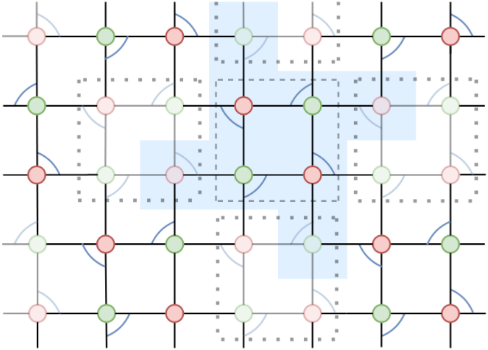
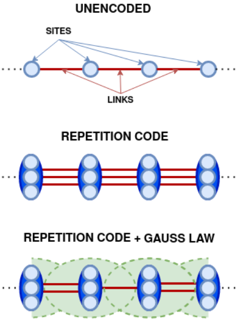
Quantum Error Correction Codes and Lattice Gauge Theories

Alessandro Roggero





DTP/TALENT School

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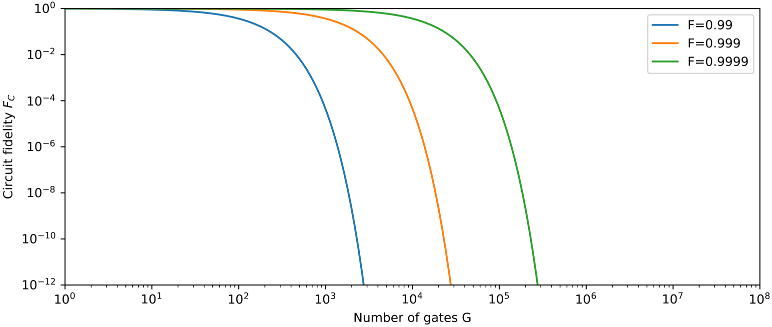
Why Quantum Error Correction?

Imagine you have some quantum algorithm and you counted how many gates you need to carry it out. How do we know if you can pull it off?

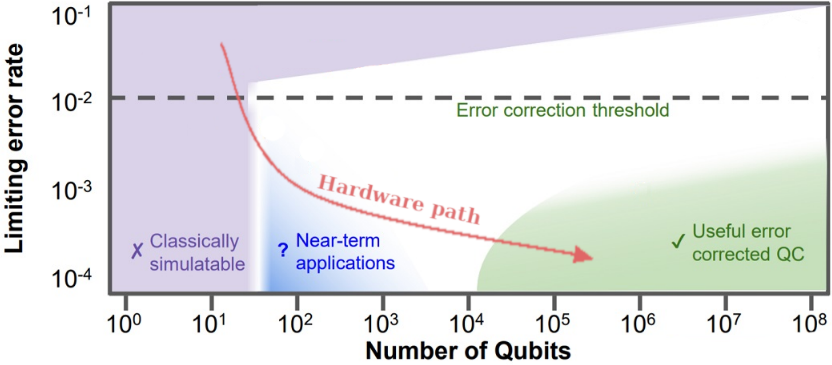
Say the number of gates you need is G and that you have access to a quantum computer that implements your gates with error at most ϵ.

- the **fidelity** of your gate is given by $F_g = (1-\epsilon)$
- the fidelity of the whole circuit is $F_C = F_g^G = (1-\epsilon)^G \approx e^{-\epsilon G}$



Where are we right now?

image adapted from Google AI



- inverse error rate gives the total coherence "time" τ_{coh}

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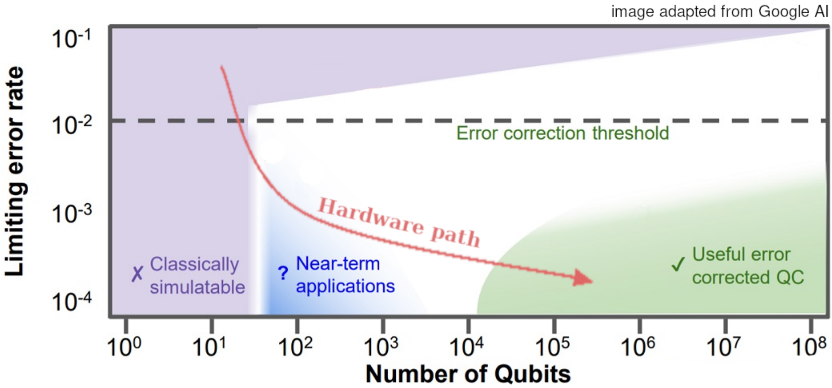

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Threshold Theorem(s) Ben-Or, Aharonov, Kitaev, Knill, Gottesman,. . .

When rate below threshold can extend $\tau_{coh}^{eff} \to \infty$ with $polylog(N)$ effort

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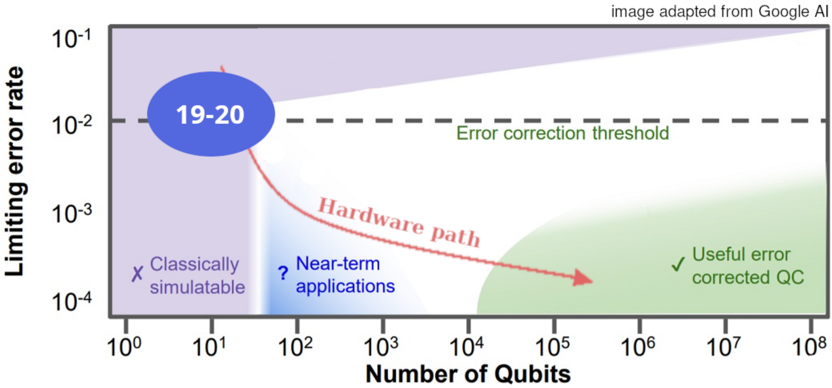

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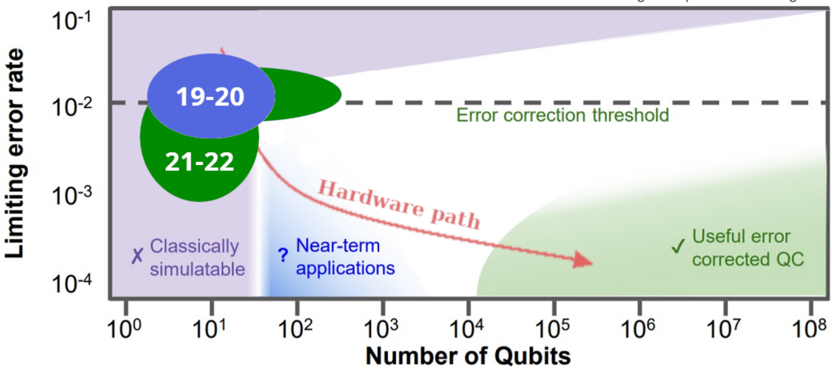

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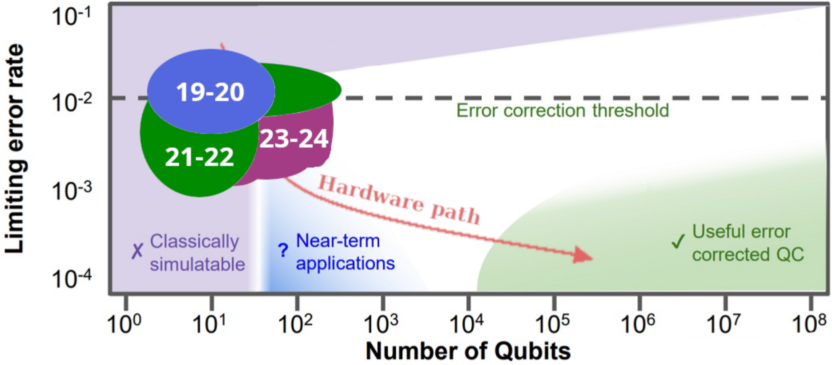

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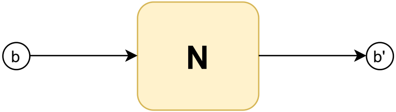
Classical error correction



- 0 error occurred: $b' = b$
- 1 error occurred: $b' = \bar{b}$

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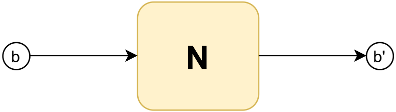

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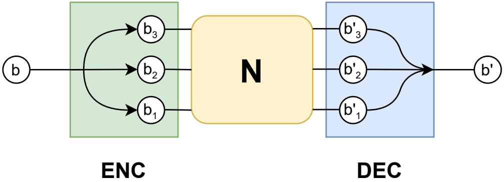

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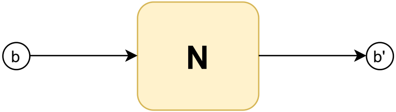

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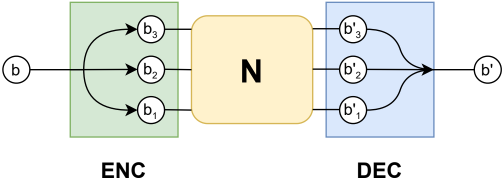

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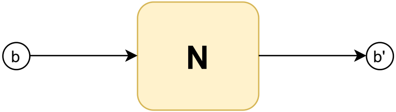

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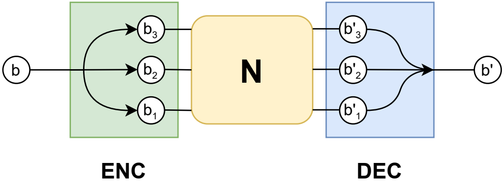

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Cicle repeated multiple times as we carry out operations on the information
$\Rightarrow$ requires to make copies of b at any given point in the execution

Quantum Error Correction

1. since we cannot make copies of unknown states, we need to enlarge the original Hilbert space and look at state evolution there
2. identify **logical states** as those invariant under a set of symmetries
   - this takes the place of the classical map $|b\rangle \rightarrow |bbb\rangle$
3. measure the symmetry operators in order to detect and fix errors
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2. group elements:
   $g_1 = Z_1 Z_2 \quad g_2 = Z_2 Z_3$
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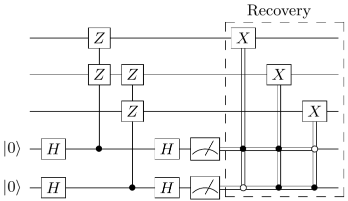
3. 

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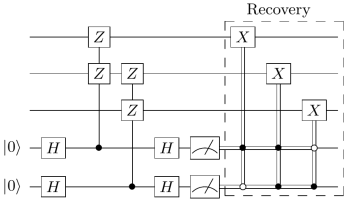
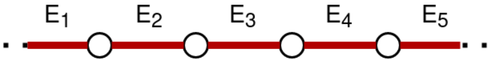
3. 

If we do this for every degree of freedom (eg. qubit) we turn any quantum computation into a gauge theory

Simple example: U(1) pure gauge



Symmetry generator connecting links across site

$$\hat{G}_l = \hat{E}_{l+1} - \hat{E}_l$$

$$\hat{G}_l|phys\rangle = 0 \quad \Rightarrow \quad \exp(i\theta\hat{G}_l)|phys\rangle = |phys\rangle$$

Focusing only on two links we will have then

$$|\Psi_{l,l+1}\rangle = \sum_{\epsilon_A,\epsilon_B} \Psi_{\epsilon_A,\epsilon_B} |\epsilon_A\rangle_l \otimes |\epsilon_B\rangle_{l+1}$$

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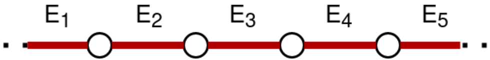

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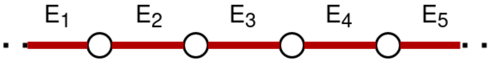

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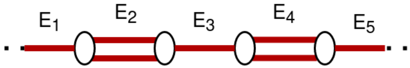
In order to obtain the full repetition code we add one link every other one

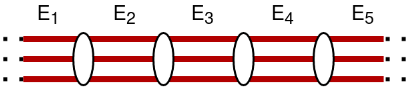


$$\left|\Psi^{odd}_{l,l+1}\right\rangle_L = \sum_{\epsilon} \Psi_\epsilon |\epsilon\rangle_l \otimes |\epsilon\rangle_{l+1} \otimes |\epsilon\rangle_{l+1}$$

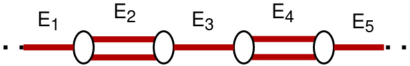
$$\left|\Psi^{even}_{l,l+1}\right\rangle_L = \sum_{\epsilon} \Psi_\epsilon |\epsilon\rangle_l \otimes |\epsilon\rangle_l \otimes |\epsilon\rangle_{l+1}$$

Simple example: qubit cost for U(1) pure gauge

Cost for a system with 2N links (L qubits per link)



6NL qubits



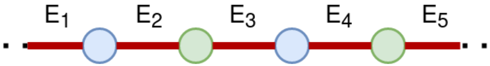
3NL qubits

This code can only correct X errors $\longrightarrow$ code concatenation

$[[9,1,3]] \times 2NL$	$[[5,1,3]] \times 2NL$	Gauss Law
18 NL	10 NL	9 NL

Gauss code is 10% cheaper than best possible single qubit code

Adding fermions in a U(1) theory



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$\hat{G}_l = \hat{E}_{l+1} - \hat{E}_l - q_l \hat{n}_l$

Focusing only on two links and the site in between we will have then

$$|\Psi_{l,l+1}\rangle = \sum_{\epsilon_A,\epsilon_B} \sum_{n=0,1} \Psi^n_{\epsilon_A,\epsilon_B} |\epsilon_A\rangle_l \otimes |n\rangle \otimes |\epsilon_B\rangle_{l+1}$$

$$\longrightarrow |\Psi_{l,l+1}\rangle_{phys} = \sum_{\epsilon} \sum_{n=0,1} \Psi^n_{\epsilon} |\epsilon\rangle_l \otimes |n\rangle \otimes |\epsilon + nq_l\rangle_{l+1}$$

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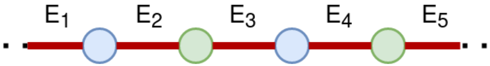

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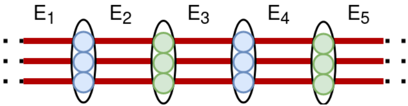
We can bring this in the required form if we can perform the map

$$\hat{W}_l |\Psi_{l,l+1}\rangle_{phys} = \sum_{\epsilon} \sum_{n=0,1} \Psi^n_{\epsilon} |\epsilon\rangle_l \otimes |n\rangle \otimes |\epsilon\rangle_{l+1}$$

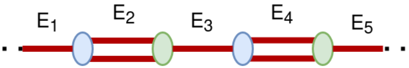
Operation is easy to do transversally for 1 qubit per link ($U(1) \approx \mathbb{Z}_2$)

Adding fermions in a truncated U(1) or $\mathbb{Z}_2$ theory

Cost for a system with 2N sites and 2N links (1 qubit per link)



12N qubits



5N qubits
(sites don't need encoding)

This code can only correct X errors $\longrightarrow$ code concatenation

$[[9,1,3]] \times 4N$	$[[5,1,3]] \times 4N$	Gauss Law
36 N	20 N	15 N

Gauss code is 25% cheaper than best possible single qubit code

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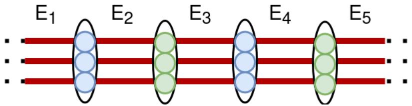

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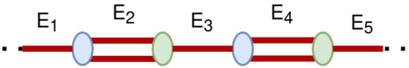

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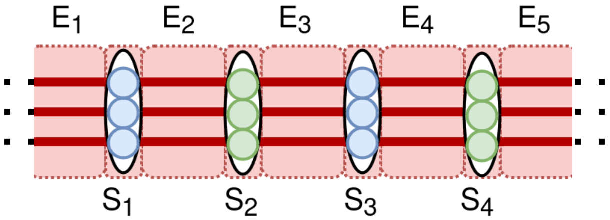
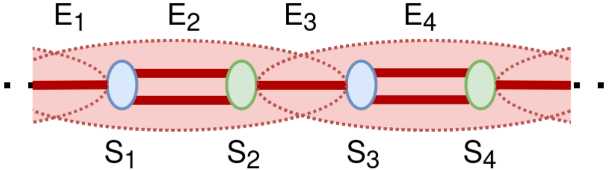
ADDITIONAL BONUS: some "two qubit" operations are cheaper too

Where's the catch?

More conventional error correction schemes that encode individual logical qubits are likely able to sustain a higher error rate





If the error rate is sufficiently low one can use an even sparser Gauss code

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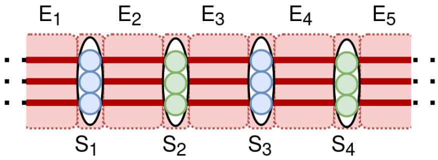

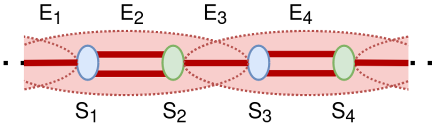

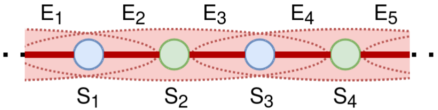

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Scaling up in dimension

Spagnoli, AR, Wiebe, arXiv:2405.19293 (2024)

Working with a low enough error rate so that we expect at most one error on a patch containing one site, the $2D$ sites sorrounding it and the links connected to them we can build an error correcting code with parameters

$$[[3(M + DM), DM, 3]]$$

Out of the $M + DM$ initial degrees of freedom, the M gauge constraints leave only DM logical ones. With staggered fermions $M = 2N$.

D	$[[9,1,3]] \times 2(1+D)N$	$[[5,1,3]] \times 2(1+D)N$	Gauss Law
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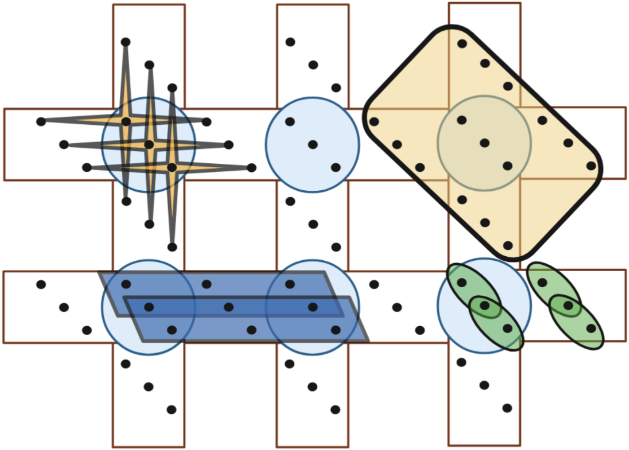
If we allow for non local stabilizers for phase flips, we can lower this down

$$[[M+DM+O(\log(M+DM)), DM, 3]]$$

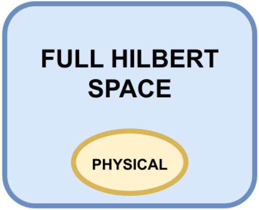
Scaling up in dimension: example with $D = 2$

Spagnoli, AR, Wiebe, arXiv:2405.19293 (2024)



Multiple options to perform the full encoding

Hilbert space for gauge theories



Physical states live in a tiny subspace of the natural Hilbert space we consider when writing down a lattice gauge theory

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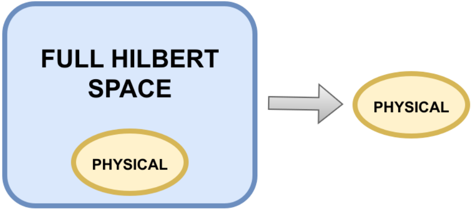

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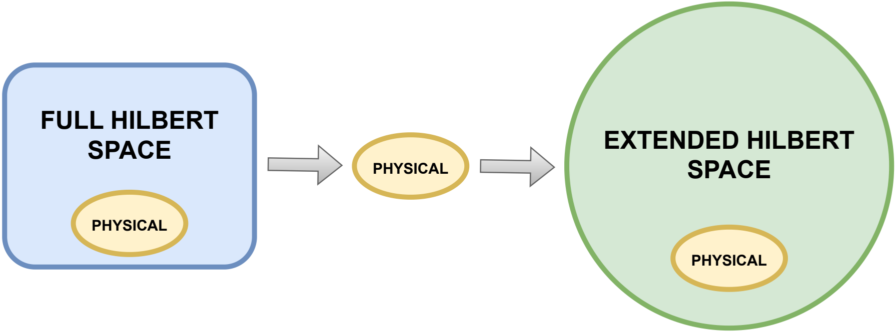

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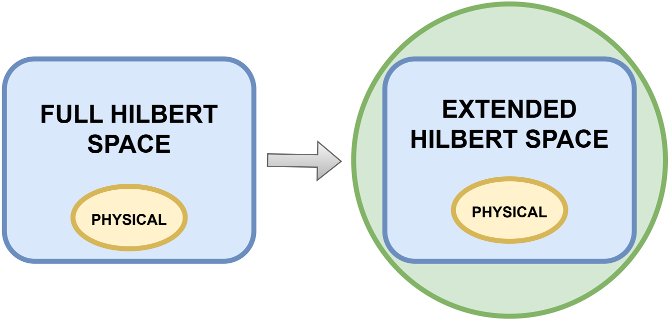

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Real time simulations with gauge covariant codes

For real time evolution we need access to exponentials of Pauli operators

$$H = \sum_k c_k P_k \quad \Rightarrow \quad e^{-itH} \approx \prod_k e^{-itc_k P_k}$$

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Real time simulations with gauge covariant codes

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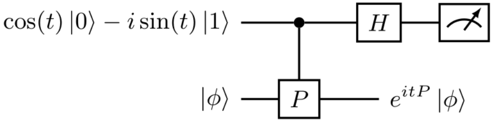

- controlled Pauli can be done transversally between the two codes
- arbitrary rotations need to be done only on the ancilla code
- with a second ancilla this can be done deterministically

Working directly in the physical space

Can we remove redundant degrees of freedom and work directly in the physical space? Often this leads to non-localities

- integrating out the links in a 1D theory leads to a theory of fermions with all-to-all "Coulomb" interactions (cost $O(N^2)$)
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When we use a quantum error correcting code the logical states can be manipulated using a complete set of logical operations

- logical states

$$|0\rangle_L = |000\rangle \quad |1\rangle_L = |111\rangle$$

- stabilizers

$$s_1 = Z_1 Z_2 \quad s_2 = Z_2 Z_3$$

- logical operations

$$\overline{X} = X_1 X_2 X_3 \quad \overline{Z} = Z_1 Z_2 Z_3$$

or the simpler versions

$$\overline{Z} = Z_1 = Z_2 = Z_3$$

Working directly in the physical space

Consider for instance a $\mathbb{Z}_2$ theory in one spatial dimension

$$H = m\sum_l (-1)^l \psi_l^\dagger \psi_l + \epsilon \sum_l (\psi_l^\dagger Q_l \psi_{l+1} + \psi_{l+1}^\dagger Q_l^\dagger \psi_l) + 2\lambda_E \sum_l P_l$$

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In terms of logical operations of our error correcting code we find

$$H = \frac{m}{2}\sum_l \overline{Z}_{l-1}\overline{Z}_l + \frac{\epsilon}{2}\sum_l \left(1 + \overline{Z}_{l-1}\overline{Z}_{l+1}\right)\overline{X}_l + 2\lambda_E \sum_l \overline{Z}_l$$

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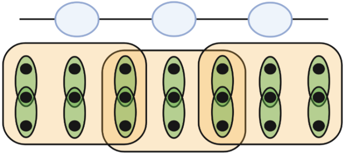
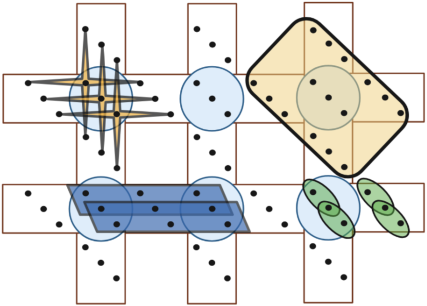
We just found a dual spin model to the $\mathbb{Z}_2$ gauge theory which is both local and free of any gauge redundancy.

Summary and future steps

Rajput, AR, Wiebe, npj QI (2023), Spagnoli, AR, Wiebe, arXiv:2405.19293 (2024)





- QEC codes exploit gauge symmetries: can we use this to design more efficient QEC codes when our goal is to simulate a gauge theory?
- Simple repetition code works for $\mathbb{Z}_2$ (also in higher spatial dimensions)
- Quantum Error Correction formulation allows to find dual theories
- Work in progress on encoding for links with larger local Hilbert space
- What other discrete symmetries can be encoded this way?

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