Linear Inequalities and Their Applications

Jensen’s Inequality: Theory and Real-World Applications

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Jensen's Inequality: From Theory to Practice

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Introduction

Agenda

What are Convex Functions?

Definition

A function $f : I \to \mathbb{R}$ is **convex** if for all $x, y \in I$ and $\lambda \in [0, 1]$:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Examples of Convex Functions

- $f(x) = x^2$ (Parabola)
- $f(x) = e^x$ (Exponential)
- $f(x) = |x|$ (Absolute value)
- $f(x) = -\log x$ (for $x > 0$)

Concave Functions

A function is **concave** if $-f$ is convex. Examples: $\log x$, $\sqrt{x}$, $\sin x$ on $[0, \pi]$.

Jensen's Inequality

Jensen's Inequality: Formal Statement

Theorem (Jensen's Inequality)
Let $f : I \to \mathbb{R}$ be a convex function on an interval I. Then for any $x_1, x_2, \ldots, x_n \in I$ and weights $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$, we have:

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

For Concave Functions
If f is concave, the inequality reverses:

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq \sum_{i=1}^{n} \lambda_i f(x_i)$$

Example

Let $f(x) = x^2$ (convex), $x_1 = 2$, $x_2 = 8$, and $\lambda_1 = \lambda_2 = 0.5$.

Left side:

$$f\left(\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 8\right) = f(5) = 5^2 = 25$$

Right side:

$$\frac{1}{2}f(2) + \frac{1}{2}f(8) = \frac{1}{2} \cdot 4 + \frac{1}{2} \cdot 64 = 2 + 32 = 34$$

$25 \leq 34$ Jensen's inequality holds!

Geometric Interpretation

The chord lies above the graph for convex functions. The weighted average of function values is greater than or equal to the function of the weighted average.

Mathematical Applications

Application 1: Arithmetic Mean - Geometric Mean Inequality

Theorem (AM-GM Inequality)
For positive real numbers $x_1, x_2, \ldots, x_n$:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}$$

Proof using Jensen.
Take $f(x) = -\log x$ (concave). By Jensen:

$$-\log\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq -\frac{\log x_1 + \cdots + \log x_n}{n}$$

Multiplying by -1 and exponentiating gives AM-GM. $\square$

Example
For $x = 4$, $y = 9$: $\frac{4+9}{2} = 6.5 \geq \sqrt{4 \cdot 9} = 6$

Application 2: Hölder's Inequality

Theorem (Hölder's Inequality)
For $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, and sequences $\{a_i\}, \{b_i\}$:

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q \right)^{1/q}$$

Proof Sketch using Jensen.
Use the convexity of e^x and appropriate substitutions. The weighted form of Jensen gives the key inequality. □

Special Case: Cauchy-Schwarz
When $p = q = 2$, we get Cauchy-Schwarz:

$$\left(\sum a_i b_i \right)^2 \leq \left(\sum a_i^2 \right) \left(\sum b_i^2 \right)$$

Application 3: Information Theory - Entropy

Shannon Entropy
For a probability distribution $\{p_i\}$:

$$H(p) = -\sum_{i=1}^{n} p_i \log p_i$$

Theorem (Maximum Entropy)
Among all discrete distributions on n outcomes, the uniform distribution maximizes entropy.

Proof.
Since $f(x) = x \log x$ is convex, by Jensen:

$$\frac{1}{n} \sum \log p_i \leq \log \left(\frac{1}{n} \sum p_i \right) = \log \left(\frac{1}{n} \right)$$

This implies $H(p) \leq \log n$, with equality when $p_i = \frac{1}{n}$. $\square$

Real-World Applications

Life Scenario 1: Investment Portfolio

The Problem
An investor has two investment options:

- Option A: Guaranteed 5% return
- Option B: 50% chance of 0%, 50% chance of 10% return

Expected return for both: $0.5 \times 0 + 0.5 \times 0.10 = 5\%$

Jensen's Insight
Utility of money $U(x) = \log x$ (concave due to risk aversion).

For Option A: $U(1.05) = \log(1.05) \approx 0.0488$

For Option B: $0.5U(1.0) + 0.5U(1.10) = 0.5 \times 0 + 0.5 \times 0.0953 \approx 0.0477$

$0.0488 > 0.0477$ $\boxed{?}$ Certainty equivalent is better!

Conclusion
Risk-averse investors prefer guaranteed returns due to concave utility.

Life Scenario 2: Exam Performance

The Scenario
A student takes 10 quizzes throughout a semester. Their scores:

70, 75, 80, 85, 90, 65, 95, 88, 92, 78

Average score: $\frac{70+75+80+85+90+65+95+88+92+78}{10} = 81.8$

Jensen's Application
Let $f(x)$ be the "grade transformation" (e.g., curved grading).

If f is convex (favoring high scores):

$$f(\text{average}) \leq \text{average of } f(\text{scores})$$

If f is concave (favoring consistency):

$$f(\text{average}) \geq \text{average of } f(\text{scores})$$

Example
Square transformation ($f(x) = x^2$, convex): Average of squares = 6741.4, Square of average = 6691.24 $\Rightarrow$ 6691.24 < 6741.4

Further Exploration

Advanced Topics

- **Generalized Jensen**: For infinite measures, conditional expectations
- **Operator Jensen**: For matrices and operators
- **Discrete vs Continuous**: Integral forms of Jensen
- **Probabilistic Jensen**: For random variables and martingales

Recommended Resources

- Hardy, Littlewood, Pólya - ”Inequalities”
- Steele - ”The Cauchy-Schwarz Master Class”
- Cover & Thomas - ”Elements of Information Theory”
- Boyd & Vandenberghe - ”Convex Optimization”

Questions?

Thank You!