ $\tau_{1/2}$ (Gazit, Navratil, & Quaglioni))
 $\Lambda=400$ MeV: $c_D=-0.39$, $c_E=-0.27$ (adjusted to ${}^4\text{He}$)

Neutron matter



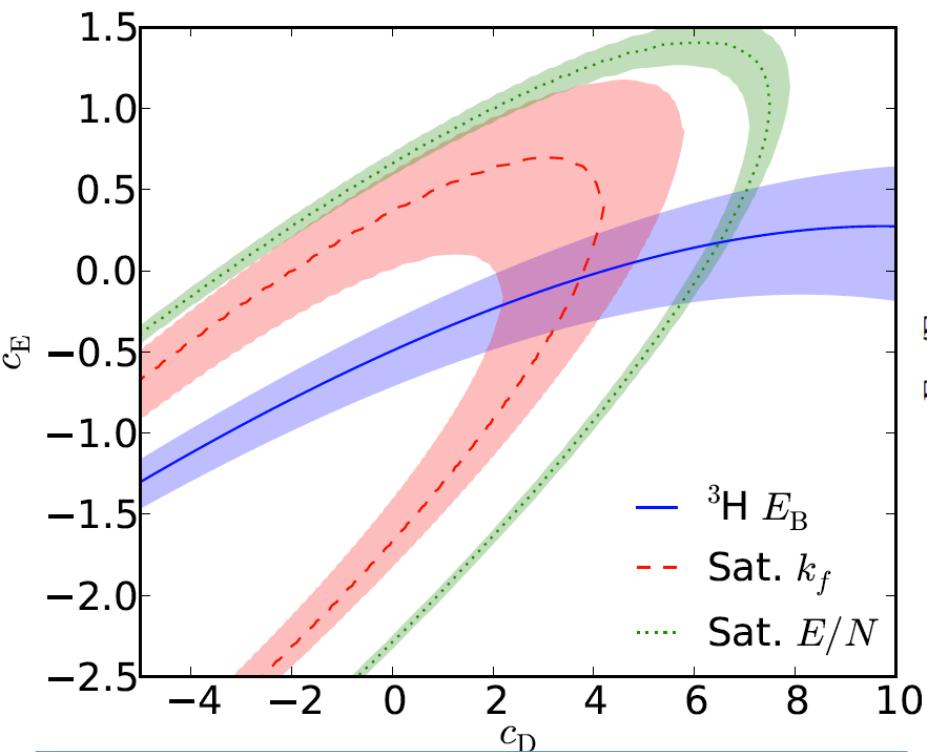
Neutron matter is perturbative (small differences between MBPT2 and coupled clusters)
3NFs act repulsively in neutron matter and NNLO_{opt}
Error bands from variation of cutoffs and level of sophistication in treating 3NFs

Symmetric nuclear matter



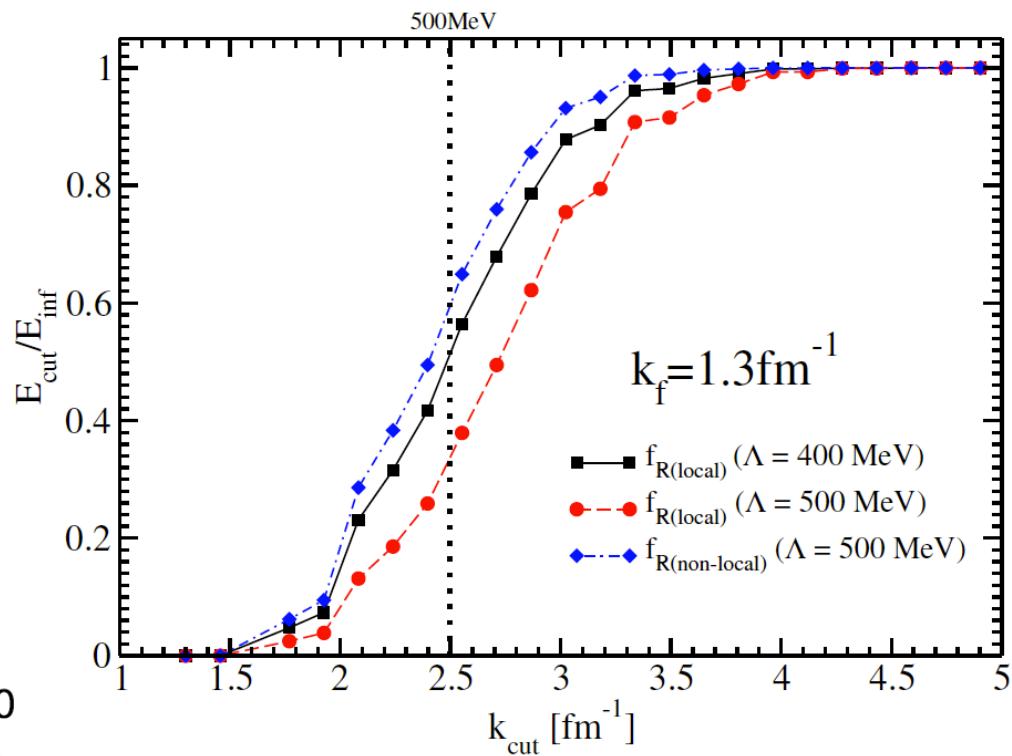
Nuclear matter is not perturbative (larger differences between MBPT2 and coupled clusters)
3NFs act repulsively in nuclear matter and NNLO_{opt}
⌚ Regularization scheme dependence of 3NF; sensitivity to sophistication in treatment of 3NFs

Understanding the 3NF at NNLO



5% error bands for saturation k_f , E/N and binding energy of ${}^3\text{H}$

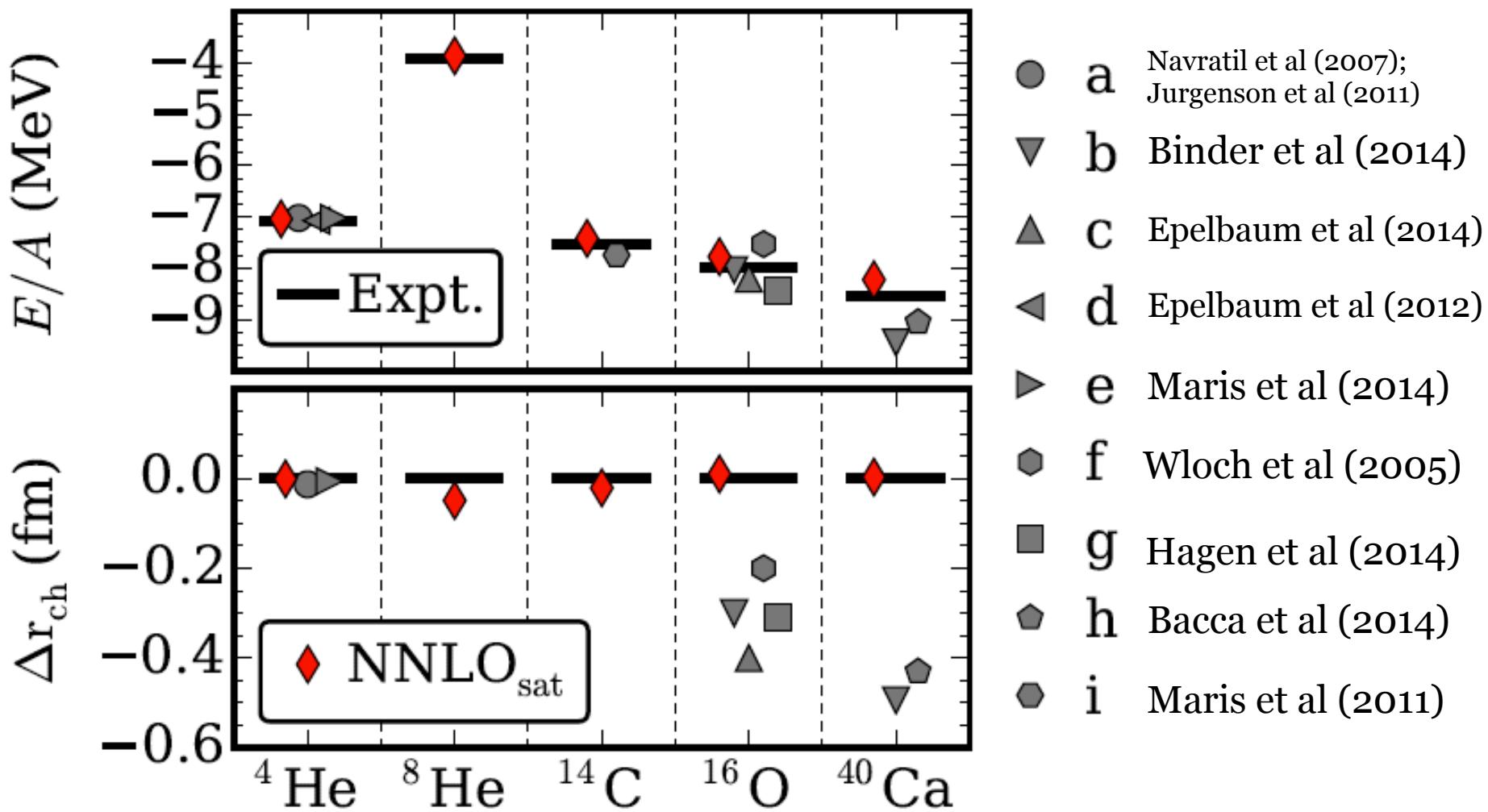
⌚ Variation of c_D and c_E not sufficient to simultaneously bind light nuclei and nuclear matter



Cutoff dependent fraction of residual 3NF contribution to MBPT2 energy per particle in SNM around saturation density.

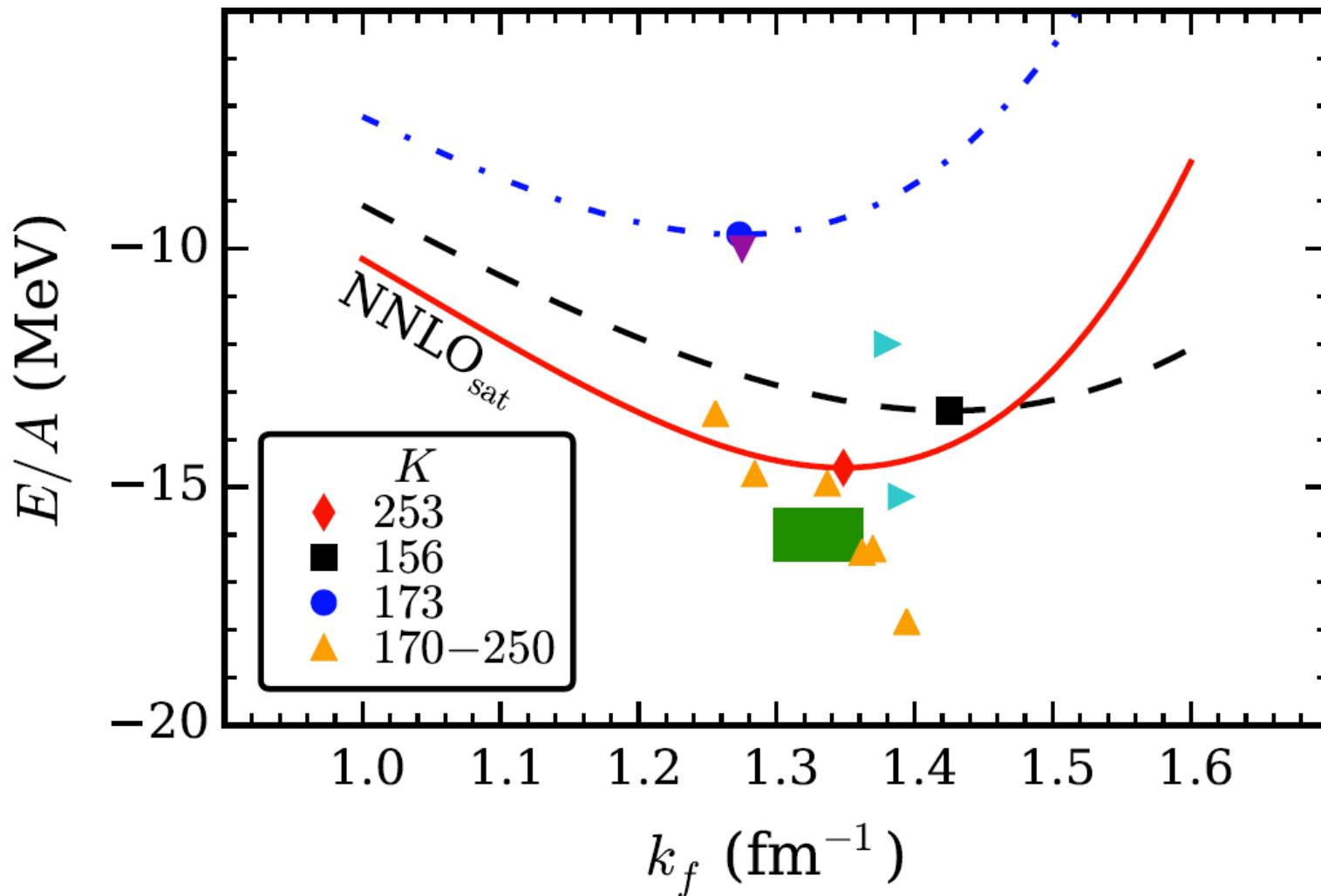
Local regulators converge slower than non-local regulators

Accurate nuclear binding energies and radii from a chiral interaction



Our solution: simultaneous optimization of NN and 3NFs with input from selected nuclei up to $A \sim 25$ ($NNLO_{sat}$). A. Ekström *et al*, Phys. Rev. C **91**, 051301(R) (2015)

Nuclear matter from NNLO_{sat}



Nuclear matter saturation curves for NNLO_{sat} and other interactions.
Hagen et al (2014); Carbone et al (2013); Coraggio et al 2014;
Hebeler et al PRC 2011.