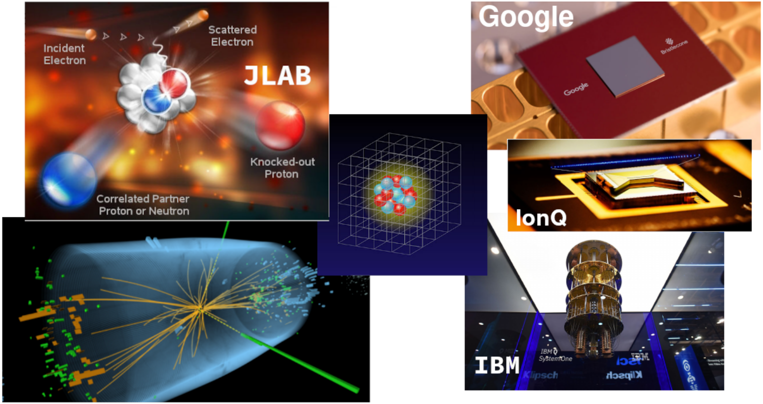
Tuesday lectures: measurements and time-evolution

Alessandro Roggero



Trento Institute for Fundamental Physics and Applications

DNP/TALENT School

ECT* – 17 June, 2025

What is a Quantum Computer?

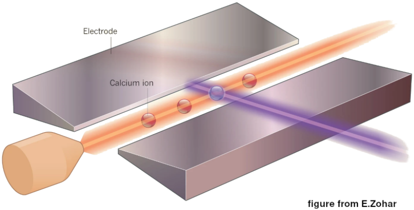


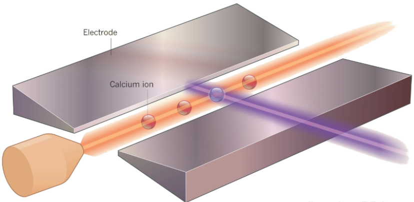
figure from E.Zohar

A Quantum Computer is a controllable quantum many-body system that allows to enact unitary transformations on an initial state ρ_0

$$\rho_0 \to U\rho_0 U^\dagger$$

n degrees of freedom so $\rho \in \mathcal{H}^{\otimes n}$

What is a Quantum Computer?



A Quantum Computer is a controllable quantum many-body system that allows to enact unitary transformations on an initial state ρ_0

$$\rho_0 \rightarrow U\rho_0 U^\dagger$$

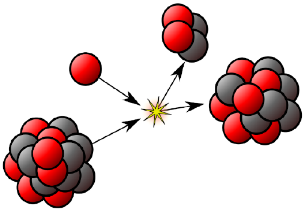
n degrees of freedom so $\rho \in \mathcal{H}^{\otimes n}$

In a Quantum Simulation we want to use this freedom to describe the time-evolution of a closed system

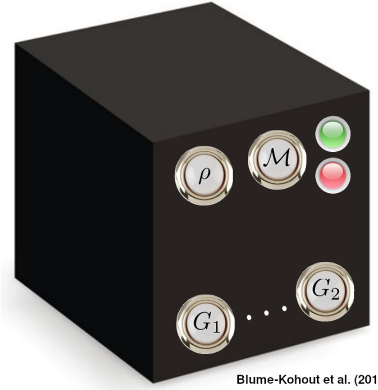
$$\rho(t) \rightarrow U(t)\rho_0 U(t)^\dagger$$

described by some Hamiltonian

$$U(t) = \exp\left(itH\right) \ .$$



Black box model for a quantum computer

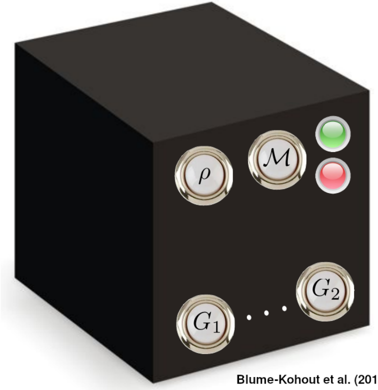


Blume-Kohout et al. (2013)

Box contains n qubits (2-level sys.) together with a set of buttons

- initial state preparation ρ
- projective measurement $\mathcal{M}$
- quantum operations G_k

Black box model for a quantum computer



Blume-Kohout et al. (2013)

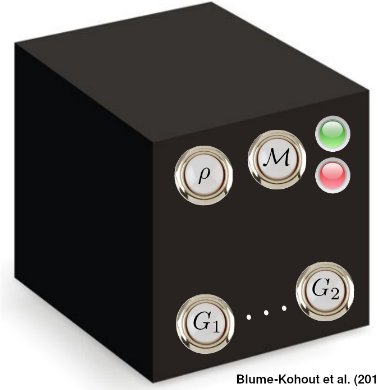
Box contains n qubits (2-level sys.) together with a set of buttons

- initial state preparation ρ
- projective measurement $\mathcal{M}$
- quantum operations G_k

Solovay–Kitaev Theorem

We can build a **universal** black box with only a **finite number** of buttons

Black box model for a quantum computer



Blume-Kohout et al. (2013)

Box contains n qubits (2-level sys.) together with a set of buttons

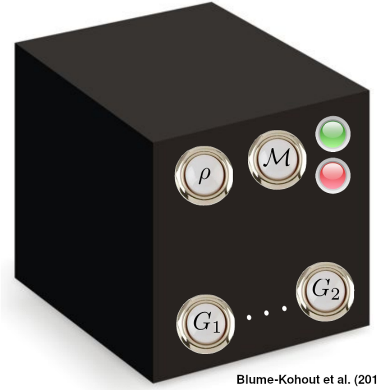
- initial state preparation ρ
- projective measurement $\mathcal{M}$
- quantum operations G_k

Solovay–Kitaev Theorem

We can build a **universal** black box with only a **finite number** of buttons

Lloyd (1996) We can simulate time evolution of local Hamiltonians

Black box model for a quantum computer



Blume-Kohout et al. (2013)

Box contains n qubits (2-level sys.) together with a set of buttons

- initial state preparation ρ
- projective measurement $\mathcal{M}$
- quantum operations G_k

Solovay–Kitaev Theorem

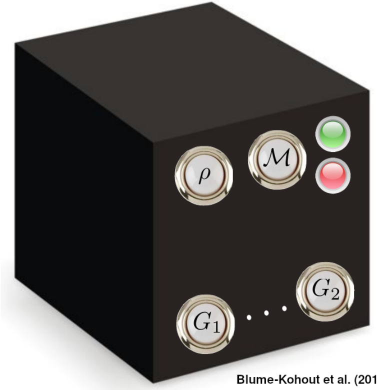
We can build a **universal** black box with only a **finite number** of buttons

Lloyd (1996) We can simulate time evolution of local Hamiltonians

1. discretize the physical problem

$$|\Psi(0)\rangle \rightarrow |\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$$

Black box model for a quantum computer



Blume-Kohout et al. (2013)

Box contains n qubits (2-level sys.) together with a set of buttons

- initial state preparation ρ
- projective measurement $\mathcal{M}$
- quantum operations G_k

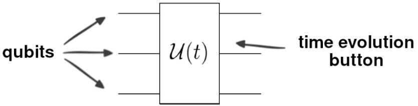
Solovay–Kitaev Theorem

We can build a **universal** black box with only a **finite number** of buttons

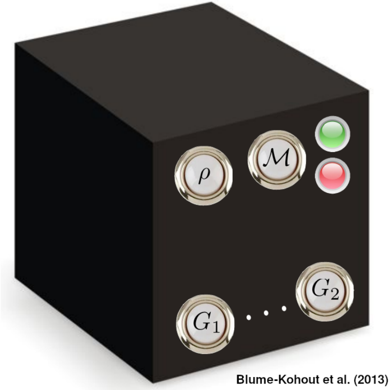
Lloyd (1996) We can simulate time evolution of local Hamiltonians

1. discretize the physical problem
2. map physical states to bb states

$$|\Psi(0)\rangle \rightarrow |\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$$



Black box model for a quantum computer



Blume-Kohout et al. (2013)

Box contains n qubits (2-level sys.) together with a set of buttons

- initial state preparation ρ
- projective measurement $\mathcal{M}$
- quantum operations G_k

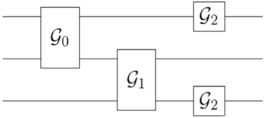
Solovay–Kitaev Theorem

We can build a **universal** black box with only a **finite number** of buttons

Lloyd (1996) We can simulate time evolution of local Hamiltonians

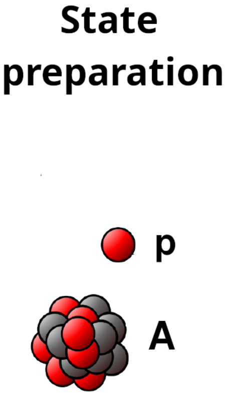
1. discretize the physical problem
2. map physical states to bb states
3. push correct button sequence

$$|\Psi(0)\rangle \rightarrow |\Psi(t)\rangle = e^{-iHt}|\Psi(0)\rangle$$



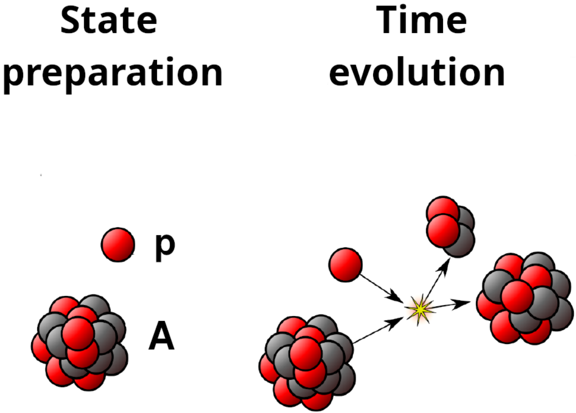
Steps in a quantum simulation

The quantum simulation of a nuclear reaction requires three main steps



Steps in a quantum simulation

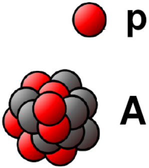
The quantum simulation of a nuclear reaction requires three main steps

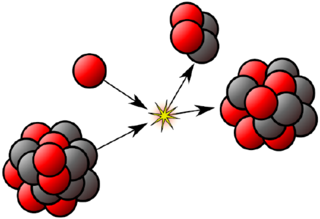


Steps in a quantum simulation

The quantum simulation of a nuclear reaction requires three main steps

State preparation | **Time evolution** | **Measurement**





Today we will focus on the last twp steps: time-evolution and measurement

Quick recap of quantum gates

single-qubit gates

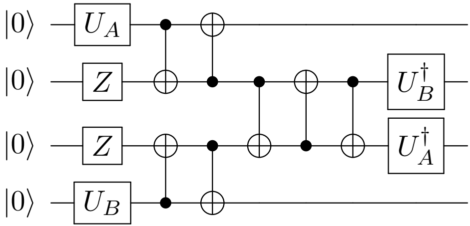
$$-\boxed{R_{\hat{n}}(\theta)}- = \exp\left(i\theta\frac{\hat{n}\cdot\vec{\sigma}}{2}\right)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -\boxed{X}-$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\boxed{Y}-$$

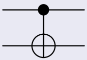
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\boxed{Z}-$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = -\boxed{S}-$$



two-qubit entangling gate

CNOT =



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$|\Phi_0\rangle = a\,|00\rangle + b\,|01\rangle + c\,|10\rangle + d\,|11\rangle$$

$$|\Phi_1\rangle = a\,|00\rangle + b\,|01\rangle + c\,|11\rangle + d\,|10\rangle$$

EXERCISE: show that $\forall U_A, U_B$ the output of the circuit above is $|0000\rangle$

Quick recap of quantum gates II

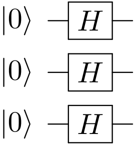
Hadamard Gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- rotates between Z and X basis

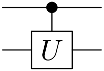
$$\left.\begin{aligned} H|0\rangle &= |+\rangle \\ H|1\rangle &= |-\rangle \end{aligned}\right\} \qquad X|\pm\rangle = \pm|\pm\rangle$$

- generates uniform superposition



$$H^{\otimes 3}|0\rangle = \frac{1}{\sqrt{2^3}}\sum_{k=0}^{2^3-1}|k\rangle$$

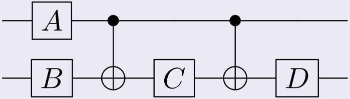
Generic controlled unitary



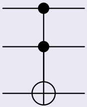
$$= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix}$$

Single qubit U

Barenco et al. (1995)



Controlled CNOT: Toffoli



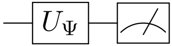
$=$ [6 CNOT $+$ 9 single qubit]$*$

* see eg. Nielsen & Chuang

Measuring an observable: single qubit case

Computational basis is eigenbasis of Z so that, if $|\Psi\rangle = U_\Psi |0\rangle$, we have

$$\langle\Psi|Z|\Psi\rangle = |\langle 0|\Psi\rangle|^2 - |\langle 1|\Psi\rangle|^2 \equiv \ |0\rangle$$



Measuring an observable: single qubit case

Computational basis is eigenbasis of Z so that, if $|\Psi\rangle = U_\Psi |0\rangle$, we have

$$\langle\Psi|Z|\Psi\rangle = |\langle 0|\Psi\rangle|^2 - |\langle 1|\Psi\rangle|^2 \equiv \begin{quantikz} \lstick{$|0\rangle$} & \gate{U_\Psi} & \meter{} \end{quantikz}$$

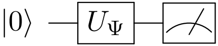
We now need to repeat calculation M times to estimate the probabilities

$$P(0) = |\langle 0|\Psi\rangle|^2 \sim \frac{\sum_k \delta_{s_k,0}}{M} \qquad Var\left[P(0)\right] \sim \frac{v_0}{M} \longrightarrow 0\ .$$

Measuring an observable: single qubit case

Computational basis is eigenbasis of Z so that, if $|\Psi\rangle = U_\Psi |0\rangle$, we have

$$\langle\Psi|Z|\Psi\rangle = |\langle 0|\Psi\rangle|^2 - |\langle 1|\Psi\rangle|^2 \equiv$$

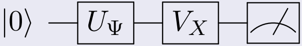


We now need to repeat calculation M times to estimate the probabilities

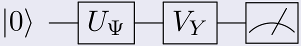
$$P(0) = |\langle 0|\Psi\rangle|^2 \sim \frac{\sum_k \delta_{s_k,0}}{M} \qquad Var\left[P(0)\right] \sim \frac{v_0}{M} \longrightarrow 0\ .$$

Other expectation values accessible by basis transformation

$X = V_X Z V_X^\dagger$



$Y = V_Y Z V_Y^\dagger$



- for X we can use $X = V_X Z V_X^\dagger$ where V_X is the Hadamard

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- for Y we can use $Y = SXS^\dagger$ so that $V_Y = SV_X = SH$

Measuring an observable: the Pauli group

Given a state $|\Psi\rangle$ defined over n qubits and an encoded operator

$$O = \sum_{k=1}^{N_K} c_k P_k \quad P_k \in \{(\mathbb{1}, X, Y, Z)^{\otimes n}\}$$

we want to measure the expectation value $\langle\Psi|O|\Psi\rangle$ [McClean et al. (2014)].

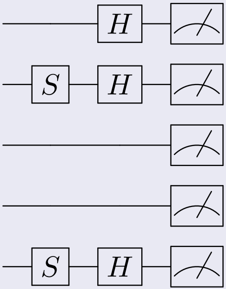
Measuring an observable: the Pauli group

Given a state $|\Psi\rangle$ defined over n qubits and an encoded operator

$$O = \sum_{k=1}^{N_K} c_k P_k \quad P_k \in \{(\mathbb{1}, X, Y, Z)^{\otimes n}\}$$

we want to measure the expectation value $\langle\Psi|O|\Psi\rangle$ [McClean et al. (2014)].

Example: $X_0 Y_1 Z_2 Z_3 Y_4$



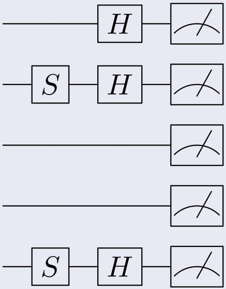
Measuring an observable: the Pauli group

Given a state $|\Psi\rangle$ defined over n qubits and an encoded operator

$$O = \sum_{k=1}^{N_K} c_k P_k \quad P_k \in \{(\mathbb{1}, X, Y, Z)^{\otimes n}\}$$

we want to measure the expectation value $\langle\Psi|O|\Psi\rangle$ [McClean et al. (2014)].

Example: $X_0 Y_1 Z_2 Z_3 Y_4$



- $\forall k$ perform M experiments to get $\langle P_k\rangle$ with

$$Var[P_k] \sim \frac{\langle P_k^2\rangle - \langle P_k\rangle^2}{M} = \frac{1 - \langle P_k\rangle^2}{M}$$

- we can now evaluate $\langle O\rangle$ with variance

$$Var[O] = \sum_{k=1}^{N_K} |c_k|^2 Var[P_k]$$

$\Rightarrow$ total error $\propto \sqrt{N_K/M}$.

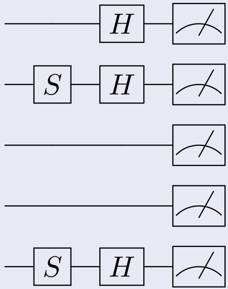
Measuring an observable: the Pauli group

Given a state $|\Psi\rangle$ defined over n qubits and an encoded operator

$$O = \sum_{k=1}^{N_K} c_k P_k \quad P_k \in \{(\mathbb{1}, X, Y, Z)^{\otimes n}\}$$

we want to measure the expectation value $\langle\Psi|O|\Psi\rangle$ [McClean et al. (2014)].

Example: $X_0 Y_1 Z_2 Z_3 Y_4$



- naive estimator has total error $\propto \sqrt{N_K/M}$
- we can measure multiple terms together!

$$X_0 Y_1 Z_2 Z_3 Y_4 \begin{cases} X_0 Y_1 \mathbb{1}_2 Z_3 Y_4 \\ X_0 Y_1 \mathbb{1}_2 \mathbb{1}_3 Y_4 \\ \dots \\ \mathbb{1}_0 Y_1 \mathbb{1}_2 \mathbb{1}_3 \mathbb{1}_4 \\ X_0 X_1 Z_2 Z_3 X_4 \\ \dots \end{cases} \Rightarrow \epsilon_{tot} \propto \sqrt{\frac{N_G}{M}}$$

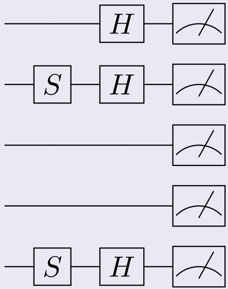
Measuring an observable: the Pauli group

Given a state $|\Psi\rangle$ defined over n qubits and an encoded operator

$$O = \sum_{k=1}^{N_K} c_k P_k \qquad P_k \in \{(\mathbb{1}, X, Y, Z)^{\otimes n}\}$$

we want to measure the expectation value $\langle\Psi|O|\Psi\rangle$ [McClean et al. (2014)].

Example: $X_0 Y_1 Z_2 Z_3 Y_4$



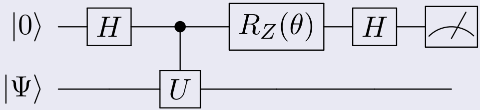
- naive estimator has total error $\propto \sqrt{N_K/M}$
- we can measure multiple terms together!

$$X_0 Y_1 Z_2 Z_3 Y_4 \begin{cases} X_0 Y_1 \mathbb{1}_2 Z_3 Y_4 \\ X_0 Y_1 \mathbb{1}_2 \mathbb{1}_3 Y_4 \\ \cdots \\ \mathbb{1}_0 Y_1 \mathbb{1}_2 \mathbb{1}_3 \mathbb{1}_4 \\ X_0 X_1 Z_2 Z_3 X_4 \\ \cdots \end{cases} \Rightarrow \epsilon_{tot} \propto \sqrt{\frac{N_G}{M}}$$

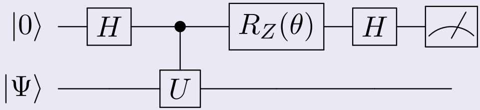
- can do much better $M \approx \frac{\sqrt{N_K}}{\epsilon}$ [see 2111.09283]

Measuring an observable: Hadamard test

Kitaev (1995)



Measuring an observable: Hadamard test

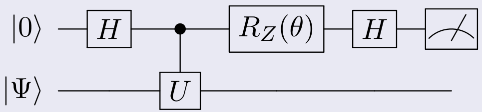


Kitaev (1995)

When $\theta = 0$ we have:

1. $|\Phi_0\rangle = |0\rangle \otimes |\Psi\rangle$
2. $|\Phi_1\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |\Psi\rangle$
3. $|\Phi_2\rangle = \frac{|0\rangle\otimes|\Psi\rangle}{\sqrt{2}} + \frac{|1\rangle\otimes U|\Psi\rangle}{\sqrt{2}}$
4. $|\Phi_3\rangle = \frac{|0\rangle\otimes(\mathbb{1}+U)|\Psi\rangle}{2} + \frac{|1\rangle\otimes(\mathbb{1}-U)|\Psi\rangle}{2}$

Measuring an observable: Hadamard test



Kitaev (1995)

When $\theta = 0$ we have:

1. $|\Phi_0\rangle = |0\rangle \otimes |\Psi\rangle$
2. $|\Phi_1\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} \otimes |\Psi\rangle$
3. $|\Phi_2\rangle = \frac{|0\rangle\otimes|\Psi\rangle}{\sqrt{2}} + \frac{|1\rangle\otimes U|\Psi\rangle}{\sqrt{2}}$
4. $|\Phi_3\rangle = \frac{|0\rangle\otimes(\mathbb{1}+U)|\Psi\rangle}{2} + \frac{|1\rangle\otimes(\mathbb{1}-U)|\Psi\rangle}{2}$

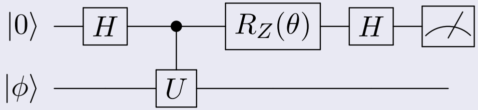
Result of ancilla measurement

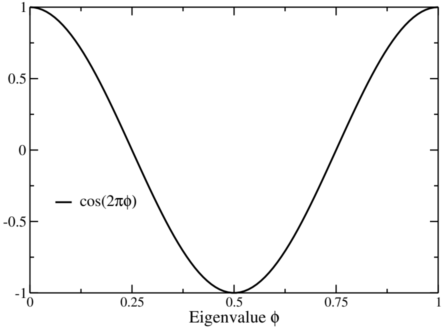
$$\langle Z\rangle_a = \frac{\langle\Psi|\left(U + U^\dagger\right)|\Psi\rangle}{2} = \mathcal{R}\langle\Psi|U|\Psi\rangle$$

EXERCISE: find the proper angle θ needed to measure the imaginary part

EXAMPLE: eigenvalue estimation

Take a unitary U and an eigenvector $|\phi\rangle$ so that: $U|\phi\rangle = e^{i2\pi\phi}|\phi\rangle$

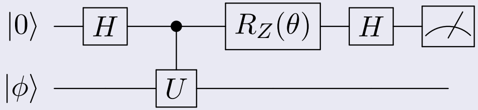


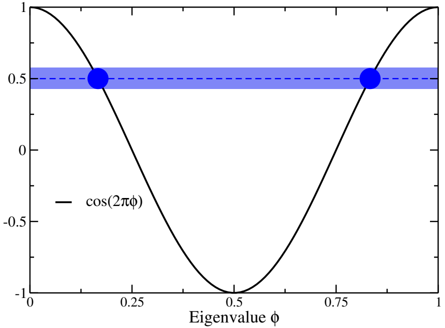


- for $\theta = 0$: $\langle Z\rangle_a = cos(2\pi\phi)$

EXAMPLE: eigenvalue estimation

Take a unitary U and an eigenvector $|\phi\rangle$ so that: $U|\phi\rangle = e^{i2\pi\phi}|\phi\rangle$



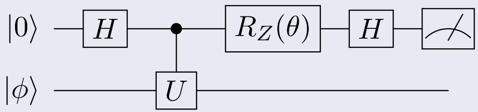


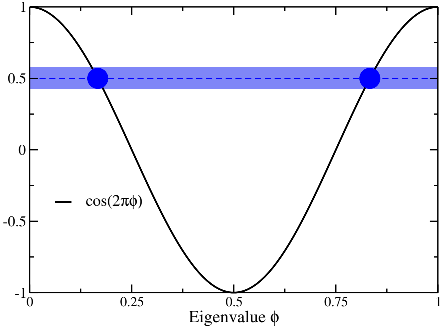
- for $\theta = 0$: $\langle Z\rangle_a = cos(2\pi\phi)$
- error δ with $M \propto 1/\delta^2$ samples:

$$Var[Z_a] \sim \frac{1}{M}$$

EXAMPLE: eigenvalue estimation

Take a unitary U and an eigenvector $|\phi\rangle$ so that: $U|\phi\rangle = e^{i2\pi\phi}|\phi\rangle$





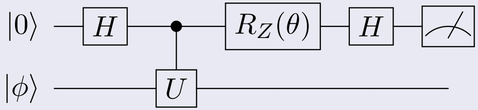
- for $\theta = 0$: $\langle Z\rangle_a = cos(2\pi\phi)$
- error δ with $M \propto 1/\delta^2$ samples:

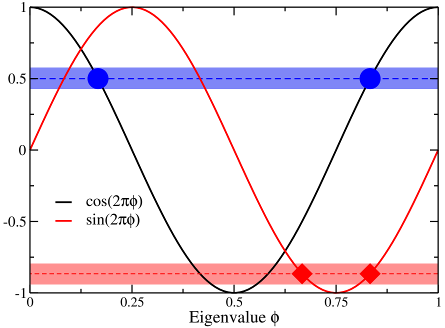
$$Var[Z_a] \sim \frac{1}{M}$$

- not enough to separate $(\phi, 1-\phi)$

EXAMPLE: eigenvalue estimation

Take a unitary U and an eigenvector $|\phi\rangle$ so that: $U|\phi\rangle = e^{i2\pi\phi}|\phi\rangle$





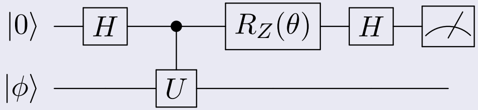
- for $\theta = 0$: $\langle Z\rangle_a = cos(2\pi\phi)$
- error δ with $M \propto 1/\delta^2$ samples:

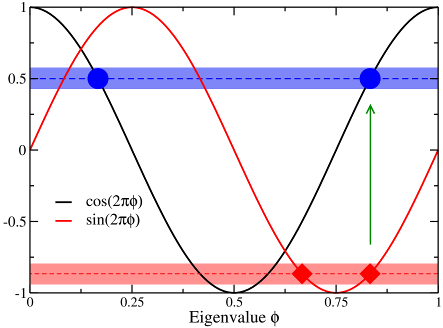
$$Var[Z_a] \sim \frac{1}{M}$$

- not enough to separate $(\phi, 1-\phi)$
- for $\theta = \theta_{ex}$: $\langle Z\rangle_a = sin(2\pi\phi)$

EXAMPLE: eigenvalue estimation

Take a unitary U and an eigenvector $|\phi\rangle$ so that: $U|\phi\rangle = e^{i2\pi\phi}|\phi\rangle$





- for $\theta = 0$: $\langle Z\rangle_a = cos(2\pi\phi)$
- error δ with $M \propto 1/\delta^2$ samples:

$$Var[Z_a] \sim \frac{1}{M}$$

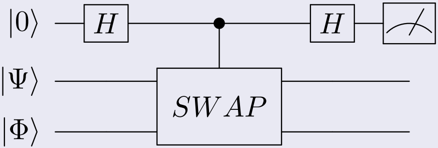
- not enough to separate $(\phi, 1-\phi)$
- for $\theta = \theta_{ex}$: $\langle Z\rangle_a = sin(2\pi\phi)$

EXAMPLE 2: the SWAP test

- State Tomography: reconstruction of state $|\Psi\rangle$ costs $O(N)$ samples
- State Overlap: we can compute $|\langle\Psi|\Phi\rangle|^2$ using only $O(\log(N))$ gates

EXAMPLE 2: the SWAP test

- State Tomography: reconstruction of state $|\Psi\rangle$ costs $O(N)$ samples
- State Overlap: we can compute $|\langle\Psi|\Phi\rangle|^2$ using only $O(\log(N))$ gates

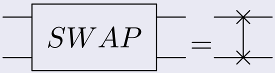


Buhrman, Cleve, Watrous & de Wolf (2001)

$$\Rightarrow \langle Z\rangle_a = |\langle\Psi|\Phi\rangle|^2$$

The SWAP gate

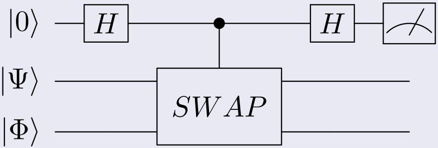
$$\text{SWAP}\,|\Psi\rangle \otimes |\Phi\rangle = |\Phi\rangle \otimes |\Psi\rangle$$



$$2 \text{ qubits} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

EXAMPLE 2: the SWAP test

- State Tomography: reconstruction of state $|\Psi\rangle$ costs $O(N)$ samples
- State Overlap: we can compute $|\langle\Psi|\Phi\rangle|^2$ using only $O(\log(N))$ gates

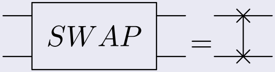


Buhrman, Cleve, Watrous & de Wolf (2001)

$$\Rightarrow \langle Z\rangle_a = |\langle\Psi|\Phi\rangle|^2$$

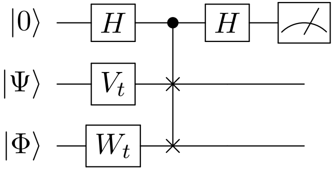
The SWAP gate

$$\text{SWAP}\,|\Psi\rangle\otimes|\Phi\rangle = |\Phi\rangle\otimes|\Psi\rangle$$



$$2 \text{ qubits} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Why should we care?



$$\Rightarrow M(\Psi\leftrightarrow\Phi) = \left|\langle\Psi|V_t^\dagger W_t|\Phi\rangle\right|^2$$

Efficient transition matrix element!

What is a Quantum Computer?

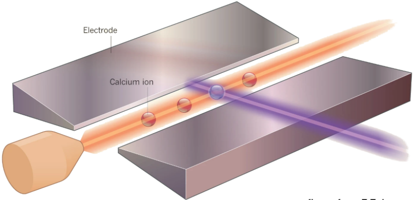


figure from E.Zohar

A Quantum Computer is a controllable quantum many-body system that allows to enact unitary transformations on an initial state ρ_0

$$\rho_0 \rightarrow U \rho_0 U^\dagger$$

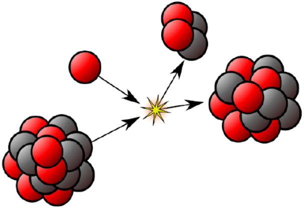
n degrees of freedom so $\rho \in \mathcal{H}^{\otimes n}$

In a Quantum Simulation we want to use this freedom to describe the time-evolution of a closed system

$$\rho(t) \rightarrow U(t) \rho_0 U(t)^\dagger$$

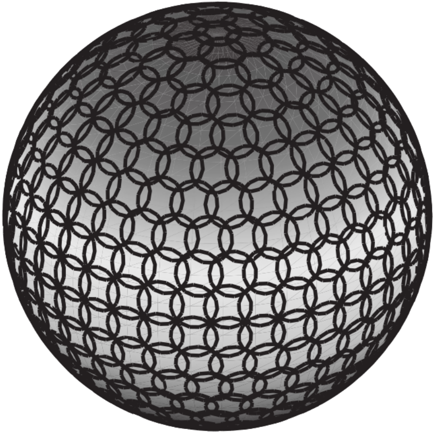
described by some Hamiltonian

$$U(t) = \exp\left(itH\right) \ .$$



Can we always do this?

image from Nielsen&Chuang



Any unitary operation can be thought as the time evolution operator for some (Hermitian) Hamiltonian

$$U \quad \leftrightarrow \quad e^{iH}$$

A simple counting argument shows that for a fixed choice of universal buttons (quantum gates) there are unitary operations on n qubits which will require $O(2^n)$ operations

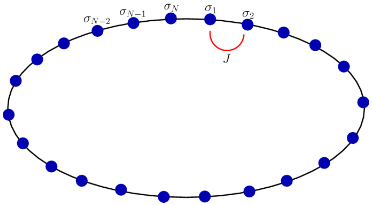
We can find Hamiltonians whose time evolution cannot be simulated **efficiently**

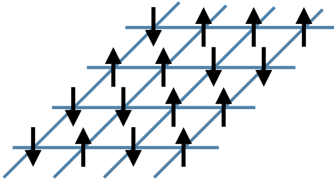
Efficient Hamiltonian Simulation

Hamiltonians encountered in physics have usually structure, like locality

$$H^{1D}_{Ising} = J\sum_{i=1}^{N} Z_i Z_{i+1} + h\sum_{i=1}^{N} X_i$$

$$H^{1D}_{Heis} = J\sum_{i=1}^{N} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1}$$





$$H^{2D}_{Ising} = J\sum_{\langle i,j\rangle} Z_i Z_j + h\sum_{i} X_i$$

$$H^{2D}_{Heis} = J\sum_{\langle i,j\rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j$$

All these situations are examples of 2-local spin Hamiltonians

Quantum Simulation of k-local Hamiltonians

- locality constraints number of terms appearing in the Hamiltonian
- one can approximate full evolution with products of evolutions

$$e^{it(A+B)} = e^{itA}e^{itB} + \mathcal{O}\left(t^2\|[A,B]\|\right)$$

- locality constrains how expensive any individual term can be

Quantum Simulation of k-local Hamiltonians

- locality constraints number of terms appearing in the Hamiltonian
- one can approximate full evolution with products of evolutions

$$e^{it(A+B)} = e^{itA}e^{itB} + \mathcal{O}\left(t^2 \|[A,B]\|\right)$$

- locality constrains how expensive any individual term can be

S. LLOYD (1996): k-local hamiltonians can be simulated efficiently

Consider a system of n qubits and a k-local Hamiltonian $H = \sum_j^{N_j} h_j$ where each term h_j acts on at most $k = \mathcal{O}(1)$ qubits at a time for $N_j = \mathcal{O}(\text{poly}(n))$, then using the Trotter-Suzuki decomposition

$$\left\| U(\tau) - \prod_j^{N_j} \exp\left(i\tau h_j\right) \right\| \leq C\tau^2$$

we can implement $U(\tau)$ with error ϵ using $\mathcal{O}\left(\text{poly}(\tau, 1/\epsilon, n) 4^k\right)$ gates.

Explicit example: Ising chain

Let's consider for instance a simple 1D Ising chain with Hamiltonian

$$H = J \sum_{i=1}^{N-1} Z_i Z_{i+1} + h \sum_{i=1}^{N} X_i$$

Since $[Z_i, X_i] \neq 0$ the following decomposition is only an approximation

$$e^{-itH} \approx e^{-iJtZ_1Z_2} e^{-iJtZ_2Z_3} \dots e^{-iJtZ_{N-1}Z_N} e^{-ithX_1} e^{-ithX_2} \dots e^{-ithX_N}$$

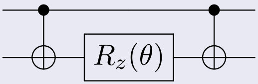
How good is the approximation above? If we collect all the two-qubit Zs in H_Z and the transverse fields in H_X we have

$$\left\| e^{-itH} - e^{-itH_Z} e^{-itH_X} \right\| \leq \frac{t^2}{2} \left\| [H_Z, H_X] \right\|$$

- for more check out: Childs et al. 1912.08854

EXERCISE

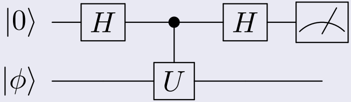
Show that $U_2 = e^{-i\frac{\theta}{2} Z_1 Z_2}$ is



EXAMPLE 3: Can we apply a non-unitary operation?

YES, but only with some probability

- this can be useful for example if the transition matrix element we considered before is genereated by a non unitary operator



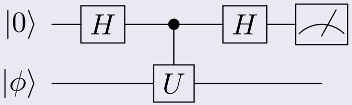
$$\Rightarrow \quad |0\rangle \otimes \frac{\mathbb{1}+U}{2} |\phi\rangle + |1\rangle \otimes \frac{\mathbb{1}-U}{2} |\phi\rangle$$

- we will measure $|0\rangle$ with $P_0 = \frac{1}{2}\left(1 + \mathcal{R}\langle\phi|U|\phi\rangle\right) \Rightarrow |\phi_0\rangle = \dfrac{\mathbb{1}+U}{2\sqrt{P_0}} |\phi\rangle$

EXAMPLE 3: Can we apply a non-unitary operation?

YES, but only with some probability

- this can be useful for example if the transition matrix element we considered before is genereated by a non unitary operator



$$\Rightarrow \quad |0\rangle \otimes \frac{\mathbb{1}+U}{2}|\phi\rangle + |1\rangle \otimes \frac{\mathbb{1}-U}{2}|\phi\rangle$$

- we will measure $|0\rangle$ with $P_0 = \frac{1}{2}\left(1 + \mathcal{R}\langle\phi|U|\phi\rangle\right) \Rightarrow |\phi_0\rangle = \dfrac{\mathbb{1}+U}{2\sqrt{P_0}}|\phi\rangle$

Concrete example: projection operators

If we take U to be the reflection around $|\psi\rangle$, like $U = (2|\psi\rangle\langle\psi| - \mathbb{1})$, we find

$$P_0 = |\langle\phi|\psi\rangle|^2 \quad \Rightarrow \quad |\phi_0\rangle = \frac{|\psi\rangle\langle\psi|}{\sqrt{P_0}}|\phi\rangle = |\psi\rangle$$