The beloved commutator gym

Morten's brute force approach

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We have defined the following unperturbed Hamiltonian

$$\hat{H}_0 = \xi \sum_{p,\sigma} (p-1) \hat{a}^\dagger_{p\sigma} \hat{a}_{p\sigma}, \tag{1}$$

and interacting part as

$$\hat{V} = -\frac{1}{2} g \sum_{p,q} \hat{a}^\dagger_{p+} \hat{a}^\dagger_{p-} \hat{a}_{q-} \hat{a}_{q+}, \tag{2}$$

with the full Hamiltonian being given by $\hat{H} = \hat{H}_0 + \hat{V}$. The spin operator is defined as

$$\hat{S}_z = \frac{1}{2} \sum_{p,\sigma} \sigma \hat{a}^\dagger_{p\sigma} \hat{a}_{p\sigma} \tag{3}$$

$$\hat{S}^2 = \hat{S}_z^2 + \frac{1}{2} \left(\hat{S}_+ \hat{S}_- + \hat{S}_- \hat{S}_+ \right) \tag{4}$$

$$\hat{S}_\pm = \sum_p \hat{a}^\dagger_{p\pm} \hat{a}_{p\mp}. \tag{5}$$

We define the pair creation and annihilation operators $\hat{P}^+_p$ and $\hat{P}^-_p$ as

$$\hat{P}^+_p = \hat{a}^\dagger_{p+} \hat{a}^\dagger_{p-}, \qquad \hat{P}^-_p = \hat{a}_{p-} \hat{a}_{p+}. \tag{6}$$

Yes, I did code all this Latex stuff, don't ask me why

We compute first the commutator between $\hat{H}_0$ and $\hat{S}_z$ using (1) and (3):

$$\begin{aligned}
\left[\hat{H}_0, \hat{S}_z\right] &= \left[\xi \sum_{p,\sigma} (p-1) \hat{a}^\dagger_{p\sigma} \hat{a}_{p\sigma}, \frac{1}{2} \sum_{p,\sigma} \sigma \hat{a}^\dagger_{p\sigma} \hat{a}_{p\sigma} \right] \\
&= \frac{\xi}{2} \sum_{p,\sigma} \sum_{p',\sigma'} (p-1) \sigma' \left[a^\dagger_{p\sigma} \hat{a}_{p\sigma}, \hat{a}^\dagger_{p'\sigma'} \hat{a}_{p'\sigma'} \right].
\end{aligned}$$

We continue

$$\begin{aligned}
\left[a^{\dagger}_{p\sigma}\hat{a}_{p\sigma}, a^{\dagger}_{p'\sigma'}\hat{a}_{p'\sigma'}\right] &= a^{\dagger}_{p\sigma}\hat{a}_{p\sigma}\hat{a}^{\dagger}_{p'\sigma'}\hat{a}_{p'\sigma'} - a^{\dagger}_{p'\sigma'}\hat{a}_{p'\sigma'}\hat{a}^{\dagger}_{p\sigma}\hat{a}_{p\sigma} \\
&= a^{\dagger}_{p\sigma}\hat{a}_{p\sigma}\hat{a}^{\dagger}_{p'\sigma'}\hat{a}_{p'\sigma'} - a^{\dagger}_{p'\sigma'}\left(\delta_{pp'}\delta_{\sigma\sigma'} - \hat{a}^{\dagger}_{p\sigma}\hat{a}_{p'\sigma'}\right)\hat{a}_{p\sigma} \\
&= a^{\dagger}_{p\sigma}\hat{a}_{p\sigma}\hat{a}^{\dagger}_{p'\sigma'}\hat{a}_{p'\sigma'} - a^{\dagger}_{p'\sigma'}\hat{a}_{p\sigma}\delta_{pp'}\delta_{\sigma\sigma'} + \hat{a}^{\dagger}_{p\sigma}a^{\dagger}_{p'\sigma'}\hat{a}_{p\sigma}\hat{a}_{p'\sigma'} \\
&= a^{\dagger}_{p\sigma}\hat{a}_{p\sigma}\hat{a}^{\dagger}_{p'\sigma'}\hat{a}_{p'\sigma'} - a^{\dagger}_{p'\sigma'}\hat{a}_{p\sigma}\delta_{pp'}\delta_{\sigma\sigma'} + \hat{a}^{\dagger}_{p\sigma}\left(\delta_{pp'}\delta_{\sigma\sigma'} - \hat{a}_{p\sigma}a^{\dagger}_{p'\sigma'}\right)\hat{a}_{p'\sigma'} \\
&= a^{\dagger}_{p'\sigma'}\hat{a}_{p\sigma}\delta_{pp'}\delta_{\sigma\sigma'} - a^{\dagger}_{p\sigma}\hat{a}_{p'\sigma'}\delta_{pp'}\delta_{\sigma\sigma'}.
\end{aligned}$$

This gives

$$\begin{aligned}
\left[\hat{H}_0, \hat{S}_z\right] &= \frac{\xi}{2}\sum_{p,\sigma}\sum_{p',\sigma'}(p-1)\sigma'\left(a^{\dagger}_{p'\sigma'}\hat{a}_{p\sigma}\delta_{pp'}\delta_{\sigma\sigma'} - a^{\dagger}_{p\sigma}\hat{a}_{p'\sigma'}\delta_{pp'}\delta_{\sigma\sigma'}\right) \\
&= \frac{\xi}{2}\sum_{p,\sigma}(p-1)\sigma\left(a^{\dagger}_{p\sigma}\hat{a}_{p\sigma} - a^{\dagger}_{p\sigma}\hat{a}_{p\sigma}\right) \\
&= 0.
\end{aligned}$$

Then we compute the commutator between $\hat{V}$ and $\hat{S}_z$, using $\hat{V}$ from Eq. (2):

$$\begin{aligned}
\left[\hat{V}, \hat{S}_z\right] &= \left[-\frac{1}{2}g\sum_{p,q}\hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}, \frac{1}{2}\sum_{p,\sigma}\sigma\hat{a}^{\dagger}_{p\sigma}\hat{a}_{p\sigma}\right] \\
&= -\frac{1}{4}g\sum_{p,q}\sum_{p',\sigma}\sigma\left[\hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}, \hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma}\right].
\end{aligned}$$

Using the commutation relation between the creation and annihilation operators we obtain

$$\begin{aligned}
\left[\hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}, \hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma}\right] &= \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma} - \hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma}\hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+} \\
&= \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma} - \hat{a}^{\dagger}_{p'\sigma}\left(\delta_{pp'}\delta_{\sigma+} - \hat{a}^{\dagger}_{p+}\hat{a}_{p'\sigma}\right)\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+} \\
&= \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma} - \hat{a}^{\dagger}_{p'\sigma}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma+} - \\
&\quad \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+} \\
&= \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma} - \hat{a}^{\dagger}_{p'\sigma}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma+} - \\
&\quad \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p'\sigma}\left(\delta_{pp'}\delta_{\sigma-} - \hat{a}^{\dagger}_{p-}\hat{a}_{p'\sigma}\right)\hat{a}_{q-}\hat{a}_{q+} \\
&= \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma} - \hat{a}^{\dagger}_{p'\sigma}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma+} - \\
&\quad \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma-} - \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{q-}\hat{a}_{q+}\hat{a}_{p'\sigma} \\
&= \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma} - \hat{a}^{\dagger}_{p'\sigma}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma+} - \\
&\quad \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma-} - \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\left(\delta_{qp'}\delta_{\sigma-} - \hat{a}_{q-}\hat{a}^{\dagger}_{p'\sigma}\right)\hat{a}_{q+}\hat{a}_{p'\sigma} \\
&= \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{p'\sigma} - \hat{a}^{\dagger}_{p'\sigma}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma+} - \\
&\quad \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma-} - \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q+}\hat{a}_{p'\sigma}\delta_{qp'}\delta_{\sigma-} + \\
&\quad \hat{a}^{\dagger}_{p+}\hat{a}^{\dagger}_{p-}\hat{a}_{q-}\hat{a}^{\dagger}_{p'\sigma}\hat{a}_{q+}\hat{a}_{p'\sigma}.
\end{aligned}$$

Only one change is needed now

$$\begin{aligned}\left[\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q+},\hat{a}^\dagger_{p'\sigma}\hat{a}_{p'\sigma}\right] &= -\hat{a}^\dagger_{p'\sigma}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma+} - \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p'\sigma}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma-} - \\ &\quad \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q+}\hat{a}_{p'\sigma}\delta_{qp'}\delta_{\sigma-} + \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{p'\sigma}\delta_{qp'}\delta_{\sigma+} \\ &= \hat{a}^\dagger_{p-}\hat{a}^\dagger_{p'\sigma}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma+} - \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p'\sigma}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma-} - \\ &\quad \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q+}\hat{a}_{p'\sigma}\delta_{qp'}\delta_{\sigma-} + \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{p'\sigma}\delta_{qp'}\delta_{\sigma+}.\end{aligned}$$

If we then insert this in $\hat{V}$ and $\hat{S}_z$ above we obtain

$$\begin{aligned}\left[\hat{H}_0,\hat{S}_z\right] &= -\frac{1}{4}g\sum_{p,q}\sum_{p',\sigma}\sigma\left(\hat{a}^\dagger_{p-}\hat{a}^\dagger_{p'\sigma}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma+} - \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p'\sigma}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'}\delta_{\sigma-} - \right. \\ &\quad \left. \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q+}\hat{a}_{p'\sigma}\delta_{qp'}\delta_{\sigma-} + \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{p'\sigma}\delta_{qp'}\delta_{\sigma+}\right) \\ &= -\frac{1}{4}g\sum_{p,q}\left(\hat{a}^\dagger_{p-}\hat{a}^\dagger_{p+}\hat{a}_{q-}\hat{a}_{q+} - \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q+} - \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q+}\hat{a}_{q-} + \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q+}\right) \\ &= 0.\end{aligned}$$

Then we move to the commutator between $\hat{H}_0$ and $\hat{S}^2$. Using Eqs. (1), (3), (4) and (5):

$$\left[\hat{H}_0,\hat{S}^2\right] = \left[\hat{H}_0,\hat{S}_z^2+\frac{1}{2}\left(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\right)\right] = \frac{1}{2}\left(\left[\hat{H}_0,\hat{S}_+\hat{S}_-\right] + \left[\hat{H}_0,\hat{S}_-\hat{S}_+\right]\right),$$

where we have used the fact that $\hat{H}_0$ commutes with $\hat{S}_z$ and thereby $\hat{S}_z^2$. The other operators give rise to

$$\begin{aligned}\left[\hat{H}_0,\hat{S}_+\hat{S}_-\right] &= \left[\xi\sum_{p,\sigma}(p-1)\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma},\left(\sum_p \hat{a}^\dagger_{p+}\hat{a}_{p-}\right)\left(\sum_p \hat{a}^\dagger_{p-}\hat{a}_{p+}\right)\right] \\ &= \xi\sum_{p,\sigma}\sum_{p',p''}(p-1)\left[\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma},\hat{a}^\dagger_{p'+}\hat{a}_{p'-}\hat{a}^\dagger_{p''-}\hat{a}_{p''+}\right].\end{aligned}$$

We use again the commutation relation between creation and annihilation operators and find

$$\begin{aligned}\left[\hat{H}_0,\hat{S}_+\hat{S}_-\right] &= -\xi\sum_{p,\sigma}\sum_{p',p''}(p-1)\left[\hat{a}^\dagger_{p'+}\hat{a}_{p'-}\hat{a}^\dagger_{p''-}\hat{a}_{p''+},\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma}\right] \\ &= -\xi\sum_{p,\sigma}\sum_{p',p''}(p-1)\left(\hat{a}^\dagger_{p'-}\hat{a}^\dagger_{p\sigma}\hat{a}_{p''-}\hat{a}_{p''+}\delta_{pp'}\delta_{\sigma+} - \hat{a}^\dagger_{p'+}\hat{a}^\dagger_{p\sigma}\hat{a}_{p''-}\hat{a}_{p''+}\delta_{pp'}\delta_{\sigma-} - \right. \\ &\quad \left. \hat{a}^\dagger_{p'+}\hat{a}^\dagger_{p'-}\hat{a}_{p''+}\hat{a}_{p\sigma}\delta_{pp''}\delta_{\sigma-} + \hat{a}^\dagger_{p'+}\hat{a}^\dagger_{p'-}\hat{a}_{p''-}\hat{a}_{p\sigma}\delta_{pp''}\delta_{\sigma+}\right) \\ &= -\xi\sum_{p',p''}(p-1)\left(\hat{a}^\dagger_{p'-}\hat{a}^\dagger_{p'+}\hat{a}_{p''-}\hat{a}_{p''+} - \hat{a}^\dagger_{p'+}\hat{a}^\dagger_{p'-}\hat{a}_{p''-}\hat{a}_{p''+} - \right. \\ &\quad \left. \hat{a}^\dagger_{p'+}\hat{a}^\dagger_{p'-}\hat{a}_{p''+}\hat{a}_{p''-} + \hat{a}^\dagger_{p'+}\hat{a}^\dagger_{p'-}\hat{a}_{p''-}\hat{a}_{p''+}\right) \\ &= 0.\end{aligned}$$

We proceed then with the commutator between $\hat{H}_0$ and $\hat{S}_-\hat{S}_+$. Here we need

$$\left[\hat{S}_+, \hat{S}_-\right] = 2\hat{S}_z \quad \Rightarrow \quad \hat{S}_-\hat{S}_+ = \hat{S}_+ S_- - 2\hat{S}_z.$$

Using this relation and the fact that $\hat{S}_+\hat{S}_-$ commutes with $\hat{H}_0$, we can find the commutator between $\hat{H}_0$ and $\hat{S}^2$:

$$\begin{aligned}
\left[\hat{H}_0, \hat{S}^2\right] &= \frac{1}{2}\left(\left[\hat{H}_0, \hat{S}_+\hat{S}_-\right] + \left[\hat{H}_0, \hat{S}_-\hat{S}_+\right]\right) \\
&= \frac{1}{2}\left(\left[\hat{H}_0, \hat{S}_+ S_- - \hat{S}_z\right]\right) \\
&= 0,
\end{aligned}$$

where we have also used that $\hat{H}_0$ commutes with $\hat{S}_z$.

Finally, we need to find the corresponding commutator between $\hat{V}$ and $\hat{S}^2$. Using Eq. (4), and that $\hat{S}_z$ og $\hat{S}_z^2$ commute with $\hat{V}$ and

$$\begin{aligned}
\left[\hat{V}, \hat{S}^2\right] &= \left[\hat{V}, \hat{S}_z^2 + \frac{1}{2}\left(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+\right)\right] = \left[\hat{V}, \hat{S}_z^2 + \hat{S}_+\hat{S}_- - \hat{S}_z\right] \\
&= \left[V, \hat{S}_+\hat{S}_-\right] = \left(\left[V, \hat{S}_+\right]\hat{S}_- + \hat{S}_+\left[\hat{V}, \hat{S}_-\right]\right),
\end{aligned}$$

combined with $\hat{S}_- = \hat{S}_+^\dagger$ and $\hat{V}^\dagger = \hat{V}$, we find

$$\left[\hat{V}, \hat{S}^2\right] = \left[V, \hat{S}_+\right]\hat{S}_- - \hat{S}_+\left[\hat{V}, \hat{S}_+\right]^\dagger = \left[V, \hat{S}_+\right]\hat{S}_- - \left(\left[V, \hat{S}_+\right]\hat{S}_-\right)^\dagger.$$

We need to find $[V, \hat{S}_+]$. Inserting for $\hat{V}$ using Eq. (2) and for $\hat{S}_+$ from Eq. (5) we can calculate

$$\left[\hat{V}, \hat{S}_+\right] = \left[-\frac{1}{2}g\sum_{p,q}\hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+}, \sum_p \hat{a}_{p+}^\dagger\hat{a}_{p-}\right] = -\frac{1}{2}g\sum_{p,p',q}\left[\hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+}, \hat{a}_{p'+}^\dagger\hat{a}_{p'-}\right].$$

Let us then look at the commutator between the creation and annihilation operators

$$\begin{aligned}
\left[\hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+}, \hat{a}_{p'+}^\dagger\hat{a}_{p'-}\right] &= \hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+}\hat{a}_{p'+}^\dagger\hat{a}_{p'-} - \hat{a}_{p'+}^\dagger\hat{a}_{p'-}\hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+} \\
&= \hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+}\hat{a}_{p'+}^\dagger\hat{a}_{p'-} - \hat{a}_{p+}^\dagger\hat{a}_{p'+}^\dagger\hat{a}_{p'-}\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+} \\
&= \hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+}\hat{a}_{p'+}^\dagger\hat{a}_{p'-} - \hat{a}_{p+}^\dagger\hat{a}_{p'+}^\dagger\left(\delta_{pp'} - \hat{a}_{p-}^\dagger\hat{a}_{p'-}\right)\hat{a}_{q-}\hat{a}_{q+} \\
&= \hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+}\hat{a}_{p'+}^\dagger\hat{a}_{p'-} - \hat{a}_{p+}^\dagger\hat{a}_{p'+}^\dagger\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'} + \\
&\quad \hat{a}_{p+}^\dagger\hat{a}_{p'+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{p'-}\hat{a}_{q-}\hat{a}_{q+} \\
&= \hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{q+}\hat{a}_{p'+}^\dagger\hat{a}_{p'-} - \hat{a}_{p+}^\dagger\hat{a}_{p'+}^\dagger\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'} + \\
&\quad \hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{p'+}^\dagger\hat{a}_{q+}\hat{a}_{p'-} \\
&= -\hat{a}_{p+}^\dagger\hat{a}_{p'+}^\dagger\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'} + \hat{a}_{p+}^\dagger\hat{a}_{p-}^\dagger\hat{a}_{q-}\hat{a}_{p'-}\delta_{qp'}.
\end{aligned}$$

Inserting back in $[\hat{V}, \hat{S}_+]$:

$$\left[\hat{V}, \hat{S}_+\right] = -\frac{1}{2}g \sum_{p,p',q} \left(\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{p'-}\delta_{qp'} - \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p'+}\hat{a}_{q-}\hat{a}_{q+}\delta_{pp'} \right)$$
$$= -\frac{1}{2}g \sum_{p,q} \left(\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q-} + \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p+}\hat{a}_{q+}\hat{a}_{q-} \right).$$

We insert then this expression in $[\hat{V}, \hat{S}^2]$ and employing Eq. (5) again for for $S_\pm$ we arrive at

$$\begin{aligned}\left[\hat{V}, \hat{S}^2\right] = -\frac{g}{2} \sum_{p,p',q} \Big(&\left[\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q-} + \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p+}\hat{a}_{q+}\hat{a}_{q-}\right]\hat{a}^\dagger_{p'-}\hat{a}_{p'+} - \\ &\hat{a}^\dagger_{p'+}\hat{a}_{p'-}\left[\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q-} + \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p+}\hat{a}_{q+}\hat{a}_{q-}\right]^\dagger\Big) \\ = -\frac{g}{2} \sum_{p,p',q} \Big(&\left[\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q-} + \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p+}\hat{a}_{q+}\hat{a}_{q-}\right]\hat{a}^\dagger_{p'-}\hat{a}_{p'+} - \\ &\hat{a}^\dagger_{p'+}\hat{a}_{p'-}\left[\hat{a}^\dagger_{q-}\hat{a}^\dagger_{q-}\hat{a}_{p-}\hat{a}_{p+} + \hat{a}^\dagger_{q-}\hat{a}^\dagger_{q+}\hat{a}_{p+}\hat{a}_{p+}\right]\Big).\end{aligned}$$

Further manipulations result in

$$\begin{aligned}\left[\hat{V}, \hat{S}^2\right] = -\frac{g}{2} \sum_{p,p',q} \Big(&\left[\hat{a}^\dagger_{p+}\hat{a}^\dagger_{p-}\hat{a}_{q-}\hat{a}_{q-} + \hat{a}^\dagger_{p+}\hat{a}^\dagger_{p+}\hat{a}_{q+}\hat{a}_{q-}\right]\hat{a}^\dagger_{p'-}\hat{a}_{p'+} - \\ &\hat{a}^\dagger_{p'+}\hat{a}_{p'-}\left[\hat{a}^\dagger_{q-}\hat{a}^\dagger_{q-}\hat{a}_{p-}\hat{a}_{p+} + \hat{a}^\dagger_{q-}\hat{a}^\dagger_{q+}\hat{a}_{p+}\hat{a}_{p+}\right]\Big).\end{aligned}$$

A commutator between two operators is also an operator. When such an operator acts on a state we see that

$$\left[\hat{V}, \hat{S}^2\right] = 0.$$

It is straightforward to see from Eq. (6) that the interaction of Eq. (2) can be written as

$$\hat{V} = -\frac{1}{2}g \sum_{p,q} \hat{P}^+_p \hat{P}^-_q,$$

and since $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ is given by Eq. (1), we have

$$\hat{H} = \sum_{p,\sigma}(p-1)\hat{a}^\dagger_{p\sigma}\hat{a}_{p\sigma} - \frac{1}{2}g \sum_{p,q} \hat{P}^+_p \hat{P}^-_q, \tag{7}$$

with $\xi = 1$.

The Hamiltonian matrix

The aim here is to compute the Hamiltonian matrix for a system with no broken pairs. This means that every level p contains two particles, one with spin up and one with spin down. We employ the basis

$$|\Phi_0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\Phi_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \dots \quad |\Phi_5\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

where

$$\begin{aligned}
|\Phi_0\rangle &= \hat{a}^\dagger_{2+}\hat{a}^\dagger_{2-}\hat{a}^\dagger_{1+}\hat{a}^\dagger_{1-}|0\rangle = \hat{P}_2^+\hat{P}_1^+|0\rangle, \\
|\Phi_1\rangle &= \hat{P}_3^+\hat{P}_1^+|0\rangle, \\
|\Phi_2\rangle &= \hat{P}_4^+\hat{P}_1^+|0\rangle, \\
|\Phi_3\rangle &= \hat{P}_3^+\hat{P}_2^+|0\rangle, \\
|\Phi_4\rangle &= \hat{P}_4^+\hat{P}_2^+|0\rangle, \\
|\Phi_5\rangle &= \hat{P}_4^+\hat{P}_3^+|0\rangle.
\end{aligned}$$

We are now going to compute $\langle\Phi_i|\hat{H}|\Phi_j\rangle$ using the Hamiltonian of Eq. (7). The one-body operator acts only on the diagonal and results in terms proportional with $(p-1)$. The interaction will excite or deexcite a pair of particles from level q to level p. Using this it is easy to see that the Hamilotnian matrix becomes

$$\hat{H} = \begin{pmatrix}
2-g & -g/2 & -g/2 & -g/2 & -g/2 & 0 \\
-g/2 & 4-g & -g/2 & -g/2 & 0 & -g/2 \\
-g/2 & -g/2 & 6-g & 0 & -g/2 & -g/2 \\
-g/2 & -g/2 & 0 & 6-g & -g/2 & -g/2 \\
-g/2 & 0 & -g/2 & -g/2 & 8-g & -g/2 \\
0 & -g/2 & -g/2 & -g/2 & -g/2 & 10-g
\end{pmatrix}.$$

This matrix can easily be diagonalized using for example Octave or Matlab or python.