DTP/TALENT 2025: Exercises for first week

June 16, 2025

In this first hands-on session the main goal is to familiarize yourself with how to write quantum circuits and run them. Some of the examples we'll see use a software package named Qiskit developed by Ibold but there are other options, like Google's Cirq. For those of you new to quantum circuits you can find an introductory notebook *1_getting_started_with_qiskit.ipynb* in the *Exercises* folder.

1 First exercise set

The exercises we present each week are meant to build the basis for the possible project we will work. See in particular the third lecture today.

1.1 Ex1: One-qubit basis and Pauli matrices

Write a function which sets up a one-qubit basis and apply the various Pauli matrices to these basis states.

1.2 Ex2: Hadamard and Phase gates

Apply the Hadamard and Phase gates to the same one-qubit basis states and study their actions on these states.

1.3 Ex3: Traces of operators

Prove that the trace is cyclic, that is for three operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, we have

$$\mathrm{Tr}\{\mathbf{ABC}\} = \mathrm{Tr}\{\mathbf{CAB}\} = \mathrm{Tr}\{\mathbf{BCA}\}.$$

1.4 Ex4: Exponentiated operators

Let $\mathbf{A}$ be an operator on a vector space satisfying $\mathbf{A}^2 = 1$ and α any real constant. Show that

$$\exp\{\imath\alpha\mathbf{A}\} = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!}\mathbf{A}^n = \mathbf{I}\cos\alpha + \imath\mathbf{A}\sin\alpha.$$

Does this apply to the Pauli matrices?

1.5 Ex5: Hamiltonians rewritten in terms of simple Pauli matrices

We consider a simple 2×2 real Hamiltonian consisting of a diagonal part H_0 and off-diagonal part H_I, playing the roles of a non-interactive one-body and interactive two-body part respectively. Defined through their matrix elements, we express them in the Pauli basis $|0\rangle$ and $|1\rangle$

$$H = H_0 + H_I$$

$$H_0 = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \qquad H_I = \lambda \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

Where $\lambda \in [0,1]$ is a coupling constant parameterizing the strength of the interaction.

1.6 Rewriting in terms of Pauli matrices

Define

$$E_+ = \frac{E_1 + E_2}{2}, \qquad E_- = \frac{E_1 - E_2}{2}$$

show that by combining the identity and Z Pauli matrix, this can be expressed as

$$H_0 = E_+ I + E_- Z$$

1.7 The interaction part

For H_1 we use the same trick to fill the diagonal, defining $V_+ = (V_{11}+V_{22})/2$, $V_- = (V_{11}-V_{22})/2$. From the hermiticity requirements of H, we note that $V_{12} = V_{21} \equiv V_o$. Use this to simplify the problem to a simple X term.

$$H_I = V_+ I + V_- Z + V_o X$$

1.8 Measurement basis

For the above system show that the Pauli X matrix can be rewritten in terms of the Hadamard matrices and the Pauli Z matrix, that is

$$X = HZH.$$

1.9 More exercises

1. Write a circuit that prepares the single qubit state

$$|\Psi(\theta)\rangle = \cos(\theta) |0\rangle + \sin(\theta) |1\rangle , \tag{1}$$

and measures the expectation value of Z on the resulting state.

- you can use a R_y rotation to perform this (see here for definitions)
- perform the calculation in two ways: first by extracting the full *state_vector* and then performing a fixed number of shots and reconstructing the expectation value from those
- it might also be helpful to implement everything directly in NumPy in order to have a reference value to compare to
- make a plot as a function of θ of the function

$$\langle Z(\theta)\rangle = \langle \Psi(\theta)|Z|\Psi(\theta)\rangle \,, \qquad (2)$$

extracted in the various ways.

2. Try to show that the following circuit gives the same final state regardless on the choice of the single-qubit gates U_A and U_B
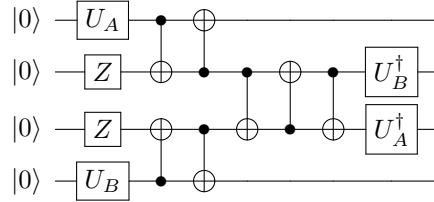
- try to choose a simple pair of gates U_A and U_B so that you can find the final state easily. What do you get?
- [HINT] in order to understand what is happening you might want to consider the following sub-circuit
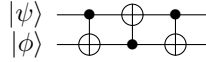
where $|\psi\rangle$ and $|\phi\rangle$ are generic single-qubit states. What is this doing?

3. Quantum teleportation is an interesting communication protocol where information is transmitted between a sender, we'll call her Sandy, and a receiver, will call him Rick, through a combination of entanglement and classical communication. The protocol works as follows:

- Sandy has one qubit, we'll call B_Ψ, in a state that she wants to send to Rick. For this example we'll use $|\Psi(\theta)\rangle$ from Eq. (1) above
- before the protocol began, Sandy and Rick prepare a two qubit Bell state $|B\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. By keeping one qubit each, let's call B_S and B_R, they can now use as a resource for communication.
- the protocol now proceeds as follows:
  (a) Sandy entangles her qubit B_Ψ with her part of the Bell state on qubit B_S. The total state on qubits (B_Ψ, B_S, B_R) is then

$$\frac{\cos(\theta)}{\sqrt{2}}|0\rangle \otimes (|00\rangle + |11\rangle) + \frac{\sin(\theta)}{\sqrt{2}}|1\rangle \otimes (|10\rangle + |01\rangle) \quad (3)$$

(b) she applies an Hadamard gate to her qubit B_Ψ obtaining

$$\frac{|00\rangle}{2} \otimes (\cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle) + \frac{|01\rangle}{2} \otimes (\cos(\theta)\,|1\rangle + \sin(\theta)\,|0\rangle)$$
$$\frac{|10\rangle}{2} \otimes (\cos(\theta)\,|0\rangle - \sin(\theta)\,|1\rangle) + \frac{|11\rangle}{2} \otimes (\cos(\theta)\,|1\rangle - \sin(\theta)\,|0\rangle)$$

$$(4)$$

(c) she measures their qubits and send the outcomes to Rick
(d) Rick can then recover the unknown state $|\Psi\rangle$ on his qubit B_R by performing on it a specific Pauli transformation depending on the measurement outcome
  - no operation if results is $(B_\Psi, B_S) = (0,0)$
  - Pauli X if results is $(B_\Psi, B_S) = (0,1)$
  - Pauli Z if results is $(B_\Psi, B_S) = (1,0)$
  - Pauli XZ if results is $(B_\Psi, B_S) = (1,1)$

Try to show that indeed the state on Rick's qubit B_R is the correct one. Then try to implement this protocol with Qiskit. For instance you can try to measure qubit B_R after the whole protocol has ended and compare the results with those you would've expected if you had the correct state $|\Psi(\theta)\rangle$. You can use the following code-snippet to get started

```python
# defines a circuit with 3 qubits and 3 classical bits
telep_circ = QuantumCircuit(3,3)
# stores labels for the 3 qubits
quantum_bits = telep_circ.qubits
# stores labels for the 3 classical bits (for mesaurements)
classical_bits = telep_circ.clbits
# performs CNOT to entangle qubits and Hadamard on B_psi
telep_circ.cx(quantum_bits[0],quantum_bits[1])
telep_circ.h(quantum_bits[0])
# measurement
telep_circ.barrier()
telep_circ.measure(quantum_bits[0],classical_bits[0])
telep_circ.measure(quantum_bits[1],classical_bits[1])
# classical control
telep_circ.barrier()
## apply a Z to qubit B_R if measure on c_psi is 1.
with telep_circ.if_test((classical_bits[0],1)):
    telep_circ.z(quantum_bits[2])
## apply an X to qubit B_R if measure on c_S is 1.
with telep_circ.if_test((classical_bits[1],1)):
    telep_circ.x(quantum_bits[2])
telep_circ.draw()
```

4. if you already know everything about Qiskit and quantum teleportation try to use your skills to understand what this circuit is doing

$$|0\rangle \quad —H—\ —S—\ \bullet \quad —XZ—$$
$$|\phi\rangle \quad —\oplus—\ —\measuredangle—$$

(5)
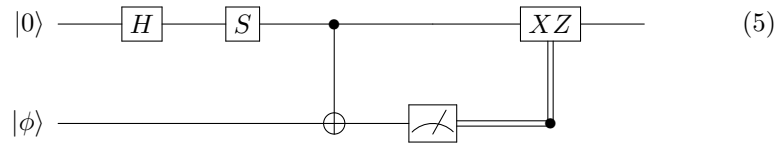
- try to understand what state is left on the first qubit
- if instead of the gates S we used $T = \sqrt{S}$, what classically controlled operation should we use to get the same functionality?
- why would a circuit like this be useful?
- can you find a way to achieve the same functionality but without measuring the second qubit at all?
- and how about a version without any controls after the measurement?