QUANTUM COMPUTATION OF LATTICE GAUGE THEORIES

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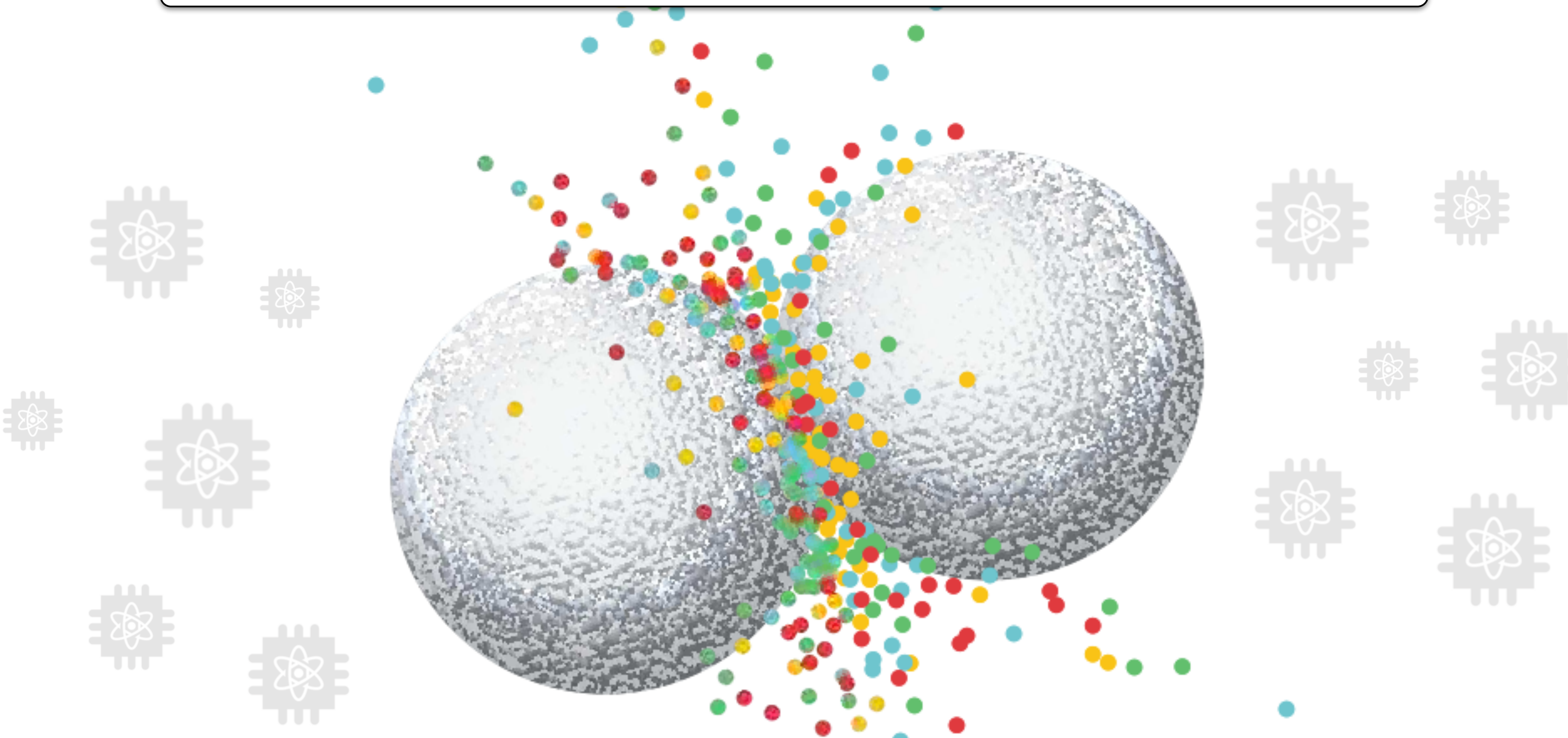


DTP-TALENT COURSE ON QUANTUM COMPUTING FOR NUCLEAR PHYSICS
ECT*, TRENTO, ITALY
JUNE 2025

OUTLINE OF PART I:
HAMILTONIAN FORMULATION OF LATTICE GAUGE THEORIES

i) Hamiltonian vs. Lagrangian formulation of LGTs

ii) Kogut-Susskind formulation: Basis states, Hilbert space, and constraints

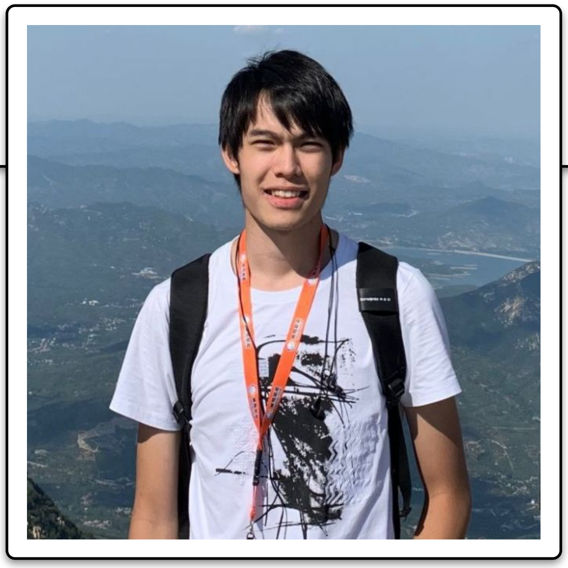
- An Abelian case: U(1) LGT
- A non-Abelian case: SU(2) LGT

iii) Kogut-Susskind formulation: Hamiltonian

iv) A variety of formulations: a brief overview

v) Classical Hamiltonian-simulation methods: a brief discussion

TA: Chung-Chun Hsieh



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	Path integral (Lagrangian)	Hamiltonian
Degrees of freedom	Fields and their derivatives	Fields and their conjugate variables
Spacetime signature	Often Euclidean	Minkowski
Starting point	$\mathcal{L}[\varphi, \partial\varphi]$	$\hat{H}[\hat{\varphi}, \hat{\pi}]$
Hilbert space	Not explicitly constructed/relevant	Built out of $O^\dagger \vert\text{vac.}\rangle^*$ $^*\vert\text{vac.}\rangle = \vert\text{empty state}\rangle$
Expectation values	$\frac{1}{\mathcal{Z}}\int \mathcal{D}\varphi \; e^{-S} O$	$\langle\psi\vert\hat{O}\vert\psi\rangle$
Dynamical quantities	Sometimes accessible with indirect methods, e.g., Luescher method.	In principle accessible: $\langle\psi\vert e^{i\hat{H}t}\hat{O}e^{-i\hat{H}t}\vert\psi\rangle$
Computational methods	Monte Carlo, etc.	Classical Hamiltonian methods like exact diag., tensor networks/ quantum simulation
Computational challenge	Sign and signal-to-noise problem for real-time quantities and finite-density systems.	Exponential scaling of the Hilbert space with the number of DOF.

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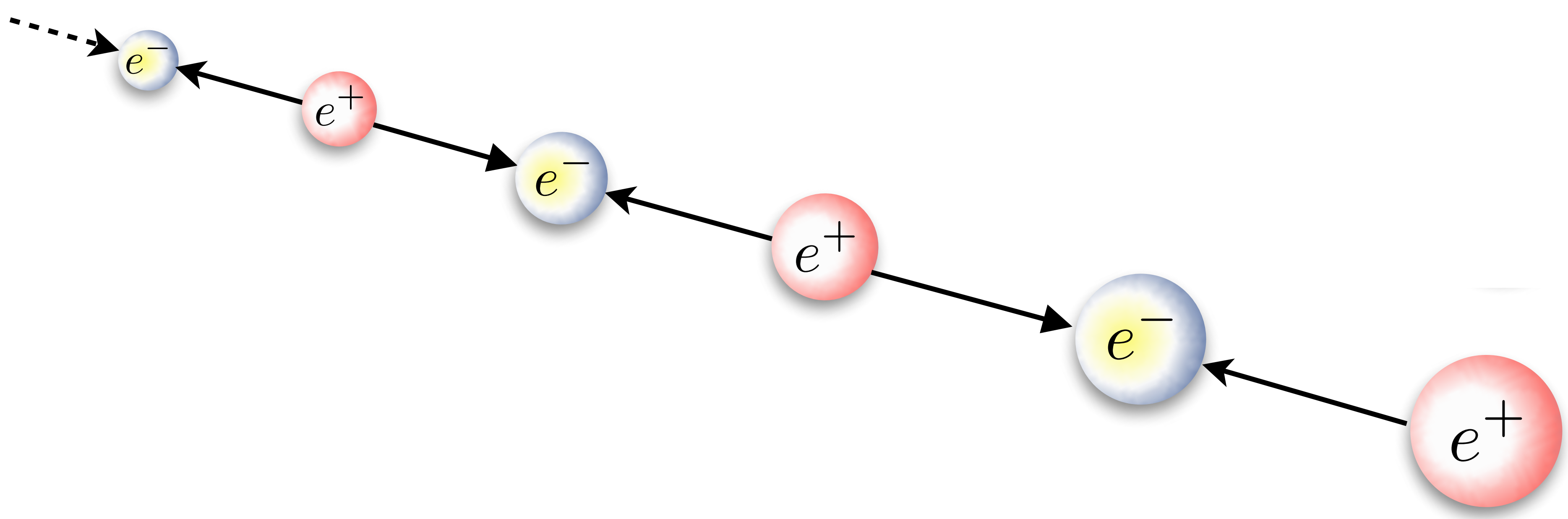
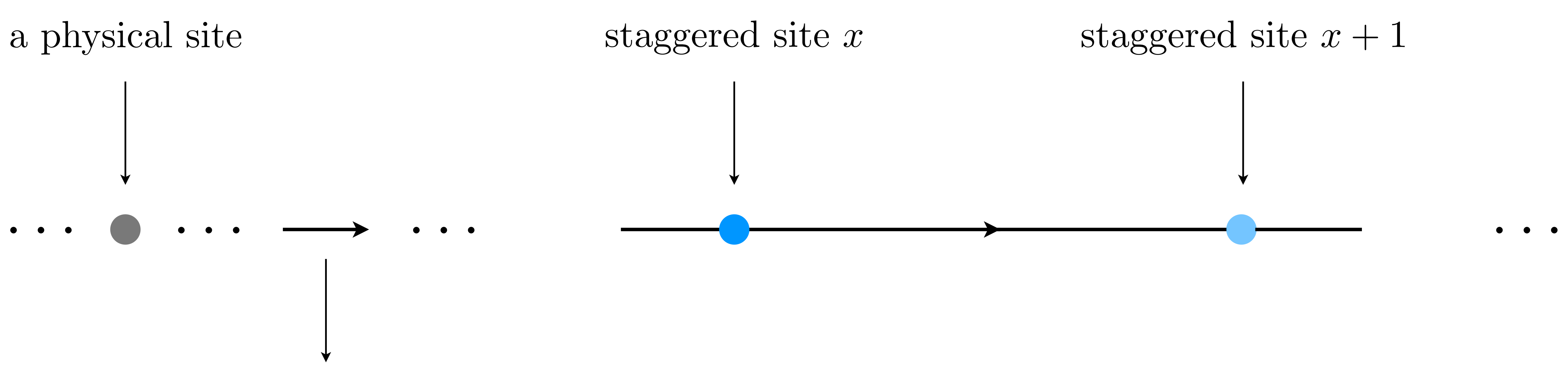
FOCUSING ON A SIMPLE EXAMPLE: THE 1+1 DIMENSIONAL QUANTUM ELECTRODYNAMICS COUPLED TO MATTER (SCHWINGER MODEL)

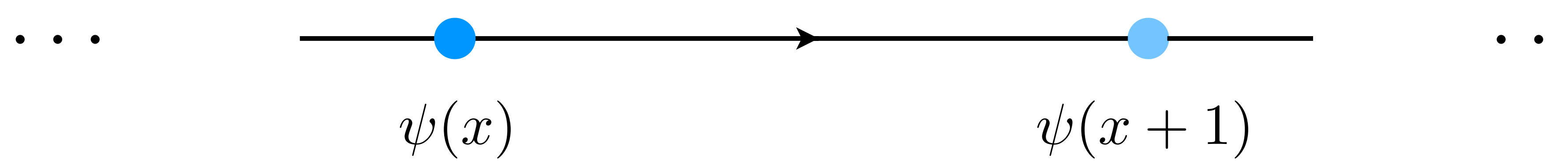


THE BASICS



Staggering: Multi-component Dirac fermion field is split to single-component fermion fields each occupying one site.

THE BASICS



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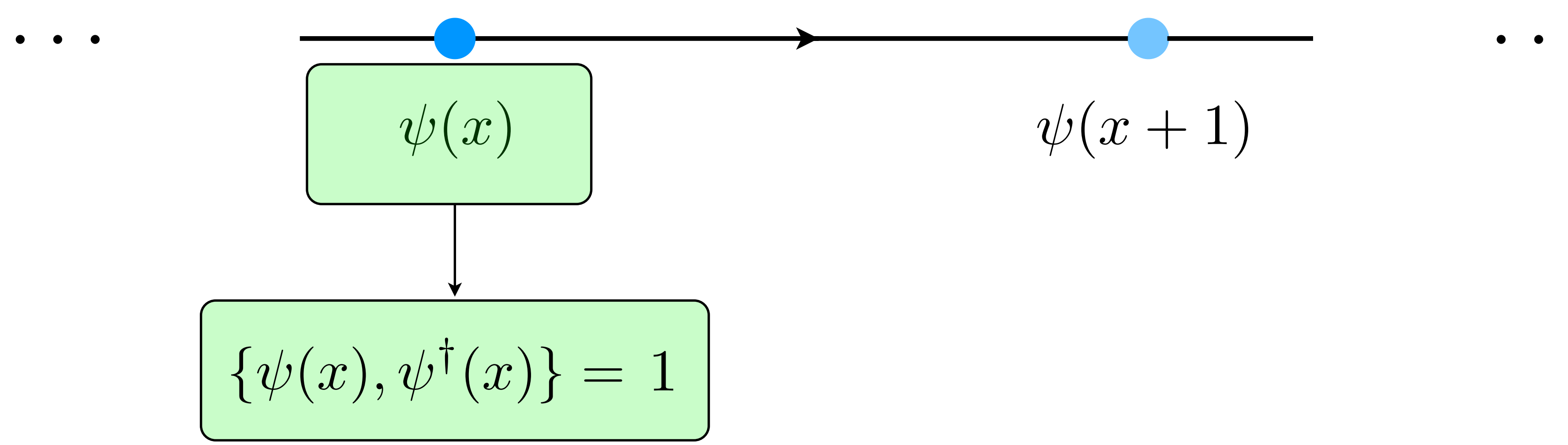
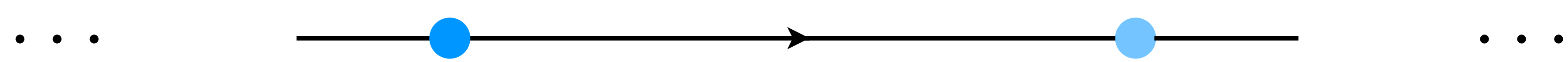


All other anticommutations are zero.

THE BASICS



$$\psi|f\rangle = (1-\delta_{f,0})|f-1\rangle$$
$$\psi^\dagger|f\rangle = (1-\delta_{f,1})|f+1\rangle$$

$\psi(x)$

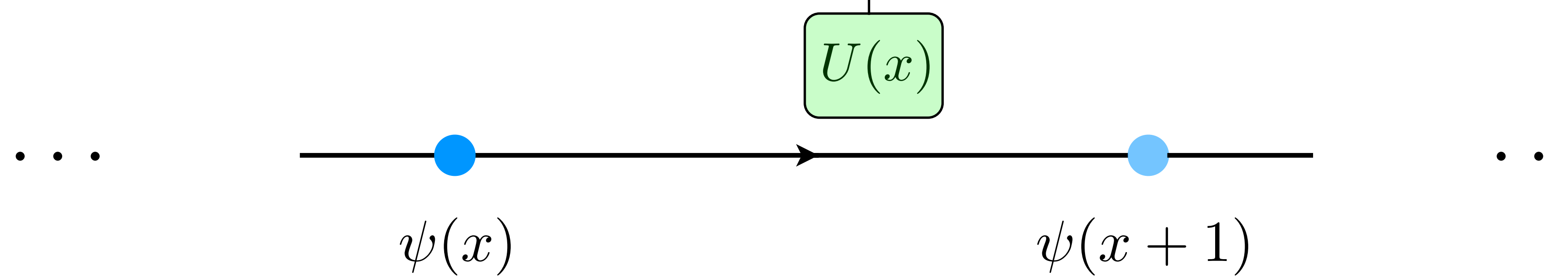
$\psi(x+1)$

Fermions can have occupation number zero or one.

THE BASICS

Picking the temporal gauge, and introducing the link variable *U*

$$U(x) = e^{iagA(x)} \quad (A_0 = 0,\ A_1 = A)$$

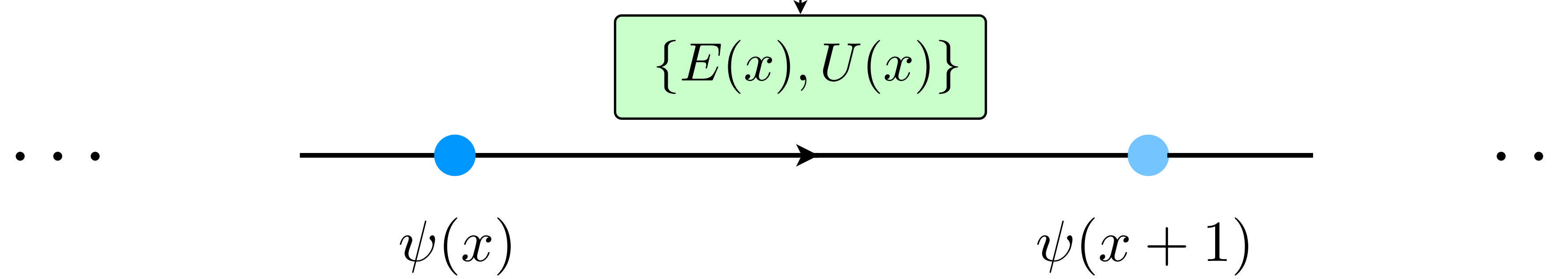


THE BASICS

E and *U* are conjugate variable pairs.
Not simultaneously diagonalizable!

$$[E(x), agA(x)] = \frac{1}{i}$$

or: $$[E, U] = U$$

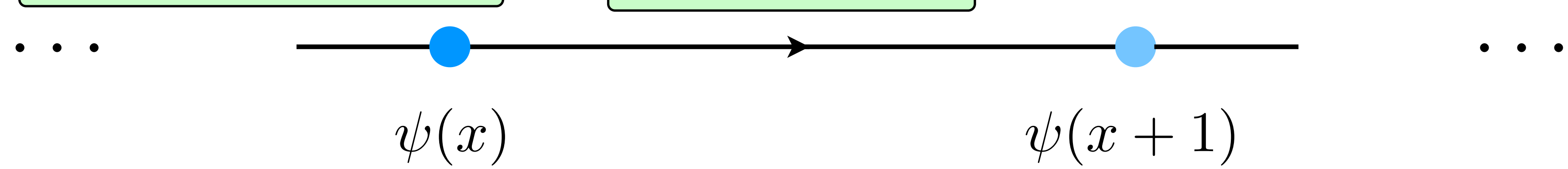


THE BASICS

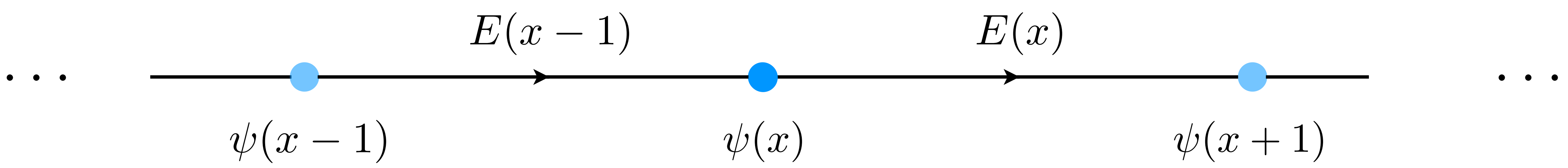
A discrete infinite-dimensional Hilbert space of a 1D quantum rotor: $E \in \mathbb{Z}$

$$\hat{E}|E\rangle = E|E\rangle$$
$$\hat{U}|E\rangle = |E+1\rangle$$

$$\{E(x), U(x)\}$$



THE BASICS

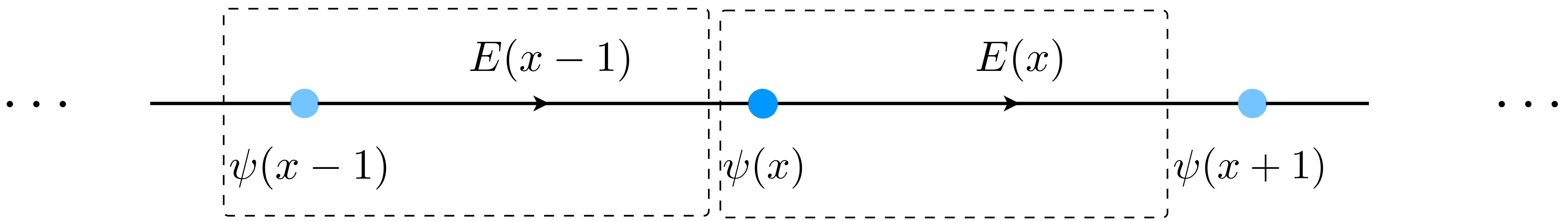


THE BASICS

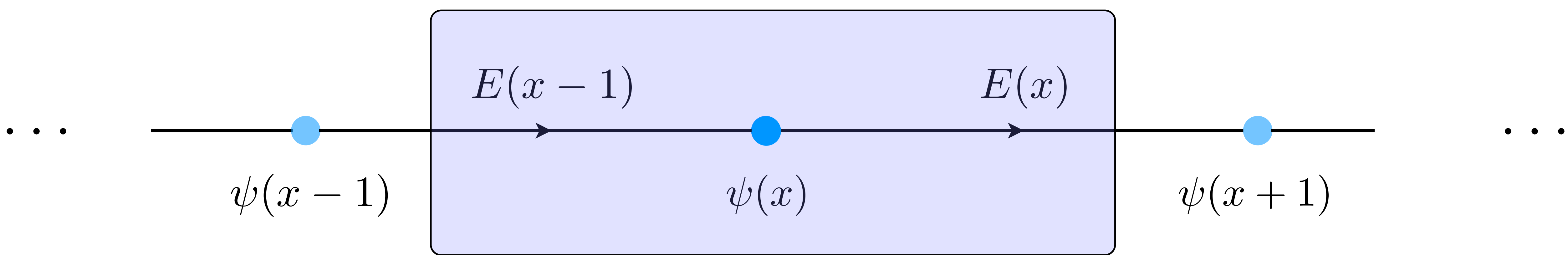
Therefore Hilbert space is spanned by the basis states:

$$\boxed{\cdots (|f\rangle_{x-1} \otimes |E\rangle_{x-1}) \otimes (|f\rangle_{x} \otimes |E\rangle_{x}) \cdots}$$

However, not all states are physical!



THE BASICS



$$G(x)|\psi\rangle_{\text{phys.}} = 0$$

$$G(x) = E(x) - E(x-1) + \psi^\dagger(x)\psi(x) - \frac{1-(-1)^x}{2}$$

Gauss's law constraint stating that the flux of the electric field is equal to the staggered electric charge.

EXAMPLE

Consider a two-site theory with periodic boundary conditions. Impose a cutoff $\Lambda = 1$ on the electric field such that $E \in [-\Lambda, \Lambda]$.

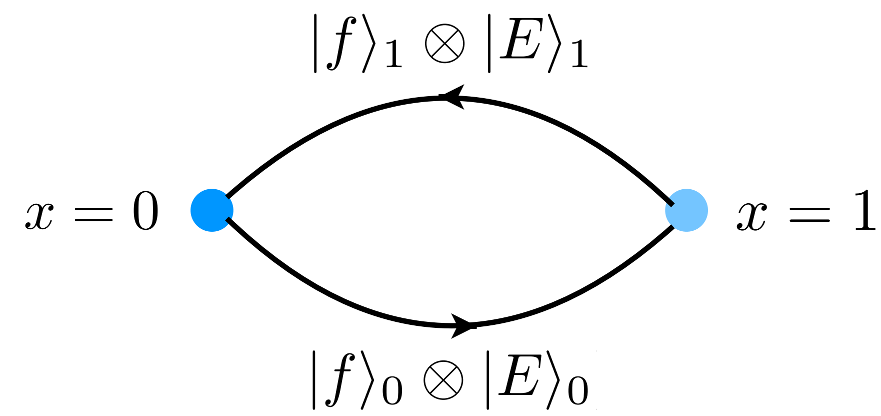
a) How many basis states are there?

b) What are the physical states? Identify the particle content of states.

c) What is the value of the total electric charge for each state?

EXAMPLE

Consider a two-site theory with periodic boundary conditions. Impose a cutoff $\Lambda = 1$ on the electric field such that $E \in [-\Lambda, \Lambda]$.



a) How many basis states are there?

There are $2^2 \times 3^2 = 36$ basis states.

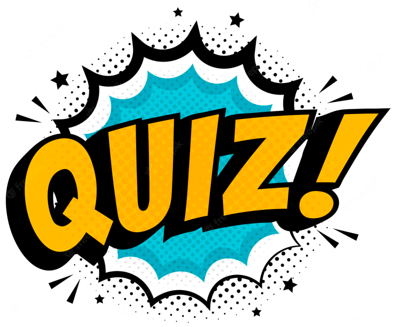
b) What are the physical states? Identify the particle content of states.

There are only 5 states consistent with the Gauss's law:

$(\lvert 0\rangle_0 \otimes \lvert -1\rangle_0) \otimes (\lvert 1\rangle_1 \otimes \lvert -1\rangle_1)$	No matter	
$(\lvert 0\rangle_0 \otimes \lvert 0\rangle_0) \otimes (\lvert 1\rangle_1 \otimes \lvert 0\rangle_1)$	No matter	
$(\lvert 0\rangle_0 \otimes \lvert 1\rangle_0) \otimes (\lvert 1\rangle_1 \otimes \lvert 1\rangle_1)$	No matter	
$(\lvert 1\rangle_0 \otimes \lvert 0\rangle_0) \otimes (\lvert 0\rangle_1 \otimes \lvert 1\rangle_1)$	e^-	e^+
$(\lvert 1\rangle_0 \otimes \lvert -1\rangle_0) \otimes (\lvert 0\rangle_1 \otimes \lvert 0\rangle_1)$	e^-	e^+

c) What is the value of the total electric charge for each state?

Recall that $Q(x) = -\psi^\dagger(x)\psi(x) + \dfrac{1-(-1)^x}{2}$, so $Q(0) + Q(1) = 0$ for all physical states.



How many different electric-charge sectors exist for lattice Schwinger model with periodic boundary conditions (with no background charges)? What about with open boundary conditions?

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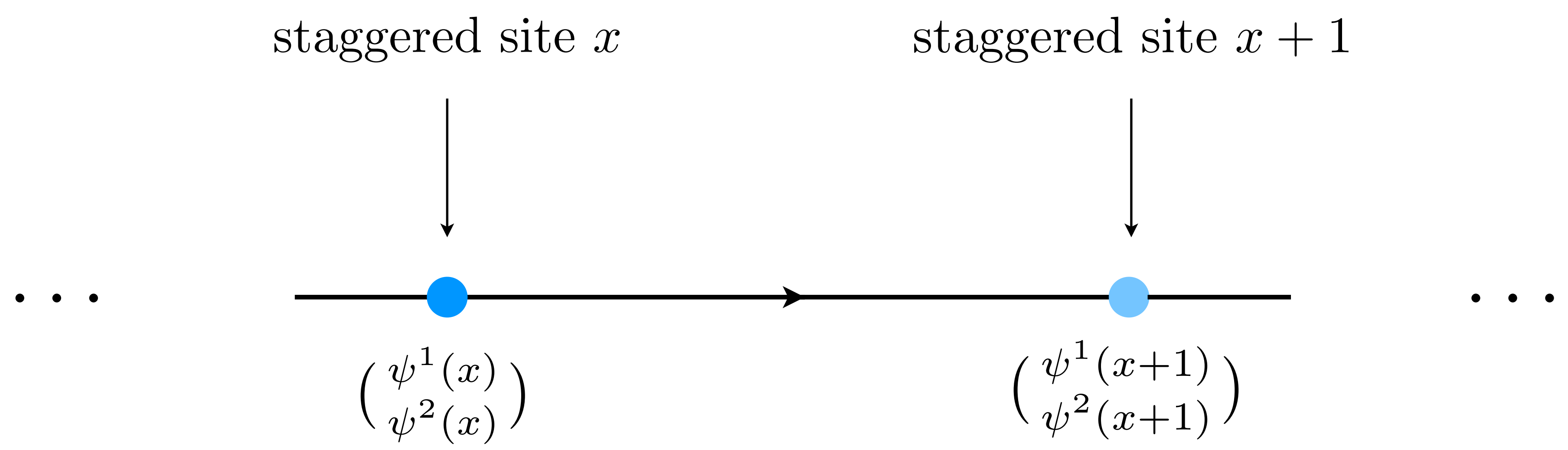
EXAMPLE: LET US CONSIDER THE CASE OF SU(2) LGT IN 1+1 D.

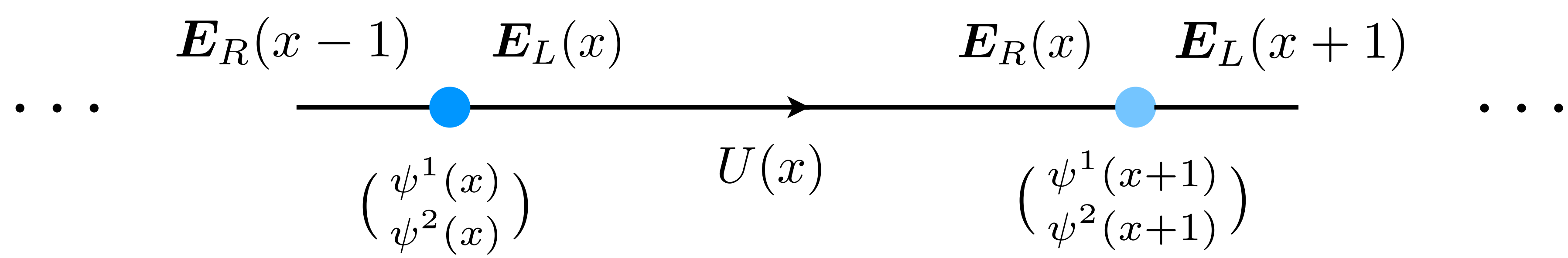
ZD, Raychowdhury, and Shaw, Phys. Rev. D 104, 074505, arXiv:2009.11802 [hep-lat]

THE BASICS



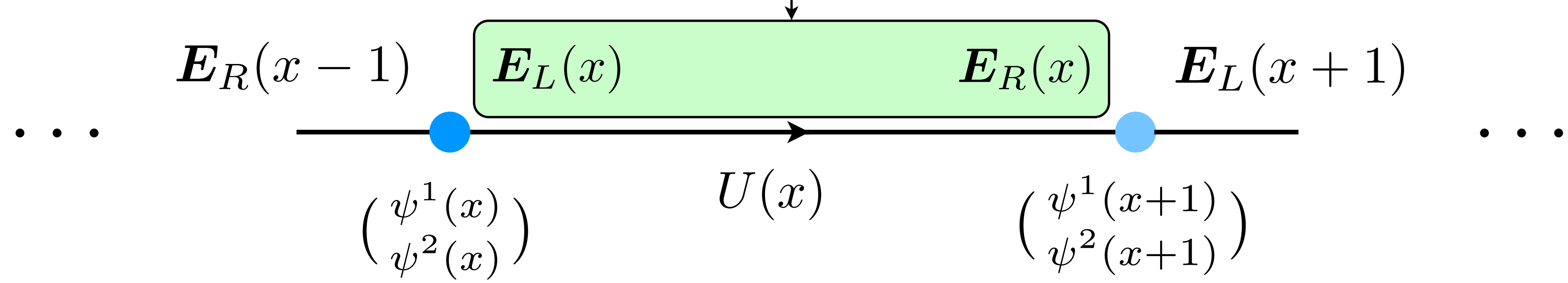
staggered site x
staggered site $x+1$
$\begin{pmatrix} \psi^1(x) \\ \psi^2(x) \end{pmatrix}$
$\begin{pmatrix} \psi^1(x{+}1) \\ \psi^2(x{+}1) \end{pmatrix}$

THE BASICS


$\boldsymbol{E}_R(x-1)$
$\boldsymbol{E}_L(x)$
$\boldsymbol{E}_R(x)$
$\boldsymbol{E}_L(x+1)$
$U(x)$
$\binom{\psi^1(x)}{\psi^2(x)}$
$\binom{\psi^1(x+1)}{\psi^2(x+1)}$

THE BASICS

$$
\begin{aligned}
[\hat{E}_L^a, \hat{E}_L^b] &= -i\epsilon^{abc}\hat{E}_L^c, \\
[\hat{E}_R^a, \hat{E}_R^b] &= i\epsilon^{abc}\hat{E}_R^c, \\
[\hat{E}_L^a, \hat{E}_R^b] &= 0,
\end{aligned}
$$



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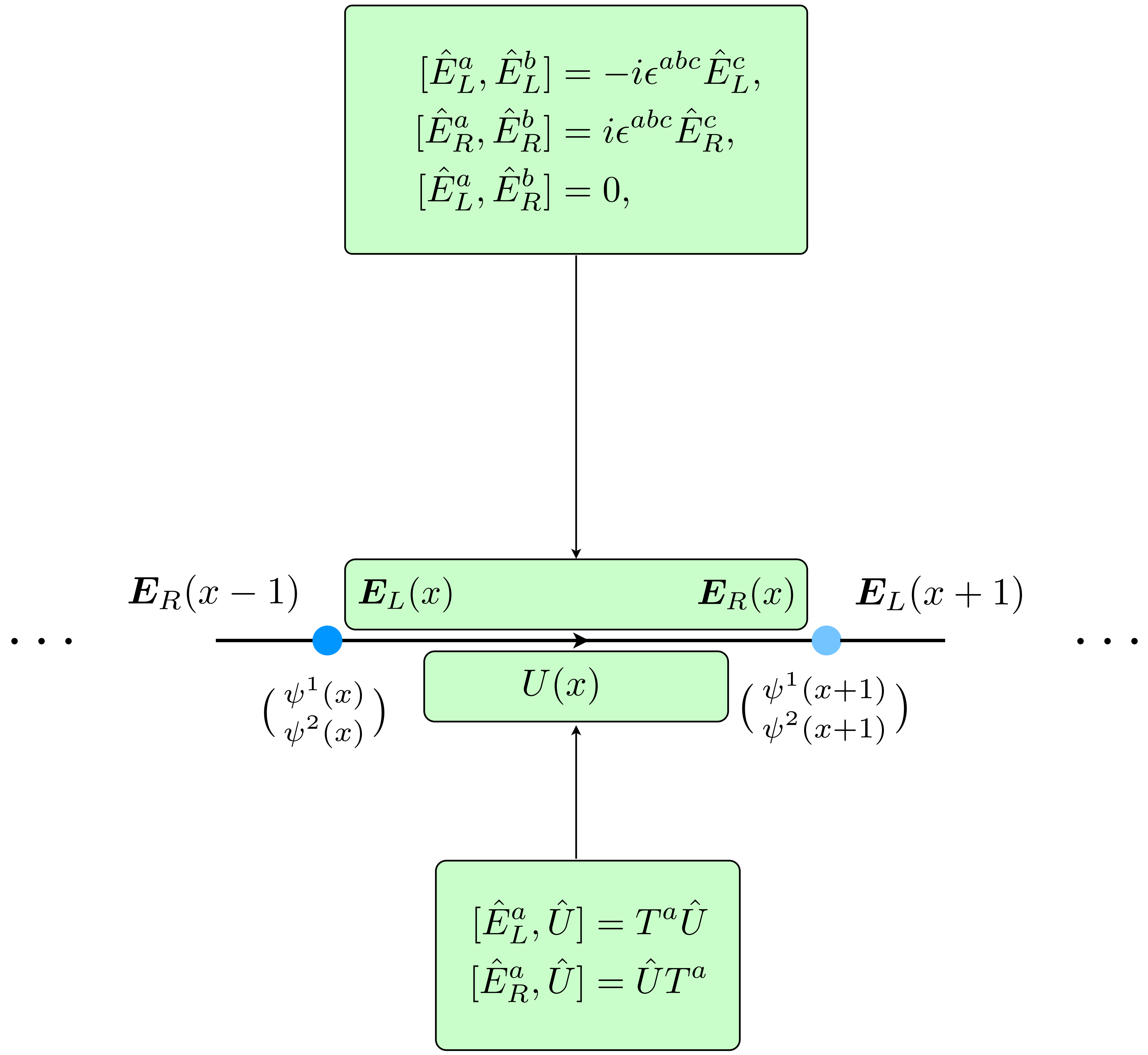
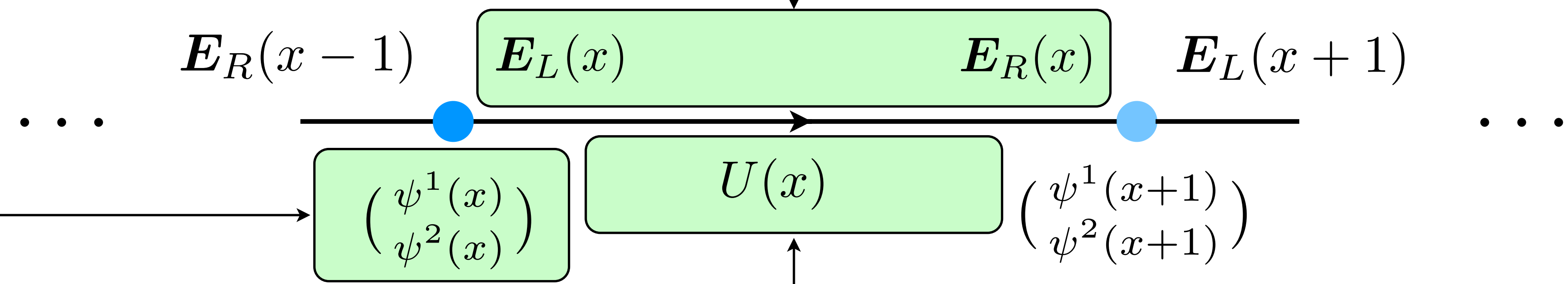


$[\hat{E}^a_L, \hat{E}^b_L] = -i\epsilon^{abc}\hat{E}^c_L,$
$[\hat{E}^a_R, \hat{E}^b_R] = i\epsilon^{abc}\hat{E}^c_R,$
$[\hat{E}^a_L, \hat{E}^b_R] = 0,$
$\boldsymbol{E}_R(x-1)$
$\boldsymbol{E}_L(x)$
$\boldsymbol{E}_R(x)$
$\boldsymbol{E}_L(x+1)$
$\cdots$
$\cdots$
$\binom{\psi^1(x)}{\psi^2(x)}$
$U(x)$
$\binom{\psi^1(x+1)}{\psi^2(x+1)}$
$[\hat{E}^a_L, \hat{U}] = T^a\hat{U}$
$[\hat{E}^a_R, \hat{U}] = \hat{U}T^a$

THE BASICS

$$
\begin{aligned}
[\hat{E}^a_L, \hat{E}^b_L] &= -i\epsilon^{abc}\hat{E}^c_L, \\
[\hat{E}^a_R, \hat{E}^b_R] &= i\epsilon^{abc}\hat{E}^c_R, \\
[\hat{E}^a_L, \hat{E}^b_R] &= 0,
\end{aligned}
$$



$$
\begin{aligned}
\hat{\rho}^a(x) &\equiv \psi^\dagger(x) T^a \psi(x), \\
[\hat{\rho}^a, \psi] &= -T^a \psi.
\end{aligned}
$$

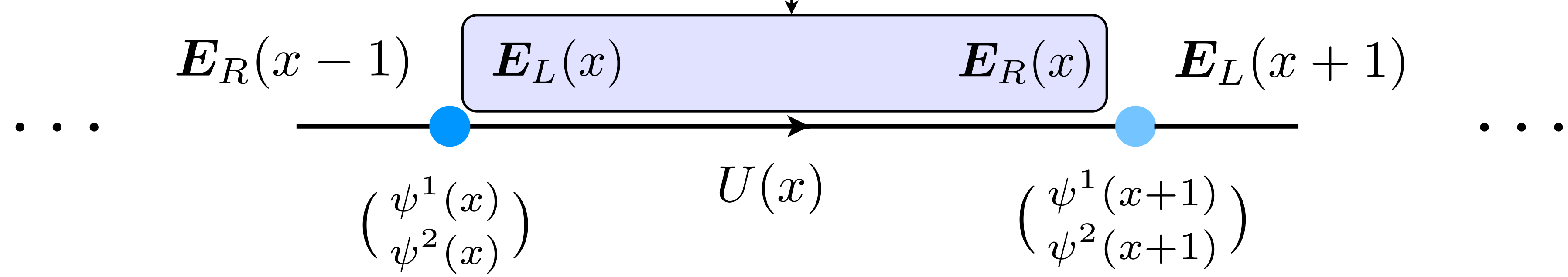
SU(2) fermionic charges

$$
\begin{aligned}
[\hat{E}^a_L, \hat{U}] &= T^a \hat{U} \\
[\hat{E}^a_R, \hat{U}] &= \hat{U} T^a
\end{aligned}
$$

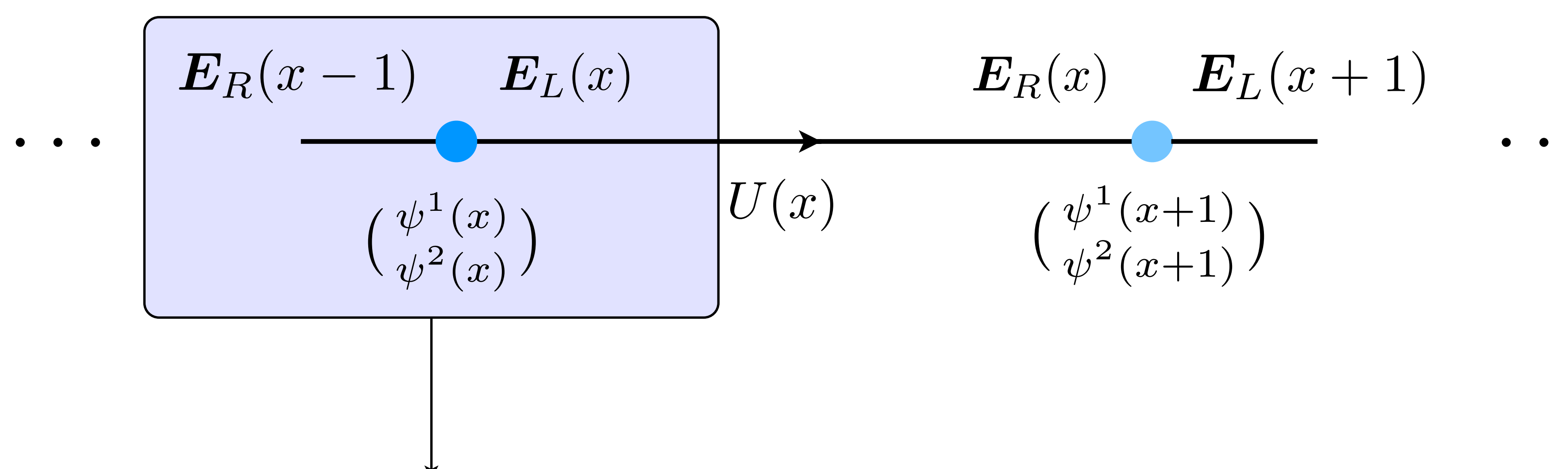
THE BASICS

An `Abelian' Gauss's law

$$\hat{\boldsymbol{E}}_L^2 = \hat{\boldsymbol{E}}_R^2$$



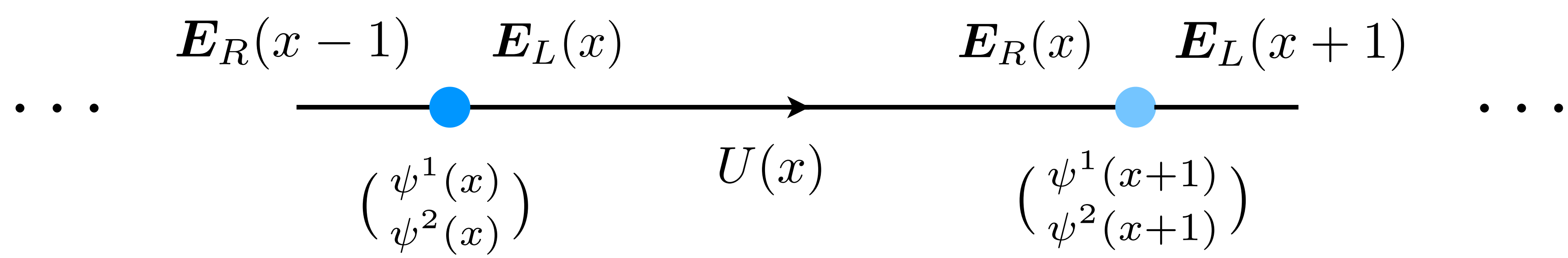
THE BASICS



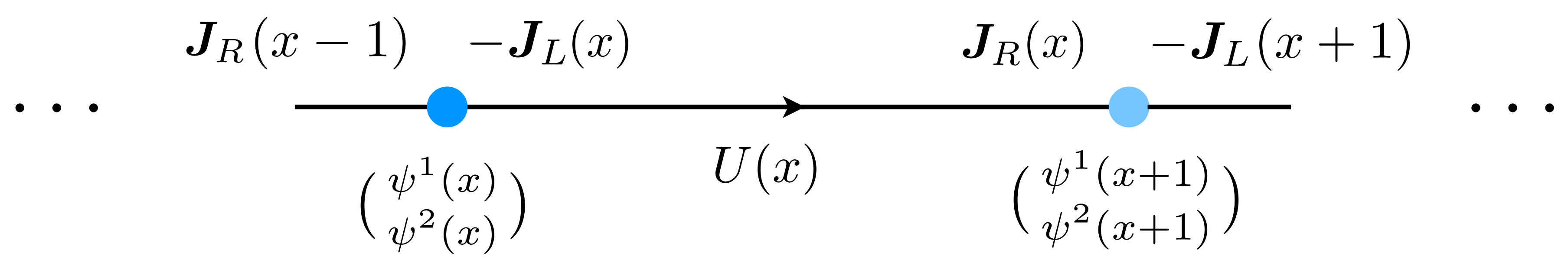
$$\hat{G}^a(x) = -\hat{E}^a_L(x) + \hat{E}^a_R(x-1) + \hat{\rho}^a(x)$$

Three Gauss's law operators

ANGULAR MOMENTUM BASIS OF SU(2) LGT



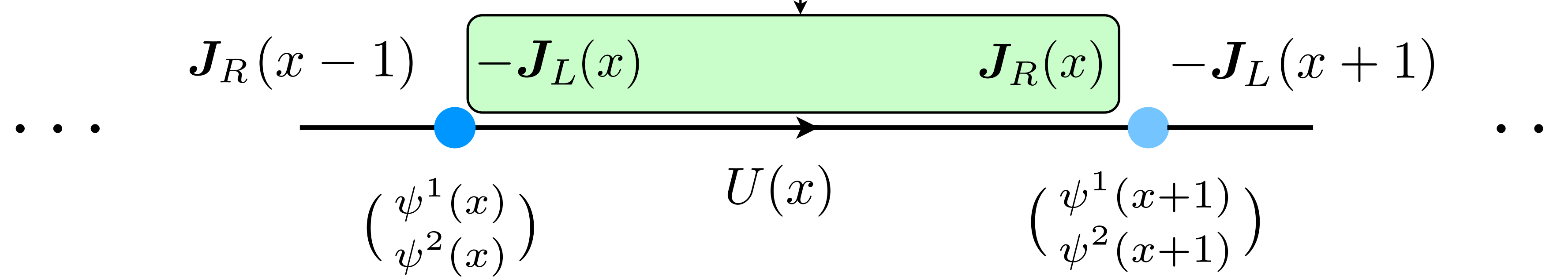
ANGULAR MOMENTUM BASIS OF SU(2) LGT



ANGULAR MOMENTUM BASIS OF SU(2) LGT

$$\hat{\boldsymbol{J}}_R^2\,|J_R,m_R\rangle = J_R(J_R+1)\,|J_R,m_R\rangle$$
$$\hat{\boldsymbol{J}}_L^2\,|J_L,m_L\rangle = J_L(J_L+1)\,|J_L,m_L\rangle$$

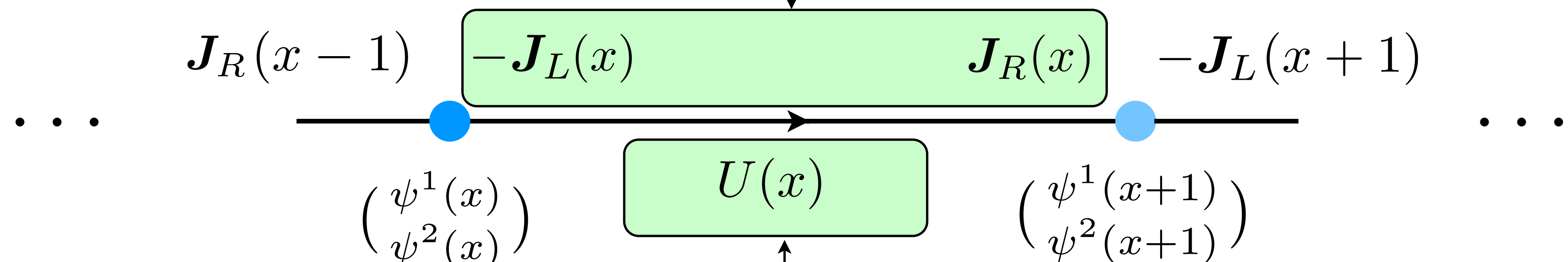
$$\hat{J}_R^3\,|J_R,m_R\rangle = m_R\,|J_R,m_R\rangle$$
$$\hat{J}_L^3\,|J_L,m_L\rangle = m_L\,|J_L,m_L\rangle$$



ANGULAR MOMENTUM BASIS OF SU(2) LGT

$$\hat{\boldsymbol{J}}_R^2 \, |J_R, m_R\rangle = J_R(J_R+1)\, |J_R, m_R\rangle$$
$$\hat{\boldsymbol{J}}_L^2 \, |J_L, m_L\rangle = J_L(J_L+1)\, |J_L, m_L\rangle$$

$$\hat{J}_R^3 \, |J_R, m_R\rangle = m_R \, |J_R, m_R\rangle$$
$$\hat{J}_L^3 \, |J_L, m_L\rangle = m_L \, |J_L, m_L\rangle$$



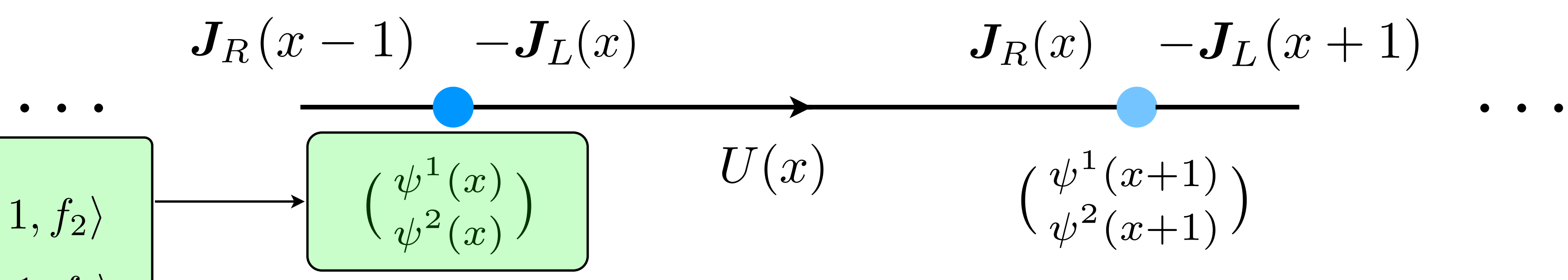
$$\hat{U}^{(\alpha,\beta)}(x)\, |J_L, m_L\rangle^{(x)} \otimes |J_R, m_R\rangle^{(x)} =$$

$$\sum_{j=J\pm\frac{1}{2}} \sqrt{\frac{2J+1}{2j+1}}\, \langle J, m_L; \frac{1}{2}, \alpha | j, m_L+\alpha\rangle \langle J, m_R; \frac{1}{2}, \beta | j, m_R+\beta\rangle \, |j, m_L+\alpha\rangle^{(x)} \otimes |j, m_R+\beta\rangle^{(x)}$$

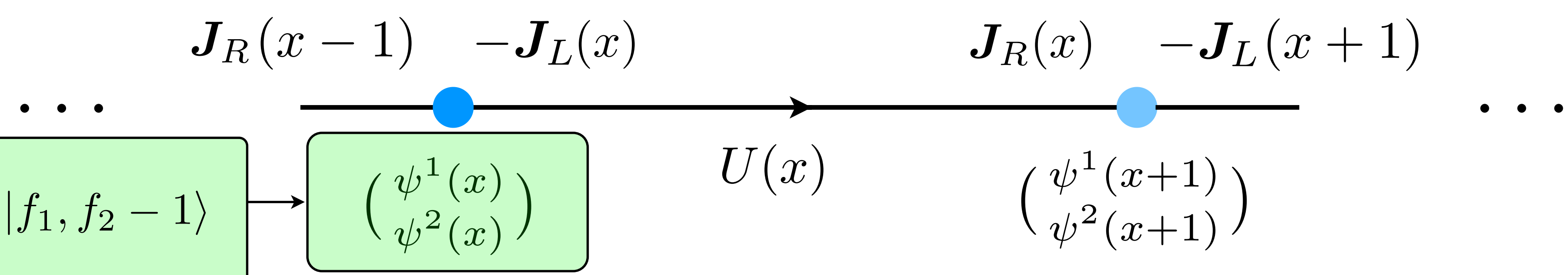
ANGULAR MOMENTUM BASIS OF SU(2) LGT

$$\psi^1 \left| f_1, f_2 \right\rangle = (1 - \delta_{f_1,0}) \left| f_1 - 1, f_2 \right\rangle$$

$$\psi^{1^\dagger} \left| f_1, f_2 \right\rangle = (1 - \delta_{f_1,1}) \left| f_1 + 1, f_2 \right\rangle$$



ANGULAR MOMENTUM BASIS OF SU(2) LGT



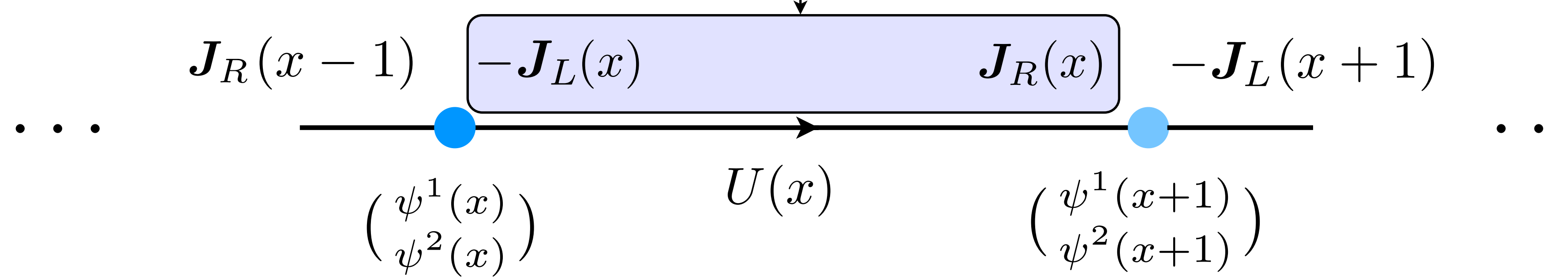
$$\psi_2|f_1,f_2\rangle = (-1)^{f_1}(1-\delta_{f_2,0})|f_1,f_2-1\rangle$$

$$\psi_2^\dagger|f_1,f_2\rangle = (-1)^{f_1}(1-\delta_{f_2,1})|f_1,f_2+1\rangle$$

PHYSICAL CONSTRAINTS

An `Abelian' Gauss's law

$$\hat{\boldsymbol{J}}_L^2 = \hat{\boldsymbol{J}}_R^2$$



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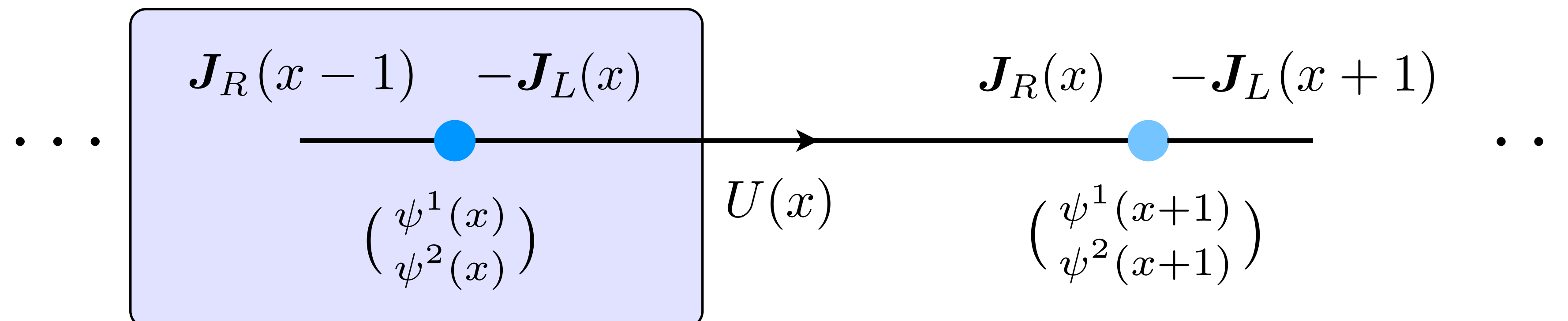
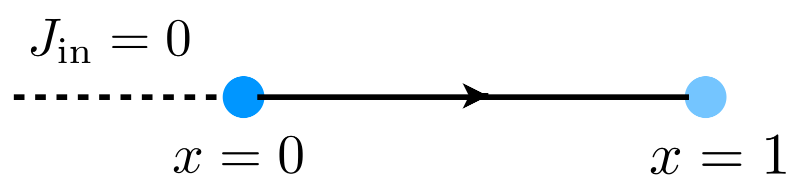


$$\left[\hat{J}_L^a(x) + \hat{J}_R^a(x-1) + \frac{1}{2}\psi^\dagger(x)\tau^a\psi(x)\right]|\phi\rangle_{(\mathrm{KS})} = 0$$

Non-Abelian Gauss's laws

EXAMPLE

Consider a two-site theory with open boundary conditions. Impose a cutoff $\Lambda = 1/2$ on the total angular momentum on each link such that only $J = 0, 1/2$ values are allowed. The incoming angular momentum is set to zero.



a) How many basis states are there?

b) What are the physical states in the sector with $\nu = 1$ where $\nu \equiv \frac{1}{2} \sum_x \psi^\dagger(x)\psi(x)$?

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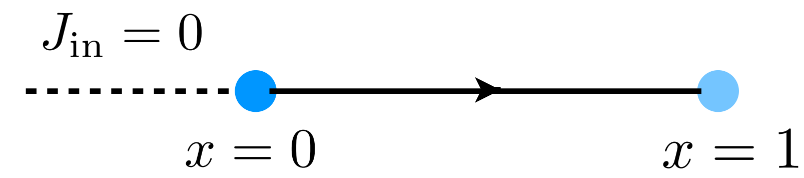

a) How many basis states are there?

There are $4^2 \times 5 = 80$ basis states (4 fermionic states $|f_1, f_2\rangle$ at each site and 5 angular momentum states $|J, m_L\rangle \otimes |J, m_L\rangle$ on the only link.).

b) What are the physical states in the sector with $\nu = 1$ where $\nu \equiv \frac{1}{2}\sum_x \psi^\dagger(x)\psi(x)$?

1) $\left[|0,0\rangle\,|0,0\rangle\,|0,0\rangle\right]^{(0)} \otimes \left[|0,0\rangle\,|1,1\rangle\,|0,0\rangle\right]^{(1)}$

2) $$\frac{1}{2}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|0,1\rangle\,|0,0\rangle\right]^{(1)}$$
$$-\frac{1}{2}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|1,0\rangle\,|0,0\rangle\right]^{(1)}$$
$$-\frac{1}{2}\left[|0,0\rangle\,|0,1\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|0,1\rangle\,|0,0\rangle\right]^{(1)}$$
$$+\frac{1}{2}\left[|0,0\rangle\,|0,1\rangle\,|\tfrac{1}{2},\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},-\tfrac{1}{2}\rangle\,|1,0\rangle\,|0,0\rangle\right]^{(1)}$$

3) $$\frac{1}{\sqrt{6}}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|1,0\rangle\,|1,-1\rangle\right]^{(1)}$$
$$-\frac{1}{2\sqrt{3}}\left[|0,0\rangle\,|1,0\rangle\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle\right]^{(0)} \otimes \left[|\tfrac{1}{2},\tfrac{1}{2}\rangle\,|0,1\rangle\,|1,0\rangle\right]^{(1)}$$
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4) $\left[|0,0\rangle\,|1,1\rangle\,|0,0\rangle\right]^{(0)} \otimes \left[|0,0\rangle\,|0,0\rangle\,|0,0\rangle\right]^{(1}$

OUTLINE OF PART I:
HAMILTONIAN FORMULATION OF LATTICE GAUGE THEORIES

i) Hamiltonian vs. Lagrangian formulation of LGTs

ii) Kogut-Susskind formulation: Basis states, Hilbert space, and constraints

- An Abelian case: U(1) LGT
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KOGUT-SUSSKIND HAMILTONIAN

$$H^{(\mathrm{KS})} = \boxed{H_I^{(\mathrm{KS})}} + H_E^{(\mathrm{KS})} + H_M^{(\mathrm{KS})}$$

$$H_I^{(\mathrm{KS})} = \frac{1}{2a} \sum_x \left[\psi^\dagger(x) \hat{U}(x) \psi(x+1) + \mathrm{h.c.} \right]$$

Interaction term

[Hopping terms in the y and z directions with site-dependent coupling if you consider > 1+1 D.]

KOGUT-SUSSKIND HAMILTONIAN

$$H^{(\mathrm{KS})} = H_I^{(\mathrm{KS})} + \boxed{H_E^{(\mathrm{KS})}} + H_M^{(\mathrm{KS})}$$

$$\boxed{H_E^{(\mathrm{KS})} = \frac{g^2 a}{2} \sum_x \hat{\boldsymbol{E}}(x)^2}$$

Electric field term

KOGUT-SUSSKIND HAMILTONIAN

$$H^{(\mathrm{KS})} = H_I^{(\mathrm{KS})} + H_E^{(\mathrm{KS})} + \boxed{H_M^{(\mathrm{KS})}}$$

$$H_M^{(\mathrm{KS})} = m \sum_x (-1)^x \psi^\dagger(x)\psi(x)$$

Mass term

KOGUT-SUSSKIND HAMILTONIAN

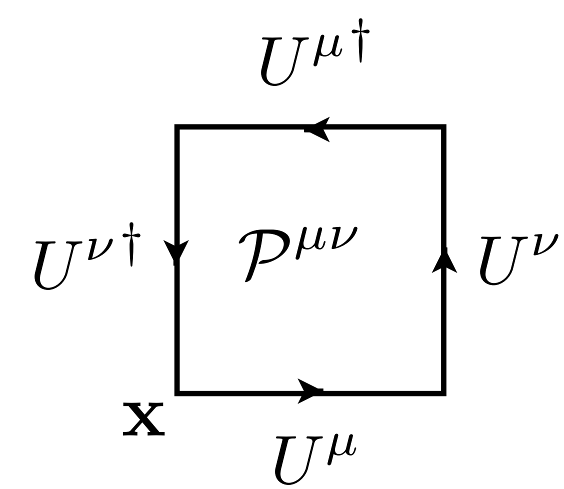
Only in d+1 D with d>1

$$H^{(\mathrm{KS})} = H_I^{(\mathrm{KS})} + H_E^{(\mathrm{KS})} + H_M^{(\mathrm{KS})} + \boxed{H_B^{(\mathrm{KS})}}$$

$$\boxed{H_B^{(KS)} = \frac{a^d}{a^4 g^2} \sum_{\mathbf{x}} \mathrm{Tr}\left[2 - \mathcal{P}_{\mu\nu}(\mathbf{x}) - \mathcal{P}_{\mu\nu}^{\dagger}(\mathbf{x})\right]}$$

Magnetic or plaquette term

[$\propto B^2$ in the continuum limit ($a \to 0$).]



EXAMPLE

What is the vacuum of the Kogut-Susskind Hamiltonian in the U(1) case in the strong-coupling limit ($g \to \infty$)? Consider both massless and massive fermions.

EXAMPLE

What is the vacuum of the Kogut-Susskind Hamiltonian in the U(1) case in the strong-coupling limit ($g \to \infty$)? Consider both massless and massive fermions.

Since the electric-field term dominates in the strong-coupling limit, the vacuum corresponds to no electric field flux. In the **massless** limit, the vacuum is degenerate and consists of either

$$(|1\rangle_0 \otimes |0\rangle_0) \otimes (|0\rangle_1 \otimes |0\rangle_1) \otimes (|1\rangle_2 \otimes |0\rangle_2) \otimes (|0\rangle_2 \otimes |0\rangle_2)\cdots$$

or

$$(|0\rangle_0 \otimes |0\rangle_0) \otimes (|1\rangle_1 \otimes |0\rangle_1) \otimes (|0\rangle_2 \otimes |0\rangle_2) \otimes (|1\rangle_2 \otimes |0\rangle_2)\cdots$$

since only these two states are consistent with Gauss's law (with no mass, even and odd labeling of the sites is arbitrary).

In the **massive** limit, the degeneracy is lifted and the state with the least energy is that with the lowest mass term, which is the second option above with mass term equal to $-\frac{N}{2}m$ where N is the number of staggered sites (even and odd labeling is no longer arbitrary).

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PURELY FERMIONIC FORMULATION (ONLY IN 1+1 D AND WITH OPEN BCs)

EXAMPLE

Show that the Schwinger model Hamiltonian becomes:

$$H = \frac{1}{2a}\sum_x \left[\psi^\dagger(x)\psi(x+1) + \text{h.c.}\right] + \frac{a}{2}\sum_x \left\{\varepsilon_0 - \sum_{y=0}^{x}\left[\psi^\dagger(y)\psi(y) - \frac{1-(-1)^y}{2}\right]\right\}^2 + m\sum_x (-1)^x \psi^\dagger(x)\psi(x)$$

with open boundary conditions where ε_0 denote a fixed incoming electric field. This means that local fermion-boson formulation is replaced by a non-local fermionic formulation.

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with open boundary conditions where ε_0 denote a fixed incoming electric field. This means that local fermion-boson formulation is replaced by a non-local fermionic formulation.

i) Let us first transform to a gauge where $U = \mathbb{I}$:

$$\psi(x) \to \psi'(x) = (\prod_{y<x} U(y))\psi(x)$$

$$\psi^\dagger(x) \to {\psi'}^\dagger(x) = \psi^\dagger(x)(\prod_{y<x} U(y))^\dagger$$

$$U(x) \to U'(x) = (\prod_{y<x} U(y))U(x)(\prod_{y<x+1} U(y))^\dagger = \mathbb{I}$$

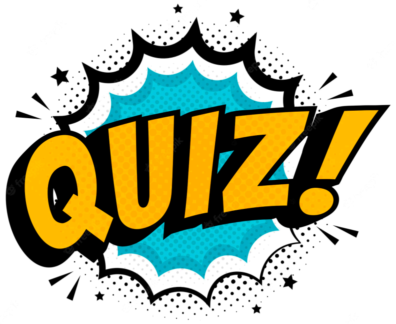
ii) Now exploit the Gauss's law to rewrite $E(x)$ in terms of the matter charge $Q(x)$:

$$E(0) = \varepsilon_0 + Q(0)$$

$$E(1) = E(0) + Q(1) = \varepsilon_0 + Q(0) + Q(1)$$

$$E(x) = E(x-1) + Q(x) = \varepsilon_0 + \sum_{y\leq x} Q(y)$$

i) and ii) give directly the fermionic Hamiltonian above given the definition of $Q(x)$.



Why can we not fully remove the gauge fields in a theory with periodic boundary conditions? What about higher dimensions?

PURELY FERMIONIC FORMULATION FOR THE SU(2) LGT IN 1+1 D

$$H^{(\mathrm{KS})} = \boxed{H_I^{(\mathrm{KS})}} + H_E^{(\mathrm{KS})} + H_M^{(\mathrm{KS})}$$

$$H_I^{(\mathrm{KS})} = \frac{1}{2a}\sum_x \left[\psi^\dagger(x)\hat{U}(x)\psi(x+1) + \mathrm{h.c.}\right]$$

There is a gauge transformation to gauge $U' = \mathbb{I}$.

$$H_I^{(\mathrm{F})} = \frac{1}{2a}\sum_x \left[\psi^{\dagger\prime}(x)\psi'(x+1) + \mathrm{h.c.}\right]$$

Interaction term

PURELY FERMIONIC FORMULATION FOR THE SU(2) LGT IN 1+1 D

$$H^{(\mathrm{KS})} = H_I^{(\mathrm{KS})} + \boxed{H_E^{(\mathrm{KS})}} + H_M^{(\mathrm{KS})}$$

$$H_E^{(\mathrm{KS})} = \frac{g^2 a}{2} \sum_x \hat{\boldsymbol{E}}(x)^2$$

Using Gauss's laws and transforming to gauge $U' = \mathbb{I}$:

$$H_E^{(\mathrm{F})} = \frac{g^2 a}{2} \sum_x \sum_{a=1}^{3} \left[\epsilon_0^a + \sum_{y=0}^{x} \psi^{\dagger'}(y) T^a \psi'(y) \right]^2$$

Electric field term

PURELY FERMIONIC FORMULATION FOR THE SU(2) LGT IN 1+1 D

$$H^{(\mathrm{KS})} = H_I^{(\mathrm{KS})} + H_E^{(\mathrm{KS})} + \boxed{H_M^{(\mathrm{KS})}}$$

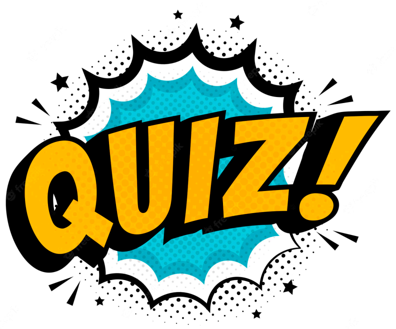
$$H_M^{(\mathrm{KS})} = m \sum_x (-1)^x \psi^\dagger(x)\psi(x)$$

Transforming to gauge $U' = \mathbb{I}$:

$$H_M^{(\mathrm{F})} = m \sum_x (-1)^x \psi^{\dagger'}(x)\psi'(x)$$

Mass term

…all this only works in 1+1 D and with open boundary conditions.



In the example of the two-site theory in the SU(2) model, we found out that there are 4 physical basis states in the $\nu = 1$ sector with open boundary conditions with $J_{\text{in}} = 0$ and with up to $J = 1/2$ angular momentum. How many physical basis states are there in the fully fermionic formulation? Do you see a mismatch? How do you explain it? Convince yourself that the two theories have exactly the same spectrum.

IN GENERAL, MANY HAMILTONIAN FORMULATIONS OF GAUGE THEORIES EXIST…WHICH ONE TO PICK?

So far, we only considered the **electric or irreducible representation (irrep) basis** in which only H_E is diagonal!

$$H^{(\mathrm{KS})} = \boxed{H_I^{(\mathrm{KS})}} + \boxed{H_E^{(\mathrm{KS})}} + H_M^{(\mathrm{KS})} + \boxed{H_B^{(\mathrm{KS})}}$$

In the **group-element basis**, only H_I and H_B are diagonal.

In the **magnetic or dual basis** H_B is diagonal.

$$H^{(\mathrm{KS})} = \boxed{H_I^{(\mathrm{KS})}} + \boxed{H_E^{(\mathrm{KS})}} + H_M^{(\mathrm{KS})} + \boxed{H_B^{(\mathrm{KS})}}$$

The imposition of Gauss's law is simplest in the electric-field basis. On the other hand, toward the continuum limit ($ag \to 0$), many electric field excitations need to be kept and a large Hilbert space may need to be considered.

Recall $\boxed{H_E \propto g^2}$ while $\boxed{H_B \propto 1/g^2}$.

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EXAMPLES OF **CLASSICAL-COMPUTING METHODS**

i) Analytical methods such as **strong-coupling expansion**
See e.g., Bank et al, PRD 13, 1043 (1976). It has limited scope but can give intuitive insight.

ii) Exact numerical methods, i.e., **exact diagonalization** (ED)
It is ideal but costly and often unpractical even for small LGT problems.

iii) Approximate numerical methods such as **tensor-networks** (TN)
Very efficient for certain quantities such as low-energy spectrum, but limited in scope when entanglement grows beyond area law, such as in real-time problems. Limited studies in higher dimensions. See e.g. this Ph.D. thesis by S. Kuhn for an overview of TN methods in high-energy physics.

STAY TUNED TO THE NEXT PART FOR
QUANTUM-COMPUTING METHODS!

TO BE CONTINUED…

QUESTIONS?