# DTP/TALENT 2025: Exercises for first week

June 16, 2025

In this first hands-on session the main goal is to familiarize yourself with how to write quantum circuits and run them. Some of the examples we'll see use a software package named Qiskit developed by Ibold but there are other options, like Google's Cirq. For those of you new to quantum circuits you can find an introductory notebook *1_getting_started_with_qiskit.ipynb* in the *Exercises* folder.

## 1 First exercise set

The exercises we present each week are meant to build the basis for the possible project we will work. See in particular the third lecture today.

### 1.1 Ex1: One-qubit basis and Pauli matrices

Write a function which sets up a one-qubit basis and apply the various Pauli matrices to these basis states.

### 1.2 Ex2: Hadamard and Phase gates

Apply the Hadamard and Phase gates to the same one-qubit basis states and study their actions on these states.

### 1.3 Ex3: Traces of operators

Prove that the trace is cyclic, that is for three operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, we have

$$\mathrm{Tr}\{\mathbf{ABC}\} = \mathrm{Tr}\{\mathbf{CAB}\} = \mathrm{Tr}\{\mathbf{BCA}\}.$$

### 1.4 Ex4: Exponentiated operators

Let $\mathbf{A}$ be an operator on a vector space satisfying $\mathbf{A}^2 = 1$ and $\alpha$ any real constant. Show that

$$\exp\{\imath\alpha\mathbf{A}\} = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!}\mathbf{A}^n = \mathbf{I}\cos\alpha + \imath\mathbf{A}\sin\alpha.$$

Does this apply to the Pauli matrices?

## 1.5 Ex5: Hamiltonians rewritten in terms of simple Pauli matrices

We consider a simple $2 \times 2$ real Hamiltonian consisting of a diagonal part $H_0$ and off-diagonal part $H_I$, playing the roles of a non-interactive one-body and interactive two-body part respectively. Defined through their matrix elements, we express them in the Pauli basis $|0\rangle$ and $|1\rangle$

$$H = H_0 + H_I$$
$$H_0 = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \qquad H_I = \lambda \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

Where $\lambda \in [0, 1]$ is a coupling constant parameterizing the strength of the interaction.

## 1.6 Rewriting in terms of Pauli matrices

Define

$$E_+ = \frac{E_1 + E_2}{2}, \qquad E_- = \frac{E_1 - E_2}{2}$$

show that by combining the identity and $Z$ Pauli matrix, this can be expressed as

$$H_0 = E_+ I + E_- Z$$

## 1.7 The interaction part

For $H_1$ we use the same trick to fill the diagonal, defining $V_+ = (V_{11}+V_{22})/2, V_- = (V_{11}-V_{22})/2$. From the hermiticity requirements of $H$, we note that $V_{12} = V_{21} \equiv V_o$. Use this to simplify the problem to a simple $X$ term.

$$H_I = V_+ I + V_- Z + V_o X$$

## 1.8 Measurement basis

For the above system show that the Pauli $X$ matrix can be rewritten in terms of the Hadamard matrices and the Pauli $Z$ matrix, that is

$$X = HZH.$$

## 1.9 More exercises

1. Write a circuit that prepares the single qubit state

$$|\Psi(\theta)\rangle = \cos(\theta) |0\rangle + \sin(\theta) |1\rangle \ , \tag{1}$$

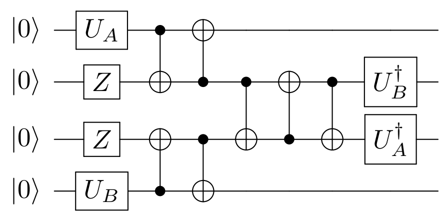
and measures the expectation value of $Z$ on the resulting state.

- you can use a $R_y$ rotation to perform this (see here for definitions)
- perform the calculation in two ways: first by extracting the full *state_vector* and then performing a fixed number of shots and reconstructing the expectation value from those
- it might also be helpful to implement everything directly in NumPy in order to have a reference value to compare to
- make a plot as a function of $\theta$ of the function

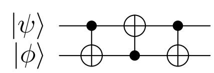
$$\langle Z(\theta)\rangle = \langle \Psi(\theta)|Z|\Psi(\theta)\rangle \ , \tag{2}$$

  extracted in the various ways.

2. Try to show that the following circuit gives the same final state regardless on the choice of the single-qubit gates $U_A$ and $U_B$




- try to choose a simple pair of gates $U_A$ and $U_B$ so that you can find the final state easily. What do you get?
- [HINT] in order to understand what is happening you might want to consider the following sub-circuit




  where $|\psi\rangle$ and $|\phi\rangle$ are generic single-qubit states. What is this doing?

3. Quantum teleportation is an interesting communication protocol where information is transmitted between a sender, we'll call her Sandy, and a receiver, will call him Rick, through a combination of entanglement and classical communication. The protocol works as follows:

- Sandy has one qubit, we'll call $B_\Psi$, in a state that she wants to send to Rick. For this example we'll use $|\Psi(\theta)\rangle$ from Eq. (1) above
- before the protocol began, Sandy and Rick prepare a two qubit Bell state $|B\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. By keeping one quibit each, let's call $B_S$ and $B_R$, they can now use as a resource for communication.
- the protocol now proceeds as follows:
  (a) Sandy entangles her qubit $B_\Psi$ with her part of the Bell state on qubit $B_S$. The total state on qubits $(B_\Psi, B_S, B_R)$ is then

$$\frac{\cos(\theta)}{\sqrt{2}}\,|0\rangle \otimes (|00\rangle + |11\rangle) + \frac{\sin(\theta)}{\sqrt{2}}\,|1\rangle \otimes (|10\rangle + |01\rangle) \tag{3}$$

(b) she applies an Hadamard gate to her qubit $B_\Psi$ obtaining

$$\frac{|00\rangle}{2} \otimes (\cos(\theta)|0\rangle + \sin(\theta)|1\rangle) + \frac{|01\rangle}{2} \otimes (\cos(\theta)|1\rangle + \sin(\theta)|0\rangle)$$
$$\frac{|10\rangle}{2} \otimes (\cos(\theta)|0\rangle - \sin(\theta)|1\rangle) + \frac{|11\rangle}{2} \otimes (\cos(\theta)|1\rangle - \sin(\theta)|0\rangle) \tag{4}$$

(c) she measures their qubits and send the outcomes to Rick

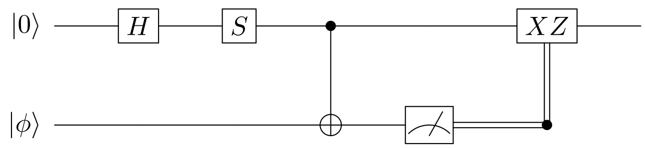
(d) Rick can then recover the unknown state $|\Psi\rangle$ on his qubit $B_R$ by performing on it a specific Pauli transformation depending on the measurement outcome

- no operation if results is $(B_\Psi, B_S) = (0, 0)$
- Pauli X if results is $(B_\Psi, B_S) = (0, 1)$
- Pauli Z if results is $(B_\Psi, B_S) = (1, 0)$
- Pauli XZ if results is $(B_\Psi, B_S) = (1, 1)$

Try to show that indeed the state on Rick's qubit $B_R$ is the correct one. Then try to implement this protocol with Qiskit. For instance you can try to measure qubit $B_R$ after the whole protocol has ended and compare the results with those you would've expected if you had the correct state $|\Psi(\theta)\rangle$. You can use the following code-snippet to get started

```
# defines a circuit with 3 qubits and 3 classical bits
telep_circ = QuantumCircuit(3,3)
# stores labels for the 3 qubits
quantum_bits = telep_circ.qubits
# stores labels for the 3 classical bits (for mesaurements)
classical_bits = telep_circ.clbits
# performs CNOT to entangle qubits and Hadamard on B_psi
telep_circ.cx(quantum_bits[0],quantum_bits[1])
telep_circ.h(quantum_bits[0])
# measurement
telep_circ.barrier()
telep_circ.measure(quantum_bits[0],classical_bits[0])
telep_circ.measure(quantum_bits[1],classical_bits[1])
# classical control
telep_circ.barrier()
## apply a Z to qubit B_R if measure on c_psi is 1.
with telep_circ.if_test((classical_bits[0],1)):
    telep_circ.z(quantum_bits[2])
## apply an X to qubit B_R if measure on c_S is 1.
with telep_circ.if_test((classical_bits[1],1)):
    telep_circ.x(quantum_bits[2])
telep_circ.draw()
```

4. if you already know everything about Qiskit and quantum teleportation try to use your skills to understand what this circuit is doing




(5)

   - try to understand what state is left on the first qubit
   - if instead of the gates $S$ we used $T = \sqrt{S}$, what classically controlled operation should we use to get the same functionality?
   - why would a circuit like this be useful?
   - can you find a way to achieve the same functionality but without measuring the second qubit at all?
   - and how about a version without any controls after the measurement?

5. Try to simulate the time-evolution of two Ising spins with Hamiltonian

$$H = JZ_1Z_2 + h(X_1 + X_2) \ ,$$

on a simple input state like e.g. $|10\rangle$. You can use e.g. $J = h = 1$

   - write a simple Numpy code that does the evolution exactly and use it as a reference
   - implement a first order Trotter approximation

$$U_1(t,r) = \left(e^{-iJ\frac{t}{r}Z_1Z_2}e^{-ih\frac{t}{r}X_1}e^{-ih\frac{t}{r}X_2}\right)^r$$

     and compare the results with the exact one. For instance you can compare the expectation values of some Pauli operators on both qubits.

   - Try to change the number of steps $r$ so that the error at some reasonably long time is only a few percent. Is this value of $r$ comparable to what the error estimate predicts?
   - you can try with a second-order formula like

$$U_2(t,r) = \left(e^{-ih\frac{t}{2r}X_1}e^{-ih\frac{t}{2r}X_2}e^{-iJ\frac{t}{r}Z_1Z_2}e^{-ih\frac{t}{2r}X_1}e^{-ih\frac{t}{2r}X_2}\right)^r \ ,$$

     how much better is this?

   - try to look at both product formulas, can you see why the first order one is better than what it should be? After you came out with a reason for this, try to have a look here.